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SPATIAL STATIONARITY, ERGODICITY, AND CLT FOR PARABOLIC ANDERSON MODEL WITH DELTA INITIAL CONDITION IN DIMENSION $d \geq 1$*

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Abstract. Suppose that $\{u(t,x)\}_{t>0,x\in\mathbb{R}^d}$ is the solution to a d-dimensional parabolic Anderson model with delta initial condition and driven by a Gaussian noise that is white in time and has a spatially homogeneous covariance given by a nonnegative-definite measure f which satisfies Dalang's condition. Let $\boldsymbol{p}_t(x) := (2\pi t)^{-d/2}\exp\{-\|x\|^2/(2t)\}$ denote the standard Gaussian heat kernel on $\mathbb{R}^d$. We prove that for all $t > 0$, the process $U(t) := \{u(t,x)/\boldsymbol{p}_t(x) : x \in \mathbb{R}^d\}$ is stationary using the Feynman–Kac formula and is ergodic under the additional condition $\hat{f}\{0\} = 0$, where $\hat{f}$ is the Fourier transform of f. Moreover, using the Malliavin–Stein method, we investigate various central limit theorems (CLTs) for $U(t)$ based on the quantitative analysis of f. In particular, when f is given by the Riesz kernel, i.e., $f(\mathrm{d}x) = \|x\|^{-\beta}\mathrm{d}x$, we obtain a multiple phase transition for the CLT for $U(t)$ from $\beta \in (0,1)$ to $\beta = 1$ to $\beta \in (1, d \wedge 2)$.

Key words. parabolic Anderson model, ergodicity, central limit theorem, stationarity, Malliavin calculus, Stein method

AMS subject classifications. 60H15, 60H07, 60F05

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1. Introduction.

Consider the following *parabolic Anderson model*:

$$\begin{cases}\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + u(t,x)\eta(t,x) & \text{for } (t,x) \in (0,\infty)\times\mathbb{R}^d\\ \text{subject to} \quad u(0) = \delta_0, & \end{cases} \tag{1.1}$$

where η denotes a centered, generalized Gaussian random field with

$$\mathrm{E}[\eta(t,x)\eta(s,y)] = \delta_0(t-s)f(x-y) \qquad [s,t \geq 0,\ x,y \in \mathbb{R}^d]$$

for a nonzero, nonnegative-definite, tempered Borel measure f on $\mathbb{R}^d$. As in Walsh [24], by a "solution" to (1.1) we mean a solution to the integral equation,

(1.2)
$$u(t,x) = \boldsymbol{p}_t(x) + \int_{(0,t)\times\mathbb{R}^d} \boldsymbol{p}_{t-s}(x-y)u(s,y)\,\eta(\mathrm{d}s\,\mathrm{d}y) \quad \text{a.s. for all } t>0 \text{ and } x\in\mathbb{R}^d,$$

where $\boldsymbol{p}_t(x)$ denotes the heat kernel; that is,

$$\boldsymbol{p}_t(x) = (2\pi t)^{-d/2}\mathrm{e}^{-\|x\|^2/(2t)} \qquad \text{for } t>0 \text{ and } x \in \mathbb{R}^d.$$

The existence and uniqueness problem for (1.1) and of its variations has been studied extensively by many authors [3,5,10]. In the present particular setting, it is easy to see

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that (1.2) has a (unique) predictable solution u iff there exists a (unique) predictable solution U to the following:

$$U(t\,,x) = 1 + \int_{(0,t)\times\mathbb{R}^d} \frac{\boldsymbol{p}_{t-s}(x-y)\boldsymbol{p}_s(y)}{\boldsymbol{p}_t(x)}\, U(s\,,y)\,\eta(\mathrm{d}s\,\mathrm{d}y), \tag{1.3}$$

where the pairing $(u\,,U)$ is given by

$$U(t\,,x) := \frac{u(t\,,x)}{\boldsymbol{p}_t(x)} \qquad \text{for } t>0 \text{ and } x\in\mathbb{R}^d. \tag{1.4}$$

It is possible to check directly that

$$\frac{\boldsymbol{p}_{t-s}(a)\boldsymbol{p}_s(b)}{\boldsymbol{p}_t(a+b)} = \boldsymbol{p}_{s(t-s)/t}\left(b - \frac{s}{t}(a+b)\right) \quad \text{for all } 0<s<t \text{ and } a,b\in\mathbb{R}^d. \tag{1.5}$$

In fact, both sides represent the probability density of $(X_{t-s}\,,X_s)$ where X denotes a Brownian bridge that emanates from zero and is conditioned to reach $a+b$ at time t.

With the preceding in mind, (1.3) can be recast as the following linear integral equation:

$$U(t\,,x) = 1 + \int_{(0,t)\times\mathbb{R}^d} \boldsymbol{p}_{s(t-s)/t}\left(y - \frac{s}{t}x\right) U(s\,,y)\,\eta(\mathrm{d}s\,\mathrm{d}y). \tag{1.6}$$

In order to present the basic existence, uniqueness result for (1.6), hence also (1.1), let us introduce the following function $\Upsilon : (0\,,\infty)\to(0\,,\infty]$:

$$\Upsilon(\beta) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \frac{\hat{f}(\mathrm{d}y)}{\beta + \|y\|^2} \qquad \text{for all } \beta>0, \tag{1.7}$$

where $\hat{f}$ denotes the Fourier transform of f.

Then we have the following result, which is a variation on a celebrated theorem of Dalang [10] to the linear setting of (1.1), started at initial measure δ_0.

THEOREM 1.1. *Suppose $\Upsilon(\beta)<\infty$ for one, hence all, $\beta>0$. Then, the integral equation* (1.6) *has a solution $U = \{U(t\,,x)\}_{t>0,x\in\mathbb{R}^d}$ that is a predictable random field. Moreover, U is the only predictable solution to* (1.6) *that satisfies the following for all $\varepsilon\in(0\,,1)$, $t>0$, and $k\geq 2$:*

$$\sup_{x\in\mathbb{R}^d} \mathrm{E}\left(|U(t\,,x)|^k\right) \leq \left(\frac{2}{\varepsilon}\right)^k \exp\left\{\frac{tk}{4}\Upsilon^{-1}\left(\frac{1-\varepsilon}{4z_k^2}\right)\right\} := c_{t,k}, \tag{1.8}$$

where z_k denotes the optimal constant in the Burkholder–Davis–Gundy (BDG) inequality for continuous $L^k(\Omega)$-martingales. Finally, $U(t) := \{U(t\,,x)\}_{x\in\mathbb{R}^d}$ is a stationary random field for every $t>0$, and $\lim_{t\to 0} U(t\,,x) = 1$ in $L^k(\Omega)$ for every $x\in\mathbb{R}^d$ and for all $k\geq 2$.

From now on, we always assume the following:

$$\Upsilon(1)<\infty \quad \text{and} \quad f(\mathbb{R}^d)>0. \tag{1.9}$$

Thanks to (1.4) and Theorem 1.1, the finiteness of $\Upsilon(1)$ implies that (1.1) has a predictable solution u that uniquely satisfies that $u(t\,,x) = (1+o(1))\boldsymbol{p}_t(x)$ in $L^k(\Omega)$ as $t\to 0$ for every $x\in\mathbb{R}^d$ and for all $k\geq 2$. Furthermore, the strict positivity of the total mass of f is assumed merely to avoid degeneracies in (1.1). Before we delve deeper into that topic, however, let us pause and make a few remarks.

Remark. When $d = 1$ and η denotes space-time white noise [$f = \delta_0$], the existence and uniqueness of u, hence also U, are especially well known; see, for example, [3]. In that case, the stationarity of $U(t)$ was proved first by Amir, Corwin, and Quastel [1], who used the fact that $\{\eta(t\,,x)\}_{t,x}$ has the same law as $\{\eta(t\,,x+at)\}_{t,x}$ for all $a \in \mathbb{R}$. Our proof of stationarity relies on the Feynman–Kac formula and works in the present much more general setting.

Remark. It is possible to prove, using ideas from Dalang [10], that the $(d+1)$-parameter random field U has a version that is continuous in $L^k(\Omega)$ for every $k \geq 2$. In turn, this fact and a suitable extension of Doob's separability theory (see Doob [13]) together show that U has a measurable version that solves (1.6). From now on, we always choose this version of U (and denote it also by U).

With (1.9) in place and the above remarks under way, we return to the topic at hand and present the first novel contribution of this paper.

THEOREM 1.2. *If $\hat{f}\{0\} = 0$, then $U(t)$ is ergodic for all $t > 0$.*

According to Theorem 1.1 in Chen et al. [6], the condition $\hat{f}\{0\} = 0$ determines the spatial ergodicity of the solution to (1.1) with flat initial condition. In the case of delta initial condition, $\hat{f}\{0\} = 0$ also implies the spatial ergodicity of U according to Theorem 1.2. For each fixed $N \geq \mathrm{e}$, we introduce the spatial average

$$\mathcal{S}_{N,t} = \frac{1}{N^d}\int_{[0,N]^d} [U(t,x) - 1]\,\mathrm{d}x. \tag{1.10}$$

Then, condition $\hat{f}\{0\} = 0$, Theorem 1.2, and the ergodic theorem together imply the following law of large numbers: For every $t > 0$,

$$\lim_{N\to\infty} \mathcal{S}_{N,t} = 0 \qquad \text{a.s. and in } L^k(\Omega) \text{ for all } k \geq 2.$$

The main result of this paper is a corresponding central limit theorem (CLT), which turns out to hold in the strongest possible sense of convergence in total variation. Let Z denote the standard Gaussian random variable, and recall that the total variation distance between random variables X and Y on $\mathbb{R}$ is defined as

$$d_{\mathrm{TV}}(X\,,Y) = \sup |\mathrm{P}(X \in B) - \mathrm{P}(Y \in B)|,$$

where the supremum is take over all Borel subsets B of $\mathbb{R}$.

Recall that the condition $f(\mathbb{R}^d) < \infty$ implies a CLT for the spatial averages of the solution to (1.1) with flat/constant initial data [7, Theorem 1.1]. The situation is much more involved in the present setting where the initial condition is a delta mass. In this setting, we first must analyze the asymptotic behavior of $\mathrm{Var}(\mathcal{S}_{N,t})$ under different assumptions on the covariance measure f. In the case of a flat initial condition, the condition $f(\mathbb{R}^d) < \infty$ implies that the variance of the spatial average of the solution is of the order N^{-d} as $N \to \infty$; see [7, Proposition 5.2]. By contrast, we will see in section 5 that, in the present setting, the normalization of $\mathrm{Var}(\mathcal{S}_{N,t})$ depends on the detailed structure of the covariance measure f, as well as on the spatial dimension d. Moreover, in order to prove the CLT, we appeal to the Malliavin–Stein method (see Proposition 2.1 below), from which we will deduce how the covariance measure f characterizes the CLT for the spatial average of $U(t)$ in various ways. In the case of a flat initial condition, it has been proved in [9, Theorem 2.4] that the convergence rate for CLT in terms of total variation is $N^{-d/2}$, while for the delta initial condition, the convergence rate for CLT is determined not only by spatial dimension d but also by the behavior of f.

We start by introducing the following quantity associated with f:

$$\mathcal{R}(f) := \frac{1}{\pi^d}\int_0^\infty \mathrm{d}s \int_{\mathbb{R}^d} \hat{f}(\mathrm{d}z) \prod_{j=1}^{d} \frac{1-\cos(sz_j)}{(sz_j)^2}. \tag{1.11}$$

The following theorem states that the finiteness of $\mathcal{R}(f)$ ensures the CLT for the spatial average of $U(t)$ and the convergence rate is $N^{-1/2}$ regardless of the spatial dimension d.

THEOREM 1.3. *If $\mathcal{R}(f) < \infty$, then for all fixed $t > 0$ there exists $C = C(t) > 0$ such that*

$$d_{\mathrm{TV}}\left(\frac{\mathcal{S}_{N,t}}{\sqrt{\mathrm{Var}(\mathcal{S}_{N,t})}}\,,\mathrm{Z}\right) \le \frac{C}{\sqrt{N}} \qquad \textit{for every } N \ge \mathrm{e}.$$

The asymptotic behavior of $\mathrm{Var}(\mathcal{S}_{N,t})$ will be discussed in detail in Theorem 5.1 below. It follows from that analysis and from Theorem 1.3 that if $\mathcal{R}(f) < \infty$, then

$$\sqrt{N}\mathcal{S}_{N,t} = \frac{1}{N^{d-(1/2)}}\int_{[0,N]^d} [U(t\,,x)-1]\,\mathrm{d}x \xrightarrow{\mathrm{d}} \mathrm{N}(0\,, t\mathcal{R}(f)) \qquad \text{as } N \to \infty,$$

where "$\xrightarrow{\mathrm{d}}$" denotes convergence in distribution.

We will see in Lemma 5.9 below that $\mathcal{R}(f) < \infty$ only if $d \ge 2$. Thus, the preceding CLT has no content in dimension one. When $d = 1$, we are able to derive a CLT under the additional constraint $f(\mathbb{R}) < \infty$. According to Theorem 1.1 in Chen et al. [7], the finiteness condition $f(\mathbb{R}) < \infty$ implies a CLT for the solution to (1.1) with flat initial condition. The same holds in the present setting of delta initial condition, except the rate is different (and so are many of the underlying arguments).

THEOREM 1.4 ($d = 1$). *If $f(\mathbb{R}) < \infty$ and $d = 1$, then for all fixed $t > 0$ there exists $C = C(t) > 0$ such that*

$$d_{\mathrm{TV}}\left(\frac{\mathcal{S}_{N,t}}{\sqrt{\mathrm{Var}(\mathcal{S}_{N,t})}}\,,\mathrm{Z}\right) \le C\sqrt{\frac{\log N}{N}} \qquad \textit{for all } N \ge \mathrm{e}.$$

In particular, Theorem 1.4 and Theorem 5.2 below together imply that if $d = 1$ and $f = a\delta_0$ for some $a > 0$, then

$$\sqrt{\frac{N}{\log N}}\,\mathcal{S}_{N,t} = \frac{1}{\sqrt{N\log N}}\int_0^N [U(t,x)-1]\,\mathrm{d}x \xrightarrow{\mathrm{d}} \mathrm{N}(0\,, 2tf(\mathbb{R})) = \mathrm{N}(0\,, 2ta) \text{ as } N \to \infty.$$

On the other hand, if the measure f is finite, as well as a Rajchman measure,[1] then

$$\sqrt{\frac{N}{\log N}}\,\mathcal{S}_{N,t} = \frac{1}{\sqrt{N\log N}}\int_0^N [U(t\,,x)-1]\,\mathrm{d}x \xrightarrow{\mathrm{d}} \mathrm{N}(0\,, tf(\mathbb{R})) \quad \text{as } N \to \infty.$$

The above results give a more or less comprehensive idea of the CLT for $U(t)$ when $\mathcal{R}(f) < \infty$, especially when the measure f is in addition finite. By contrast

[1]We recall that a finite measure f is *Rajchman* if its Fourier transform $\mathbb{R}^d \ni x \mapsto \hat{f}(x) := \int_{\mathbb{R}^d} \mathrm{e}^{ix\cdot y} f(\mathrm{d}y)$ vanishes at infinity; that is, $\lim_{\|x\|\to\infty} \hat{f}(x) = 0$. Lyons [19] discusses a survey of the rich subject of Rajchman measures. Note that, in the present setting, $\hat{f} : \mathbb{R}^d \to \mathbb{R}$ is a nonnegative, nonnegative-definite, uniformly bounded, and continuous function.

with this case, there does not seem to be a canonical description of a CLT when $\mathcal{R}(f) = \infty$. This condition occurs for any ambient dimension d, for example, when f is given by a Riesz kernel; see Remark 5.11 below. In the following, we will present the CLT specifically in the case that f is given by a Riesz kernel that satisfies Dalang's condition, $\Upsilon(1) < \infty$, that is, when $f(\mathrm{d}x) = \|x\|^{-\beta}\,\mathrm{d}x$ where $0 < \beta < 2 \wedge d$. In contrast with what happens in the case that the initial condition is flat (see Huang et al. [17]), the CLT for U undergoes a multiple phase transition from $\beta \in (0\,,1)$ to $\beta = 1$ to $\beta \in (1\,, d \wedge 2)$.

THEOREM 1.5. *If $f(\mathrm{d}x) = \|x\|^{-\beta}\mathrm{d}x$ for some $\beta \in (0\,, d \wedge 2)$, then for all fixed $t > 0$ there exists $C = C(t) > 0$ such that for all $N \geq \mathrm{e}$,*

$$d_{\mathrm{TV}}\left(\frac{\mathcal{S}_{N,t}}{\sqrt{\mathrm{Var}(\mathcal{S}_{N,t})}}\,, \mathrm{Z}\right) \leq \begin{cases} CN^{-\beta/2} & \text{if } \beta \in (0\,,1),\\ C\sqrt{\log(N)/N} & \text{if } \beta = 1,\\ CN^{-(2-\beta)/2} & \text{if } \beta \in (1\,,2). \end{cases}$$

As a consequence of Theorem 1.5 and Theorem 5.4 below, we obtain the following CLTs:

(A) if $0 < \beta < 1$, then

$$N^{\beta/2}\mathcal{S}_{N,t} = \frac{1}{N^{d-(\beta/2)}}\int_{[0,N]^d} [U(t\,,x) - 1]\,\mathrm{d}x \xrightarrow{\mathrm{d}} \mathrm{N}(0\,, t\sigma_{0,\beta,d}) \quad \text{as } N \to \infty;$$

(B) if $\beta = 1$, then

$$\sqrt{\frac{N}{\log N}}\,\mathcal{S}_{N,t} = \frac{1}{N^{d-(1/2)}\sqrt{\log N}}\int_{[0,N]^d} [U(t\,,x) - 1]\,\mathrm{d}x \xrightarrow{\mathrm{d}} \mathrm{N}(0\,, t\sigma_{1,\beta,d}) \quad \text{as } N \to \infty; \text{ and}$$

(C) if $1 < \beta < 2 \wedge d$, then

$$N^{1-(\beta/2)}\mathcal{S}_{N,t} = \frac{1}{N^{d-1+(\beta/2)}}\int_{[0,N]^d} [U(t\,,x) - 1]\,\mathrm{d}x \xrightarrow{\mathrm{d}} \mathrm{N}(0\,, t^{2-\beta}\sigma_{2,\beta,d}) \quad \text{as } N \to \infty,$$

where $\sigma_{0,\beta,d}$, $\sigma_{1,\beta,d}$, and $\sigma_{2,\beta,d}$ are nondegenerate and defined explicitly in Theorem 5.4.

The logarithmic correction that appears in Theorems 1.4 and 1.5 [$\beta = 1$] is related to the transition functions of the Brownian bridge; see (1.5). Indeed, the conditional probability density $\boldsymbol{p}_{s(t-s)/t}(sx/t)$ becomes t/s after a change of variable in x. The resulting singularity at $s = 0$ ultimately give rises to the $\log N$ factor in Theorems 1.4 and 1.5.

Remark. The convergence rates for the total variation distance in Theorems 1.3, 1.4, and 1.5 are natural. Indeed, one can observe that in each case the convergence rate for the total variation distance is of the same order as $\sqrt{\mathrm{Var}(\mathcal{S}_{N,t})}$ as $N \to \infty$; see Theorems 5.1, 5.2, and 5.4. A similar phenomenon can be observed in the context of spatial CLT for other related SPDEs [8, 9, 12, 15, 16, 17, 23]. See [22] for recent advances on the parabolic Anderson model driven by a Gaussian noise that is colored in both its space and time variables.

Remark. One can follow the method in [8] to prove the functional CLT in time corresponding to the CLTs below Theorems 1.3, 1.4, and 1.5, respectively. For instance, one can use the argument in [8, Proposition 4.1] to compute the covariance of the limit Gaussian process and then prove the convergence of finite dimensional distributions and tightness. We leave these for interested readers.

The organization of this paper is as follows. We establish the well-posedness and spatial stationarity for the solution to (1.6) in Theorem 1.1 in section 3. The ergodicity property in Theorem 1.2 is proved in section 4. Section 5 is devoted to analyzing the asymptotic behavior of the variance of spatial average. Moreover, we present the estimates on total variation distance in Theorems 1.3, 1.4, and 1.5 in section 6. And the last section is an appendix that contains a few technical lemmas that are used throughout the paper.

Let us conclude the introduction by setting forth some notation that will be used throughout. We write "$g_1(x) \lesssim g_2(x)$ for all $x \in X$" when there exists a real number L such that $g_1(x) \le Lg_2(x)$ for all $x \in X$. Alternatively, we might write "$g_2(x) \gtrsim g_1(x)$ for all $x \in X$." By "$g_1(x) \asymp g_2(x)$ for all $x \in X$" we mean that $g_1(x) \lesssim g_2(x)$ for all $x \in X$ and $g_2(x) \lesssim g_1(x)$ for all $x \in X$. Finally, "$g_1(x) \propto g_2(x)$ for all $x \in X$" means that there exists a real number L such that $g_1(x) = Lg_2(x)$ for all $x \in X$. For every $Z \in L^k(\Omega)$, we write $\|Z\|_k$ instead of the more cumbersome $\|Z\|_{L^k(\Omega)}$.

2. Preliminaries.

2.1. The BDG inequality. Let us collect a few facts about the optimal constants $\{z_k\}_{k\ge 2}$ of the BDG inequality.

First, recall from the BDG inequality that for every continuous $L^2(\Omega)$-martingale $\{M_t\}_{t\ge 0}$,

$$\mathrm{E}\left(|M_t|^k\right) \le z_k^k \mathrm{E}\left(\langle M\rangle_t^{k/2}\right) \qquad \text{for all } t \ge 0 \text{ and } k \ge 2.$$

Davis [11] has shown that every z_k is the largest positive root of a certain special function, in particular, that z_k is the largest positive root of the monic Hermite polynomial He_k when k is an even integer. These remarks and the appendix of Carlen and Krée [2] together imply the following:

$$z_2 = 1, \quad z_4 = \sqrt{3+\sqrt{6}} \approx 2.334, \quad \text{and} \quad \sup_{k\ge 2} \frac{z_k}{\sqrt{k}} = \lim_{k\to\infty} \frac{z_k}{\sqrt{k}} = 2. \tag{2.1}$$

Moreover, the special case where the martingale M is Brownian motion shows us that

$$z_k \ge \|\mathrm{N}(0\,,1)\|_k = \sqrt{2}\left[\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{k+1}{2}\right)\right]^{1/k} \qquad \text{for all } k \ge 2. \tag{2.2}$$

Therefore, we learn from the Stirling formula that z_k is bounded from above and from below by nondegenerate multiples of $\sqrt{k}$, uniformly for all $k \ge 2$.

2.2. The Clark–Ocone formula. Define $\mathcal{H}_0$ to be the reproducing kernel Hilbert space, spanned by all real-valued functions on $\mathbb{R}^d$, that corresponds to the inner product $\langle \phi\,,\psi\rangle_{\mathcal{H}_0} := \langle \phi\,,\psi * f\rangle_{L^2(\mathbb{R}^d)}$, and set $\mathcal{H} := L^2(\mathbb{R}_+ \times \mathcal{H}_0)$. The Gaussian family $\{W(h)\}_{h\in\mathcal{H}}$ formed by the Wiener integrals

$$W(h) = \int_{\mathbb{R}_+\times\mathbb{R}^d} h(s\,,x)\,\eta(\mathrm{d}s\,\mathrm{d}x) \qquad [h \in \mathcal{H}]$$

defines an *isonormal Gaussian process* on the Hilbert space $\mathcal{H}$. In this framework we can develop the Malliavin calculus (see, for instance, [20]). We denote by D the Malliavin derivative operator and by δ the corresponding divergence operator whose domain in $L^2(\Omega)$ is denoted by $\mathrm{Dom}[\delta]$.

Let $\{\mathcal{F}_s\}_{s\ge 0}$ denote the filtration generated by the infinite dimensional white noise $t \mapsto \eta(t)$; that is, $\mathcal{F}_t$ is the filtration generated by all Wiener integrals of the form $\int_{(0,t)\times\mathbb{R}^d} \phi\,\mathrm{d}\eta$ as ϕ ranges over all test functions of rapid decrease (which are easily seen to be dense in $\mathcal{H}$). A basic idea used in this paper is the Clark–Ocone formula (see [6, Proposition 6.3]),

$$F = \mathrm{E}[F] + \int_{\mathbb{R}_+\times\mathbb{R}^d} \mathrm{E}\left[D_{s,y}F \mid \mathcal{F}_s\right] \eta(\mathrm{d}s\,\mathrm{d}z), \tag{2.3}$$

valid a.s. for every random variable F in the Gaussian Sobolev space $\mathbb{D}^{1,2}$. Using Jensen's inequality for conditional expectation, this equality leads immediately to the Poincaré-type inequality

$$|\operatorname{Cov}(F\,,G)| \le \int_0^\infty \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^d} f(\mathrm{d}y')\ \ \|D_{s,y}F\|_2\, \|D_{s,y'+y}G\|_2 \tag{2.4}$$

for $F, G \in \mathbb{D}^{1,2}$ provided that DF and DG are real-valued random variables.

2.3. The Malliavin–Stein method. Recall that the total variation distance between two Borel probability measures μ and ν on $\mathbb{R}$ is defined as

$$d_{\mathrm{TV}}(\mu\,,\nu) = \sup |\mu(B) - \nu(B)|,$$

where the supremum is taken over all Borel subsets B of $\mathbb{R}$. We might abuse notation and write $d_{\mathrm{TV}}(F\,,G)$, $d_{\mathrm{TV}}(F\,,\nu)$, or $d_{\mathrm{TV}}(\mu\,,G)$ instead of $d_{\mathrm{TV}}(\mu\,,\nu)$ whenever the laws of F and G are respectively μ and ν.

A combination of Malliavin calculus and Stein's method for normal approximations leads to the following bound on the total variation distance (see [21, Theorem 8.2.1]).

Proposition 2.1. *Suppose $F \in \mathbb{D}^{1,2}$ satisfies $\mathrm{E}[F^2] = 1$ and $F = \delta(v)$ for some element v in the domain in $L^2(\Omega)$ of the divergence operator δ. Then,*

$$d_{\mathrm{TV}}(F\,, \mathrm{N}(0\,,1)) \le 2\sqrt{\operatorname{Var}\left(\langle DF\,, v\rangle_{\mathcal{H}}\right)}. \tag{2.5}$$

3. Existence, uniqueness, and stationarity: Proof of Theorem 1.1. The proof of Theorem 1.1 follows a route that is nowadays standard. Therefore, we sketch the bulk argument, enough to make sure that the numerology of (1.8) is explained in sufficient detail. Also, the proof does require one technical lemma that we state and prove next. The following identity will be used several times later on:

$$\left(\boldsymbol{p}_r * f\right)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{-r\|y\|^2/2}\mathrm{e}^{ix\cdot y}\hat{f}(\mathrm{d}y) \quad \text{for all } r>0 \text{ and } x\in\mathbb{R}^d. \tag{3.1}$$

Since $\boldsymbol{p}_r$ is a test function of rapid decrease for every $r>0$, the above identity follows from the very definition of $\hat{f}$.

Recall the function Υ defined in (1.7).

Lemma 3.1. *$\int_0^t \exp\{-\beta\{s\wedge(t-s)\}\}(\boldsymbol{p}_{2s(t-s)/t} * f)(0)\,\mathrm{d}s \le 4\Upsilon(2\beta)$ for every $t,\beta>0$.*

Proof. We apply the identity (3.1) with $r = 2s(t-s)/t$ in order to find that

$$\begin{aligned}\left(\boldsymbol{p}_{2s(t-s)/t} * f\right)(0) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{-s(t-s)\|y\|^2/t}\hat{f}(\mathrm{d}y)\\ &\le \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \exp\left(-\frac{s\wedge(t-s)}{2}\|y\|^2\right)\hat{f}(\mathrm{d}y),\end{aligned}$$

using the elementary fact that $s(t-s)/t \geq \frac{1}{2}[s \wedge (t-s)]$. Integration and symmetry together imply

$$\int_0^t \mathrm{e}^{-\beta\{s\wedge(t-s)\}} \left(\boldsymbol{p}_{2s(t-s)/t} * f\right)(0)\,\mathrm{d}s \leq \frac{2}{(2\pi)^d}\int_0^{t/2} \mathrm{e}^{-\beta s}\,\mathrm{d}s \int_{\mathbb{R}^d} \mathrm{e}^{-s\|y\|^2/2}\hat{f}(\mathrm{d}y).$$

The bound $\int_0^{t/2}(\cdots) \leq \int_0^{\infty}(\cdots)$ yields the lemma. ∎

With Lemma 3.1 under way, we can start the proof of Theorem 1.1.

Proof of Theorem 1.1 *(part* 1*): Existence and uniqueness.* Throughout the proof, define

$$\beta_{\varepsilon,k} := \frac{1}{2}\Upsilon^{-1}\left(\frac{1-\varepsilon}{4z_k^2}\right). \tag{3.2}$$

We begin by proving existence and uniqueness.

The proof of existence and uniqueness works by Picard iteration, as is customary, and uses ideas from Foondun and Khoshnevisan [14] in order to establish the moment bound (1.8) and uniqueness.

Define for all $t > 0$ and $x \in \mathbb{R}^d$, $U_0(t\,,x) := 1$ and

$$U_{n+1}(t\,,x) = 1 + \int_{(0,t)\times\mathbb{R}^d} \boldsymbol{p}_{s(t-s)/t}\left(y - \frac{s}{t}x\right) U_n(s\,,y)\,\eta(\mathrm{d}s\,\mathrm{d}y), \tag{3.3}$$

valid for every $n \in \mathbb{Z}_+$. Define, for all $t > 0$, $n \in \mathbb{N}$, and $x \in \mathbb{R}^d$,

$$\mathcal{D}_n(t\,,x) := U_n(t\,,x) - U_{n-1}(t\,,x) \quad \text{and} \quad \mathcal{E}_n(t) := \sup_{a\in\mathbb{R}^d} \|\mathcal{D}_n(t\,,a)\|_k^2.$$

We first observe that, because of the semigroup property of the heat kernel,

$$\begin{aligned}\|\mathcal{D}_1(t\,,x)\|_k^2 &= \left\|\int_{(0,t)\times\mathbb{R}^d} \boldsymbol{p}_{s(t-s)/t}\left(y - \frac{s}{t}x\right)\eta(\mathrm{d}s\,\mathrm{d}y)\right\|_k^2\\ &\leq z_k^2\int_0^t \mathrm{d}s\int_{\mathbb{R}^d}\mathrm{d}y\int_{\mathbb{R}^d} f(\mathrm{d}y')\ \boldsymbol{p}_{s(t-s)/t}(y)\boldsymbol{p}_{s(t-s)/t}(y'+y)\\ &= z_k^2\int_0^t \mathrm{d}s\int_{\mathbb{R}^d} f(\mathrm{d}w)\ \boldsymbol{p}_{2s(t-s)/t}(w).\end{aligned}$$

Therefore, we may appeal to Lemma 3.1 to find that for every $\beta > 0$,

$$\begin{aligned}\|\mathcal{D}_1(t\,,x)\|_k^2 &\leq z_k^2\mathrm{e}^{\beta t}\int_0^t \mathrm{e}^{-\beta\{s\wedge(t-s)\}}\mathrm{d}s\int_{\mathbb{R}^d} f(\mathrm{d}w)\ \boldsymbol{p}_{2s(t-s)/t}(w)\\ &\leq 4z_k^2\mathrm{e}^{\beta t}\Upsilon(2\beta).\end{aligned} \tag{3.4}$$

Next, we might observe that

$$
\begin{aligned}
&\|\mathcal{D}_{n+1}(t\,,x)\|_k^2\\
&\quad= \left\| \int_{(0,t)\times\mathbb{R}^d} \boldsymbol{p}_{s(t-s)/t}\left(y-\frac{s}{t}x\right)\mathcal{D}_n(s\,,y)\,\eta(\mathrm{d}s\,\mathrm{d}y)\right\|_k^2\\
&\quad\le z_k^2\int_0^t \mathrm{d}s\int_{\mathbb{R}^d}\mathrm{d}y\int_{\mathbb{R}^d} f(\mathrm{d}w)\ \boldsymbol{p}_{s(t-s)/t}\left(y-\frac{s}{t}x\right)\boldsymbol{p}_{s(t-s)/t}\\
&\qquad\times\left(w+y-\frac{s}{t}x\right)\|\mathcal{D}_n(s\,,y)\mathcal{D}_n(s\,,w+y)\|_{k/2}\\
&\quad\le z_k^2\int_0^t[\mathcal{E}_n(s)]^2\,\mathrm{d}s\int_{\mathbb{R}^d}\mathrm{d}y\int_{\mathbb{R}^d} f(\mathrm{d}w)\ \boldsymbol{p}_{s(t-s)/t}\left(y-\frac{s}{t}x\right)\boldsymbol{p}_{s(t-s)/t}\left(w+y-\frac{s}{t}x\right)\\
&\quad= z_k^2\int_0^t \mathcal{E}_n(s)\,\mathrm{d}s\int_{\mathbb{R}^d} f(\mathrm{d}w)\ \boldsymbol{p}_{2s(t-s)/t}(w).
\end{aligned}
$$

Since the right-hand side does not depend on x, we may optimize to find that

$$
\begin{aligned}
\mathrm{e}^{-\beta t}\mathcal{E}_{n+1}(t) &\le z_k^2\mathrm{e}^{-\beta t}\int_0^t \mathcal{E}_n(s)\,\mathrm{d}s\int_{\mathbb{R}^d} f(\mathrm{d}w)\,\boldsymbol{p}_{2s(t-s)/t}(w)\\
&= z_k^2\int_0^t \mathrm{e}^{-\beta\{s\vee(t-s)\}}\mathcal{E}_n(s)\mathrm{e}^{-\beta\{s\wedge(t-s)\}}\,\mathrm{d}s\int_{\mathbb{R}^d} f(\mathrm{d}w)\,\boldsymbol{p}_{2s(t-s)/t}(w)\\
&\le z_k^2\int_0^t \mathrm{e}^{-\beta s}\mathcal{E}_n(s)\mathrm{e}^{-\beta\{s\wedge(t-s)\}}\,\mathrm{d}s\int_{\mathbb{R}^d} f(\mathrm{d}w)\,\boldsymbol{p}_{2s(t-s)/t}(w).
\end{aligned}
$$

In particular, set

$$\mathcal{F}_n(t\,,\beta) := \sup_{s\in(0,t]}\left[\mathrm{e}^{-\beta s}\mathcal{E}_n(s)\right] \quad \text{for all } n\in\mathbb{N} \text{ and } t,\beta>0$$

in order to deduce from Lemma 3.1 that $\mathcal{F}_{n+1}(t\,,\beta)\le 4z_k^2\Upsilon(2\beta)\mathcal{F}_n(t\,,\beta)$. Plug in $\beta=\beta_{\varepsilon,k}$, defined in (3.2), to find inductively that

$$\mathcal{F}_{n+1}(t\,,\beta_{\varepsilon,k})\le(1-\varepsilon)\mathcal{F}_n(t\,,\beta_{\varepsilon,k})\le\cdots\le(1-\varepsilon)^n\mathcal{F}_1(t\,,\beta_{\varepsilon,k}).$$

Now, we can read off from (3.4) that $\mathcal{F}_1(t\,,\beta_{\varepsilon,k}) = \sup_{s\in(0,t]}[\exp\{-\beta_{\varepsilon,k}s\}\mathcal{E}_1(s)] \le 4z_k^2\Upsilon(2\beta_{\varepsilon,k}) = 1-\varepsilon$. This yields $\mathcal{F}_{n+1}(t\,,\beta_{\varepsilon,k})\le(1-\varepsilon)^{n+1}$, and hence

$$\sup_{x\in\mathbb{R}^d}\|U_{n+1}(t\,,x)-U_n(t\,,x)\|_k^2\le(1-\varepsilon)^{n+1}\mathrm{e}^{\beta_{\varepsilon,k}t} \qquad \text{for all } t>0 \text{ and } n\in\mathbb{Z}_+.$$

At this point, standard arguments imply that $U(t\,,x):=\lim_{n\to\infty}U_n(t\,,x)$ exists in $L^k(\Omega)$ for every $k\ge 2$ and solves (1.1). Moreover,

$$
\begin{aligned}
\|U(t\,,x)\|_k &\le \|U_0(t\,,x)\|_k+\sum_{n=0}^\infty\|U_{n+1}(t\,,x)-U_n(t\,,x)\|_k\le 1+\mathrm{e}^{\beta_{\varepsilon,k}t/2}\sum_{m=1}^\infty(1-\varepsilon)^{m/2}\\
&\le\frac{\exp\left(\beta_{\varepsilon,k}t/2\right)}{1-\sqrt{1-\varepsilon}}.
\end{aligned}
$$

Since $1-\sqrt{1-\varepsilon}\ge\varepsilon/2$, this proves (1.8). ∎

Proof of Theorem 1.1 *(part* 2*): Stationarity.* For every $\varepsilon > 0$ we define a new Gaussian noise η^ε via its Wiener integrals,

$$\int_{(0,t)\times\mathbb{R}^d} \varphi(y)\, \eta^\varepsilon(\mathrm{d}s\,\mathrm{d}y) := \int_{(0,t)\times\mathbb{R}^d} (\varphi * \boldsymbol{p}_\varepsilon)(y)\, \eta(\mathrm{d}s\,\mathrm{d}y) \qquad \text{for all } t > 0 \text{ and } \varphi \in \mathcal{H}_0.$$

Because of the semigroup property of the heat kernel, η^ε is a generalized Gaussian random field with

$$\mathrm{Cov}\left[\eta^\varepsilon(t\,,x)\ ,\ \eta^\varepsilon(s\,,y)\right] = \delta_0(t-s) f_\varepsilon(x-y), \quad \text{where } f_\varepsilon := \boldsymbol{p}_{2\varepsilon} * f.$$

As is customary in distribution theory, the rapidly decreasing test function f_ε is identified with a positive-definite tempered measure (also denoted by f_ε) that, among many other things, satisfies (1.9). In fact, the total mass of the measure f_ε is merely the total integral of the function f_ε, which is $f(\mathbb{R}^d)$. Let Υ_ε be defined as in (1.7), but with f replaced by f_ε, in order to see immediately that $\Upsilon_\varepsilon \le \Upsilon$ pointwise. Thus, the already-proved portion of Theorem 1.1 applies to show that the stochastic PDE

$$\begin{aligned} &\partial_t u^\varepsilon = \tfrac{1}{2}\Delta u^\varepsilon + u^\varepsilon \eta^\varepsilon \qquad \text{on } (0\,,\infty)\times\mathbb{R}^d,\\ &\text{subject to } u^\varepsilon(0) = \delta_0 \qquad \text{on } \mathbb{R}^d, \end{aligned}$$

has a predictable random-field solution u^ε that is unique subject to

$$\sup_{(t,x,\varepsilon)\in(0,T)\times\mathbb{R}^d\times(0,\infty)} \|u^\varepsilon(t\,,x)/\boldsymbol{p}_t(x)\|_k < \infty \qquad \text{for every } T > 0 \text{ and } k \ge 2.$$

Let us expand on this a little as follows: For every $z \in \mathbb{R}^d$, consider the SPDE

$$\begin{cases} \partial_t u^\varepsilon(t\,,x;\,z) = \tfrac{1}{2}\Delta_x u^\varepsilon(t\,,x\,;z) + u^\varepsilon(t\,,x\,;z)\eta^\varepsilon(t\,,x) \qquad \text{on } (0\,,\infty)\times\mathbb{R}^d,\\ \text{subject to } u^\varepsilon(0\,,\bullet\,;z) = \delta_z(\bullet) \qquad \text{on } \mathbb{R}^d. \end{cases}$$

Then we can apply the same argument that was used in the already-proved portion of Theorem 1.1 in order to establish the existence of a random-field solution $u^\varepsilon(\bullet\,,\bullet\,;z)$ to the preceding, one for every $z \in \mathbb{R}^d$, that is unique among all that satisfy

$$L_{T,k} := \sup_{(t,x,z,\varepsilon)\in(0,T)\times\mathbb{R}^d\times\mathbb{R}^d\times(0,\infty)} \left\| \frac{u^\varepsilon(t\,,x\,;z)}{\boldsymbol{p}_t(x-z)} \right\|_k < \infty \qquad \text{for every } T > 0 \text{ and } k \ge 2.$$

We remark that $u^\varepsilon(t\,,x\,;0) = u^\varepsilon(t\,,x)$ for all $t, \varepsilon > 0$ and $x \in \mathbb{R}^d$.

Let $U^\varepsilon(t\,,x) := u^\varepsilon(t\,,x)/\boldsymbol{p}_t(x)$ and $U^\varepsilon(t\,,x\,;z) := u^\varepsilon(t\,,x\,;z)/\boldsymbol{p}_t(x-z)$ for all $t > 0$, $\varepsilon \in (0\,,1)$, and $x, z \in \mathbb{R}^d$; confer with (1.4). The method of Dalang [10] can be used to show also that $(t\,,x\,,z) \mapsto U^\varepsilon(t\,,x\,;z)$—hence also $(t\,,x\,,z) \mapsto u^\varepsilon(t\,,x\,;z)$—is continuous in $L^k(\Omega)$ for every $k \ge 2$ and $\varepsilon > 0$. We skip the details and mention only that, in particular, u^ε and U^ε both have Lebesgue-measurable versions for every $\varepsilon > 0$, which we always use.

Choose and fix an arbitrary nonrandom function $v_0 \in L^\infty(\mathbb{R}^d)$ to see from linearity that

$$v^\varepsilon(t\,,x) := \int_{\mathbb{R}^d} u^\varepsilon(t\,,x\,;z) v_0(z)\,\mathrm{d}z \qquad [t > 0,\ x \in \mathbb{R}^d] \tag{3.5}$$

is the unique predictable solution to the SPDE

$$\begin{cases} \partial_t v^\varepsilon(t\,,x) = \tfrac{1}{2}\Delta v^\varepsilon(t\,,x) + v^\varepsilon(t\,,x)\eta^\varepsilon(t\,,x) \qquad \text{for } (t\,,x) \in (0\,,\infty)\times\mathbb{R}^d,\\ \text{subject to } v^\varepsilon(0) = v_0 \qquad \text{on } \mathbb{R}^d, \end{cases}$$

that satisfies

$$L'_{T,k,\varepsilon} := \sup_{(t,x)\in(0,T)\times\mathbb{R}^d} \|v^\varepsilon(t\,,x)\|_k < \infty \qquad \text{for every } T,\varepsilon > 0 \text{ and } k \geq 2.$$

Recall that v^ε has the following mild formulation:

$$\begin{aligned} v^\varepsilon(t\,,x) &= (\boldsymbol{p}_t * v_0)(x) + \int_{(0,t)\times\mathbb{R}^d} \boldsymbol{p}_{t-s}(y-x) v^\varepsilon(s\,,y)\,\eta^\varepsilon(\mathrm{d}s\,\mathrm{d}y) \\ &= (\boldsymbol{p}_t * v_0)(x) + \int_{(0,t)\times\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \boldsymbol{p}_{t-s}(y-x) v^\varepsilon(s\,,y) \boldsymbol{p}_\varepsilon(y-z)\,\mathrm{d}y \right) \eta(\mathrm{d}s\,\mathrm{d}z), \end{aligned}$$

thanks to a stochastic Fubini argument, which we skip. The spatial correlation function f_ε of η^ε clearly is in $\mathscr{S}(\mathbb{R}^d)$ and hence is bounded; in fact,

$$f_\varepsilon(x) = (\boldsymbol{p}_{2\varepsilon} * f)(x) \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon\|y\|^2} \hat{f}(\mathrm{d}y) < \infty \text{ for all } \varepsilon \in (0\,,1) \text{ and } x \in \mathbb{R}^d\text{; see (3.1).}$$

Let B denote a standard Brownian motion that is independent of η, and let E_B and E_η denote, respectively, the conditional expectation operators given B and η. According to general theory (see Hu and Nualart [18, Proposition 5.2]), v^ε has a Feynman–Kac representation

$$v^\varepsilon(t\,,x) = \mathrm{E}_B \left[v_0(B_t + x) \exp\left(\int_{(0,t)\times\mathbb{R}^d} \boldsymbol{p}_\varepsilon\left(y - x - B_{t-s}\right) \eta(\mathrm{d}s\,\mathrm{d}y) - \tfrac{1}{2} t f_\varepsilon(0) \right) \right].$$

Define

$$B_s^{t,w} := B_s - \frac{s}{t}(B_t - w) \qquad \text{for all } s \in [0\,,t] \text{ and } t > 0 \text{ and } w \in \mathbb{R}^d.$$

We can see that $B^{t,w}$ is a Brownian bridge on $[0\,,t]$, conditioned to go from the space-time point $(0\,,0)$ to the space-time point $(t\,,w)$. And in fact,

$$\boldsymbol{p}_{s(t-s)/t}\left(y - \frac{s}{t}x\right) \text{ of (1.6) is the probability density of } B_s^{t,x} \text{ at } y.$$

Because $\{B_s^{t,w}\}_{s\in[0,t]}$ is independent of B_t, we may disintegrate and write

$$v^\varepsilon(t\,,x) = \int_{\mathbb{R}^d} \boldsymbol{p}_t(z-x) \mathrm{E}_B \left[\exp\left(\int_{(0,t)\times\mathbb{R}^d} \boldsymbol{p}_\varepsilon\big(y - x - B_{t-s}^{t,z}\big) \eta(\mathrm{d}s\,\mathrm{d}y) - \tfrac{1}{2} t f_\varepsilon(0) \right) \right] v_0(z)\,\mathrm{d}z.$$

We compare the above to (3.5) in order to deduce from the fact that $v_0 \in L^\infty(\mathbb{R}^d)$ is arbitrary that the following is a version of $u^\varepsilon(t\,,x\,;z)$:

$$u^\varepsilon(t\,,x\,;z) = \boldsymbol{p}_t(z-x) \mathrm{E}_B \left[\exp\left(\int_{(0,t)\times\mathbb{R}^d} \boldsymbol{p}_\varepsilon\big(y - x - B_{t-s}^{t,z}\big) \eta(\mathrm{d}s\,\mathrm{d}y) - \tfrac{1}{2} t f_\varepsilon(0) \right) \right].$$

We adopt this version of $u^\varepsilon(t\,,x\,;z)$ (rather than the old ones). Set $z = 0$ to see that we have adopted the following versions of $u^\varepsilon(t\,,x)$ and $U^\varepsilon(t\,,x)$:

$$\begin{aligned} u^\varepsilon(t\,,x) &= \boldsymbol{p}_t(x) \mathrm{E}_B \left[\exp\left(\int_{(0,t)\times\mathbb{R}^d} \boldsymbol{p}_\varepsilon\big(y - x - B_{t-s}^{t,0}\big) \eta(\mathrm{d}s\,\mathrm{d}y) - \tfrac{1}{2} t f_\varepsilon(0) \right) \right], \text{ and hence} \\ U^\varepsilon(t\,,x) &= \mathrm{E}_B \left[\exp\left(\int_{(0,t)\times\mathbb{R}^d} \boldsymbol{p}_\varepsilon\big(y - x - B_{t-s}^{t,0}\big) \eta(\mathrm{d}s\,\mathrm{d}y) - \tfrac{1}{2} t f_\varepsilon(0) \right) \right]. \end{aligned}$$

According to the Malliavin-calculus method of Hu and Nualart [18] (see also [5, Theorem 1.9(2)]), $\lim_{\varepsilon\to 0} U^\varepsilon(t\,,x) = U(t\,,x)$ in $L^2(\Omega)$ for every $t>0$ and $x\in\mathbb{R}^d$. Therefore, our goal of proving the stationarity of $U(t)$ would follow once we demonstrate the stationarity of $U^\varepsilon(t)$ for every $t,\varepsilon>0$. But that is not hard to do. Indeed, by the Itô–Walsh isometry for stochastic integrals,

$$
\begin{aligned}
&\mathrm{E}_\eta\left[\int_{(0,t)\times\mathbb{R}^d} \boldsymbol{p}_\varepsilon\big(y-a-B^{t,0}_{t-s}\big)\eta(\mathrm{d}s\,\mathrm{d}y)\times\int_{(0,t)\times\mathbb{R}^d} \boldsymbol{p}_\varepsilon\big(y-b-B^{t,0}_{t-s}\big)\eta(\mathrm{d}s\,\mathrm{d}y)\right]\\
&\quad=\int_0^t \mathrm{d}s\int_{\mathbb{R}^d}\mathrm{d}y\int_{\mathbb{R}^d} f(\mathrm{d}z)\ \boldsymbol{p}_\varepsilon\big(y-a-B^{t,-a}_{t-s}\big)\boldsymbol{p}_\varepsilon\big(z+y-b-B^{t,-b}_{t-s}\big)\\
&\quad=t\int_{\mathbb{R}^d} f(\mathrm{d}z)\ \boldsymbol{p}_{2\varepsilon}\,(z-b+a)\qquad\text{[semigroup property]}\\
&\quad=t\,(\boldsymbol{p}_{2\varepsilon}*f)\,(b-a)\,,
\end{aligned}
$$

which proves the asserted stationarity of $U^\varepsilon(t)$ for every $t,\varepsilon>0$. □

Remark 3.2. As a consequence of the Feynman–Kac formula, we can see immediately that $U(t\,,x)\ge 0$ a.s. for all $t>0$ and $x\in\mathbb{R}^d$.

Proof of Theorem 1.1 *(part* 3*): Behavior near* $t=0$. We now complete the proof by showing that $\lim_{t\to 0} U(t\,,x)=1$ in $L^k(\Omega)$ for every $x\in\mathbb{R}^d$. By stationarity, it suffices to consider only the case that $x=0$. Now in accord with (1.6) and (1.8), there exists a real number K such that, uniformly for all $t\in(0\,,1)$,

$$
\begin{aligned}
&\mathrm{E}\left(|U(t\,,0)-1|^k\right)\\
&\quad\le \mathrm{E}\int_0^t\mathrm{d}s\int_{\mathbb{R}^d}\mathrm{d}y\int_{\mathbb{R}^d} f(\mathrm{d}w)\ \boldsymbol{p}_{s(t-s)/t}(y)\boldsymbol{p}_{s(t-s)/t}(w+y)\|U(s\,,y)U(s\,,w+y)\|_{k/2}\\
&\quad\le K\int_0^t\mathrm{d}s\int_{\mathbb{R}^d}\mathrm{d}y\int_{\mathbb{R}^d} f(\mathrm{d}w)\ \boldsymbol{p}_{s(t-s)/t}(y)\boldsymbol{p}_{s(t-s)/t}(w+y)\\
&\quad= K\int_0^t\left(\boldsymbol{p}_{2s(t-s)/t}*f\right)(0)\,\mathrm{d}s\le K\mathrm{e}^{\beta t}\int_0^t \mathrm{e}^{-\beta\{s\wedge(t-s)\}}\left(\boldsymbol{p}_{2s(t-s)/t}*f\right)(0)\,\mathrm{d}s\\
&\quad\le 4K\mathrm{e}^{\beta t}\Upsilon(2\beta)\qquad\text{for all }\beta>0.
\end{aligned}
$$

Set $\beta=1/t$ to find that $\mathrm{E}(|U(t\,,0)-1|^k)\le 4K\mathrm{e}\Upsilon(2/t)\to 0$ as $t\to 0$, owing to the dominated convergence theorem, (1.7), and the theorem's condition that $\Upsilon(\beta)<\infty$ for one, hence all, $\beta>0$. This concludes the proof. □

4. Ergodicity: Proof of Theorem 1.2.

The following bound on the Malliavin derivative of $U(t\,,x)$ is a key technical result of the paper. Among other things, it also plays a central role in our proof of Theorem 1.2.

PROPOSITION 4.1. *Choose and fix $k\ge 2$, $t>0$, and $x\in\mathbb{R}^d$. Then, $U(t\,,x)\in\cap_{k\ge 2}\mathbb{D}^{1,k}$, and for almost every $(s\,,y)\in(0\,,t)\times\mathbb{R}^d$,*

$$
\begin{aligned}
\|D_{s,y}U(t\,,x)\|_k &\le \frac{64}{7}\exp\left\{\frac{t}{2}\left[\beta_{7/8,k}+\frac{1}{2}\Upsilon^{-1}\left(\frac{1}{32z_k^2}\right)\right]\right\}\boldsymbol{p}_{s(t-s)/t}\left(y-\frac{s}{t}x\right)\\
&:= C_{t,k}\,\boldsymbol{p}_{s(t-s)/t}\left(y-\frac{s}{t}x\right),
\end{aligned}\tag{4.1}
$$

where $\beta_{\varepsilon,k}$ was defined in (3.2).

Because of (1.4), $U(t\,,x) \in \cap_{k\geq 2}\mathbb{D}^{1,k}$ iff $u(t\,,x) \in \cap_{k\geq 2}\mathbb{D}^{1,k}$. Thus, portions of the above are already included in the work of Chen, Hu, and Nualart [4, Proposition 5.1]. The point here is mainly the explicit bound for the moments of the Malliavin derivative of $U(t\,,x)$.

Remark 4.2. Properties of the Malliavin derivative, and (1.5), together imply that the inequality of Proposition 4.1 is equivalent to the following:

$$\begin{aligned}\|D_{s,y}u(t\,,x)\|_k &\leq \frac{64}{7}\exp\left\{\frac{t}{2}\left[\beta_{7/8,k}+\frac{1}{2}\Upsilon^{-1}\left(\frac{1}{32z_k^2}\right)\right]\right\}\boldsymbol{p}_t(x)\boldsymbol{p}_{s(t-s)/t}\left(y-\frac{s}{t}x\right)\\ &= \frac{64}{7}\exp\left\{\frac{t}{2}\left[\beta_{7/8,k}+\frac{1}{2}\Upsilon^{-1}\left(\frac{1}{32z_k^2}\right)\right]\right\}\boldsymbol{p}_{t-s}(x-y)\boldsymbol{p}_s(y).\end{aligned}$$

The proof of Proposition 4.1 requires some notation and two intervening lemmas.

Define $u_0(t\,,x)=\boldsymbol{p}_t(x)$, and iteratively let

$$u_{n+1}(t\,,x)=\boldsymbol{p}_t(x)+\int_{(0,t)\times\mathbb{R}^d}\boldsymbol{p}_{t-r}(x-z)u_n(r\,,z)\,\eta(\mathrm{d}r\,\mathrm{d}z) \tag{4.2}$$

for every $n\in\mathbb{Z}_+$. It is easy to see that, for every $n\geq 2$, $u_n(t\,,x)=\boldsymbol{p}_t(x)U_n(t\,,x)$, where U_n was defined in the proof of Theorem 1.1, and denotes the nth stage in the Picard iteration approximation of U. It follows from the proof of Theorem 1.1 that $u_n(t\,,x)$ converges to $u(t\,,x)=\boldsymbol{p}_t(x)U(t\,,x)$ in $L^k(\Omega)$ as $n\to\infty$ for every $k\geq 2$. It also follows from basic properties of the Malliavin derivative that a.s.,

$$D_{s,y}u_{n+1}(t\,,x)=\boldsymbol{p}_{t-s}(x-y)u_n(s\,,y)+\int_{(s,t)\times\mathbb{R}^d}\boldsymbol{p}_{t-r}(x-z)D_{s,y}u_n(r\,,z)\,\eta(\mathrm{d}r\,\mathrm{d}z) \tag{4.3}$$

for almost every $(s\,,y)\in(0\,,t)\times\mathbb{R}^d$ and all $n\in\mathbb{Z}_+$ for which the right-hand side is well defined. The following shows inductively that indeed the right-hand side is well defined for every n and provides a bound on its $L^k(\Omega)$-norms.

LEMMA 4.3. *Choose and fix $n\in\mathbb{N}$, $k\geq 2$, $t>0$, and $x\in\mathbb{R}^d$, and let $\beta:=\beta_{7/8,k}$, as defined in* (3.2). *Then,*

$$\|D_{s,y}u_n(t\,,x)\|_k\leq\alpha_n\mathrm{e}^{\beta(t-s)/2}\boldsymbol{p}_{t-s}(x-y)\boldsymbol{p}_s(y) \tag{4.4}$$

for almost every $(s\,,y)\in(0\,,t)\times\mathbb{R}^d$, where

$$\alpha_1:=\sup_{m\in\mathbb{Z}_+}\sup_{x\in\mathbb{R}^d}\sup_{s\in(0,t]}\|U_m(s\,,x)\|_k<\infty\quad\text{and}\quad\alpha_n:=\left(\sqrt{8}\left[1-2^{-n}\right]+2^{-n}\right)\alpha_1\leq 4\alpha_1$$

for the random fields $\{U_n\}_{n=0}^\infty$ defined in the proof of Theorem 1.1.

The fact that α_1 is finite is a consequence of the proof of (1.8). In fact, the proof of Theorem 1.1 (with $\varepsilon=7/8$) shows that

$$\alpha_1\leq\frac{16}{7}\exp\left\{\frac{t}{4}\Upsilon^{-1}\left(\frac{1}{32z_k^2}\right)\right\}. \tag{4.5}$$

Proof of Lemma 4.3. We proceed to prove (4.4) by using induction on n.

Because $D_{s,y}u_0(t\,,x)=0$, it follows from (4.3) that $\|D_{s,y}u_1(t\,,x)\|_k\leq\alpha_1\boldsymbol{p}_{t-s}(x-y)\boldsymbol{p}_s(y)$. In particular, (4.4) holds for $n=1$. Next, we suppose (4.4) is true for some integer $n \geq 1$ and proceed to prove that it is true when n is replaced by $n+1$. With this aim in mind, observe using the BDG inequality that

$$\mathcal{E}_{n+1} = \mathcal{E}_{n+1}(s\,, y\,, t\,, x\,, k) := \|D_{s,y} u_{n+1}(t\,, x)\|_k^2$$

satisfies

$$\begin{aligned}
\mathcal{E}_{n+1} &\leq 2\alpha_1^2 \left[\boldsymbol{p}_{t-s}(x-y)\boldsymbol{p}_s(y)\right]^2 + 2z_k^2 \int_s^t \mathrm{d}r \int_{\mathbb{R}^d} \mathrm{d}z \int_{\mathbb{R}^d} f(\mathrm{d}z') \\
&\quad \times \boldsymbol{p}_{t-r}(x-z)\boldsymbol{p}_{t-r}(x-z-z') \left\|D_{s,y}u_n(r\,,z)\right\|_k \left\|D_{s,y}u_n(r\,,z'+z)\right\|_k \\
&\leq 2\alpha_1^2 \left[\boldsymbol{p}_{t-s}(x-y)\boldsymbol{p}_s(y)\right]^2 + 2z_k^2\alpha_n^2 \left[\boldsymbol{p}_s(y)\right]^2 \int_0^{t-s} \mathrm{e}^{\beta r}\,\mathrm{d}r \int_{\mathbb{R}^d} \mathrm{d}z \int_{\mathbb{R}^d} f(\mathrm{d}z') \\
&\quad \times \boldsymbol{p}_{t-s-r}(x-z)\boldsymbol{p}_r(z-y)\boldsymbol{p}_{t-s-r}(x-z-z')\boldsymbol{p}_r(z'+z-y),
\end{aligned}$$

thanks to the induction hypothesis and a change of variables ($r \leftrightarrow r-s$). Apply (1.5) in order to find that

$$\begin{aligned}
\mathcal{E}_{n+1} &\leq 2\alpha_1^2 \left[\boldsymbol{p}_{t-s}(x-y)\boldsymbol{p}_s(y)\right]^2 \\
&\quad + 2z_k^2\alpha_n^2 \left[\boldsymbol{p}_{t-s}(x-y)\boldsymbol{p}_s(y)\right]^2 \int_0^{t-s} \mathrm{e}^{\beta\{r\vee(t-s-r)\}}\,\mathrm{d}r \int_{\mathbb{R}^d} \mathrm{d}z \int_{\mathbb{R}^d} f(\mathrm{d}z') \\
&\quad \times \boldsymbol{p}_{r(t-s-r)/(t-s)}\left(z-y-\frac{r}{t-s}(x-y)\right)\boldsymbol{p}_{r(t-s-r)/(t-s)} \\
&\quad \times \left(z'+z-y-\frac{r}{t-s}(x-y)\right) \\
&= \left[\boldsymbol{p}_{t-s}(x-y)\boldsymbol{p}_s(y)\right]^2 \left\{2\alpha_1^2 + 2z_k^2\alpha_n^2 \int_0^{t-s} \mathrm{e}^{\beta\{r\vee(t-s-r)\}} \left(\boldsymbol{p}_{2r(t-s-r)/(t-s)} * f\right)(0)\,\mathrm{d}r\right\},
\end{aligned}$$

where we have appealed to the semigroup property of the heat kernel for the last line. Take square roots and apply the simple inequality $(|a|+|b|)^{1/2} \leq |a|^{1/2}+|b|^{1/2}$—valid for all $a, b \in \mathbb{R}$—to see that

$$\frac{\|D_{s,y}u_{n+1}(t\,,x)\|_k}{\boldsymbol{p}_{t-s}(x-y)\boldsymbol{p}_s(y)} \leq \sqrt{2}\alpha_1 + \alpha_n \left\{2z_k^2 \int_0^{t-s} \mathrm{e}^{\beta\{r\vee(t-s-r)\}} \left(\boldsymbol{p}_{2r(t-s-r)/(t-s)} * f\right)(0)\,\mathrm{d}r\right\}^{1/2}.$$

Since $r \vee (t-s-r) = t-s-\{r \wedge (t-s-r)\}$, this proves that

$$\begin{aligned}
&\frac{\|D_{s,y}u_{n+1}(t\,,x)\|_k}{\boldsymbol{p}_{t-s}(x-y)\boldsymbol{p}_s(y)} \\
&\quad \leq \sqrt{2}\alpha_1 + \alpha_n \mathrm{e}^{\beta(t-s)/2} \left\{2z_k^2 \int_0^{t-s} \mathrm{e}^{-\beta\{r\wedge(t-s-r)\}} \left(\boldsymbol{p}_{2r(t-s-r)/(t-s)} * f\right)(0)\,\mathrm{d}r\right\}^{1/2} \\
&\quad \leq \sqrt{2}\alpha_1 + \alpha_n \mathrm{e}^{\beta(t-s)/2}\sqrt{8z_k^2\Upsilon(2\beta)} \qquad \text{(see Lemma 3.1)} \\
&\quad \leq \sqrt{2}\alpha_1 + \tfrac{1}{2}\alpha_n \mathrm{e}^{\beta(t-s)/2} \leq \alpha_{n+1}\mathrm{e}^{\beta(t-s)/2},
\end{aligned}$$

thanks to the definition (3.2) of $\beta = \beta_{7/8,k}$ and the readily checkable fact that $\alpha_{n+1} = \sqrt{2}\alpha_1 + \frac{1}{2}\alpha_n$. This proves (4.4) with n replaced by $n+1$ and concludes the inductive stage of the argument. ∎

Our next technical lemma implies, inductively, that $u_n(t\,,x) \in \mathbb{D}^{1,2}$ for every $n \in \mathbb{N}$.

LEMMA 4.4. *There exist real numbers $A, B > 0$ such that*

$$\sup_{n\in\mathbb{N}}\sup_{x\in\mathbb{R}^d} \mathrm{E}\left(\|Du_n(t\,,x)\|_{\mathcal{H}}^2\right) \le At^{-d}\mathrm{e}^{Bt} \qquad \textit{for all } t>0.$$

Proof. We compute directly, using Lemma 4.3, as follows:

$$\begin{aligned}\mathrm{E}\left(\|Du_n(t\,,x)\|_{\mathcal{H}}^2\right) &= \int_0^t \mathrm{d}s\int_{\mathbb{R}^d}\mathrm{d}y\int_{\mathbb{R}^d} f(\mathrm{d}y')\ \mathrm{E}\left[D_{s,y}u_n(t\,,x)D_{s,y+y'}u_n(t\,,x)\right]\\ &\le \int_0^t \mathrm{d}s\int_{\mathbb{R}^d}\mathrm{d}y\int_{\mathbb{R}^d} f(\mathrm{d}y')\ \|D_{s,y}u_n(t\,,x)\|_2\,\|D_{s,y+y'}u_n(t\,,x)\|_2\\ &\le c\int_0^t \mathrm{d}s\int_{\mathbb{R}^d}\mathrm{d}y\int_{\mathbb{R}^d} f(\mathrm{d}y')\ \boldsymbol{p}_{t-s}(x-y)\boldsymbol{p}_s(y)\boldsymbol{p}_{t-s}(x-y-y')\boldsymbol{p}_s(y+y'),\end{aligned}$$

where $c := 16\alpha_1^2\exp(\beta_{7/8,2}t)$, using the constants of Lemma 4.3. Note that α_1 depends on t, and in fact Theorem 1.1 ensures that $c \le c_1\exp(c_2t)$ where c_1 and c_2 do not depend on t. Apply (1.5) to see that

$$\begin{aligned}&\mathrm{E}\left(\|Du_n(t\,,x)\|_{\mathcal{H}}^2\right)\\ &\quad\le c_1\mathrm{e}^{c_2t}[\boldsymbol{p}_t(x)]^2\int_0^t \mathrm{d}s\int_{\mathbb{R}^d}\mathrm{d}y\int_{\mathbb{R}^d} f(\mathrm{d}y')\ \boldsymbol{p}_{s(t-s)/t}\left(y-\frac{s}{t}x\right)\boldsymbol{p}_{s(t-s)/t}\left(y+y'-\frac{s}{t}x\right)\\ &\quad= c_1\mathrm{e}^{c_2t}[\boldsymbol{p}_t(x)]^2\int_0^t\left(\boldsymbol{p}_{2s(t-s)/t}*f\right)(0)\,\mathrm{d}s\\ &\quad\le \frac{c_1\mathrm{e}^{(1+c_2)t}}{(2\pi t)^d}\int_0^t \mathrm{e}^{-\{s\wedge(t-s)\}}\left(\boldsymbol{p}_{2s(t-s)/t}*f\right)(0)\,\mathrm{d}s.\end{aligned}$$

Since $\Upsilon(2)<\infty$, an appeal to Lemma 3.1 completes the proof. ◻

We are in position to verify Proposition 4.1.

Proof of Proposition 4.1. The proof is similar to that of [6, Theorem 6.4]. Choose and fix $k\ge 2$, $t>0$, and $x\in\mathbb{R}^d$. Thanks to (1.4) and (1.5), the proposition's assertion is equivalent to the following inequality, valid for a.e. $(s\,,y)\in(0\,,t)\times\mathbb{R}^d$:

$$\|D_{s,y}u(t\,,x)\|_k \le \frac{64}{7}\exp\left\{\frac{t}{2}\left[\beta_{7/8,k}+\frac{1}{2}\Upsilon^{-1}\left(\frac{1}{32z_k^2}\right)\right]\right\}\boldsymbol{p}_{t-s}(x-y)\boldsymbol{p}_s(y),$$

We will prove the above reformulation of the proposition.

Thanks to Lemma 4.4 and closeablility properties of the Malliavin derivative operator (see Nualart [20]), it follows that, after possibly moving to subsequence, $Du_n(t\,,x)$ converges to $Du(t\,,x)$ in the weak topology of $L^2(\Omega\,;\mathcal{H})$. Then, we use a smooth approximation $\{\psi_\varepsilon\}_{\varepsilon>0}$ to the identity in $\mathbb{R}_+\times\mathbb{R}^d$ and apply Fatou's lemma and duality for L^k-spaces in order to find that, for almost every $(s\,,y)\in(0\,,t)\times\mathbb{R}^d$ and for all $k\ge 2$,

$$\begin{aligned}\|D_{s,y}u(t\,,x)\|_k &\le \limsup_{\varepsilon\to0}\left\|\int_{\mathbb{R}_+\times\mathbb{R}^d} D_{s',y'}u(t\,,x)\psi_\varepsilon(s-s',y-y')\,\mathrm{d}s'\mathrm{d}y'\right\|_k\\ &\le \limsup_{\varepsilon\to0}\sup_{\|G\|_{k/(k-1)}\le1}\left|\int_{\mathbb{R}_+\times\mathbb{R}^d}\mathrm{E}\left[GD_{s',y'}u(t\,,x)\right]\psi_\varepsilon(s-s',y-y')\,\mathrm{d}s'\mathrm{d}y'\right|.\end{aligned}$$

Choose and fix a random variable $G\in L^2(\Omega)$ such that $\|G\|_{k/(k-1)}\le 1$. We can find an unbounded subsequence $n(1)<n(2)<\cdots$ of positive integers such that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}_+\times\mathbb{R}^d} \mathrm{E}\left[GD_{s',y'}u(t\,,x)\right]\psi_\varepsilon(s-s',y-y')\,\mathrm{d}s'\mathrm{d}y'\right|\\
&\quad=\lim_{\ell\to\infty}\left|\int_{\mathbb{R}_+\times\mathbb{R}^d} \mathrm{E}\left[GD_{s',y'}u_{n(\ell)}(t\,,x)\right]\psi_\varepsilon(s-s',y-y')\,\mathrm{d}s'\mathrm{d}y'\right|\\
&\quad\le\limsup_{\ell\to\infty}\int_{\mathbb{R}_+\times\mathbb{R}}\left\|D_{s',y'}u_{n(\ell)}(t\,,x)\right\|_k\psi_\varepsilon(s-s',y-y')\,\mathrm{d}s'\mathrm{d}y'\\
&\quad\le\sup_{n\in\mathbb{N}}\alpha_n\int_{(0,t)\times\mathbb{R}^d}\mathrm{e}^{\beta(t-s')/2}\boldsymbol{p}_{t-s'}(x-y')\boldsymbol{p}_{s'}(y')\psi_\varepsilon(s-s',y-y')\,\mathrm{d}s'\mathrm{d}y'\\
&\quad\le 4\alpha_1\mathrm{e}^{\beta t/2}\int_{(0,t)\times\mathbb{R}^d}\boldsymbol{p}_{t-s'}(x-y')\boldsymbol{p}_{s'}(y')\psi_\varepsilon(s-s',y-y')\,\mathrm{d}s'\mathrm{d}y';
\end{aligned}
$$

see Lemma 4.3. Let $\varepsilon\to 0$ and appeal to (4.5) in order to finish. ◻

The second, and final, step of the proof of Theorem 1.2 is a Poincaré-type inequality for certain nonlinear functionals of U. In order to describe that inequality, let us first choose and fix points $\zeta^1,\ldots,\zeta^k\in\mathbb{R}^d$ and bounded Lipschitz-continuous functions $g_1,\ldots,g_k\in C_b^1(\mathbb{R})$ such that

$$g_j(0)=0 \quad\text{and}\quad \mathrm{Lip}(g_j)=1 \qquad \text{for every } j=1,\ldots,k. \tag{4.6}$$

Then define for every $t,N>0$ and $x\in\mathbb{R}^d$,

$$\mathcal{G}(t\,,x):=\prod_{j=1}^k g_j\left(U(t\,,x+\zeta^j)\right). \tag{4.7}$$

Lemma 4.5. *Choose and fix an integer $k\ge 2$, points $x,\zeta^1,\ldots,\zeta^k\in\mathbb{R}^d$, and functions $g_1,\ldots,g_k\in C_b^1(\mathbb{R})$ that satisfy* (4.6). *Then, there exists a real number $A=A(t\,,k\,,g_1\,,\ldots,g_k)$ given by* (4.8) *below such that*

$$|\mathrm{Cov}\left(\mathcal{G}(t\,,0)\,,\mathcal{G}(t\,,x)\right)|\le A^2\sum_{j_0=1}^k\sum_{j_1=1}^k\int_0^t\left(\boldsymbol{p}_{2s(t-s)/t}*f\right)\left(\frac{s}{t}(x+\zeta^{j_0}-\zeta^{j_1})\right)\mathrm{d}s.$$

Proof. By the chain rule of Malliavin calculus (see Nualart [20]),

$$
\begin{aligned}
&D_{s,z}\mathcal{G}(t\,,x)\\
&\quad=\mathbf{1}_{(0,t)}(s)\sum_{j_0=1}^k\left(\prod_{\substack{j=1\\ j\ne j_0}}^k g_j\left(U(t\,,x+\zeta^j)\right)\right)g'_{j_0}\left(U(t\,,x+\zeta^{j_0})\right)D_{s,z}U(t\,,x+\zeta^{j_0})
\end{aligned}
$$

for almost every $(s\,,z)\in(0\,,t)\times\mathbb{R}^d$. Therefore, Proposition 4.1 ensures that

$$
\begin{aligned}
\left\|D_{s,z}\mathcal{G}(t\,,x)\right\|_k&\le\mathbf{1}_{(0,t)}(s)\max_{1\le j\le k}\sup_{a\in\mathbb{R}}|g_j(a)|^{k-1}\sum_{j_0=1}^k\left\|D_{s,z}U(t\,,x+\zeta^{j_0})\right\|_k\\
&\le A\mathbf{1}_{(0,t)}(s)\sum_{j_0=1}^k\boldsymbol{p}_{s(t-s)/t}\left(z-\frac{s}{t}(x+\zeta^{j_0})\right)
\end{aligned}
$$

with

$$A := \frac{64}{7} \exp\left\{ \frac{t}{2} \left[\beta_{7/8,k} + \frac{1}{2} \Upsilon^{-1}\left(\frac{1}{32 z_k^2} \right) \right] \right\} \max_{1 \le j \le k} \sup_{a \in \mathbb{R}} |g_j(a)|^{k-1}. \tag{4.8}$$

It follows from the Poincaré inequality (2.4) that $|\operatorname{Cov}(\mathcal{G}(t\,,x)\,,\mathcal{G}(t\,,0))|$ is bounded from above by

$$A^2 \sum_{j_0=1}^{k} \sum_{j_1=1}^{k} \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^d} f(\mathrm{d}y')\, \boldsymbol{p}_{s(t-s)/t}\left(y - \frac{s}{t}(x + \zeta^{j_0}) \right) \boldsymbol{p}_{s(t-s)/t}\left(y' + y - \frac{s}{t}\zeta^{j_1} \right).$$

Apply the semigroup property of the heat kernel together with Fubini's theorem to finish. ∎

Proof of Theorem 1.2. Define

$$V_N(t) := \operatorname{Var}\left(\frac{1}{N^d} \int_{[0,N]^d} \mathcal{G}(t,x)\mathrm{d}x \right) \text{ and } \mathcal{G}(x) := \prod_{j=1}^{k} g_j(U(t\,,x+\zeta^j)) \quad \text{for all } x \in \mathbb{R}^d,$$

where $\mathcal{G}(t,x)$ has been defined in (4.7) and the bounded functions $g_1, \ldots, g_k$ therein satisfy (4.6). Since $U(t)$ is stationary (Theorem 1.1), [6, Lemma 7.2] implies the desired ergodicity provided that we prove that, for all $t > 0$,

$$\lim_{N \to \infty} V_N(t) = 0. \tag{4.9}$$

For every real number $N > 0$, define the functions

$$I_N(x) := N^{-d} \mathbf{1}_{[0,N]^d}(x) \quad \text{and} \quad \tilde{I}_N(x) = I_N(-x) \qquad \text{for } x \in \mathbb{R}^d. \tag{4.10}$$

By Lemma 4.5,

$$\begin{aligned} V_N(t) &= \frac{1}{N^{2d}} \int_{[0,N]^d} \mathrm{d}x \int_{[0,N]^d} \mathrm{d}y \operatorname{Cov}[\mathcal{G}(t\,,x)\,,\mathcal{G}(t\,,y)] \\ &\le \frac{A^2}{N^{2d}} \sum_{j_0=1}^{k} \sum_{j_1=1}^{k} \int_0^t \mathrm{d}s \int_{[0,N]^d} \mathrm{d}x \int_{[0,N]^d} \mathrm{d}y \ \left(\boldsymbol{p}_{2s(t-s)/t} * f \right) \left(\frac{s}{t}(x - y + \zeta^{j_0} - \zeta^{j_1}) \right) \\ &= A^2 \sum_{j_0=1}^{k} \sum_{j_1=1}^{k} \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}x \left(I_N * \tilde{I}_N \right)(x) \left(f * \boldsymbol{p}_{2s(t-s)/t} \right) \left(\frac{s}{t}(x + \zeta^{j_0} - \zeta^{j_1}) \right). \end{aligned}$$

Therefore, (7.2) implies that

$$V_N(t) \le \frac{k^2 A^2}{\pi^d} \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \hat{f}(\mathrm{d}y)\, \mathrm{e}^{-\frac{s(t-s)}{t}\|y\|^2} \prod_{j=1}^{d} \frac{1 - \cos(Nsy_j/t)}{(Nsy_j/t)^2}.$$

The quantity $\prod_{j=1}^d \{1 - \cos(Nsy_j/t)\}/(Nsy_j/t)^2$ is bounded above by 2^{-d} and converges to zero as $N \to \infty$ for each $s > 0$ and $y \neq 0$. Since $\hat{f}\{0\} = 0$, the dominated convergence theorem implies that $\lim_{N\to\infty} V_N(t) = 0$, taking into account that

$$\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \hat{f}(\mathrm{d}y)\, \mathrm{e}^{-\frac{s(t-s)}{t}\|y\|^2} < \infty,$$

which follows from Dalang's condition $\Upsilon(1) < \infty$. This proves (4.9), whence follows ergodicity. ∎

5. Asymptotic variance. Recall the spatial average $\mathcal{S}_{N,t}$ and the quantity $\mathcal{R}(f)$ are defined in (1.10) and (1.11), respectively.

THEOREM 5.1 ($d \geq 1$). *For all* $t > 0$,

$$\lim_{N\to\infty} N\mathrm{Var}(\mathcal{S}_{N,t}) = t\,\mathcal{R}(f). \tag{5.1}$$

The quantity on the right-hand side is strictly positive and (5.1) *holds whenever* $\mathcal{R}(f)$ *is finite or infinite.*

According to the criteria in Proposition 5.8 and Lemma 5.9 below, the value of $\mathcal{R}(f)$ could be finite or infinite. For example, if $f(\mathrm{d}x) = \boldsymbol{p}_1(x)\mathrm{d}x$ and $d \geq 2$, then $\mathcal{R}(f) < \infty$; if f is given by the Riesz kernel, i.e., $f(\mathrm{d}x) = \|x\|^{-\beta}\mathrm{d}x, 0 < \beta < 2 \wedge d$, then $\mathcal{R}(f) = \infty$. Moreover, it is easy to deduce from Lemma 5.9 below that, in the case that $d = 1$, $\mathcal{R}(f)$ is always infinite, which might suggest that the above $1/N$ rate of decay of $\mathrm{Var}(\mathcal{S}_{N,t})$ is not the right one in one dimension. Indeed, this is the case. And the following result identifies the correct rate canonically as $N^{-1}\log N$ in dimension one.

THEOREM 5.2 ($d = 1$). *Assume* $f(\mathbb{R}) < \infty$. *Then for all* $t > 0$,

$$tf(\mathbb{R}) \leq \liminf_{N\to\infty} \frac{N}{\log N}\,\mathrm{Var}(\mathcal{S}_{N,t}) \leq \limsup_{N\to\infty} \frac{N}{\log N}\,\mathrm{Var}(\mathcal{S}_{N,t}) \leq 2tf(\mathbb{R}). \tag{5.2}$$

Both bounds are sharp in the following sense:

1. *If* $f = a\delta_0$ *for some* $a > 0$, *then* $\mathrm{Var}(\mathcal{S}_{N,t}) \sim 2tf(\mathbb{R})N^{-1}\log N$ *as* $N \to \infty$.
2. *If* $\lim_{x\to\infty} \hat{f}(x) = 0$, *then* $\mathrm{Var}(\mathcal{S}_{N,t}) \sim tf(\mathbb{R})N^{-1}\log N$ *as* $N \to \infty$.

Remark 5.3. The condition in item 2 of Theorem 5.2 is a well-known one. Indeed, finite Borel measures whose Fourier transforms vanish at infinity are called *Rajchman measures*. See Lyons [19] for the background and rich history of the work on Rajchman measures in classical harmonic analysis.

We now turn to the Riesz kernel case. Define

$$\varphi(y) := \frac{1-\cos y}{y^2} \quad \text{for all } y \in \mathbb{R}\setminus\{0\}, \tag{5.3}$$

and $\varphi(0) := 1/2$ to preserve continuity.

THEOREM 5.4 (Riesz kernel). *Assume* $f(\mathrm{d}x) = \|x\|^{-\beta}\mathrm{d}x$ *and* $\hat{f}(\mathrm{d}x) = \kappa_{\beta,d}\,\|x\|^{\beta-d}\mathrm{d}x$, *where* $0 < \beta < 2 \wedge d$ *and* $\kappa_{\beta,d}$ *is a positive constant depending on* β *and* d.

1. *If* $0 < \beta < 1$, *then*

$$\lim_{N\to\infty} N^{\beta}V_N(t) = \frac{t}{1-\beta}\int_{[-1,1]^d} \|z\|^{-\beta}\prod_{i=1}^{d}(1-|z_i|)\mathrm{d}z := t\,\sigma_{0,\beta,d}. \tag{5.4}$$

2. *If* $1 = \beta < 2 \wedge d$, *then*

$$\lim_{N\to\infty} \frac{N}{\log N}V_N(t) = \frac{2t\,\kappa_{1,d}}{\pi^d}\int_{\mathbb{R}^d} \|z\|^{1-d}\prod_{j=1}^{d}\varphi(z_j)\mathrm{d}z := t\,\sigma_{1,\beta,d}. \tag{5.5}$$

3. *If* $1 < \beta < 2 \wedge d$, *then*

$$\begin{aligned}\lim_{N\to\infty} N^{2-\beta}V_N(t) &= \frac{t^{2-\beta}\,\kappa_{\beta,d}}{\pi^d}\int_{\mathbb{R}^d} \|z\|^{2-\beta-d}\prod_{j=1}^{d}\varphi(z_j)\mathrm{d}z\int_0^\infty r^{\beta-2}\mathrm{e}^{-r}\mathrm{d}r \\ &:= t^{2-\beta}\sigma_{2,\beta,d}.\end{aligned} \tag{5.6}$$

We now begin to work toward proving the above theorems. First, we denote

$$V_N(t) := \mathrm{Var}(\mathcal{S}_{N,t}) = \int_{\mathbb{R}^d} \left(I_N * \tilde{I}_N\right)(x)\chi_t(x)\,\mathrm{d}x, \tag{5.7}$$

where I_N and $\tilde{I}_N$, defined in (4.10), are given by $I_N(x) := N^{-d}\mathbf{1}_{[0,N]^d}(x)$ and $\tilde{I}_N(x) = I_N(-x)$ for $x \in \mathbb{R}^d$, and for every $N, t > 0$ and $x \in \mathbb{R}^d$,

$$\chi_t(x) := \mathrm{Cov}\left[U(t\,,0)\ ,\ U(t\,,x)\right]. \tag{5.8}$$

Now we begin to establish a series of supporting lemmas.

LEMMA 5.5 ($d \geq 1$). *Let χ be defined by* (5.8). *Then, for every $t > 0$ and $x \in \mathbb{R}^d$,*

$$\chi_t(x) = \int_0^t \left(\boldsymbol{p}_{2s(t-s)/t} * f\right)(sx/t)\,\mathrm{d}s + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} f(\mathrm{d}y)\ \boldsymbol{p}_{2s(t-s)/t}\left(y - \frac{s}{t}x\right)\chi_s(y).$$

Proof. Apply (1.6) and elementary properties of the Walsh integral to see that

$$\begin{aligned}
&\mathrm{E}\left[U(t\,,0)U(t\,,x)\right]\\
&= 1 + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y' \int_{\mathbb{R}^d} f(\mathrm{d}y)\ \boldsymbol{p}_{s(t-s)/t}(y')\boldsymbol{p}_{s(t-s)/t}\left(y + y' - \frac{s}{t}x\right)\mathrm{E}\left[U(s\,,y')U(s\,,y+y')\right]\\
&= 1 + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y' \int_{\mathbb{R}^d} f(\mathrm{d}y)\ \boldsymbol{p}_{s(t-s)/t}(y')\boldsymbol{p}_{s(t-s)/t}\left(y + y' - \frac{s}{t}x\right)\mathrm{E}\left[U(s\,,0)U(s\,,y)\right],
\end{aligned}$$

owing to the stationarity (Theorem 1.1). This and the semigroup property of the heat kernel together imply the lemma since $\mathrm{E}[U(t\,,0)U(t\,,x)] = \chi_t(x) + 1$. ∎

Our second supporting lemma describes the behavior of χ_t as $t \to 0$.

LEMMA 5.6 ($d \geq 1$). $\lim_{t\downarrow 0}\chi_t(x) = 0$ *uniformly for all $x \in \mathbb{R}^d$.*

Proof. It is easy to deduce from Lemma 5.5 and positivity of the solution (see Remark 3.2) that

$$\chi_t(x) \geq \int_0^t \left(\boldsymbol{p}_{2s(t-s)/t} * f\right)(sx/t)\,\mathrm{d}s \geq 0 \qquad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d. \tag{5.9}$$

Now, the Cauchy–Schwarz inequality and stationarity together ensure that $\chi_t(x) \leq \chi_t(0)$. Therefore, it suffices to prove that $\chi_t(0) \to 0$ as $t \downarrow 0$. Theorem 1.1 ensures that $C := \sup_{t\in(0,1)}\chi_t(0) = \sup_{t\in(0,1)}\sup_{x\in\mathbb{R}^d}\chi_t(x) < \infty$. Therefore, we deduce from Lemma 5.5 that

$$\chi_t(0) \leq (1 + C)\int_0^t \left(\boldsymbol{p}_{2s(t-s)/t} * f\right)(0)\,\mathrm{d}s.$$

Since $\chi_t(0) \leq (1 + C)4\mathrm{e}^{\beta t}\Upsilon(2\beta)$ for every $\beta, t > 0$ (Lemma 3.1), it follows that

$$\limsup_{t\to 0}\chi_t(0) \leq (1 + C)4\lim_{\beta\to\infty}\Upsilon(2\beta) = 0.$$

This concludes the proof. ∎

In light of (5.7) and Lemma 5.5, we write

$$V_N(t) = \mathrm{Var}(\mathcal{S}_{N,t}) = V_N^{(1)}(t) + V_N^{(2)}(t), \tag{5.10}$$

where

$$V_N^{(1)}(t) = \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}x \, \left(I_N * \tilde{I}_N\right)(x) \left(\boldsymbol{p}_{2s(t-s)/t} * f\right)(sx/t), \tag{5.11}$$

$$V_N^{(2)}(t) = \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}x \, \left(I_N * \tilde{I}_N\right)(x) \int_{\mathbb{R}^d} f(\mathrm{d}y) \, \boldsymbol{p}_{2s(t-s)/t}\left(y - \frac{s}{t}x\right) \chi_s(y). \tag{5.12}$$

As we will see, the main contribution for the asymptotic behavior of $V_N(t)$ is $V_N^{(1)}(t)$, thanks to Lemma 5.6.

5.1. Analysis in dimension $d \geq 2$. The primary goal of this section is to prove Theorem 5.1. Therefore, in this section, we will not assume that $d \geq 2$ unless we say so explicitly. Recall the function φ defined in (5.3).

LEMMA 5.7 ($d \geq 1$). *For every $N, t > 0$,*

$$\begin{aligned} V_N^{(1)}(t) &= \int_{\mathbb{R}^d} \left(I_N * \tilde{I}_N\right)(x) \, \mathrm{d}x \int_0^t \mathrm{d}s \left(\boldsymbol{p}_{2s(t-s)/t} * f\right)(sx/t) \\ &= \frac{t}{N\pi^d} \int_0^N \mathrm{d}s \int_{\mathbb{R}^d} \hat{f}(\mathrm{d}z) \, \mathrm{e}^{-t\|z\|^2(1-s/N)s/N} \prod_{j=1}^d \varphi(z_j s). \end{aligned}$$

Proof. By (7.1),

$$\begin{aligned} &\int_{\mathbb{R}^d} \left(I_N * \tilde{I}_N\right)(x) \, \mathrm{d}x \int_0^t \mathrm{d}s \left(\boldsymbol{p}_{2(t-s)/t} * f\right)(sx/t) \\ &\quad = \frac{1}{\pi^d} \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \hat{f}(\mathrm{d}z) \, \mathrm{e}^{-s(t-s)\|z\|^2/t} \prod_{j=1}^d \varphi(Nz_j s/t) \\ &\quad = \frac{t}{N\pi^d} \int_0^N \mathrm{d}s \int_{\mathbb{R}^d} \hat{f}(\mathrm{d}z) \, \mathrm{e}^{-t\|z\|^2(1-s/N)s/N} \prod_{j=1}^d \varphi(z_j s), \end{aligned}$$

where in the second equality we use change of variable ($s \to st/N$). ◻

Before we prove Theorem 5.1, we give some estimates on the quantity $\mathcal{R}(f)$.

PROPOSITION 5.8 ($d \geq 1$). *Recall $\mathcal{R}(f)$ from* (1.11)*. Then,*

$$2^{1-2d} \int_0^\infty f\left([-r\,,r]^d\right) \frac{\mathrm{d}r}{r^d} \leq \mathcal{R}(f) \leq \int_0^\infty f\left([-r\,,r]^d\right) \frac{\mathrm{d}r}{r^d}.$$

Proof. We observe that $\prod_{j=1}^d \varphi(z_j r) = 2^{-d} r^{-d} [(I_1 * \tilde{I}_1)(\bullet/r)]\hat{}\,(z)$ for all $z \in \mathbb{R}^d$ and $r > 0$. Hence we can write

$$\mathcal{R}(f) = \frac{1}{(2\pi)^d} \int_0^\infty \frac{\mathrm{d}r}{r^d} \int_{\mathbb{R}^d} \hat{f}(\mathrm{d}z) \, \left(\widehat{\left(I_1 * \tilde{I}_1\right)(\bullet/r)}\right)(z).$$

Denote $\phi_r = (I_1 * \tilde{I}_1)(\bullet/r)$ for every fixed $r > 0$. Choose a nonnegative smooth function ψ with compact support such that $\int_{\mathbb{R}^d} \psi(x)\mathrm{d}x = 1$. For $0 < \varepsilon < 1$, define $\psi_\varepsilon(x) = \varepsilon^{-d}\psi(x/\varepsilon)$ for all $x \in \mathbb{R}^d$. It is clear that $\psi_\varepsilon * \phi_r$ has compact support uniformly for all $0 < \varepsilon < 1$ and $\sup_{0<\varepsilon<1} \sup_{x\in\mathbb{R}^d} (\psi_\varepsilon * \phi_r)(x) < \infty$. Moreover, we have $\sup_{0<\varepsilon<1} \sup_{x\in\mathbb{R}^d} |\hat{\psi}_\varepsilon(x)| \leq 1$ and $\lim_{\varepsilon\to 0} \hat{\psi}_\varepsilon(x) = 1$ for all $x \in \mathbb{R}^d$. Using these facts and that f is locally integrable as a tempered distribution and $\int_{\mathbb{R}^d} \hat{\phi}_r(x)\hat{f}(\mathrm{d}x) < \infty$ by Dalang's condition, we obtain that for every fixed $r > 0$,

$$\begin{aligned}\int_{\mathbb{R}^d} \phi_r(x)f(\mathrm{d}x) &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} \left(\psi_\varepsilon * \phi_r\right)(x)f(\mathrm{d}x) = \lim_{\varepsilon\to 0}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{\psi}_\varepsilon(x)\hat{\phi}_r(x)\hat{f}(\mathrm{d}x)\\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{\phi}_r(x)\hat{f}(\mathrm{d}x),\end{aligned}$$

where the first and third equalities hold by the dominated convergence theorem and the second by the definition of the Fourier transform and the property $\widehat{\psi_\varepsilon * \phi_r} = \hat{\psi}_\varepsilon\hat{\phi}_r$.

Therefore,

$$\mathcal{R}(f) = \int_0^\infty \frac{\mathrm{d}r}{r^d}\int_{\mathbb{R}^d}\left(I_1 * \tilde{I}_1\right)(z/r)f(\mathrm{d}z).$$

Now appealing to the inequality $2^{-d}\mathbf{1}_{[-1/2,1/2]^d} \le I_1 * \tilde{I}_1 \le \mathbf{1}_{[-1,1]^d}$ (see [6, (3.17)]), we obtain

$$2^{1-2d}\int_0^\infty f\left([-r\,,r]^d\right)\frac{\mathrm{d}r}{r^d} \le \mathcal{R}(f) \le \int_0^\infty f\left([-r\,,r]^d\right)\frac{\mathrm{d}r}{r^d},$$

which completes the proof. ∎

Now we can prove Theorem 5.1.

Proof of Theorem 5.1. By Proposition 5.8, it is clear that $\mathcal{R}(f)$ is strictly positive since we assume $f(\mathbb{R}^d) > 0$ throughout the paper. Let us proceed with the proof of (5.1).

Assume $\mathcal{R}(f) < \infty$ first. By Lemma 5.7 and the dominated convergence theorem,

$$\lim_{N\to\infty} NV_N^{(1)}(t) = \lim_{N\to\infty} N\int_{\mathbb{R}^d}\left(I_N * \tilde{I}_N\right)(x)\mathrm{d}x\int_0^t \mathrm{d}s\left(\boldsymbol{p}_{2(t-s)/t} * f\right)(sx/t) = t\mathcal{R}(f).$$

In light of (5.12), it remains to prove that

$$\lim_{N\to\infty} N\int_{\mathbb{R}^d}\left(I_N * \tilde{I}_N\right)(x)\,\mathrm{d}x\int_0^t \mathrm{d}s\int_{\mathbb{R}^d} f(\mathrm{d}y)\;\boldsymbol{p}_{2s(t-s)/t}\left(y - \frac{s}{t}x\right)\chi_s(y) = 0. \tag{5.13}$$

By the Cauchy–Schwarz inequality and stationarity, $\chi_t(x) \le \chi_t(0)$ for all $t > 0$ and $x \in \mathbb{R}^d$. Therefore,

$$\begin{aligned}&\int_{\mathbb{R}^d}\left(I_N * \tilde{I}_N\right)(x)\,\mathrm{d}x\int_0^t \mathrm{d}s\int_{\mathbb{R}^d} f(\mathrm{d}y)\;\boldsymbol{p}_{2s(t-s)/t}\left(y - \frac{s}{t}x\right)\chi_s(y)\\ &\quad\le \int_0^t \chi_s(0)\,\mathrm{d}s\int_{\mathbb{R}^d}\left(I_N * \tilde{I}_N\right)(x)\,\mathrm{d}x\int_{\mathbb{R}^d} f(\mathrm{d}y)\;\boldsymbol{p}_{2s(t-s)/t}\left(y - \frac{s}{t}x\right)\end{aligned}$$

for every $N, t > 0$. Repeat the computation of Lemma 5.7 to find that, for every $N, t > 0$,

$$\begin{aligned}&\int_{\mathbb{R}^d}\left(I_N * \tilde{I}_N\right)(x)\,\mathrm{d}x\int_0^t \mathrm{d}s\int_{\mathbb{R}^d} f(\mathrm{d}y)\;\boldsymbol{p}_{2s(t-s)/t}\left(y - \frac{s}{t}x\right)\chi_s(y)\\ &\quad\le \frac{t}{N\pi^d}\int_0^N \mathrm{d}s\,\chi_{st/N}(0)\int_{\mathbb{R}^d}\hat{f}(\mathrm{d}z)\,\mathrm{e}^{-t\|z\|^2(1-s/N)s/N}\prod_{j=1}^d \varphi(z_j s).\end{aligned} \tag{5.14}$$

Since $\sup_{0<r\le t}\chi_r(0)<\infty$ and $\lim_{N\to\infty}\chi_{st/N}(0)=0$ for all $s>0$ (see Lemma 5.6), the equality (5.14) and the dominated convergence theorem together imply (5.13). This completes the proof of the theorem when $\mathcal{R}(f)<\infty$.

We now assume that $\mathcal{R}(f)=\infty$ and aim to prove (5.1). Thanks to (5.9) and Lemma 5.5, $\chi_t(x)\ge\int_0^t(\boldsymbol{p}_{2s(t-s)/t}*f)(sx/t)\,\mathrm{d}s$. Therefore, (5.10) and Lemma 5.7 together imply that, for every $N,t>0$,

$$N\mathrm{Var}(\mathcal{S}_{N,t})\ge\frac{t}{\pi^d}\int_0^N\mathrm{d}s\int_{\mathbb{R}^d}\hat f(\mathrm{d}z)\,\mathrm{e}^{-t\|z\|^2(1-s/N)s/N}\prod_{j=1}^d\varphi(z_js).$$

Now we apply Fatou's lemma to conclude $\liminf_{N\to\infty}N\mathrm{Var}(\mathcal{S}_{N,t})\ge t\mathcal{R}(f)=\infty$. This implies (5.13). ∎

In the following, we give some criteria for the finiteness of $\mathcal{R}(f)$.

LEMMA 5.9. *If $d=1$, $\mathcal{R}(f)=\infty$. If $d\ge2$, $\mathcal{R}(f)<\infty$ is equivalent to one of the following:*

1. $\int_0^\infty r^{-d}f([-r\,,r]^d)\,\mathrm{d}r<\infty$;
2. $\int_{\mathbb{R}^d}\|x\|^{1-d}f(\mathrm{d}x)<\infty$;
3. $\int_{\mathbb{R}^d}\|z\|^{-1}\hat f(\mathrm{d}z)<\infty$.

Proof. Let $d=1$. According to (1.9), there exists $R>0$ such that $f([-R,R])>0$. Hence by Proposition 5.8,

$$\mathcal{R}(f)\ge\frac12\int_0^\infty r^{-1}f([-r\,,r])\,\mathrm{d}r\ge f([-R\,,R])\int_R^\infty r^{-1}\,\mathrm{d}r=\infty.$$

Assume $d\ge2$. By Proposition 5.8, we only need to prove that items 1, 2, and 3 are equivalent. Let $B_r=\{x\in\mathbb{R}^d:\|x\|\le r\}$ to see that $B_r\subseteq[-r\,,r]^d\subseteq B_{r\sqrt d}$, whence

$$\int_0^\infty\frac{f(B_r)}{r^d}\,\mathrm{d}r\le\int_0^\infty f\left([-r\,,r]^d\right)\frac{\mathrm{d}r}{r^d}\le(\sqrt d)^{d-1}\int_0^\infty\frac{f(B_r)}{r^d}\,\mathrm{d}r.$$

This proves the equivalence of 1 and 2 since Fubini's theorem ensures that

$$\int_0^\infty\frac{f(B_r)}{r^d}\,\mathrm{d}r=\frac{1}{d-1}\int_{\mathbb{R}^d}\frac{f(\mathrm{d}x)}{\|x\|^{d-1}}.$$

Next, we prove the equivalence of 1 and 3. We observe that for all $r>0$ and $z\in\mathbb{R}^d$,

$$\prod_{j=1}^d\frac{\sin^2(rz_j)}{(rz_j)^2}=2^{-d}r^{-d}\left[\left(\mathbf{1}_{[-1,1]^d}*\mathbf{1}_{[-1,1]^d}\right)(\bullet/r)\right]\hat{}\,(z).$$

Using the same approximation argument as in the proof of Proposition 5.8, we have

$$\begin{aligned}\int_0^\infty\!\mathrm{d}r\int_{\mathbb{R}^d}\hat f(\mathrm{d}z)\prod_{j=1}^d\frac{\sin^2(rz_j)}{(rz_j)^2}&=2^{-d}\int_0^\infty\frac{\mathrm{d}r}{r^d}\int_{\mathbb{R}^d}\hat f(\mathrm{d}z)\left[\left(\mathbf{1}_{[-1,1]^d}*\mathbf{1}_{[-1,1]^d}\right)(\bullet/r)\right]\hat{}\,(z)\\&=\pi^d\int_0^\infty\frac{\mathrm{d}r}{r^d}\int_{\mathbb{R}^d}f(\mathrm{d}z)\left(\mathbf{1}_{[-1,1]^d}*\mathbf{1}_{[-1,1]^d}\right)(z/r).\end{aligned}$$

Now we apply the inequality $\mathbf{1}_{[-1,1]^d}\le\mathbf{1}_{[-1,1]^d}*\mathbf{1}_{[-1,1]^d}\le2^d\mathbf{1}_{[-2,2]^d}$ and use Lemma 5.10 to conclude the equivalence of 1 and 3. ∎

LEMMA 5.10. *The following relation holds:*

$$\int_0^\infty \prod_{j=1}^d \frac{\sin^2(rz_j)}{(rz_j)^2}\,\mathrm{d}r \asymp \|z\|^{-1}.$$

Proof. On one hand, we can write

$$\prod_{j=1}^d \frac{\sin^2(rz_j)}{(rz_j)^2} \le \prod_{j=1}^d (1\wedge (r|z_j|)^{-2}) \le 1\wedge (r \max_{1\le j\le d}|z_j|)^{-2} \le 1\wedge (d^{-1/2}r\|z\|)^{-2},$$

which implies

$$\int_0^\infty \prod_{j=1}^d \frac{\sin^2(rz_j)}{(rz_j)^2}\,\mathrm{d}r \le \|z\|^{-1}\int_0^\infty \mathrm{d}r\,(1\wedge (d^{-1}r^{-2})).$$

On the other hand,

$$\int_0^\infty \prod_{j=1}^d \frac{\sin^2(rz_j)}{(rz_j)^2}\,\mathrm{d}r \ge \int_{r\|z\|\le 1} \prod_{j=1}^d \frac{\sin^2(rz_j)}{(rz_j)^2}\,\mathrm{d}r \ge \|z\|^{-1}\inf_{0<|x|\le 1}\left(\frac{\sin x}{x}\right)^{2d}.$$ ∎

Remark 5.11. From item 2 of Lemma 5.9, we deduce that $\mathcal{R}(f)=\infty$ if f is given by a Riesz kernel that satisfies Dalang's condition, $\Upsilon(1)<\infty$, i.e., $f(\mathrm{d}x)=\|x\|^{-\beta}\mathrm{d}x$ for some $0<\beta<d\wedge 2$.

5.2. Analysis in dimension $d=1$. Set $d=1$ and repeat the computations in the proof of Lemmas 5.5 and 5.7 to see that

$$\begin{aligned}\mathrm{Var}(\mathcal{S}_{N,t}) &= \frac{t}{\pi N}\int_0^t \frac{\mathrm{d}r}{r}\int_{-\infty}^\infty \mathrm{d}z\;\varphi(z)\mathrm{e}^{-\frac{t(t-r)}{r}\frac{z^2}{N^2}}\hat{f}\left(\frac{tz}{Nr}\right)\\ &\quad+\int_0^t \mathrm{d}s\int_{\mathbb{R}} f(\mathrm{d}y)\,\chi_s(y)\int_{\mathbb{R}^d}(I_N*\tilde{I}_N)(x)\boldsymbol{p}_{2s(t-s)/t}\left(y-\frac{s}{t}x\right)\mathrm{d}x\\ &:= V_N^{(1)}(t)+V_N^{(2)}(t),\end{aligned} \tag{5.15}$$

where φ and $\chi_s(y)$ are defined in (5.3) and (5.8), respectively.

LEMMA 5.12. *For all $t>0$, $V_N^{(2)}(t)=o(\log(N)/N)$ as $N\to\infty$.*

Proof. Choose and fix $\varepsilon>0$. Since $\sup_{y\in\mathbb{R}^d}\chi_s(y)=\chi_s(0)$, we apply Lemma 7.1 and the change of variables $z\mapsto tz/(Ns)$ to see that

$$\begin{aligned}\frac{N}{\log N}V_N^{(2)}(t) &\le \frac{t}{\pi\log N}\int_{\mathbb{R}}\mathrm{d}z\;\varphi(z)\int_0^t \mathrm{d}s\;\frac{\chi_s(0)}{s}\,\hat{f}\left(\frac{tz}{Ns}\right)\exp\left\{-\frac{t(t-s)}{N^2 s}z^2\right\}\\ &\le \frac{tf(\mathbb{R})}{\pi\log N}\int_{\mathbb{R}}\mathrm{d}z\;\varphi(z)\int_0^t \mathrm{d}s\;\frac{\chi_s(0)}{s}\exp\left\{-\frac{t(t-s)}{N^2 s}z^2\right\}\\ &:= T_{2,1}+T_{2,2},\end{aligned}$$

where

$$\begin{aligned}T_{2,1} &= \frac{tf(\mathbb{R})}{\pi\log N}\int_{\mathbb{R}}\mathrm{d}z\;\varphi(z)\int_0^t \frac{\mathrm{d}s}{s}\,\mathbf{1}_{\{s\le tN^{-\varepsilon}\}}\chi_s(0)\exp\left\{-\frac{t(t-s)}{N^2 s}z^2\right\},\\ T_{2,2} &= \frac{tf(\mathbb{R})}{\pi\log N}\int_{\mathbb{R}}\mathrm{d}z\;\varphi(z)\int_0^t \frac{\mathrm{d}s}{s}\,\mathbf{1}_{\{s> tN^{-\varepsilon}\}}\chi_s(0)\exp\left\{-\frac{t(t-s)}{N^2 s}z^2\right\}.\end{aligned}$$

By Lemma A.1 in Chen et al. [8], for all $\varepsilon > 0$,

$$T_{2,1} \le \frac{7t\log_+(1/t)f(\mathbb{R})}{\pi} \sup_{0\le s\le tN^{-\varepsilon}} \chi_s(0) \int_{\mathbb{R}} \mathrm{d}z\ \varphi(z)\log_+(1/|z|).$$

Hence by Lemma 5.6, for all $\varepsilon > 0$,

$$\limsup_{N\to\infty} T_{2,1} = 0. \tag{5.16}$$

Similarly, using Theorem 1.1 and the fact that $\int_{\mathbb{R}} \varphi(z)\mathrm{d}z = \pi$, we deduce

$$T_{2,2} \le tf(\mathbb{R}) \sup_{0\le s\le t} \chi_s(0) \frac{\log t - \log(tN^{-\varepsilon})}{\log N}. \tag{5.17}$$

Therefore, we conclude from (5.16) and (5.17) that, for all $\varepsilon > 0$,

$$\limsup_{N\to\infty} \frac{N}{\log N} V_N^{(2)}(t) \le tf(\mathbb{R}) \sup_{0\le s\le t} \chi_s(0)\varepsilon,$$

which proves this lemma by letting $\varepsilon \to 0$. ∎

Proof of Theorem 5.2. In the case that $f = \delta_0$, item 1 of Theorem 5.2 was proved in Chen et al. [8]. The same proof works for the more general f of the form $a\delta_0$. Therefore, we prove only (5.2) and item 2.

We recall that, from Lemma 7.1,

$$V_N^{(1)}(t) = \frac{t}{N\pi} \int_{\mathbb{R}} \mathrm{d}z\ \varphi(z) \int_0^t \frac{\mathrm{d}s}{s} \exp\left\{-\frac{t(t-s)}{N^2 s} z^2\right\} \hat{f}\left(\frac{tz}{Ns}\right).$$

Since $\hat{f}$ is maximized at 0,

$$V_N^{(1)}(t) \le \frac{tf(\mathbb{R})}{\pi N} \int_{\mathbb{R}} \mathrm{d}z\ \varphi(z) \int_0^t \frac{\mathrm{d}s}{s} \exp\left\{-\frac{t(t-s)}{N^2 s} z^2\right\}.$$

Hence Lemma A.1 of [8] and Lemma 5.12 imply the third inequality in (5.2).

On the other hand, using change of variables $s = trN^{-2}$,

$$\begin{aligned} V_N^{(1)}(t) &= \frac{t}{\pi N} \int_0^{N^2} \frac{\mathrm{d}r}{r} \int_{\mathbb{R}} \mathrm{d}z\ \varphi(z) \exp\left\{-tz^2\left[\frac{1-(r/N^2)}{r}\right]\right\} \hat{f}\left(\frac{zN}{r}\right) \\ &= \frac{t}{\pi N}(T_{1,1} + T_{1,2}), \end{aligned} \tag{5.18}$$

where

$$\begin{aligned} T_{1,1} &:= \int_0^1 \frac{\mathrm{d}r}{r} \int_{\mathbb{R}} \mathrm{d}z\ \varphi(z) \exp\left\{-tz^2\left[\frac{1-(r/N^2)}{r}\right]\right\} \hat{f}\left(\frac{zN}{r}\right), \\ T_{1,2} &:= \int_1^{N^2} \frac{\mathrm{d}r}{r} \int_{\mathbb{R}} \mathrm{d}z\ \varphi(z) \exp\left\{-tz^2\left[\frac{1-(r/N^2)}{r}\right]\right\} \hat{f}\left(\frac{zN}{r}\right). \end{aligned}$$

It is easy to see that, for all $N \ge 1$ and for all $a > 0$

$$\begin{aligned} \int_0^1 \exp\left\{-a\left[\frac{1-(r/N^2)}{r}\right]\right\} \frac{\mathrm{d}r}{r} &\le \int_0^1 \exp\left\{-a\left[\frac{1-r}{r}\right]\right\} \frac{\mathrm{d}r}{r} \\ &= \mathrm{e}^a \int_a^\infty \frac{\mathrm{e}^{-s}\,\mathrm{d}s}{s} \le \log_+(\mathrm{e}/a), \end{aligned}$$

where $\log_+(x) = \log(\mathrm{e} + x)$ for $x > 0$. Since $\sup \hat{f} = \hat{f}(0) = f(\mathbb{R})$, it follows that

$$T_{1,1} \leq f(\mathbb{R}) \int_{\mathbb{R}} \varphi(z) \log_+ \left(\frac{\mathrm{e}}{tz^2} \right) \mathrm{d}z < \infty.$$

Therefore,

$$\limsup_{N\to\infty} \frac{T_{1,1}}{\log N} = 0. \tag{5.19}$$

So all of the asymptotic behavior of $V_N^{(1)}(t)$ is captured via the asymptotic behavior of $T_{1,2}$. Now

$$\begin{aligned} T_{1,2} &= \int_{-\infty}^{\infty} \mathrm{d}z \; \varphi(z) \int_{1/N^2}^{1} \frac{\mathrm{d}s}{s} \exp\left\{ -tz^2 \left[\frac{1-s}{sN^2} \right] \right\} \hat{f}\left(\frac{z}{sN} \right) \\ &= T_{1,2,1} + T_{1,2,2}, \end{aligned}$$

where

$$\begin{aligned} T_{1,2,1} &:= \int_{-\log N}^{\log N} \mathrm{d}z \; \varphi(z) \int_{1/N^2}^{1} \frac{\mathrm{d}s}{s} \exp\left\{ -tz^2 \left[\frac{1-s}{sN^2} \right] \right\} \hat{f}\left(\frac{z}{sN} \right), \\ T_{1,2,2} &:= \int_{|z|>\log N} \mathrm{d}z \; \varphi(z) \int_{1/N^2}^{1} \frac{\mathrm{d}s}{s} \exp\left\{ -\frac{tz^2}{N^2} \left[\frac{1-s}{s} \right] \right\} \hat{f}\left(\frac{z}{sN} \right). \end{aligned}$$

Now,

$$0 \leq T_{1,2,2} \leq f(\mathbb{R}) \log(N^2) \int_{|z|>\log N} \mathrm{d}z \; \left(\frac{1-\cos z}{z^2} \right) = o(\log N). \tag{5.20}$$

So all of the asymptotic behavior of $V_N^{(1)}(t)$ is captured via the asymptotic behavior of $T_{1,2,1}$. To study that term, we rescale one more time (but slightly differently from before) in order to see that

$$\begin{aligned} T_{1,2,1} &= \int_{-\log N}^{\log N} \mathrm{d}z \; \varphi(z) \int_{1/N}^{N} \frac{\mathrm{d}r}{r} \exp\left\{ -\frac{tz^2}{N} \left[\frac{1}{r} - \frac{1}{N} \right] \right\} \hat{f}\left(\frac{z}{r} \right) \\ &\geq \exp\left\{ -\frac{t|\log N|^2}{N} \right\} \int_{-\log N}^{\log N} \mathrm{d}z \; \varphi(z) \int_{1}^{N} \frac{\mathrm{d}r}{r} \hat{f}\left(\frac{z}{r} \right). \end{aligned}$$

Hence,

$$\begin{aligned} T_{1,2,1} &\geq (1 + o(1)) \int_{-\log N}^{\log N} \mathrm{d}z \; \left(\frac{1-\cos z}{z^2} \right) \int_{(\log N)^2}^{N} \frac{\mathrm{d}r}{r} \hat{f}\left(\frac{z}{r} \right) \\ &= (f(\mathbb{R}) + o(1)) \int_{-\log N}^{\log N} \mathrm{d}z \; \left(\frac{1-\cos z}{z^2} \right) \int_{(\log N)^2}^{N} \frac{\mathrm{d}r}{r} \\ &= (f(\mathbb{R}) + o(1)) \int_{-\infty}^{\infty} \left(\frac{1-\cos z}{z^2} \right) \mathrm{d}z \; \log N \\ &= (\pi f(\mathbb{R}) + o(1)) \log N. \end{aligned}$$

This proves that

$$\pi f(\mathbb{R}) \leq \liminf_{N\to\infty} \frac{1}{\log N} T_{1,2,1}. \tag{5.21}$$

Therefore, Lemma 5.12 and the relations (5.18), (5.19), (5.20), and (5.21) prove the first inequality in (5.2).

It remains to prove item 2. We assume that $\hat{f}$ vanishes at infinity. Combining Lemma 5.12 and the above arguments, the problem is reduced to the following:

$$\limsup_{N\to\infty} \frac{T_{1,2,1}}{\log N} \leq \pi f(\mathbb{R}). \tag{5.22}$$

With this in mind, let us recall from the definition of $T_{1,2,1}$ that

$$T_{1,2,1} \leq \int_{-\log N}^{\log N} \varphi(z)\,\mathrm{d}z \int_{1/N}^{N} \frac{\mathrm{d}r}{r}\ \hat{f}(z/r).$$

Because

$$\int_{-\log N}^{\log N} \varphi(z)\,\mathrm{d}z\ \int_{(\log N)^2}^{N} \frac{\mathrm{d}r}{r}\ \hat{f}(z/r) \leq \pi f(\mathbb{R}) \int_{(\log N)^2}^{N} \frac{\mathrm{d}r}{r} \sim \pi f(\mathbb{R}) \log N,$$

as $N \to \infty$, this and symmetry reduce our goal (5.22) to proving that, when $\hat{f}$ vanishes at infinity,

$$\int_0^{\log N} \varphi(z)\,\mathrm{d}z\ \int_{1/N}^{(\log N)^2} \frac{\mathrm{d}r}{r}\ \hat{f}(z/r) = o(\log N) \qquad \text{as } N \to \infty.$$

Since $\int_{1/\sqrt{\log N}}^{(\log N)^2} r^{-1}\,\mathrm{d}r = o(\log N)$, we can further reduce our goal to proving the following: When $\hat{f}$ vanishes at infinity,

$$\int_0^{\log N} \varphi(z)\,\mathrm{d}z\ \int_{1/N}^{1/\sqrt{\log N}} \frac{\mathrm{d}r}{r}\ \hat{f}(z/r) = o(\log N) \qquad \text{as } N \to \infty.$$

But this is so since (1)

(5.23)
$$\int_{1/(\log N)^{1/4}}^{\log N} \varphi(z)\,\mathrm{d}z\ \int_{1/N}^{1/\sqrt{\log N}} \frac{\mathrm{d}r}{r}\ \hat{f}(z/r) \leq \pi \sup_{w\geq(\log N)^{1/4}} \hat{f}(w) \log N = o(\log N),$$

and (2) because $\varphi \leq 1$,

$$\int_0^{1/(\log N)^{1/4}} \varphi(z)\,\mathrm{d}z\ \int_{1/N}^{1/\sqrt{\log N}} \frac{\mathrm{d}r}{r}\ \hat{f}(z/r) \leq \frac{f(\mathbb{R})}{(\log N)^{1/4}} \int_{1/N}^{1/\sqrt{\log N}} \frac{\mathrm{d}r}{r} = o(\log N).$$

This proves item 2. ∎

5.3. Analysis of Riesz kernel case. We now aim to prove Theorem 5.4. Assume $f(\mathrm{d}x) = \|x\|^{-\beta}\mathrm{d}x$ and $\hat{f}(\mathrm{d}x) = \kappa_{\beta,d}\,\|x\|^{\beta-d}\mathrm{d}x$, where $0 < \beta < 2 \wedge d$ and $\kappa_{\beta,d}$ is a positive constant depending on β and d. In this case, we first provide another supporting lemma on the behavior of $\chi_t(x)$ as $x \to \infty$.

LEMMA 5.13. *Recall* (5.8). *For all* $t > 0$, $\lim_{x\to\infty} \chi_t(x) = 0$.

Proof. By the Poincaré inequality (2.4),

$$\begin{aligned}
|\chi_t(x)| &= |\operatorname{Cov}(U(t\,,0)\,,U(t\,,x))| \\
&\le \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} f(\mathrm{d}y) \int_{\mathbb{R}^d} \mathrm{d}y' \, \|D_{s,y'}U(t\,,0)\|_2 \|D_{s,y+y'}U(t\,,x)\|_2 \\
&\le C_{t,2}^2 \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} f(\mathrm{d}y) \int_{\mathbb{R}^d} \mathrm{d}y' \, \boldsymbol{p}_{s(t-s)/s}\,(y')\, \boldsymbol{p}_{s(t-s)/s}\left(y' + y - \frac{s}{t}x\right) \\
&= C_{t,2}^2 \int_0^t \mathrm{d}s \int_{\mathbb{R}^2} f(\mathrm{d}y)\, \boldsymbol{p}_{2s(t-s)/s}\left(y - \frac{s}{t}x\right) = \int_0^t \mathrm{d}s \left(\boldsymbol{p}_{2s(t-s)/s} * f\right)\left(\frac{s}{t}x\right),
\end{aligned}$$

where in the second inequality we use Proposition 4.1 and in the first equality we use semigroup property. Now we apply (3.1) to see that

$$\begin{aligned}
|\chi_t(x)| &\le C_{t,2}^2 \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \hat{f}(\mathrm{d}z)\, \exp\left\{-\frac{s(t-s)\|z\|^2}{t} + i\left(\frac{s}{t}\right) z\cdot x\right\} \\
&= \kappa_{\beta,d} C_{t,2}^2 \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}z \, \|z\|^{\beta-d} \exp\left\{-\frac{s(t-s)\|z\|^2}{t} + i\left(\frac{s}{t}\right) z\cdot x\right\}.
\end{aligned}$$

Since $\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}z\, \|z\|^{\beta-d} \exp\{-s(t-s)\|z\|^2/t\} < \infty$, the dominated convergence theorem and the Riemann–Lebesgue lemma together imply that $\lim_{x\to\infty}\chi_t(x) = 0$. ∎

Proof of Theorem 5.4, *part* 1*:* $0 < \beta < 1$. Let $\psi(x) := \prod_{i=1}^d (1 - |x_i|)$ for all $x \in \mathbb{R}^d$. We observe that $(I_N * \tilde{I})(x) = N^{-d}\psi(x/N)\mathbf{1}_{[-N,N]^d}(x)$ for all $x \in \mathbb{R}^d$. Recall (5.11) and (5.12). Since $f(\mathrm{d}x) = \|x\|^{-\beta}\mathrm{d}x$, we can write

$$\begin{aligned}
V_N^{(1)}(t) &= \frac{1}{N^d}\int_{[-N,N]^d} \mathrm{d}x\, \psi(x/N) \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\, \|y\|^{-\beta} \boldsymbol{p}_{2s(t-s)/t}\left(y - \frac{s}{t}x\right), \\
V_N^{(2)}(t) &= \frac{1}{N^d}\int_{[-N,N]^d} \mathrm{d}x\, \psi(x/N) \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\, \|y\|^{-\beta} \boldsymbol{p}_{2s(t-s)/t}\left(y - \frac{s}{t}x\right)\chi_s(y).
\end{aligned}$$

The term $V_N^{(1)}(t)$ can be expressed as

$$\begin{aligned}
V_N^{(1)}(t) &= \frac{1}{N^d}\int_{[-N,N]^d} \mathrm{d}x\ \psi(x/N) \int_0^t \mathrm{d}s\, \mathrm{E}\left(\left\|\sqrt{\frac{2s(t-s)}{t}}Z - \frac{s}{t}x\right\|^{-\beta}\right) \\
&= N^{-\beta}\int_{[-1,1]^d} \mathrm{d}z\ \psi(z) \int_0^t \mathrm{d}s\ \mathrm{E}\left(\left\|\frac{1}{N}\sqrt{\frac{2s(t-s)}{t}}Z - \frac{s}{t}z\right\|^{-\beta}\right),
\end{aligned}$$

where we have made the change of variable $x = Nz$ and Z denotes a d-dimensional standard normal random variable. An easy exercise shows that $\lim_{N\to\infty}(\boldsymbol{p}_{1/N} * \|\cdot\|^{-\beta})(x) = \|x\|^{-\beta}$ for all $x \in \mathbb{R}^d \setminus \{0\}$, which implies that for any $s \in (0\,,t]$ and $z \in \mathbb{R}^d \setminus \{0\}$,

$$\lim_{N\to\infty} \mathrm{E}\left(\left\|\frac{1}{N}\sqrt{2s(t-s)/t}Z - \frac{s}{t}z\right\|^{-\beta}\right) = t^\beta s^{-\beta}\|z\|^{-\beta}.$$

Moveover, according to Lemma 3.1 of [17],

$$\sup_{N\ge1} \mathrm{E}\left(\left\|\frac{1}{N}\sqrt{\frac{2s(t-s)}{t}}Z - \frac{s}{t}z\right\|^{-\beta}\right) \le Ct^\beta s^{-\beta}\|z\|^{-\beta}.$$

Because $\beta < 1$, the dominated convergence theorem implies that

$$\lim_{N\to\infty} N^\beta V_N^{(1)}(t) = \int_{[-1,1]^d} \mathrm{d}z\, \psi(z) \int_0^t \mathrm{d}s\ t^\beta s^{-\beta} \|z\|^{-\beta} < \infty.$$

Finally to complete the proof of (5.4) it suffices to show that

$$\lim_{N\to\infty} N^\beta V_N^{(2)}(t) = 0. \tag{5.24}$$

Using the same arguments as before and recalling $\chi_s(y)$ in (5.8), we can write

$$N^\beta V_N^{(2)}(t) \le \int_{[-1,1]^d} \mathrm{d}z\, \psi(z) \int_0^t \mathrm{d}s\ \mathrm{E}\left[\chi_s\left(\sqrt{\frac{2s(t-s)}{t}}Z - \frac{s}{t}Nz\right) \left\|\frac{1}{N}\sqrt{2s(t-s)/t}Z - \frac{s}{t}z\right\|^{-\beta}\right].$$

Thus, we can conclude (5.24) from the fact that $\chi_s(\sqrt{2s(t-s)/t}Z - (sNz)/t)$ is uniformly bounded (Theorem 1.1) and converges to zero almost surely as $N \to \infty$ (Lemma 5.13). ∎

Before we move on to proving part 2 and part 3, we express the quantities $V_N^{(1)}(t)$ and $V_N^{(2)}(t)$ using $\hat{f}(\mathrm{d}x) = \kappa_{\beta,d}\|x\|^{\beta-d}\mathrm{d}x$. In fact, from (5.11) and using the identity (7.1), we see that

$$\begin{aligned} V_N^{(1)}(t) &= \frac{\kappa_{\beta,d}}{\pi^d} \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{e}^{-s(t-s)\|z\|^2/t} \prod_{j=1}^d \frac{1-\cos(Nz_j s/t)}{(Nz_j s/t)^2} \|z\|^{\beta-d}\mathrm{d}z \\ &= \frac{\kappa_{\beta,d}}{\pi^d N^\beta} \int_0^t \mathrm{d}s\, \frac{t^\beta}{s^\beta} \int_{\mathbb{R}^d} \mathrm{d}z\ \|z\|^{\beta-d} \prod_{j=1}^d \varphi(z_j) \exp\left\{-\frac{t(t-s)}{N^2 s}\|z\|^2\right\} \\ &= \frac{t\,\kappa_{\beta,d}}{\pi^d N^\beta} \int_{\mathbb{R}^d} \mathrm{d}z\ \|z\|^{\beta-d} \prod_{j=1}^d \varphi(z_j) \int_0^\infty \mathrm{d}r\,(1+r)^{\beta-2} \exp\left\{-\frac{rt\|z\|^2}{N^2}\right\} \qquad (5.25)\\ &= \frac{t^{2-\beta}\,\kappa_{\beta,d}}{\pi^d N^{2-\beta}} \int_{\mathbb{R}^d} \mathrm{d}z\ \|z\|^{2-\beta-d} \prod_{j=1}^d \varphi(z_j) \int_0^\infty \mathrm{d}r\ \left(\frac{t\|z\|^2}{N^2}+r\right)^{\beta-2} \mathrm{e}^{-r}, \qquad (5.26) \end{aligned}$$

where φ is defined in (5.3) and we use change of variables in the last three equalities. Similarly, using change of variables and (5.12)

$$\begin{aligned} V_N^{(2)}(t) &\le \frac{t\,\kappa_{\beta,d}}{\pi^d N^\beta} \int_{\mathbb{R}^d} \mathrm{d}z\ \|z\|^{\beta-d} \prod_{j=1}^d \varphi(z_j) \int_0^\infty \mathrm{d}r\,(1+r)^{\beta-2}\mathrm{e}^{-r\cdot\frac{t\|z\|^2}{N^2}} \chi_{t(1+r)^{-1}}(0) \\ &= \frac{t^{2-\beta}\,\kappa_{\beta,d}}{\pi^d N^{2-\beta}} \int_{\mathbb{R}^d} \mathrm{d}z\ \|z\|^{2-\beta-d} \prod_{j=1}^d \varphi(z_j) \int_0^\infty \mathrm{d}r\ \left(\frac{t\|z\|^2}{N^2}+r\right)^{\beta-2} \\ &\quad \times \mathrm{e}^{-r}\chi_{t(1+rN^2/(t\|z\|^2))^{-1}}(0). \end{aligned} \tag{5.27}$$

Proof of Theorem 5.4, *part* 2: $\beta = 1$. Using (5.26) with $\beta = 1$, we have

(5.28)
$$\frac{N}{\log N} V_N^{(1)}(t) = \frac{\kappa_{1,d}}{\pi^d} \int_{\mathbb{R}^d} \mathrm{d}z \; \|z\|^{1-d} \prod_{j=1}^d \varphi(z_j) \frac{t}{\log N} \int_0^\infty \mathrm{d}r \, \left(\frac{t\|z\|^2}{N^2} + r \right)^{-1} \mathrm{e}^{-r}.$$

According to Lemma A.1 of Chen et al. [8], we have

$$\frac{t}{\log N} \int_0^\infty \mathrm{d}r \, \left(\frac{t\|z\|^2}{N^2} + r \right)^{-1} \mathrm{e}^{-r} = \frac{t}{\log N} \int_0^t \exp\left(-\frac{(t-s)t}{s} \cdot \frac{\|z\|^2}{N^2} \right) \frac{\mathrm{d}s}{s}$$
$$\leq 7t \log_+(1/t) \log_+(1/\|z\|) \quad \text{for all } N \geq \mathrm{e}, \tag{5.29}$$

where $\log_+(a) = \log(\mathrm{e} + a)$ for $a > 0$, and

$$\lim_{N\to\infty} \frac{t}{\log N} \int_0^\infty \mathrm{d}r \, \left(\frac{t\|z\|^2}{N^2} + r \right)^{-1} \mathrm{e}^{-r} = 2t \qquad \text{for all } z \in \mathbb{R}^d \setminus \{0\}. \tag{5.30}$$

Therefore, since $\int_{\mathbb{R}^d} \|z\|^{1-d} \prod_{j=1}^d \varphi(z_j) \log_+(1/\|z\|)\mathrm{d}z < \infty$, by (5.28)–(5.30) and the dominated convergence theorem,

$$\lim_{N\to\infty} \frac{N}{\log N} V_N^{(1)}(t) = \frac{2t\,\kappa_{1,d}}{\pi^d} \int_{\mathbb{R}^d} \|z\|^{1-d} \prod_{j=1}^d \varphi(z_j)\mathrm{d}z.$$

In light of (5.5), it suffices to prove

$$\lim_{N\to\infty} \frac{N}{\log N} V_N^{(2)}(t) = 0. \tag{5.31}$$

Similarly, letting $\beta = 1$ in (5.27),

$$\frac{N}{\log N} V_N^{(2)}(t) \leq \frac{\kappa_{1,d}}{\pi^d} \int_{\mathbb{R}^d} \mathrm{d}z \; \|z\|^{1-d} \prod_{j=1}^d \varphi(z_j) \frac{t}{\log N} \int_0^\infty \mathrm{d}r \, \left(\frac{t\|z\|^2}{N^2} + r \right)^{-1} \times \mathrm{e}^{-r} \chi_{t(1+rN^2/(t\|z\|^2))^{-1}}(0). \tag{5.32}$$

Choose and fix $0 < \varepsilon < 2$. We see from (5.32) and (5.29) that

$$\frac{N}{\log N} V_N^{(2)}(t) \leq \frac{\kappa_{1,d}}{\pi^d} \sup_{0\leq s\leq t} \chi_s(0) \int_{\mathbb{R}^d} \mathrm{d}z \; \|z\|^{1-d} \prod_{j=1}^d \varphi(z_j) \frac{t}{\log N} \int_0^{N^{-\varepsilon}} \mathrm{d}r \, \left(\frac{t\|z\|^2}{N^2} + r \right)^{-1}$$
$$+ \frac{\kappa_{1,d}}{\pi^d} 7t \log_+(1/t) \int_{\mathbb{R}^d} \mathrm{d}z \; \|z\|^{1-d} \prod_{j=1}^d \varphi(z_j) \log_+(1/\|z\|) \sup_{0\leq s\leq t^2\|z\|^2/N^{2-\varepsilon}} \chi_s(0).$$

Letting $N \to \infty$ and using Lemma 5.6 and the dominated convergence theorem, we conclude that for every $0 < \varepsilon < 2$,

$$\limsup_{N\to\infty} \frac{N}{\log N} V_N^{(2)}(t) \leq \frac{t(2-\varepsilon)\kappa_{1,d}}{\pi^d} \sup_{0\leq s\leq t} \chi_s(0) \int_{\mathbb{R}^d} \mathrm{d}z \; \|z\|^{1-d} \prod_{j=1}^d \varphi(z_j).$$

Since the choice of $0 < \varepsilon < 2$ is arbitrary, we let $\varepsilon \to 2$ to obtain (5.31). This proves (5.5). ∎

Proof of Theorem 5.4*, part* 3*:* $1 < \beta < 2$. Recall (5.26). Under the condition $1 < \beta < 2$, we have $\int_{\mathbb{R}^d} \|z\|^{2-\beta-d} \prod_{j=1}^d \varphi(z_j)\mathrm{d}z < \infty$ and $\int_0^\infty r^{\beta-2}\mathrm{e}^{-r}\mathrm{d}r < \infty$. Hence by the dominated convergence theorem,

$$\lim_{N\to\infty} N^{2-\beta} V_N^{(1)}(t) = \frac{t^{2-\beta}\,\kappa_{\beta,d}}{\pi^d} \int_{\mathbb{R}^d} \|z\|^{2-\beta-d} \prod_{j=1}^d \varphi(z_j)\mathrm{d}z \int_0^\infty r^{\beta-2}\mathrm{e}^{-r}\mathrm{d}r. \tag{5.33}$$

Moreover, from (5.27), Lemma 5.6, and the dominated convergence theorem

$$\begin{aligned}
&\limsup_{N\to\infty} N^{2-\beta} V_N^{(2)}(t) \\
&\le \frac{t^{2-\beta}\,\kappa_{\beta,d}}{\pi^d} \int_{\mathbb{R}^d} \mathrm{d}z\ \|z\|^{2-\beta-d} \prod_{j=1}^d \varphi(z_j) \int_0^\infty \mathrm{d}r\,\mathrm{e}^{-r} \lim_{N\to\infty} \left(\frac{t\|z\|^2}{N^2} + r\right)^{\beta-2} \chi_{t(1+rN^2/(t\|z\|^2))^{-1}}(0) \\
&= 0,
\end{aligned}$$

which together with (5.33) proves (5.6). ◻

6. Total variation distance. In this section, we will estimate the total variation distance and prove Theorems 1.3–1.5.

We recall that

$$\mathcal{S}_{N,t} = \frac{1}{N^d}\int_{[0,N]^d} [U(t\,,x)-1]\,\mathrm{d}x \quad \text{and} \quad V_N(t) = \mathrm{Var}(\mathcal{S}_{N,t}).$$

We can estimate the total variation distance between the normalized random variable

$$\widetilde{\mathcal{S}}_{N,t} := \mathcal{S}_{N,t}/\sqrt{V_N(t)}$$

and an $\mathrm{N}(0\,,1)$ random variable Z using the inequality (2.5). According to the inequality (2.5), we need to express the random variable $\widetilde{\mathcal{S}}_{N,t}$ as a divergence or as an Itô–Walsh stochastic integral. From (1.6) we obtain $\widetilde{\mathcal{S}}_{N,t} = V_N(t)^{-1/2}\delta(v_N)$, where

$$v_N(s,y) = \frac{1}{N^d} U(s\,,y) \int_{[0,N]^d} \boldsymbol{p}_{s(t-s)/t}\left(y - \frac{s}{t}x\right)\mathrm{d}x. \tag{6.1}$$

In this way, inequality (2.5) yields

$$d_{\mathrm{TV}}(\widetilde{\mathcal{S}}_{N,t}, Z) \le \frac{2}{V_N(t)}\sqrt{\mathrm{Var}\left(\langle D\mathcal{S}_{N,t}, v_N\rangle_{\mathcal{H}}\right)}. \tag{6.2}$$

The Malliavin derivative of $\mathcal{S}_{N,t}$ can be computed as follows:

$$\begin{aligned}
D_{s,y}\mathcal{S}_{N,t} &= \frac{1}{N^d}\left(\int_{[0,N]^d} \boldsymbol{p}_{s(t-s)/t}\left(y - \frac{s}{t}x\right)\mathrm{d}x\right) U(s\,,y) \\
&\quad + \frac{1}{N^d}\int_{(s,t)\times\mathbb{R}^d} \left(\int_{[0,N]^d} \boldsymbol{p}_{r(t-r)/t}\left(w - \frac{s}{t}x\right)\mathrm{d}x\right) D_{s,y}U(r\,,w)\,\eta(\mathrm{d}r,\mathrm{d}w).
\end{aligned} \tag{6.3}$$

From (6.1) and (6.3), we obtain

$$\begin{aligned}
\langle D\mathcal{S}_{N,t}\,, v_N\rangle_{\mathcal{H}} &= \frac{1}{N^{2d}}\int_0^t \mathrm{d}s \int_{\mathbb{R}^{2d}} f(\mathrm{d}z)\mathrm{d}y \int_{[0,N]^{2d}} \mathrm{d}x\mathrm{d}x' \\
&\qquad \boldsymbol{p}_{s(t-s)/t}\left(y - \frac{s}{t}x\right) \boldsymbol{p}_{s(t-s)/t}\left(y + z - \frac{s}{t}x\right) U(s\,,y)U(s\,,y+z) \\
&\quad + \frac{1}{N^{2d}}\int_0^t \int_{\mathbb{R}^d} \eta(\mathrm{d}r\,,\mathrm{d}w) \int_0^r \mathrm{d}s \int_{\mathbb{R}^{2d}} f(\mathrm{d}z)\mathrm{d}y \int_{[0,N]^{2d}} \mathrm{d}x\mathrm{d}x' \\
&\qquad \boldsymbol{p}_{r(t-r)/t}\left(w - \frac{r}{t}x\right) \boldsymbol{p}_{s(t-s)/t}\left(y + z - \frac{s}{t}x'\right) U(s\,,y+z) D_{s,y}U(r\,,w),
\end{aligned}$$

where we use the stochastic Fubini's theorem in the second equality. As a consequence,

$$\mathrm{Var}\left(\langle D\mathcal{S}_{N,t}, v_N\rangle_{\mathcal{H}}\right) \leq \frac{2}{N^{4d}}\left(\Phi_N^{(1)} + \Phi_N^{(2)}\right), \tag{6.4}$$

where

$$\begin{aligned}\Phi_N^{(1)} = &\int_{[0,t]^2} \mathrm{d}s_1\mathrm{d}s_2 \int_{\mathbb{R}^{4d}} f(\mathrm{d}z_1)f(\mathrm{d}z_2)\mathrm{d}y_1\mathrm{d}y_2\\ &\times \int_{[0,N]^{4d}} \mathrm{d}x_1\mathrm{d}x_1'\mathrm{d}x_2\mathrm{d}x_2'\, \boldsymbol{p}_{s_1(t-s_1)/t}\left(y_1 - \frac{s_1}{t}x_1\right)\\ &\times \boldsymbol{p}_{s_1(t-s_1)/t}\left(y_1 + z_1 - \frac{s_1}{t}x_1'\right)\boldsymbol{p}_{s_2(t-s_2)/t}\left(y_2 - \frac{s_2}{t}x_2\right)\boldsymbol{p}_{s_2(t-s_2)/t}\left(y_2 + z_2 - \frac{s_2}{t}x_2'\right)\\ &\times \mathrm{Cov}\left(U(s_1\,,y_1)U(s_1\,,y_1+z_1)\,,U(s_2\,,y_2)U(s_2\,,y_2+z_2)\right),\end{aligned}$$

and

$$\begin{aligned}\Phi_N^{(2)} = &\int_0^t \mathrm{d}r \int_{[0,r]^2} \mathrm{d}s_1\mathrm{d}s_2 \int_{\mathbb{R}^{6d}} f(\mathrm{d}b)\mathrm{d}w f(\mathrm{d}z_1)\mathrm{d}y_1 f(\mathrm{d}z_2)\mathrm{d}y_2 \int_{[0,N]^{4d}} \mathrm{d}x_1\mathrm{d}x_1'\mathrm{d}x_2\mathrm{d}x_2'\\ &\times \boldsymbol{p}_{r(t-r)/t}\left(w - \frac{r}{t}x_1\right)\boldsymbol{p}_{s_1(t-s_1)/t}\left(y_1 + z_1 - \frac{s_1}{t}x_1'\right)\\ &\times \boldsymbol{p}_{r(t-r)/t}\left(w + b - \frac{r}{t}x_2\right)\boldsymbol{p}_{s_2(t-s_2)/t}\left(y_2 + z_2 - \frac{s_2}{t}x_2'\right)\\ &\times \mathrm{E}\left[U(s_1\,,y_1+z_1)D_{s_1,y_1}U(r\,,w)U(s_2\,,y_2+z_2)D_{s_2,y_2}U(r\,,w+b)\right].\end{aligned}$$

We are going to estimate the terms $\Phi_N^{(1)}$ and $\Phi_N^{(2)}$. Using the Poincaré inequality (2.4), we can write

$$\begin{aligned}\Phi_N^{(1)} \leq &\int_{[0,t]^2} \mathrm{d}s_1\mathrm{d}s_2 \int_0^{s_1\wedge s_2} \mathrm{d}r \int_{\mathbb{R}^{6d}} f(\mathrm{d}z_1)f(\mathrm{d}z_2)f(\mathrm{d}b)\mathrm{d}a\mathrm{d}y_1\mathrm{d}y_2 \int_{[0,N]^{4d}} \mathrm{d}x_1\mathrm{d}x_1'\mathrm{d}x_2\mathrm{d}x_2'\\ &\times \boldsymbol{p}_{s_1(t-s_1)/t}\left(y_1 - \frac{s_1}{t}x_1\right)\boldsymbol{p}_{s_1(t-s_1)/t}\left(y_1 + z_1 - \frac{s_1}{t}x_1'\right)\\ &\times \boldsymbol{p}_{s_2(t-s_2)/t}\left(y_2 - \frac{s_2}{t}x_2\right)\boldsymbol{p}_{s_2(t-s_2)/t}\left(y_2 + z_2 - \frac{s_2}{t}x_2'\right)\\ &\times \Big(\left\|D_{r,a}U(s_1\,,y_1)\right\|_4\left\|U(s_1\,,y_1+z_1)\right\|_4 + \left\|U(s_1\,,y_1)\right\|_4\left\|D_{r,a}U(s_1\,,y_1+z_1)\right\|_4\Big)\\ &\times \Big(\left\|D_{r,a+b}U(s_2\,,y_2)\right\|_4\left\|U(s_2\,,y_2+z_2)\right\|_4 + \left\|U(s_2\,,y_2)\right\|_4\left\|D_{r,a+b}U(s_2\,,y_2+z_2)\right\|_4\Big).\end{aligned}$$

The estimates (1.8) and (4.1) and the semigroup property yield

$$\begin{aligned}\Phi_N^{(1)} \leq 4C_{t,4}^2c_{t,4}^2 &\int_{[0,t]^2} \mathrm{d}s_1\mathrm{d}s_2 \int_0^{s_1\wedge s_2} \mathrm{d}r \int_{\mathbb{R}^{5d}} f(\mathrm{d}z_1)f(\mathrm{d}z_2)f(\mathrm{d}b)\mathrm{d}y_1\mathrm{d}y_2 \int_{[0,N]^{4d}} \mathrm{d}x_1\mathrm{d}x_1'\mathrm{d}x_2\mathrm{d}x_2'\\ &\times \boldsymbol{p}_{s_1(t-s_1)/t}\left(y_1 - \frac{s_1}{t}x_1\right)\boldsymbol{p}_{s_1(t-s_1)/t}\left(y_1 + z_1 - \frac{s_1}{t}x_1'\right)\\ &\times \boldsymbol{p}_{s_2(t-s_2)/t}\left(y_2 - \frac{s_2}{t}x_2\right)\boldsymbol{p}_{s_2(t-s_2)/t}\left(y_2 + z_2 - \frac{s_2}{t}x_2'\right)\\ &\times \Bigg[\boldsymbol{p}_{r(s_1-r)/s_1+r(s_2-r)/s_2}\left(b - \frac{r}{s_2}y_2 + \frac{r}{s_1}y_1\right) + \boldsymbol{p}_{r(s_1-r)/s_1+r(s_2-r)/s_2}\\ &\quad\times\left(b - \frac{r}{s_2}(y_2+z_2) + \frac{r}{s_1}y_1\right)\\ &\quad + \boldsymbol{p}_{r(s_1-r)/s_1+r(s_2-r)/s_2} \times \left(b - \frac{r}{s_2}y_2 + \frac{r}{s_1}(y_1+z_1)\right)\\ &\quad + \boldsymbol{p}_{r(s_1-r)/s_1+r(s_2-r)/s_2}\left(b - \frac{r}{s_2}(y_2+z_2) + \frac{r}{s_1}(y_1+z_1)\right)\Bigg].\end{aligned}$$

By symmetry, we conclude that

$$\begin{aligned}\Phi_N^{(1)} \le 16C_{t,4}^2 c_{t,4}^2 \int_{[0,t]^2} \mathrm{d}s_1 \mathrm{d}s_2 \int_0^{s_1\wedge s_2} \mathrm{d}r \int_{\mathbb{R}^{5d}} f(\mathrm{d}z_1) f(\mathrm{d}z_2) f(\mathrm{d}b) \mathrm{d}y_1 \mathrm{d}y_2 \int_{[0,N]^{4d}} \mathrm{d}x_1 \mathrm{d}x_1' \mathrm{d}x_2 \mathrm{d}x_2' \\ \times \boldsymbol{p}_{s_1(t-s_1)/t}\left(y_1 - \frac{s_1}{t}x_1\right) \boldsymbol{p}_{s_1(t-s_1)/t}\left(y_1 + z_1 - \frac{s_1}{t}x_1'\right) \\ \times \boldsymbol{p}_{s_2(t-s_2)/t}\left(y_2 - \frac{s_2}{t}x_2\right) \boldsymbol{p}_{s_2(t-s_2)/t}\left(y_2 + z_2 - \frac{s_2}{t}x_2'\right) \\ \times \boldsymbol{p}_{r(s_1-r)/s_1 + r(s_2-r)/s_2}\left(b - \frac{r}{s_2}y_2 + \frac{r}{s_1}y_1\right).\end{aligned} \tag{6.5}$$

As for $\Phi_N^{(2)}$, similarly, by the Cauchy–Schwarz inequality and the estimates (1.8) and (4.1), one sees that

$$\begin{aligned}\Phi_N^{(2)} &\le C_{t,4}^2 c_{t,4}^2 \int_0^t \mathrm{d}r \int_{[0,r]^2} \mathrm{d}s_1 \mathrm{d}s_2 \int_{\mathbb{R}^{6d}} f(\mathrm{d}b) \mathrm{d}w f(\mathrm{d}z_1) \mathrm{d}y_1 f(\mathrm{d}z_2) \mathrm{d}y_2 \int_{[0,N]^{4d}} \mathrm{d}x_1 \mathrm{d}x_1' \mathrm{d}x_2 \mathrm{d}x_2' \\ &\quad \times \boldsymbol{p}_{r(t-r)/t}\left(w - \frac{r}{t}x_1\right) \boldsymbol{p}_{s_1(t-s_1)/t}\left(y_1 + z_1 - \frac{s_1}{t}x_1'\right) \boldsymbol{p}_{r(t-r)/t}\left(w + b - \frac{r}{t}x_2\right) \\ &\quad \times \boldsymbol{p}_{s_2(t-s_2)/t}\left(y_2 + z_2 - \frac{s_2}{t}x_2'\right) \boldsymbol{p}_{s_1(r-s_1)/r}\left(y_1 - \frac{s_1}{r}w\right) \boldsymbol{p}_{s_2(r-s_2)/r}\left(y_2 - \frac{s_2}{r}(w+b)\right) \\ &= C_{t,4}^2 c_{t,4}^2 \int_0^t \mathrm{d}r \int_{[0,r]^2} \mathrm{d}s_1 \mathrm{d}s_2 \int_{\mathbb{R}^{4d}} f(\mathrm{d}b) \mathrm{d}w f(\mathrm{d}z_1) f(\mathrm{d}z_2) \int_{[0,N]^{4d}} \mathrm{d}x_1 \mathrm{d}x_1' \mathrm{d}x_2 \mathrm{d}x_2' \\ &\quad \times \boldsymbol{p}_{r(t-r)/t}\left(w - \frac{r}{t}x_1\right) \boldsymbol{p}_{s_1(t-s_1)/t + s_1(r-s_1)/r}\left(z_1 - \frac{s_1}{t}x_1' + \frac{s_1}{r}w\right) \\ &\quad \times \boldsymbol{p}_{r(t-r)/t}\left(w + b - \frac{r}{t}x_2\right) \boldsymbol{p}_{s_2(t-s_2)/t + s_2(r-s_2)/r}\left(z_2 - \frac{s_2}{t}x_2' + \frac{s_2}{r}(w+b)\right),\end{aligned} \tag{6.6}$$

where we use a semigroup property in the equality.

In the following, we will prove Theorems 1.3–1.5 separately. The identity below will be used several times later on:

$$\boldsymbol{p}_t(\sigma x) = \sigma^{-d}\boldsymbol{p}_{t/\sigma^2}(x) \quad \text{for all } x \in \mathbb{R}^d \text{ and } t,\, \sigma > 0. \tag{6.7}$$

6.1. Proof of Theorem 1.3.

Proof of Theorem 1.3. With the notation introduced in (6.4) and according to Theorem 5.1, it suffices to show that

$$N^{-4d+3}\left(\Phi_N^{(1)} + \Phi_N^{(1)}\right) \le C \tag{6.8}$$

for all $N \ge \mathrm{e}$ and for some constant C depending on t.

We will start with the expression for $\Phi_N^{(1)}$ given in (6.5). Using the elementary relation

$$\boldsymbol{p}_\sigma(x)\boldsymbol{p}_\sigma(y) = 2^d \boldsymbol{p}_{2\sigma}(x+y)\boldsymbol{p}_{2\sigma}(x-y), \qquad \sigma > 0,\ x, y \in \mathbb{R}^d, \tag{6.9}$$

we can write

$$\begin{aligned}&\boldsymbol{p}_{s_1(t-s_1)/t}\left(y_1 - \frac{s_1}{t}x_1\right) \boldsymbol{p}_{s_1(t-s_1)/t}\left(y_1 + z_1 - \frac{s_1}{t}x_1'\right) \\ &\qquad = 2^d \boldsymbol{p}_{2s_1(t-s_1)/t}\left(2y_1 + z_1 - \frac{s_1}{t}(x_1 + x_1')\right) \boldsymbol{p}_{2s_1(t-s_1)/t}\left(z_1 - \frac{s_1}{t}(x_1' - x_1)\right) \\ &\qquad = \boldsymbol{p}_{s_1(t-s_1)/(2t)}\left(y_1 + \frac{z_1}{2} - \frac{s_1}{2t}(x_1 + x_1')\right) \boldsymbol{p}_{2s_1(t-s_1)/t}\left(z_1 - \frac{s_1}{t}(x_1' - x_1)\right),\end{aligned}$$

where in the second equality we used the scaling property (6.7). In the same way, we obtain

$$\begin{aligned}&\boldsymbol{p}_{s_2(t-s_2)/t}\left(y_2-\frac{s_2}{t}x_2\right)\boldsymbol{p}_{s_2(t-s_2)/t}\left(y_2+z_2-\frac{s_2}{t}x_2'\right)\\&\quad=\boldsymbol{p}_{s_2(t-s_2)/(2t)}\left(y_2+\frac{z_2}{2}-\frac{s_2}{2t}(x_2+x_2')\right)\boldsymbol{p}_{2s_2(t-s_2)/t}\left(z_2-\frac{s_2}{t}(x_2'-x_2)\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{L}:=&\int_{\mathbb{R}^{2d}}\mathrm{d}y_1\mathrm{d}y_2\ \boldsymbol{p}_{s_1(t-s_1)/t}\left(y_1-\frac{s_1}{t}x_1\right)\boldsymbol{p}_{s_1(t-s_1)/t}\left(y_1+z_1-\frac{s_1}{t}x_1'\right)\boldsymbol{p}_{s_2(t-s_2)/t}\left(y_2-\frac{s_2}{t}x_2\right)\\&\times\boldsymbol{p}_{s_2(t-s_2)/t}\left(y_2+z_2-\frac{s_2}{t}x_2'\right)\boldsymbol{p}_{r(s_1-r)/s_1+r(s_2-r)/s_2}\left(b-\frac{r}{s_2}y_2+\frac{r}{s_1}y_1\right)\\=&\left(\frac{s_1}{r}\right)^d\boldsymbol{p}_{s_1(t-s_1)/t}\left(z_1-\frac{s_1}{t}(x_1'-x_1)\right)\boldsymbol{p}_{s_2(t-s_2)/t}\left(z_2-\frac{s_2}{t}(x_2'-x_2)\right)\\&\times\int_{\mathbb{R}^{2d}}\mathrm{d}y_1\mathrm{d}y_2\ \boldsymbol{p}_{s_1(t-s_1)/(2t)}\left(y_1+\frac{z_1}{2}-\frac{s_1}{2t}(x_1+x_1')\right)\boldsymbol{p}_{s_2(t-s_2)/(2t)}\\&\times\left(y_2+\frac{z_2}{2}-\frac{s_2}{2t}(x_2+x_2')\right)\boldsymbol{p}_{(s_1/r)^2[r(s_1-r)/s_1+r(s_2-r)/s_2]}\left(\frac{s_1}{r}b-\frac{s_1}{s_2}y_2+y_1\right).\end{aligned}$$

With the notation

$$\mathcal{M}=\boldsymbol{p}_{2s_1(t-s_1)/t}\left(z_1-\frac{s_1}{t}(x_1'-x_1)\right)\boldsymbol{p}_{2s_2(t-s_2)/t}\left(z_2-\frac{s_2}{t}(x_2'-x_2)\right)$$

integrating in y_1 and using the semigroup property yields

$$\begin{aligned}\mathcal{L}=&\left(\frac{s_1}{r}\right)^d\mathcal{M}\int_{\mathbb{R}^d}\mathrm{d}y_2\ \boldsymbol{p}_{s_2(t-s_2)/(2t)}\left(y_2+\frac{z_2}{2}-\frac{s_2}{2t}(x_2+x_2')\right)\\&\times\boldsymbol{p}_{s_1(t-s_1)/(2t)+(s_1/r)^2[r(s_1-r)/s_1+r(s_2-r)/s_2]}\left(\frac{s_1}{r}b-\frac{s_1}{s_2}y_2-\frac{z_1}{2}+\frac{s_1}{2t}(x_1+x_1')\right)\\=&\left(\frac{s_2}{r}\right)^d\mathcal{M}\int_{\mathbb{R}^d}\mathrm{d}y_2\ \boldsymbol{p}_{s_2(t-s_2)/(2t)}\left(y_2+\frac{z_2}{2}-\frac{s_2}{2t}(x_2+x_2')\right)\\&\times\boldsymbol{p}_{(s_2/s_1)^2\{s_1(t-s_1)/(2t)+(s_1/r)^2[r(s_1-r)/s_1+r(s_2-r)/s_2]\}}\\&\times\left(\frac{s_2}{r}b-y_2-\frac{s_2}{2s_1}z_1+\frac{s_2}{2t}(x_1+x_1')\right),\end{aligned}$$

where in the second equality we used the scaling property (6.7). Integrating in y_2 and using the semigroup property we finally get

$$\mathcal{L}=\left(\frac{s_2}{r}\right)^d\mathcal{M}\,\boldsymbol{p}_{\alpha_1}\left(\frac{s_2}{r}b-\frac{s_2}{2s_1}z_1+\frac{s_2}{2t}(x_1+x_1'-x_2-x_2')+\frac{z_2}{2}\right),$$

where

$$\alpha_1=\frac{s_2(t-s_2)}{2t}+\left(\frac{s_2}{s_1}\right)^2\left\{\frac{s_1(t-s_1)}{2t}+\left(\frac{s_1}{r}\right)^2\left[\frac{r(s_1-r)}{s_1}+\frac{r(s_2-r)}{s_2}\right]\right\}.$$

A further application of the scaling property (6.7) yields

$$\mathcal{L}=\mathcal{M}\,\boldsymbol{p}_{\alpha_2}\left(b-\frac{r}{2s_1}z_1+\frac{r}{2s_2}z_2+\frac{r}{2t}(x_1+x_1'-x_2-x_2')\right),$$

where

$$\alpha_2 = \left(\frac{r}{s_2}\right)^2 \alpha_1 = \frac{r^2(t-s_2)}{2ts_2} + \frac{r^2(t-s_1)}{2ts_1} + \frac{r(s_1-r)}{s_1} + \frac{r(s_2-r)}{s_2}.$$

Making the change of variables $x_i \to Nx_i$ we obtain

$$\begin{aligned}
&N^{-4d+3}\Phi_N^{(1)} \\
&\quad \le N^3 16 C_{t,4}^2 c_{t,4}^2 \int_{[0,1]^{4d}} \mathrm{d}x_1 \mathrm{d}x_1' \mathrm{d}x_2 \mathrm{d}x_2' \int_{\mathbb{R}^{3d}} f(\mathrm{d}z_1) f(\mathrm{d}z_2) f(\mathrm{d}b) \int_{[0,t]^2} \mathrm{d}s_1 \mathrm{d}s_2 \int_0^{s_1\wedge s_2} \mathrm{d}r \\
&\qquad \times \boldsymbol{p}_{2s_1(t-s_1)/t}\left(z_1 - \frac{Ns_1}{t}(x_1'-x_1)\right) \boldsymbol{p}_{2s_2(t-s_2)/t}\left(z_2 - \frac{Ns_2}{t}(x_2'-x_2)\right) \\
&\qquad \times \boldsymbol{p}_{\alpha_2}\left(b - \frac{r}{2s_1}z_1 + \frac{r}{2s_2}z_2 + \frac{Nr}{2t}(x_1+x_1'-x_2-x_2')\right).
\end{aligned}$$

With a further change of variables $s_1 = \frac{t}{N}r_1$, $s_2 = \frac{t}{N}r_2$, $r = \frac{t}{N}\sigma$, we can write

$$\begin{aligned}
N^{-4d+3}\Phi_N^{(1)} &= 16 C_{t,4}^2 c_{t,4}^2 t^3 \int_{[0,1]^{4d}} \mathrm{d}x_1 \mathrm{d}x_1' \mathrm{d}x_2 \mathrm{d}x_2' \int_{\mathbb{R}^{3d}} f(\mathrm{d}z_1) f(\mathrm{d}z_2) f(\mathrm{d}b) \int_{[0,N]^2} \mathrm{d}r_1 \mathrm{d}r_2 \\
&\quad \times \boldsymbol{p}_{\frac{2tr_1}{N}(1-\frac{r_1}{N})}\left(z_1 - r_1(x_1'-x_1)\right) \boldsymbol{p}_{\frac{2tr_2}{N}(1-\frac{r_1}{N})}\left(z_2 - r_2(x_2'-x_2)\right) \\
&\quad \times \int_0^{r_1\wedge r_2} \mathrm{d}\sigma\, \boldsymbol{p}_{\gamma_{3,N}}\left(b - \frac{\sigma}{2r_1}z_1 + \frac{\sigma}{2r_2}z_2 + \frac{\sigma}{2}(x_1+x_1'-x_2-x_2')\right),
\end{aligned}$$

where

$$\gamma_{3,N} = \frac{t\sigma^2}{2N}\left(\frac{1}{r_1} + \frac{1}{r_2} - \frac{2}{N}\right) + \frac{t\sigma}{N}\left(2 - \frac{\sigma}{r_1} - \frac{\sigma}{r_2}\right).$$

We also set

$$\gamma_{1,N} = \frac{2tr_1}{N}\left(1 - \frac{r_1}{N}\right), \qquad \gamma_{2,N} = \frac{2tr_2}{N}\left(1 - \frac{r_2}{N}\right).$$

With the notation $y_1 = r_1(x_1'-x_1)$, $y_2 = r_2(x_2'-x_2)$, $y_3 = \frac{\sigma}{2}(x_1+x_1'-x_2-x_2')$, the Fourier transform of the function

$$\Psi_1(z_1,z_2,b) := \boldsymbol{p}_{\gamma_{1,N}}(z_1-y_1)\, \boldsymbol{p}_{\gamma_{2,N}}(z_2-y_2)\, \boldsymbol{p}_{\gamma_{3,N}}\left(b - \frac{\sigma}{2r_1}z_1 + \frac{\sigma}{2r_2}z_2 + y_3\right)$$

is given by

$$\begin{aligned}
\hat{\Psi}_1(\xi_1,\xi_2,\xi_3) &= \exp\left(-\frac{\gamma_{1,N}}{2}\left\|\xi_1 + \frac{\sigma}{2r_1}\xi_3\right\|^2 - \frac{\gamma_{2,N}}{2}\left\|\xi_2 - \frac{\sigma}{2r_2}\xi_3\right\|^2 - \frac{\gamma_{3,N}}{2}\|\xi_3\|^2\right) \\
&\quad \times \exp\left(i\left(\xi_1 + \frac{\sigma}{2r_1}\xi_3\right)\cdot y_1 + i\left(\xi_2 - \frac{\sigma}{2r_2}\xi_3\right)\cdot y_2 - i\xi_3\cdot y_3\right).
\end{aligned}$$

Notice that

$$\begin{aligned}
&\left(\xi_1 + \frac{\sigma}{2r_1}\xi_3\right)\cdot y_1 + \left(\xi_2 - \frac{\sigma}{2r_2}\xi_3\right)\cdot y_2 - \xi_3\cdot y_3 \\
&\quad = -x_1\cdot(r_1\xi_1 + \sigma\xi_3) - x_2\cdot(r_2\xi_2 - \sigma\xi_3) + x_1'\cdot(r_1\xi_1) + x_2'\cdot(r_2\xi_2).
\end{aligned}$$

Set

$$\begin{aligned}
\Delta_1(\xi_1,\xi_2,\xi_3) &:= \int_{[0,1]^{4d}} \mathrm{d}x_1 \mathrm{d}x_1' \mathrm{d}x_2 \mathrm{d}x_2' \\
&\qquad \exp\left(i\left(-x_1\cdot(r_1\xi_1+\sigma\xi_3) - x_2\cdot(r_2\xi_2-\sigma\xi_3) + x_1'\cdot(r_1\xi_1) + x_2'\cdot(r_2\xi_2)\right)\right).
\end{aligned}$$

Then, Parseval's identity implies that

$$\begin{aligned}
&N^{-4d+3}\Phi_N^{(1)}\\
&\quad\le 16C_{t,4}^2c_{t,4}^2t^3\frac{1}{(2\pi)^{3d}}\int_{[0,N]^2}\mathrm{d}r_1\mathrm{d}r_2\int_0^{r_1\wedge r_2}\mathrm{d}\sigma\int_{\mathbb{R}^{3d}}\hat f(\mathrm{d}\xi_1)\hat f(\mathrm{d}\xi_2)\hat f(\mathrm{d}\xi_3)\Delta_1(\xi_1,\xi_2,\xi_3)\\
&\qquad\times\exp\left(-\frac{\gamma_{1,N}}{2}\left\|\xi_1+\frac{\sigma}{2r_1}\xi_3\right\|^2-\frac{\gamma_{2,N}}{2}\left\|\xi_2-\frac{\sigma}{2r_2}\xi_3\right\|^2-\frac{\gamma_{3,N}}{2}\|\xi_3\|^2\right)\\
&\quad\le C\int_{[0,\infty)^3}\mathrm{d}\sigma\mathrm{d}r_1\mathrm{d}r_2\int_{\mathbb{R}^{3d}}\hat f(\mathrm{d}\xi_1)\hat f(\mathrm{d}\xi_2)\hat f(\mathrm{d}\xi_3)|\Delta_1(\xi_1,\xi_2,\xi_3)|.
\end{aligned}$$

Taking into account that $\mathcal{R}(f)<\infty$, which is equivalent by Lemma 5.9 to $\int_{\mathbb{R}^d}\|z\|^{-1}\hat f(\mathrm{d}z)<\infty$, it suffices to show that

$$\int_{[0,\infty)^3}\mathrm{d}\sigma\mathrm{d}r_1\mathrm{d}r_2|\Delta_1(\xi_1,\xi_2,\xi_3)|\le C(\|\xi_1\|\|\xi_2\|\|\xi_3\|)^{-1}\tag{6.10}$$

for some constant C not depending on t. We have

$$\begin{aligned}
&|\Delta_1(\xi_1,\xi_2,\xi_3)|\\
&\quad=\prod_{j=1}^d\frac{|\mathrm{e}^{-ir_1\xi_1^j}-1|}{r_1|\xi_1^j|}\frac{|\mathrm{e}^{-ir_2\xi_2^j}-1|}{r_2|\xi_2^j|}\frac{|\mathrm{e}^{i(r_1\xi_1^j+\sigma\xi_3^j)}-1|}{|r_1\xi_1^j+\sigma\xi_3^j|}\frac{|\mathrm{e}^{i(r_2\xi_2^j-\sigma\xi_3^j)}-1|}{|r_2\xi_2^j-\sigma\xi_3^j|}\\
&\quad\le 2^{4d}\prod_{j=1}^d(1\wedge(r_1|\xi_1^j|)^{-1})(1\wedge(r_2|\xi_2^j|)^{-1})(1\wedge|r_1\xi_1^j+\sigma\xi_3^j|^{-1})(1\wedge|r_2\xi_2^j-\sigma\xi_3^j|^{-1}).
\end{aligned}$$

For any $x\in\mathbb{R}^d$, we have

$$\prod_{i=1}^d\left(1\wedge|x^j|^{-1}\right)\le 1\wedge\left(\max_{1\le j\le d}|x_j|\right)^{-1}\le 1\wedge d^{-1/2}\|x\|^{-1}\le 1\wedge\|x\|^{-1}.\tag{6.11}$$

As a consequence,

$$|\Delta_1(\xi_1,\xi_2,\xi_3)|\le(1\wedge(r_1\|\xi_1\|)^{-1})(1\wedge(r_2\|\xi_2\|)^{-1})(1\wedge\|r_1\xi_1+\sigma\xi_3\|^{-1})(1\wedge\|r_2\xi_2-\sigma\xi_3\|^{-1}),$$

which implies

$$\begin{aligned}
&\int_{[0,\infty)^3}\mathrm{d}\sigma\mathrm{d}r_1\mathrm{d}r_2\ |\Delta(\xi_1,\xi_2,\xi_3)|\le C(\|\xi_1\|\|\xi_2\|\|\xi_3\|)^{-1}\\
&\qquad\times\int_{[0,\infty)^3}\mathrm{d}x\mathrm{d}y\mathrm{d}z\ (1\wedge x^{-1})(1\wedge y^{-1})(1\wedge\|xe_1+ze_3\|^{-1})(1\wedge\|ye_2-ze_3\|^{-1}),
\end{aligned}$$

where the last inequality follows from a change of variable, and e_i, $i=1,2,3$, are unit vectors. Note that

$$\|xe_1+ze_3\|^2=x^2+z^2+2xz\langle e_1,e_3\rangle\ge x^2+z^2-2xz=(x-z)^2.$$

Therefore,

$$\begin{aligned}
&\int_{[0,\infty)^3}\mathrm{d}x\mathrm{d}y\mathrm{d}z\ (1\wedge x^{-1})(1\wedge y^{-1})(1\wedge\|xe_1+ze_3\|^{-1})(1\wedge\|ye_2-ze_3\|^{-1})\\
&\qquad\le\int_{\mathbb{R}^3}\mathrm{d}x\mathrm{d}y\mathrm{d}z\ (1\wedge|x|^{-1})(1\wedge|y|^{-1})(1\wedge|x-z|^{-1})(1\wedge|y-z|^{-1}).
\end{aligned}$$

Finally, applying Hölder and Young's inequality, we obtain

$$\int_{\mathbb{R}^3} \mathrm{d}x\mathrm{d}y\mathrm{d}z\ (1\wedge|x|^{-1})(1\wedge|y|^{-1})(1\wedge|x-z|^{-1})(1\wedge|y-z|^{-1})$$
$$\le \|(1\wedge|\bullet|^{-1})*(1\wedge|\bullet|^{-1})\|^2_{L^2(\mathbb{R})} \le \|1\wedge|\bullet|^{-1}\|^4_{L^{4/3}(\mathbb{R})} < \infty.$$

Let us turn now to the analysis of $\Phi_N^{(2)}$ given in (6.6). Because the variable w appears in four heat kernels and three of them have different variances, we cannot proceed as in the case of $\Phi_N^{(1)}$. Then, we start making the changes of variables without integrating in w. The first change of variables is $x_i \to Nx_i$, which yields

$$\begin{aligned}
&N^{-4d+3}\Phi_N^{(2)}\\
&\quad= N^3C_{t,4}^2c_{t,4}^2\int_0^t \mathrm{d}r\int_{[0,r]^2}\mathrm{d}s_1\mathrm{d}s_2\int_{\mathbb{R}^{4d}} f(\mathrm{d}b)\mathrm{d}wf(\mathrm{d}z_1)f(\mathrm{d}z_2)\int_{[0,1]^{4d}}\mathrm{d}x_1\mathrm{d}x_1'\mathrm{d}x_2\mathrm{d}x_2'\\
&\qquad\times \boldsymbol{p}_{r(t-r)/t}\left(w-\frac{rN}{t}x_1\right)\boldsymbol{p}_{s_1(t-s_1)/t+s_1(r-s_1)/r}\left(z_1-\frac{s_1N}{t}x_1'+\frac{s_1}{r}w\right)\\
&\qquad\times \boldsymbol{p}_{r(t-r)/t}\left(w+b-\frac{rN}{t}x_2\right)\boldsymbol{p}_{s_2(t-s_2)/t+s_2(r-s_2)/r}\left(z_2-\frac{s_2N}{t}x_2'+\frac{s_2}{r}(w+b)\right).
\end{aligned}$$

Next we make the change of variables $s_1=\frac{t}{N}r_1$, $s_2=\frac{t}{N}r_2$, and $r=\frac{t}{N}\sigma$, in order to obtain

$$\begin{aligned}
&N^{-4d+3}\Phi_N^{(2)}\\
&\quad= t^3C_{t,4}^2c_{t,4}^2\int_{[0,N]} d\sigma\int_{[0,\sigma]^2}\mathrm{d}r_1\mathrm{d}r_2\int_{\mathbb{R}^{4d}} f(\mathrm{d}b)\mathrm{d}wf(\mathrm{d}z_1)f(\mathrm{d}z_2)\int_{[0,1]^{4d}}\mathrm{d}x_1\mathrm{d}x_1'\mathrm{d}x_2\mathrm{d}x_2'\\
&\quad\times \boldsymbol{p}_{\frac{t\sigma}{N}(1-\frac{\sigma}{N})}(w-\sigma x_1)\,\boldsymbol{p}_{\frac{tr_1}{N}(1-\frac{r_1}{N})+\frac{tr_1}{N\sigma}(\sigma-r_1)}\left(z_1-r_1x_1'+\frac{r_1}{\sigma}w\right)\\
&\quad\times \boldsymbol{p}_{\frac{t\sigma}{N}(1-\frac{\sigma}{N})}(w+b-\sigma x_2)\,\boldsymbol{p}_{\frac{tr_2}{N}(1-\frac{r_2}{N})+\frac{tr_2}{N\sigma}(\sigma-r_2)}\left(z_2-r_2x_2'+\frac{r_2}{\sigma}(w+b)\right).
\end{aligned}$$

To simplify the presentation, we set

$$\gamma_{0,N}=\frac{t\sigma}{N}\left(1-\frac{\sigma}{N}\right),\qquad \gamma_{1,N}=\frac{tr_1}{N}\left(1-\frac{r_1}{N}\right)+\frac{tr_1}{N\sigma}(\sigma-r_1)$$

and

$$\gamma_{2,N}=\frac{tr_2}{N}\left(1-\frac{r_2}{N}\right)+\frac{tr_2}{N\sigma}(\sigma-r_2).$$

With the change of variables $z=w-\sigma x_1$ and the notation $y_1=x_1'-x_1$, $y_2=x_2'-x_1$, and $y_3=x_2-x_1$, we can write

$$\begin{aligned}
\Psi_2(z_1,z_2,b) &:= \int_{\mathbb{R}^d}\mathrm{d}w\ \boldsymbol{p}_{\gamma_{0,N}}(w-\sigma x_1)\,\boldsymbol{p}_{\gamma_{1,N}}\left(z_1-r_1x_1'+\frac{r_1}{\sigma}w\right)\\
&\qquad\times \boldsymbol{p}_{\gamma_{0,N}}(w+b-\sigma x_2)\,\boldsymbol{p}_{\gamma_{2,N}}\left(z_2-r_2x_2'+\frac{r_2}{\sigma}(w+b)\right)\\
&= \int_{\mathbb{R}^d}\mathrm{d}z\ \boldsymbol{p}_{\gamma_{0,N}}(z)\,\boldsymbol{p}_{\gamma_{1,N}}\left(z_1-r_1y_1+\frac{r_1}{\sigma}z\right)\\
&\qquad\times \boldsymbol{p}_{\gamma_{0,N}}(z+b-\sigma y_3)\,\boldsymbol{p}_{\gamma_{2,N}}\left(z_2-r_2y_2+\frac{r_2}{\sigma}(z+b)\right).
\end{aligned}$$

The Fourier transform of the function $\Psi_2(z_1, z_2, b)$ is equal to

$$\begin{aligned}
&\hat{\Psi}_2(\xi_1, \xi_2, \xi_3)\\
&\quad = \int_{\mathbb{R}^d} \mathrm{d}z \boldsymbol{p}_{\gamma_{0,N}}(z) \exp\left(-\frac{1}{2}\gamma_{1,N}\|\xi_1\|^2 - \frac{1}{2}\gamma_{0,N}\left\|\xi_2 - \frac{r_2}{\sigma}\xi_3\right\|^2 - \frac{1}{2}\gamma_{2,N}\|\xi_3\|^2\right)\\
&\qquad \times \exp\left(i\xi_1 \cdot \left(r_1 y_1 - \frac{r_1}{\sigma}z\right) + i\left(\xi_2 - \frac{r_2}{\sigma}\xi_3\right)\cdot(\sigma y_3 - z) + i\xi_3 \cdot \left(r_2 y_2 - \frac{r_2}{\sigma}z\right)\right)\\
&\quad = \exp\left(-\frac{1}{2}\gamma_{1,N}\|\xi_1\|^2 - \frac{1}{2}\gamma_{0,N}\left\|\xi_2 - \frac{r_2}{\sigma}\xi_3\right\|^2 - \frac{1}{2}\gamma_{2,N}\|\xi_3\|^2 - \frac{1}{2}\gamma_{0,N}\left\|\frac{r_1}{\sigma}\xi_1 + \xi_2\right\|^2\right)\\
&\qquad \times \exp\left(ir_1\xi_1 \cdot y_1 + i\left(\sigma\xi_2 - r_2\xi_3\right)\cdot y_3 + ir_2\xi_3 \cdot y_2\right).
\end{aligned}$$

Set

$$\begin{aligned}
\Delta_2(\xi_1, \xi_2, \xi_3) := \int_{[0,1]^{4d}} &\mathrm{d}x_1 \mathrm{d}x_1' \mathrm{d}x_2 \mathrm{d}x_2'\\
&\exp\left(ir_1\xi_1 \cdot (x_1' - x_1) + i\left(\sigma\xi_2 - r_2\xi_3\right)\cdot(x_2 - x_1) + ir_2\xi_3 \cdot (x_2' - x_1)\right).
\end{aligned}$$

Then, Parseval's identity implies that

$$\begin{aligned}
&N^{-4d+3}\Phi_N^{(1)}\\
&\quad = C_{t,4}^2 c_{t,4}^2 t^3 \frac{1}{(2\pi)^{3d}} \int_{[0,N]} d\sigma \int_{[0,\sigma]^2} \mathrm{d}r_1 \mathrm{d}r_2 \int_{\mathbb{R}^{3d}} \hat{f}(\mathrm{d}\xi_1)\hat{f}(\mathrm{d}\xi_2)\hat{f}(\mathrm{d}\xi_3)\Delta_1(\xi_1, \xi_2, \xi_3)\\
&\qquad \times \exp\left(-\frac{1}{2}\gamma_{1,N}\|\xi_1\|^2 - \frac{1}{2}\gamma_{0,N}\left\|\xi_2 - \frac{r_2}{\sigma}\xi_3\right\|^2 - \frac{1}{2}\gamma_{2,N}\|\xi_3\|^2 - \frac{1}{2}\gamma_{0,N}\left\|\frac{r_1}{\sigma}\xi_1 + \xi_2\right\|^2\right)\\
&\quad \le C \int_{[0,\infty)^3} \mathrm{d}\sigma \mathrm{d}r_1 \mathrm{d}r_2 \int_{\mathbb{R}^{3d}} \hat{f}(\mathrm{d}\xi_1)\hat{f}(\mathrm{d}\xi_2)\hat{f}(\mathrm{d}\xi_3)|\Delta_2(\xi_1, \xi_2, \xi_3)|.
\end{aligned}$$

Taking into account that $\mathcal{R}(f) < \infty$, which is equivalent by Lemma 5.9 to $\int_{\mathbb{R}^d} \|z\|^{-1} \hat{f}(\mathrm{d}z) < \infty$, it suffices to show that

$$\int_{[0,\infty)^3} \mathrm{d}\sigma \mathrm{d}r_1 \mathrm{d}r_2 |\Delta_2(\xi_1, \xi_2, \xi_3)| \le C(\|\xi_1\|\|\xi_2\|\|\xi_3\|)^{-1} \tag{6.12}$$

for some constant C not depending on t. Taking into account that

$$\begin{aligned}
|\Delta_2(\xi_1, \xi_2, \xi_3)| &= \prod_{j=1}^d \frac{|\mathrm{e}^{-i(r_1\xi_1^j + r_2\xi_3^j|} - 1}{|r_1\xi_1^j + r_2\xi_3^j|} \frac{|\mathrm{e}^{ir_1\xi_1^j} - 1|}{|r_1\xi_1^j|} \frac{|\mathrm{e}^{i(\sigma\xi_2^j - r_2\xi_3^j)} - 1|}{|\sigma\xi_2^j - r_2\xi_3^j|} \frac{|\mathrm{e}^{ir_2\xi_3^j} - 1|}{|r_2\xi_3^j|}\\
&\le (1 \wedge \|r_1\xi_1 + r_2\xi_3\|^{-1})(1 \wedge \|r_1\xi_1\|^{-1})(1 \wedge \|\sigma\xi_2 - r_2\xi_3\|^{-1})(1 \wedge \|r_2\xi_3\|^{-1}),
\end{aligned}$$

the proof of (6.12) can be done by the same arguments as in the proof of (6.10). The proof of Theorem 1.3 is now complete. ∎

6.2. Proof of Theorem 1.4.

Proof of Theorem 1.4. By Theorem 5.2 and Proposition 2.1, we need to show that there exists a constant $C > 0$ such that for all $N \ge \mathrm{e}$,

$$\mathrm{Var}\left(\langle D\mathcal{S}_{N,t}, v_N\rangle_{\mathcal{H}}\right) \le C\left(\frac{\log N}{N}\right)^3. \tag{6.13}$$

We recall the decomposition (6.4) of $\mathrm{Var}\left(\langle D\mathcal{S}_{N,t}, v_N\rangle_{\mathcal{H}}\right)$.

Estimation of $\Phi_N^{(1)}$. According to (6.5), we integrate x_1' and x_2' on $\mathbb{R}$ and obtain

$$\begin{aligned}\Phi_N^{(1)} &\le 16C_{t,4}^2c_{t,4}^2 f(\mathbb{R})^2\int_{[0,t]^2}\mathrm{d}s_1\mathrm{d}s_2\,\frac{t^2}{s_1s_2}\int_0^{s_1\wedge s_2}\mathrm{d}r\int_{\mathbb{R}^3}f(\mathrm{d}b)\mathrm{d}y_1\mathrm{d}y_2\int_{[0,N]^2}\mathrm{d}x_1\mathrm{d}x_2\\&\quad\times \boldsymbol{p}_{s_1(t-s_1)/t}\left(y_1-\frac{s_1}{t}x_1\right)\boldsymbol{p}_{s_2(t-s_2)/t}\left(y_2-\frac{s_2}{t}x_2\right)\boldsymbol{p}_{r(s_1-r)/s_1+r(s_2-r)/s_2}\\&\quad\times\left(b-\frac{r}{s_2}y_2+\frac{r}{s_1}y_1\right)\\&=16C_{t,4}^2c_{t,4}^2 f(\mathbb{R})^2\int_{[0,t]^2}\mathrm{d}s_1\mathrm{d}s_2\,\frac{t^2}{s_2}\int_0^{s_1\wedge s_2}\mathrm{d}r\,\frac{1}{r}\int_{\mathbb{R}^3}f(\mathrm{d}b)\mathrm{d}y_1\mathrm{d}y_2\int_{[0,N]^2}\mathrm{d}x_1\mathrm{d}x_2\\&\quad\times \boldsymbol{p}_{s_1(t-s_1)/t}\left(y_1-\frac{s_1}{t}x_1\right)\boldsymbol{p}_{s_2(t-s_2)/t}\left(y_2-\frac{s_2}{t}x_2\right)\\&\quad\times \boldsymbol{p}_{[r(s_1-r)/s_1+r(s_2-r)/s_2]/(r^2/s_1^2)}\left(\frac{s_1}{r}b-\frac{s_1}{s_2}y_2+y_1\right),\end{aligned}$$

where in the equality we use property (6.7) with $d=1$. Hence, by the semigroup property, we see that

$$\begin{aligned}\Phi_N^{(1)} &\le 16C_{t,4}^2c_{t,4}^2 f(\mathbb{R})^2\int_{[0,t]^2}\mathrm{d}s_1\mathrm{d}s_2\,\frac{t^2}{s_2}\int_0^{s_1\wedge s_2}\mathrm{d}r\,\frac{1}{r}\int_{\mathbb{R}^2}f(\mathrm{d}b)\mathrm{d}y_2\int_{[0,N]^2}\mathrm{d}x_1\mathrm{d}x_2\\&\quad\times \boldsymbol{p}_{s_2(t-s_2)/t}\left(y_2-\frac{s_2}{t}x_2\right)\boldsymbol{p}_{s_1(t-s_1)/t+[r(s_1-r)/s_1+r(s_2-r)/s_2]/(r^2/s_1^2)}\\&\quad\times\left(\frac{s_1}{r}b-\frac{s_1}{s_2}y_2+\frac{s_1}{t}x_1\right).\end{aligned}$$

We repeat the use of (6.7) with $d=1$ and the semigroup property to obtain

$$\begin{aligned}\Phi_N^{(1)} &\le 16C_{t,4}^2c_{t,4}^2 f(\mathbb{R})^2\int_{[0,t]^2}\mathrm{d}s_1\mathrm{d}s_2\,\frac{t^2}{s_1}\int_0^{s_1\wedge s_2}\mathrm{d}r\,\frac{1}{r}\int_{\mathbb{R}^2}f(\mathrm{d}b)\mathrm{d}y_2\int_{[0,N]^2}\mathrm{d}x_1\mathrm{d}x_2\\&\quad\times \boldsymbol{p}_{s_2(t-s_2)/t}\left(y_2-\frac{s_2}{t}x_2\right)\boldsymbol{p}_{[s_1(t-s_1)/t+[r(s_1-r)/s_1+r(s_2-r)/s_2]/(r^2/s_1^2)]/(s_1^2/s_2^2)}\\&\quad\times\left(\frac{s_2}{r}b-y_2+\frac{s_2}{t}x_1\right)\\&=16C_{t,4}^2c_{t,4}^2 f(\mathbb{R})^2\int_{[0,t]^2}\mathrm{d}s_1\mathrm{d}s_2\,\frac{t^2}{s_1}\int_0^{s_1\wedge s_2}\mathrm{d}r\,\frac{1}{r}\int_{\mathbb{R}}f(\mathrm{d}b)\int_{[0,N]^2}\mathrm{d}x_1\mathrm{d}x_2\\&\quad\times \boldsymbol{p}_{s_2(t-s_2)/t+[s_1(t-s_1)/t+[r(s_1-r)/s_1+r(s_2-r)/s_2]/(r^2/s_1^2)]/(s_1^2/s_2^2)}\left(\frac{s_2}{r}b+\frac{s_2}{t}x_1-\frac{s_2}{t}x_2\right)\\&=16C_{t,4}^2c_{t,4}^2 f(\mathbb{R})^2\int_{[0,t]^2}\mathrm{d}s_1\mathrm{d}s_2\,\frac{t^2}{s_1s_2}\int_0^{s_1\wedge s_2}\mathrm{d}r\int_{\mathbb{R}}f(\mathrm{d}b)\int_{[0,N]^2}\mathrm{d}x_1\mathrm{d}x_2\,\boldsymbol{p}_{2r(t-r)/t}\\&\quad\times\left(b+\frac{r}{t}(x_1-x_2)\right),\end{aligned}$$

where in the second equality we use the relation

$$\begin{aligned}2r(t-r)/t&=[s_2(t-s_2)/t+[s_1(t-s_1)/t+[r(s_1-r)/s_1+r(s_2-r)/s_2]/(r^2/s_1^2)]/\\&\qquad(s_1^2/s_2^2)]/(s_2^2/r^2).\end{aligned}$$

Now, using the notation $I_N = \frac{1}{N}1_{[0,N]}$ and Plancherel's identity, we conclude that

$$\begin{aligned}
\Phi_N^{(1)} &\le 16C_{t,4}^2 c_{t,4}^2 N^2 f(\mathbb{R})^2 \int_{[0,t]^2} \mathrm{d}s_1\mathrm{d}s_2\, \frac{t^2}{s_1 s_2} \int_0^{s_1\wedge s_2} \mathrm{d}r\, \left(I_N * \tilde{I}_N * \left(f * \boldsymbol{p}_{2r(t-r)/t}\left(\frac{r}{t}(\cdot)\right)\right)\right)(0)\\
&= 16C_{t,4}^2 c_{t,4}^2 \frac{N^2}{\pi} f(\mathbb{R})^2 \int_{[0,t]^2} \mathrm{d}s_1\mathrm{d}s_2\, \frac{t^3}{r s_1 s_2} \int_0^{s_1\wedge s_2} \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z\, \frac{1-\cos(Nz)}{N^2 z^2} \hat{f}\left(\frac{tz}{r}\right) \mathrm{e}^{-\frac{t(t-r)}{r} z^2}\\
&\le 16C_{t,4}^2 c_{t,4}^2 \frac{N}{\pi} f(\mathbb{R})^3 \int_{[0,t]^2} \mathrm{d}s_1\mathrm{d}s_2\, \frac{t^3}{r s_1 s_2} \int_0^{s_1\wedge s_2} \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z\, \frac{1-\cos(z)}{z^2} \mathrm{e}^{-\frac{t(t-r)}{r}\frac{z^2}{N^2}}\\
&= 32C_{t,4}^2 c_{t,4}^2 \frac{N}{\pi} f(\mathbb{R})^3 \int_{0\le s_1\le s_2\le t} \mathrm{d}s_1\mathrm{d}s_2\mathrm{d}r\, \frac{t^3}{r s_1 s_2} \int_0^{s_1\wedge s_2} \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z\, \frac{1-\cos(z)}{z^2} \mathrm{e}^{-\frac{t(t-r)}{r}\frac{z^2}{N^2}}.
\end{aligned}$$

Integrating in the variables s_1 and s_2 yields

$$\Phi_N^{(1)} \le 32C_{t,4}^2 c_{t,4}^2 \frac{N}{\pi} f(\mathbb{R})^3 \int_0^t \mathrm{d}r \frac{t^3}{r}\left(\log\left(\frac{t}{r}\right)\right)^2 \int_{\mathbb{R}} \mathrm{e}^{-\frac{t(t-r)}{r}\frac{z^2}{N^2}} \varphi(z)\mathrm{d}z,$$

where we recall that $\varphi(z) = (1-\cos z)/z^2$. Making the change of variables $\frac{t-r}{r} = \theta$ allows us to write

$$\Phi_N^{(1)} \le 32C_{t,4}^2 c_{t,4}^2 \frac{N}{\pi} t^3 f(\mathbb{R})^3 \int_{\mathbb{R}} \varphi(z)\mathrm{d}z \int_0^\infty \mathrm{d}\theta \frac{1}{\theta+1} (\log(\theta+1))^2 \mathrm{e}^{-\frac{t\theta z^2}{N^2}}.$$

Integrating by parts and using the fact that

$$\left(\frac{1}{3}(\log(\theta+1))^3 \mathrm{e}^{-\frac{t\theta z^2}{N^2}}\right)_{\theta=0}^{\theta=\infty} = 0,$$

we obtain

$$\begin{aligned}
\Phi_N^{(1)} &\le 32C_{t,4}^2 c_{t,4}^2 \frac{N}{3\pi} t^3 \int_{\mathbb{R}} \varphi(z)\mathrm{d}z \int_0^\infty \mathrm{d}\theta\, (\log(\theta+1))^3 \mathrm{e}^{-\frac{t\theta z^2}{N^2}} \frac{tz^2}{N^2}\\
&= 32C_{t,4}^2 c_{t,4}^2 \frac{N}{3\pi} t^3 \int_{\mathbb{R}} \varphi(z)\mathrm{d}z \int_0^\infty \mathrm{d}\theta \left(\log\left(\frac{N^2}{tz^2}\theta+1\right)\right)^3 \mathrm{e}^{-\theta}.
\end{aligned}$$

Using the inequality

$$\begin{aligned}
\log\left(\frac{N^2}{tz^2}\theta+1\right) &\le 2\log N + \log(\theta+1) + \log\left(\frac{1}{t}+1\right) + \log\left(\frac{1}{z^2}+1\right)\\
&\le \left(2\log N + \log\left(\frac{1}{t}+1\right)\right)\left(1+\log(\theta+1)+\log\left(\frac{1}{z^2}+1\right)\right),
\end{aligned}$$

and taking into account that

$$C := \int_{\mathbb{R}} \varphi(z)\mathrm{d}z \int_0^\infty \mathrm{d}\theta \left(1+\log(\theta+1)+\log\left(\frac{1}{z^2}+1\right)\right)^3 e^{-\theta} < \infty,$$

we finally get

$$\Phi_N^{(1)} \lesssim C_{t,4}^2 c_{t,4}^2 t^3 N \left(2\log N + \log\left(\frac{1}{t}+1\right)\right)^3,$$

which provides the desired estimate.

Estimation of $\Phi_N^{(2)}$. Recall the estimate in (6.6). Notice that we should not integrate the variables x_1' and x_2' on the whole real line, because this would produce a factor $(s_1s_2)^{-1}$ which is not integrable on $[0,r]^2$. For this reason, we choose to integrate the variables x_1 and x_2 on $\mathbb{R}$ and we obtain, using (6.7) with $d=1$,

$$\begin{aligned}
\Phi_N^{(2)} &\le C_{t,4}^2 c_{t,4}^2 \int_0^t \mathrm{d}r\, \frac{t^2}{r^2} \int_{[0,r]^2} \mathrm{d}s_1 \mathrm{d}s_2 \int_{\mathbb{R}^4} f(\mathrm{d}b)\mathrm{d}w f(\mathrm{d}z_1) f(\mathrm{d}z_2) \int_{[0,N]^2} \mathrm{d}x_1'\mathrm{d}x_2' \\
&\quad \times \boldsymbol{p}_{s_1(t-s_1)/t+s_1(r-s_1)/r}\left(z_1 - \frac{s_1}{t}x_1' + \frac{s_1}{r}w\right) \boldsymbol{p}_{s_2(t-s_2)/t+s_2(r-s_2)/r} \\
&\quad \times \left(z_2 - \frac{s_2}{t}x_2' + \frac{s_2}{r}(w+b)\right) \\
&= C_{t,4}^2 c_{t,4}^2 \int_0^t \mathrm{d}r\, \frac{t^2}{s_1 s_2} \int_{[0,r]^2} \mathrm{d}s_1 \mathrm{d}s_2 \int_{\mathbb{R}^3} f(\mathrm{d}b) f(\mathrm{d}z_1) f(\mathrm{d}z_2) \int_{[0,N]^2} \mathrm{d}x_1'\mathrm{d}x_2' \\
&\quad \times \boldsymbol{p}_\alpha \left(\frac{r}{s_2}z_2 - \frac{r}{s_1}z_1 + b - \frac{r}{t}(x_2' - x_1')\right) \\
&= C_{t,4}^2 c_{t,4}^2 \int_0^t \mathrm{d}r\, \frac{t^2}{s_1 s_2} \int_{[0,r]^2} \mathrm{d}s_1 \mathrm{d}s_2 \int_{\mathbb{R}^2} f(\mathrm{d}z_1) f(\mathrm{d}z_2) \int_{[0,N]^2} \mathrm{d}x_1'\mathrm{d}x_2' \\
&\quad \times (f * \boldsymbol{p}_\alpha)\left(\frac{r}{t}(x_2'-x_1') + \frac{r}{s_1}z_1 - \frac{r}{s_2}z_2\right),
\end{aligned}$$

where

$$\alpha = [s_1(t-s_1)/t + s_1(r-s_1)/r]/(s_1^2/r^2) + [s_2(t-s_2)/t + s_2(r-s_2)/r]/(s_2^2/r^2).$$

Using $I_N = \frac{1}{N}1_{[0,N]}$, we write

$$\begin{aligned}
\Phi_N^{(2)} &\le N^2 C_{t,4}^2 c_{t,4}^2 \int_0^t \mathrm{d}r\, \frac{t^2}{s_1 s_2} \int_{[0,r]^2} \mathrm{d}s_1 \mathrm{d}s_2 \int_{\mathbb{R}^2} f(\mathrm{d}z_1) f(\mathrm{d}z_2) \\
&\quad \times \left(I_N * \tilde{I}_N * \left((f * \boldsymbol{p}_\alpha)\left(\frac{r}{t}(\cdot) + \frac{r}{s_1}z_1 - \frac{r}{s_2}z_2\right)\right)\right)(0).
\end{aligned}$$

We apply Plancherel's identity to conclude that

$$\begin{aligned}
&\Phi_N^{(2)} \\
&\le \frac{N^2}{\pi} C_{t,4}^2 c_{t,4}^2 \int_0^t \mathrm{d}r\, \frac{t^3}{r s_1 s_2} \int_{[0,r]^2} \mathrm{d}s_1 \mathrm{d}s_2 \int_{\mathbb{R}^2} f(\mathrm{d}z_1) f(\mathrm{d}z_2) \int_{\mathbb{R}} \mathrm{d}z \\
&\quad \times \frac{1-\cos(Nz)}{N^2 z^2} \mathrm{e}^{iz\left(\frac{t}{s_2}z_2 - \frac{t}{s_1}z_1\right)} \hat{f}\left(\frac{tz}{r}\right) \mathrm{e}^{-\frac{\alpha t^2}{2r^2}z^2} \\
&= \frac{N}{\pi} C_{t,4}^2 c_{t,4}^2 \int_0^t \mathrm{d}r\, \frac{t^3}{r s_1 s_2} \int_{[0,r]^2} \mathrm{d}s_1 \mathrm{d}s_2 \int_{\mathbb{R}} \mathrm{d}z\, \frac{1-\cos(z)}{z^2} \hat{f}\left(\frac{tz}{s_2}\right) \hat{f}\left(-\frac{tz}{s_1}\right) \hat{f}\left(\frac{tz}{r}\right) \mathrm{e}^{-\frac{\alpha t^2}{2r^2}\frac{z^2}{N^2}} \\
&\le \frac{N}{\pi} C_{t,4}^2 c_{t,4}^2 f(\mathbb{R})^3 \int_0^t \mathrm{d}r\, \frac{t^3}{r s_1 s_2} \int_{[0,r]^2} \mathrm{d}s_1 \mathrm{d}s_2 \int_{\mathbb{R}} \mathrm{d}z\, \frac{1-\cos(z)}{z^2} \mathrm{e}^{-\frac{\alpha t^2}{2r^2}\frac{z^2}{N^2}}.
\end{aligned}$$

Denote

$$\sigma := \frac{\alpha t^2}{r^2} = t(t-s_1)/s_1 + t^2(r-s_1)/(rs_1) + t(t-s_2)/s_2 + t^2(r-s_2)/(rs_2).$$

Recalling $\varphi(z) = (1-\cos(z))/z^2$, we can write

$$\begin{aligned}\Phi_N^{(2)} &\le \frac{N}{\pi} C_{t,4}^2 c_{t,4}^2 f(\mathbb{R})^3 \int_{\mathbb{R}} \varphi(z) \mathrm{d}z \int_0^t \int_{[0,r]^2} \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}r \frac{t^3}{r s_1 s_2} \mathrm{e}^{-\frac{\sigma z^2}{2N^2}} \\ &= \frac{N}{\pi} C_{t,4}^2 c_{t,4}^2 f(\mathbb{R})^3 \int_{\mathbb{R}} \varphi(z) \mathrm{d}z \int_0^t \mathrm{d}r \frac{t^3}{r} \left(\int_0^r \mathrm{d}s \frac{1}{s} \mathrm{e}^{-\frac{[t(t-s)/s + t^2(r-s)/(rs)]z^2}{2N^2}} \right)^2.\end{aligned}$$

Making the change of variables $(r-s)/s = \theta$ yields

$$\int_0^r \mathrm{d}s \frac{1}{s} \mathrm{e}^{-\frac{[t(t-s)/s + t^2(r-s)/(rs)]z^2}{2N^2}} = \int_0^\infty \frac{1}{1+\theta} \mathrm{e}^{-\frac{tz^2}{2N^2}(2\theta t + t - r)/r)} \mathrm{d}\theta.$$

As a consequence,

$$\Phi_N^{(2)} \le \frac{N}{\pi} C_{t,4}^2 c_{t,4}^2 t^3 f(\mathbb{R})^3 \int_{\mathbb{R}} \varphi(z) \mathrm{d}z \int_0^t \frac{1}{r} \mathrm{e}^{-\frac{z^2}{N^2} t(t-r)/r} \left(\int_0^\infty \frac{1}{1+\theta} \mathrm{e}^{-\frac{t^2 z^2 \theta}{rN^2}} \mathrm{d}\theta \right)^2 \mathrm{d}z \mathrm{d}r.$$

With the further change of variable $\frac{t-r}{r} = \xi$, we obtain

$$\begin{aligned}\Phi_N^{(2)} &\le \frac{N}{\pi} C_{t,4}^2 c_{t,4}^2 t^3 f(\mathbb{R})^3 \int_{\mathbb{R}} \varphi(z) \mathrm{d}z \int_0^\infty \frac{1}{1+\xi} e^{-\frac{tz^2\xi}{N^2}} \left(\int_0^\infty \frac{1}{1+\theta} \mathrm{e}^{-\frac{t(\xi+1)z^2\theta}{N^2}} \mathrm{d}\theta \right)^2 \mathrm{d}z \mathrm{d}\xi \\ &\le \frac{N}{\pi} C_{t,4}^2 c_{t,4}^2 t^3 f(\mathbb{R})^3 \int_{\mathbb{R}} \varphi(z) \mathrm{d}z \left(\int_0^\infty \frac{1}{1+\theta} \mathrm{e}^{-\frac{tz^2\theta}{N^2}} \mathrm{d}\theta \right)^3 \mathrm{d}z \\ &= \frac{N}{\pi} C_{t,4}^2 c_{t,4}^2 t^3 f(\mathbb{R})^3 \int_{\mathbb{R}} \varphi(z) \mathrm{d}z \left(\int_0^\infty \frac{1}{\theta + \frac{tz^2}{N^2}} \mathrm{e}^{-\theta} \mathrm{d}\theta \right)^3 \mathrm{d}z.\end{aligned}$$

We have

$$\begin{aligned}\int_0^\infty \frac{1}{\theta + \frac{tz^2}{N^2}} \mathrm{e}^{-\theta} \mathrm{d}\theta &\le \int_1^\infty \mathrm{e}^{-\theta} \mathrm{d}\theta + \int_0^1 \frac{1}{\theta + \frac{tz^2}{N^2}} \mathrm{d}\theta = \mathrm{e}^{-1} + \log\left(1 + \frac{N^2}{tz^2}\right) \\ &\le \mathrm{e}^{-1} + 2\log N + \log(1+1/t) + \log(1+z^{-2}).\end{aligned}$$

Taking into account that

$$\int_{\mathbb{R}} \varphi(z)(1+\log(1+z^{-2}))^3 \mathrm{d}z < \infty,$$

we obtain the desired estimate for the term $\Phi_N^{(2)}$. This completes the proof of the estimate (6.13). ∎

6.3. Proof of Theorem 1.5.

6.3.1. Estimation of $\Phi_N^{(1)}$. Recalling (6.5) and using change of variables $y_1 - \frac{s_1}{t}x_1 = \alpha_1$, $y_2 - \frac{s_2}{t}x_2 = \alpha_2$, $y_1 + z_1 - \frac{s_1}{t}x_1' = \alpha_3$, $y_2 + z_2 - \frac{s_2}{t}x_2' = \alpha_4$, $b - \frac{r}{s_2}y_2 + \frac{r}{s_1}y_1 = \alpha_5$, yields that

$$\begin{aligned}\Phi_N^{(1)} \le\ & 16 C_{t,4}^2 c_{t,4}^2 \int_{[0,t]^2} \mathrm{d}s_1 \mathrm{d}s_2 \int_0^{s_1 \wedge s_2} \mathrm{d}r \int_{\mathbb{R}^{5d}} \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \mathrm{d}\alpha_4 \mathrm{d}\alpha_5 \int_{[0,N]^{4d}} \mathrm{d}x_1 \mathrm{d}x_1' \mathrm{d}x_2 \mathrm{d}x_2' \\ & \times \boldsymbol{p}_{s_1(t-s_1)/t}(\alpha_1) \boldsymbol{p}_{s_2(t-s_2)/t}(\alpha_2) \boldsymbol{p}_{s_1(t-s_1)/t}(\alpha_3) \boldsymbol{p}_{s_2(t-s_2)/t}(\alpha_4) \\ & \times \boldsymbol{p}_{r(s_1-r)/s_1 + r(s_2-r)/s_2}(\alpha_5) \\ & \times \|\alpha_3 - \alpha_1 - \frac{s_1}{t}(x_1 - x_1')\|^{-\beta} \|\alpha_4 - \alpha_2 - \frac{s_2}{t}(x_2 - x_2') \\ & \times \|^{-\beta} \|\alpha_5 + \frac{r}{s_2}\alpha_2 - \frac{r}{s_1}\alpha_1 + \frac{r}{t}(x_2 - x_1)\|^{-\beta}.\end{aligned}$$

Let Z_1, Z_2, Z_3, Z_4, Z_5 be independent and identically distributed $\mathrm{N}(0,1)$. We can write

$$\begin{aligned}
(6.14)\quad \Phi_N^{(1)} \le\ & 16C_{t,4}^2 c_{t,4}^2 \int_{[0,t]^2} \mathrm{d}s_1\mathrm{d}s_2 \int_0^{s_1\wedge s_2} \mathrm{d}r \int_{[0,N]^{4d}} \mathrm{d}x_1\mathrm{d}x_1'\mathrm{d}x_2\mathrm{d}x_2' \\
& \times \mathrm{E}\Bigg[\left\| \sqrt{s_1(t-s_1)/t}Z_3 - \sqrt{s_1(t-s_1)/t}Z_1 - \frac{s_1}{t}(x_1-x_1') \right\|^{-\beta} \\
& \times \left\| \sqrt{s_2(t-s_2)/t}Z_4 - \sqrt{s_2(t-s_2)/t}Z_2 - \frac{s_2}{t}(x_2-x_2') \right\|^{-\beta} \\
& \times \Bigg\| \sqrt{r(s_1-r)/s_1 + r(s_2-r)/s_2}Z_5 + \frac{r}{s_2}\sqrt{s_2(t-s_2)/t}Z_2 \\
& \quad - \frac{r}{s_1}\sqrt{s_1(t-s_1)/t}Z_1 + \frac{r}{t}(x_2-x_1) \Bigg\|^{-\beta} \Bigg] \\
=\ & 16C_{t,4}^2 c_{t,4}^2 N^{4d-3\beta} \int_{[0,t]^2} \mathrm{d}s_1\mathrm{d}s_2 \int_0^{s_1\wedge s_2} \mathrm{d}r \left(\frac{t}{r}\right)^\beta \left(\frac{t}{s_1}\right)^\beta \left(\frac{t}{s_2}\right)^\beta \\
& \times \int_{[0,1]^{4d}} \mathrm{d}x_1\mathrm{d}x_1'\mathrm{d}x_2\mathrm{d}x_2' \\
& \times \mathrm{E}\Bigg[\left\| \frac{t}{Ns_1}\sqrt{s_1(t-s_1)/t}Z_3 - \frac{t}{Ns_1}\sqrt{s_1(t-s_1)/t}Z_1 - (x_1-x_1') \right\|^{-\beta} \\
& \times \left\| \frac{t}{Ns_2}\sqrt{s_2(t-s_2)/t}Z_4 - \frac{t}{Ns_2}\sqrt{s_2(t-s_2)/t}Z_2 - (x_2-x_2') \right\|^{-\beta} \\
& \times \Bigg\| \frac{t}{Nr}\sqrt{r(s_1-r)/s_1 + r(s_2-r)/s_2}Z_5 + \frac{t}{Ns_2}\sqrt{s_2(t-s_2)/t}Z_2 \\
& \quad - \frac{t}{Ns_1}\sqrt{s_1(t-s_1)/t}Z_1 + (x_2-x_1) \Bigg\|^{-\beta} \Bigg],
\end{aligned}$$

where in the second equality we have made a change of variables.

Case 1. $0<\beta<1$. Applying Lemma 3.1 of [17] to the random variables Z_5, Z_4, Z_3 in this order, we see that the spatial integral in (6.14) is bounded above by

$$\begin{aligned}
& C\,\mathrm{E}\Bigg[\int_{[0,1]^{4d}} \mathrm{d}x_1\mathrm{d}x_1'\mathrm{d}x_2\mathrm{d}x_2' \left\| \frac{t}{Ns_2}\sqrt{s_2(t-s_2)/t}Z_2 - \frac{t}{Ns_1}\sqrt{s_1(t-s_1)/t}Z_1 + (x_2-x_1) \right\|^{-\beta} \\
& \quad \times \left\| \frac{t}{Ns_1}\sqrt{s_1(t-s_1)/t}Z_1 + (x_1-x_1') \right\|^{-\beta} \left\| \frac{t}{Ns_2}\sqrt{s_2(t-s_2)/t}Z_2 + (x_2-x_2') \right\|^{-\beta} \Bigg] \\
& \le C\,\mathrm{E}\Bigg[\int_{[-1,1]^{3d}} \mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3 \left\| \frac{t}{Ns_2}\sqrt{s_2(t-s_2)/t}Z_2 - \frac{t}{Ns_1}\sqrt{s_1(t-s_1)/t}Z_1 + y_3 \right\|^{-\beta} \\
& \qquad \times \left\| \frac{t}{Ns_1}\sqrt{s_1(t-s_1)/t}Z_1 + y_1 \right\|^{-\beta} \left\| \frac{t}{Ns_2}\sqrt{s_2(t-s_2)/t}Z_2 + y_2 \right\|^{-\beta} \Bigg] \\
& \le C \left(\sup_{z\in\mathbb{R}^d} \int_{[-1,1]^d} \|z+y\|^{-\beta}\mathrm{d}y \right)^3 = C' < \infty,
\end{aligned}$$

where in the first inequality we use a change of variables and in the second inequality we use the fact (see also [17, (3.10)])

$$(6.15)\qquad \sup_{z\in\mathbb{R}^d} \int_{[-1,1]^d} \|z+y\|^{-\beta}\mathrm{d}y < \infty.$$

Denote

$$A_t = \int_{[0,t]^2} \mathrm{d}s_1\mathrm{d}s_2 \int_0^{s_1\wedge s_2} \mathrm{d}r \left(\frac{t}{r}\right)^\beta \left(\frac{t}{s_1}\right)^\beta \left(\frac{t}{s_2}\right)^\beta .$$

Condition $\beta < 1$ implies $A_t < \infty$. Therefore, in the case $0 < \beta < 1$, we conclude that

$$\Phi_N^{(1)} \le C' C_{t,4}^2 c_{t,4}^2 A_t \, N^{4d-3\beta}. \tag{6.16}$$

Case 2. $1 < \beta < 2$. Recall (6.14). Applying Lemma 7.2 to Z_5, Z_4, Z_3, using the change of variables ($x_1' = x_1 - y_1$, $x_2' = x_2 - y_2$, $x_1 = x_2 - y_3$) and the fact that for all $c_1 > 0$ and $z \in \mathbb{R}^d$

$$\begin{aligned} \int_{[-1,1]^d} c_1 \wedge \|z + y_1\|^{-\beta} \mathrm{d}y_1 &\le 2^d \left(c_1 \wedge \int_{[-1,1]^d} \|z+y_1\|^{-\beta}\mathrm{d}y_1 \right) \\ &\lesssim c_1 \wedge 1, \qquad \text{see (6.15)}, \end{aligned} \tag{6.17}$$

we obtain that

$$\begin{aligned} \Phi_N^{(1)} \lesssim C_{t,4}^2 c_{t,4}^2 N^{4d-3\beta} \int_{[0,t]^2} \mathrm{d}s_1\mathrm{d}s_2 \int_0^{s_1\wedge s_2} \mathrm{d}r \left[\left[N^\beta (s_1(t-s_1)/t)^{-\beta/2} \right] \wedge \left(\frac{t}{s_1}\right)^\beta \right] \\ \times \left[\left[N^\beta (s_2(t-s_2)/t)^{-\beta/2} \right] \wedge \left(\frac{t}{s_2}\right)^\beta \right] \\ \times \left[\left[N^\beta (r(s_1-r)/s_1 + r(s_2-r)/s_2)^{-\beta/2} \right] \wedge \left(\frac{t}{r}\right)^\beta \right]. \end{aligned}$$

The change of variables $s_1 \to \frac{ts_1}{N^2}$, $s_2 \to \frac{ts_2}{N^2}$, and $r \to \frac{tr}{N^2}$ allows us to write

$$\begin{aligned} \Phi_N^{(1)} \lesssim N^{4d+3\beta-6} \int_{[0,N]^2} \mathrm{d}s_1\mathrm{d}s_2 \int_0^{s_1\wedge s_2} \mathrm{d}r \left[\left[(s_1(1-s_1/N^2))^{-\beta/2} \right] \wedge s_1^{-\beta} \right] \\ \times \left[\left[(s_2(1-s_2/N^2))^{-\beta/2} \right] \wedge s_2^{-\beta} \right] \left[\left[r^{-\beta/2} \left(1 - \frac{r}{2s_1} - \frac{r}{2s_2} \right)^{-\beta/2} \right] \wedge r^{-\beta} \right]. \end{aligned}$$

For the integral in the variable r we make the further change of variables $r(\frac{1}{2s_1} + \frac{1}{2s_2}) = \lambda$ in order to obtain

$$\begin{aligned} \Phi_N^{(1)} \lesssim N^{4d+3\beta-6} \int_{[0,N^2]^2} \mathrm{d}s_1\mathrm{d}s_2 \left[\left[(s_1(1-s_1/N^2))^{-\beta/2} \right] \wedge s_1^{-\beta} \right] \\ \times \left[\left[(s_2(1-s_2/N^2))^{-\beta/2} \right] \wedge s_2^{-\beta} \right] \\ \times \left(\frac{1}{2s_1} + \frac{1}{2s_2} \right)^{\beta-1} \int_0^1 \mathrm{d}\lambda \, \lambda^{-\frac{\beta}{2}} \left(\left[\left(\frac{1}{2s_1} + \frac{1}{2s_2} \right)^{-\beta/2} (1-\lambda)^{-\frac{\beta}{2}} \right] \wedge \lambda^{-\frac{\beta}{2}} \right). \end{aligned} \tag{6.18}$$

From (6.18), we apply Lemma 7.4 to conclude that in the case $1 < \beta < 2$,

$$\Phi_N^{(1)} \lesssim N^{4d+3\beta-6}. \tag{6.19}$$

Case 3. $\beta = 1$. Notice that the estimate in (6.18) still holds for $\beta = 1$. Now we apply Lemma 7.5 to conclude that in the case $\beta = 1$

$$\Phi_N^{(1)} \lesssim N^{4d-3}(\log N)^3. \tag{6.20}$$

6.3.2. Estimation of $\Phi_N^{(2)}$. Recall (6.6). Using the change of variables $w-\frac{r}{t}x_1=\alpha_1$, $w+b-\frac{r}{t}x_2=\alpha_2$, $z_1-\frac{s_1}{t}x_1'+\frac{s_1}{r}w=\alpha_3$, $z_2-\frac{s_2}{t}x_2'+\frac{s_2}{r}(w+b)=\alpha_4$, we obtain

$$\begin{aligned}\Phi_N^{(2)} &\le C_{t,4}^2 c_{t,4}^2\int_0^t \mathrm{d}r\int_{[0,r]^2}\mathrm{d}s_1\mathrm{d}s_2\int_{\mathbb{R}^{4d}}\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\mathrm{d}\alpha_4\int_{[0,N]^{4d}}\mathrm{d}x_1\mathrm{d}x_1'\mathrm{d}x_2\mathrm{d}x_2'\\
&\quad\times \boldsymbol{p}_{r(t-r)/t}(\alpha_1)\,\boldsymbol{p}_{r(t-r)/t}(\alpha_2)\,\boldsymbol{p}_{s_1(t-s_1)/t+s_1(r-s_1)/r}(\alpha_3)\,\boldsymbol{p}_{s_2(t-s_2)/t+s_2(r-s_2)/r}(\alpha_4)\\
&\quad\times\left\|\alpha_2-\alpha_1+\frac{r}{t}(x_2-x_1)\right\|^{-\beta}\left\|\alpha_3-\frac{s_1}{r}\alpha_1+\frac{s_1}{t}(x_1'-x_1)\right\|^{-\beta}\\
&\quad\times\left\|\alpha_4-\frac{s_2}{r}\alpha_2+\frac{s_2}{t}(x_2'-x_2)\right\|^{-\beta}\\
&= C_{t,4}^2 c_{t,4}^2\int_0^t \mathrm{d}r\int_{[0,r]^2}\mathrm{d}s_1\mathrm{d}s_2\int_{[0,N]^{4d}}\mathrm{d}x_1\mathrm{d}x_1'\mathrm{d}x_2\mathrm{d}x_2'\\
&\quad\times \mathrm{E}\Bigg[\left\|\sqrt{\frac{r(t-r)}{t}}\,Z_2-\sqrt{\frac{r(t-r)}{t}}\,Z_1+\frac{r}{t}(x_2-x_1)\right\|^{-\beta}\\
&\quad\times\left\|\sqrt{\frac{s_1(t-s_1)}{t}+\frac{s_1(r-s_1)}{r}}\,Z_3-\frac{s_1}{t}\sqrt{\frac{r(t-r)}{t}}\,Z_1+\frac{s_1}{t}(x_1'-x_1)\right\|^{-\beta}\\
&\quad\times\left\|\sqrt{\frac{s_2(t-s_2)}{t}+\frac{s_2(r-s_2)}{r}}\,Z_4-\frac{s_2}{t}\sqrt{\frac{r(t-r)}{t}}\,Z_2+\frac{s_2}{t}(x_2'-x_2)\right\|^{-\beta}\Bigg].\end{aligned}$$

Now, using a change of variables yields that

(6.21)

$$\begin{aligned}\Phi_N^{(2)} &\le C_{t,4}^2 c_{t,4}^2 N^{4d-3\beta}\int_0^t \mathrm{d}r\int_{[0,r]^2}\mathrm{d}s_1\mathrm{d}s_2\left(\frac{t}{r}\right)^{\beta}\left(\frac{t}{s_1}\right)^{\beta}\left(\frac{t}{s_2}\right)^{\beta}\int_{[0,1]^{4d}}\mathrm{d}x_1\mathrm{d}x_1'\mathrm{d}x_2\mathrm{d}x_2'\\
&\quad\times \mathrm{E}\Bigg[\left\|\frac{t}{Nr}\sqrt{\frac{r(t-r)}{t}}\,Z_2-\frac{t}{Nr}\sqrt{\frac{r(t-r)}{t}}\,Z_1+(x_2-x_1)\right\|^{-\beta}\\
&\quad\times\left\|\frac{t}{Ns_1}\sqrt{\frac{s_1(t-s_1)}{t}+\frac{s_1(r-s_1)}{r}}\,Z_3-\frac{1}{N}\sqrt{\frac{r(t-r)}{t}}\,Z_1+(x_1'-x_1)\right\|^{-\beta}\\
&\quad\times\left\|\frac{t}{Ns_2}\sqrt{\frac{s_2(t-s_2)}{t}+\frac{s_2(r-s_2)}{r}}\,Z_4-\frac{1}{N}\sqrt{\frac{r(t-r)}{t}}\,Z_2+(x_2'-x_2)\right\|^{-\beta}\Bigg].\end{aligned}$$

Case 1. $0<\beta<1$. We first apply Lemma 3.1 of [17] for Z_4, Z_3, then use a change of variables and (6.15) to conclude

$$\Phi_N^{(2)} \le C\,\bar{A}_t C_{t,4}^2 c_{t,4}^2 N^{4d-3\beta}, \tag{6.22}$$

where $\bar{A}_t=\int_0^t \mathrm{d}r\int_{[0,r]^2}\mathrm{d}s_1\mathrm{d}s_2\ (t/r)^{\beta}(t/s_1)^{\beta}(t/s_2)^{\beta}<\infty$ since $\beta<1$.

Case 2. $1<\beta<2$. In this case, recalling (6.21), we proceed in the following order: applying Lemma 7.2 for Z_4, Z_3, using the change of variables ($x_1'=y_1+x_1$, $x_2'=y_2+x_2$) and (6.17), then applying Lemma 7.2 for Z_2 and using change of variables $x_2=y_3+x_1$ and (6.17), to obtain that

(6.23)

$$\begin{aligned}
\Phi_N^{(2)} &\lesssim C_{t,4}^2 c_{t,4}^2 N^{4d-3\beta} \int_0^t \mathrm{d}r \int_{[0,r]^2} \mathrm{d}s_1 \mathrm{d}s_2 \left[\left[N^\beta (r(t-r)/t)^{-\beta/2}\right] \wedge (t/r)^\beta\right] \\
&\quad \times \left[\left[N^\beta (s_1(t-s_1)/t + s_1(r-s_1)/r)^{-\beta/2}\right] \wedge (t/s_1)^\beta\right] \\
&\quad \times \left[\left[N^\beta (s_2(t-s_2)/t + s_2(r-s_2)/r)^{-\beta/2}\right] \wedge (t/s_2)^\beta\right] \\
&\lesssim N^{4d-3\beta} \int_0^1 \mathrm{d}r \int_{[0,r]^2} \mathrm{d}s_1 \mathrm{d}s_2 \left[\left[N^\beta (r(1-r))^{-\beta/2}\right] \wedge r^{-\beta}\right] \\
&\quad \times \left[\left[N^\beta (s_1(r-s_1)/r)^{-\beta/2}\right] \wedge s_1^{-\beta}\right] \left[\left[N^\beta (s_2(r-s_2)/r)^{-\beta/2}\right] \wedge s_2^{-\beta}\right] \\
&= N^{4d-3\beta} \int_0^1 \mathrm{d}r \left[\left[N^\beta (r(1-r))^{-\beta/2}\right] \wedge r^{-\beta}\right] \left[\int_0^r \left[\left[N^\beta (s(r-s)/r)^{-\beta/2}\right] \wedge s^{-\beta}\right] \mathrm{d}s\right]^2,
\end{aligned}$$

where the second inequality follows by a change of variables. Using a change of variables again, we see from (6.23) that

$$\begin{aligned}
\Phi_N^{(2)} &\lesssim N^{4d-3\beta} \int_0^1 \mathrm{d}r \left[\left[N^\beta (r(1-r))^{-\beta/2}\right] \wedge r^{-\beta}\right] r^{2-2\beta} \\
&\quad \times \left[\int_0^1 \left[\left[(Nr^{1/2})^\beta (s(1-s))^{-\beta/2}\right] \wedge s^{-\beta}\right] \mathrm{d}s\right]^2 \\
&= 2N^{4d+3\beta-6} \int_0^N \mathrm{d}\alpha\, \alpha^{5-4\beta} \left[\left[(\alpha^2(1-\alpha^2/N^2))^{-\beta/2}\right] \wedge \alpha^{-2\beta}\right] \\
&\quad \times \left[\int_0^1 \left[\left[\alpha^\beta (s(1-s))^{-\beta/2}\right] \wedge s^{-\beta}\right] \mathrm{d}s\right]^2. \qquad (6.24)
\end{aligned}$$

We apply Lemma 7.6 to conclude that in the case $1 < \beta < 2$,

$$\Phi_N^{(2)} \lesssim N^{4d+3\beta-6}. \qquad (6.25)$$

Case 3. $\beta = 1$. Notice that the estimate in (6.24) still holds for $\beta = 1$. We apply Lemma 7.7 to conclude that in the case $\beta = 1$

$$\Phi_N^{(2)} \lesssim N^{4d-3} (\log N)^3. \qquad (6.26)$$

6.3.3. Proof of Theorem 1.5.

Proof of Theorem 1.5. Recall (6.2) and (6.4). The case $0 < \beta < 1$ follows from Theorem 5.4, item 1, (6.16), and (6.22); the case $\beta = 1$ follows from Theorem 5.4, item 2, (6.20), and (6.26); the case $1 < \beta < 2$ follows from Theorem 5.4, item 3, (6.19), and (6.25). ∎

7. Appendix.

LEMMA 7.1. *Let I_N and $\tilde{I}_N$ be defined in* (4.10). *Then for all $s < t$ and $w \in \mathbb{R}^d$,*

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{d}x \left(I_N * \tilde{I}_N\right)(x) \left(f * \boldsymbol{p}_{2s(t-s)/t}\right) \left(\frac{s}{t}x + w\right) \\
&= \frac{1}{\pi^d} \int_{\mathbb{R}^d} \mathrm{e}^{-s(t-s)\|z\|^2/t} \prod_{j=1}^d \frac{1-\cos(Nz_j s/t)}{(Nz_j s/t)^2} \mathrm{e}^{iz\cdot w} \hat{f}(\mathrm{d}z) \qquad (7.1) \\
&\le \frac{1}{\pi^d} \int_{\mathbb{R}^d} \mathrm{e}^{-s(t-s)\|z\|^2/t} \prod_{j=1}^d \frac{1-\cos(Nz_j s/t)}{(Nz_j s/t)^2} \hat{f}(\mathrm{d}z). \qquad (7.2)
\end{aligned}$$

Proof. Clearly, it suffices to prove (7.1), which is a consequence of the identity (3.1) and the fact that the Fourier transform of $I_N * \tilde{I}_N$ is $2^d \prod_{j=1}^d \frac{1-\cos(Nz_j)}{(Nz_j)^2}$. ∎

LEMMA 7.2. *Let* $\mathrm{Z} \sim \mathrm{N}(0,1)$. *There exists a constant* $C > 0$ *such that for all* $s > 0$ *and* $y \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} \boldsymbol{p}_s(x+y)\|x\|^{-\beta}\mathrm{d}x = \mathrm{E}\left[\|\sqrt{s}\mathrm{Z}+y\|^{-\beta}\right] \le C\left(s^{-\beta/2} \wedge \|y\|^{-\beta}\right). \tag{7.3}$$

Proof. Since the convolution between $\boldsymbol{p}_s$ and $\|\cdot\|^{-\beta}$ is nonnegative-definite and maximized at 0, using a change of variable we can write

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \boldsymbol{p}_s(x+y)\|x\|^{-\beta}\mathrm{d}x = \int_{\mathbb{R}^d} \boldsymbol{p}_s(x)\|x\|^{-\beta}\mathrm{d}x = s^{-\beta/2}\int_{\mathbb{R}^d} \boldsymbol{p}_1(x)\|x\|^{-\beta}\mathrm{d}x.$$

This together with Lemma 3.1 of [17] implies (7.3). ∎

LEMMA 7.3. *Fix* $1 \le \beta < 2$. *Then we have for all* $\alpha > 0$,

$$\int_0^1 \lambda^{-\beta/2}\left(\left[\alpha^\beta(1-\lambda)^{-\beta/2}\right] \wedge \lambda^{-\beta/2}\right)\mathrm{d}\lambda \asymp \begin{cases} \mathbf{1}_{\{0<\alpha<1\}}\alpha + \mathbf{1}_{\{\alpha\ge1\}}\log\alpha, & \beta = 1,\\ \mathbf{1}_{\{0<\alpha<1\}}\alpha^\beta + \mathbf{1}_{\{\alpha\ge1\}}\alpha^{2\beta-2}, & 1<\beta<2.\end{cases} \tag{7.4}$$

Proof. We observe that

$$\alpha^\beta(1-\lambda)^{-\beta/2} \le \lambda^{-\beta/2} \Leftrightarrow \lambda \le \frac{1}{1+\alpha^2}.$$

Hence,

$$\begin{aligned}&\int_0^1 \lambda^{-\beta/2}\left(\left[\alpha^\beta(1-\lambda)^{-\beta/2}\right] \wedge \lambda^{-\beta/2}\right)\mathrm{d}\lambda\\ &\quad= \alpha^\beta\int_0^{1/(1+\alpha^2)} \lambda^{-\beta/2}(1-\lambda)^{-\beta/2}\mathrm{d}\lambda + \int_{1/(1+\alpha^2)}^1 \lambda^{-\beta}\mathrm{d}\lambda.\end{aligned}$$

Case 1. $\beta = 1$. In this case, for $0<\alpha<1$,

$$\alpha\int_0^{1/(1+\alpha^2)} \lambda^{-1/2}(1-\lambda)^{-1/2}\mathrm{d}\lambda \asymp \alpha$$

and

$$\int_{1/(1+\alpha^2)}^1 \lambda^{-1}\mathrm{d}\lambda = \log(1+\alpha^2) \asymp \alpha^2.$$

On the other hand, for $\alpha \ge 1$,

$$\alpha\int_0^{1/(1+\alpha^2)} \lambda^{-1/2}(1-\lambda)^{-1/2}\mathrm{d}\lambda \asymp \alpha\int_0^{1/(1+\alpha^2)} \lambda^{-1/2}\mathrm{d}\lambda \asymp 1$$

and

$$\int_{1/(1+\alpha^2)}^1 \lambda^{-1}\mathrm{d}\lambda = \log(1+\alpha^2) \asymp \log\alpha.$$

This proves the first part of (7.4).

Case 2. $1 < \beta < 2$. In this case, for $0 < \alpha < 1$,

$$\alpha^\beta \int_0^{1/(1+\alpha^2)} \lambda^{-\beta/2}(1-\lambda)^{-\beta/2}\mathrm{d}\lambda \asymp \alpha^\beta$$

and

$$\int_{1/(1+\alpha^2)}^1 \lambda^{-\beta}\mathrm{d}\lambda = \frac{1}{\beta-1}((1+\alpha^2)^{\beta-1}-1) \asymp \alpha^2.$$

On the other hand, for $\alpha \geq 1$,

$$\alpha^\beta \int_0^{1/(1+\alpha^2)} \lambda^{-\beta/2}(1-\lambda)^{-\beta/2}\mathrm{d}\lambda \asymp \alpha^\beta \int_0^{1/(1+\alpha^2)} \lambda^{-\beta/2}\mathrm{d}\lambda \asymp \alpha^{2\beta-2}.$$

This proves the second part of (7.4) and hence completes the proof. ∎

Lemma 7.4. *Fix* $1 < \beta < 2$. *Then*

(7.5)

$$\sup_{N\geq \mathrm{e}} \int_{[0,N^2]^2} \mathrm{d}s_1\mathrm{d}s_2 \left[\left[(s_1(1-s_1/N^2))^{-\beta/2}\right] \wedge s_1^{-\beta}\right] \left[\left[(s_2(1-s_2/N^2))^{-\beta/2}\right] \wedge s_2^{-\beta}\right] \\ \times \left(\frac{1}{2s_1}+\frac{1}{2s_2}\right)^{\beta-1} \int_0^1 \mathrm{d}\lambda\, \lambda^{-\frac{\beta}{2}} \left(\left[\left(\frac{1}{2s_1}+\frac{1}{2s_2}\right)^{-\beta/2}(1-\lambda)^{-\frac{\beta}{2}}\right] \wedge \lambda^{-\frac{\beta}{2}}\right) < \infty.$$

Proof. Applying Lemma 7.3 (second part) with $\alpha = (\frac{1}{2s_1}+\frac{1}{2s_2})^{-1/2}$, for all $N \geq \mathrm{e}$, the above integral is bounded above by a constant times

$$\int_{[0,N^2]^2} \mathrm{d}s_1\mathrm{d}s_2 \left[\left[(s_1(1-s_1/N^2))^{-\beta/2}\right] \wedge s_1^{-\beta}\right] \left[\left[(s_2(1-s_2/N^2))^{-\beta/2}\right] \wedge s_2^{-\beta}\right] \\ \times \left(\frac{1}{2s_1}+\frac{1}{2s_2}\right)^{\beta-1} \left(\mathbf{1}_{\{\frac{1}{2s_1}+\frac{1}{2s_2}>1\}}\left(\frac{1}{2s_1}+\frac{1}{2s_2}\right)^{-\beta/2} + \mathbf{1}_{\{\frac{1}{2s_1}+\frac{1}{2s_2}\leq 1\}}\left(\frac{1}{2s_1}+\frac{1}{2s_2}\right)^{1-\beta}\right).$$

For $(s_1, s_2) \in [0,1]^2$, the above integrand is bounded above by

$$s_1^{-\beta/2}(1-1/N^2)^{-\beta/2}s_2^{-\beta/2}(1-1/N^2)^{-\beta/2}\left(\frac{1}{2s_1}+\frac{1}{2s_2}\right)^{\beta/2-1},$$

whence for all $N \geq \mathrm{e}$ the integral over $[0,1]^2$ is bounded above by

$$(1-1/\mathrm{e}^2)^{-\beta}\int_{[0,1]^2} s_1^{-\beta/2}s_2^{-\beta/2}\left(\frac{1}{2s_1}+\frac{1}{2s_2}\right)^{\beta/2-1}\mathrm{d}s_1\mathrm{d}s_2 \\ \lesssim \int_0^1 s_2^{-\beta/2}\mathrm{d}s_2 \int_0^1 s_1^{-\beta/2}s_1^{1-\beta/2}\mathrm{d}s_1 < \infty.$$

Moreover, for $(s_1, s_2) \in [0,1] \times (1, N^2]$, the integrand is bounded above by a constant times

$$s_1^{-\beta/2}(1-1/N^2)^{-\beta/2}s_2^{-\beta}\left[\left(\frac{1}{2s_1}+\frac{1}{2s_2}\right)^{\beta/2-1}+1\right] \\ \lesssim (1-1/\mathrm{e}^2)^{-\beta/2}\left(s_1^{1-\beta}s_2^{-\beta}+s_1^{-\beta/2}s_2^{-\beta}\right),$$

whence for all $N \geq \mathrm{e}$ the integral over $[0,1] \times (1, N^2]$ is bounded above by a constant times

$$\int_0^1 s_1^{1-\beta} \mathrm{d}s_1 \int_1^\infty s_2^{-\beta} \mathrm{d}s_2 + \int_0^1 s_1^{-\beta/2} \mathrm{d}s_1 \int_1^\infty s_2^{-\beta} \mathrm{d}s_2 < \infty.$$

Similarly, the integral over $(1, N^2] \times [0,1]$ is also finite uniformly for $N \geq \mathrm{e}$.

Furthermore, for $(s_1, s_2) \in (1, N^2]^2$, the integrand is bounded above by $s_1^{-\beta} s_2^{-\beta}$, which implies that the integral over $(1, N^2]^2$ is also finite uniformly for $N \geq \mathrm{e}$. The proof is complete. ◻

LEMMA 7.5. *There exists a constant $C > 0$ such that for all $N \geq \mathrm{e}$*

$$\begin{aligned}(7.6)\quad &\int_{[0,N^2]^2} \mathrm{d}s_1 \mathrm{d}s_2 \left[\left[(s_1(1 - s_1/N^2))^{-1/2}\right] \wedge s_1^{-1}\right] \left[\left[(s_2(1 - s_2/N^2))^{-1/2}\right] \wedge s_2^{-1}\right] \\ &\quad \times \int_0^1 \mathrm{d}\lambda\, \lambda^{-\frac{1}{2}} \left(\left[\left(\frac{1}{2s_1} + \frac{1}{2s_2} \right)^{-1/2} (1-\lambda)^{-\frac{1}{2}} \right] \wedge \lambda^{-\frac{1}{2}} \right) \\ &\leq C\, (\log N)^3.\end{aligned}$$

Proof. The proof is similar to that of Lemma 7.4.

Applying Lemma 7.3 (first part) with $\alpha = (\frac{1}{2s_1} + \frac{1}{2s_2})^{-1/2}$, for all $N \geq \mathrm{e}$, the above integral is bounded above by a constant times

$$\begin{aligned}&\int_{[0,N^2]^2} \mathrm{d}s_1 \mathrm{d}s_2 \left[\left[(s_1(1 - s_1/N^2))^{-1/2}\right] \wedge s_1^{-1}\right] \left[\left[(s_2(1 - s_2/N^2))^{-1/2}\right] \wedge s_2^{-1}\right] \\ &\times \left(\mathbf{1}_{\{\frac{1}{2s_1} + \frac{1}{2s_2} > 1\}} \left(\frac{1}{2s_1} + \frac{1}{2s_2} \right)^{-1/2} + \mathbf{1}_{\{\frac{1}{2s_1} + \frac{1}{2s_2} \leq 1\}} \log \left[\left(\frac{1}{2s_1} + \frac{1}{2s_2} \right)^{-1/2} \right] \right).\end{aligned}$$

For $(s_1, s_2) \in [0,1]^2$, the above integrand is bounded above by

$$s_1^{-1/2} (1 - 1/N^2)^{-1/2} s_2^{-1/2} (1 - 1/N^2)^{-1/2} \left(\frac{1}{2s_1} + \frac{1}{2s_2} \right)^{-1/2},$$

whence for all $N \geq \mathrm{e}$ the integral over $[0,1]^2$ is bounded above by a constant times

$$\begin{aligned}&(1 - 1/\mathrm{e}^2)^{-1} \int_{[0,1]^2} s_1^{-1/2} s_2^{-1/2} \left(\frac{1}{2s_1} + \frac{1}{2s_2} \right)^{-1/2} \mathrm{d}s_1 \mathrm{d}s_2 \\ &\quad \lesssim \int_0^1 s_2^{-1/2} \mathrm{d}s_2 \int_0^1 s_1^{-1/2} s_1^{1/2} \mathrm{d}s_1 < \infty.\end{aligned}$$

Moreover, for $(s_1, s_2) \in [0,1] \times (1, N^2]$, the integrand is bounded above by

$$\begin{aligned}&s_1^{-1/2} (1 - 1/N^2)^{-1/2} s_2^{-1} \left[\left(\frac{1}{2s_1} + \frac{1}{2s_2} \right)^{-1/2} + \log \left[\left(\frac{1}{2s_1} + \frac{1}{2s_2} \right)^{-1/2} \right] \right] \\ &\quad \lesssim (1 - 1/\mathrm{e}^2)^{-1/2} \left(s_2^{-1} + s_1^{-1/2} s_2^{-1} \log(2s_2) \right),\end{aligned}$$

whence for all $N \geq \mathrm{e}$ the integral over $[0,1] \times (1, N^2]$ is bounded above by a constant times

$$\int_1^{N^2} s_2^{-1} \mathrm{d}s_2 + \int_0^1 s_1^{-1/2} \mathrm{d}s_1 \int_1^{N^2} \frac{\log(2s_2)}{s_2} \mathrm{d}s_2 \asymp (\log N)^2.$$

Similarly, the integral over $(1, N^2] \times [0,1]$ is also bounded above by $C(\log N)^2$ for $N \geq \mathrm{e}$.

Furthermore, the integral over $(1, N^2]^2$ is bounded above by

$$\begin{aligned}
&\int_{(1,N^2]^2} \mathrm{d}s_1 \mathrm{d}s_2\, s_1^{-1} s_2^{-1} \log\left[\left(\frac{1}{2s_1} + \frac{1}{2s_2}\right)^{-1/2}\right] \\
&= 2\int_1^{N^2} \mathrm{d}s_1 \int_1^{s_1} \mathrm{d}s_2\, s_1^{-1} s_2^{-1} \log\left[\left(\frac{1}{2s_1} + \frac{1}{2s_2}\right)^{-1/2}\right] \\
&\leq \int_1^{N^2} \mathrm{d}s_1 \int_1^{s_1} \mathrm{d}s_2\, s_1^{-1} s_2^{-1} \log s_1 = \int_1^{N^2} \frac{(\log s_1)^2}{s_1} \mathrm{d}s_1 \asymp (\log N)^3.
\end{aligned}$$

The proof is complete. ∎

LEMMA 7.6. *Fix $1 < \beta < 2$. Then,*

$$\sup_{N \geq \mathrm{e}} \int_0^N \mathrm{d}\alpha\, \alpha^{5-4\beta} \left[\left[(\alpha^2(1-\alpha^2/N^2))^{-\beta/2}\right] \wedge \alpha^{-2\beta}\right] \left(\int_0^1 \left[\left[\alpha^\beta (s(1-s))^{-\beta/2}\right] \wedge s^{-\beta}\right] \mathrm{d}s\right)^2 < \infty. \tag{7.7}$$

Proof. By Lemma 7.3, for all $N \geq \mathrm{e}$, the above integral is bounded above by a constant times

$$\begin{aligned}
&\int_0^1 \mathrm{d}\alpha\, \alpha^{5-2\beta} \left[\left[(\alpha^2(1-1/N^2))^{-\beta/2}\right] \wedge \alpha^{-2\beta}\right] \\
&\quad + \int_1^N \mathrm{d}\alpha\, \alpha \left[\left[(\alpha^2(1-\alpha^2/N^2))^{-\beta/2}\right] \wedge \alpha^{-2\beta}\right] \\
&\leq (1 - 1/\mathrm{e}^2)^{-\beta/2} \int_0^1 \alpha^{5-3\beta} \mathrm{d}\alpha + \int_1^\infty \alpha^{1-2\beta} \mathrm{d}\alpha < \infty. \qquad ∎
\end{aligned}$$

LEMMA 7.7. *There exists a constant $C > 0$ such that for all $N \geq \mathrm{e}$,*

$$\int_0^N \mathrm{d}\alpha\, \alpha \left[\left[(\alpha^2(1-\alpha^2/N^2))^{-1/2}\right] \wedge \alpha^{-2}\right] \left(\int_0^1 \left[\left[\alpha (s(1-s))^{-1/2}\right] \wedge s^{-1}\right] \mathrm{d}s\right)^2 \tag{7.8}$$

$$\leq C(\log N)^3. \tag{7.9}$$

Proof. Thanks to Lemma 7.3 (first part), the integral in (7.8) is bounded above by a constant times

$$(1-1/\mathrm{e}^2)^{-1/2} \int_0^1 \alpha^{1-1+2} \mathrm{d}\alpha + \int_1^N \alpha^{1-2} (\log \alpha)^2 \mathrm{d}\alpha \asymp (\log N)^3. \qquad ∎$$

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