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The Matroid Cup Game

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ABSTRACT

In this paper we introduce the *Matroid Cup Game*. This cup game generalizes the One- and p-Cup Games. We show that a natural greedy strategy maintains max fill $O(\log n)$ in the Matroid Cup Game with mild resource augmentation. Further, we introduce a new objective for cup game: the *total water*, which is the amount of water across all cups. We develop a novel Emptier strategy that maintain $O(rn)$ total water *without resource augmentation*. We also reveal a novel relationship between max fill and total water, which leads to an intuitive analysis of the max fill bounds in the One- and p-Cup Games.

1. Introduction

The cup game is an elegant model that studies how closely a discrete process can simulate a continuous process. Cup games naturally arise in the analysis of data structures [3,9,10], online algorithms [5] and queuing systems [1,6,13]. They model a situation where we want some algorithm/policy to "keep up" with a more powerful adversary.

The most basic cup game is the *One Cup Game*. In the One Cup Game, there are n initially empty cups, and two players: the Emptier and the Filler. In each round of the game, first the Filler (the adversary) distributes one unit of water among the n cups. Then the Emptier chooses one cup to remove one unit of water from. If the chosen cup has less than one unit of water, then the Emptier removes all water from the cup. In this paper we always assume that the Filler and Emptier are both *adaptive* and can observe the amount of water in each cup. Further, the Filler's and Emptier's decision must be made online with no knowledge of future rounds.

Note that the Filler is more powerful than the Emptier, because the former can add water fractionally, while the latter is restricted to removing water integrally. Our goal is to give a strategy for the Emptier so that the Emptier can "keep up" with the Filler. The most extensively studied goal is to minimize the fill of the most-filled cup over *all* rounds of the game, which we call the *max fill*. Note that in general the One Cup Game can proceed for infinitely many rounds.

Perhaps the most natural strategy for the Emptier is the *Greedy Strategy*, where in each round the Emptier chooses the most-filled cup to remove one unit from. It is known that the Greedy Strategy maintains max fill $O(\log n)$ for the One Cup Game [1,10]. The proof technique is to show that Greedy maintains a system of carefully-crafted invariants, which imply the $O(\log n)$ bound. Further, there exists a strategy for the *Filler* such that any Emptier must incur max fill $\Omega(\log n)$ [1,10]. Note that this lower bound holds against any Emptier strategy – even a randomized one – assuming that the Filler is adaptive and can observe the current cup state. In the regime where the Filler is non-adaptive and cannot observe the cups, a randomized Emptier can achieve $O(\log\log n)$ max fill for polynomially many rounds [7,12].

Recently, it was shown that the analogous greedy strategy for the *p-Cup Game* also maintains max fill $O(\log n)$ [12]. Further, they showed that there exists a Filler strategy such that any Emptier must incur max fill $\Omega(\log(n-p))$. The p-Cup Game differs from the One Cup Game only in the constraints placed on the Filler and Emptier. Here, we have a parameter $p \in \mathbb{N}$ such that in each round the Filler distributes p units of water among the n cups, adding at most one unit per cup, and then the Emptier chooses p distinct cups to remove up to one unit from. Then the Greedy Strategy for the p-Cup Game is to pick the p fullest cups to remove one unit from.

1.1. Matroid Cup Game

Our first contribution is the introduction of the *Matroid Cup Game*, which defines a cup game subject to a given matroid.

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Definition 1.1 *(Matroid).* A matroid is a set family $\mathcal{M} \subset 2^X$ for some finite ground set X satisfying the following properties:

1. $\emptyset \in \mathcal{M}$.
2. If $A \in \mathcal{M}$ and $B \subset A$, then $B \in \mathcal{M}$.
3. (Basis Exchange Property) If $A, B \in \mathcal{M}$ and $|A| > |B|$, then there exists $x \in A \setminus B$ such that $B \cup \{x\} \in \mathcal{M}$.

The sets in $\mathcal{M}$ are called *independent sets*. The *rank* of a matroid is the size of the largest independent set. A *basis* of a matroid is a maximal (inclusion-wise) independent set.

Further, we recall that the *independent set polytope* for a given matroid is the convex hull of all indicator vectors of its independent sets, so the matroid polytope is a subset of $\mathbb{R}^X$.

Definition 1.2 *(Matroid Cup Game).* We are given a matroid on the ground set of n cups. All n cups are initially empty. In each round of the game, first the Filler chooses a point in the independent set polytope to add to the cups. Then the Emptier chooses an independent set of cups to remove one unit of water from. If some of the chosen cups have less than one unit of water, then the Emptier removes all water from those cups.

Note that the natural extension of the Greedy Strategy for the One- and p-Cup Games is the following:

Definition 1.3 *(Greedy strategy for Matroid Cup Game).* The Greedy Strategy for the Emptier is to use the following algorithm on each Emptier turn: Let f be the current fill vector such that $f(i)$ is the fill of cup i. Sort the cups is descending order of fill so $f(1) \geq \cdots \geq f(n)$. Then for $i = 1$ to n, choose the ith cup to remove one unit from if it forms an independent set with the set of all previously chosen cups (in this turn.)

We now discuss some applications of the Matroid Cup Game.

- p**-Cup Game**: We use a uniform matroid on the n cups where the ground set is the n cups, and a set of cups is independent if and only if it has size at most p. It is immediate that this cup game as well as the Greedy Strategy coincide with the p-Cup Game.
- **Hierarchical Cup Game**: This cup game is constrained by a laminar matroid. A laminar matroid is associated with a laminar family $\mathcal{F}$ of subsets of cups: for any $A \neq B \in \mathcal{F}$, $A \subseteq B$, $B \subseteq A$, or $A \cap B = \emptyset$. Further, each subset A in $\mathcal{F}$ is associated a capacity $\mu(A)$. A subset S is independent if and only if $|S \cap A| \leq \mu(A)$ for all $A \in \mathcal{F}$. Laminar matroids can be used to model data transfer in a tree network from the root to the leaves. To see this, consider a tree where there is a node corresponding to each set A in $\mathcal{F}$ and the node can transfer up to $\mu(A)$ messages at a time. Then, the root can send a message to each leaf in S simultaneously if and only if S is independent. Tree network structure is a popular networking model for problems arising in practice [2,11].
- **Matching Cup Game**: This cup game is constrained by a transversal matroid. In this cup game there is a bipartite graph where one side is the n cups and the other side V is a set of vertices disjoint from the cups. A subset of cups S is declared independent if and only if there is a matching saturating all cups in S. This cup game can model switch networks, and so-called co-flow found in cluster computing [4,8].

For more examples of matroids, we refer the reader to [14].

1.2. Our results

Our first result is an asymptotically optimal upper bound on the max fill for the Matroid Cup Game with mild resource augmentation. As mentioned, minimizing the max fill is perhaps the most popular objective considered in the cup games. Under ϵ-resource augmentation, the Filler is weakened by filling $(1-\epsilon)$ units of independent sets in each round.

Definition 1.4 *(Resource augmentation for Matroid Cup Game).* Let $\epsilon \in (0, 1)$. We define the Matroid Cup Game with ϵ-resource augmentation as follows. As in the original Matroid Cup Game, we are given a matroid $\mathcal{M}$ on the n cups with associated independent set polytope $\mathcal{P} \subset \mathbb{R}^n$. The only difference is that in each round, the Filler is only allowed to add $x \in (1-\epsilon)\mathcal{P}$, where $(1-\epsilon)\mathcal{P} = \{x \in \mathbb{R}^n \mid \frac{1}{1-\epsilon} x \in \mathcal{P}\}$.

Theorem 1.5. *(Section 2) Let $\epsilon(n) \in (1/poly(n), 1)$. In the Matroid Cup Game on n cups with $\epsilon(n)$-resource augmentation, the Greedy Strategy maintains max fill $O(\frac{1}{\epsilon(n)} \log n)$.*

For constant $\epsilon(n)$, this is essentially the best result one can hope for, as it is known that the One Cup game, which is a special case of the Matroid Cup Game, has $\Omega(\log n)$ max fill for some Filler Strategy even with resource augmentation.

We continue to study the Matroid Cup Game without resource augmentation. Unfortunately, cup games become much more challenging without weakening the Filler. Indeed, until very recently no non-trivial bounds have been known for the p-Cup Game without resource augmentation [12].

Our second result is a non-trivial bound on the *total water*, which is the total amount of water in all cups, without resource augmentation. Note that it is not obvious to keep total water bounded as the game runs over an infinite number of rounds.

Theorem 1.6. *(Section 3) There exists an Emptier Strategy for the Matroid Cup Game that maintains total water $O(rn)$ (without resource augmentation) where r is the rank of the given matroid.*

Our final result is an interesting relationship between max fill and total water. This relationship allows us to reduce the problem of bounding the max fill to that of bounding total water, under the Greedy Strategy. This is useful as it is often easier to bound the total water rather than directly bounding the max fill. For example, it is easy to see that Greedy maintains total water $O(n)$ for the One Cup Game.

Theorem 1.7. *(Section 4) Suppose the Greedy Strategy maintains total water $\alpha n'$ for the Matroid Cup Game, where n' is the maximum number of non-empty cups over every turn of the game. Then the Greedy Strategy maintains max fill $2\alpha \log_2 n'$ for all $n' \geq 2$.*

Unfortunately we do not know how to apply the reduction for the Matroid Cup Game as we do not even know if the Greedy algorithm maintains finite total water without resource augmentation. Using this reduction we however obtain a more intuitive analysis of the max fill bound for the One- and p-Cup Games although it gives a weaker bound for the p-Cup game. Specifically, using this approach, we obtain $O(\log n)$ and $O(p \log n)$ bounds on the max fill for the One-cup and p-cup games, respectively. We hope that this reduction will find more applications for other cup games in the future.

1.3. Technical overview

For each of our three main results, we illustrate the main ideas on the One Cup Game.

Bounding the Max Fill with Resource Augmentation. We give a potential function proof that bounds the max fill. In particular, we consider an exponential potential function, that is roughly of the form: $\Phi(f) = \sum_{i=1}^{n} e^{f(i)}$, where $f(i)$ is the current fill of the ith cup. Note that in the One Cup Game, ϵ-resource augmentation corresponds to the Filler adding $(1-\epsilon)$ units. Importantly, the Greedy Emptier always removes one unit from the fullest cup. In terms of the potential function, the Emptier will always decrease the largest exponent by one, whereas the Filler can increase all exponents by $1-\epsilon$ in total. If the largest exponent is large enough, say $O(\log n)$, we can show that the Emptier's decrease of Φ will always overpower the Filler's increase.

To extend this idea to the Matroid Cup Game, we need to extend our argument to the case where Greedy chooses more than one cup. That is, each exponent that Greedy chooses is better than $1-\epsilon$ units of exponents that the Filler chooses.

Bounding the Total Water without Resource Augmentation. It is informative to see how the Greedy Strategy maintains total water $O(n)$ in the One Cup Game. Consider a round of the One Cup Game when the total water is at least n in the beginning. Then the Filler adds one unit. Now by averaging over the at most n non-empty cups, there exists a cup with at least one unit of water. By definition of Greedy, the Emptier will choose to remove water from a cup with at least one unit. In this round, the Filler added one unit, and the Emptier removed one unit, so the total water is unchanged.

To extend to the Matroid Cup Game, we change our "coordinate system." Instead of looking at the water in each cup, we consider the water in each *independent set*. Thus, for each independent set, we have a non-negative coefficient, which is the amount of water in that independent set. We can view an Emptier Strategy as picking an independent set, and trying to reduce its coefficient by one. The Emptier succeeds in maintaining the total water in the One Cup Game when it can successfully remove one unit. Similarly, in the Matroid Cup Game, the Emptier succeeds when it can remove one unit from each cup in an independent set. Thus, instead of averaging over cups as in the One Cup Game, we can average over independent sets to find one with coefficient at least one to remove from.

Reducing Max Fill to Total Water. We use the fact that the Greedy Strategy maintains total water $O(n)$ in the One Cup Game to bound the max fill. Because the total water is always $O(n)$, there are always at most $n/2$ cups with at least c units of water for some constant $c > 0$. Thus, if we look only at the dynamics of the water above height c in the original game, we can see that in this "subgame", the Filler adds at most one unit in each round, and the Emptier is still Greedy. Further, the max fill in the original game is at most the max fill of the subgame plus c.

Importantly, in the subgame, there are always at most $n/2$ non-empty cups, so we can similarly bound the total water in the subgame by $O(n/2)$. It follows, there are at most $n/4$ cups in the subgame with at least c units of water (equivalently, at most $n/4$ cups in the original game with at least $2c$ units of water.) Then, we can consider a "sub-subgame" of at most $n/4$ non-empty cups such that the max fill in the original game is at most the max fill of the subgame plus $2c$. Continuing this argument inductively gives that in the original game has at most one cup with at least $c\log n$ units of water. The water above height $c\log n$ is like an instance of the One Cup Game on a single cup, where it is trivial to achieve max fill $O(1)$. Thus, the max fill in the original game is $O(\log n)$. This reduction from max fill to total water by considering sub-games generalizes straightforwardly to the Matroid Cup Game.

1.4. Preliminaries

Throughout this paper, we will represent the fills of all cups in a *fill vector*, which is a vector in $\mathbb{R}^n$ such that the ith entry is the amount of water in the ith cup. Also, for a vector $v \in \mathbb{R}^n$, we let $v(i)$ denote the ith entry of v for all $i \in [n]$.

Given a fill vector $f \in \mathbb{R}^n$, we denote the max fill of f by $max(f) = \max_i f(i)$, which is the amount of water in the most-filled cup in f. We denote the total water of f by $tot(f) = \sum_i f(i)$, which is the sum of water across all cups. Then the max fill/total water of an instance of the Matroid Cup Game is the largest max fill/total water achieved by any fill vector f of the game.

We say a *round* of the Matroid Cup Game consists of a Filler turn and the subsequent Emptier turn. Thus, a *turn* of the Matroid Cup Game is either a Filler or Emptier action.

2. Resource augmentation for Matroid Cup Game

In this section, we prove that the *Greedy Emptier* (Definition 1.3) maintains max fill $O(\log n)$ in the Matroid Cup Game using *resource augmentation* (Definition 1.4.)

Recall that in the original Matroid Cup Game, the Filler can add $x \in \mathcal{P}$, but in the ϵ-resource augmentation game, the Filler choice is scaled down by a $(1-\epsilon)$-factor. By weakening the Emptier, we can show that Greedy maintains bounded max fill.

The key technical lemma we will use states that the cups chosen by the Greedy Emptier are "elementwise better" than the cups chosen fractionally by the Filler.

Lemma 2.1. *Let $\mathcal{P} \subset \mathbb{R}^n$ be the independent set polytope of some matroid $\mathcal{M}$ and $w \in \mathbb{R}^n_+$ any weight vector. Let $x^* \in \{0,1\}^n$ be the indicator of the max weight independent set of $\mathcal{M}$ with respect to weights w. Then for any $x \in \mathcal{P}$, there exists a (fractional) assignment of the units of x to the units of x^* given by values $y(i,j) \in [0,1]$ (denoting assigning $y(i,j)$ units of $x(i)$ to $x^*(j)$) for all $i \in [n]$, $j \in S$ such that:*

1. *For all $i \in [n]$, $\sum_{j\in S} y(i,j) = x(i)$, so every $x(i)$ is assigned fully.*
2. *For all $j \in S$, $\sum_{i\in[n]} y(i,j) \le 1$, so every $x^*(j) = 1$ is assigned at most one unit.*
3. *For all $i \in [n]$, $j \in S$, if $y(i,j) > 0$, then $w(i) \le w(j)$.*

Proof. Without loss of generality, assume $w(1) \ge \cdots \ge w(n)$, and Greedy considers elements in order $1, \ldots, n$. Then suppose the max weight independent set is indexed by $1 = i_1 < \cdots < i_r \le n$, where r is the rank of matroid $\mathcal{M}$.

Further, for any set of cups $S \subset [n]$, let $rank(S)$ denote the size of the largest independent set contained in S. In our proof, we use the standard fact about the independent set polytope $\mathcal{P}$ that for all $x \in \mathcal{P}$ and any $S \subset [n]$, we have $\sum_{i\in S} x(i) \le rank(S)$ (see e.g. [14].)

For all $j \in [r-1]$, we define the interval $I_j = [1, i_{j+1})$. We claim that $\sum_{i\in I_j} x(i) \le j$ for all j. Suppose not, so there exists some j with $\sum_{i\in I_j} x(i) > j$. Because $x \in \mathcal{P}$, this implies $rank(I_j) \ge j+1$. However, Greedy picks only j elements from I_j. Thus, using the basis exchange property, Greedy should have picked a $(j+1)$th element from I_j instead of picking its next element from $[n] \setminus I_j$. This is a contradiction.

Using that $\sum_{i\in I_j} x(i) \le j$ for all $j \in [r-1]$, it is immediate that assigning the one unit of x with highest weight to $x^*(i_1)$, then the next one unit to $x^*(i_2)$, and so on gives the desired fractional assignment. □

Using this lemma, we give a potential function argument to bound the max fill.

Theorem 2.2 *(Theorem 1.5, restated). Let $\epsilon(n) \in (1/poly(n), 1)$. Then in the Matroid Cup Game on n cups with $\epsilon(n)$-resource augmentation, the Greedy Strategy maintains max fill $O(\frac{1}{\epsilon(n)} \log n)$.*

Proof. We consider the exponential potential function $\Phi(f) = \sum_{i\in[n]} \int_0^{f(i)} (1+\epsilon(n))^{\lceil x \rceil} dx$ for any fill vector $f \in \mathbb{R}^n$. It suffices to prove that at the beginning of every round with fill, say f, we have $\Phi(f) = poly(n)$. This implies that for all i, $\int_0^{f(i)} (1+\epsilon(n))^{\lceil x \rceil} dx = poly(n) \Rightarrow f(i) = O(\log_{1+\epsilon(n)} n) = O(\frac{1}{\epsilon(n)} \log n)$.

We show that $\Phi(f) = poly(n)$ at the beginning of each round by induction on the rounds of the game. At the beginning of the game, $f = 0$, so $\Phi(f) = 0$. Now consider an arbitrary round $f \to \bar{f} \to f'$, where f is the beginning of the round, $\bar{f}$ is after the Filler's turn, and f' is after the Emptier's turn. We may assume inductively that $\Phi(f) = poly(n)$, and we wish to show that $\Phi(f') = poly(n)$ as well. Suppose the Filler adds $x \in (1-\epsilon(n))\mathcal{P}$ to the cups in $f \to \bar{f}$. Without loss of generality, we may assume $\bar{f}(1) \geq \cdots \geq \bar{f}(n)$. There are two cases to consider:

Case 1: $\bar{f}(1) < 1$. In this case $\bar{f}(i) < 1$ for all i so $\Phi(f') \leq \Phi(\bar{f}) \leq (1+\epsilon(n))n = poly(n)$.

Case 2: $\bar{f}(1) \geq 1$. Let the Emptier's choice of cups be indexed by $1 = i_1 < \cdots < i_r \leq n$ with corresponding indicator vector x^*. Note that this is exactly the max weight independent set with respect to weights $\bar{f}$. Applying Lemma 2.1 to $\frac{1}{1-\epsilon(n)} x \in \mathcal{P}$ and weight vector $\bar{f}$, for all $j \in [r]$, we assign $x^*(i_j)$ to at most one unit of $\frac{1}{1-\epsilon(n)} x$ of smaller fill in $\bar{f}$. This corresponds to assigning $x^*(i_j)$ with at most $(1-\epsilon(n))$ units of x of smaller fill.

Now consider any $j \in [r]$ such that $\bar{f}(i_j) \geq 1$ (note that such a j exists because we are in Case 2.) Then in $\bar{f} \to f'$, the Emptier removes one unit from cup i_j, which decreases Φ by at least $1 \cdot (1+\epsilon(n))^{\lceil \bar{f}(i_j) \rceil - 1}$. Similarly, in $f \to \bar{f}$ the at most $(1-\epsilon(n))$ units of x that are assigned to $x^*_{i_j}$ increase Φ by at most $(1-\epsilon(n))(1+\epsilon(n))^{\lceil \bar{f}_{i_j} \rceil}$. Thus, the net change in potential in $f \to f'$ due to removing $x^*(i_j)$ and adding the assigned Filler units is at most:

$$(1-\epsilon(n))(1+\epsilon(n))^{\lceil \bar{f}(i_j) \rceil} - (1+\epsilon(n))^{\lceil \bar{f}(i_j) \rceil - 1}$$
$$= -\epsilon(n)^2 (1+\epsilon(n))^{\lceil \bar{f}(i_j) \rceil - 1},$$

which we note is negative, so the potential only decreases due to such j's.

Now we consider $j \in [r]$ such that $\bar{f}(i_j) < 1$. For these cups, the Emptier is unable to remove a full unit. However, using our fractional assignment, all of the Filler actions that are assigned to some i_j with $\bar{f}(i_j) < 1$ add water to cups whose fill in $\bar{f}$ is at most one. The increase in potential due to *all* such Filler actions is at most $(1+\epsilon(n))n$.

To complete the proof, we show that either $\bar{f}(i_1) = \bar{f}(1)$ is large enough such that the decrease in potential due to $x^*(i_1)$ overpowers the increase due to these cups with less than one unit, or $\bar{f}(1)$ is small enough such that $\phi(f') = poly(n)$. For the former case, we require:

$$(1+\epsilon(n))n \leq \epsilon(n)^2 (1+\epsilon(n))^{\lceil \bar{f}(1) \rceil - 1}.$$

Recall that $\epsilon(n) > 1/poly(n)$, so it suffices for $\bar{f}(1)$ to satisfy:

$$(1+\epsilon(n))^{\lceil \bar{f}(1) \rceil - 1} \geq (1+\epsilon(n)) poly(n),$$

so if $\bar{f}(1) = \Omega(\log_{1+\epsilon(n)} n)$, then the potential does not increase in this round.

Otherwise, $\bar{f}(1) = O(\log_{1+\epsilon(n)} n)$ so $\Phi(f') \leq \Phi(\bar{f}) \leq n \int_0^{O(\log_{1+\epsilon(n)} n)} (1+\epsilon(n))^{\lceil x \rceil} dx = poly(n)$, as required. □

3. Max Coefficient Strategy for Matroid Cup Game

In this section, we develop a *non-Greedy* strategy for the Matroid Cup Game on a rank r matroid (without resource augmentation) that maintains total water $O(rn)$. We call our strategy the Max Coefficient Strategy. This provides the first non-trivial guarantee for any strategy for the Matroid Cup Game. Our strategy and analysis utilize a polyhedral interpretation of the Matroid Cup Game.

Consider an instance of the Matroid Cup Game on rank r matroid $\mathcal{M}$ with independent set polytope $\mathcal{P} \subset \mathbb{R}^n$. Thus, in each round the Filler adds a point $x \in \mathcal{P}$ to the cups, so x is a convex combination of independent sets of $\mathcal{M}$. We recall that a *basis* of $\mathcal{M}$ is a maximal (inclusion-wise) independent set. By definition, every independent set contained in some basis, and the properties of matroids imply that every basis has size r.

3.1. Max Coefficient Strategy

We develop a strategy assuming that in each round, the Filler adds a convex combination of *bases* to the cups (not just independent sets.) Note that we can always extend each Filler choice to be a convex combination of bases. This only increases the amount of water the Filler adds to each cup.

Our key technical tool is the following well-known variant of Carathéodory's Theorem for the conic hull. For vectors $v_1, \ldots, v_\ell \in \mathbb{R}^n$, we let $cone(v_1, \ldots, v_\ell) = \{\sum_i \lambda_i v_i \mid \lambda_i \geq 0 \quad \forall i\}$ denote the set of all non-negative linear combinations of $v_1, \ldots, v_\ell$ (i.e. the conic hull of $v_1, \ldots, v_\ell$.) The proof is a standard dimension argument, which we include for completeness.

Lemma 3.1. *Let $v_1, \ldots, v_\ell \in \mathbb{R}^n$. If $v \in cone(v_1, \ldots, v_\ell)$, then v can be written as the conic combination of at most n vectors from $v_1, \ldots, v_\ell$.*

Proof. Because $v \in cone(v_1, \ldots, v_\ell)$, by re-indexing the v_i's, there exist $s \leq \ell$ and $\lambda_1, \ldots, \lambda_s > 0$ such that $v = \sum_{i\in[s]} \lambda_i v_i$. If $s \leq n$, then we are done. Otherwise, $s > n$, so the set $\{v_1, \ldots, v_s\} \subset \mathbb{R}^n$ is not linearly independent.

Then there exist $\mu_i \in \mathbb{R}$ for all $i \in [s]$ such that $v_s = \sum_{i\in[s-1]} \mu_i v_i$. Because $\lambda_i > 0$ for all $i \in [s]$, there exist some $\epsilon > 0$ such that in the representation:

$$v = \sum_{i\in[s]} \lambda_i v_i$$
$$= \sum_{i\in[s-1]} \lambda_i v_i + (\lambda_s - \epsilon) v_s + \epsilon \sum_{i\in[s-1]} \mu_i v_i$$
$$= \sum_{i\in[s-1]} (\lambda_i - \epsilon \mu_i) v_i + (\lambda_s - \epsilon) v_s,$$

every weight is still non-negative. Further, we can choose $\epsilon > 0$ such that at least one weight becomes zero. Thus, we have found another representation of v as a conic combination of $v_1, \ldots, v_\ell$ with strictly smaller support. Repeat this procedure until $s \leq n$. □

Now consider a Matroid Cup Game on n cups with matroid $\mathcal{M}$. Let $\mathcal{B}$ be the collection of bases of $\mathcal{M}$. We will conflate a basis with its indicator vector in $\mathbb{R}^n$, so $cone(\mathcal{B})$ is the set of all non-negative combinations of indicators of bases. Using the above lemma, our new Emptier strategy is the following (MAX COEFFICIENT.) In words, the Emptier maintains a representation of the current fill vector as a non-negative linear combination of bases. Thus, each basis has a non-negative coefficient, λ_B. In each Emptier turn, we remove water from the max coefficient basis as long as it has coefficient at least one.

Algorithm 1: MAX COEFFICIENT.

Assume $f \in cone(\mathcal{B})$.
Compute a representation $f = \sum_{B\in\mathcal{B}} \lambda_B 1(B)$ with $\lambda_B \geq 0$ for all B and $\lambda_B > 0$ for at most n bases (guaranteed by Lemma 3.1.)
If $\max_{B\in\mathcal{B}} \lambda_B \geq 1$, then choose the basis B' achieving this maximum to empty.
Otherwise, do nothing.

We note that the Max Coefficient Emptier is not equivalent to the Greedy Emptier, because the former deliberately idles some turns, and even if the Max Coefficient Emptier does choose to empty some cups, then its choice does not agree with Greedy in general.

3.2. Analysis of Max Coefficient Strategy

The purpose of idling turns is to guarantee that $f \in cone(\mathcal{B})$ holds throughout the game.

Lemma 3.2. *Consider a Matroid Cup Game, where the Filler always adds a convex combination of bases, and the Emptier uses the Max Coefficient Strategy. Then $f \in cone(\mathcal{B})$ at the beginning of each turn.*

Proof. We prove the lemma by induction on each turn. Initially, $f = 0$, so clearly $f \in cone(\mathcal{B})$. Because the Filler always adds a convex combination of bases, it is clear that every Filler turn maintains $f \in cone(\mathcal{B})$. For an Emptier turn, if the Emptier does nothing, then clearly the invariant is maintained.

Finally, suppose the Emptier chooses to remove water from basis B'. Then we may assume inductively that $f \in cone(\mathcal{B})$, so $f = \sum_{B\in\mathcal{B}} \lambda_B 1(B)$ with $\lambda_B \geq 0$ for all $B \in \mathcal{B}$, where $1(B)$ is the indicator vector of basis B. By definition of the Max Coefficient Emptier, we have $\lambda_{B'} \geq 1$. It follows, every cup in B' has at least one unit of water, and removing water from B' corresponds to decreasing its coefficient by one, so the invariant is maintained. □

Thus, as long as the Filler always adds a convex combination of bases, the Max Coefficient Strategy is well-defined. Now, we use the properties of the representation of $f \in cone(\mathcal{B})$ to show the main result of this section.

Theorem 3.3. *Consider a Matroid Cup Game, where the Filler always adds a convex combination of bases. Then the Max Coefficient Strategy maintains total water $O(rn)$.*

Proof. By Lemma 3.2, the Max Coefficient Strategy is well-defined. By assumption, in each round of the game, the Filler adds r units of water. Now suppose $tot(f) \geq rn$ at the beginning of the Emptier's turn. It follows, $tot(f) = \sum_{B\in\mathcal{B}} \lambda_B r \Rightarrow \sum_{B\in\mathcal{B}} \lambda_B \geq n$. Note that we use the fact that every basis has size r.

Here, we use the representation guaranteed by Lemma 3.1. Note that at most n of the coefficients λ_B are non-zero, so by averaging, the max coefficient basis, say B', satisfies $\lambda_{B'} \geq 1$. By definition of the Max Coefficient Emptier, in this Emptier turn, we remove one unit from each cup of B' (so r units overall.) To summarize, if $tot(f) \geq rn$ at the beginning of the Emptier's turn, then in this round the total water does not increase. We conclude, the only way the total water can increase is if $tot(f) < rn$. □

Then to prove Theorem 1.6, we need to remove the assumption that the Filler always adds a convex combination of bases. To do this, the Emptier can always extend the Filler's action to be a convex combination of bases, and proceed as if the Filler had actually added that extra water.

4. Bounding the max fill via bounding the total water

The goal of this section is to prove Theorem 1.7, which allows us to reduce the task of bounding the max fill to bounding the total water. The structure of our proof is induction on the number of non-empty cups n'. Suppose we can upper bound the total water by $\alpha n'$. Then, there are always at most $n'/2$ cups with at least 2α units of water by averaging. The key idea of our reduction is to show that the cups with at least 2α units of water behave like a cup game with $n'/2$ non-empty cups. To formalize this idea, we define an *instance* of the Matroid Cup Game and a *truncated instance*.

Definition 4.1 *(Instance of Matroid Cup Game).* An instance of the Matroid Cup Game is given by a sequence of Filler actions, $x_1, x_2, \ldots, x_t$ and Emptier actions, $y_1, y_2, \ldots, y_t$ for all rounds t of the game such that x_t is a point in the independent set polytope and y_t is an integral point in the independent set polytope (encoding the cups the Emptier chooses in this round.)

Definition 4.2 *(Truncated instance of Matroid Cup Game).* Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ define an instance of the Matroid Cup Game. Then for any $h \geq 0$, the h-truncated cup game is defined by the Filler actions $x'_1, x'_2, \ldots$ and the same Emptier actions as in the original game.

The x'_t-vectors are defined as follows. Let f_t be the fill vector at the beginning of round t in the original game. Then for all cups $i \in [n]$, we define:

$$x'_t(i) = \begin{cases} x_t(i) & , f_t(i) \geq h \\ \max(f_t(i) + x_t(i) - h, 0) & , f_t(i) < h \end{cases}.$$

It is trivial to verify that the x'_t-vectors are feasible actions for the Filler as long as the x_t-vectors are also feasible, so any h-truncated instance is also an instance of the Matroid Cup Game. The h-truncated cup game is defined to keep track of water above height h in the original game.

Lemma 4.3. *Let G be an instance of the Matroid Cup Game and G' the h-truncated instance of G. Then for all turns and all cups i, we have $f'(i) = \max(f(i) - h, 0)$, where f and f' are the fill vectors for G and G' at this turn, respectively.*

Proof. The proof is by induction on the turns of the Filler/Emptier in G and G'. In the base case, f and f' are both zero at the beginning of G and G', so the lemma holds.

Now consider an arbitrary Filler turn and let f and f' denote the fill vectors at the beginning of this turn. We may assume inductively that $f'(i) = \max(f(i) - h, 0)$ for all i. In this turn, say the Filler in G adds x to f and the Filler in G' adds x' to f'. For any cup i, if $f(i) \geq h$, then $x'(i) = x(i)$, so $f'(i) + x'(i) = f(i) + x(i) - h$, as required. Otherwise $f(i) < h$, so $f'(i) + x'(i) = 0 + \max(f(i) + x(i) - h, 0)$, as required.

It remains to consider an arbitrary Emptier turn. Because the Emptier in G' copies the Emptier in G, it is easy to see that every Emptier turn maintains the inductive hypothesis. □

Crucially, the truncated game also preserves the Greedy Strategy, so the truncated game is indeed another instance of the Matroid Cup Game with a Greedy Emptier. This allows us to induct on truncated games.

Lemma 4.4. *Let G be an instance of the Matroid Cup Game and G' the h-truncated instance of G. If the Emptier is Greedy in G, then it is also Greedy in G'.*

Proof. By Lemma 4.3, at every Emptier turn the fill vector of G' preserves the ordering of the cups (from most full to least full) in the fill vector of G. It follows by definition of the Greedy Strategy in G that the same choice of cups by the Emptier corresponds to a valid run of Greedy in G'. □

Composing the above two lemmas gives our proof of Theorem 1.7.

Theorem 4.5 *(Theorem 1.7, restated). Suppose the Greedy Strategy maintains total water $\alpha n'$ for the Matroid Cup Game, where n' is the maximum number of non-empty cups over every turn of the game. Then the Greedy Strategy maintains max fill $2\alpha \log_2 n'$ for all $n' \geq 2$.*

Proof. We prove the theorem by induction on $n' \geq 2$. Without loss of generality, we may assume n' is a power of two. In the base case, $n' = 2$, so the max fill is trivially upper bounded by the total water, which is $2\alpha = 2\alpha \log_2 n'$.

Now we consider $n' > 2$, so let G be an instance of the Matroid Cup Game with at most n' non-empty cups such that the Emptier is Greedy. Then by assumption, the total water is at most $\alpha n'$. It follows, there are always at most $n'/2$ cups of G with at least 2α units of water. Thus, the 2α-truncated game of G, say G', has at most $n'/2$ non-empty cups. Lemma 4.4 gives that the Emptier in G' is also Greedy. Applying the inductive hypothesis to G', the max fill in G' is at most $2\alpha \log_2 n'/2$.

Finally, we relate the max fill of G to the max fill of G'. By Lemma 4.3, the max fill of G is at most the max fill of G' plus 2α, so the max fill of G is at most $2\alpha + 2\alpha \log_2 n'/2 = 2\alpha \log_2 n'$. □

4.1. Applications of Theorem 1.7

Now we use Theorem 1.7 to give simple analyses of Greedy for the One- and p-Cup Games. It will be convenient to consider a slightly modified Matroid Cup Game instead of the Matroid Cup Game with at most n' non-empty cups. We recall that the *restriction* of matroid $\mathcal{M}$ to subset of the ground set S is the matroid on ground set S, where $T \subset S$ is independent in the restriction if and only if T is independent in $\mathcal{M}$.

Definition 4.6 *(Dynamic Matroid Cup Game).* We are given a matroid $\mathcal{M}$ on ground set $[N]$. There are $n \leq N$ cups, initially empty. Let $C \subset [N]$ index the cups. In each round of the game, first for each empty cup $i \in C$, the Filler can update C by replacing i with any $j \in [N] \setminus C$. Then the round proceeds as in the standard Matroid Cup Game, but on the restriction of $\mathcal{M}$ on C.

Note that by definition of the Dynamic Matroid Cup Game, C always contains all non-empty cups.

Lemma 4.7. *An instance of the Matroid Cup Game with at most n' non-empty cups is equivalent to an instance of the Dynamic Matroid Cup Game on n' cups.*

Proof. Let G denote an instance of the Matroid Cup Game. We construct an instance G' of the Dynamic Matroid Cup Game on the same matroid and ground set.

Consider any round of G. Throughout both turns of this round, at most n' cups will be non-empty. In particular, the Filler/Emptier will only add/remove from a subset of at most n' cups. Thus, in the corresponding round of G', at the beginning we can update C to contain this subset of at most n' cups. After this update, we can copy the Filler and Emptier actions of G in G'. Thus, the fill vectors of G and G' agree at every turn. □

The previous lemma implies that for the p-Cup Game (i.e. the Matroid Cup Game on a uniform matroid), an instance with at most n' non-empty cups is equivalent to an instance of the Dynamic p-Cup Game on n' cups. Further, note that an instance of the Dynamic p-Cup Game on n' cups is equivalent to an instance of the *standard* p-Cup Game on n' cups. Thus, we have the following convenient corollary of Theorem 1.7 for the p-Cup Game.

Corollary 4.8. *Suppose the Greedy Strategy maintains total water αn for the p-Cup Game on n cups for all $n \geq 2$. Then the Greedy Strategy maintains max fill $2\alpha \log_2 n$ for all $n \geq 2$.*

One Cup Game. Using the corollary, we give a simple proof that Greedy maintains max fill $O(\log n)$ in the One Cup Game, which is the best possible upper bound. We do so by bounding the total water.

Lemma 4.9. *The Greedy Strategy maintains total water $O(n)$ for the One Cup Game on n cups for all n.*

Proof. Suppose $tot(f) \geq n$ at the beginning of the Emptier's turn. Then there exists some cup with at least one unit. By definition of Greedy, the Emptier will choose such a cup in this round, and thus will reduce the total water by one. The Filler can increase the total water by at most one in each round, so in this round, the total water cannot increase. We conclude, the only way the total water can increase is if $tot(f) < n$. □

Thus, composing Corollary 4.8 and Lemma 4.9, we obtain:

Corollary 4.10. *The Greedy Strategy maintains max fill $O(\log n)$ for the One Cup Game on n cups for all $n \geq 2$.*

***p*-Cup Game.** For the p-Cup Game, the analysis of total water is slightly more involved. Our proof strategy is to show that if the total water is at least $O(pn)$ at the beginning of the Emptier's turn, then the Emptier can always find p cups to remove one unit of water from. The main concern is if all of the water is concentrated in less than p cups. The next lemma suggests that this is not the case under the Greedy Strategy.

Lemma 4.11. *Suppose the Emptier follows the Greedy strategy in the p-Cup Game. If in some round, the max fill increases from less than k at the beginning to at least k at the end, then the beginning of the round must have total water at least $(p+1)(k-1)$.*

Proof. Consider such a round, $f \to \bar{f} \to f'$, where f is the fill vector at the beginning of the round, $\bar{f}$ is after the Filler's turn, and f' is the end of the round. Thus, we assume $max(f) < k$ and $max(f') \geq k$.

By definition of Greedy, we claim that there must be at least $p+1$ cups with at least k units each in $\bar{f}$. Assume for contradiction that this is not the case, so there are at most p cups with at least k units each in $\bar{f}$. Then in $\bar{f} \to f'$, the Emptier will remove water from each of these cups. Note that $max(\bar{f}) \leq max(f) + 1 < k+1$, so after removing water from each of these cups, they have less than

k units. Thus, these cups cannot achieve the max fill in f', which is at least k, so there must exist another cup with at least k units in $\bar{f}$. This is a contradiction.

It follows that there must be at least $p+1$ cups with at least $k-1$ units in f, because the Filler can add at most one unit to each cup in $f \to \bar{f}$. This immediately implies $tot(f) \geq (p+1)(k-1)$ total water. □

Using the above lemma, we are ready to bound the total water in the p-Cup Game. The main idea of the proof is that if the total water increases in some Emptier turn, then the Emptier could not find p cups to remove one unit from. This suggests that the total water is concentrated in the top p cups. If the total water is very large, then we can use Lemma 4.11 to derive a contradiction.

Lemma 4.12. *The Greedy Strategy maintains total water $O(pn)$ for the p-Cup Game on n cups for all n.*

Proof. We may assume $p \leq n$ or else the lemma is trivial. Assume for contradiction that there exists some round where the total water increases to $(p+1)(k-1)$ or more, where we define $k = n + p + 2 = O(n)$. We consider the first round where the total water increases to this quantity. Thus, we consider the first such round $f \to \bar{f} \to f'$, where f is the beginning, $\bar{f}$ is after the Filler's turn, and f' is the end such that $tot(f) < (p+1)(k-1)$ and $tot(f') \geq (p+1)(k-1)$.

Because the total water increases from f to f', in this round the Emptier must have removed less than p units of water from $\bar{f}$ to f'. By definition of Greedy, this implies that every cup outside the top p in f' has at most one unit of water. Thus, the top p cups of f' have total water at least $(p+1)(k-1) - n$. By averaging, $max(f') \geq \frac{(p+1)(k-1)-n}{p}$.

Recall that we consider the first such round where the total water increases beyond $(p+1)(k-1)$, so the contrapositive of Lemma 4.11 implies that $max(f) \geq k$ or $max(f') < k$. However, we claim that it cannot be the case that $max(f) \geq k$.

To see this, assume for contradiction that $max(f) \geq k$, so there must exist a round before $f \to \bar{f} \to f'$ where the max fill increased from less than k to at least k. By Lemma 4.11, at the beginning of this earlier round we must have total water at least $(p+1)(k-1)$. This contradicts our assumption that $f \to \bar{f} \to f'$ is the first round where the total water increases to at least $(p+1)(k-1)$.

The previous paragraph implies $max(f') < k$. Combining our upper and lower bounds on the max fill of f' gives:

$$\frac{(p+1)(k-1)-n}{p} \leq k.$$

Solving for k gives $k \leq n + p + 1$, which contradicts our choice of k. In conclusion, the total water never increases to $(p+1)(k-1) = O(pn)$. □

Again, composing Corollary 4.8 and Lemma 4.12, we can bound the max fill of the p-Cup Game.

Corollary 4.13. *The Greedy Strategy maintains max fill $O(p \log n)$ for the p-Cup Game on n cups for all $n \geq 2$.*

Matroid Cup Game. We comment that Theorem 1.7 requires that we upper bound the total water achieved by the Greedy Emptier, so in particular, the fact that Max Coefficient maintains total water $O(rn)$ does not imply that it also maintains max fill $O(r \log n)$.

5. Discussion

In this paper, we developed novel tools towards analyzing more general cup games. In light of our new Emptier Strategy for the Matroid Cup Game and our reduction from bounding the max fill to bounding the total water, the most natural open question is if the Greedy Strategy also maintains total water $O(rn)$ in the Matroid Cup Game. Intuitively, Greedy is the “best possible” Emptier strategy, so if another Emptier Strategy maintains total water $O(rn)$, then Greedy should as well. If we could obtain the refined bound of $O(rn')$ on the total water of Greedy, where n' is the max number of non-empty cups, then this would immediately imply a $O(r \log n)$ upper bound on the max fill for the Matroid Cup Game. This would be the first upper bound on the max fill in the Matroid Cup Game without resource augmentation.

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