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On the Classification of Normal Stein Spaces and Finite Ball Quotients With Bergman–Einstein Metrics

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We study the Bergman metric of a finite ball quotient $\mathbb{B}^n/\Gamma$, where $n \geq 2$ and $\Gamma \subseteq \mathrm{Aut}(\mathbb{B}^n)$ is a finite, fixed point free, abelian group. We prove that this metric is Kähler–Einstein if and only if Γ is trivial, that is, when the ball quotient $\mathbb{B}^n/\Gamma$ is the unit ball $\mathbb{B}^n$ itself. As a consequence, we characterize the unit ball among normal Stein spaces with isolated singularities and abelian fundamental groups in terms of the existence of a Bergman–Einstein metric.

1 Introduction

Since the introduction of the Bergman kernel in [3, 4] and the subsequent groundbreaking work by Kobayashi [20] and Fefferman [11], the study of the Bergman kernel and metric has been a central subject in several complex variables and complex geometry. A general problem of fundamental importance seeks to characterize complex analytic spaces in terms of geometric properties of their Bergman metrics. The Bergman kernel of the unit ball $\mathbb{B}^n \subseteq \mathbb{C}^n$, for example, is explicitly known,

$$K_{\mathbb{B}^n}(z,\bar{w}) = \frac{n!}{\pi^n}\frac{1}{(1-\langle z,\bar{w}\rangle)^{n+1}}, \quad \langle z,\bar{w}\rangle = \sum_{j=1}^{n} z_j\bar{w}_j,$$

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and it is routine to verify that the Bergman metric,

$$(g_{\mathbb{B}^n})_{i\bar{j}} = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K_{\mathbb{B}^n}(z, \bar{z}),$$

is *Kähler–Einstein*, that is, has Ricci curvature equal to a constant multiple of the metric tensor; indeed, the Bergman metric of the unit ball has constant holomorphic sectional curvature, which implies the Kähler–Einstein property. A well-known conjecture posed by S.-Y. Cheng [7] in 1979 asserts that the Bergman metric of a bounded, strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary is Kähler–Einstein if and only if the domain is biholomorphic to the unit ball $\mathbb{B}^n$. There are also variations of this conjecture in terms of other canonical metrics; see, for example, Li [22–24] and the references therein.

The aforementioned Cheng conjecture was confirmed by S. Fu–B. Wong [14] and S. Nemirovski–R. Shafikov [26] in the 2D case and by X. Huang and the 2nd author [18] in higher dimensions. X. Huang and X. Li [16] recently generalized this result to Stein manifolds with strongly pseudoconvex boundary as follows: *The only Stein manifold with smooth and compact strongly pseudoconvex boundary for which the Bergman metric is Kähler–Einstein is the unit ball $\mathbb{B}^n$ (up to biholomorphism)*. These results lead naturally to the question of whether a similar characterization of $\mathbb{B}^n$ holds in the setting of normal Stein spaces *with possible singularities*; see Conjecture 1.4 in [17]. In this paper, we provide strong evidence that this is the case. The following two theorems establish the 1st results that the authors are aware of characterizing the unit ball among normal Stein spaces with possible singularities in terms of the existence of a Bergman–Einstein metric.

Theorem 1.1. Let V be an n-dimensional Stein space in $\mathbb{C}^N$ with $N > n \geq 2$, and $G = V \cap \mathbb{B}^N$. Assume that every point in $\overline{G}$ is a smooth point of V, except for finitely many normal singularities in G, and that G has a smooth boundary. Then the Bergman metric of G is Kähler–Einstein if and only if G is biholomorphic to $\mathbb{B}^n$.

Theorem 1.2. Let V be an n-dimensional Stein space in $\mathbb{C}^N$ with $N > n \geq 2$ and $\Omega \subseteq \mathbb{C}^N$ a bounded strongly pseudoconvex domain with smooth and real-algebraic boundary. Write $G = V \cap \Omega$. Assume every point in $\overline{G}$ is a smooth point of V, except for finitely many normal singularities in G, and that G has a smooth boundary. Then the following are equivalent:

(i) G is biholomorphic to $\mathbb{B}^n$.

(ii) The fundamental group of the regular part of G is abelian and the Bergman metric of G is Kähler–Einstein.

Remark 1.3. As we will see in the proof (Section 3), if G itself is assumed to be bounded in Theorem 1.2, then the boundedness assumption on Ω can be dropped.

We shall utilize the work by Lichtblau [25] (see also F. Forstnerič [13] and D'Angelo–Lichtblau [9]) and X. Huang [15], as well as methods from [18], [16], and [10] to reduce the proofs of Theorems 1.1 and 1.2 to that of the following theorem, which is one of the main results in the paper.

Theorem 1.4. Let Γ be a finite abelian subgroup of $\mathrm{Aut}(\mathbb{B}^n)$, $n \geq 2$, and assume Γ is fixed point free. Then the Bergman metric of $\mathbb{B}^n/\Gamma$ is Kähler–Einstein if and only if Γ is the trivial group.

Here a subgroup Γ of $\mathrm{Aut}(\mathbb{B}^n)$ is called *fixed point free* if the only element $\gamma \in \Gamma$ with a fixed point on $\partial\mathbb{B}^n$ is the identity. The fixed point free condition on Γ guarantees that the quotient space $\mathbb{B}^n/\Gamma$ has smooth boundary (see [13]). Moreover, as we shall see in Section 4, an abelian fixed point free finite group Γ is in fact cyclic.

To prove Theorem 1.4, it suffices to show that if Γ is not the trivial group, that is, $\Gamma \neq \{\mathrm{id}\}$, then the Bergman metric is not Kähler–Einstein. For that, we shall use the transformation formula for the Bergman kernel under branched holomorphic coverings of complex analytic spaces; see Theorem 2.3 below. A crucial step in the proof is to reduce the non-Einstein condition to several combinatorial inequalities. The proofs of these combinatorial inequalities are technical and will be given in a separate section; see Section 5.

We remark that the analogue of Theorem 1.4 is not true in the case $n = 1$. If we denote the unit disk in $\mathbb{C}$ by $\mathbb{D}$ $(= \mathbb{B}^1)$, then one readily verifies that any finite subgroup $\Gamma \subseteq \mathrm{Aut}(\mathbb{D})$ must be fixed point free and cyclic. Nevertheless, in this case, X. Huang and X. Li [16] proved the very interesting result that the Bergman metric of $\mathbb{D}/\Gamma$ always has constant Gaussian curvature, which is equivalent to being Kähler–Einstein in the one-dimensional case.

The paper is organized as follows. Section 2 recalls some preliminaries on the Bergman metric and finite ball quotients. In Section 3, we prove that Theorems 1.1 and 1.2 follow from Theorem 1.4. Theorem 1.4 is then proved in Section 4, except for some combinatorial lemmas used in the proof that are left to Section 5.

2 Preliminaries

2.1 The Bergman kernel

In this subsection, we will briefly review some properties of the Bergman kernel and metric on a complex manifold. More details can be found in [20] and [21].

Let M be an n-dimensional complex manifold. Let $L^2_{(n,0)}(M)$ denote the space of L^2-integrable $(n,0)$-forms on M, equipped with the inner product

$$(\varphi,\psi)_{L^2(M)} := i^{n^2}\int_M \varphi\wedge\overline{\psi}, \quad \varphi,\psi\in L^2_{(n,0)}(M). \tag{2.1}$$

Define the *Bergman space* of M to be

$$A^2_{(n,0)}(M) := \left\{\varphi\in L^2_{(n,0)}(M) : \varphi \text{ is a holomorphic } (n,0)\text{-form on } M\right\}. \tag{2.2}$$

Assume $A^2_{(n,0)}(M)\neq\{0\}$. Then $A^2_{(n,0)}(M)$ is a separable Hilbert space. Taking any orthonormal basis $\{\varphi_k\}_{k=1}^q$ of $A^2_{(n,0)}(M)$ with $1\leq q\leq\infty$, we define the *Bergman kernel (form)* of M to be

$$K_M(x,\bar{y}) = i^{n^2}\sum_{k=1}^q \varphi_k(x)\wedge\overline{\varphi_k(y)}.$$

Then, $K_M(x,\bar{x})$ is a real-valued, real-analytic form of degree (n,n) on M and is independent of the choice of orthonormal basis.

The Bergman kernel form remains unchanged if we remove a proper complex analytic subvariety, as the following theorem from [20] shows:

Theorem 2.1 ([20]). If M' is a domain in an n-dimensional complex manifold M and if $M-M'$ is a complex analytic subvariety of M of complex dimension $\leq n-1$, then

$$K_M(x,\bar{y}) = K_{M'}(x,\bar{y}) \quad \text{for any } x,y\in M'.$$

This theorem suggests the following generalization of the Bergman kernel form to complex analytic spaces.

Definition 2.2. Let M be a reduced complex analytic space, and let $V \subseteq M$ denote its set of singular points. The Bergman kernel form of M is defined as

$$K_M(x,\bar{y}) = K_{M-V}(x,\bar{y}) \quad \text{for any } x,y \in M-V,$$

where K_{M-V} denotes the Bergman kernel form of the complex manifold consisting of regular points of M.

Let N_1, N_2 be two complex manifolds of dimension n. Let $\gamma : N_1 \to M$ and $\tau : N_2 \to M$ be holomorphic maps. The pullback of the Bergman kernel $K_M(x,\bar{y})$ of M to $N_1 \times N_2$ is defined in the standard way. That is, for any $z \in N_1, w \in N_2$,

$$\big((\gamma,\tau)^* K_M\big)(z,\bar{w}) = \sum_{k=1}^{q} \gamma^*\varphi_k(z) \wedge \overline{\tau^*\varphi_k(w)}.$$

In terms of local coordinates, we may write the Bergman kernel form of M as

$$K_M(x,\bar{y}) = \widetilde{K}_M(x,\bar{y}) dx_1 \wedge \cdots dx_n \wedge d\overline{y_1} \wedge \cdots \wedge d\overline{y_n}, \tag{2.3}$$

where the function $\widetilde{K}_M(x,\bar{y})$ depends on the choice of local coordinates. We then have

$$\big((\gamma,\tau)^* K_M\big)(z,\bar{w}) = \widetilde{K}_M(\gamma(z),\overline{\tau(w)})\, J_\gamma(z)\, \overline{J_\tau(w)}\, dz_1 \wedge \cdots dz_n \wedge d\overline{w_1} \wedge \cdots \wedge d\overline{w_n}, \tag{2.4}$$

where J_γ and J_τ are the Jacobian determinants of the maps γ and τ, respectively. In particular, we observe that the kernel function $\widetilde{K}_M(x,\bar{y})$ transforms according to the usual biholomorphic invariance formula under changes of local coordinates.

Let M be as in Definition 2.2. Assume $K_M(x,\bar{x})$ is nowhere vanishing (on the set of regular points of M, where it is defined). We define a Hermitian $(1,1)$-form on the regular part of M by

$$\omega_M := i\,\partial\bar{\partial} \log K_M(x,\bar{x}) := i\,\partial\bar{\partial} \log \widetilde{K}_M(x,\bar{x}). \tag{2.5}$$

Here $\widetilde{K}_M$ is as in (2.3). It is easy to verify that this form is independent of the choice of local coordinates used to determine the function $\widetilde{K}_M(x,\bar{x})$. The Bergman metric on M is the metric induced by ω_M (when it indeed induces a positive definite metric on the regular part of M).

We recall the Bergman kernel transformation formula in [10] for (possibly branched) covering maps of complex analytic spaces. This formula generalizes a classical theorem of Bell ([1], [2]; see also [6]):

Theorem 2.3. Let M_1 and M_2 be two complex analytic sets. Let $V_1 \subseteq M_1$ and $V_2 \subseteq M_2$ be proper analytic subvarieties such that $M_1 - V_1, M_2 - V_2$ are complex manifolds of the same dimension. Assume that $f : M_1 - V_1 \to M_2 - V_2$ is a finite ($m-$sheeted) holomorphic covering map. Let Γ be the deck transformation group for the covering map (with $|\Gamma| = m$), and denote by K_i the Bergman kernels of M_i for $i = 1, 2$. Then the Bergman kernel forms transform according to

$$\sum_{\gamma \in \Gamma} (\mathrm{id}, \gamma)^* K_1 = (f, f)^* K_2 \quad \text{on } (M_1 - V_1) \times (M_1 - V_1),$$

where $\mathrm{id} : M_1 \to M_1$ is the identity map.

2.2 Finite ball quotients

In this subsection, we recall the canonical realization of a finite ball quotient due to H. Cartan [5]. Let $\mathbb{B}^n$ denote the unit ball in $\mathbb{C}^n$ and $\mathrm{Aut}(\mathbb{B}^n)$ its (biholomorphic) automorphism group. Let Γ be a finite subgroup of $\mathrm{Aut}(\mathbb{B}^n)$. Assume Γ is fixed point free; that is, assume no $\gamma \in \Gamma - \{\mathrm{id}\}$ has any fixed points on $\partial \mathbb{B}^n$. As the unitary group $U(n)$ is a maximal compact subgroup of $\mathrm{Aut}(\mathbb{B}^n)$, by basic Lie group theory, there exists some $\psi \in \mathrm{Aut}(\mathbb{B}^n)$ such that $\Gamma \subseteq \psi^{-1} \cdot U(n) \cdot \psi$. Thus without loss of generality, we can assume $\Gamma \subseteq U(n)$, that is, Γ is a finite unitary subgroup. The origin $0 \in \mathbb{C}^n$ is then always a fixed point of every element in Γ. Moreover, the fixed point free condition on Γ is equivalent to the assertion that every $\gamma \in \Gamma - \{\mathrm{id}\}$ has no other fixed point than 0. We also note that, by the fixed point free condition, the action of Γ on $\partial \mathbb{B}^n$ is properly discontinuous and $\partial \mathbb{B}^n / \Gamma$ is a smooth manifold. We remark that such groups Γ are well understood. The classification of the fixed point free finite unitary groups appears in the book [28] as part of the solution to the problem of finding all spaces of constant curvature.

By a theorem of H. Cartan [5], the quotient $\mathbb{C}^n / \Gamma$ can be realized as a normal algebraic subvariety V in some $\mathbb{C}^N$. Let $\mathcal{A}$ denote the algebra of Γ-invariant holomorphic polynomials, that is,

$$\mathcal{A} := \left\{ p \in \mathbb{C}[z_1, \cdots, z_n] : p \circ \gamma = p \text{ for all } \gamma \in \Gamma \right\}.$$

By Hilbert's basis theorem, $\mathcal{A}$ is finitely generated. Moreover, we can find a minimal set of homogeneous polynomials $\{p_1, \cdots, p_N\} \subseteq \mathcal{A}$ such that every $p \in \mathcal{A}$ can be expressed in the form

$$p(z) = q(p_1(z), \cdots, p_N(z)),$$

where q is some holomorphic polynomial in $\mathbb{C}^N$. The map $Q := (p_1, \cdots, p_N) : \mathbb{C}^n \to \mathbb{C}^N$ is proper and induces a homeomorphism of $\mathbb{C}^n/\Gamma$ onto $V := Q(\mathbb{C}^n)$. As Q is a proper holomorphic polynomial map, V is an algebraic variety. The restriction of Q to the unit ball $\mathbb{B}^n$ maps $\mathbb{B}^n$ properly onto a relatively compact domain $\Omega \subseteq V$. In this way, $\mathbb{B}^n/\Gamma$ is realized as Ω. Following [27], we call such Q the *basic map* associated to Γ. The ball quotient $\Omega = \mathbb{B}^n/\Gamma$ is nonsingular if and only if the group Γ is generated by *reflections*, that is, elements of finite order in $U(n)$ that fix a complex subspace of dimension $n-1$ in $\mathbb{C}^n$ (see [27]); thus, if Γ is fixed point free and nontrivial, then $\Omega = \mathbb{B}^n/\Gamma$ must have singularities. Moreover, Ω has smooth boundary if and only if Γ is fixed point free (see [13] for more results along this line).

3 Proof of Theorems 1.1 and 1.2

In this section, we prove that Theorems 1.1 and 1.2 follow from Theorem 1.4; see Section 3.2 and 3.1, respectively.

3.1 Proof of Theorem 1.2

The implication (i) $\Longrightarrow$ (ii) in Theorem 1.2 is trivial. We therefore only need to prove the converse. Let G be as in Theorem 1.2 and assume the conditions in (ii) hold. To prove (i), assuming that Theorem 1.4 has been proved, we proceed in three steps.

Step 1. It follows from the assumption that the boundary ∂G is strongly pseudoconvex. We first prove that the boundary ∂G is indeed spherical. Recall that a CR hypersurface M of dimension $2n-1$ is said to be spherical if it is locally CR diffeomorphic, near every point, to an open piece of the unit sphere $S^{2n-1} \subseteq \mathbb{C}^n$. The proof uses the Kähler–Einstein assumption, the localization of the Bergman kernel, Fefferman's expansion of the Bergman kernel, as well as CR invariant theory. Since the detailed proof for this step is contained in [16] (see Theorem 1.1 in [16]). We will omit the proof here.

Step 2. In this step, we prove that G is biholomorphic to a ball quotient $\mathbb{B}^n/\Gamma$ for some finite fixed point free subgroup $\Gamma \subseteq U(n)$. Since we know ∂G is spherical from Step 1 and ∂G is contained in a real algebraic hypersurface in $\mathbb{C}^N$, it follows from Corollary 3.3 in [15] that ∂G is CR equivalent to CR spherical space form S^{2n-1}/Γ where $\Gamma \subseteq U(n)$ is as above. More precisely, there is an algebraic CR map $F : S^{2n-1} \to \partial G$, which is a finite covering map. From this one can further prove that G is biholomorphic to $\mathbb{B}^n/\Gamma$. The proof of this is identical with Step 3 in Section 5 of [10]. The general setting of [10] is in dimension $n = 2$, but, as pointed out in Remark 5.4 in [10], this argument works for all dimensions. The argument shows that F extends to a proper, holomorphic branched covering map from $\mathbb{B}^n$ onto G, which realizes G as the ball quotient $\mathbb{B}^n/\Gamma$. In particular, G is biholomorphic to $\mathbb{B}^n/\Gamma$ as claimed. Since $\Gamma \subseteq U(n)$ is fixed point free, either G has one unique singular point at $F(0)$ when $\Gamma \neq \{\mathrm{id}\}$ or G is smooth when $\Gamma = \{\mathrm{id}\}$. In the former case, $F : \mathbb{B}^n - \{0\} \to G - \{F(0)\}$ is a smooth covering map whose group of deck transformations is Γ, and in the latter case, F extends as a biholomorphism $\mathbb{B}^n \to G$.

Step 3. By the conclusion in Step 2, the fundamental group of the regular part of G is isomorphic to Γ. By assumption in (ii), Γ is abelian. Moreover, the biholomorphism between G and $\mathbb{B}^n/\Gamma$ gives an isometry between the Bergman metrics of G and $\mathbb{B}^n/\Gamma$. By assumption in (ii) again, the Bergman metric of $\mathbb{B}^n/\Gamma$ is Kähler–Einstein. Thus, by Theorem 1.4, Γ must then be the trivial group $\{\mathrm{id}\}$. Hence G is biholomorphic to $\mathbb{B}^n$.

3.2 Proof of Theorem 1.1

We now prove Theorem 1.1, under the assumption that Theorems 1.4 and 1.2 have been proved. The "if" implication is trivial, and we only need to prove the converse. Thus, we assume that G is as in Theorem 1.1 and the Bergman metric of G is Kähler–Einstein, and we shall prove G is biholomorphic to $\mathbb{B}^n$. By copying the argument in Step 1 and Step 2 in Section 3.1, we conclude that there is an algebraic CR map F from S^{2n-1} to $\partial G \subseteq \partial \mathbb{B}^N = S^{2N-1}$, which is a finite covering map. In particular, the map F induces a smooth, nonconstant CR map from the spherical space form S^{2n-1}/Γ, for some finite fixed point free subgroup $\Gamma \subseteq \mathrm{Aut}(\mathbb{B}^n)$, to S^{2N-1} (see [25], [9], and [8]). Since Γ is a finite subgroup of $\mathrm{Aut}(\mathbb{B}^n)$, by basic Lie group theory as above, Γ is contained in a conjugate of the unitary group $U(n)$. Then it follows from [25] that the finite subgroup Γ is abelian; and thus so is the fundamental group of the regular part of G (Indeed, by Theorem 8 in [9], Γ is conjugate to one from a short list of special cyclic subgroups of $U(n)$). Now, Theorem 1.1 follows from Theorem 1.2.

Remark 3.1. Theorem 1.1 can be also proved without going through the full depth of Theorems 1.2 and 1.4. By [9], there are only three cases in the short list of special cyclic subgroups of $U(n)$ that need to be considered. One of these is generated by a multiple of the identity and the proof in this case is much simpler. The other two cases have either only two or three distinct eigenvalues, and the proof also simplifies, although to a lesser extent.

4 Proof of Theorem 1.4

In this section, we shall prove Theorem 1.4. It suffices to prove the Bergman metric of $\mathbb{B}^n/\Gamma$ cannot be Kähler–Einstein if $\Gamma \subseteq \mathrm{Aut}(\mathbb{B}^n)$ is nontrivial, abelian, and fixed point free. We will prove this by contradiction. Thus, we suppose $\Gamma \subseteq \mathrm{Aut}(\mathbb{B}^n)$ is abelian and fixed point free, $\Gamma \neq \{\mathrm{id}\}$, and the Bergman kernel of $\Omega = \mathbb{B}^n/\Gamma$ is Kähler–Einstein. As before, we know Γ is contained in a conjugate of $U(n)$. Thus, without loss of generality, we will assume $\Gamma \subseteq U(n)$.

We shall split our proof into three subsections. Section 4.1 reduces the Kähler–Einstein condition of the Bergman metric to a functional equation (see equation (4.8)) for general finite, fixed point free groups $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$. In Section 4.2, we focus on the case where the group Γ is additionally assumed to be abelian, and simplify the equation further into a rather explicit one (see equation (4.13)). After that, in Section 4.3, we take the Taylor expansion of both sides of the equation. By carefully comparing the lowest order Taylor terms, we conclude that they can never match up due to some combinatorial inequalities. The proofs of these inequalities are then given in Section 5, which concludes the proof of Theorem 1.4.

4.1 The Kähler–Einstein equation on finite ball quotients

Since any two realizations of $\mathbb{B}^n/\Gamma$ are biholomorphic, we can use H. Cartan's canonical realization of $\mathbb{B}^n/\Gamma$, which was discussed in Section 2.2. Thus, let $Q : \mathbb{C}^n \to \mathbb{C}^N$ be the basic map realizing $\mathbb{B}/\Gamma$ as a domain $\Omega := Q(\mathbb{B}^n)$ in the n-dimensional algebraic variety $Q(\mathbb{C}^n)$, as explained in Section 2.2. Set

$$Z := \{z \in \mathbb{C}^n : \text{the Jacobian of } Q \text{ at } z \text{ is not of full rank}\}.$$

Note that in fact $Z = \{0\}$ by the fixed point free condition and nontriviality of Γ (see [5] and [10]). We denote by K_Ω and $K_{\mathbb{B}^n}$ the Bergman kernel forms of Ω and $\mathbb{B}^n$, respectively. By the transformation formula in Theorem 2.3, they are related by

$$\sum_{\gamma\in\Gamma} (\mathrm{id}, \gamma)^* K_{\mathbb{B}^n} = (Q, Q)^* K_\Omega \quad \text{on } (\mathbb{B}^n - Z) \times (\mathbb{B}^n - Z), \tag{4.1}$$

where $\mathrm{id} : \mathbb{B}^n \to \mathbb{B}^n$ is the identity map. We note that

$$Q^*(i\partial\overline{\partial}\log K_\Omega) = i\partial\overline{\partial}\log((Q,Q)^*K_\Omega).$$

Furthermore, we also note that the Kähler–Einstein condition is a local property and that Q is a local biholomorphism (on $\mathbb{B}^n - Z$). It follows that the Bergman metric of Ω is Kähler–Einstein if and only if the logarithm of the left hand side of (4.1), restricted to the diagonal $w = z$, gives the potential function of a Kähler–Einstein metric on $\mathbb{B}^n - Z$.

Recall the notation $\langle u, v\rangle = \sum_{i=1}^n u_i v_i$ for two column vectors $u = (u_1, \cdots, u_n)^\intercal$, $v = (v_1, \cdots, v_n)^\intercal$. Set $d_\gamma := \det\gamma$ for $\gamma \in U(n)$. The left hand side of (4.1), in the standard coordinates z, w of $\mathbb{C}^n$, equals

$$\sum_{\gamma\in\Gamma}(\mathrm{id},\gamma)^*K_{\mathbb{B}^n} = \frac{n!}{\pi^n}\sum_{\gamma\in\Gamma}\frac{\overline{d_\gamma}}{(1-\langle z,\overline{\gamma w}\rangle)^{n+1}}dz_1\wedge\ldots dz_n\wedge d\overline{w}_1\wedge\ldots\wedge d\overline{w}_n,$$

where $z, w \in \mathbb{B}^n$ are regarded as column vectors and the elements of Γ as unitary matrices. We introduce the function

$$\varphi(z,\overline{w}) := \sum_{\gamma\in\Gamma}\frac{\overline{d_\gamma}}{(1-\langle z,\overline{\gamma w}\rangle)^{n+1}},$$

and note that $\varphi(z,\overline{z})$ is real analytic on $\mathbb{B}^n$.

Remark 4.1. We also note that the above formula of φ can be regarded as a kind of invariant average over the group of the Bergman kernel of the ball (up to constants). Moreover, by the fixed point freeness of Γ, except for $\gamma = \mathrm{id}$, every term $\frac{\overline{d_\gamma}}{(1-\langle z,\overline{\gamma z}\rangle)^{n+1}}$ extends real analytically to a neighborhood of $\overline{\mathbb{B}^n}$.

Next, by the preceding discussion, we conclude that the Bergman metric of Ω is Kähler–Einstein if and only if $\varphi = \varphi(z,\bar{z})$ is the potential function of a Kähler–Einstein metric, that is, for $z \in \mathbb{B}^n - Z$ and some constant $c_1 \in \mathbb{R}$,

$$-\partial\overline{\partial}\log\Phi(z,\overline{z}) = -c_1\,\partial\overline{\partial}\log\varphi(z,\overline{z}), \tag{4.2}$$

where $\Phi = \det(g_{i\bar{j}})$ with $g_{i\bar{j}} = \partial_{z_i}\partial_{\overline{z_j}}\log\varphi$. (We remark that one can use the result by Klembeck [19] to find the value of c_1, but this value will also come out directly from our arguments below.) The equation (4.2) is equivalent to the statement that $\log\Phi - c_1\log\varphi$ is pluriharmonic on $\mathbb{B}^n - Z$. Consequently, since $Z = \{0\}$ and $n \geq 2$ so that $\mathbb{B}^n - Z$ is simply connected, there exists some holomorphic function h on $\mathbb{B}^n - Z$ such that

$$\log \Phi(z,\overline{z}) - c_1 \log \varphi(z,\overline{z}) = h(z) + \overline{h(z)}.$$

By Hartogs's extension theorem, again since $n \geq 2$, we may assume h is holomorphic on $\mathbb{B}^n$.

Lemma 4.2. The function h is constant. Furthermore, $h + \overline{h} = n \ln(n+1)$ and $c_1 = 1$.

Proof. This lemma is in fact proved in [16] using ideas from [14]. For the reader's convenience, we also sketch a proof here. We give a slightly different proof in order to avoid some tedious computations.

Set $g = e^{2h}$. Then g is holomorphic in $\mathbb{B}^n$ and $|g| = e^{h+\bar{h}} > 0$. We first study the boundary behavior of g.

Claim. $\lim_{|z|\to 1} |g| = a$ for some constant $0 \leq a \leq \infty$.

Proof of the claim. Note that

$$|g| = e^{h+\bar{h}} = \frac{\Phi}{\varphi^{c_1}}. \tag{4.3}$$

We also note that

$$\begin{aligned} \frac{n!}{\pi^n}\varphi(z,\bar{z}) =& \frac{n!}{\pi^n}\left(\frac{1}{(1-|z|^2)^{n+1}} + \sum_{\gamma\in\Gamma,\gamma\neq \mathrm{id}} \frac{\overline{d_\gamma}}{(1-\langle z,\overline{\gamma z}\rangle)^{n+1}}\right) \\ :=& \frac{n!}{\pi^n}\frac{1}{(1-|z|^2)^{n+1}} + T(z,\bar{z}), \end{aligned} \tag{4.4}$$

where $T(z,\bar{z})$ is real analytic in a neighborhood of $\overline{\mathbb{B}^n}$, as observed in Remark 4.1. In particular, the asymptotic singular part of $\frac{n!}{\pi^n}\varphi$ as $z \to \partial\mathbb{B}^n$ is the same as that of the Bergman kernel of $\mathbb{B}^n$. Let J be the Monge–Ampère type operator as defined in (4.7). With the preceding observation and the well known formula

$$\Phi = \det\Big(\partial_{z_i}\partial_{\overline{z_j}} \log(\varphi)\Big) = \frac{J(\varphi)}{\varphi^{n+1}},$$

a simple calculation yields that the most singular part of Φ (as $z \to \partial\mathbb{B}^n$) is identical with that of the volume form of the Bergman metric on $\mathbb{B}^n$. More precisely,

$$\Phi = \frac{(n+1)^n}{(1-|z|^2)^{n+1}} + \widehat{\Phi}, \tag{4.5}$$

where $\widehat{\Phi}$ is real analytic in $\mathbb{B}^n - Z$ and satisfies $(1-|z|^2)^{n+1}\widehat{\Phi} \to 0$ as $|z| \to 1$. Then by (4.3), (4.4), and (4.5), we see

$$\lim_{|z|\to 1} |g| = (n+1)^n \lim_{|z|\to 1} \frac{(1-|z|^2)^{(n+1)c_1}}{(1-|z|^2)^{n+1}}. \tag{4.6}$$

Thus, depending on c_1, we have $\lim_{|z|\to 1} |g| = a$ for some $0 \le a \le \infty$. This proves the claim. ■

But g is a nowhere vanishing holomorphic function in $\mathbb{B}^n$. We apply the maximum modulus principle to g and $\frac{1}{g}$, respectively, and use (4.6) to obtain $c_1 = 1$. And in this case by (4.6), $a = (n+1)^n$. Applying the maximum principle again, we see $|g| \equiv a = (n+1)^n$. This implies g and thus h are constant functions, and $h + \overline{h} \equiv n \ln(n+1)$. The proof of the lemma is finished. ■

We define the Monge–Ampère type operator J as follows (note that it differs by a sign from the standard operator introduced by Fefferman [12]):

$$J(\varphi) := \det \begin{pmatrix} \varphi & \varphi_{\overline{z_j}} \\ \varphi_{z_i} & \varphi_{z_i\overline{z_j}} \end{pmatrix}, \tag{4.7}$$

We use Lemma 4.2 and the well-known formula $\Phi = J(\varphi)/\varphi^{n+1}$ to further simplify (4.2) into

$$J(\varphi)(z,\overline{z}) = (n+1)^n \varphi^{n+2}(z,\overline{z}) \tag{4.8}$$

for $z \in \mathbb{B}^n - Z$. Since both sides of (4.8) are in fact real analytic in $\mathbb{B}^n$, we see (4.8) holds on $\mathbb{B}^n$ by continuity. We pause here to observe that if Γ is such that $\varphi(0,0) \neq 0$, then it follows that $\log\varphi$ extends as the potential of a Kähler–Einstein metric in the whole unit ball $\mathbb{B}^n$, which by uniqueness of the Cheng–Yau metric, can be used to directly conclude that $\Gamma = \{\mathrm{id}\}$; this was previously observed in [16, Corollary 5.4]. Now, let us compute $J(\varphi)$. Clearly, we have

$$\varphi_{z_i} = (n+1)\sum_{\gamma\in\Gamma} \frac{\overline{d_\gamma}\cdot\overline{(\gamma z)_i}}{(1-\langle z, \overline{\gamma z}\rangle)^{n+2}}, \qquad \varphi_{\overline{z_j}} = (n+1)\sum_{\gamma\in\Gamma} \frac{\overline{d_\gamma}\cdot(z^{\intercal}\overline{\gamma})_j}{(1-\langle z, \overline{\gamma z}\rangle)^{n+2}},$$

where $(\gamma z)_i$ denotes the i-th entry of the column vector γz and similarly $(z^{\intercal}\overline{\gamma})_j$ denotes the j-th entry of the row vector $z^{\intercal}\overline{\gamma}$. By differentiating both sides one more time, we obtain

$$\varphi_{z_i\overline{z_j}} =(n+1)\sum_{\gamma\in\Gamma}\overline{d_\gamma}\cdot\frac{\overline{\gamma_{ij}}(1-\langle z,\overline{\gamma z}\rangle)+(n+2)\overline{(\gamma z)_i}(z^{\mathsf{T}}\overline{\gamma})_j}{(1-\langle z,\overline{\gamma z}\rangle)^{n+3}},$$

where γ_{ij} is the (i,j) component of the matrix γ.

For each $\gamma\in\Gamma, 0\le j\le n$, we define a column vector-valued function $\xi_j(\gamma):\mathbb{B}^n\to\mathbb{C}^{n+1}$ in the variables $(z,\overline{z})$ as follows:

$$\xi_0(\gamma)(z,\overline{z}) := \begin{pmatrix} 1-\langle z,\gamma\overline{z}\rangle \\ (n+1)\gamma\overline{z}\end{pmatrix} \quad\text{and}\quad \xi_j(\gamma)(z,\overline{z}) := \begin{pmatrix} z^{\mathsf{T}}(\gamma)_j \\ \frac{(\gamma)_j(1-\langle z,\gamma\overline{z}\rangle)+(n+2)\gamma\overline{z}(z^{\mathsf{T}}(\gamma)_j)}{1-\langle z,\gamma\overline{z}\rangle}\end{pmatrix} \text{ for} 1\le j\le n,$$

where $(\gamma)_j$ is the j-th column vector of the matrix γ. Given any $(n+1)$ (possibly repeated) elements $\gamma_0,\cdots,\gamma_n$ in Γ, we define a matrix-valued function $A(\gamma_0,\cdots,\gamma_n):\mathbb{B}^n\to\mathbb{C}^{(n+1)^2}$ as follows:

$$A(\gamma_0,\cdots,\gamma_n)=\begin{pmatrix}\xi_0(\gamma_0) & \cdots & \xi_n(\gamma_n)\end{pmatrix}.$$

We emphasize that the map $A(\gamma_0,\cdots,\gamma_n)$ sends a point $z\in\mathbb{B}^n$ to an $(n+1)\times(n+1)$ matrix. We then expand the determinant in (4.7) by multi-linearity with respect to columns. We obtain the following formula:

$$J(\varphi)=\sum_{\gamma_0,\cdots,\gamma_n\in\Gamma}\frac{(n+1)^n\,\overline{d_{\gamma_0}}\cdots\overline{d_{\gamma_n}}}{\prod_{i=0}^n(1-\langle z,\overline{\gamma_i z}\rangle)^{n+2}}\det\big(A(\overline{\gamma_0},\cdots,\overline{\gamma_n})\big). \tag{4.9}$$

4.2 Abelian group case

From now on, we will assume that Γ is a finite, abelian, fixed point free subgroup of $U(n)$.

Lemma 4.3. Let $\Gamma\subseteq U(n)$ be a finite abelian group. If Γ is also fixed point free, then it is cyclic of order m for some $m\in\mathbb{N}$. Furthermore, after replacing Γ by an appropriate conjugate of itself, there are primitive m–th roots of unity $\varepsilon_1,\cdots,\varepsilon_n$ such that the diagonal matrix $\operatorname{diag}(\varepsilon_1,\cdots,\varepsilon_n)$ generates Γ.

Proof. The result is well known; see, for example, [28]. For the convenience of readers, we will sketch a proof here. Since $\Gamma\subseteq U(n)$ is abelian, one can simultaneously diagonalize the matrices in Γ. Thus, replacing Γ by an appropriate conjugate of Γ, we can assume Γ consists of diagonal matrices. Consider the group homomorphism $\pi_1:\Gamma\to U(1)$ defined by $\pi_1(\gamma):=\gamma_{11}$ for $\gamma=\operatorname{diag}(\gamma_{11},\cdots,\gamma_{nn})$. We claim $\pi:\Gamma\to\pi_1(\Gamma)$

is an isomorphism. Indeed, let $\gamma \in \Gamma$ be such that $\pi_1(\gamma) = 1$. Then 1 is an eigenvalue of γ, and by the fixed point free condition, γ must be the identity. This proves the claim. But $\pi_1(\Gamma) \subseteq U(1)$ is finite, thus it must be cyclic. So is Γ and this proves the 1st assertion. To prove the 2nd assertion, write γ for a generator of Γ. By our assumption, γ is a diagonal matrix, and again the fixed-point free condition guarantees that all diagonal elements of γ have the same order m. This establishes the 2nd assertion. ∎

By Lemma 4.3, we can assume $\Gamma = \{\gamma, \gamma^2, \cdots, \gamma^m = \mathrm{id}\}$ for some generator

$$\gamma = \begin{pmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_n \end{pmatrix}.$$

Here $m \geq 2$ by the nontriviality of Γ. And $\varepsilon_1, \cdots, \varepsilon_n$ are primitive m-th roots of unity. By setting $\varepsilon := \varepsilon_1$, for $1 \leq j \leq n$ we can write ε_j in the form of

$$\varepsilon_j = \varepsilon^{t_j}, \quad \text{for some} 1 \leq t_j \leq m-1 \text{ with} \gcd(t_j, m) = 1.$$

By renumbering the coordinates, we can assume

$$1 = t_1 \leq t_2 \leq \cdots \leq t_n \leq m-1.$$

For any $\gamma \in \Gamma \subseteq U(1) \times \cdots \times U(1)$, note that $\gamma^{-1} = \overline{\gamma}^{\intercal} = \overline{\gamma}$. Hence, we can replace all $\overline{\gamma_j}$ by γ_j in the sum in $J(\varphi)$ and obtain

$$J(\varphi) = \sum_{\gamma_0, \cdots, \gamma_n \in \Gamma} \frac{(n+1)^n \, d_{\gamma_0} \cdots d_{\gamma_n}}{\prod_{i=0}^n (1 - \langle z, \gamma_i \overline{z} \rangle)^{n+2}} \det\big(A(\gamma_0, \cdots, \gamma_n)\big).$$

Write $\gamma_j = \gamma^{k_j}$ for some $0 \leq k_j \leq m-1$. Then $d_{\gamma_j} = \det \gamma_j = \varepsilon^{k_j(\sum_{i=1}^n t_i)}$. Choose $z = z^* := (z_1, 0, \cdots, 0)^{\intercal}$ with $|z_1| < 1$ and set $x = z^* \cdot \overline{z^*} = z_1 \overline{z_1} < 1$. Then at z^*, we have

$$\det\big(A(\gamma_0, \cdots, \gamma_n)\big) = \begin{vmatrix} 1 - \varepsilon^{k_0} x & z_1 \varepsilon^{k_1} & 0 \\ (n+1)\varepsilon^{k_0} \overline{w_1} & \frac{\varepsilon^{k_1}(1-\varepsilon^{k_1}x) + (n+2)\varepsilon^{2k_1}x}{1-\varepsilon^{k_1}x} & 0 \\ 0 & 0 & \begin{pmatrix} \varepsilon_2^{k_2} & & \\ & \ddots & \\ & & \varepsilon_n^{k_n} \end{pmatrix} \end{vmatrix}$$

$$= \varepsilon^{k_1 + \sum_{j=2}^n k_2 t_2} \Big(1 - (n+2)\varepsilon^{k_0} x + (n+2) \frac{1 - \varepsilon^{k_0} x}{1 - \varepsilon^{k_1} x} \varepsilon^{k_1} x\Big).$$

In the following, we use the notation

- $T = (t_1, \cdots, t_n)$, where $t_1 = 1$.
- $K = (k_0, k_1, \cdots, k_n)$ and $K' = (k_1, \cdots, k_n)$.
- $|T| = \sum_{j=1}^{n} t_j$ and $|K'| = \sum_{j=1}^{n} k_j$.

Using these notations, we have, at $z = z^*$,

$$J(\varphi)(z^*, \overline{z^*}) = (n+1)^n \sum_{k_0,\cdots,k_n=0}^{m-1} \frac{\varepsilon^{|K|\cdot|T|+K'\cdot T}}{\prod_{i=0}^{n}(1-\varepsilon^{k_i}x)^{n+2}} \Big(1-(n+2)\varepsilon^{k_0}x+(n+2)\frac{1-\varepsilon^{k_0}x}{1-\varepsilon^{k_1}x}\,\varepsilon^{k_1}x\Big).$$

If we set

$$\begin{aligned} \mathrm{I} :=& \sum_{k_0,\cdots,k_n=0}^{m-1} \frac{\varepsilon^{|K|\cdot|T|+K'\cdot T}}{\prod_{i=0}^{n}(1-\varepsilon^{k_i}x)^{n+2}}, \\ \mathrm{II} :=& -(n+2) \sum_{k_0,\cdots,k_n=0}^{m-1} \frac{\varepsilon^{|K|\cdot|T|+K'\cdot T+k_0}x}{\prod_{i=0}^{n}(1-\varepsilon^{k_i}x)^{n+2}}, \\ \mathrm{III} :=& (n+2) \sum_{k_0,\cdots,k_n=0}^{m-1} \frac{\varepsilon^{|K|\cdot|T|+K'\cdot T+k_1}x}{\prod_{i=0}^{n}(1-\varepsilon^{k_i}x)^{n+2}}\,\frac{1-\varepsilon^{k_0}x}{1-\varepsilon^{k_1}x}, \end{aligned}$$

then

$$J(\varphi)(z^*, \overline{z^*}) = (n+1)^n\big(\mathrm{I}+\mathrm{II}+\mathrm{III}\big). \tag{4.10}$$

We pause to introduce the following definition and lemmas. Let ε be as above. Write $\mathbb{D}$ for the open unit disk in $\mathbb{C}$.

Definition 4.4. Let $t \in \mathbb{Z}, p \in \mathbb{Z}^+$. Define $f_{t,p} : \mathbb{D} \to \mathbb{C}$ as

$$f_{t,p}(x) := \sum_{k=0}^{m-1} \frac{1}{\varepsilon^{tk}(\varepsilon^k - x)^p}.$$

Lemma 4.5. The following holds:

$$f'_{t,p}(x) = p f_{t,p+1}(x).$$

In general, for $j \geq 2$, $f^{(j)}_{t,p}(x) = p(p+1)\cdots(p+j-1)f_{t,p+j}(x)$.

Proof. Note

$$f'_{t,p}(x) = \sum_{k=0}^{m-1} \frac{p}{\varepsilon^{tk}(\varepsilon^k - x)^{p+1}} = pf_{t,p+1}(x).$$

This proves the 1st statement. The latter assertion follows from the 1st statement and an inductive argument. ■

Lemma 4.6. The following hold:

(1)

$$f_{t,p}(0) = \begin{cases} 0 & \text{if} m \nmid (t+p), \\ m & \text{if } m \mid (t+p). \end{cases}$$

(2) For $j \geq 1$,

$$f^{(j)}_{t,p}(0) = \begin{cases} 0 & \text{if } m \nmid (t+p+j), \\ m \prod_{i=0}^{i=j-1}(p+i) & \text{if } m \mid (t+p+j). \end{cases}$$

Proof. To prove part (1), we note that

$$f_{t,p}(0) = \sum_{k=0}^{m-1} \varepsilon^{-k(t+p)}.$$

Then the result in part (1) follows directly by the fact that ε is a primitive m-th root of unity. Part (2) follows from part (1) and Lemma 4.5. ■

Lemma 4.7. The following holds:

$$\sum_{k=0}^{m-1} \frac{\varepsilon^{tk}}{(1-\varepsilon^k x)^p} = f_{t-p,p}(x).$$

Proof.

$$\sum_{k=0}^{m-1} \frac{\varepsilon^{tk}}{(1-\varepsilon^k x)^p} = \sum_{k=0}^{m-1} \frac{1}{\varepsilon^{tk}(1-\varepsilon^{-k}x)^p} = \sum_{k=0}^{m-1} \frac{1}{\varepsilon^{(t-p)k}(\varepsilon^k - x)^p} = f_{t-p,p}(x),$$

where the 1st equality follows from the fact that ε is a primitive m-th root of unity. ■

Now, using the above notation and Lemma 4.7, we shall express $J(\varphi)(z^*, \overline{z^*})$ in terms of $f_{t,p}$.

$$\begin{aligned}
\mathrm{I} =& \sum_{k_0,\cdots,k_n=0}^{m-1} \frac{\varepsilon^{|K|\cdot|T|+K'\cdot T}}{\prod_{i=0}^{n}(1-\varepsilon^{k_i}x)^{n+2}} \\
=& \sum_{k_0=0}^{m-1} \frac{\varepsilon^{k_0|T|}}{(1-\varepsilon^{k_0}x)^{n+2}} \sum_{k_1=0}^{m-1} \frac{\varepsilon^{k_1(|T|+t_1)}}{(1-\varepsilon^{k_1}x)^{n+2}} \cdots \sum_{k_n=0}^{m-1} \frac{\varepsilon^{k_n(|T|+t_n)}}{(1-\varepsilon^{k_n}x)^{n+2}} \\
=& f_{|T|-(n+2),n+2}(x) f_{|T|+t_1-(n+2),n+2}(x) \cdots f_{|T|+t_n-(n+2),n+2}(x).
\end{aligned}$$

$$\begin{aligned}
\mathrm{II} =& -(n+2)x \sum_{k_0,\cdots,k_n=0}^{m-1} \frac{\varepsilon^{|K|\cdot|T|+K'\cdot T+k_0}}{\prod_{i=0}^{n}(1-\varepsilon^{k_i}x)^{n+2}} \\
=& -(n+2)x \sum_{k_0=0}^{m-1} \frac{\varepsilon^{k_0(|T|+1)}}{(1-\varepsilon^{k_0}x)^{n+2}} \sum_{k_1=0}^{m-1} \frac{\varepsilon^{k_1(|T|+t_1)}}{(1-\varepsilon^{k_1}x)^{n+2}} \cdots \sum_{k_n=0}^{m-1} \frac{\varepsilon^{k_n(|T|+t_n)}}{(1-\varepsilon^{k_n}x)^{n+2}} \\
=& -(n+2)x f_{|T|-(n+1),n+2}(x) f_{|T|+t_1-(n+2),n+2}(x) \cdots f_{|T|+t_n-(n+2),n+2}(x).
\end{aligned}$$

$$\begin{aligned}
\mathrm{III} =& (n+2)x \sum_{k_0,\cdots,k_n=0}^{m-1} \frac{\varepsilon^{|K|\cdot|T|+K'\cdot T+k_1}}{\prod_{i=0}^{n}(1-\varepsilon^{k_i}x)^{n+2}} \frac{1-\varepsilon^{k_0}x}{1-\varepsilon^{k_1}x} \\
=& (n+2)x \sum_{k_0=0}^{m-1} \frac{\varepsilon^{k_0|T|}}{(1-\varepsilon^{k_0}x)^{n+1}} \sum_{k_1=0}^{m-1} \frac{\varepsilon^{k_1(|T|+t_1+1)}}{(1-\varepsilon^{k_1}x)^{n+3}} \sum_{k_2=0}^{m-1} \frac{\varepsilon^{k_n(|T|+t_2)}}{(1-\varepsilon^{k_2}x)^{n+2}} \cdots \sum_{k_n=0}^{m-1} \frac{\varepsilon^{k_n(|T|+t_n)}}{(1-\varepsilon^{k_n}x)^{n+2}} \\
=& (n+2)x f_{|T|-(n+1),n+1}(x) f_{|T|+t_1-(n+2),n+3}(x) f_{|T|+t_2-(n+2),n+2}(x) \cdots f_{|T|+t_n-(n+2),n+2}(x).
\end{aligned}$$

Set

$$\begin{aligned}
P :=& f_{|T|-(n+2),n+2} f_{|T|-(n+1),n+2} - (n+2)x\big(f^2_{|T|-(n+1),n+2} - f_{|T|-(n+1),n+1} f_{|T|-(n+1),n+3}\big), \\
Q :=& f_{|T|+t_2-(n+2),n+2} \cdots f_{|T|+t_n-(n+2),n+2}.
\end{aligned} \tag{4.11}$$

By (4.10) and the fact $t_1 = 1$, we conclude that $J(\varphi)(z^*, \overline{z^*})$ can be written as

$$J(\varphi)(z^*, \overline{z^*}) = (n+1)^n P(x)\, Q(x). \tag{4.12}$$

Moreover, at $z = z^* = (z_1, 0, \cdots, 0)^\intercal$, we can simplify φ as

$$\varphi(z^*, \overline{z^*}) = \sum_{\gamma\in\Gamma} \frac{\overline{d_\gamma}}{(1 - \langle z^*, \overline{\gamma z^*}\rangle)^{n+1}} = \sum_{\gamma\in\Gamma} \frac{d_\gamma}{(1 - \langle z^*, \gamma\overline{z^*}\rangle)^{n+1}} = \sum_{k=0}^{m-1} \frac{\varepsilon^{k|T|}}{(1 - \varepsilon^k x)^{n+1}}$$

$$= f_{|T|-(n+1),n+1}(x).$$

The 2nd equality here is due to the fact that $\Gamma \subseteq U(1) \times \cdots \times U(1) \subseteq U(n)$, as also explained above. By the above expression for φ and (4.12), we conclude that at $z = z^*$, the Kähler–Einstein equation (4.8) is reduced to, for $x \in [0, 1) \subseteq \mathbb{R}$,

$$f^{n+2}_{|T|-(n+1),n+1}(x) = P(x)Q(x), \tag{4.13}$$

where P, Q are defined in (4.11). Since both sides of (4.13) are holomorphic in $\mathbb{D}$, we conclude that (4.13) in fact holds for all $x \in \mathbb{D}$.

4.3 Reduction to combinatorial inequalities

We shall take the Taylor expansion of both sides in (4.13) at $x = 0$. By comparing the Taylor coefficients, we shall prove that (4.13) cannot hold if $m = |\Gamma| \geq 2$ and $n \geq 2$, which will establish Theorem 1.4. We shall proceed by dividing the proof into several cases.

Case I. $m \mid |T|$.

As $m \mid |T|$, $m \nmid |T| + 1$. Lemma 4.6 yields that

$$f_{|T|-(n+1),n+1}(0) = m, \qquad f_{|T|-(n+1),n+2}(0) = 0.$$

Therefore, at $x = 0$,

$$f^{n+2}_{|T|-(n+1),n+1}(0) = m^{n+2} \neq 0 = P(0) \cdot Q(0),$$

which implies that the Kähler–Einstein equation (4.13) does not hold.

Case II. $m \nmid |T|$ and $m \mid |T| + 1$.

In this case, we have

$$m \nmid |T| + 2, \cdots, m \nmid |T| + m,\ m \mid |T| + m + 1.$$

We take the Taylor expansion of $f_{|T|-(n+2),n+2}$ at $x = 0$.

$$
\begin{aligned}
f_{|T|-(n+2),n+2}(x) &= \sum_{j=0}^{m+1} \frac{f^{(j)}_{|T|-(n+2),n+2}(0)}{j!} x^j + O(m+2) \\
&= \binom{n+2}{1} mx + \binom{n+m+2}{m+1} mx^{m+1} + O(m+2).
\end{aligned} \tag{4.14}
$$

Here for a holomorphic function h in a neighborhood $U \subseteq \mathbb{C}$ of 0, we say h is $O(j), j \geq 1$, if $h^{(i)}(0) = 0$ for all $0 \leq i < j$. The last equality follows from Lemma 4.6.

Similarly, we also have

$$
\begin{aligned}
f_{|T|-(n+1),n+2}(x) =& m + \binom{m+n+1}{m} mx^m + O(m+1), \\
f_{|T|-(n+1),n+1}(x) =& \binom{n+1}{1} mx + \binom{m+n+1}{m+1} mx^{m+1} + O(m+2), \\
f_{|T|-(n+1),n+3}(x) =& \binom{m+n+1}{m-1} mx^{m-1} + O(m).
\end{aligned}
$$

By (4.11), it follows that

$$
\begin{aligned}
P =& \left(\binom{n+2}{1} mx + \binom{n+m+2}{m+1} mx^{m+1} \right) \left(m + \binom{m+n+1}{m} mx^m \right) \\
& - (n+2)x \left(m + \binom{m+n+1}{m} mx^m \right)^2 \\
& + (n+2)x \left(\left(\binom{n+1}{1} mx + \binom{m+n+1}{m+1} mx^{m+1} \right) \binom{m+n+1}{m-1} mx^{m-1} + O(m+2) \right. \\
=& m^2 \binom{n+m+1}{m} x^{m+1} \left(-n - 2 + \frac{m+n+2}{m+1} + (n+1)m \right) + O(m+2) \\
=& \frac{m^4(n+1)}{m+1} \binom{n+m+1}{m} x^{m+1} + O(m+2).
\end{aligned}
$$

Recall that

$$
Q = f_{|T|+t_2-(n+2),n+2} \cdots f_{|T|+t_n-(n+2),n+2},
$$

where $1 = t_1 \leq t_2 \leq \cdots \leq t_n \leq m-1$. Let $1 \leq a \leq n$ be such that

$$
1 = t_1 = \cdots = t_a < t_{a+1} \leq \cdots \leq t_n.
$$

When $a = n$, the above means that all t_j's equal 1. Now for $1 \leq j \leq a$, we have

$$f_{|T|+t_j-(n+2),n+2}(x) = f_{|T|-(n+1),n+2}(x) = m + O(1).$$

And for $a+1 \leq j \leq n$, $|T|+1 < |T|+t_j < |T|+m+1$. We get, by a similar computation as in (4.14),

$$f_{|T|+t_j-(n+2),n+2}(x) = \binom{n+m+2-t_j}{m+1-t_j} m x^{m+1-t_j} + O(m+2-t_j).$$

Thus,

$$Q = m^{n-1}\binom{m+n+2-t_{a+1}}{m+1-t_{a+1}} \cdots \binom{m+n+2-t_n}{m+1-t_n} x^{(m+1)(n-a)-(t_{a+1}+\cdots+t_n)} + \text{h.o.t.},$$

where h.o.t. denotes the higher order term. Combining this with the Taylor expansion of P, the lowest order term in PQ at $x = 0$ is

$$\frac{m^{n+3}(n+1)}{m+1}\binom{n+m+1}{m}\binom{m+n+2-t_{a+1}}{m+1-t_{a+1}} \cdots \binom{m+n+2-t_n}{m+1-t_n} x^{(m+1)(n-a+1)-\sum_{j=a+1}^{n} t_j}. \tag{4.15}$$

When $a = n$, $\sum_{j=a+1}^{n} t_j$ is a null sum and equals zero. Furthermore, we have

$$f^{n+2}_{|T|-(n+1),n+1} = (n+1)^{n+2} m^{n+2} x^{n+2} + \text{h.o.t}. \tag{4.16}$$

Suppose that the Kähler–Einstein equation (4.13) holds. Then $f^{n+2}_{|T|-(n+1),n+1}$ and PQ must share the same Taylor expansion at $x = 0$. In particular, their lowest order terms, where the former is found in (4.16) and the latter in (4.15), must have the same degree, that is, $n+2 = (m+1)(n-a) - \sum_{j=a+1}^{n} t_j$. In this case, however, the coefficients of the lowest order terms do not match by the following lemma.

Lemma 4.8. Suppose $m, n \geq 2$, $1 \leq a \leq n$, and $1 = t_1 = \cdots = t_a < t_{a+1} \leq \cdots \leq t_n \leq m-1$. If $n+2 = (m+1)(n-a+1) - \sum_{j=a+1}^{n} t_j$, then

$$(n+1)^{n+1}(m+1) > m\binom{n+m+1}{m}\binom{m+n+2-t_{a+1}}{m+1-t_{a+1}} \cdots \binom{m+n+2-t_n}{m+1-t_n}.$$

In the case $a = n$, that is, all t_j's equal 1, the above is reduced to the following: if $m = n+1$, then

$$(n+1)^{n+1}(m+1) > m\binom{n+m+1}{m}.$$

This is a contradiction and we thus conclude the Kähler–Einstein equation (4.13) does not hold. The proof of Lemma 4.8 is left to Section 5.

Case III. $m \nmid |T|, m \nmid |T|+1, \cdots, m \nmid |T|+k-1$ and $m \mid |T|+k$ for some $2 \le k < m$.

We follow the same procedure as in Case II. Similarly, as in (4.14), by using Lemma 4.6, we have

$$\begin{aligned} f_{|T|-(n+2),n+2}(x) &= \sum_{j=0}^{k+m} \frac{f^{(j)}_{|T|-(n+2),n+2}(0)}{j!} x^j + O(x^{k+m+1}) \\ &= \binom{n+k+1}{k} m x^k + \binom{n+k+m+1}{k+m} m x^{k+m} + O(k+m+1), \end{aligned}$$

and

$$\begin{aligned} f_{|T|-(n+1),n+2}(x) &= \binom{n+k}{k-1} m x^{k-1} + \binom{k+m+n}{k+m-1} m x^{k+m-1} + O(k+m), \\ f_{|T|-(n+1),n+1}(x) &= \binom{n+k}{k} m x^k + \binom{n+k+m}{k+m} m x^{k+m} + O(k+m+1), \\ f_{|T|-(n+1),n+3}(x) &= \binom{n+k}{k-2} m x^{k-2} + \binom{n+k+m}{k+m-2} m x^{k+m-2} + O(k+m-1). \end{aligned}$$

By (4.11), it follows that

$$P = \binom{n+k}{k-1}\binom{n+k+m}{n} \frac{m^4}{k} x^{2k+m-1} + \text{h.o.t.} \tag{4.17}$$

Now we turn to the computation of the leading term in Q. Recall that $1 = t_1 \le t_2 \le \cdots \le t_n \le m-1$. We shall divide the computation into two subcases: $k < t_n$ and $k \ge t_n$.

Subcase III (a). $k < t_n$.

Since $k \ge 2$, there exists some $1 \le a \le n-1$ such that

$$1 = t_1 \le \cdots \le t_a \le k < t_{a+1} \le \cdots \le t_n \le m-1.$$

For $1 \le j \le a$, as $|T| + 1 \le |T| + t_j \le |T| + k$, by the Taylor expansion and Lemma 4.6, we have

$$f_{|T|+t_j-(n+2),n+2}(x) = \binom{n+1+k-t_j}{k-t_j} mx^{k-t_j} + O(k-t_j+1).$$

For $a+1 \le j \le n$, it follows that $|T| + k < |T| + t_j < |T| + k + m$. Thus, by the Taylor expansion and Lemma 4.6,

$$f_{|T|+t_j-(n+2),n+2}(x) = \binom{n+1+m+k-t_j}{m+k-t_j} mx^{m+k-t_j} + O(m+k-t_j+1).$$

By (4.11), we obtain

$$Q = \prod_{j=2}^{a} \binom{n+1+k-t_j}{k-t_j} mx^{k-t_j} \cdot \prod_{j=a+1}^{n} \binom{n+1+m+k-t_j}{m+k-t_j} mx^{m+k-t_j} + \text{h.o.t.}$$

Therefore, (4.17) and the above equality yield the leading term in the Taylor expansion of PQ at $x=0$ as

$$\frac{m^{n+3}}{k}\binom{n+k}{k-1}\binom{n+k+m}{n}\prod_{j=2}^{a}\binom{n+1+k-t_j}{k-t_j} \cdot \prod_{j=a+1}^{n}\binom{n+1+m+k-t_j}{m+k-t_j} \cdot x^s,$$

where

$$s = (n+1)k + m - 1 - \sum_{j=2}^{a} t_j + \sum_{j=a+1}^{n} (m-t_j). \tag{4.18}$$

On the other hand, the left hand side of (4.13) satisfies

$$f_{|T|-(n+1),n+1}^{n+2} = \binom{n+k}{k}^{n+2} m^{n+2} x^{k(n+2)} + \text{h.o.t.}$$

Suppose that the Kähler–Einstein equation (4.13) holds. Then $f_{|T|-(n+1),n+1}^{n+2}$ and PQ must share the same Taylor expansion at $x=0$. In particular, their lowest order terms have the same degree, that is, $s = k(n+2)$, which in view of (4.18) implies that

$$k = m - \sum_{j=1}^{a} t_j + \sum_{j=a+1}^{n} (m-t_j). \tag{4.19}$$

In this case, the coefficients of the lowest terms are, however, unequal by the following lemma.

Lemma 4.9. Suppose $1 \le a \le n-1$, $2 \le k \le m-1$ and $1 = t_1 \le \cdots \le t_a \le k < t_{a+1} \le \cdots \le t_n \le m-1$. If (4.19) holds, then

$$k\binom{n+k}{k}^{n+2} > m\binom{n+k+m}{n}\prod_{j=1}^{a}\binom{n+1+k-t_j}{k-t_j}\cdot\prod_{j=a+1}^{n}\binom{n+1+m+k-t_j}{m+k-t_j}.$$

This is a contradiction and we thus conclude that (4.13) does not hold. We will leave the proof of Lemma 4.9 to Section 5.

Subcase III (b). $k \ge t_n$.

In this case, $|T|+1 \le |T|+t_j \le |T|+k$ for all $1 \le j \le n$. Thus we have, by the Taylor expansion,

$$Q = \prod_{j=2}^{n}\binom{n+1+k-t_j}{k-t_j} m x^{k-t_j} + \text{h.o.t.}$$

Note that all other terms in (4.13) have the same Taylor expansions as in case III (a). As before, in order to disprove (4.13), it is sufficient to verify the following lemma, whose proof is also delayed to Section 5.

Lemma 4.10. Let $2 \le k \le m-1$ and $n \ge 2$. Let $1 = t_1 \le \cdots \le t_n \le k$. If

$$k = m - \sum_{j=1}^{n} t_j,$$

then

$$k\binom{n+k}{k}^{n+2} > m\binom{n+k+m}{n}\prod_{j=1}^{n}\binom{n+1+k-t_j}{k-t_j}.$$

5 Proof of the Combinatorial Lemmas

In this section, we shall prove Lemmas 4.8, 4.9, and 4.10.

5.1 Proof of Lemma 4.8

For the reader's convenience, we restate Lemma 4.8 here.

Lemma 5.1. Suppose $m, n \geq 2$, $1 \leq a \leq n$, and $1 = t_1 = \cdots = t_a < t_{a+1} \leq \cdots \leq t_n \leq m-1$. If $n+2 = (m+1)(n-a+1) - \sum_{j=a+1}^{n} t_j$, then

$$(n+1)^{n+1}(m+1) > m\binom{n+m+1}{m}\binom{m+n+2-t_{a+1}}{m+1-t_{a+1}}\cdots\binom{m+n+2-t_n}{m+1-t_n}. \tag{5.1}$$

Proof. We divide the proof into two cases.

Case I. $n = 2$.

In this case, by the assumption of Lemma 4.8, we have $4 = (m+1)(3-a) - \sum_{j=a+1}^{n} t_j$ and $1 \leq a \leq 2$.

Suppose $a = 1$. Then $2(m+1) = 4 + t_2 \leq m+3$, which yields $m \leq 1$. This contradicts the assumption $m \geq 2$. Thus we have $a = 2$. It follows that $t_2 = 1$ and $m = 3$. A straightforward computation shows

$$\text{LHS of (5.1)} = 108 > 60 = \text{RHS of (5.1)}.$$

So this case is verified.

Case II. $n \geq 3$.

We first prove the following elementary combinatorial inequality, which will be used in the proof.

Lemma 5.2. For any integers $n, k \geq 3$, we have

$$\binom{n+k}{k-1} < (n+1)^{k-1}. \tag{5.2}$$

Proof.

$$\binom{n+k}{k-1} \cdot (n+1)^{-(k-1)} = \prod_{t=1}^{k-1} \frac{(n+1+t)}{t \cdot (n+1)} = \frac{(n+2)(n+3)}{2(n+1)^2} \cdot \prod_{t=3}^{k-1} \frac{(n+1+t)}{t \cdot (n+1)}.$$

Since $n \geq 3$,

$$2(n+1)^2 - (n+2)(n+3) = n^2 - n - 4 \geq 2 > 0,$$

which implies that

$$\frac{(n+2)(n+3)}{2(n+1)^2} < 1$$

When $t \geq 3$,

$$t(n+1) - (n+1+t) = n(t-1) - 1 \geq 2n - 1 > 0,$$

which implies that

$$\frac{(n+1+t)}{t \cdot (n+1)} < 1.$$

The result therefore follows. ■

Recall that $1 \leq t_j \leq m-1$ for any $1 \leq j \leq n$. By applying (5.2) with $k = m+2-t_j$, we get

$$\binom{m+n+2-t_j}{m+1-t_j} < (n+1)^{(m+1-t_j)}.$$

Thus,

$$\text{RHS of (5.1)} < m(n+1)^{m+\sum_{j=a+1}^{n}(m+1-t_j)} = m(n+1)^{(n+1)} < \text{LHS of (5.1)}.$$

So the proof is complete also in Case II. ■

5.2 Proof of Lemma 4.9 and Lemma 4.10

We will prove a slightly more general result.

Lemma 5.3. Let k, m, n be integers such that $1 \leq k \leq m-1$ and $n \geq 2$. Let $\lambda = (\lambda_1, \cdots \lambda_n) \in \mathbb{Z}^n$ satisfy $\lambda_j \leq k$ for each $1 \leq j \leq n$. If

$$m - k = \sum_{j=1}^{n} \lambda_j,$$

then

$$k\binom{n+k}{k}^{n+2} > m\binom{n+k+m}{n}\prod_{j=1}^{n}\binom{n+1+k-\lambda_j}{k-\lambda_j}. \tag{5.3}$$

Clearly, Lemma 4.9 follows from 5.3 by taking $(\lambda_1, \cdots, \lambda_n) = (t_1, \cdots, t_a, t_{a+1} - m, \cdots, t_n - m)$. Lemma 4.10 follows from Lemma 5.3 by taking $(\lambda_1, \cdots, \lambda_n) = (t_1, \cdots, t_n)$.

Proof of Lemma 5.3. We divide the proof into several steps.

Step 1. We show that it is actually sufficient to prove (5.3) for $0 \le \lambda_j \le k$ for all $1 \le j \le n$.

We begin this step with the following elementary combinatorial lemma.

Lemma 5.4. Let $n \in \mathbb{N}$ and let s, t be integers such that $s+1 < t \le k$. Then we have

$$\binom{n+1+k-s}{k-s}\binom{n+1+k-t}{k-t} < \binom{n+1+k-(s+1)}{k-(s+1)}\binom{n+1+k-(t-1)}{k-(t-1)}. \tag{5.4}$$

Proof. A straightforward computation gives

$$\frac{\text{LHS of}(5.4)}{\text{RHS of}(5.4)} = \frac{n+1+k-s}{k-s} \cdot \frac{k+1-t}{n+2+k-t}.$$

Note that

$$(n+1+k-s)(k+1-t) - (k-s)(n+2+k-t) = (n+1)(s+1-t) < 0.$$

The result thus follows immediately. ■

Now we fix m, n, k and apply Lemma 5.4 to the product $\left(\prod_{j=1}^{n} \frac{n+1+k-\lambda_j}{k-\lambda_j}\right)$ in the right hand side of (5.3). Suppose $\lambda_{j_1} < 0$ for some $1 \le j_1 \le n$. Since $\sum_{j=1}^{n} \lambda_j = m - k \ge 1$, there is some $1 \le j_2 \le n$ such that $\lambda_{j_2} > 0$. We change $\binom{n+1+k-\lambda_{j_1}}{k-\lambda_{j_1}}\binom{n+1+k-\lambda_{j_2}}{k-\lambda_{j_2}}$ to $\binom{n+1+k-(\lambda_{j_1}+1)}{k-(\lambda_{j_1}+1)}\binom{n+1+k-(\lambda_{j_2}-1)}{k-(\lambda_{j_2}-1)}$, that is, use $\lambda_{j_1}+1$ as the new λ_{j_1} and use $\lambda_{j_2}-1$ as the new λ_{j_2}. Then the sum $\sum_{j=1}^{n} \lambda_j$ is still equal to $m-k$, and the value of the right hand side of (5.3) becomes larger. We keep doing this if there is some $\lambda_j < 0$ for some $1 \le j \le n$. Then we finally get $0 \le \lambda_j \le k$ for all $1 \le j \le n$ and $m-k = \sum_{j=1}^{n} \lambda_j$ still holds; and the process will not make the value of the right hand side of (5.3) smaller. So we only need to prove (5.3) with the additional condition $\lambda_j \ge 0$ for all $1 \le j \le n$.

From now on, we will assume $\lambda_j \ge 0$ for all $1 \le j \le n$. As $\sum_{j=1}^{n} \lambda_j = m-k \ge 1$, without loss of generality, we can further assume $\lambda_1 \ge 1$.

Step 2. We show that it is actually sufficient to prove (5.3) for $\lambda_1 = 1$ and $\lambda_2 = \cdots = \lambda_n = 0$.

For simplicity, we denote the right hand side of (5.3) by F:

$$F(n,k,\lambda) := m\binom{n+k+m}{n}\prod_{j=1}^{n}\binom{n+1+k-\lambda_j}{k-\lambda_j}, \tag{5.5}$$

where $m = k + \sum_{j=1}^{n} \lambda_j$ and $\lambda = (\lambda_1, \cdots, \lambda_n)$. The function F has the following property.

Lemma 5.5. Suppose $n, k \geq 1$ and $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$ with $0 \leq \lambda_j \leq k$ for each $1 \leq j \leq n$. If $\lambda_{j_1} \geq 1$ for some $1 \leq j_1 \leq n$, then

$$F(n, k, \lambda) \leq F(n, k, \lambda - e_{j_1}), \tag{5.6}$$

where $e_{j_1} = (0, \cdots, 0, 1, 0 \cdots, 0)$ is the unit vector along the j_1-th direction in $\mathbb{R}^n$.

Consequently, if $\lambda_1 \geq 1$, and all other λ_j's are nonnegative, then

$$F(n, k, \lambda) \leq F(n, k, e_1).$$

Proof. We cancel the common combinatorial factors in (5.6), and write it as

$$m\binom{n+k+m}{n}\binom{n+1+k-\lambda_{j_1}}{k-\lambda_{j_1}} \leq (m-1)\binom{n+k+m-1}{n}\binom{n+2+k-\lambda_{j_1}}{k-\lambda_{j_1}+1},$$

where $m = k + \sum_{j=1}^{n} \lambda_j$.

By expanding the remaining combinatorial terms and further canceling common factors, we deduce that (5.6) is equivalent to

$$m \cdot \frac{n+k+m}{k+m} \leq (m-1) \cdot \frac{n+2+k-\lambda_{j_1}}{k-\lambda_{j_1}+1}.$$

Clearly, the right hand side is increasing with respect to $1 \leq \lambda_{j_1} \leq k$. Thus, it is sufficient to prove

$$m \cdot \frac{n+k+m}{k+m} \leq (m-1) \cdot \frac{n+k+1}{k}, \tag{5.7}$$

A straightforward computation shows that

$$\begin{aligned}
(5.7) \iff & m \cdot \frac{n}{k+m} + 1 \leq (m-1) \cdot \frac{n+1}{k} \\
\iff & mnk + k^2 + mk \leq mnk + mk - nk - k + (m^2 - m)(n+1) \\
\iff & k^2 + nk + k \leq (m^2 - m)(n+1).
\end{aligned}$$

The last inequality follows immediately by the fact $m = k + \sum_{j=1}^{n} \lambda_j \geq k + 1$. So the proof is finished. ■

Thus, with the help of Lemma 5.5, it suffices to prove (5.3) for $\lambda_1 = 1$, $\lambda_2 = \cdots = \lambda_n = 0$ and $m = k+1$. That is, we only need to show that

$$k\binom{n+k}{k}^{n+2} > (k+1)\binom{n+2k+1}{n}\binom{n+k}{k-1}\binom{n+k+1}{k}^{n-1}. \tag{5.8}$$

Step 3. We complete the proof of Lemma 5.3, by proving (5.8) for any $n \geq 2$, $k \geq 1$. Let us further simplify (5.8) to the following equivalent inequalities.

$$\begin{aligned}(5.8) \iff & k \cdot \frac{(n+k)!^2}{n!^2\, k!^2} > (k+1) \cdot \frac{(n+2k+1)!}{n!\,(2k+1)!} \cdot \frac{k}{n+1} \cdot \frac{(n+k+1)^{n-1}}{(n+1)^{n-1}} \\ \iff & \frac{(n+1)^n}{(k+1)(n+k+1)^{n-1}} \cdot \frac{(n+k)!^2\,(2k+1)!}{n!\,k!^2\,(n+2k+1)!} > 1.\end{aligned}$$

Denote the left hand side term in the last inequality by $L(n,k)$, that is,

$$L(n,k) := \frac{(n+1)^n}{(k+1)(n+k+1)^{n-1}} \cdot \frac{(n+k)!^2\,(2k+1)!}{n!\,k!^2\,(n+2k+1)!}.$$

It remains to prove $L(n,k) > 1$ for $n \geq 2$, $k \geq 1$.

Lemma 5.6. Given nonnegative integers n, k, we have

$$L(n,k) \leq L(n+1,k). \tag{5.9}$$

Proof. Set $Q(n,k) := L(n+1,k)/L(n,k)$. Then

$$Q(n,k) = \frac{(n+2)^{n+1}}{(n+1)^{n+1}} \cdot \frac{(n+k+1)^{n+1}}{(n+k+2)^n(n+2k+2)}.$$

Regarding k as a real variable in $[0,\infty)$, we take the logarithmic derivative of $Q(n,k)$ with respect to k:

$$\begin{aligned}\frac{\partial \log Q(n,k)}{\partial k} =& \frac{n+1}{n+k+1} - \frac{n}{n+k+2} - \frac{2}{n+2k+2} \\ =& \frac{2n+k+2}{(n+k+1)(n+k+2)} - \frac{2}{n+2k+2} \\ =& \frac{nk}{(n+k+1)(n+k+2)(n+2k+2)}.\end{aligned}$$

It follows that for given $n \geq 0$, $Q(n,k)$ is increasing with respect to $k \geq 0$. Thus, for $n, k \geq 0$, we have

$$Q(n,k) \geq Q(n,0) = 1.$$

That is the desired result. ■

Now, for $n \geq 2, k \geq 1$, Lemma 5.6 yields that

$$\begin{aligned} L(n,k) \geq L(2,k) &= \frac{3^2}{(k+1)(k+3)} \cdot \frac{(k+2)!^2\,(2k+1)!}{2!\,k!^2\,(2k+3)!} \\ &= \frac{9(k+1)^2(k+2)^2}{2(k+1)(k+3)(2k+2)(2k+3)} \\ &= \frac{9(k^2+4k+4)}{4(2k^2+9k+9)} > 1. \end{aligned}$$

The proof of Lemma 5.3 is now complete. ■

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