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Khintchine-type theorems for values of subhomogeneous functions at integer points

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Abstract
This work has been motivated by recent papers that quantify the density of values of generic quadratic forms and other polynomials at integer points, in particular ones that use Rogers' second moment estimates. In this paper, we establish such results in a very general framework. Given any subhomogeneous function (a notion to be defined) $f : \mathbb{R}^n \to \mathbb{R}$, we derive a necessary and sufficient condition on the approximating function ψ for guaranteeing that a generic element $f \circ g$ in the G-orbit of f is ψ-approximable; that is, $|f \circ g(\mathbf{v})| \le \psi(\|\mathbf{v}\|)$ for infinitely many $\mathbf{v} \in \mathbb{Z}^n$. We also deduce a sufficient condition in the case of uniform approximation. Here G can be any closed subgroup of $\mathrm{ASL}_n(\mathbb{R})$ satisfying certain axioms that allow for the use of Rogers-type estimates.

Keywords Oppenheim conjecture · Metric Diophantine approximation · Geometry of numbers · Counting lattice points · ψ-Approximability

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1 Introduction

Let f be an indefinite and nondegenerate quadratic form in $n \geq 3$ real variables that is not a real multiple of a quadratic form with rational coefficients. The Oppenheim–Davenport Conjecture, proved in a breakthrough paper by Margulis [29], states that 0 is an accumulation point of $f(\mathbb{Z}^n)$: in other words, for any $\varepsilon > 0$,

$$\text{there exist infinitely many } \mathbf{v} \in \mathbb{Z}^n \text{ with } |f(\mathbf{v})| \leq \varepsilon. \tag{1.1}$$

Margulis' approach, via the dynamics of unipotent flows on homogeneous spaces, was not effective: given $\varepsilon > 0$, it did not give any bound on the length of the shortest integer vector $\mathbf{v}$ for which (1.1) holds. Effective versions were later established for any $n \geq 5$ [6,10] using methods from analytic number theory, but these methods are not applicable to the most difficult case $n = 3$. One of the difficulties in establishing effective variants of Margulis' Theorem is proving the aforesaid bounds for *any* quadratic form f as above. This difficulty is attenuated when one seeks to prove such bounds only for *generic* f as above (with respect to the natural measure class). Recently, such effective generic results have been proved both in the original setting of quadratic forms and in related settings of other homogeneous polynomials; for example, see [4,5,8,12,15–18,23,28,30].

In order to describe some of the aforementioned results in greater detail and to lay the foundation for our own work in the present paper, let us introduce some definitions. Given a norm ν on $\mathbb{R}^n$, a function $f : \mathbb{R}^n \to \mathbb{R}$, and a function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ (to which we shall refer as an *approximating function*), let us say that f is (ψ, ν)-*approximable* if ε in the right-hand side of (1.1) can be replaced by $\psi\big(\nu(\mathbf{x})\big)$. Equivalently, f is (ψ, ν)-approximable if card $\big(\mathbb{Z}^n \cap A_{f,\psi,\nu}\big) = \infty$, where

$$A_{f,\psi,\nu} := \left\{\mathbf{x} \in \mathbb{R}^n : |f(\mathbf{x})| \leq \psi\big(\nu(\mathbf{x})\big)\right\}. \tag{1.2}$$

The above definition is a way to quantify the density of $f(\mathbb{Z}^n)$ at 0 in terms of the approximating function ψ. We note that this definition is dependent also on the chosen norm; under some mild assumptions, however, we shall see that this is not significant for our purposes. Every specific example that we consider in this paper will satisfy these mild assumptions. It is also clear that the definition of (ψ, ν)-approximability also makes sense when ψ is defined only for all sufficiently large nonnegative real numbers; however, it is convenient to assume that the domain of ψ is all of $\mathbb{R}_{\geq 0}$ by arbitrarily extending the function, if necessary. We shall sometimes tacitly do so.

Consider first the special case

$$\psi(z) = \varphi_s(z) := z^{-s}, \tag{1.3}$$

where $s \geq 0$ is arbitrary. Let ν be any norm on $\mathbb{R}^n$. It was recently shown by Athreya and Margulis [4, Theorem 1.1] that for every $p, q \in \mathbb{Z}_{\geq 1}$ with $p+q = n \geq 3$, almost every (with respect to the natural measure class) nondegenerate real quadratic form Q of signature (p, q) is (φ_s, ν)-approximable for every $s < n-2$. Previously this was established by Ghosh, Gorodnik, and Nevo for $n = 3$ [15]; see also the work of Bourgain [8], which deals with generic ternary diagonal forms. Similar results were obtained in [4,16,23]. For instance [23, Theorem 1] generalizes [4, Theorem 1.1] as follows: let

$$f(\mathbf{x}) := \sum_{j=1}^{p} x_j^d - \sum_{k=p+1}^{n} x_k^d, \tag{1.4}$$

where $p, q \in \mathbb{Z}_{\geq 1},\ \ p+q = n \geq 3,$ and $0 < d < n$ is an even integer; then for any real s with $0 < s < n-d$, almost every polynomial in the $\mathrm{SL}_n(\mathbb{R})$-orbit of f is (φ_s, ν)-approximable.

We note that in all the aforementioned papers, a property stronger than ψ-approximability has been established. Let us denote $\mathbb{Z}^n_{\neq 0} := \mathbb{Z}^n \smallsetminus \{\mathbf{0}\}$, and say that f is *uniformly* (ψ, ν)*-approximable* if for every sufficiently large $T \in \mathbb{R}_{>0}$, there exists $\mathbf{v} \in \mathbb{Z}^n_{\neq 0}$ with

$$\nu(\mathbf{v}) \leq T \text{ and } |f(\mathbf{v})| \leq \psi(T).$$

In other words, if for any $\varepsilon, T \in \mathbb{R}_{>0}$, we set

$$B_{f,\varepsilon,\nu,T} := \left\{\mathbf{x} \in \mathbb{R}^n : |f(\mathbf{x})| \leq \varepsilon \text{ and } \nu(\mathbf{x}) \leq T\right\} = A_{f,\varepsilon,\nu} \cap \{\mathbf{x} \in \mathbb{R}^n : \nu(\mathbf{x}) \leq T\}$$

(here, the ε in $A_{f,\varepsilon,\nu}$ stands for the constant function $\psi \equiv \varepsilon$), then f is uniformly (ψ, ν)-approximable if and only if for every sufficiently large $T \in \mathbb{R}_{>0}$, the set $B_{f,\psi(T),\nu,T}$ contains a nonzero integer vector. See, for instance, [39, §1.1] for a discussion of asymptotic versus uniform approximation in metric number theory, and [25,26] for some recent results in uniform metric Diophantine approximation. ("Asymptotic approximation" is the sort of approximation that we have simply called "approximation" so far in this paper.) It is easy to verify that if the approximating function ψ is nonincreasing and f does not represent 0 nontrivially, then the uniform (ψ, ν)-approximability of f implies its (ψ, ν)-approximability. All the aforementioned papers actually provide conditions sufficient for the uniform (φ_s, ν)-approximability of generic elements of the $\mathrm{SL}_n(\mathbb{R})$-orbit of a given polynomial. For instance, [23, Theorem 1] states that for f as in (1.4) and for any $s < n-d$, almost every polynomial in the $\mathrm{SL}_n(\mathbb{R})$-orbit of f is uniformly (φ_s, ν)-approximable.

In this paper, we establish a generalization of the aforementioned results under the mild conditions on f and ψ to which we previously alluded. Furthermore, our methods allow us to generalize to the case of vector-valued functions with no additional effort. We now introduce these conditions, which will require some more notation and terminology. Now and hereafter, we shall denote by n an arbitrary element of $\mathbb{Z}_{\geq 2}$ and by ℓ an arbitrary element of $\mathbb{Z}_{\geq 1}$.

Definition 1.1 We define a non-strict partial order $\leq$ on $\mathbb{R}^\ell$ as follows. For any $\mathbf{x} = (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell$ and any $\mathbf{y} = (y_1, \ldots, y_\ell) \in \mathbb{R}^\ell$, write $\mathbf{x} \leq \mathbf{y}$ if and only if for each $j \in \{1, \ldots, \ell\}$, one has $x_j \leq y_j$.

Definition 1.2 Let

$$f = (f_1, \ldots, f_\ell) : \mathbb{R}^n \to \mathbb{R}^\ell \quad \text{and} \quad \psi = (\psi_1, \ldots, \psi_\ell) : \mathbb{R}_{\geq 0} \to (\mathbb{R}_{>0})^\ell$$

be given, and let ν be an arbitrary norm on $\mathbb{R}^n$.

- We abuse notation and write $|f|$ to denote the function $(|f_1|, \ldots, |f_\ell|) : \mathbb{R}^n \to \mathbb{R}^\ell$.
- We define $A_{f,\psi,\nu} := \left\{\mathbf{x} \in \mathbb{R}^n : |f(\mathbf{x})| \leq \psi\big(\nu(\mathbf{x})\big)\right\}$.
- For any $T \in \mathbb{R}_{>0}$ and any $\boldsymbol{\varepsilon} \in (\mathbb{R}_{>0})^\ell$, we define

$$B_{f,\boldsymbol{\varepsilon},\nu,T} := \left\{\mathbf{x} \in \mathbb{R}^n : |f(\mathbf{x})| \leq \boldsymbol{\varepsilon} \text{ and } \nu(\mathbf{x}) \leq T\right\}.$$

- We say that f is (ψ, ν)-*approximable* if $A_{f,\psi,\nu} \cap \mathbb{Z}^n$ has infinite cardinality.
- We say that f is *uniformly* (ψ, ν)-*approximable* if $B_{f,\psi(T),\nu,T} \cap \mathbb{Z}^n_{\neq 0} \neq \varnothing$ for each sufficiently large $T \in \mathbb{R}_{>0}$.
- We say that f is *subhomogeneous* if f is Borel measurable and there exists a constant $d = d_f \in \mathbb{R}_{>0}$ such that for each $t \in (0, 1)$ and each $\mathbf{x} \in \mathbb{R}^n$ one has $|f(t\mathbf{x})| \leq t^d |f(\mathbf{x})|$.
- We say that ψ is *regular* if ψ is Borel measurable and there exist real numbers $a = a_\psi \in \mathbb{R}_{>1}$ and $b = b_\psi \in \mathbb{R}_{>0}$ such that for each $z \in \mathbb{R}_{>0}$ one has $b_\psi \psi(z) \leq \psi(a_\psi z)$.
- We say that ψ is *nonincreasing* if each component function of ψ is nonincreasing in the usual sense.

Note that subhomogeneity is our only assumption on f; in particular, f need not be a polynomial or even continuous. See [14, Definition 2.2] for another instance of using the regularity assumption on the approximating function in the context of Diophantine approximation.

Now and henceforth, we shall denote by m the Lebesgue measure on a Euclidean space of any dimension. (The dimension will be clear from the context.) The following is a special case of our main results, Theorems 3.4 and 3.8.

Theorem 1.3 *Let η and ν be arbitrary norms on $\mathbb{R}^n$, let $f : \mathbb{R}^n \to \mathbb{R}^\ell$ be subhomogeneous, and let $\psi : \mathbb{R}_{\geq 0} \to (\mathbb{R}_{>0})^\ell$ be regular and nonincreasing. Then*

(i) *If $m\left(A_{f,\psi,\eta}\right)$ is finite (resp., infinite), then $f \circ g$ is (ψ, ν)-approximable for Haar almost no (resp., almost every) $g \in \mathrm{SL}_n(\mathbb{R})$.*

(ii) *Suppose that $\sum_{k=1}^{\infty} \dfrac{1}{m(B_{f,\psi(2^k),\eta,2^k})} < \infty$; then $f \circ g$ is uniformly (ψ, ν)-approximable for Haar almost every $g \in \mathrm{SL}_n(\mathbb{R})$.*

Part (i) is consistent with many other results in Diophantine approximation, where the finitude versus infinitude of the volume of a certain set provides a necessary and sufficient condition for the existence of finitely versus infinitely many solutions of certain inequalities almost everywhere. That being said, it seems remarkable that so very little needs to be assumed in order to have such a result. Moreover, a byproduct of Theorem 1.3(i) is that, under the above assumptions on f and ψ, the finitude versus infinitude of $m\left(A_{f,\psi,\nu}\right)$ does not depend on the choice of the norm ν. This is stated explicitly in Lemma 3.1 below.

We shall show in Sect. 4 that Theorem 1.3 implies the following result, a special case of Corollary 4.1 that concerns the approximability of a function that is essentially a generalized indefinite quadratic form:

Corollary 1.4 *Let $d \in \mathbb{R}_{\geq 1}$, and fix any $p, q \in \mathbb{Z}_{\geq 1}$ with $p + q = n$. Let ν be a norm on $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by*

$$f(\mathbf{x}) := \sum_{j=1}^{p} \left|x_j\right|^d - \sum_{k=p+1}^{n} |x_k|^d . \tag{1.5}$$

Let $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ be regular and nonincreasing. The following then holds:

(i) *If $\int_1^\infty \psi(z) z^{n-(d+1)}\, dz$ is finite (resp., infinite), then $f \circ g$ is (ψ, ν)-approximable for almost no (resp., almost every) $g \in \mathrm{SL}_n(\mathbb{R})$.*
(ii) *Suppose that*

$$\sum_{k=1}^{\infty} \frac{1}{k\psi(2^k)} < \infty \quad \text{if } d = n;$$

$$\sum_{k=1}^{\infty} \frac{1}{2^{(n-d)k}\psi(2^k)} < \infty \quad \text{if } d < n.$$

Then $f \circ g$ is uniformly (ψ, ν)-approximable for almost every $g \in \mathrm{SL}_n(\mathbb{R})$.

Since $d \geq 1$ in (1.5) is assumed to be arbitrary as opposed to an even integer as in (1.4), the above corollary generalizes the aforementioned work of Athreya–Margulis and Kelmer–Yu. In particular, we can conclude that for ν and f as in Corollary 1.4 and for almost every $g \in \mathrm{SL}_n(\mathbb{R})$, the function $f \circ g$ is

- (φ_{n-d}, ν)-approximable (the critical exponent case), and
- uniformly (ψ, ν)-approximable, where $\psi(z) = \dfrac{(\log z)^{1+\varepsilon}}{z^{n-d}}$ for an arbitrary $\varepsilon > 0$ (the critical exponent case with a logarithmic correction).

We note that for any regular and nonincreasing $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ and any $d \in \mathbb{R}_{>n}$, the integral in Corollary 1.4(i) converges because it is majorized by

$$\psi(1) \int_1^\infty z^{n-(d+1)}\, dz < \infty;$$

in this case, almost every element in the $\mathrm{SL}_n(\mathbb{R})$ orbit of f in (1.5) is not (ψ, ν)-approximable and hence is not uniformly (ψ, ν)-approximable. Other applications of Theorem 1.3 can be found in Sect. 4.

Historically, there have been several different approaches to this circle of problems. In particular, the papers [16] and [15] continue the line of thought behind Margulis' proof of the Oppenheim Conjecture, reducing the problem to studying the action of the stabilizer of the function f on the space of lattices, and using ergodic properties of the action to establish quantitative density of $f(\mathbb{Z}^n)$. In the present paper, however, we follow the methods of [4,23], which have their origin in the work of Rogers and Schmidt [31,32,34] and involve studying the asymptotics of the number of lattice points of generic lattices in families of subsets of $\mathbb{R}^n$. One of the advantages of the approach taken in this paper is that it makes it possible to significantly generalize the setting. In particular, one can work with vector-valued functions $f = (f_1, \ldots, f_\ell) : \mathbb{R}^n \to \mathbb{R}^\ell$, and can consider specific subsets of $\mathbb{Z}^n$, for example the set of all primitive integer points $\mathbb{Z}^n_{\mathrm{pr}}$.

It is also worth mentioning that the aforementioned papers were dealing with the density of $f(\mathbb{Z}^n)$ in $\mathbb{R}$, not just at zero. In other words, for various examples of polynomials f, these papers presented conditions depending on $s \in \mathbb{R}_{>0}$ sufficient for showing that for every $\xi \in \mathbb{R}$, almost every $g \in \mathrm{SL}_n(\mathbb{R})$, and every sufficiently large $T \in \mathbb{R}_{>0}$ there exists $\mathbf{v} \in \mathbb{Z}^n_{\neq 0}$ for which

$$\nu(\mathbf{v}) \leq T \text{ and } |\xi - f(g\mathbf{v})| \leq T^{-s}. \tag{1.6}$$

See, for instance, the two recent papers [17,18] of Ghosh–Kelmer–Yu. We discuss a possible approach to this case, the inhomogeneous one, in Sect. 5.1, and plan to address it in a forthcoming paper.

Theorems 3.4 and 3.8, our main results, are essentially a generalization of Theorem 1.3 to a class of groups that act on $\mathbb{R}^n$ and satisfy certain axioms, which $\mathrm{SL}_n(\mathbb{R})$ happens to satisfy. Another example of such a group is $\mathrm{Sp}_n(\mathbb{R})$, the group of symplectic linear isomorphisms of $\mathbb{R}^n$ when $n \in \mathbb{Z}_{>0}$ is even, or the group $\mathrm{ASL}_n(\mathbb{R}) := \mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ of unimodular affine isomorphisms of $\mathbb{R}^n$. For the infinite measure case of Theorem 3.4, we actually obtain a quantitative version when we stipulate that the element g lie in an arbitrary fixed compactum of the group.

Let us briefly delineate the structure of this paper. In Sect. 2, we define a class of groups that satisfy certain axioms conducive to proving our main Diophantine results. The utility of these axioms is that they enable us to prove generic counting results in certain spaces of lattices; our approach is a generalization of the method developed by Schmidt in [34]. Using the axioms on f and ψ that have already been introduced, we then proceed in Sect. 3 to transfer the results concerning the space of lattices to those concerning Diophantine approximation. In Sect. 4, we then discuss specific examples of subhomogeneous f to obtain conditions for approximability in terms of the convergence or divergence of certain infinite series or improper integrals, as in Corollary 1.4.

Possible examples with which we do not concern ourselves here abound: one can, for example, take f to be a system of several quadratic forms or a pair consisting of a quadratic and a linear form, as in the papers [5,19,20]. It also appears very likely that one could use [21, Proposition 5.2 and Theorem 6.1] to prove S-arithmetic analogues over $\mathbb{Q}$ of the results of this paper. Further possible extensions and open questions are mentioned in Sect. 5.

2 Counting results for generic lattices

Let G be a closed subgroup of $\mathrm{ASL}_n(\mathbb{R})$, and let Γ be the subgroup of G defined by

$$\Gamma := \{g \in G : g\mathbb{Z}^n = \mathbb{Z}^n\}. \tag{2.1}$$

Now and hereafter, we assume that Γ is a lattice in G; that is, Γ is a discrete subgroup of G whose covolume in G is finite. (In each particular example of such a group G that we shall consider, the subgroup Γ will indeed be a lattice in G.) Set $X := G/\Gamma$. Notice that we then have a well-defined bijection between X and $\{g\mathbb{Z}^n : g \in G\}$ that is given by $g\Gamma \longleftrightarrow g\mathbb{Z}^n$. We therefore identify X with $\{g\mathbb{Z}^n : g \in G\}$, and we equip X with the quotient topology.

Now let $\mathcal{P}$ be any Γ-invariant subset of $\mathbb{Z}^n$. Given any $\Lambda \in X$, fix any $g \in G$ for which $\Lambda = g\mathbb{Z}^n$; then define $\Lambda_{\mathcal{P}} := g\mathcal{P}$. Then $\Lambda_{\mathcal{P}}$ is well-defined because $\mathcal{P}$ is Γ-invariant.

Given any function $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, we define its *$\mathcal{P}$-Siegel transform* $\widehat{f}^{\mathcal{P}} : X \to [0, \infty]$ by

$$\widehat{f}^{\mathcal{P}}(\Lambda) := \sum_{\mathbf{v} \in \Lambda_{\mathcal{P}}} f(\mathbf{v}).$$

We equip G with the left Haar measure μ_G that is normalized so that any fundamental Borel set in G for X has μ_G-measure equal to 1. We then let μ_X be the left G-invariant Borel probability measure on X that is induced from μ_G in the canonical manner. Note that if f is Borel measurable, then $\widehat{f}^{\mathcal{P}}$ is μ_X-measurable.

Let us now introduce the axioms on G to which we alluded at the end of the introduction.

Definition 2.1 Let G and $\mathcal{P}$ be as above.

(i) We say that G is of *$\mathcal{P}$-Siegel type* if there exists a constant $c = c_{\mathcal{P}} \in \mathbb{R}_{>0}$ such that for any bounded and compactly supported Borel measurable function $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ we have

$$\int_X \widehat{f}^{\mathcal{P}} \, d\mu_X = c \int_{\mathbb{R}^n} f \, dm. \tag{2.2}$$

(ii) Let $r \in \mathbb{R}_{\geq 1}$ be given. We say that G is of *$(\mathcal{P}, r)$-Rogers type* if there exists a constant $D = D_{\mathcal{P},r} \in \mathbb{R}_{>0}$ such that for any bounded Borel $E \subset \mathbb{R}^n$ with $m(E) > 0$ we have

$$\left\| \widehat{\mathbb{1}_E}^{\mathcal{P}} - \left(\int_X \widehat{\mathbb{1}_E}^{\mathcal{P}} \, d\mu_X \right) \mathbb{1}_X \right\|_r \leq D \cdot m(E)^{1/r}. \tag{2.3}$$

Remark 2.2 (i) The definition of $\mathcal{P}$-Siegel type is nothing more than the assertion that a variant of the Siegel Mean Value Theorem—-first proved by Siegel in the context of $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ in the seminal paper [36]—-holds for the $\mathcal{P}$-Siegel transform on X. Using Lebesgue's Monotone Convergence Theorem, it is easy to see that if G is of $\mathcal{P}$-Siegel type, then (2.2) holds for any L^1 function $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. Similarly, if there exists $r \in [1, \infty)$ for which G is of $(\mathcal{P}, r)$-Rogers type, then (2.3) is satisfied for any (not necessarily bounded) Borel $E \subset \mathbb{R}^n$ of finite measure.

(ii) Assuming that G is of $\mathcal{P}$-Siegel type, the assumption of $(\mathcal{P}, 2)$-Rogers type is equivalent to the assumption that for any bounded Borel $E \subset \mathbb{R}^n$, the variance of the random variable $\widehat{\mathbb{1}_E}^{\mathcal{P}}$ is bounded from above by a uniform scalar multiple of the expectation of $\widehat{\mathbb{1}_E}^{\mathcal{P}}$. This condition was used by Schmidt to great effect in [34]; see a remark after Theorem 2.9 below. The definition of $(\mathcal{P}, r)$-Rogers type for arbitrary $r \in [1, \infty)$ is a natural generalization of this condition.

(iii) Notice that if G is of $\mathcal{P}$-Siegel type, then G is of $(\mathcal{P}, 1)$-Rogers type.

Before we provide some examples of groups that satisfy the various Siegel and Rogers type axioms, let us record and prove some simple facts that will be helpful going forward.

Proposition 2.3 *(Logarithmic Convexity of L^p Norms) Let (Y, μ) be a measure space. Let $r, t \in \mathbb{R}_{\geq 1}$ and $\theta \in (0, 1)$ be arbitrary. Set $s := \left(\frac{\theta}{r} + \frac{1-\theta}{t} \right)^{-1} \geq 1$. For each $f \in L^r(Y, \mu) \cap L^t(Y, \mu)$ we then have*

$$\|f\|_s \leq \|f\|_r^{\theta} \cdot \|f\|_t^{1-\theta}.$$

Proof This is a well-known special case of the Riesz–Thorin interpolation theorem. For proofs of this special case and the general theorem, see [13, Proposition 7.37] and [13, Theorem 7.38], respectively. □

Corollary 2.4 *Suppose that the group G is of $(\mathcal{P}, 1)$-Rogers type and that there exists $s \in \mathbb{R}_{>1}$ for which G is of $(\mathcal{P}, s)$-Rogers type. Then for each $r \in (1, s)$ the group G is of $(\mathcal{P}, r)$-Rogers type.*

Proof Let $r \in (1, s)$. Fix $\theta \in (0, 1)$ for which $\frac{1}{r} = \frac{\theta}{1} + \frac{1-\theta}{s}$. Let $D_1 = D_{\mathcal{P},1}$ and $D_s = D_{\mathcal{P},s}$ be as in Definition 2.1. Let $E \subset \mathbb{R}^n$ be a bounded Borel set. The foregoing proposition implies

$$\begin{aligned} \left\| \widehat{\mathbb{1}_E}^{\mathcal{P}} - \left(\int_X \widehat{\mathbb{1}_E}^{\mathcal{P}} d\mu_X \right) \mathbb{1}_X \right\|_r &\leq \left\| \widehat{\mathbb{1}_E}^{\mathcal{P}} - \left(\int_X \widehat{\mathbb{1}_E}^{\mathcal{P}} d\mu_X \right) \mathbb{1}_X \right\|_1^{\theta} \cdot \left\| \widehat{\mathbb{1}_E}^{\mathcal{P}} - \left(\int_X \widehat{\mathbb{1}_E}^{\mathcal{P}} d\mu_X \right) \mathbb{1}_X \right\|_s^{1-\theta} \\ &\leq D_1^{\theta} m(E)^{\theta} \cdot D_s^{1-\theta} m(E)^{\frac{1-\theta}{s}} = D_1^{\theta} D_s^{1-\theta} m(E)^{1/r}. \end{aligned}$$

□

In this paper, the examples of G that we shall consider are $\mathrm{ASL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$, and also $\mathrm{Sp}_n(\mathbb{R})$ when n is even. When $G = \mathrm{ASL}_n(\mathbb{R})$, it is clear that the only Γ-invariant subset of $\mathbb{Z}^n$ is $\mathbb{Z}^n$ itself. If $G = \mathrm{SL}_n(\mathbb{R})$ or $G = \mathrm{Sp}_n(\mathbb{R})$ (for even n only in the latter case), then Γ acts transitively on $\mathbb{Z}^n_{\mathrm{pr}}$; in these cases, two obvious choices of $\mathcal{P}$ are therefore $\mathcal{P} = \mathbb{Z}^n_{\mathrm{pr}}$ and $\mathcal{P} = \mathbb{Z}^n_{\neq 0}$. We now record the various Siegel and Rogers axioms that the groups just mentioned satisfy.

In the following theorem and thereafter, ζ denotes the Euler–Riemann zeta function. Let us mention that the following theorem is a compilation of results that are by now standard in the literature.

Theorem 2.5 (i) *The group* $\mathrm{ASL}_n(\mathbb{R})$ *is of* $\mathbb{Z}^n$*-Siegel type with* $c_{\mathbb{Z}^n} = 1$.
(ii) *The group* $\mathrm{SL}_n(\mathbb{R})$ *is of* $\mathbb{Z}^n_{\mathrm{pr}}$*-Siegel type with* $c_{\mathbb{Z}^n_{\mathrm{pr}}} = 1/\zeta(n)$ *and of* $\mathbb{Z}^n_{\neq 0}$*-Siegel type with* $c_{\mathbb{Z}^n_{\neq 0}} = 1$.
(iii) *Suppose n is even. Then the group* $\mathrm{Sp}_n(\mathbb{R})$ *is of* $\mathbb{Z}^n_{\mathrm{pr}}$*-Siegel type with* $c_{\mathbb{Z}^n_{\mathrm{pr}}} = 1/\zeta(n)$ *and of* $\mathbb{Z}^n_{\neq 0}$*-Siegel type with* $c_{\mathbb{Z}^n_{\neq 0}} = 1$.

Proof (i) From Lemma 3 of [2] and the ensuing discussion therein, we see that this claim holds with $c_{\mathbb{Z}^n} = 1$.
(ii) By the main theorem in [36] and [36, (25)], it follows that for every bounded and compactly supported Riemann integrable function $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, we have

$$\int_{\mathbb{R}^n} f\, dm = \int_X \widehat{f}^{\mathbb{Z}^n_{\neq 0}}\, d\mu_X$$

and

$$\int_{\mathbb{R}^n} f\, dm = \zeta(n) \int_X \widehat{f}^{\mathbb{Z}^n_{\mathrm{pr}}}\, d\mu_X.$$

The desired results now follow from Lebesgue's Monotone Convergence Theorem.
(iii) After making the requisite changes in notation, the assertion that $\mathrm{Sp}_n(\mathbb{R})$ is of $\mathbb{Z}^n_{\mathrm{pr}}$-Siegel type with $c_{\mathbb{Z}^n_{\mathrm{pr}}} = 1/\zeta(n)$ is precisely the content of [22, (0.6)]. Let $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a compactly supported Borel measurable function. For any $k \in \mathbb{Z}$, define $f_k : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ by $f_k(\mathbf{x}) := f(k\mathbf{x})$. Then for any $\Lambda \in X$,

$$\widehat{f}^{\mathbb{Z}^n_{\neq 0}}(\Lambda) = \sum_{\mathbf{v} \in \Lambda \smallsetminus \{\mathbf{0}\}} f(\mathbf{v}) = \sum_{k=1}^{\infty} \sum_{\mathbf{v} \in \Lambda_{\mathrm{pr}}} f(k\mathbf{v}) = \sum_{k=1}^{\infty} \sum_{\mathbf{v} \in \Lambda_{\mathrm{pr}}} f_k(\mathbf{v}) = \sum_{k=1}^{\infty} \widehat{f_k}^{\mathbb{Z}^n_{\mathrm{pr}}}(\Lambda).$$

It is now easy to conclude that $\mathrm{Sp}_n(\mathbb{R})$ is of $\mathbb{Z}^n_{\neq 0}$-Siegel type with $c_{\mathbb{Z}^n_{\neq 0}} = 1$. □

Theorem 2.6 (i) *The group* $\mathrm{ASL}_n(\mathbb{R})$ *is of* $(\mathbb{Z}^n, 2)$*-Rogers type.*
(ii) *Suppose* $n \geq 3$*. Then* $\mathrm{SL}_n(\mathbb{R})$ *is of* $\left(\mathbb{Z}^n_{\mathrm{pr}}, 2\right)$*-Rogers type and of* $(\mathbb{Z}^n_{\neq 0}, 2)$*-Rogers type.*

(iii) *Suppose n is even and $n \geq 4$. Then* $\mathrm{Sp}_n(\mathbb{R})$ *is of* $\left(\mathbb{Z}^n_{\mathrm{pr}}, 2\right)$*-Rogers type and of* $(\mathbb{Z}^n_{\neq 0}, 2)$*-Rogers type.*

Proof (i) The result [2, Lemma 4] shows that for any bounded Borel $E \subset \mathbb{R}^n$, we have

$$\left\| \widehat{\mathbb{1}_E}^{\mathbb{Z}^n} - \left(\int_X \widehat{\mathbb{1}_E}^{\mathbb{Z}^n} \, d\mu_X \right) \mathbb{1}_X \right\|_2 = m(E)^{1/2}.$$

(ii) Let $E \subset \mathbb{R}^n$ be bounded and Borel. Since $\int_X \widehat{\mathbb{1}_E}^{\mathbb{Z}^n_{\mathrm{pr}}} \, d\mu_X = \frac{1}{\zeta(n)} m(E)$, a simple change of notation and a routine algebraic manipulation of [22, (0.2)] yield

$$\left\| \widehat{\mathbb{1}_E}^{\mathbb{Z}^n_{\mathrm{pr}}} - \frac{m(E)}{\zeta(n)} \mathbb{1}_X \right\|_2 \leq \sqrt{\frac{2}{\zeta(n)}} \, m(E)^{1/2}.$$

Hence, $\mathrm{SL}_n(\mathbb{R})$ is of $(\mathbb{Z}^n_{\mathrm{pr}}, 2)$-Rogers type.
Let B denote the closed Euclidean ball in $\mathbb{R}^n$ that is centered at the origin and whose measure is equal to $m(E)$. By [32, Theorem 1 and Lemma 1], it follows

$$\left\| \widehat{\mathbb{1}_E}^{\mathbb{Z}^n_{\neq 0}} \right\|_2^2 \leq \left\| \widehat{\mathbb{1}_B}^{\mathbb{Z}^n_{\neq 0}} \right\|_2^2 \leq m(E)^2 + \sum_{k,q \in \mathbb{Z}_{\neq 0} : \gcd(k,q)=1} \int_{\mathbb{R}^n} \mathbb{1}_B(k\mathbf{x}) \, \mathbb{1}_B(q\mathbf{x}) \, dm(\mathbf{x}).$$

As in the proof of [3, Theorem 2.2], we have

$$\sum_{k,q \in \mathbb{Z}_{\neq 0} : \gcd(k,q)=1} \int_{\mathbb{R}^n} \mathbb{1}_B(k\mathbf{x}) \, \mathbb{1}_B(q\mathbf{x}) \, dm(\mathbf{x}) \leq 8 \frac{\zeta(n-1)}{\zeta(n)} m(E).$$

Hence, $\mathrm{SL}_n(\mathbb{R})$ is of $(\mathbb{Z}^n_{\neq 0}, 2)$-Rogers type.

(iii) Let $E \subset \mathbb{R}^n$ be bounded and Borel. Since $\int_X \widehat{\mathbb{1}_E}^{\mathbb{Z}^n_{\mathrm{pr}}} \, d\mu_X = \frac{1}{\zeta(n)} m(E)$, a simple change of notation and a routine rearrangement of [22, (0.10)] yield

$$\left\| \widehat{\mathbb{1}_E}^{\mathbb{Z}^n_{\mathrm{pr}}} - \left(\int_X \widehat{\mathbb{1}_E}^{\mathbb{Z}^n_{\mathrm{pr}}} \, d\mu_X \right) \mathbb{1}_X \right\|_2 \leq \frac{2}{\sqrt{\zeta(n)}} \, m(E)^{1/2}.$$

Since $\int_X \widehat{\mathbb{1}_E}^{\mathbb{Z}^n_{\neq 0}} \, d\mu_X = m(E)$, a simple change of notation and a routine rearrangement of [22, (0.11)] yield

$$\left\| \widehat{\mathbb{1}_E}^{\mathbb{Z}^n_{\neq 0}} - \left(\int_X \widehat{\mathbb{1}_E}^{\mathbb{Z}^n_{\neq 0}} \, d\mu_X \right) \mathbb{1}_X \right\|_2 \leq \frac{2\zeta\left(\frac{n}{2}\right)}{\sqrt{\zeta(n)}} \, m(E)^{1/2}.$$

Before handling the case of $\mathrm{SL}_2(\mathbb{R})$, we first prove an interpolation result that we shall have to use.

Lemma 2.7 *Let G be a closed subgroup of* $\mathrm{SL}_n(\mathbb{R})$, *and let* Γ *be as in* (2.1). *Suppose further that G is of $\mathbb{Z}^n_{\mathrm{pr}}$-Siegel type with $c_{\mathbb{Z}^n_{\mathrm{pr}}} = 1/\zeta(n)$ and of $(\mathbb{Z}^n_{\mathrm{pr}}, 2)$-Rogers type. For each $r \in (1,2)$ it then follows that G is of $\left(\mathbb{Z}^n_{\neq 0}, r\right)$-Rogers type.*

Proof Arguing as in (iii) of Theorem 2.5, we conclude that G is of $\mathbb{Z}^n_{\neq 0}$-Siegel type with $c_{\mathbb{Z}^n_{\neq 0}} = 1$. Let $D = D_{\mathbb{Z}^n_{\mathrm{pr}},2} \in \mathbb{R}_{>0}$ be as in Definition 2.1. Let $A \subset \mathbb{R}^n$ be bounded and Borel. Then

$$\left\| \widehat{\mathbb{1}_A}^{\mathbb{Z}^n_{\mathrm{pr}}} - \frac{m(A)}{\zeta(n)} \mathbb{1}_X \right\|_2 \leq D\, m(A)^{1/2}.$$

Since G is of $\mathbb{Z}^n_{\mathrm{pr}}$-Siegel type with $c_{\mathbb{Z}^n_{\mathrm{pr}}} = 1/\zeta(n)$, we have

$$\left\| \widehat{\mathbb{1}_A}^{\mathbb{Z}^n_{\mathrm{pr}}} - \frac{m(A)}{\zeta(n)} \mathbb{1}_X \right\|_1 \leq \frac{2m(A)}{\zeta(n)}.$$

Let $r \in (1,2)$ be given. Set $\theta := \dfrac{2}{r} - 1$; then $\theta \in (0,1)$ and $r = \left(\dfrac{\theta}{1} + \dfrac{1-\theta}{2}\right)^{-1}$. By the logarithmic convexity of the L^p norms, one has

$$\begin{aligned}\left\| \widehat{\mathbb{1}_A}^{\mathbb{Z}^n_{\mathrm{pr}}} - \frac{m(A)}{\zeta(n)} \mathbb{1}_X \right\|_r &\leq \left\| \widehat{\mathbb{1}_A}^{\mathbb{Z}^n_{\mathrm{pr}}} - \frac{m(A)}{\zeta(n)} \mathbb{1}_X \right\|_1^{\theta} \cdot \left\| \widehat{\mathbb{1}_A}^{\mathbb{Z}^n_{\mathrm{pr}}} - \frac{m(A)}{\zeta(n)} \mathbb{1}_X \right\|_2^{1-\theta} \\ &\leq \left(\frac{2}{\zeta(n)}\right)^{\theta} m(A)^{\theta} \cdot D^{1-\theta} m(A)^{\frac{1-\theta}{2}} = \frac{2^{\theta} D^{1-\theta}}{\zeta(n)^{\theta}} m(A)^{1/r}.\end{aligned}$$

Now let $E \subset \mathbb{R}^n$ be bounded and Borel. For each $k \in \mathbb{Z}_{>0}$, let

$$E_k := \{\mathbf{x} \in \mathbb{R}^n : k\mathbf{x} \in E\}.$$

We then have

$$\begin{aligned}\left\| \widehat{\mathbb{1}_E}^{\mathbb{Z}^n_{\neq 0}} - m(E)\mathbb{1}_X \right\|_r &= \left\| \sum_{k=1}^{\infty} \left(\widehat{\mathbb{1}_{E_k}}^{\mathbb{Z}^n_{\mathrm{pr}}} - \frac{m(E_k)}{\zeta(n)} \mathbb{1}_X \right) \right\|_r \leq \sum_{k=1}^{\infty} \left\| \widehat{\mathbb{1}_{E_k}}^{\mathbb{Z}^n_{\mathrm{pr}}} - \frac{m(E_k)}{\zeta(n)} \mathbb{1}_X \right\|_r \\ &\leq \sum_{k=1}^{\infty} \frac{2^{\theta} D^{1-\theta}}{\zeta(n)^{\theta}} m(E_k)^{1/r} = \left(\frac{2^{\theta} D^{1-\theta}}{\zeta(n)^{\theta}} \sum_{k=1}^{\infty} k^{-n/r} \right) m(E)^{1/r}.\end{aligned}$$

Since $1 < r < 2 \leq n$, it follows that $D_{\mathbb{Z}^n_{\neq 0},r} := \left(\dfrac{2^{\theta} \left(D_{\mathbb{Z}^n_{\mathrm{pr}},2}\right)^{1-\theta}}{\zeta(n)^{\theta}} \sum_{k=1}^{\infty} k^{-n/r} \right) < \infty.$ □

Theorem 2.8 *The group* $\mathrm{SL}_2(\mathbb{R})$ *is of* $\left(\mathbb{Z}^n_{\mathrm{pr}}, 2\right)$*-Rogers type; for each* $r \in (1,2)$ *the group* $\mathrm{SL}_2(\mathbb{R})$ *is of* $\left(\mathbb{Z}^n_{\neq 0}, r\right)$*-Rogers type.*

Proof If E is any bounded Borel subset of $\mathbb{R}^n$ that has sufficiently large volume, then [3, (4.4)] yields

$$\left\| \widehat{\mathbb{1}_E}^{\mathbb{Z}^n_{\mathrm{pr}}} - \frac{m(E)}{\zeta(2)} \mathbb{1}_X \right\|_2 \leq 4m(E)^{1/2}.$$

This implies the first assertion by choosing the constant $D_{\mathbb{Z}^n_{\mathrm{pr}},2}$ of Definition 2.1 (ii) to be sufficiently large. The second assertion now follows at once from Lemma 2.7. □

Now that we have considered some examples of groups that satisfy the Siegel and Rogers axioms, let us state and prove the first results that make these axioms worthwhile.

Theorem 2.9 *Let G be a closed subgroup of $\mathrm{ASL}_n(\mathbb{R})$, let Γ be as in (2.1), and let $\mathcal{P}$ be a Γ-invariant subset of $\mathbb{Z}^n$. Suppose G is of $\mathcal{P}$-Siegel type with $c = c_{\mathcal{P}}$. Let E be a Borel measurable subset of $\mathbb{R}^n$.*

(i) *If $m(E) < \infty$, then $\mu_X\left(\{\Lambda \in X : \mathrm{card}\,(\Lambda_{\mathcal{P}} \cap E) < \infty\}\right) = 1$.*

For the remaining statements of this theorem, suppose in addition to the preceding hypotheses that we are given $r \in \mathbb{R}_{>1}$ for which G is of $(\mathcal{P}, r)$-Rogers type.

(ii) *Suppose $m(E) = \infty$. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and for each $t \in \mathbb{R}_{>0}$, set*

$$E_t := \{\mathbf{x} \in E : \|\mathbf{x}\| \leq t\}.$$

Then for μ_X-almost every $\Lambda \in X$, one has $\lim_{t\to\infty} \dfrac{\mathrm{card}\,(\Lambda_{\mathcal{P}} \cap E_t)}{c\, m\,(E_t)} = 1$. In particular, $\mu_X\left(\{\Lambda \in X : \mathrm{card}\,(\Lambda_{\mathcal{P}} \cap E) = \infty\}\right) = 1$.

(iii) *Let $\{F_k\}_{k\in\mathbb{Z}_{\geq 1}}$ be Borel measurable subsets of $\mathbb{R}^n$ with $0 < m(F_k) < \infty$ for each $k \in \mathbb{Z}_{\geq 1}$. Suppose $\sum_{k=1}^{\infty} m(F_k)^{1-r} < \infty$. Then the following holds: for μ_X-almost every $\Lambda \in X$, there exists some $k_\Lambda \in \mathbb{Z}_{\geq 1}$ such that for each integer $k \geq k_\Lambda$, we have $\Lambda_{\mathcal{P}} \cap F_k \neq \varnothing$.*

Proof (i) Suppose $m(E) < \infty$. Apply (2.2) to $\mathbb{1}_E : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$; this is valid in light of Remark 2.2(i). This shows that for μ_X-almost every $\Lambda \in X$, one has $\widehat{\mathbb{1}_E}^{\mathcal{P}}(\Lambda) = \mathrm{card}\,(\Lambda_{\mathcal{P}} \cap E) < \infty$.

(ii) For each $t \in \mathbb{R}_{>0}$, define $h_t : X \to \mathbb{R}_{\geq 0}$ by

$$h_t(\Lambda) := \widehat{\mathbb{1}_{E_t}}^{\mathcal{P}}(\Lambda) = \mathrm{card}\,(\Lambda_{\mathcal{P}} \cap E_t)\,.$$

For each $t \in \mathbb{R}_{>0}$, one has $\displaystyle\int_X h_t \, d\mu_X = c\, m(E_t)$. Let $D = D_{\mathcal{P},r}$ be as in Definition 2.1. Fix any $\gamma \in \mathbb{R}$ with $\gamma > (r-1)^{-1}$. For each $k \in \mathbb{Z}_{\geq 1}$, fix $t_k \in \mathbb{R}_{>0}$ for which $m\left(E_{t_k}\right) = k^\gamma$. Let $\varepsilon \in \mathbb{R}_{>0}$ be given. For each $k \in \mathbb{Z}_{\geq 1}$, it follows from Markov's inequality and the hypotheses on G that we have

$$\mu_X\left(\left\{\Lambda \in X : \left|\frac{h_{t_k}(\Lambda)}{c\, m(E_{t_k})} - 1\right| \geq \varepsilon\right\}\right) \leq \frac{1}{\varepsilon^r}\left\|\frac{h_{t_k}}{c\, m\left(E_{t_k}\right)} - \mathbb{1}_X\right\|_r^r$$
$$\leq \frac{1}{\varepsilon^r}\left(\frac{D}{c}\right)^r m\left(E_{t_k}\right)^{1-r} = \left(\frac{D}{\varepsilon\, c}\right)^r k^{\gamma(1-r)}.$$

Since $\varepsilon \in \mathbb{R}_{>0}$ is arbitrary and $\gamma(1-r) < -1$, the Borel–Cantelli lemma now implies that for μ_X-almost every $\Lambda \in X$, we have $\lim\limits_{k\to\infty} \dfrac{\operatorname{card}\left(\Lambda_{\mathcal{P}} \cap E_{t_k}\right)}{c\, m\left(E_{t_k}\right)} = 1$. For any $k \in \mathbb{Z}_{\geq 1}$ and any $t \in \left[t_k, t_{k+1}\right)$, we have

$$\frac{k^\gamma}{(k+1)^\gamma}\frac{h_{t_k}}{m(E_{t_k})} = \frac{h_{t_k}}{m(E_{t_{k+1}})} \leq \frac{h_t}{m(E_t)} \leq \frac{h_{t_{k+1}}}{m(E_{t_k})} = \frac{(k+1)^\gamma}{k^\gamma}\frac{h_{t_{k+1}}}{m(E_{t_{k+1}})}.$$

The result follows.

(iii) Let $D = D_{\mathcal{P},r}$ be as in Definition 2.1. For each $k \in \mathbb{Z}_{\geq 1}$ we have

$$\mu_X\left(\left\{\Lambda \in X : \Lambda_{\mathcal{P}} \cap F_k = \varnothing\right\}\right) \leq \mu_X\left(\left\{\Lambda \in X : \left|\widehat{\mathbb{1}_{F_k}}^{\mathcal{P}}(\Lambda) - c\, m(F_k)\right|^r \geq (c\, m(F_k))^r\right\}\right)$$
$$\leq \left\|\widehat{\mathbb{1}_{F_k}}^{\mathcal{P}} - c\, m(F_k)\mathbb{1}_X\right\|_r^r \left(c\, m(F_k)\right)^{-r}$$
$$\leq D^r m(F_k)\left(c\, m(F_k)\right)^{-r} = D^r c^{-r} m(F_k)^{1-r}.$$

The desired result now follows from the Borel–Cantelli lemma.

□

Statement (ii) of the foregoing theorem is a variation of a very general counting result due to Schmidt: see [34]. See also [37, Chapter 1, Lemma 10] for a result abstracted by Sprindžuk from the work of Schmidt. Following Sprindžuk, it is not difficult to state and prove part (ii) of the above theorem with an estimate for an error term. Let us also mention that our proof of (ii) is similar to an argument used by Durrett in his proof of [11, Chapter 1, Theorem 6.8].

Remark 2.10 Let G be a closed subgroup of $\mathrm{ASL}_n(\mathbb{R})$, let Γ be as in (2.1), and let $\mathcal{P}$ be a Γ-invariant subset of $\mathbb{Z}^n$. Suppose that G is of $\mathcal{P}$-Siegel type, and suppose that there exists $r \in \mathbb{R}_{>1}$ for which G is of $(\mathcal{P}, r)$-Rogers type. It is now easy to prove a probabilistic analogue of the Minkowski convex body theorem. Indeed, let $c = c_{\mathcal{P}}$ and $D = D_{\mathcal{P},r}$ be as in Definition 2.1; let E be a Borel subset of $\mathbb{R}^n$ with $0 < m(E) < \infty$. As in the proof of Theorem 2.9(iii), it follows that

$$\mu_X\left(\{\Lambda \in X : \Lambda_{\mathcal{P}} \cap E = \varnothing\}\right) \leq D^r c^{-r} m(E)^{1-r}.$$

This sort of result, with $r = 2$, was first established by Athreya–Margulis for $G = \mathrm{SL}_n(\mathbb{R})$ and $\mathcal{P} = \mathbb{Z}^n_{\neq 0}$ [3, Theorem 2.2], and then by Athreya for $G = \mathrm{ASL}_n(\mathbb{R})$ and $\mathcal{P} = \mathbb{Z}^n$ [2, Theorem 1].

Remark 2.11 Suppose $n \geq 2$ is arbitrary and $G = \mathrm{SL}_n(\mathbb{R})$. In [38, Corollary 2.14], Strömbergsson proves that the bound of Athreya–Margulis in [3, Theorem 2.2] is sharp. It now follows from the preceding remark that for each $r \in \mathbb{R}_{>2}$ and each subset $\mathcal{P}$ of $\mathbb{Z}^n_{\neq 0}$ that is $\Gamma = \mathrm{SL}_n(\mathbb{Z})$-invariant, we have that the group G is *not* of $(\mathcal{P}, r)$-Rogers type.

Remark 2.12 Suppose $n \geq 2$ is arbitrary and $G = \mathbb{R}^n$, which is a closed subgroup of $\mathrm{ASL}_n(\mathbb{R})$. Then $\Gamma = \mathbb{Z}^n$, and $X = \mathbb{R}^n/\mathbb{Z}^n$ is the n-dimensional torus. It is easy to see that $\mathbb{R}^n$ is then of $\mathbb{Z}^n$-Siegel type with $c_{\mathbb{Z}^n} = 1$. For each $r \in \mathbb{R}_{>1}$, however, $\mathbb{R}^n$ is is *not* of $(\mathbb{Z}^n, r)$-Rogers type. This may be seen from the spectacular impossibility of obtaining a result as in Remark 2.10. For any $\varepsilon \in (0, 1)$, define $U_\varepsilon := \mathbb{R}^{n-1} \times \left(\frac{1}{2} - \frac{\varepsilon}{2}, \frac{1}{2} + \frac{\varepsilon}{2}\right) \subseteq \mathbb{R}^{n-1} \times (0, 1)$. For each $\varepsilon \in (0, 1)$, we then have $m(U_\varepsilon) = \infty$ and $\mu_X(\{\Lambda \in X : \Lambda_{\mathbb{Z}^n} \cap U_\varepsilon = \varnothing\}) = 1 - \varepsilon$.

In the following section, we transfer our counting results for generic lattices to statements involving small values of generic functions, thereby establishing a more general version of Theorem 1.3.

3 Zero-full laws in diophantine approximation

We begin by proving two lemmata.

Lemma 3.1 *Let $f : \mathbb{R}^n \to \mathbb{R}^\ell$ be subhomogeneous, and let $\psi : \mathbb{R}_{\geq 0} \to (\mathbb{R}_{>0})^\ell$ be regular and nonincreasing. Let η and ν be any norms on $\mathbb{R}^n$, and let $s \in \mathbb{R}_{>0}$. Then $m(A_{f,s\psi,\eta}) < \infty$ if and only if $m(A_{f,\psi,\nu}) < \infty$.*

Proof Suppose without loss of generality that the image of f is a subset of $\left(\mathbb{R}_{\geq 0}\right)^\ell$. Let $a = a_\psi$, $b = b_\psi$, and $d = d_f$ be as in Definition 1.2; let $\mathbf{x} \in A_{f,\psi,\eta}$. Let $s \in \mathbb{R}_{>0}$ be given. Suppose first $s \leq 1$. Then $s^{1/d} \in (0, 1]$, and thus

$$f\big(s^{1/d}\mathbf{x}\big) \leq s f(\mathbf{x}) \leq s\psi\big(\eta(\mathbf{x})\big) \leq s\psi\big(\eta(s^{1/d}\mathbf{x})\big).$$

This proves $s^{1/d} A_{f,\psi,\eta} \subseteq A_{f,s\psi,\eta}$. Also, note that $A_{f,s\psi,\eta} \subseteq A_{f,\psi,\eta}$. Hence, the Lebesgue measure of $A_{f,s\psi,\eta}$ is finite if and only if the Lebesgue measure of $A_{f,\psi,\eta}$ is finite. Suppose next $s \geq 1$. By repeating the preceding argument with $s\psi$ in place of ψ and s^{-1} in place of s, we obtain the same conclusion for $s \geq 1$.

Now, using the equivalence of the two norms, fix $C \in \mathbb{R}_{>1}$ for which $C^{-1}\eta \leq \nu \leq C\eta$. Fix a positive integer k for which $a^k > C$. Suppose that $A_{f,\psi,\nu}$ has infinite Lebesgue measure. Let $\mathbf{x} \in A_{f,\psi,\nu}$. By a simple induction,

$$f(\mathbf{x}) \leq \psi\big(\nu(\mathbf{x})\big) \leq \psi\left(C^{-1}\eta(\mathbf{x})\right) \leq b^{-k}\psi\big(a^k C^{-1}\eta(\mathbf{x})\big) \leq b^{-k}\psi\big(\eta(\mathbf{x})\big).$$

Thus, the Lebesgue measure of $A_{f,b^{-k}\psi,\eta}$ is infinite as well. In conjunction with the foregoing and by symmetry, this completes the proof. ◻

Lemma 3.2 *Let $\psi : \mathbb{R}_{\geq 0} \to (\mathbb{R}_{>0})^\ell$ be regular and nonincreasing. Then the following holds: for any $c \in \mathbb{R}_{\geq 0}$ there exists $s \in \mathbb{R}_{>0}$ such that for each $x \in [0, c]$ and each $y \in \mathbb{R}_{>c}$, one has $\psi(y - x) \leq s\psi(y)$.*

Proof Let $a = a_\psi$ and $b = b_\psi$ be as in Definition 1.2. Let $c \in \mathbb{R}_{\geq 0}$. Define

$$s := \max_{1 \leq i \leq \ell} \left(\frac{1}{b}, \frac{\psi_i(0)}{\psi_i(\frac{ac}{a-1})} \right).$$

Let $x \in [0, c]$ and $y \in \mathbb{R}_{>c}$. We consider two cases.

- Case 1: suppose $y \leq \dfrac{ac}{a-1}$. Then

$$\psi(y - x) \leq \psi(0) \leq s\psi\left(\frac{ac}{a-1}\right) \leq s\psi(y).$$

- Case 2: suppose $y > \dfrac{ac}{a-1}$. Since $c \geq x \geq 0$ and $a - 1 > 0$, it follows $y > \dfrac{ax}{a-1}$; hence, $y - x > \dfrac{y}{a}$. Thus,

$$\psi(y - x) \leq \psi\left(\frac{y}{a}\right) \leq \frac{1}{b}\psi\left(a \cdot \frac{y}{a}\right) = \frac{1}{b}\psi(y) \leq s\psi(y).$$

This completes the proof. □

Before proving our main results, let us first augment two definitions given in §1.

Definition 3.3 In this definition, assume that we are using the same notation as in Definition 1.2. Now take an arbitrary subset $\mathcal{P}$ of $\mathbb{Z}^n$ and

- say that f is $(\psi, \nu, \mathcal{P})$-*approximable* if $A_{f,\psi,\nu} \cap \mathcal{P}$ has infinite cardinality;
- say that f is *uniformly* $(\psi, \nu, \mathcal{P})$-*approximable* if $B_{f,\psi(T),\nu,T} \cap \mathcal{P} \neq \varnothing$ for each sufficiently large $T \in \mathbb{R}_{>0}$.

Notice that by taking $\mathcal{P} = \mathbb{Z}^n_{\neq 0}$ in the above definition we recover the previously defined notions of asymptotic and uniform (ψ, ν)-approximability. We now state and prove our main result on asymptotic approximation.

Theorem 3.4 *Let G be a closed subgroup of $\mathrm{ASL}_n(\mathbb{R})$, let Γ be as in (2.1), and let $\mathcal{P}$ be a Γ-invariant subset of $\mathbb{Z}^n$. Suppose G is of $\mathcal{P}$-Siegel type. Let $c = c_{\mathcal{P}}$ be as in Definition 2.1(i). Let $\psi : \mathbb{R}_{\geq 0} \to (\mathbb{R}_{>0})^\ell$ be regular and nonincreasing; let $f : \mathbb{R}^n \to \mathbb{R}^\ell$ be subhomogeneous; let ν and η be norms on $\mathbb{R}^n$.*

(i) *Suppose $m\left(A_{f,\psi,\eta}\right) < \infty$. Then for almost every $g \in G$ the function $f \circ g$ is not $(\psi, \nu, \mathcal{P})$-approximable.*

(ii) *Suppose $m\left(A_{f,\psi,\eta}\right) = \infty$, and suppose there exists $r \in \mathbb{R}_{>1}$ for which G is of $(\mathcal{P}, r)$-Rogers type. Then for each nonempty compact subset K of G there exist constants $D_K \in \mathbb{R}_{\geq 1}$, $E_K \in \mathbb{R}_{\geq 0}$ and $J_K \in \mathbb{R}_{\geq 1}$ such that for μ_G-almost every $g \in K$ we have*

$$\limsup_{T\to\infty} \frac{\operatorname{card}\left\{\mathbf{v} \in \mathcal{P} : (f \circ g)(\mathbf{v}) \leq \psi\big(\nu(\mathbf{v})\big) \text{ and } 2D_K E_K < \nu(\mathbf{v}) \leq T\right\}}{m\left(\left\{\mathbf{t} \in \mathbb{R}^n : f(\mathbf{t}) \leq J_K \psi\big(\nu(\mathbf{t})\big) \text{ and } E_K < \nu(\mathbf{t}) \leq D_K T + E_K\right\}\right)} \leq c_{\mathcal{P}}, \tag{3.1}$$

and

$$\liminf_{T\to\infty} \frac{\operatorname{card}\left\{\mathbf{v} \in \mathcal{P} : (f \circ g)(\mathbf{v}) \leq \psi\big(\nu(\mathbf{v})\big) \text{ and } E_K D_K^{-1} < \nu(\mathbf{v}) \leq D_K E_K + D_K T\right\}}{m\left(\left\{\mathbf{t} \in \mathbb{R}^n : f(\mathbf{t}) \leq J_K^{-1} \psi\big((\nu(\mathbf{t})\big) \text{ and } 2E_K < \nu(\mathbf{t}) \leq T\right\}\right)} \geq c_{\mathcal{P}}. \tag{3.2}$$

Moreover, if $K \subseteq \mathrm{SL}_n(\mathbb{R})$, then each of the above inequalities holds with $E_K = 0$. In particular, for almost every $g \in G$ the function $f \circ g$ is $(\psi, \nu, \mathcal{P})$-approximable.

Proof Let us denote elements of $\mathrm{ASL}_n(\mathbb{R})$ by $\langle h, \mathbf{z}\rangle$, where $h \in \mathrm{SL}_n(\mathbb{R})$ and $\mathbf{z} \in \mathbb{R}^n$; that is,

$$\langle h, \mathbf{z}\rangle : \mathbb{R}^n \to \mathbb{R}^n \text{ is the affine transformation given by } \mathbf{x} \mapsto h\mathbf{x} + \mathbf{z}. \tag{3.3}$$

We suppose without loss of generality that the image of f is a subset of $\left(\mathbb{R}_{\geq 0}\right)^{\ell}$. For any $h \in \mathrm{SL}_n(\mathbb{R})$ we write $\|h\|$ to denote the operator norm of h when both the domain and codomain of h are equipped with the norm ν on $\mathbb{R}^n$ that is mentioned in the hypotheses.

Suppose that $m\left(A_{f,\psi,\eta}\right) < \infty$. Lemma 3.1 implies that for any $N \in \mathbb{Z}_{\geq 1}$ we have $m\left(A_{f,N\psi,\nu}\right) < \infty$. Theorem 2.9(i) then implies

$$\mu_X\left(\left\{\Lambda \in X : \operatorname{card}\left(\Lambda_{\mathcal{P}} \cap A_{f,N\psi,\nu}\right) = \infty\right\}\right) = 0,$$

which is equivalent to

$$\mu_G\left(\left\{g \in G : \operatorname{card}\left(g\mathcal{P} \cap A_{f,N\psi,\nu}\right) = \infty\right\}\right) = 0. \tag{3.4}$$

Let $a = a_\psi$, $b = b_\psi$, and $d = d_f$ be as in Definition 1.2. Let $g = \langle h, \mathbf{z}\rangle$ be any element of G for which

$$f \circ g \text{ is } (\psi, \nu, \mathcal{P})\text{-approximable.} \tag{3.5}$$

Let $D := \max\left\{\|h\|, \left\|h^{-1}\right\|\right\}$, and let $E := \nu(\mathbf{z})$. Let k be a nonnegative integer for which $a^k \geq D$. Let $C := b^{-k}$. Appealing to Lemma 3.2, we let $F \in \mathbb{R}_{>0}$ be a constant for which the following is true: for each $x \in [0, E]$ and each $y \in (E, \infty)$, we have $\psi(y - x) \leq F\psi(y)$. Finally, let N be any integer with $N \geq CF$.

Let $\mathbf{v}$ be an arbitrary element of the infinite set $\left\{\mathbf{x} \in \mathcal{P} \cap A_{f\circ g,\psi,\nu} : \nu(\mathbf{x}) > 2DE\right\}$. Notice that

$$\left\|h^{-1}\right\| \nu(h\mathbf{v}) \geq \nu(\mathbf{v}) > 2DE \geq 2\left\|h^{-1}\right\| E,$$

whence $\nu(h\mathbf{v}) > 2E$. Hence,

$$\nu(h\mathbf{v} + \mathbf{z}) \geq \nu(h\mathbf{v}) - \nu(\mathbf{z}) > 2E - \nu(\mathbf{z}) = E.$$

Since $(f \circ g)(\mathbf{v}) \leq \psi\big(\nu(\mathbf{v})\big)$, it follows

$$\begin{aligned} f(g\mathbf{v}) \leq \psi\big(\nu(\mathbf{v})\big) \leq \psi\left(\frac{\nu(h\mathbf{v})}{\|h\|}\right) \leq b^{-k}\psi\left(a^k\frac{\nu(h\mathbf{v})}{\|h\|}\right) \leq C\psi\big(\nu(h\mathbf{v})\big) \\ \leq C\psi\big(\nu(h\mathbf{v}+\mathbf{z}) - \nu(\mathbf{z})\big) \leq CF\psi\big(\nu(h\mathbf{v}+\mathbf{z})\big) \\ \leq N\psi\big(\nu(h\mathbf{v}+\mathbf{z})\big) = N\psi\big(\nu(g\mathbf{v})\big). \end{aligned}$$

Thus, card $\left(g\mathcal{P} \cap A_{f,N\psi,\nu}\right) = \infty$; hence, in view of (3.4), the set of $g \in G$ that satisfy (3.5) is null. This proves (i).

Suppose now that $m\left(A_{f,\psi,\eta}\right) = \infty$; suppose further that there exists $r \in \mathbb{R}_{>1}$ for which G is of $(\mathcal{P}, r)$-Rogers type. Let $\varepsilon \in \mathbb{R}_{>0}$ be given. Let K be an arbitrary nonempty compact subset of G. Since the inversion map is a homeomorphism and finite unions of compact sets are compact, we assume without loss of generality that $K = K^{-1}$. Define $\pi : \mathrm{ASL}_n(\mathbb{R}) \to \mathrm{SL}_n(\mathbb{R})$ and $\rho : \mathrm{ASL}_n(\mathbb{R}) \to \mathbb{R}^n$ by $\pi : \langle h, \mathbf{z}\rangle \mapsto h$ and $\rho : \langle h, \mathbf{z}\rangle \mapsto \mathbf{z}$. Note that π is a group homomorphism. We define

$$D_K := \sup\left\{\|h\| : h \in \pi(K)\right\}, \qquad E_K := \sup\left\{\nu(\mathbf{z}) : \mathbf{z} \in \rho(K)\right\}. \tag{3.6}$$

Note that $E_K = 0$ if and only if $K \subseteq \mathrm{SL}_n(\mathbb{R})$. Note that $D_K \geq 1$. Let $a = a_\psi$ and $b = b_\psi$ be as in Definition 1.2. Set $k := \min\left\{j \in \mathbb{Z}_{\geq 0} : a^j \geq D_K\right\}$. Set $C_K := b^{-k}$. Note that $C_K \geq 1$. Appealing to Lemma 3.2, we let $F_K \in \mathbb{R}_{\geq 1}$ be a constant for which the following is true: for each $x \in [0, E_K]$ and each $y \in (E_K, \infty)$, we have $\psi(y - x) \leq F_K\psi(y)$. Set $J_K := C_K F_K$.

Let $\langle h_1, \mathbf{z}_1\rangle \in K$ be arbitrary. Let R be any real number with $R > 2D_K E_K$. Let $\mathbf{x}$ be any element of $\mathbb{R}^n$ with $2D_K E_K < \nu(\mathbf{x}) \leq R$. Then

$$\left\|h_1^{-1}\right\| \nu(h_1\mathbf{x}) \geq \nu(\mathbf{x}) > 2D_K E_K \geq 2\left\|h_1^{-1}\right\| E_K,$$

whence $\nu(h_1\mathbf{x}) > 2E_K$. It follows

$$\nu(h_1\mathbf{x} + \mathbf{z}_1) \geq \nu(h_1\mathbf{x}) - \nu(\mathbf{z}_1) > 2E_K - \nu(\mathbf{z}_1) \geq E_K.$$

Suppose further that $f(h_1\mathbf{x}+\mathbf{z}_1) \le \psi\big(\nu(\mathbf{x})\big)$. Then

$$f(h_1\mathbf{x}+\mathbf{z}_1) \le \psi\big(\nu(\mathbf{x})\big) \le \psi\left(\frac{\nu(h_1\mathbf{x})}{\|h_1\|}\right) \le b^{-k}\psi\left(a^k\frac{\nu(h_1\mathbf{x})}{\|h_1\|}\right) \le C_K\psi\big(\nu(h_1\mathbf{x})\big)$$
$$\le C_K\psi\big(\nu(h_1\mathbf{x}+\mathbf{z}_1)-\nu(\mathbf{z}_1)\big) \le C_K F_K\psi\big(\nu(h_1\mathbf{x}+\mathbf{z}_1)\big) = J_K\psi\big(\nu(h_1\mathbf{x}+\mathbf{z}_1)\big).$$

Finally, we note that $\nu(h_1\mathbf{x}+\mathbf{z}_1) \le D_K R + E_K$. We have therefore shown

$$\langle h_1,\mathbf{z}_1\rangle\left\{\mathbf{t}\in A_{f\circ\langle h_1,\mathbf{z}_1\rangle,\psi,\nu} : 2D_K E_K < \nu(\mathbf{t}) \le R\right\} \subseteq \left\{\mathbf{t}\in A_{f,J_K\psi,\nu} : E_K < \nu(\mathbf{t}) \le D_K R + E_K\right\}. \tag{3.7}$$

By Lemma 3.1, we have $m\left(\left\{\mathbf{t}\in A_{f,J_K\psi,\nu} : \nu(\mathbf{t}) > E_K\right\}\right) = \infty$. By using (3.7) and then applying Theorem 2.9(ii), it follows that for μ_G-almost every $\langle h,\mathbf{z}\rangle \in K$ and any $\varepsilon\in\mathbb{R}_{>0}$ there exists some $T_{\langle h,\mathbf{z}\rangle}\in\mathbb{R}_{>0}$ such that for every $T \ge T_{\langle h,\mathbf{z}\rangle}$, we have

$$\begin{aligned}
&\frac{\operatorname{card}\left\{\mathbf{v}\in\mathcal{P} : (f\circ\langle h,\mathbf{z}\rangle)(\mathbf{v}) \le \psi\big(\nu(\mathbf{v})\big) \text{ and } 2D_K E_K < \nu(\mathbf{v}) \le T\right\}}{m\left(\left\{\mathbf{t}\in\mathbb{R}^n : f(\mathbf{t}) \le J_K\psi\big(\nu(\mathbf{t})\big) \text{ and } E_K < \nu(\mathbf{t}) \le D_K T + E_K\right\}\right)}\\
&\le \frac{\operatorname{card}\left\{\mathbf{w}\in\langle h,\mathbf{z}\rangle\mathcal{P} : f(\mathbf{w}) \le J_K\psi\big(\nu(\mathbf{w})\big) \text{ and } E_K < \nu(\mathbf{w}) \le D_K T + E_K\right\}}{m\left(\left\{\mathbf{t}\in\mathbb{R}^n : f(\mathbf{t}) \le J_K\psi\big(\nu(\mathbf{t})\big) \text{ and } E_K < \nu(\mathbf{t}) \le D_K T + E_K\right\}\right)}\\
&< c_{\mathcal{P}} + \varepsilon.
\end{aligned}$$

It follows that (3.1) holds for μ_G-almost every $\langle h,\mathbf{z}\rangle \in K$.

Now let $\langle h_2,\mathbf{z}_2\rangle \in K$ be arbitrary. Let R' be any real number with $R' > 2E_K$. By an argument similar to the one given for (3.7), one can show

$$\langle h_2,\mathbf{z}_2\rangle^{-1}\left\{\mathbf{t}\in A_{f,J_K^{-1}\psi,\nu} : 2E_K < \nu(\mathbf{t}) \le R'\right\} \subseteq \left\{\mathbf{t}\in A_{f\circ\langle h_2,\mathbf{z}_2\rangle,\psi,\nu} : E_K D_K^{-1} < \nu(\mathbf{t}) \le D_K(E_K+R')\right\}. \tag{3.8}$$

By Lemma 3.1, we have $m\left(\left\{\mathbf{t}\in A_{f,J_K^{-1}\psi,\nu} : \nu(\mathbf{t}) > 2E_K\right\}\right) = \infty$. By using (3.8) and then applying Theorem 2.9(ii), it follows that for μ_G-almost every $\langle h,\mathbf{z}\rangle \in K$ and any $\varepsilon\in\mathbb{R}_{>0}$ there exists some $T'_{\langle h,\mathbf{z}\rangle}\in\mathbb{R}_{>0}$ such that for every real $T \ge T'_{\langle h,\mathbf{z}\rangle}$, we have

$$\begin{aligned}
&\frac{\operatorname{card}\left\{\mathbf{v}\in\mathcal{P} : (f\circ\langle h,\mathbf{z}\rangle)(\mathbf{v}) \le \psi\big(\nu(\mathbf{v})\big) \text{ and } E_K D_K^{-1} < \nu(\mathbf{v}) \le D_K(E_K+T)\right\}}{m\left(\left\{\mathbf{t}\in\mathbb{R}^n : f(\mathbf{t}) \le J_K^{-1}\psi\big(\nu(\mathbf{t})\big) \text{ and } 2E_K < \nu(\mathbf{t}) \le T\right\}\right)}\\
&\ge \frac{\operatorname{card}\left\{\mathbf{w}\in\langle h,\mathbf{z}\rangle\mathcal{P} : f(\mathbf{w}) \le J_K^{-1}\psi\big(\nu(\mathbf{w})\big) \text{ and } 2E_K < \nu(\mathbf{w}) \le T\right\}}{m\left(\left\{\mathbf{t}\in\mathbb{R}^n : f(\mathbf{t}) \le J_K^{-1}\psi\big(\nu(\mathbf{t})\big) \text{ and } 2E_K < \nu(\mathbf{t}) \le T\right\}\right)}\\
&> c_{\mathcal{P}} - \varepsilon.
\end{aligned}$$

Thus, (3.2) holds for μ_G-almost every $\langle h,\mathbf{z}\rangle \in K$. The final statement of (ii) now follows from the σ-compactness of G and an application of Lemma 3.1. □

We now prepare to prove our results on uniform approximation. We first prove a lemma similar to Lemma 3.1.

Lemma 3.5 *Let $f : \mathbb{R}^n \to \mathbb{R}^\ell$ be subhomogeneous, with $d = d_f \in \mathbb{R}_{>0}$ as in Definition 1.2.*

(i) *Let ν be an arbitrary norm on $\mathbb{R}^n$, let $t \in (0,1)$, $T \in \mathbb{R}_{>0}$, and $\boldsymbol{\varepsilon} \in (\mathbb{R}_{>0})^\ell$. Then $t B_{f,\boldsymbol{\varepsilon},\nu,T} \subseteq B_{f,t^d\boldsymbol{\varepsilon},\nu,tT}$.*
(ii) *Let ν and η be arbitrary norms on $\mathbb{R}^n$. Then there exists $C^* = C^*_{\nu,\eta} \in \mathbb{R}_{\geq 1}$ such that for each $C \in [C^*, \infty)$, each $T \in \mathbb{R}_{>0}$, and each $\boldsymbol{\varepsilon} \in (\mathbb{R}_{>0})^\ell$, we have*

$$C^{-1} B_{f,\boldsymbol{\varepsilon},\nu,T} \subseteq B_{f,C^{-d}\boldsymbol{\varepsilon},\eta,T} \subseteq B_{f,\boldsymbol{\varepsilon},\eta,T} \subseteq C\, B_{f,C^{-d}\boldsymbol{\varepsilon},\nu,T} \subseteq C\, B_{f,\boldsymbol{\varepsilon},\nu,T}.$$

Proof Let $\mathbf{x} \in t B_{f,\boldsymbol{\varepsilon},\nu,T}$. Then, since $t^{-1}\mathbf{x} \in B_{f,\boldsymbol{\varepsilon},\nu,T}$, we have

$$\nu(\mathbf{x}) = t\nu(t^{-1}\mathbf{x}) \leq tT \quad \text{and} \quad |f(\mathbf{x})| \leq t^d |f(t^{-1}\mathbf{x})| \leq t^d \boldsymbol{\varepsilon}.$$

Hence, $\mathbf{x} \in B_{f,t^d\boldsymbol{\varepsilon},\nu,tT}$, which proves (i).

For (ii), fix $C^* = C^*_{\nu,\eta} \in \mathbb{R}_{\geq 1}$ for which $\left(C^*\right)^{-1} \eta(\cdot) \leq \nu(\cdot) \leq C^*\eta(\cdot)$. Fix any $C \in [C^*, \infty)$ and let $\mathbf{x} \in C^{-1} B_{f,\boldsymbol{\varepsilon},\nu,T}$. We have $\eta(\mathbf{x}) \leq \nu(C\mathbf{x}) \leq T$. Moreover, one has

$$|f(\mathbf{x})| \leq C^{-d}|f(C\mathbf{x})| \leq C^{-d}\boldsymbol{\varepsilon} \leq \boldsymbol{\varepsilon} \quad \Longrightarrow \quad C^{-1} B_{f,\boldsymbol{\varepsilon},\nu,T} \subseteq B_{f,C^{-d}\boldsymbol{\varepsilon},\eta,T} \subseteq B_{f,\boldsymbol{\varepsilon},\eta,T}.$$

Interchanging ν and η and then arguing similarly, one obtains

$$B_{f,\boldsymbol{\varepsilon},\eta,T} \subseteq C\, B_{f,C^{-d}\boldsymbol{\varepsilon},\nu,T} \subseteq C\, B_{f,\boldsymbol{\varepsilon},\nu,T}.$$

This completes the proof. □

Let us now introduce some definitions and then prove another lemma.

Definition 3.6 Let $f : \mathbb{R}^n \to \mathbb{R}^\ell$ and $\psi : \mathbb{R}_{\geq 0} \to (\mathbb{R}_{>0})^\ell$ be arbitrary maps. Let ν be an arbitrary norm on $\mathbb{R}^n$, and let $\mathcal{P}$ be an arbitrary subset of $\mathbb{Z}^n$. Let $t_\bullet = (t_k)_{k \in \mathbb{Z}_{\geq 1}}$ be any strictly increasing sequence of elements of $\mathbb{R}_{>0}$ with $\lim_{k\to\infty} t_k = \infty$. We say that f is *$t_\bullet$-uniformly $(\psi, \nu, \mathcal{P})$-approximable* if $B_{f,\psi(t_k),\nu,t_k} \cap \mathcal{P} \neq \varnothing$ for each sufficiently large $k \in \mathbb{Z}_{\geq 1}$.

Definition 3.7 Let $t_\bullet = (t_k)_{k \in \mathbb{Z}_{\geq 1}}$ be any strictly increasing sequence of elements of $\mathbb{R}_{>0}$ with $\lim_{k\to\infty} t_k = \infty$. We say that $t_\bullet$ is *quasi-geometric* if, in addition to the preceding, the set $\left\{\dfrac{t_{k+1}}{t_k} : k \in \mathbb{Z}_{\geq 1}\right\}$ is bounded.

Theorem 3.8 *Let G be a closed subgroup of $\mathrm{ASL}_n(\mathbb{R})$, let Γ be as in (2.1), and let $\mathcal{P}$ be a Γ-invariant subset of $\mathbb{Z}^n$. Let $t_\bullet = (t_k)_{k \in \mathbb{Z}_{\geq 1}}$ be any strictly increasing sequence of elements of $\mathbb{R}_{>0}$ with $\lim_{k\to\infty} t_k = \infty$. Suppose G is of $\mathcal{P}$-Siegel type, and suppose we are given $r \in \mathbb{R}_{>1}$ for which G is of $(\mathcal{P}, r)$-Rogers type. Let $\psi : \mathbb{R}_{\geq 0} \to (\mathbb{R}_{>0})^\ell$*

be Borel measurable, and let $f : \mathbb{R}^n \to \mathbb{R}^\ell$ *be subhomogeneous. Suppose also that there exists some norm* η *on* $\mathbb{R}^n$ *for which*

$$\sum_{k=1}^{\infty} m(B_{f,\psi(t_k),\eta,t_k})^{1-r} < \infty. \tag{3.9}$$

Let ν *be an arbitrary norm on* $\mathbb{R}^n$*. We then have the following.*

(i) *For almost every* $g \in G$ *the function* $f \circ g$ *is* $t_\bullet$*-uniformly* $(\psi, \nu, \mathcal{P})$*-approximable.*
(ii) *Suppose further that* ψ *is nonincreasing and regular, and the sequence* $t_\bullet$ *is quasi-geometric. Then for almost every* $g \in G$ *the function* $f \circ g$ *is uniformly* $(\psi, \nu, \mathcal{P})$*-approximable.*

Proof (i) Let K be a nonempty compact subset of G; as in the proof of Theorem 3.4, assume without loss of generality that $K = K^{-1}$. Define the constants D_K, E_K by (3.6). Lemma 3.5 and (3.9) imply that the series $\sum_{k=1}^{\infty} m(B_{f,\psi(t_k),\nu,t_k/(2D_K)})^{1-r}$ converges. Applying Theorem 2.9(iii), we obtain the following: For almost every $g \in G$ there exists $M_g \in \mathbb{Z}_{\geq 1}$ such that for each $k \in \mathbb{Z}$ with $k \geq M_g$ there exists some $\mathbf{v}_k \in \mathcal{P}$ with

$$\nu(g\mathbf{v}_k) \leq \frac{t_k}{2D_K} \quad \text{and} \quad |f(g\mathbf{v}_k)| \leq \psi(t_k). \tag{3.10}$$

For each such $g \in G$, we assume without loss of generality that for each $k \in \mathbb{Z}$ with $k \geq M_g$, we have $t_k > 2E_K$. If, in addition, $g \in K$, then for any $\mathbf{v}_k$ as above it now follows from (3.10) that we have $\nu(\mathbf{v}_k) \leq t_k$. Indeed, if not, then we write $g^{-1} = \langle h, \mathbf{z} \rangle$, as in (3.3), and note that

$$\begin{aligned} \nu\big(h(g\mathbf{v}_k) + \mathbf{z}\big) = \nu(\mathbf{v}_k) > t_k \implies \quad & \nu\big(h(g\mathbf{v}_k)\big) > t_k - \nu(\mathbf{z}) \geq t_k - E_K \geq \frac{t_k}{2} \\ \implies \quad & \nu(g\mathbf{v}_k) > \frac{t_k}{2D_K}, \end{aligned}$$

which is a contradiction. Therefore, for μ_G-almost every $g \in K$ and with M_g as above, it follows that for every $k \in \mathbb{Z}$ with $k \geq M_g$ there exists $\mathbf{v}_k \in \mathcal{P}$ with $|(f \circ g)(\mathbf{v}_k)| \leq \psi(t_k)$ and $\nu(\mathbf{v}_k) \leq t_k$. Hence, for μ_G-almost every $g \in K$, the function $f \circ g$ is $t_\bullet$-uniformly $(\psi, \nu, \mathcal{P})$-approximable. Since G is σ-compact, the same holds for almost every $g \in G$.

(ii) Let $a = a_\psi$ and $b = b_\psi$ be as in Definition 1.2. Fix $j \in \mathbb{Z}_{\geq 1}$ for which

$$\sup\left\{t_{k+1}/t_k : k \in \mathbb{Z}_{\geq 1}\right\} < a^j \quad \text{and} \quad \sum_{k=1}^{\infty} m(B_{f,b^j\psi(t_k),\eta,t_k})^{1-r} < \infty;$$

notice that this is indeed possible in light of Definition 3.7, Lemma 3.5, and (3.9). By statement (i), we know that for almost every $g \in G$, the function $f \circ g$ is

$t_\bullet$-uniformly $\left(b^j\psi, \nu, \mathcal{P}\right)$-approximable. Now let $h : \mathbb{R}^n \to \mathbb{R}^\ell$ be any function that is $t_\bullet$-uniformly $\left(b^j\psi, \nu, \mathcal{P}\right)$-approximable. Fix $M \in \mathbb{Z}_{\geq 1}$ such that for each $k \in \mathbb{Z}_{\geq M}$ the set $B_{h, b^j\psi(t_k), \nu, t_k} \cap \mathcal{P}$ is nonempty. Let $T \in (t_{M+2}, \infty)$ be arbitrary. Then there exists $i \in \mathbb{Z}_{\geq M+2}$ for which $t_i \leq T < t_{i+1}$. Note that there exists $\mathbf{v} \in \mathcal{P}$ with $\nu(\mathbf{v}) \leq t_i$ and $|h(\mathbf{v})| \leq b^j\psi(t_i)$. We then have $\nu(\mathbf{v}) \leq t_i \leq T$ and $|h(\mathbf{v})| \leq b^j\psi(t_i) \leq b^j b^{-j}\psi\left(a^j t_i\right) = \psi\left(a^j t_i\right) \leq \psi(t_{i+1}) \leq \psi(T)$.

□

Proof of Theorem 1.3 Theorem 1.3 is now an immediate consequence of Theorems 2.6(ii), 2.8, 3.4, and 3.8(ii) with $\mathcal{P} = \mathbb{Z}^n_{\neq 0}$ and $t_\bullet = (2^k)_{k \in \mathbb{Z}_{\geq 1}}$.

Remark 3.9 Denote by Z_n the group of scalar $n \times n$ matrices (that is, the center of $\mathrm{GL}_n(\mathbb{R})$). For any G as in Theorem 2.9 set $\widetilde{G} := G \times Z_n$; we have, for example, $\widetilde{\mathrm{SL}_n(\mathbb{R})} = \mathrm{GL}_n(\mathbb{R})$ and $\widetilde{\mathrm{ASL}_n(\mathbb{R})} = \mathrm{AGL}_n(\mathbb{R})$. It is then clear from the Fubini–Tonelli Theorem that every result of this section that was established for G also holds, *mutatis mutandis*, for $\widetilde{G}$. The same remark applies to the corollaries derived in the next section. Alternatively, the $\mathrm{GL}_n(\mathbb{R})$ analogue of our results follows easily from the corollary to Theorems 1 and 2 in [34], via an application of Lemma 3.1.

Remark 3.10 Let us note that the null and conull subsets of G in each part of Theorem 3.4 and in Theorem 3.8(ii) may be chosen independently of the norm. This is an immediate consequence of the facts that all norms on $\mathbb{R}^n$ are equivalent, that any positive multiple of a norm is a norm, that $\mathbb{Z}_{\geq 1}$ is countable and unbounded, and that $\psi = (\psi_1, \ldots, \psi_\ell) : \mathbb{R}_{\geq 0} \to (\mathbb{R}_{>0})^\ell$ is assumed to be nonincreasing in the results that were just mentioned.

In the following section, we apply Theorems 3.4 and 3.8 to investigate the orbits of several specific subhomogeneous functions f. We do so by performing several volume calculations.

4 Examples and volume calculations

Let us state the conventions that will be in force throughout this section. We shall let G denote a closed subgroup of $\mathrm{ASL}_n(\mathbb{R})$, $n \in \mathbb{Z}_{\geq 2}$, and $\mathcal{P}$ denote a Γ-invariant subset of $\mathbb{Z}^n$, where Γ is as (2.1). We shall assume G is of $\mathcal{P}$-Siegel type and that there exists $r \in \mathbb{R}_{>1}$ for which G is of $(\mathcal{P}, r)$-Rogers type. We let η denote an arbitrary norm on $\mathbb{R}^n$, and we let $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ denote an arbitrary nonincreasing and regular function.

Corollary 4.1 *Let $d \in \mathbb{R}_{\geq 1}$, and fix any $p, q \in \mathbb{Z}_{\geq 1}$ with $p + q = n$. Define $f : \mathbb{R}^n \to \mathbb{R}$ by (1.5). Define the norm ν on $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$ by*

$$\nu\big((\boldsymbol{x}, \boldsymbol{y})\big) := \max\left(\|\mathbf{x}\|_d, \|\mathbf{y}\|_d\right), \tag{4.1}$$

where $\|\cdot\|_d$ denotes the ℓ^d norm on each of the spaces $\mathbb{R}^p$ and $\mathbb{R}^q$. For each $k \in \{p, q\}$, we let v_k denote the volume of the unit ball in $\mathbb{R}^k$ and let v'_k denote the volume of the *unit sphere in $\mathbb{R}^k$ (each taken with respect to the ℓ^d norm on $\mathbb{R}^k$). Then the following hold.*

(i) *There exists some $M \in \mathbb{R}_{\geq 1}$ such that for any $T \geq S \geq M$ we have*

$$m\left(A_{f,\psi,\nu} \cap \{\mathbf{x} \in \mathbb{R}^n : S \leq \nu(\mathbf{x}) \leq T\}\right)$$
$$= \int_S^T z^{n-1}\left[\upsilon_p \upsilon_q'\left(1-\left(1-\frac{\psi(z)}{z^d}\right)^{p/d}\right)+\upsilon_q\upsilon_p'\left(1-\left(1-\frac{\psi(z)}{z^d}\right)^{q/d}\right)\right]dz.$$

(ii) *For almost every $g \in G$ the function $f \circ g$ is (resp., is not) $(\psi, \eta, \mathcal{P})$-approximable if the integral*

$$\int_1^\infty \psi(z) z^{n-(d+1)}\, dz \tag{4.2}$$

is infinite (resp., finite).

(iii) *Suppose that the series*

$$\begin{cases} \sum_{k=1}^\infty \left(k\psi(2^k)\right)^{1-r} & \text{if } d = n \\ \sum_{k=1}^\infty \left(2^{(n-d)k}\psi(2^k)\right)^{1-r} & \text{if } d \neq n \end{cases} \tag{4.3}$$

converges. Then for almost every $g \in G$ the function $f \circ g$ is uniformly $(\psi, \eta, \mathcal{P})$-approximable.

For the next example, we consider the space of products of n linearly independent linear forms on $\mathbb{R}^n$. In what follows, for each $j \in \mathbb{Z}_{\geq 0}$, we write $\log^j$ to denote the function $\mathbb{R}_{>0} \to \mathbb{R}$ given by $z \mapsto (\log z)^j$.

Corollary 4.2 *Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x_1, \ldots, x_n) := x_1 \cdots x_n$. Let ν denote the maximum norm on $\mathbb{R}^n$. Then there exists $M \in \mathbb{R}_{\geq 1}$ such that:*

(i) *For any $T \geq S \geq M$ we have*

$$m\left(A_{f,\psi,\nu} \cap \{\mathbf{x} \in \mathbb{R}^n : S \leq \nu(\mathbf{x}) \leq T\}\right) = 2^n n \int_S^T \frac{\psi(z)}{z}\left[\sum_{i=0}^{n-2} \frac{1}{i!} \log^i\left(\frac{z^n}{\psi(z)}\right)\right] dz. \tag{4.4}$$

(ii) *For almost every $g \in G$ the function $f \circ g$ is (resp., is not) $(\psi, \eta, \mathcal{P})$-approximable if the integral*

$$\int_1^\infty \frac{\psi(z)}{z} \log^{n-2}\left(\frac{z^n}{\psi(z)}\right) dz$$

is infinite (resp., finite).

(iii) *Suppose that the series*

$$\begin{cases} \sum_{k=1}^{\infty} \left(k\psi(2^k)\right)^{1-r} & \text{if } n=2 \\ \sum_{k=1}^{\infty} \left(\psi(2^k)\log^{n-1}\left(2^k\psi(2^k)^{-1/n}\right)\right)^{1-r} & \text{if } n>2 \end{cases}$$

converges. Then for almost every $g \in G$ the function $f \circ g$ is uniformly $(\psi, \eta, \mathcal{P})$-approximable.

Notice that Corollary 4.2(ii) is similar to [24, Theorem 1.11]; indeed, [24, Theorem 1.11] implies Corollary 4.2(ii) in the special case that $G = \mathrm{SL}_n(\mathbb{R})$, $\mathcal{P} = \mathbb{Z}^n_{\neq 0}$ and η is the maximum norm on $\mathbb{R}^n$. Whereas the proof of Kleinbock–Margulis in [24] relied on the Dani correspondence and the exponential mixing of the $\mathrm{SL}_n(\mathbb{R})$-action on $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$, our proof only uses the expectation and variance formulae of the Siegel transforms.

The next example is of interest because of its relation to the Khintchine–Groshev Theorem; see Remark 4.5.

Corollary 4.3 *Let $\ell \in \{1, \dots, n-1\}$ and $\mathbf{a} = (a_1, \dots, a_\ell) \in (\mathbb{R}_{>0})^\ell$ be given. Define $f : \mathbb{R}^n \to \mathbb{R}$ by*

$$f(x_1, \dots, x_n) := \max\left(|x_1|^{a_1}, \dots, |x_\ell|^{a_\ell}\right).$$

Set $a := \sum_{i=1}^{\ell} a_i^{-1}$. Let ν denote the maximum norm on $\mathbb{R}^n$. Then:

(i) *There exists some $M \in \mathbb{R}_{>0}$ such that for any $T \geq S \geq M$ we have*

$$m\left(A_{f,\psi,\nu} \cap \{\mathbf{x} \in \mathbb{R}^n : S \leq \nu(\mathbf{x}) \leq T\}\right) = 2^n(n-\ell)\int_S^T \psi(z)^a z^{n-(\ell+1)}\,dz.$$

(ii) *For almost every $g \in G$ the function $f \circ g$ is (resp., is not) $(\psi, \eta, \mathcal{P})$-approximable if the integral*

$$\int_1^\infty \psi(z)^a z^{n-(\ell+1)}\,dz$$

is infinite (resp., finite).

(iii) *Suppose that $\sum_{k=1}^{\infty}\left(2^{(n-\ell)k}\psi(2^k)^a\right)^{1-r}$ converges. Then for almost every $g \in G$ the function $f \circ g$ is uniformly $(\psi, \eta, \mathcal{P})$-approximable.*

Before proving these corollaries, let us make a few remarks.

Remark 4.4 (i) Corollary 1.4 is clearly a special case of Corollary 4.1.

(ii) The $n = 2$ case of Corollary 4.2 coincides with the $n = d = 2$ case of Corollary 4.1; however, the two volume formulas are slightly different due to the difference in the choice of norms.

(iii) In each corollary, part (i) does not require the regularity of ψ and is valid for any nonincreasing function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$.

(iv) In each of the corollaries, parts (ii) and (iii) may be used to calculate the critical exponents for asymptotic and uniform approximability, respectively: that is, the supremum of the set of all $s \in \mathbb{R}_{\geq 0}$ such that almost every element in the G-orbit of f is $(\varphi_s, \nu, \mathcal{P})$-approximable or uniformly $(\varphi_s, \nu, \mathcal{P})$-approximable, respectively, where φ_s is as in (1.3). In each corollary, one readily obtains that this supremum, if finite, is actually a maximum in the case of asymptotic approximation, and also that the critical exponents for asymptotic and uniform approximability coincide.

(v) Instead of using the sequence $\left(2^k\right)$ in part (iii) of each corollary, one may instead use any quasi-geometric sequence (t_k) that in addition is *lacunary*; that is, $\inf\left\{\dfrac{t_{k+1}}{t_k} : k \in \mathbb{Z}_{\geq 1}\right\} > 1$. In fact, it is not hard to prove that if $F : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is any Borel measurable function that satisfies some additional mild conditions, then the following are equivalent:

   (a) There exists a quasi-geometric sequence $(t_k)_{k\in\mathbb{Z}_{\geq 1}}$ for which $\sum_{k=1}^{\infty} F(t_k) < \infty$.

   (b) $\displaystyle\int_1^{\infty} \frac{F(x)}{x}\, dx < \infty.$

   (c) $\displaystyle\sum_{k=1}^{\infty} F(t_k) < \infty$ for any quasi-geometric and lacunary sequence $(t_k)_{k\in\mathbb{Z}_{\geq 1}}$.

Let us now proceed to prove the corollaries.

Proof of Corollary 4.1 Recall that the function $f : \mathbb{R}^n \to \mathbb{R}$ is defined by (1.5); that is, $f\big((\mathbf{x}, \mathbf{y})\big) = \|\mathbf{x}\|_d^d - \|\mathbf{y}\|_d^d$. For any $T \geq S \geq 0$ and with the norm ν on $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$ given by (4.1), we define

$$
\begin{aligned}
A_S^T &:= \left\{(\mathbf{x}, \mathbf{y}) \in A_{f,\psi,\nu} : S \leq \nu\left((\mathbf{x}, \mathbf{y})\right) \leq T\right\},\\
{}_{\mathbf{x}}A_S^T &:= A_S^T \cap \left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^p \times \mathbb{R}^q : \|\mathbf{y}\|_d \leq \|\mathbf{x}\|_d\right\},\\
{}_{\mathbf{y}}A_S^T &:= A_S^T \cap \left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^p \times \mathbb{R}^q : \|\mathbf{x}\|_d \leq \|\mathbf{y}\|_d\right\}.
\end{aligned}
$$

Since the function $\mathbb{R}_{>0} \to \mathbb{R}$ given by $z \mapsto z^d - \psi(z)$ is strictly increasing and unbounded from above, there exists $M \in \mathbb{R}_{\geq 1}$ such that for each $z \in [M, \infty)$, we have $z^d - \psi(z) > 0$. Now suppose that $T \geq S \geq M$. Then

$$
\begin{aligned}
m\left({}_{\mathbf{x}}A_S^T\right) &= m\left(\left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^p \times \mathbb{R}^q : \sqrt[d]{\|\mathbf{x}\|_d^d - \psi\left(\|\mathbf{x}\|_d\right)} \leq \|\mathbf{y}\|_d \leq \|\mathbf{x}\|_d \text{ and } S \leq \|\mathbf{x}\|_d \leq T\right\}\right)\\
&= v_q \int_{\{\mathbf{x}\in\mathbb{R}^p : S\leq\|\mathbf{x}\|_d\leq T\}} \left(\|\mathbf{x}\|_d^q - \left(\|\mathbf{x}\|_d^d - \psi\left(\|\mathbf{x}\|_d\right)\right)^{q/d}\right) d\mathbf{x}
\end{aligned}
$$

$$= v_q v_p' \int_S^T z^{p-1} \left(z^q - \left(z^d - \psi(z) \right)^{q/d} \right) dz$$
$$= v_q v_p' \int_S^T z^{n-1} \left(1 - \left(1 - \frac{\psi(z)}{z^d} \right)^{q/d} \right) dz.$$

By symmetry, we have

$$m\left({}_{\mathbf{y}} A_S^T \right) = v_p v_q' \int_S^T z^{n-1} \left(1 - \left(1 - \frac{\psi(z)}{z^d} \right)^{p/d} \right) dz,$$

which completes the proof of (i).

By using the Taylor expansions of the functions $x \mapsto 1-(1-x)^{p/d}$ and $x \mapsto 1-(1-x)^{q/d}$ around 0, it is easy to see that there exist some $S \in \mathbb{R}_{\geq M}$ and $C_1, C_2 \in \mathbb{R}_{>0}$ such that

$$z \in [S, \infty) \implies \left\{ 1 - \left(1 - \frac{\psi(z)}{z^d} \right)^{p/d}, 1 - \left(1 - \frac{\psi(z)}{z^d} \right)^{q/d} \right\} \subset \left[C_1 \frac{\psi(z)}{z^d}, C_2 \frac{\psi(z)}{z^d} \right]. \tag{4.5}$$

It is thus a consequence of (i) and (4.5) that $m(A_{f,\psi,\nu}) = \infty$ if and only if the integral (4.2) is infinite; hence, statement (ii) follows from Theorem 3.4. As for (iii), note that the series (4.3) diverges when $d > n$; thus, let us assume $d \leq n$. Using (i) and (4.5), we infer that there exists some $C \in \mathbb{R}_{>0}$ such that for any $T \in [S, \infty)$, we have

$$\begin{aligned} m\left(B_{f,\psi(T),\nu,T} \right) &= m\left(\left\{ \mathbf{x} \in \mathbb{R}^n : |f(\mathbf{x})| \leq \psi(T) \text{ and } \nu(\mathbf{x}) \leq T \right\} \right) \\ &\geq m\left(A_{f,\psi(T),\nu} \cap \{ \mathbf{x} \in \mathbb{R}^n : S \leq \nu(\mathbf{x}) \leq T \} \right) \\ &\geq C\psi(T) \int_S^T z^{n-(d+1)}\, dz \\ &= \begin{cases} C\psi(T)(\log T - \log S) & \text{if } d = n, \\ (n-d)^{-1} C\psi(T) \left(T^{n-d} - S^{n-d} \right) & \text{if } d < n. \end{cases} \end{aligned}$$

Thus, there exists some $C' \in \mathbb{R}_{>0}$ such that for each sufficiently large $T \in \mathbb{R}_{>0}$, we have

$$m\left(B_{f,\psi(T),\nu,T} \right) \geq \begin{cases} C'\psi(T)\log T & \text{if } d = n, \\ C'\psi(T)T^{n-d} & \text{if } d < n. \end{cases}$$

Letting $T = t_k = 2^k$ for each sufficiently large $k \in \mathbb{Z}_{\geq 1}$ and applying Theorem 3.8 implies (iii). ◻

Proof of Corollary 4.2 Recall that ν is the maximum norm on $\mathbb{R}^n$ and $f(\mathbf{x}) = x_1 \cdots x_n$. For each $k \in \{0, \dots, n-2\}$ and $z \in \mathbb{R}_{>0}$, define

$$I_k(z, \psi) := \underbrace{\int_0^z \dots \int_0^z}_{k \text{ times}} \min\left(z, \frac{\psi(z)}{z \prod_{i=1}^k y_i}\right) dy_1 \cdots dy_k;$$

when $k = 0$ here, there is no integration, and the empty product $\prod_{i=1}^0 y_i$ is equal to 1 by convention. It is easy to see that the left hand side of (4.4) is equal to $2^n n \int_S^T I_{n-2}(z)\, dz$. For each k and z as above, we now establish the following explicit formula:

$$\begin{aligned} I_k(z, \psi) &= \min\left(z^{k+1}, \frac{\psi(z)}{z}\right) \sum_{i=0}^{k} \frac{1}{i!} \max\left(0, \log\left(\frac{z^{k+2}}{\psi(z)}\right)\right)^i \\ &= \begin{cases} z^{k+1} & \text{if } \psi(z) \geq z^{k+2}; \\ \dfrac{\psi(z)}{z} \displaystyle\sum_{i=0}^{k} \frac{1}{i!} \log^i\left(\frac{z^{k+2}}{\psi(z)}\right) & \text{otherwise.} \end{cases} \end{aligned} \tag{4.6}$$

(Here, the second equality is obvious.) In this formula and in the remainder of this proof, we adopt the convention $0^0 = 1$. We now prove this formula by induction on k. The base case $k = 0$ is clear. Now suppose that for some $k \in \{1, \dots, n-3\}$, we have

$$I_{k-1}(z, \psi) = \min\left(z^k, \frac{\psi(z)}{z}\right) \sum_{i=0}^{k-1} \frac{1}{i!} \max\left(0, \log\left(\frac{z^{k+1}}{\psi(z)}\right)\right)^i;$$

then

$$\begin{aligned} I_k(z, \psi) &= \int_0^z I_{k-1}\left(z, \tfrac{1}{y}\psi\right) dy \\ &= \int_0^z \min\left(z^k, \frac{\psi(z)}{yz}\right) \sum_{i=0}^{k-1} \frac{1}{i!} \max\left(0, \log\left(\frac{yz^{k+1}}{\psi(z)}\right)\right)^i dy. \end{aligned} \tag{4.7}$$

Consider first the case $\psi(z) \geq z^{k+2}$. Then for any $0 < y \leq z$, one has

$$z^k \leq \frac{\psi(z)}{yz} \iff \frac{yz^{k+1}}{\psi(z)} \leq 1;$$

and (4.7) gives $I_k(z,\psi) = \int_0^z z^k\,dy = z^{k+1}$. If $\psi(z) < z^{k+2}$, then

$$
\begin{aligned}
I_k(z,\psi) &= \int_0^{\frac{\psi(z)}{z^{k+1}}} z^k\,dy + \int_{\frac{\psi(z)}{z^{k+1}}}^{z} \frac{\psi(z)}{yz} \sum_{i=0}^{k-1} \frac{1}{i!} \log^i\left(\frac{yz^{k+1}}{\psi(z)}\right) dy \\
&= \frac{\psi(z)}{z}\left[1 + \sum_{i=0}^{k-1} \frac{1}{i!} \int_{\frac{\psi(z)}{z^{k+1}}}^{z} \log^i\left(\frac{yz^{k+1}}{\psi(z)}\right)\frac{dy}{y}\right] \\
&= \frac{\psi(z)}{z}\left[1 + \sum_{i=0}^{k-1} \frac{1}{i!}\,\frac{1}{i+1} \log^{i+1}\left(\frac{yz^{k+1}}{\psi(z)}\right)\Bigg|_{\frac{\psi(z)}{z^{k+1}}}^{z}\right] \\
&= \frac{\psi(z)}{z} \sum_{i=0}^{k} \frac{1}{i!} \log^i\left(\frac{z^{k+2}}{\psi(z)}\right).
\end{aligned}
$$

This proves (4.6). Now fix $M \in \mathbb{R}_{\geq 1}$ such that for each $z \in \mathbb{R}_{\geq M}$, we have $\psi(z) < z^n$. Then (4.6) implies that for each $z \in \mathbb{R}_{\geq M}$, we have $I_{n-2}(z,\psi) = \frac{\psi(z)}{z} \sum_{i=0}^{n-2} \frac{1}{i!} \log^i\left(\frac{z^n}{\psi(z)}\right)$. This establishes (4.4) for any $T \geq S \geq M$ and proves (i).

In the expression $\sum_{i=0}^{n-2} \frac{1}{i!} \log^i\left(\frac{z^n}{\psi(z)}\right)$, the term that corresponds to $i = n-2$ dominates as $z \to \infty$; statement (ii) now follows from (i) and Theorem 3.4.

Finally, arguing as in the proof of Corollary 4.1(iii), we see that there exist $S \in \mathbb{R}_{\geq M}$ and $C, C' \in \mathbb{R}_{>0}$ such that for any $T \in [S,\infty)$ we have

$$
\begin{aligned}
m\left(B_{f,\psi(T),\nu,T}\right) &\geq m\left(A_{f,\psi(T),\nu} \cap \{\mathbf{x} \in \mathbb{R}^n : S \leq \nu(\mathbf{x}) \leq T\}\right) \\
&\geq C \int_S^T \frac{\psi(T)}{z} \log^{n-2}\left(\frac{z^n}{\psi(T)}\right) dz \\
&= n^{n-2} C\psi(T) \int_S^T \log^{n-2}\left(\frac{z}{\psi(T)^{1/n}}\right) \frac{dz}{z} \\
&\geq \begin{cases} C'\psi(T)(\log T - \log S) & \text{if } n = 2, \\ C'\psi(T)\left[\log^{n-1}\left(\frac{T}{\psi(T)^{1/n}}\right) - \log^{n-1}\left(\frac{S}{\psi(T)^{1/n}}\right)\right] & \text{if } n > 2. \end{cases}
\end{aligned}
$$

Thus, there exists $C'' \in \mathbb{R}_{>0}$ such that for each sufficiently large $T \in \mathbb{R}_{>0}$, we have

$$
m\left(B_{f,\psi(T),\nu,T}\right) \geq \begin{cases} C''\psi(T)\log T & \text{if } n = 2, \\ C''\psi(T)\log^{n-1}\left(\frac{T}{\psi(T)^{1/n}}\right) & \text{if } n > 2. \end{cases}
$$

Letting $T = t_k = 2^k$ for each sufficiently large $k \in \mathbb{Z}_{\geq 1}$ and applying Theorem 3.8 implies (iii).

Proof of Corollary 4.3 Recall that ν is the maximum norm on $\mathbb{R}^n$, and

$$f(x_1,\dots,x_n) = \max\left(|x_1|^{a_1},\dots,|x_\ell|^{a_\ell}\right).$$

Fix $M \in \mathbb{R}_{>0}$ such that for each $z \in \mathbb{R}_{\geq M}$ and each $i \in \{1,\dots,\ell\}$ we have $\psi(z)^{1/a_i} < M$. Take $T \geq S \geq M$, and set

$$A := A_{f,\psi,\nu} \cap \{\mathbf{x} \in \mathbb{R}^n : S \leq \nu(\mathbf{x}) \leq T\}$$

and $A_{\geq 0} := A \cap \left(\mathbb{R}_{\geq 0}\right)^n$. By symmetry, it is clear that $m(A) = 2^n\, m\left(A_{\geq 0}\right)$. For any $\mathbf{x} = (x_1,\dots,x_n) \in A_{\geq 0}$ and any $i \in \{1,\dots,\ell\}$ one has

$$|x_i| \leq \psi\left(\nu(\mathbf{x})\right)^{1/a_i} \leq \psi(S)^{1/a_i} < S \leq \nu(\mathbf{x}),$$

which implies $\nu(\mathbf{x}) \in \{|x_{\ell+1}|,\dots|x_n|\}$. Consequently,

$$A_{\geq 0} = \bigcup_{i=1,\dots,\ell;\ j=\ell+1,\dots,n} B_{ij},$$

where for any $i \in \{1,\dots,\ell\}$ and $j \in \{\ell+1,\dots,n\}$ we set

$$B_{i,j} := A_{\geq 0} \cap \left\{\mathbf{x} \in \mathbb{R}^n : \max\left(|x_1|^{a_1},\dots,|x_\ell|^{a_\ell}\right) = |x_i|^{a_i} \text{ and } \nu(\mathbf{x}) = |x_j|\right\}.$$

In other words,

$$B_{i,j} = \left\{\mathbf{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{c} S \leq x_j \leq T,\quad 0 \leq x_i \leq \psi\left(x_j\right)^{1/a_i}, \\ 0 \leq x_p \leq x_i^{a_i/a_p}\ \forall\, p \in \{1,\dots,\ell\} \smallsetminus \{i\}, \\ 0 \leq x_q \leq x_j\ \forall\, q \in \{\ell+1,\dots,n\} \smallsetminus \{j\} \end{array}\right\}.$$

Therefore,

$$\begin{aligned} m(B_{i,j}) &= \int_S^T x_j^{n-\ell-1} \int_0^{\psi(x_j)^{1/a_i}} \prod_{p\in\{1,\dots,\ell\}\smallsetminus\{i\}} x_i^{a_i/a_p}\, dx_i\, dx_j \\ \left(\text{recall the notation } a = \sum_{i=1}^{\ell} a_i^{-1}\right) &= \int_S^T x_j^{n-\ell-1} \int_0^{\psi(x_j)^{1/a_i}} x_i^{a_i(a-a_i^{-1})}\, dx_i\, dx_j \\ &= \int_S^T x_j^{n-\ell-1} \frac{1}{a_i a}\left(\psi(x_j)^{1/a_i}\right)^{a_i a} dx_j \\ &= \frac{1}{a_i a}\int_S^T z^{n-\ell-1}\psi(z)^a\, dz. \end{aligned}$$

It follows that

$$m(A)=2^n(n-\ell)\sum_{i=1}^{\ell}\frac{1}{a_i a}\int_S^T \psi(z)^a z^{n-(\ell+1)}\,dz = 2^n(n-\ell)\int_S^T \psi(z)^a z^{n-(\ell+1)}\,dz,$$

which proves (i). The other statements follow by arguing as in the proofs of Corollaries 4.1 and 4.2. □

Remark 4.5 More generally, one can take ℓ nonincreasing and regular functions $\psi_1,\ldots,\psi_\ell : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ and, using the same argument, show that for a.e. $g \in G$, the system of inequalities

$$|(g\mathbf{v})_i| \leq \psi_i\big(\nu(\mathbf{v})\big), \quad i = 1,\ldots,\ell$$

(here, $(g\mathbf{v})_i$ denotes the i^{th} component of $g\mathbf{v}$) has finitely (resp. infinitely) many solutions $\mathbf{v} \in \mathcal{P}$ if and only if the integral

$$\int_1^\infty \left[\prod_{i=1}^{\ell}\psi_i(z)\right] z^{n-(\ell+1)}\,dz$$

is finite (resp. infinite). In the case $G = \mathrm{SL}_n(\mathbb{R})$ and $\mathcal{P} = \mathbb{Z}^n_{\neq 0}$ this can also be derived from Schmidt's generalization of the Khintchine–Groshev Theorem; see [33, Theorem 2].

5 Concluding remarks

5.1 Inhomogeneous approximation

It is a natural problem to extend the methods of this paper to the inhomogeneous setting: that is, to study integer solutions of the system (1.6) for fixed ξ and almost every g, or vice versa, as is done in [17,18] for quadratic forms. Essentially this amounts to replacing the function f with $f - \xi$, thereby getting rid of the subhomogeneity condition, which is crucial for transforming the results about generic lattices to those for generic forms.

On the other hand, it is not hard to see that the assumption of Theorem 3.4 that f be subhomogeneous may be replaced with the assumption that f be Borel measurable and

- for any norms η and ν on $\mathbb{R}^n$ and any $s \in \mathbb{R}_{>0}$, the Lebesgue measure of $A_{f,s\psi,\eta}$ is finite if and only if that of $A_{f,\psi,\nu}$ is finite.

Similarly, one can weaken the subhomogeneity assumption of Theorem 3.8. This makes it possible to consider inhomogeneous problems for some classes of functions f, to be addressed in a forthcoming paper.

5.2 Counting the number of solutions

A comparison of Theorem 2.9(ii) with Theorem 3.4(ii) clearly shows a loss of information: a precise counting result for the number of lattice points in an increasing family of subsets of $\mathbb{R}^n$ turns into a rough estimate in the setting of generic subhomogeneous functions, with constants dependent on a compact subset of G. It is not clear whether, in the setting of Theorem 3.4(ii), the limit

$$\lim_{T\to\infty} \frac{\operatorname{card}\left\{\mathbf{v}\in\mathcal{P} : (f\circ g)(\mathbf{v}) \le \psi\big(\nu(\mathbf{v})\big) \text{ and } \nu(\mathbf{v}) \le T\right\}}{m\left(\left\{\mathbf{t}\in\mathbb{R}^n : f(\mathbf{t}) \le \psi\big(\nu(\mathbf{t})\big) \text{ and } \nu(\mathbf{t}) \le T\right\}\right)}$$

exists for almost every $g \in G$. It is also not clear whether any Khinchine-type results can be established without assuming the regularity of ψ.

5.3 More metric number theory

In general, the philosophy of this paper has been rooted in metric Diophantine approximation, which, in its simplest incarnation, studies the rate of approximation of typical real numbers α by rational numbers p/q. Our Khintchine-type theorems naturally give rise to many further questions. For example, in the $m\left(A_{f,\psi,\eta}\right) = \infty$ case of Theorem 1.3, one may wish to study the Hausdorff dimension of the null set consisting of all $g \in \mathrm{SL}_n(\mathbb{R})$ for which $f \circ g$ is not (ψ, ν)-approximable. In the case of a critical exponent, this might produce an analogue of badly approximable objects that constitute a set that is of full Hausdorff dimension or is winning in the sense of W. M. Schmidt [35]. Such a result is established in [27] for binary indefinite quadratic forms, that is, for the case $n = d = 2$ of Corollary 1.4. Namely, let ν be an arbitrary norm on $\mathbb{R}^2$. Then it follows from [27, Theorem 1.2] that the set

$$\{g \in \mathrm{SL}_2(\mathbb{R}) : \exists\, \varepsilon > 0 \text{ such that } f \circ g \text{ is not} (\varepsilon, \nu)\text{-approximable}\} \tag{5.1}$$

has full Hausdorff dimension. (The proof actually yields a stronger *hyperplane absolute winning* property introduced in [9] and known to imply winning, which then implies full Hausdorff dimension.) See also [1] for higher-dimensional generalizations. Note that Corollary 1.4 implies that the set (5.1) has Haar measure zero.

Alternatively, in the $m\left(A_{f,\psi,\eta}\right) < \infty$ case of Theorem 1.3, one can ask for the Hausdorff dimension of the null set consisting of all $g \in \mathrm{SL}_n(\mathbb{R})$ for which $f \circ g$ is (ψ, ν)-approximable. It seems natural to seek an analogue of the mass transference principle of Beresnevich–Velani in [7]; that being said, the lim sup sets in question have a complicated structure, and the standard techniques do not appear to be applicable.

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