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Pseudo-Edge Unfoldings of Convex Polyhedra

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Abstract
A pseudo-edge graph of a convex polyhedron K is a 3-connected embedded graph in K whose vertices coincide with those of K, whose edges are distance minimizing geodesics, and whose faces are convex. We construct a convex polyhedron K in Euclidean 3-space with a pseudo-edge graph with respect to which K is not unfoldable. The proof is based on a result of Pogorelov on convex caps with prescribed curvature, and an unfoldability obstruction for almost flat convex caps due to Tarasov. Our example, which has 340 vertices, significantly simplifies an earlier construction by Tarasov, and confirms that Dürer's conjecture does not hold for pseudo-edge unfoldings.

Keywords Edge unfolding · Dürer conjecture · Almost flat convex cap · Prescribed curvature · Weighted spanning forest · Pseudo-edge graph · Isometric embedding

Mathematics Subject Classification Primary: 52B10 · 53C45 · Secondary: 57N35 · 05C10

1 Introduction

By a *convex polyhedron* in this work we mean the boundary of the convex hull of finitely many points in Euclidean space $\mathbf{R}^3$ which do not all lie in a plane. A well-

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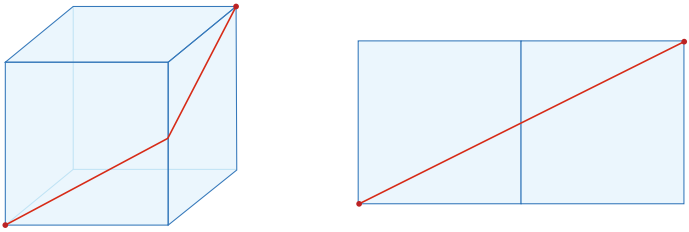


Fig. 1

known conjecture [8], attributed to the Renaissance painter Albrecht Dürer [9], states that every convex polyhedron K is *unfoldable*, i.e., it may be cut along some spanning tree of its edges and isometrically embedded into the plane $\mathbf{R}^2$. Here we study a generalization of this problem to *pseudo-edges* of K, i.e., distance minimizing geodesic segments in K connecting pairs of its vertices (see Fig. 1 for an example of a pseudo-edge which is not an actual edge). A *pseudo-edge graph* E of K is a 3-connected embedded graph composed of pseudo-edges of K, with the same vertices as those of K, and with faces which are *convex* in K, i.e., the interior angles of each face of E are less than π. Cutting K along any spanning tree T of E yields a simply connected compact surface K_T which admits an isometric immersion or *unfolding* $u_T : K_T \to \mathbf{R}^2$. If u_T is one-to-one for some T, then we say that K is *unfoldable* with respect to E. The main result of this paper is as follows:

Theorem 1.1 *There exists a convex polyhedron K with* 340 *vertices and a pseudo-edge graph with respect to which K is not unfoldable.*

Thus one may say that Dürer's conjecture does not hold in a purely intrinsic sense, since it is not possible to distinguish a pseudo-edge from an actual edge by means of local measurements within K. On the other hand, by Alexandrov's isometric embedding theorem [1], any convex polyhedron is determined up to a rigid motion by its intrinsic metric. So edges do indeed exist intrinsically, although Alexandrov's proof is not constructive and does not specify their location. A more constructive approach has been studied by Bobenko and Izmestiev [4] but that too does not yield a simple characterization for the edges. In short, the edges of convex polyhedra are not well understood from the point of view of isometric embeddings, and, in light of the above theorem, it would now be even more remarkable if the conjecture holds.

Theorem 1.1, for a polyhedron with over 19,000 vertices, was first announced in 2008 in a highly original and hitherto unpublished manuscript by Alexey Tarasov [24]. Although we do not understand all the details in that construction, since it is very complex, we can confirm that Tarasov's key ideas were correct, and utilize these in this work. These notions, which will be described below, include the obstruction for unfoldability of almost flat convex caps in Sect. 3, and the double spiral configuration in Sect. 5.

The polyhedron K in Theorem 1.1 is obtained by arranging four congruent almost flat convex caps over the faces of a regular tetrahedron. These caps have 84 interior vertices each with prescribed curvature and projection. They are constructed via a result of Pogorelov on convex caps with prescribed curvature as we describe in Sect. 2. In

Sect. 3 we study the pseudo-edges induced on a convex cap C by the edge graph G of convex subdivisions of the polygon at the base of C. Then in Sect. 4 we describe a necessary condition, due to Tarasov, for unfoldability of C in terms of spanning forests of G. Next in Sect. 5 we construct a convex subdivision of an equilateral triangle which satisfies Tarasov's criterion. Consequently, sufficiently flat convex caps constructed over this subdivision fail to be unfoldable with respect to the induced pseudo-edge graph. Finally in Sect. 6 we assemble four such caps to construct K.

Our construction differs from Tarasov's in the following respects. First, we use only two spiral paths, as opposed to three, in the subdivision of the equilateral triangle mentioned above. Second, our double spiral configuration in the center of the triangle uses far fewer vertices and thus is more transparent. Third, the corresponding convex caps we construct have planar boundaries due to our use of Pogorelov's theorem mentioned above, whereas Tarasov applies instead a related result of Alexandrov for unbounded polyhedra which does not yield precise control at the boundary of the cap. Fourth, since our caps have planar boundaries, we require only four copies of them to assemble our polyhedron, whereas Tarasov employs many more in a complex configuration.

The edge unfolding problem for convex polyhedra was first explicitly formulated by Shephard [23] in 1975, and since then has been advertised in several sources, e.g., [6,8,12,16,20,26]. The conjecture that the answer is yes, i.e., all convex polyhedra are unfoldable, appears to be first stated by Grünbaum [13] in 1991. The earliest known examples of unfoldings of convex polyhedra were drawn by Dürer [9] in 1525, all of which were nonoverlapping. Hence the unfolding problem or conjecture are often associated with his name. For more background, references, and a positive recent result see [11] where it is shown that every convex polyhedron becomes unfoldable after an affine transformation. See also O'Rourke [17,18] for other recent positive results concerning unfoldability of certain convex caps. As far as we know, Theorem 1.1 is the first hard evidence against Dürer's conjecture.

2 Convex Caps with Prescribed Boundary and Curvature

A (polyhedral) *convex cap* $C \subset \mathbf{R}^3$ is a topological disk which lies on a convex polyhedron and whose boundary ∂C lies in a plane H, while its interior $C \setminus \partial C$ is disjoint from H. The *normal cone* $N_p(C)$ of C at an interior point p is the convex cone generated by all outward normal vectors to support planes of C at p. The *unit normal cone* $\overline{N}_p(C)$ is the collection of unit vectors in $N_p(C)$. The *curvature* of C at p is defined as

$$\kappa(p) = \kappa_C(p) := \sigma(\overline{N}_p(C)),$$

where σ denotes the area measure in the unit sphere $\mathbf{S}^2$. Let $\pi : \mathbf{R}^3 \to \mathbf{R}^2$ denote the projection into the first two coordinates. A set $X \subset \mathbf{R}^3$ is a *terrain* over $\mathbf{R}^2$ provided that π is one-to-one on X, and $X \subset \mathbf{R}^2 \times [0, \infty)$. A *convex polygon* P is the convex hull of finitely many points in $\mathbf{R}^2$ which do not all lie on a line. We say that a convex cap C is *over* P provided that C is a terrain over $\mathbf{R}^2$ and $\partial C = \partial P$. We need the following result of Pogorelov [21, Lem. 1, p. 65], see also Pak's lecture notes [20, Thm. 35.7].

Lemma 2.1 (Pogorelov [21]). *Let P be a convex polygon, p_i, $i = 1, \ldots, n$, be points in the interior of P, and $\beta_i > 0$ with $\sum_i \beta_i < 2\pi$. Then there exists a unique convex cap C over P with interior vertices v_i such that $\pi(v_i) = p_i$, and $\kappa(v_i) = \beta_i$.*

A *convex subdivision* of a convex polygon P is a subdivision of P into finitely many convex polygons each of whose vertices either lies in the interior of P or coincides with a vertex of P (we assume that the interior angles of P at all its vertices are less than π). If G is the (edge) graph of a convex subdivision of P, then by an *interior vertex* p_i of G we mean a vertex of G which lies in the interior of P. We assume that the angles of incident edges of G at p_i are all less than π. We say that G is *weighted* if to each of its interior vertices there is associated a number $\alpha_i > 0$ with $\sum_i \alpha_i = 1$. Let the *total curvature* $\kappa(C)$ of a convex cap C be the sum of the curvatures of its interior vertices. Lemma 2.1 immediately yields:

Corollary 2.2 *Let P be a convex polygon, and G be the weighted graph of a convex subdivision of P, with interior vertices p_i and weights α_i. Then for any $0 < \beta < 2\pi$ there exists a convex cap C_β over P with interior vertices v_i such that $\pi(v_i) = p_i$ and $\kappa(v_i) = \beta_i := \alpha_i \beta$. In particular $\kappa(C_\beta) = \beta$.*

3 Pseudo-Edge Unfoldings of Almost Flat Convex Caps

In this section we fix P, G, and α_i to be as in Corollary 2.2, and aim to study the corresponding convex caps C_β for small β. In particular we will show that G gives rise to a unique pseudo-edge graph $\overline{G}$ of C_β (Proposition 3.4) and study the corresponding unfoldings of C_β in relation to P (Proposition 3.5).

3.1 The Induced Pseudo-Edge Graph of C_β

First we check that as $\beta \to 0$, $C_\beta \to P$. More precisely, if d_β denotes the intrinsic distance in C_β, then we have:

Lemma 3.1 *As $\beta \to 0$, $d_\beta(x, y) \to |\pi(x) - \pi(y)|$, for all $x, y \in C_\beta$.*

Proof As $\beta \to 0$, the maximum height of C_β goes to zero. If not, there exists a sequence $\beta_k \to 0$ such that the maximum height of $C_k := C_{\beta_k}$ is bounded below by $h > 0$. So, after refining the subsequence C_k further, we may assume that for some i, the height of the vertex v_i^k of C_k which projects onto p_i is bounded below by h. Let o be the point of height h above p_i, and C' be the convex cap formed by line segments connecting o to points of ∂P. Since C' lies below C_k, every support plane of C' at an interior vertex is parallel to a support plane of C_k at an interior vertex. So $\kappa(C_k) \geq \kappa(C') > 0$, which is the desired contradiction since $\kappa(C_k) = \beta_k \to 0$.

Now let L be the line segment connecting $\pi(x)$, $\pi(y)$, and $\overline{L}$ be the corresponding curve in C_β connecting x, y, such that $\pi(\overline{L}) = L$. Then $d_\beta(x, y) \leq \text{length}(\overline{L})$. But $\overline{L}$ is the graph of a convex function over L which converges to 0. Thus $\text{length}(\overline{L}) \to \text{length}(L) = |\pi(x) - \pi(y)|$. So the limit of $d_\beta(x, y)$ is not bigger than $|\pi(x) - \pi(y)|$. On the other hand $d_\beta(x, y) \geq |x - y| \geq |\pi(x) - \pi(y)|$. So $d_\beta(x, y) \to |\pi(x) - \pi(y)|$. □

A *polyhedral disk* D is a topological disk composed of a finite number of convex polygons identified along their edges. We say that D is *flat* if the total angle at each of its interior vertices is 2π. An *isometric immersion* $f: D \to \mathbf{R}^2$ is a locally one-to-one continuous map which preserves distances between points on each face of D. If f is one-to-one everywhere, then we say that it is an *isometric embedding*.

Lemma 3.2 *Let D be a flat polyhedral disk. Suppose that the total angle at each of the boundary vertices of D is less than 2π. Then there exists an isometric immersion $D \to \mathbf{R}^2$.*

Proof The angle condition along ∂D ensures that each point of D has a neighborhood which may be isometrically embedded into $\mathbf{R}^2$. Since D is simply connected, a family of these local maps may be joined to produce the desired global map, e.g., see the proof of [10, Lem. 2.2] for further details. □

By a *geodesic* in C_β we mean the image of a continuous map $\gamma: [a, b] \to C_\beta$ such that $\text{length}[\gamma] = d_\beta(\gamma(a), \gamma(b))$. For any set $X \subset \mathbf{R}^2$, and $r > 0$, $U_r(X)$ denotes the (open) set of points in $\mathbf{R}^2$ which are within a distance $< r$ of X. Further we set

$$\delta := \min \text{dist}(p_i p_j,\, p_k) \tag{1}$$

where p_i, p_j range over all pairs of adjacent vertices of G, so $p_i p_j$ indicates an edge of G (viewed as a line segment in $\mathbf{R}^2$), and p_k ranges over vertices different from p_i and p_j. Hence $\delta > 0$. By *sufficiently small*, or simply *small*, throughout this work we mean all nonzero values smaller than some positive constant. More explicitly, we say that some property holds for β sufficiently small, provided that there exists a constant $\beta_0 > 0$ such that the property holds for all $0 < \beta \le \beta_0$.

Lemma 3.3 *If β is sufficiently small, then to each edge e of G there corresponds a unique geodesic $\overline{e}$ of C_β whose end points project into the endpoints of e, and $\pi(\overline{e}) \subset U_\delta(e)$.*

Proof Let $x, y \in C_\beta$ be points which project into the end points of e, and Γ be a geodesic in C_β connecting x and y. By Lemma 3.1, $\text{length}(\Gamma) \to |\pi(x) - \pi(y)|$. Thus, for $0 < \beta \le \beta_0(e)$, $\pi(\Gamma) \subset U_\delta(e)$. We claim that Γ is unique. To this end suppose, towards a contradiction, that there exists another geodesic Γ' in C_β connecting x and y, which is different from Γ. Then again we have $\pi(\Gamma') \subset U_\delta(e)$, since by definition our geodesics are length minimizing, and so $\text{length}(\Gamma') = d_\beta(x, y) = \text{length}(\Gamma)$. Let $V \subset C_\beta$ be the region with $\pi(V) = U_\delta(e)$. Then $\Gamma, \Gamma' \subset V$. Since $\Gamma \ne \Gamma'$ and V is simply connected, there exists a simply connected domain $D \subset V$ bounded by a pair of subsegments Γ_0 and Γ_0' of Γ and Γ' respectively. Note that, by our choice of δ, see (1), V does not contain any vertices of C_β other than x and y. Thus D does not contain any vertices in its interior. So D admits an isometric immersion $f: D \to \mathbf{R}^2$ by Lemma 3.2. But, since isometries preserve geodesics, f maps Γ_0 and Γ_0' to straight line segments (which have the same end points). Hence $f(\Gamma_0) = f(\Gamma_0')$. In particular f is not locally injective at the points of ∂D where Γ_0 and Γ_0' meet, which is the desired contradiction. So Γ is indeed unique. Finally, setting $0 < \beta \le \min \beta_0(e)$, as e ranges over all edges of G, completes the proof. □

A *convex polygon* X in C_β is a region bounded by a simple closed curve composed of a finite number of geodesics meeting at angles which are less than π with respect to the interior of X. A *convex subdivision* of C_β is a subdivision into finitely many convex polygons whose interiors contain no vertices of C_β, and whose vertices are vertices of C_β. A *pseudo-edge graph* of C_β is the edge graph of a convex subdivision. By Lemma 3.3, for β sufficiently small, there exists a unique pseudo-edge graph $\overline{G}$ of C_β such that $\pi(\overline{G}) \subset U_\delta(G)$. Let G^T be the (canonical) triangulation of G given by connecting the center of mass of each nontriangular face of G to its vertices. Again by Lemma 3.3, there exists a unique triangulation $\overline{G}^T$ of $\overline{G}$ such that $\pi(\overline{G}^T) \subset U_\delta(G^T)$ and the vertices of $\overline{G}^T$ project onto the vertices of G^T. For any triangle Δ of G^T, let $\overline{\Delta}$ be the triangle of $\overline{G}^T$ whose vertices project onto the vertices of Δ. We will refer to Δ simply as a *triangle* of G, and call $\overline{\Delta}$ the corresponding triangle of $\overline{G}$. Note that, by Lemma 3.2, there exists an isometric embedding $u_\Delta \colon \overline{\Delta} \to \mathbf{R}^2$ for each triangle Δ of G.

Proposition 3.4 *For β sufficiently small, there exists a canonical homeomorphism $f \colon P \to C_\beta$ such that* (i) *f is the identity on ∂P,* (ii) *$f(G) = \overline{G}$, and* (iii) *$u_\Delta \circ f$ is an affine map on each triangle Δ of G. Furthermore, f converges to the identity map on P as $\beta \to 0$.*

Proof For any vertex p of G^T let $f(p) := \pi^{-1}(p) \cap C_\beta$. We define a mapping $g_\Delta \colon \Delta \to \Delta' := u_\Delta(\Delta)$ as follows. For any $x \in \Delta$, let (x_1, x_2, x_3) be the barycentric coordinates of x with respect to the vertices v_1, v_2, v_3 of Δ. Let $g_\Delta(x)$ be the point of Δ' whose barycentric coordinates with respect to the vertices $v_i' := u_\Delta \circ \pi^{-1}(v_i)$ of Δ' are (x_1, x_2, x_3). Finally set $f(x) := u_\Delta^{-1}(g_\Delta(x))$, where Δ is a triangle of G^T which contains x. Since C_β converges to P, as $\beta \to 0$, it follows that f converges to the identity map on P. □

3.2 Cut Forests and Unfoldings of C_β

A *tree* is a connected graph without cycles. A subgraph F of G is called a *cut forest* if (i) F is a collection of disjoint trees which contain all the vertices of G in the interior of P, and (ii) each tree of F contains exactly one vertex of ∂P; see the middle diagram in Fig. 2. By Lemma 3.3, to each cut forest F of G there corresponds a unique cut forest $\overline{F}$ of $\overline{G}$, assuming β is small. Let $C_{\beta,F}$ be the surface obtained from C_β by *cutting* it along F, i.e., take the disjoint collection of the faces of $\overline{G}$ and glue them together pairwise along all their common edges which do not belong to F.

By Lemma 3.2 there exists an isometric immersion, or *unfolding map* $u \colon C_{\beta,F} \to \mathbf{R}^2$. We assume that u fixes a designated edge e_0 of ∂C_β, and locally maps $C_{\beta,F}$ to the same side of e_0 where P lies. Let $\mathrm{cov} \colon C_{\beta,F} \to C_\beta$ be the natural covering map which sends each face of $C_{\beta,F}$ to the corresponding face of C_β, and $f \colon P \to C_\beta$ be the homeomorphism given by Proposition 3.4. Then

$$\psi := u \circ \mathrm{cov}^{-1} \circ f \tag{2}$$

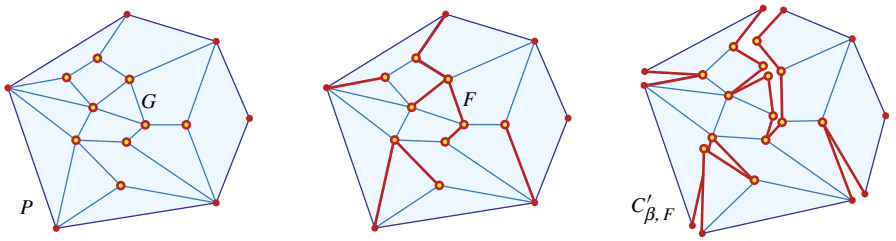


Fig. 2

is a multivalued mapping $P \to C'_{\beta,F} := u(C_{\beta,F})$. Since f converges to the identity map on P, it follows that ψ also converges to the identity on P as $\beta \to 0$. Note that ψ is single valued on $P \setminus F$ and is one-to-one on the interior of each face Φ of G. But ψ is doubly valued for points in the interior of each edge of F (where a cut occurs). Furthermore, if x is a vertex of F, then the cardinality of $\psi(x)$ is equal to the degree of x in F (see Fig. 2). For any point $x \in P$, we set

$$x' := \psi(x).$$

So x' in general indicates a set of points, not a single point. Whenever we state that x' satisfies some property, we mean that *each element* of x' satisfies that property.

Note that ψ induces a natural single-valued map ψ_Φ on each face Φ of G as follows. Let Φ' indicate the face of $C'_{\beta,F}$ corresponding to Φ, i.e., the closure of $\psi(\mathrm{int}(\Phi))$. Then we obtain a homeomorphism ψ_Φ between Φ and Φ', by setting $\psi_\Phi := \psi$ on $\mathrm{int}(\Phi)$ and extending the map continuously to the boundary of Φ. For any $x \in \Phi$, let

$$x'_\Phi := \psi_\Phi(x)$$

denote the corresponding (single) point of Φ'. Since ψ converges to the identity on P, it follows that, as $\beta \to 0, x'_\Phi \to x$ for all faces Φ of G which contain x. In other words, ψ_Φ converges pointwise to the identity map id_Φ on Φ. Thus, as ψ_Φ is continuous and Φ is compact, we conclude that

$$\psi_\Phi \to \mathrm{id}_\Phi \quad \text{uniformly as} \quad \beta \to 0 \tag{3}$$

for all faces Φ of G. Next note that for every point x in the interior of a triangle Δ of Φ we have

$$\psi_\Phi(x) = \psi(x) = u \circ \mathrm{cov}^{-1} \circ f(x) = u \circ f(x) = u_\Delta \circ f(x).$$

Also recall that, by Proposition 3.4, $u_\Delta \circ f$ is an affine map. Thus

$$\psi_\Phi \text{ is an affine map on each triangle } \Delta \subset \Phi. \tag{4}$$

These properties of ψ_Φ yield that:

Proposition 3.5 *For every $\epsilon > 0$, there exists $\beta_0(\epsilon) > 0$ such that for all $0 < \beta \leq \beta_0(\epsilon)$, $x \in P$, and faces Φ of G which contain x,*

$$|x - x'_\Phi| \leq \epsilon. \tag{5}$$

Furthermore, for any pair of points x, y which both lie in the same triangle $\Delta \subset \Phi$,

$$|(x - y) - (x'_\Phi - y'_\Phi)| \leq \epsilon |x - y|. \tag{6}$$

Proof The inequality (5) follows immediately from (3). To see (6), note that by (4) ψ_Φ may be extended to an affine mapping from $\mathbf{R}^2$ to $\mathbf{R}^2$, and thus be written as $\ell + c$ for a linear transformation $\ell\colon \mathbf{R}^2 \to \mathbf{R}^2$ and some fixed vector $c \in \mathbf{R}^2$. Thus

$$x'_\Phi - y'_\Phi = \psi_\Phi(x) - \psi_\Phi(y) = \ell(x - y).$$

If $x = y$, (6) already holds. Otherwise, we may set $z := (x - y)/|x - y|$ and divide the left hand side of (6) by $|x - y|$ to obtain

$$\frac{|(x - y) - (x'_\Phi - y'_\Phi)|}{|x - y|} = \left| \frac{x - y}{|x - y|} - \ell\left(\frac{x - y}{|x - y|}\right) \right| = |z - \ell(z)|.$$

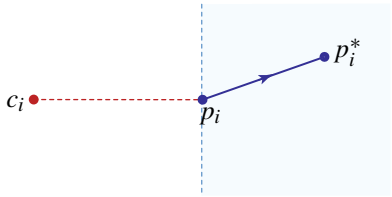
Finally note that since $\psi_\Phi \to \mathrm{id}_\Phi$, ℓ must converge to the identity map on any compact subset of $\mathbf{R}^2$. In particular, we may choose $\beta_0(\epsilon)$ so small that $|z - \ell(z)| \leq \epsilon$ for all unit vectors $z \in \mathbf{S}^1$, which completes the proof. □

4 Tarasov's Monotonicity Condition

As in the last section, let P be a convex polygon, G be the graph of a fixed convex subdivision of P with weights α_i, and C_β be the corresponding convex cap over P given by Corollary 2.2. Here we describe Tarasov's obstruction for unfoldability of C_β with respect to the induced pseudo-edge graph $\overline{G}$ given by Proposition 3.4.

By an *edge* of G we mean the line segment connecting a pair of adjacent vertices of G, and a *point* of G is any point of an edge of G. We say a pair of points are adjacent if they belong to the same edge. A *path* Γ in G is a sequence of points, each adjacent to the next, all of which are vertices except possibly the initial point. We say that Γ is *simple* if the sequence of line segments determined by its consecutive points forms a non-self-intersecting curve. If F is a cut forest of G, then each point p of F may be joined to ∂P with a unique simple path Γ_p in F, which we call the *ancestral path* of p. This induces a partial ordering on points of F as follows: we write $y \succeq x$, for $x, y \in F$ and say that y is a *descendant* of x or x is an *ancestor* of y, if $x \in \Gamma_y$. In particular note that $x \succeq x$. If $y \succeq x$ and $x \neq y$ then we say y is a *strict descendant* of x, or x is a strict ancestor of y, and write $x \succ y$. Furthermore, we adopt the following convention: for any $x \in F$, we write $i \succeq x$ provided that $p_i \succeq x$, where p_i denote the vertices of G.

Fig. 3



For any point $x \in G$, we define the *center of rotation* of x as the center of mass of its descendant vertices with respect to the weights α_i:

$$c_x := \alpha_x^{-1} \sum_{i \geq x} \alpha_i \, p_i, \quad \text{where} \quad \alpha_x := \sum_{i \geq x} \alpha_i .$$

Roughly speaking, c_x is the limit, as $\beta \to 0$, of the pivot point about which x rotates (in different directions) to generate (the elements of) x' defined in the last section (see Note 4.3).

Every interior vertex p_i of G has a unique adjacent vertex p_i^* in F which is its *parent* or first strict ancestor which is a vertex. We also refer to p_i as a *child* of p_i^*. A cut forest F of G is called *monotone* (in the sense of Tarasov), if for every interior vertex p_i of G we have

$$\langle p_i^* - p_i, \, p_i - c_i \rangle \geq 0, \quad \text{where} \quad c_i := c_{p_i}. \tag{7}$$

In other words, p_i^* must lie in the set $\langle x - p_i, \, p_i - c_i \rangle \geq 0$, which forms a half-plane when $c_i \neq p_i$; see Fig. 3. So, if $c_i \neq p_i$ and $0 \leq \measuredangle p_i^* p_i c_i \leq \pi$ denotes the angle between the vectors $p_i^* - p_i$ and $c_i - p_i$, then we have

$$\measuredangle p_i^* p_i c_i \geq \pi/2. \tag{8}$$

In particular, every parent must be further away from the center of rotation of its child than the child is:

$$|p_i^* - c_i| > |p_i - c_i|. \tag{9}$$

Hence the term "monotone". Note that (8) is equivalent to (7) whenever $c_i \neq p_i$; however, (9) is a strictly weaker notion. Below we will primarily use the form (8) of the monotonicity condition. Some other notions of monotonicity have been studied recently by O'Rourke [17,18], and Lubiw and O'Rourke [15] for cut forests of convex polyhedral disks. One of these notions, called radial monotonicity, will be invoked in Sect. 5.2 below as it is somewhat related to (8). See [11] for yet another monotonicity notion in the context of unfoldings.

Recall that $C'_{\beta,F}$ is the image of the unfolding map $u \colon C_{\beta,F} \to \mathbf{R}^2$. We say that $C'_{\beta,F}$ is *simple*, if and only if u is injective. If $C'_{\beta,F}$ is simple for some cut forest F of G, we say that C_β is *unfoldable* with respect to G. If G admits no monotone cut forests, then we say that G is *non-monotone*. The rest of this section is devoted to establishing the following result which parallels [24, Thm. 1]. Recall that by sufficiently small

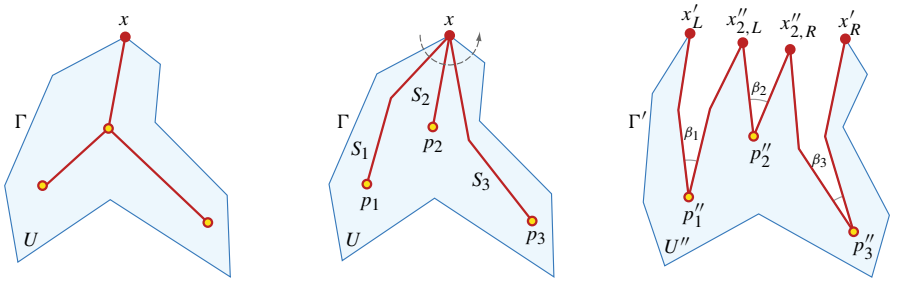


Fig. 4

throughout the paper, as we stated in Sect. 3, we mean for all values smaller than some constant.

Theorem 4.1 *If G is non-monotone, then C_β is not unfoldable with respect to $\overline{G}$, for β sufficiently small.*

We prove the above theorem via the same general approach indicated in [24], although we correct a number of errors or ambiguities, provide more details, and make many simplifications. Fix a cut forest F of G. If $x \in F$ is not a vertex, x' consists of precisely two elements: x'_R and x'_L defined as follows. Orient the edge e of F containing x from the child to the parent vertex. Then we can distinguish the faces Φ_R, Φ_L of G which lie to the right and left of e respectively. We set

$$x'_R := x'_{\Phi_R} \qquad \text{and} \qquad x'_L := x'_{\Phi_L}.$$

Let J be the $\pi/2$-clockwise rotation about the origin of $\mathbf{R}^2$, and set

$$\widetilde{c}_x := x + \frac{J(x'_R - x'_L)}{\beta_x}, \qquad \text{where} \qquad \beta_x := \alpha_x \beta = \sum_{i \succeq x} \beta_i. \tag{10}$$

The next observation parallels [24, Lem. 1].

Lemma 4.2 *For every $x \in F$, $\widetilde{c}_x \to c_x$, as $\beta \to 0$.*

Proof Let $F_x := \{y \in F \mid y \succeq x\}$, and Γ be a polygonal Jordan curve in P which encloses F_x and intersects F only at x; see the left diagram in Fig. 4. Then $\Gamma' := \psi(\Gamma)$ is a polygonal path connecting x'_R and x'_L. Let U be the region bounded by Γ which contains F_x, and $S_i \subset U$ be simple polygonal paths which connect each $p_i \succ x$ to x without intersecting each other and Γ; see the middle diagram in Fig. 4. We are going to reindex S_i and p_i as follows. Let σ be a circle centered at x with sufficiently small radius so that it intersects Γ only twice, and each S_i only once. Orient σ counterclockwise, reindex S_i, from $i = 1, \ldots, k$, in order that they intersect $\sigma \cap U$, and then reindex p_i accordingly.

Now let $S := \bigcup_i S_i$ be the resulting spanning tree for vertices of U. Then $\overline{S} := f(S)$ is a tree on $\overline{U} := f(U)$. Let $\overline{U}_{\overline{S}}$ denote the topological disk obtained by cutting $\overline{U}$

along $\overline{S}$, $\mathrm{cov}_S\colon \overline{U}_{\overline{S}} \to \overline{U}$ be the corresponding covering map, $u_S\colon \overline{U}_{\overline{S}} \to \mathbf{R}^2$ be an unfolding given by Lemma 3.2, and define the multivalued mapping $\theta\colon U \to \mathbf{R}^2$ by

$$\theta := u_S \circ \mathrm{cov}_S^{-1} \circ f.$$

Comparing this definition with that of ψ given by (2) shows that $\Gamma'' := \theta(\Gamma)$ is congruent to $\Gamma' := \psi(\Gamma)$. Indeed Γ', Γ'' are determined, up to a translation, by the edge lengths of Γ and its interior angles with respect to U. So we may assume that $\Gamma'' = \Gamma'$ (by choosing u_S appropriately, or composing it with an isometry of $\mathbf{R}^2$ which caries Γ'' to Γ'). Now note that θ converges to the identity map on U, just as ψ does by Proposition 3.5. Thus, if we set $x'' := \theta(x)$, then $x'' \to x$, as $\beta \to 0$.

Each p_i in U has a single image p_i'' under θ, while there are two images of x under θ corresponding to each S_i, which are denoted by $x''_{i,L}$ and $x''_{i,R}$; see the right diagram in Fig. 4. These may be defined similar to the way we defined x'_L and x'_R, by extending S to a triangulation of U. We claim that

$$\widetilde{c}_x = \alpha_x^{-1} \sum_{i \succeq x} \alpha_i \widetilde{p}_i, \quad \text{where} \quad \widetilde{p}_i := x + \frac{J(x''_{i,R} - x''_{i,L})}{\beta_i}. \tag{11}$$

Indeed, since $\Gamma'' = \Gamma'$, and due to our reindexing of S_i, we have $x'_L = x''_{1,L}$, $x'_R = x''_{k,R}$, and $x''_{i,R} = x''_{i+1,L}$, for $1 \le i \le k-1$. Thus, since $\beta_i = \beta\alpha_i = \beta_x \alpha_x^{-1} \alpha_i$,

$$\begin{aligned} x'_R - x'_L &= \sum_{i \succeq x} (x''_{i,R} - x''_{i,L}) = \sum_{i \succeq x} \beta_i J(x - \widetilde{p}_i) \\ &= J\left(\beta_x x - \sum_{i \succeq x} \beta_i \widetilde{p}_i\right) = \beta_x J(x - \widetilde{c}_x). \end{aligned}$$

Applying J to the far left and right sides of the last expression yields (11). Next note that, since $\measuredangle x''_{i,L} p_i'' x''_{i,R} = \beta_i$, elementary trigonometry yields that

$$p_i'' = x''_{i,M} + \frac{J(x''_{i,R} - x''_{i,L})}{2\tan(\beta_i/2)}, \quad \text{where} \quad x''_{i,M} = \frac{x''_{i,L} + x''_{i,R}}{2};$$

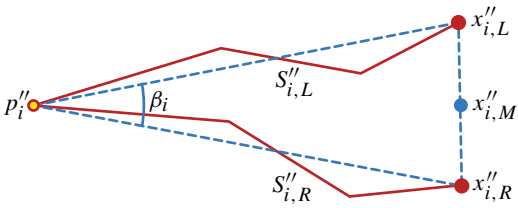
see Fig. 5. Thus we have

$$\widetilde{p}_i - p_i = (x''_{i,M} - p_i'')(1 - 2\tan(\beta_i/2)/\beta_i) + (x - x''_{i,M}) + (p_i'' - p_i).$$

Since the right hand side vanishes, as $\beta \to 0$, it follows that $\widetilde{p}_i \to p_i$, and consequently $\widetilde{c}_x \to c_x$ as desired. □

Note 4.3 Let $R_{p,\theta}\colon \mathbf{R}^2 \to \mathbf{R}^2$ denote the clockwise rotation about the point p by the angle θ. As we discussed in the proof of Lemma 4.2, $x''_{i,R} = R_{p_i'',\beta_i}(x''_{i,L})$. So, since $p_i'' \to p_i$, as $\beta \to 0$,

Fig. 5



$$x'_R = R_{p''_1,\beta_1} \circ \cdots \circ R_{p''_k,\beta_k}(x'_L) \to R_{p_1,\beta_1} \circ \cdots \circ R_{p_k,\beta_k}(x'_L) = R_{p_x,\theta_x}(x'_L),$$

for some $p_x \in \mathbf{R}^2$ and $\theta_x \in [0, 2\pi)$. It is known that [19, Lem. 1], as $\beta \to 0$, $\theta \to \sum_{i=1}^{k} \beta_i = \sum_{i \geq x} \beta_i = \beta_x$, and

$$p_x \to \beta_x^{-1} \sum_{i=1}^{k} \beta_i p_i = \alpha_x^{-1} \sum_{i \geq x} \alpha_i p_i = c_x.$$

So c_x is the limit of the cumulative pivot point of descendant vertices of x, which is the justification for the term "center of rotation". See also [5] for a study of unfoldings of flat regions of convex polyhedra.

Proof of Theorem 4.1 Fix a cut forest F of G. Then there exists an interior vertex p_i of G which does not satisfy (7) and will be fixed henceforth. Let Φ_R and Φ_L be faces of G which lie to the right and left of the oriented edge $p_i p_i^*$ respectively, see Fig. 6. We will show that, for β sufficiently small (i.e. $0 < \beta \leq \beta_0(F)$), a point of Φ'_R lies in the interior of Φ'_L. So $C'_{\beta,F}$ is not simple. Since G admits only finitely many cut forests, this will complete the proof (once we let β be smaller than the minimum value of $\beta_0(F)$ as F ranges over all cut forests of G). To start, fix $\lambda > 0$ so small that

$$\langle p_i^* - p_i, p_i - c_i \rangle \leq -\lambda |p_i^* - p_i|. \tag{12}$$

Let x be an interior point of $p_i p_i^*$ such that

$$|x - p_i| \leq \lambda/2, \tag{13}$$

and note that x does not depend on β. Next set

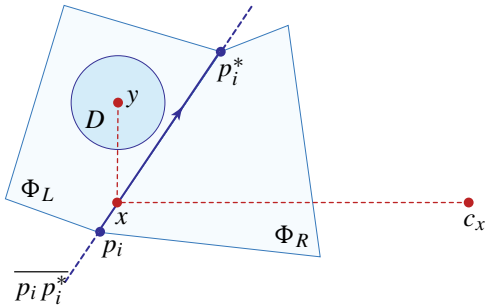
$$y := x + \beta_x J(x - c_x), \qquad r := \beta_x \lambda/2,$$

and let D be the disk of radius r centered at y. We will show that for β small: $|y' - x'_R| < r$, and $D \subset \Phi_L$. Thus $x'_R \in \text{int}(D') \subset \text{int}(\Phi'_L)$, as desired.

By the triangle inequality,

$$|y' - x'_R| \leq |(x'_R - x'_L) - (y - x)| + |(y - x) - (y' - x'_L)|.$$

Fig. 6



By Lemma 4.2 we may choose β so small that $|\widetilde{c}_x - c_x| < \lambda/4$. Then, by (10),

$$|(x'_R - x'_L) - (y - x)| = |\beta_x J(x - \widetilde{c}_x) - \beta_x J(x - c_x)| = \beta_x |\widetilde{c}_x - c_x| < \beta_x \lambda/4.$$

Set $\eta := \operatorname{diam}(P)$. Note that, as x lies in the interior of $p_i p_i^*$, we may choose β so small that y lies in the triangle Δ of Φ_L which rests on $p_i p_i^*$. Thus, by Proposition 3.5, we can make sure that

$$|(y - x) - (y' - x'_L)| \le (\lambda/(4\eta))|y - x| = (\lambda/(4\eta))\beta_x |x - c_x| \le \beta_x \lambda/4,$$

since $c_x \in P$ and therefore $|x - c_x| \le \eta$. The last three displayed expressions yield that $|y' - x'_R| < \beta_x \lambda/2 = r$, as claimed.

As $\beta \to 0$, we have $y \to x$ and $r \to 0$. Thus, for β small, $D \subset \Phi_R \cup \Phi_L$. Since y lies on the left side of the oriented line $\overline{p_i p_i^*}$ passing through p_i and p_i^*, it follows that $y \in \Phi_L$. So it remains to check that $\operatorname{dist}(y, \overline{p_i p_i^*}) \ge r$. By definition, $c_x = c_{p_i} = c_i$. Thus, by (12) and (13),

$$\langle x - c_x, p_i^* - p_i \rangle = \langle x - p_i, p_i^* - p_i \rangle + \langle p_i - c_i, p_i^* - p_i \rangle \le -(\lambda/2)|p_i^* - p_i|.$$

So

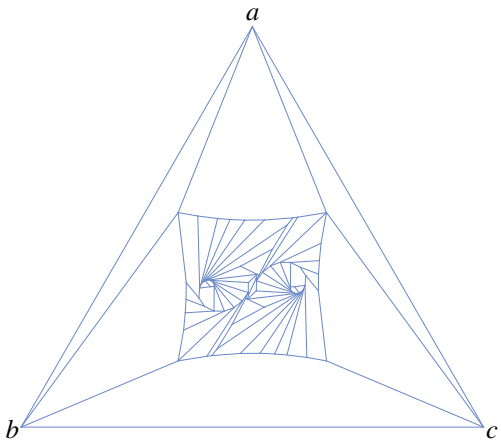
$$\cos(\measuredangle c_x x p_i^*) = -\frac{\langle x - c_x, p_i^* - p_i \rangle}{|x - c_x||p_i^* - p_i|} \ge \frac{\lambda/2}{|x - c_x|} = \frac{\beta_x \lambda/2}{\beta_x |x - c_x|} = \frac{r}{|y - x|},$$

which yields $\operatorname{dist}(y, \overline{p_i p_i^*}) = \sin(\measuredangle y x p_i^*)|y - x| = \cos(\measuredangle c_x x p_i^*)|y - x| \ge r$, and completes the proof. □

5 A Non-Monotone Convex Subdivision of the Equilateral Triangle

A convex subdivision of a convex polygon is *weighted* if the corresponding graph G is weighted, as defined in Sect. 3. Further the subdivision, or its graph G, is *non-monotone* provided that G admits no monotone cut forests, as defined in Sect. 4. In this section we show:

Fig. 7



Theorem 5.1 *The equilateral triangle admits a non-monotone weighted convex subdivision with* 84 *interior vertices.*

Earlier Tarasov [24] had constructed a non-monotone subdivision of the equilateral triangle with over 500 vertices. Here we simplify that construction as follows.

5.1 Coordinates and Weights

The edge graph G of our subdivision is illustrated in Fig. 7, and a larger depiction of the subgraph of G spanned by its vertices in the interior of the triangle appears in Fig. 8. We call this subgraph the *square*. We assume that the triangle has vertices $a := (0, 140\sqrt{3} - 85)$, $b := (-140, -85)$, and $c := (140, -85)$. The square is symmetric with respect to reflection through $o := (0, 0)$ and has 84 vertices. We label half of these vertices by $p_i, i = 1, \ldots, 42$, as shown in Fig. 8. The coordinates of these points are listed in Table 1, as well as in an accompanying *Mathematica* notebook [2] that we have provided. The other vertices of the square are the reflection of p_i, and will be denoted by $p_{-i} := -p_i$. We assume that $p_{\pm 1}$ have equal weights, and the weights of all other vertices are arbitrarily small.

5.2 Main Properties

As in [17], we say a path $\Gamma = (v_1, \ldots, v_n)$ of G is *radially monotone* with respect to a point x provided that

$$\measuredangle(v_{i+1}, v_i, x) \geq \pi/2 \tag{14}$$

for $i = 1, \ldots, n-1$. We say a path in G is *maximal* if it ends on the boundary of G. Our subdivision has been designed so that it has two important features, as expressed in the following lemmas and illustrated in Fig. 10.

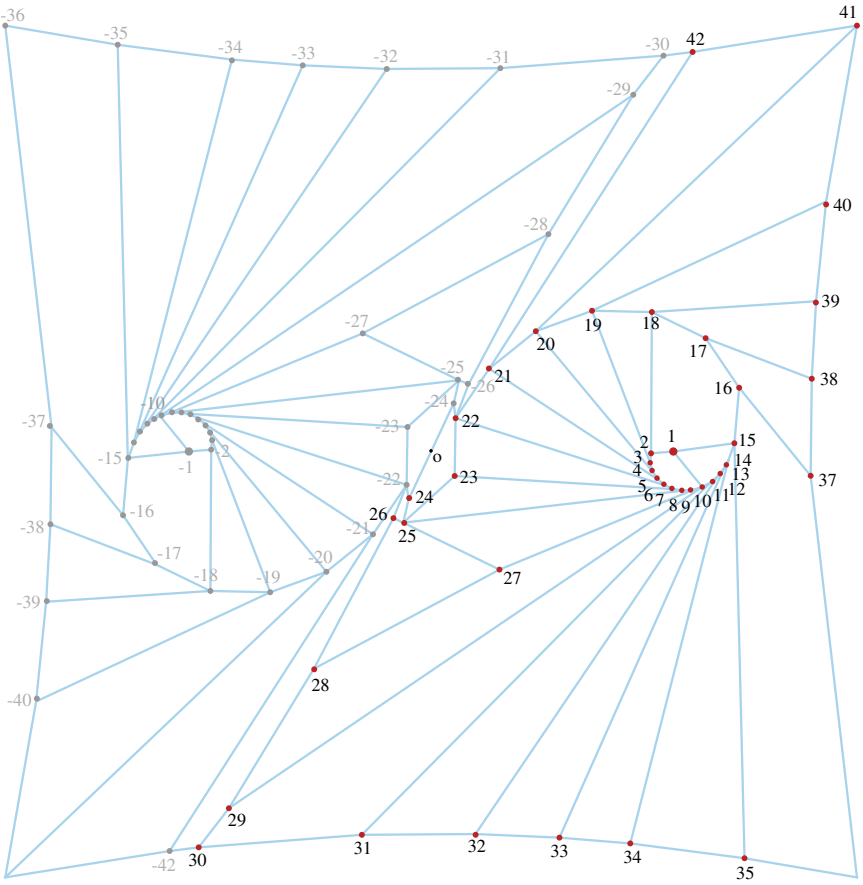


Fig. 8

Lemma 5.2 *Let Γ be a path in G which originates at p_1 and is radially monotone with respect to p_1. Then Γ must be a subpath of*

$$(p_1, p_\ell, p_{\ell+1}, \ldots, p_{21}, p_{22}, p_{-24}, p_{24}, p_{25}, p_{26}, p_{28}, p_{29}, p_{30}, p_{-42}, p_{-41}, q), \tag{15}$$

where $\ell = 2$, 10, or 15, and q is a vertex of the triangle.

Proof The only vertices of G which are adjacent to p_1 are p_ℓ, where $\ell = 2$, 10, or 15; see the left diagram in Fig. 9. Thus the second vertex of Γ must be p_ℓ. Suppose that $\ell = 2$. Other than p_1, the only vertices adjacent to p_2 are p_{18} and p_3. One quickly checks that $\measuredangle(p_3, p_2, p_1) \geq \pi/2$ while $\measuredangle(p_{18}, p_2, p_1) < \pi/2$. Thus p_3 (or $p_{\ell+1}$) is the only choice for the next vertex of Γ. Similarly, to complete the proof, it suffices to check that, if we denote the vertices of the proposed path (15) by $v_i, i = 1, \ldots, n$, then v_{i+1} is the only vertex w adjacent to v_i such that $\measuredangle(w, v_i, p_1) \geq \pi/2$. Finally, once the path reaches a vertex of the triangle, then it cannot be extended further, since the angles of the triangle are all obtuse. We omit the computations since they are trivial,

Table 1

1 : (25.5,0)	**2** : (23.1,−0.1)	**3** : (23.1,−1.1)	**4** : (23.3,−2.1)	**5** : (23.9,−2.9)
6 : (24.7,−3.5)	**7** : (25.6,−3.9)	**8** : (26.6,−4.1)	**9** : (27.6,−4)	**10**: (28.5,−3.7)
11 : (29.4,−3.2)	**12** : (30.2,−2.6)	**13** : (30.8,−1.8)	**14** : (31.3,−0.9)	**15** : (31.9,0.8)
16 : (32.4,6.7)	**17** : (28.9,11.9)	**18** : (23.2,14.7)	**19** : (16.9,14.8)	**20** : (11,12.6)
21 : (6.1,8.7)	**22** : (2.5,3.5)	**23** : (2.4,−2.5)	**24** : (−2.3,−5)	**25** : (−2.9,−7.5)
26 : (−4,−7)	**27** : (7.2,−12.4)	**28** : (−12.3,−23)	**29** : (−21.3,−37.6)	**30** : (−24.5,−41.7)
31 : (−7.3,−40.4)	**32** : (4.6,−40.3)	**33** : (13.5,−40.7)	**34** : (21,−41.3)	**35** : (33,−42.8)
36 : (44.9,−44.9)	**37** : (40,−2.5)	**38** : (40,7.6)	**39** : (40.6,15.8)	**40** :(41.6,26.3)
41 : (44.9,44.9)	**42** : (27.5,42.1)			

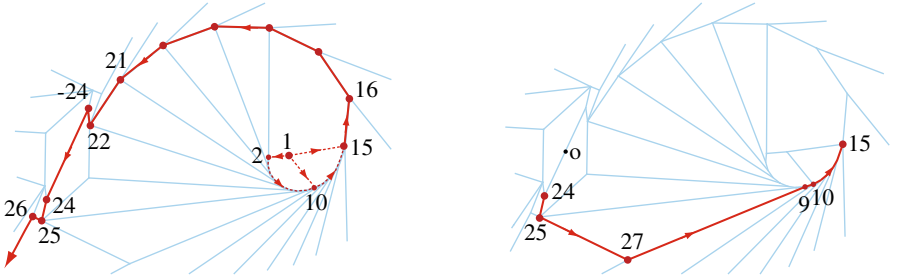


Fig. 9

and refer the reader instead to the accompanying *Mathematica* notebook [2], where all the computations have been recorded. □

Lemma 5.3 *Let* Γ *be a maximal path in* G *which originates at* p_{24} *and is radially monotone with respect to* o*. Then*

$$\Gamma = (p_{24}, p_{25}, p_{27}, p_9, p_{10}, \ldots, p_{15}, \ldots).$$

In particular Γ *contains* p_{15}*.*

Proof As in the proof of Lemma 5.2, one may easily check that the only possible choice for the successor of p_{24} in Γ, which would satisfy (14), is p_{25}. Similarly, one may recover all other vertices as well; see the right diagram in Fig. 9. This involves a series of trivial computations which are included in the accompanying *Mathematica* notebook [2]. □

5.3 Proof of Theorem 5.1

We claim that the subdivision generated by the weighted graph G described above is non-monotone. Assume, towards a contradiction, that G admits a monotone cut forest F (as defined in Sect. 3.2). Then each vertex p_i of the square part of G has a (unique) ancestral path in F, which we denote by Γ_i. Recall that the final vertex of Γ_i must be

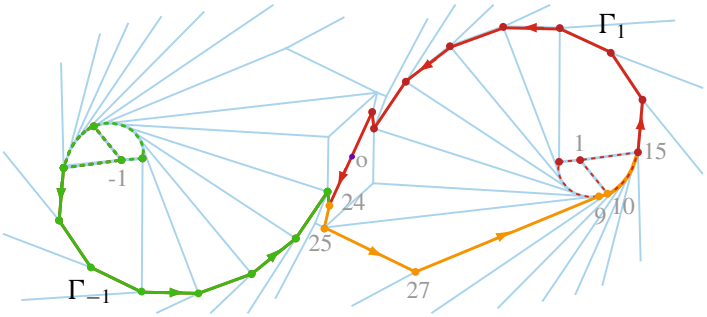


Fig. 10

a vertex of the triangle, while all other vertices, which we call the interior vertices of Γ_i, lie in the interior of the triangle, or the square part.

First note that Γ_1 and Γ_{-1} must share an interior vertex. If not, then p_{-1} cannot be a descendant of any interior vertex of Γ_1. Consequently, the center of rotation of each interior vertex of Γ_1 remains arbitrarily close to p_1, since all vertices of G other than $p_{\pm 1}$ have arbitrarily small weights by assumption. It follows then (via condition (8)) that Γ_1 is almost radially monotone with respect to p_1, i.e., vertices of Γ_1 satisfy condition (14) for $x = p_1$ and $\pi/2 - \epsilon$, for arbitrarily small $\epsilon > 0$. Thus Γ_1 must be radially monotone with respect to p_1, since there are only finitely many paths in G, and by convexity there exist radially monotone paths, with respect to any given point, which emanate from any given vertex of G. So, by Lemma 5.2, Γ_1 contains $p_{\pm 24}$. By symmetry, Γ_{-1} must contain these vertices as well, since the interior vertices of Γ_{-1} are the reflections of interior vertices of Γ_1. So Γ_1 and Γ_{-1} must join at some interior vertex.

Let p_m be the first (interior) vertex where Γ_1 and Γ_{-1} join. Then, as we described above, the subpath of Γ_1 from p_1 to p_m will be radially monotone with respect to p_1. So, by Lemma 5.2, it must be a (proper) subpath of (15). Similarly, by symmetry, the subpath of Γ_{-1} from p_{-1} to p_m must be the reflection of a subpath of (15). Hence the only possibilities are: $m = 24$ or $m = -24$. After a reflection, we may assume that $m = 24$; see Fig. 10. Then p_1 and p_{-1} are both descendants of p_{24}; therefore, the center of rotation of any vertex of Γ_{24} is arbitrarily close to the center of mass of p_1 and p_{-1}, which is o. So Γ_{24} is radially monotone with respect to o. Consequently, by Lemma 5.3, Γ_{24} contains p_{15}. But Γ_{24} is a subpath of Γ_1, which already passes through p_{15} prior to reaching p_{24}. Thus Γ_1 contains a cycle, which is the desired contradiction, and completes the proof of Theorem 5.1.

Theorem 5.1 together with Theorem 4.1 now immediately yields:

Corollary 5.4 *There exists a convex cap C over the equilateral triangle with* 84 *interior vertices and a pseudo-edge graph with respect to which C is not unfoldable. Furthermore, the total curvature of C may be arbitrarily small.*

6 Proof of Theorem 1.1

We need only one more observation. A *simple arc* Γ in a topological space X is the image of a continuous mapping $\gamma : [a, b] \to X$, which is one-to-one on (a, b). We say Γ is a *loop* provided that $\gamma(a) = \gamma(b)$. The following basic fact is also used in [3] to construct examples of un-unfoldable polyhedra. We omit the proof since it is fairly trivial (e.g. it follows by considering the different combinations).

Lemma 6.1 *Let $E \subset \mathbf{S}^2$ be an embedded graph which is isomorphic to the edge graph of a tetrahedron. Suppose there exists a simple arc Γ_i in each face Φ_i of E whose end points are distinct vertices of Φ_i, and whose interior lies in the interior of Φ_i. Then $\Gamma := \bigcup_i \Gamma_i$ contains a loop.*

Now we are ready to prove the main result of this work:

Proof of Theorem 1.1 Let C be the convex cap over the equilateral triangle given by Corollary 5.4. By Lemma 3.1 we may assume that the curvature $\kappa(C)$ is so small that the total angles of C at each of its boundary vertices is less than $2\pi/3$. Let C_i, $i = 1, \ldots, 4$, be congruent copies of C positioned over the faces of a regular tetrahedron, $K := \bigcup_i C_i$, and E be the union of the pseudo-edges E_i of C_i. Since C_i have 84 interior vertices each, K has 340 vertices. We claim that K is not unfoldable with respect to E. To see this suppose that T is a spanning tree of E, and let F_i be the closure of the restriction of T to the interior of C_i. If each F_i contains an arc connecting a pair of boundary vertices of C_i, then, by Lemma 6.1, T must contain a loop (or a cycle) which is not possible. Thus F_j must form a spanning forest of E_j for some $1 \le j \le 4$. Consequently, by Corollary 5.4, the unfolding of C_j with respect to F_j is not simple. Hence the unfolding of K with respect to T is not simple, which completes the proof. □

Note 6.2 The obvious question at the conclusion of this work is whether the above construction may yield a counterexample to (the original form of) Dürer's conjecture. The answer would depend on whether the partition of the equilateral triangle in Sect. 5, or some variation of it, can be *lifted* to a convex cap, i.e., whether there exists a convex cap over the equilateral triangle whose edges (not only vertices) project onto the edges of the partition. If so, these caps would generate an edge un-unfoldable polyhedron when assembled on the faces of a tetrahedron, as described above. It is well known that not every convex partition of a convex polygon can be lifted [7, p. 56]. A necessary and sufficient condition, called the *Maxwell–Cremona correspondence* [14,22,25], is that the edges of the subdivision admit an equilibrium stress, which can be expressed as a system of linear equations. Thus, to produce a counterexample to Dürer's conjecture (if one exists) it would suffice to find a non-monotone subdivision of the equilateral triangle which satisfies these equations. Alternatively, if no such subdivision exists, then that would yield more evidence in support of the conjecture.

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