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CERCIGNANI-LAMPIS BOUNDARY IN THE BOLTZMANN THEORY

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Abstract. The Boltzmann equation is a fundamental kinetic equation that describes the dynamics of dilute gas. In this paper we study the local well-posedness of the Boltzmann equation in bounded domain with the Cercignani-Lampis boundary condition, which describes the intermediate reflection law between diffuse reflection and specular reflection via two accommodation coefficients. We prove the local-in-time well-posedness of the equation by establishing an L^∞ estimate. In particular, for the L^∞ bound we develop a new decomposition on the boundary term combining with repeated interaction through the characteristic. Moreover, under some constraints on the wall temperature and the accommodation coefficients, we construct a unique steady solution of the Boltzmann equation.

1. **Introduction.** In this paper we consider the classical Boltzmann equation, which describes the dynamics of dilute particles. Denoting $F(t,x,v)$ the phase-space-distribution function of particles at time t, location $x \in \Omega$ moving with velocity $v \in \mathbb{R}^3$, the equation writes:

$$\partial_t F + v \cdot \nabla_x F = Q(F,F)\,. \tag{1}$$

The collision operator Q describes the binary collisions between particles:

$$\begin{aligned} &Q(F_1,F_2)(v) = Q_{\text{gain}} - Q_{\text{loss}} = Q_{\text{gain}}(F_1,F_2) - \nu(F_1)F_2 \\ &:= \iint_{\mathbb{R}^3\times\mathbb{S}^2} B(v-u,\omega)F_1(u')F_2(v')d\omega du - F_2(v)\left(\iint_{\mathbb{R}^3\times\mathbb{S}^2} B(v-u,\omega)F_1(u)d\omega du\right). \end{aligned} \tag{2}$$

In the collision process, we assume the energy and momentum are conserved. We denote the post-velocities:

$$u' = u - [(u-v)\cdot\omega]\omega, \qquad v' = v + [(u-v)\cdot\omega]\omega\,, \tag{3}$$

then they satisfy:

$$u' + v' = u + v\,, \quad |u'|^2 + |v'|^2 = |u|^2 + |v|^2\,. \tag{4}$$

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In equation (2), B is called the collision kernel which is given by

$$B(v-u,\omega) = |v-u|^{\mathcal{K}} q_0\Big(\frac{v-u}{|v-u|}\cdot\omega\Big)$$
$$\text{with} \quad -3 < \mathcal{K} \le 1\,, \quad 0 \le q_0\Big(\frac{v-u}{|v-u|}\cdot\omega\Big) \le C\Big|\frac{v-u}{|v-u|}\cdot\omega\Big|\,.$$

To describe the boundary condition for F, we denote the collection of coordinates on phase space at the boundary:

$$\gamma := \{(x,v) \in \partial\Omega \times \mathbb{R}^3\}.$$

And we denote $n = n(x)$ as the outward normal vector at $x \in \Omega$. We split the boundary coordinates γ into the incoming (γ_-) and the outgoing (γ_+) set:

$$\gamma_\mp := \{(x,v) \in \partial\Omega \times \mathbb{R}^3 : n(x)\cdot v \lessgtr 0\}.$$

The boundary condition determines the distribution on γ_-, and shows how particles back-scattered into the domain. In our model, we use the scattering kernel $R(u \to v; x, t)$:

$$F(t,x,v)|n(x)\cdot v| = \int_{n(x)\cdot u>0} R(u\to v;x,t)F(t,x,u)\{n(x)\cdot u\}du, \qquad \text{on } \gamma_-\,. \tag{5}$$

Physically, $R(u \to v; x, t)$ represents the probability of a molecule striking in the boundary at $x \in \partial\Omega$ with velocity u, and to be sent back to the domain with velocity v at the same location x and time t. There are many models for it. In [3, 4] Cercignani and Lampis proposed a generalized scattering kernel that encompasses pure diffusion and pure reflection molecules via two accommodation coefficients $r_\perp$ and $r_\parallel$. Their model writes:

$$\begin{aligned} &R(u\to v;x,t)\\ &:= \frac{1}{r_\perp r_\parallel(2-r_\parallel)\pi/2}\frac{|n(x)\cdot v|}{(2T_w(x))^2}\\ &\times \exp\left(-\frac{1}{2T_w(x)}\left[\frac{|v_\perp|^2+(1-r_\perp)|u_\perp|^2}{r_\perp}+\frac{|v_\parallel-(1-r_\parallel)u_\parallel|^2}{r_\parallel(2-r_\parallel)}\right]\right)\\ &\times I_0\left(\frac{1}{2T_w(x)}\frac{2(1-r_\perp)^{1/2}v_\perp u_\perp}{r_\perp}\right), \end{aligned} \tag{6}$$

where $T_w(x)$ is the wall temperature for $x \in \partial\Omega$ and

$$I_0(y) := \pi^{-1}\int_0^\pi e^{y\cos\phi}d\phi\,.$$

In the formula, $v_\perp$ and $v_\parallel$ denote the normal and tangential components of the velocity respectively:

$$v_\perp = v\cdot n(x)\,, \quad v_\parallel = v - v_\perp n(x)\,. \tag{7}$$

Similarly $u_\perp = u\cdot n(x)$ and $u_\parallel = u - u_\perp n(x)$.

There are a few properties the Cercignani-Lampis(C-L) model satisfies, including:

- the reciprocity property:

$$R(u\to v;x,t) = R(-v\to -u;x,t)\frac{e^{-|v|^2/(2T_w(x))}}{e^{-|u|^2/(2T_w(x))}}\frac{|n(x)\cdot v|}{|n(x)\cdot u|}\,, \tag{8}$$

- the normalization property(see the proof in appendix)

$$\int_{n(x)\cdot v<0} R(u\to v;x,t)dv = 1\,. \tag{9}$$

The normalization (9) property immediately leads to null-flux condition for F:

$$\int_{\mathbb{R}^3} F(t,x,v)\{n(x)\cdot v\}dv = 0\,, \quad \text{for } x\in\partial\Omega. \tag{10}$$

This condition guarantees the conservation of total mass:

$$\int_{\Omega\times\mathbb{R}^3} F(t,x,v)dvdx = \int_{\Omega\times\mathbb{R}^3} F(0,x,v)dvdx \text{ for all } t\geq 0\,. \tag{11}$$

Remark 1. The C-L model is an extension of the following classical diffuse boundary condition. The distribution function and scattering kernel are given by:

$$\begin{aligned} F(t,x,v) &= \frac{2}{\pi(2T_w(x))^2}e^{-\frac{|v|^2}{2T_w(x)}}\int_{n(x)\cdot u>0} F(t,x,u)\{n(x)\cdot u\}du \text{ on } (x,v)\in\gamma_-, \\ R(u\to v;x,t) &= \frac{2}{\pi(2T_w(x))^2}e^{-\frac{|v|^2}{2T_w(x)}}|n(x)\cdot v|. \end{aligned} \tag{12}$$

It corresponds to the scattering kernel in (6) with $r_\perp = 1, r_\parallel = 1$.

Other basic boundary conditions can be considered as a special case with singular R: specular reflection boundary condition:

$$F(t,x,v) = F(t,x,\mathfrak{R}_x v) \text{ on } (x,v)\in\gamma_-, \quad \mathfrak{R}_x v = v - 2n(x)(n(x)\cdot v),$$

$$R(u\to v;x,t) = \delta(u-\mathfrak{R}_x v),$$

where $r_\perp = 0, r_\parallel = 0$.

Bounce-back reflection boundary condition:

$$F(t,x,v) = F(t,x,-v) \text{ on } (x,v)\in\gamma_-,$$

$$R(u\to v;x,t) = \delta(u+v),$$

where $r_\perp = 0, r_\parallel = 2$.

Here we mention the Maxwell boundary condition, which is another classical model describes the intermediate reflection law. The scattering kernel is given by the convex combination of the diffuse and specular scattering kernel:

$$R(u\to v) = c\frac{2}{\pi(2T_w(x))^2}e^{-\frac{|v|^2}{2T_w(x)}}|n(x)\cdot v| + (1-c)\delta(u-\mathfrak{R}_x v), \quad 0\leq c\leq 1.$$

Compared with the C-L boundary condition, the Maxwell boundary condition does not cover the combination with the bounce back boundary condition. Such combination is covered in the C-L boundary condition with $r_\parallel > 1$. Moreover, the C-L boundary condition represents a smooth transition from the diffuse to the specular. The Maxwell boundary condition represents the convex combination of the Maxwellian and the dirac δ function. Here we show the graphs for both boundary condition in the two dimension for comparison. We assume the particles are moving towards the boundary with velocity $u = (u_\parallel, u_\perp) = (2,-2)$, thus the boundary condition is given by

$$\big[F(t,x,v)|n(x)\cdot v|\big]\Big|_{\gamma_-} = \int_{n(x)\cdot u>0} R(u\to v)\delta\Big(u-(2,-2)\Big)|n(x)\cdot u|du.$$

Then the distribution function $F(t,x,v)|_{\gamma_-}$ for both boundary condition can be viewed as the following graphs:

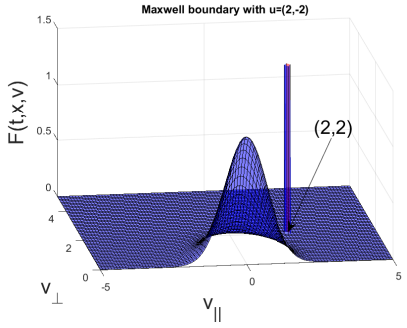

FIGURE 1. Maxwell boundary condition with $c = 1/2$.

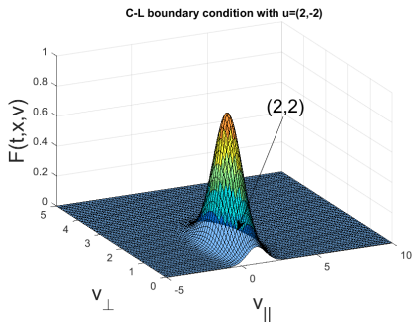

FIGURE 2. C-L boundary condition with $r_\perp = r_\parallel = 1/2$.

Moreover, we show the graphs for the distribution function $F|_{\gamma_-}$ with C-L boundary condition with smaller accommodation coefficients.

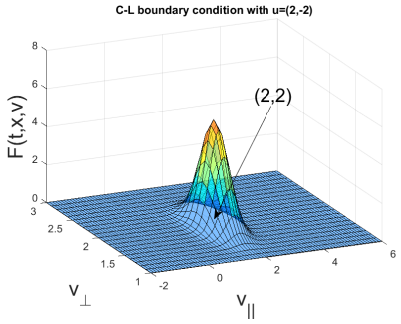

FIGURE 3. C-L boundary condition with $r_\perp = r_\parallel = 1/10$.

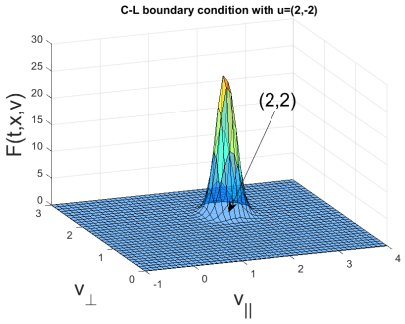

FIGURE 4. C-L boundary condition with $r_\perp = r_\parallel = 1/30$.

Figure 2 shows a smoother transition since the particles begin to concentrate toward to the point $(2,2)$. Meanwhile Figure 1 represents the phenomena that half particles are specular reflected and half particles are diffusive. When we take smaller accommodation coefficient, Figure 3 and Figure 4 demonstrate that the distribution function $F(t,x,v)|_{\gamma_-}$ gradually concentrate on $(2,2)$. Moreover, the z-coordinate shows that the C-L scattering kernel indeed tends to a dirac δ function as the accommodation coefficients become smaller.

Due to the generality of the C-L model, it has been vastly used in many applications. There are other derivations of C-L model besides the original one, and we refer interested readers to [6, 3, 2]. Also there have been many application of this model in recent years, on the rarefied gas flow in [19, 20, 25, 26, 27]; extension to the gas surface interaction model in fluid dynamics [22, 21, 30]; on the linearized Boltzmann equation in [11, 29, 23, 10]; on S-model kinetic equation in [28] etc.

1.1. **Main result.** We assume that the domain is C^2. Denote the maximum wall temperature:

$$T_M := \max\{T_w(x)\} < \infty\,. \tag{13}$$

Define the global Maxwellian using the maximum wall temperature:

$$\mu := e^{-\frac{|v|^2}{2T_M}}\,, \tag{14}$$

and weight F with it: $F = \sqrt{\mu} f$, then f satisfies

$$\partial_t f + v\cdot\nabla_x f = \Gamma(f,f)\,, \tag{15}$$

where the collision operator becomes:

$$\Gamma(f_1,f_2) = \Gamma_{\text{gain}}(f_1,f_2) - \nu(F_1)F_2/\mu = \frac{1}{\sqrt{\mu}} Q_{\text{gain}}(\sqrt{\mu}f_1,\sqrt{\mu}f_2) - \nu(F_1)f_2\,. \tag{16}$$

By the reciprocity property (8), the boundary condition for f becomes, for $(x,v)\in\gamma_-$,

$$f(t,x,v)|n(x)\cdot v| = \frac{1}{\sqrt{\mu}}\int_{n(x)\cdot u>0} R(u\to v;x,t) f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}du$$

$$= \frac{1}{\sqrt{\mu}}\int_{n(x)\cdot u>0} R(-v\to -u;x,t)\frac{e^{-|v|^2/(2T_w(x))}}{e^{-|u|^2/(2T_w(x))}} f(t,x,u)\sqrt{\mu(u)}\frac{|n(x)\cdot v|}{|n(x)\cdot u|}\{n(x)\cdot u\}du.$$

Thus

$$f(t,x,v)|_{\gamma_-} = e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|v|^2}\int_{n(x)\cdot u>0} f(t,x,u)e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}d\sigma(u,v). \tag{17}$$

Here we denote

$$d\sigma(u,v) := R(-v\to -u;x,t)du, \tag{18}$$

the probability measure in the space $\{(x,u), n(x)\cdot u>0\}$ (well-defined due to (9)).

Denote

$$w_\theta := e^{\theta|v|^2}, \tag{19}$$

$$\langle v\rangle := \sqrt{|v|^2+1}. \tag{20}$$

Theorem 1.1. *Assume $\Omega\subset\mathbb{R}^3$ is bounded and C^2. Let $0<\theta<\frac{1}{4T_M}$. Assume*

$$0<r_\perp\le 1,\quad 0<r_\parallel<2\,, \tag{21}$$

$$\frac{\min(T_w(x))}{T_M} > \max\Big(\frac{1-r_\parallel}{2-r_\parallel}, \frac{\sqrt{1-r_\perp}-(1-r_\perp)}{r_\perp}\Big), \tag{22}$$

where the T_M is defined in (13).

If $F_0=\sqrt{\mu}f_0\ge 0$ and f_0 satisfies the following estimate:

$$\|w_\theta f_0\|_\infty < \infty, \tag{23}$$

then there exists a unique solution $F(t,x,v)=\sqrt{\mu}f(t,x,v)\ge 0$ to (1) and (5) in $[0,t_\infty]\times\Omega\times\mathbb{R}^3$ with

$$t_\infty = t_\infty(\|w_\theta f_0\|_\infty, r_\perp, r_\parallel, \theta, T_M, \min\{T_w(x)\}, \Omega).$$

Moreover, the solution $F=\sqrt{\mu}f$ satisfies

$$\sup_{0\le t\le t_\infty}\|w_\theta e^{-|v|^2 t} f(t)\|_\infty \lesssim \|w_\theta f_0\|_\infty. \tag{24}$$

Remark 2. In Theorem 1.1 the accommodation coefficient can be any number that does not correspond to the dirac δ case. So far we are not able cover the specular reflection, we will explain the difficulty in section 1.3. Also we cover all the range for $\mathcal{K}$ in the collision kernel B in (2). We derive (24) and existence using the sequential argument. Assumption (23) is used to obtain the estimate (24) for the sequence solution, which is the key factor to prove the theorem.

Remark 3. In Theorem 1 we do not restrict the wall temperature to have a small fluctuation. Instead we have a more relax condition (22). In particular, for the pure diffuse reflection, i.e, $r_{\|} = r_{\perp} = 1$, we do not have any constraint for the temperature(except $T_M < \infty, \min(T_w(x)) > 0$).

Remark 4. There has been a lot of studies for Boltzmann equation in many aspects, the global solution [13, 12, 18, 17]; regularity estimate [14, 15, 1, 5]; the steady solution [8, 9, 7].

So far we are only able to prove the local well-posedness with the C-L boundary condition. There are several obstacles to construct the global solution with the C-L boundary condition for arbitrary accommodation coefficient.

To obtain the global solution of the Boltzmann equation [13] developed the $L^2 - L^\infty$ bootstrap and derive the time decay and continuous solution of the linearized Boltzmann equation with various boundary condition. In particular, for the diffuse boundary condition with constant wall temperature, [13] used the L^2 estimate on the boundary

$$\int_{n(x)\cdot u<0} f^2(t,x,u)|_{\gamma_-}|n(x)\cdot u|du \le \int_{n(x)\cdot u>0} f^2(t,x,u)|n(x)\cdot u|du,$$

$$\text{with } f|_{\gamma_-} = c_\mu\sqrt{\mu}\int_{n(x)\cdot u>0} f(t,x,u)\sqrt{\mu}|n(x)\cdot u|du. \tag{25}$$

Here c_μ is the normalization constant such that $c_\mu\mu|n\cdot u|du$ is a probability measure. To be more specific, the diffuse boundary condition can be regarded as a projection $P_\gamma f = f|_{\gamma_-}$. Then

$$\int_{n(x)\cdot u>0}(f - P_\gamma f)^2|n(x)\cdot u|du = \int_{n(x)\cdot u>0} f^2|n(x)\cdot u|du - \int_{n(x)\cdot u>0} P_\gamma f^2|n(x)\cdot u|du \ge 0.$$

However, for the C-L boundary condition, such L^2 inequality does not work. We can not regard the boundary condition (17) as a projection because of the new probability measure $d\sigma(u,v)$ in (18).

Another method to obtain the global solution is to use the entropy inequality. [12] used the entropy inequality and the $L^1 - L^\infty$ bootstrap to derive the bounded solution of the linearized Boltzmann equation with periodic boundary condition. To adapt the entropy method in bounded domain, [24] used the Jensen inequality for the *Darrozès-Guiraud information* with Maxwell boundary condition. To be more specific, we define $\mathcal{E}$ as the *Darrozès-Guiraud information*:

$$\mathcal{E} := \int_{\gamma_+} h\Big(\frac{F}{c_\mu\mu}\Big)c_\mu\mu(u)|n(x)\cdot u|du - h\Big(\int_{\gamma_+}\frac{F}{c_\mu\mu}c_\mu\mu(u)|n(x)\cdot u|du\Big), \quad h(s) = s\log s.$$

Since $c_\mu\mu(u)|n(x)\cdot u|du$ is a probability measure then $\mathcal{E} \ge 0$ by the Jensen inequality and thus the entropy inequality follows. For the C-L boundary condition, such inequality does not work since the probability measure is given by $d\sigma(u,v)$ (18), which is different from $c_\mu\mu(u)|n(x)\cdot u|du$.

Even though the global solution is not available for arbitrary accommodation coefficient, we are able to construct the steady and global solution when the coefficients are closed to 1. This means that we require the boundary condition to be closed to the diffuse boundary condition. We will discuss the steady solution in the following section.

1.2. Beside the local-in-time well-posedness, we can establish the stationary solution under some constraints. The steady problem is given as

$$v \cdot \nabla_x F = Q(F, F), \quad (x, v) \in \Omega \times \mathbb{R}^3 \tag{26}$$

with F satisfying the C-L boundary condition.

We use the short notation μ_0 to denote the global Maxwellian with temperature T_0,

$$\mu_0 := \frac{1}{2\pi (T_0)^2} \exp\Big(- \frac{|v|^2}{2T_0} \Big).$$

Denote L as the standard linearized Boltzmann operator

$$\begin{aligned} Lf &:= -\frac{1}{\sqrt{\mu_0}} \big[Q(\mu_0, \sqrt{\mu_0} f) + Q(\sqrt{\mu_0} f, \mu_0) \big] = \nu(v) f - Kf \\ &= \nu(v) f - \int_{\mathbb{R}^3} k(v, v_*) f(v_*) dv_*, \end{aligned} \tag{27}$$

with the collision frequency $\nu(v) \equiv \iint_{\mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, w) \mu_0(v_*) dw dv_* \sim \{1 + |v|\}^{\mathcal{K}}$ for $-3 < \mathcal{K} \leq 1$. Finally we define

$$P_\gamma f(x, v) := c_\mu \sqrt{\mu_0(v)} \int_{n(x) \cdot u > 0} f(x, u) \sqrt{\mu_0(u)} (n(x) \cdot u) du, \tag{28}$$

where c_μ is the normalization constant.

Corollary 1. *For given $T_0 > 0$, there exists $\delta_0 > 0$ such that if*

$$\sup_{x \in \partial\Omega} |T_w(x) - T_0| < \delta_0, \quad \max\{|1 - r_\perp|, |1 - r_\parallel|\} < \delta_0, \tag{29}$$

then there exists a non-negative solution $F_s = \mu_0 + \sqrt{\mu_0} f_s \geq 0$ with $\iint_{\Omega \times \mathbb{R}^3} f_s \sqrt{\mu_0} dx dv = 0$ to the steady problem (26). And for all $0 \leq \zeta < \frac{1}{4 + 2\delta_0}$, $\beta > 4$,

$$\| \langle v \rangle^\beta e^{\zeta |v|^2} f_s \|_\infty + |\langle v \rangle^\beta e^{\zeta |v|^2} f_s|_\infty \lesssim \delta_0 \ll 1.$$

If $\mu_0 + \sqrt{\mu_0} g_s$ with $\iint_{\Omega \times \mathbb{R}^3} g_s \sqrt{\mu_0} dx dv = 0$ is another solution such that $\| \langle v \rangle^\beta g_s \|_\infty + |\langle v \rangle^\beta g_s|_\infty \ll 1$ for $\beta > 4$, then $f_s \equiv g_s$.

Corollary 2. *For $0 < \zeta < \frac{1}{4 + 2\delta_0}$, set $\beta = 0$, and for $\zeta = 0$, set $\beta > 4$ where $\delta_0 > 0$ is in Corollary 1. There exists $\lambda > 0$ and $\varepsilon_0 > 0$, depending on δ_0, such that if $\iint_{\Omega \times \mathbf{R}^3} f_0 \sqrt{\mu_0} = \iint_{\Omega \times \mathbf{R}^3} f_s \sqrt{\mu_0} = 0$, and if*

$$\| \langle v \rangle^\beta e^{\zeta |v|^2} [f(0) - f_s] \|_\infty + |\langle v \rangle^\beta e^{\zeta |v|^2} [f(0) - f_s]|_\infty \leq \varepsilon_0, \tag{30}$$

then there exists a unique non-negative solution $F(t) = \mu_0 + \sqrt{\mu_0} f(t) \geq 0$ to the dynamical problem (1) with boundary condition (5), (6). And we have

$$\begin{aligned} &\| \langle v \rangle^\beta e^{\zeta |v|^2} [f(t) - f_s] \|_\infty + |\langle v \rangle^\beta e^{\zeta |v|^2} [f(t) - f_s]|_\infty \\ \lesssim\ & e^{-\lambda t} \big\{ \| \langle v \rangle^\beta e^{\zeta |v|^2} [f(0) - f_s] \|_\infty + |\langle v \rangle^\beta e^{\zeta |v|^2} [f(0) - f_s]|_\infty \big\}. \end{aligned}$$

Remark 5. Different to the accommodation coefficient with almost no constraint in Theorem 1.1, in Corollary 1, Corollary 2 we need to restrict these two coefficients to be close to 1 in (29). To be more specific, we require the C-L boundary to be close to the diffuse boundary condition.

In this paper we show the proof for the hard sphere case where $0 \leq \mathcal{K} \leq 1$. We can establish the same result for the soft potential case($-3 < \mathcal{K} < 0$) using the argument provided in [7].

1.3. **Difficulty and proof strategy.** For proving the local well-posedness we focus on establishing L^∞ estimate. In particular, for the L^∞ estimate we trace back along the characteristic until it hits the boundary or the initial datum. Thus we derive a new trajectory formula with C-L boundary condition in (17). Before tracing back to $t = 0$ there will be repeated interaction with the boundary, which creates a multiple integral due to the boundary condition (5). We present the formula in Lemma 2.2.

To understand this multiple integral we define $v_k, v_{k-1}, \cdots, v_1$ in Definition 2.1. The v_i represents the integral variable at i-th interaction with the boundary. For the diffuse reflection (12) with constant wall temperature, the boundary condition for $f = F/\sqrt{\mu}$ is given by (25). Thus at the i-th interaction the boundary condition is given by

$$f(v_{i-1}) = c_\mu \sqrt{\mu(v_{i-1})} \int_{n\cdot v_i>0} f(v_i)\sqrt{\mu(v_i)}|n \cdot v_i| dv_i.$$

If we further trace back $f(v_i)$ in the integrand along the trajectory until the next interaction we have

$$f(v_i) = c_\mu \sqrt{\mu(v_i)} \int_{n\cdot v_{i+1}>0} f(v_{i+1})\sqrt{\mu(v_{i+1})}|n \cdot v_{i+1}| dv_{i+1}.$$

Thus the integral over v_i becomes

$$\int_{n\cdot v_i>0} c_\mu \mu(v_i)|n \cdot v_i| dv_i.$$

The integrand for v_i is symmetric for all $1 \leq i < k$ and not affected by the other variables. Moreover, $c_\mu \mu(v_i)|n \cdot v_i| dv_i$ is probability measure. Thus we can apply Fubini's theorem to compute this multiple integral. But for the C-L boundary condition (5) (6), the integrand is a function of both v and u, as a result the probability measure is not symmetric for v_i. We are not free to apply the Fubini's theorem, which brings difficulty in bounding the trajectory formula. To be more specific, we need to compute the integral with the fixed order $v_k, v_{k-1}, \cdots v_1$. We start from the integral of v_k. By (17), the integral of v_k is

$$\int_{n(x)\cdot v_k>0} e^{-[\frac{1}{4T_M} - \frac{1}{2T_w(x)}]|v_k|^2} d\sigma(v_k, v_{k-1}). \tag{31}$$

When $r_\perp, r_\parallel \neq 0$, unlike the diffuse case, we can not decompose $d\sigma(v_k, v_{k-1})$ in (18) (6) into a product of a function of v_k and a function of v_{k-1}. Thus the integral ends up with a function of v_{k-1}, which will be included as a part of the integral over v_{k-1}. This justifies that the order of the integral can not be changed. Also the integral of v_i is affected by the variables $v_{i+1}, v_{i+2}, \cdots v_k$. Thus we have to compute the multiple integral with fixed order from v_k to v_1.

In fact, (31) can be computed explicitly as $e^{c|v_{k-1}|^2}$(Lemma 4.2,Lemma 4.3) and thus the integral for the variable v_{k-1} has exactly the same form as (31). This

allows us to inductively derive an upper bound for this multiple integral. We present the induction result in Lemma 2.3.

With an upper bound for the trajectory formula another difficulty in the L^∞ estimate is the measure $\mathbf{1}_{\{t_k>0\}}$. We need to show that this measure is small when k is large so that the L^∞ estimate follows by bounding a finite fold integral.

For this purpose [13, 1] decompose γ_+ into the subspace

$$\gamma_+^\delta = \{u \in \gamma_+ : |n \cdot u| > \delta, |u| \le \delta^{-1}\}.$$

For diffuse case (12) the boundary condition for f is given by (25). We can derive that there can be only finite number of v_j belong to γ_+^δ under the constraint that $t < \infty$. Meanwhile, by (25) the integral over $\gamma_+ \backslash \gamma_+^\delta$ is a small magnitude number $O(\delta)$. When k(times of interaction with boundary) is large enough one can obtain a large power of $O(\delta)$. The smallness of the measure $\mathbf{1}_{\{t_k>0\}}$ follows by this large power.

However, for our C-L boundary condition, the integrand is given by (17) (6), which contains the term $e^{-|v_\parallel-(1-r_\parallel)u_\parallel|^2}$ in (18). If we apply the standard decomposition the integral over $\gamma_+\backslash\gamma_+^\delta$ is no longer a small number $O(\delta)$. This is because even $|v_\parallel| \gg 1$, $|v_\parallel - (1-r_\parallel)u_\parallel|$ still depends on $u_\parallel$.

A key observation is that when $|v_\parallel|$ is large enough, if $|v_\parallel - (1-r_\parallel)u_\parallel| < \delta^{-1}$, we can obtain $|u_\parallel| \ge |v_\parallel| + \delta$ using $1 - r_\parallel < 1$. We take $1 - r_\parallel = 1/2$ as example. If $|v_\parallel - \frac{1}{2}u_\parallel| < \delta^{-1}$, we take $|v_\parallel| \ge 3\delta^{-1}$. Then we have

$$\frac{1}{2}|u_\parallel| > |v_\parallel| - \delta^{-1} > \frac{1}{2}|v_\parallel| + \frac{1}{2}\delta^{-1}, \quad |u_\parallel| > |v_\parallel| + \delta^{-1}.$$

For $1 - r_\parallel \neq 1/2$, we can choose a different number that depends on $1 - r_\parallel$ to keep this property.

Now we suppose the "bad" case $|v_\parallel - (1-r_\parallel)u_\parallel| < \delta^{-1}$ happens for a large amount of times. By the discussion above, for the multiple integral with order $v_k, \cdots, v_1$ we get an extremely huge velocity $|v_i|$ with some $i < k$. The integral with $d\sigma(v_i, v_{i-1})$ will be extremely small once $|v_{i-1}|$ is small. This will provide the key decaying factor to cancel all the other growth terms and prove the smallness of the measure $\mathbf{1}_{\{t_k>0\}}$. The other one is the"good" case $|v_\parallel - (1-r_\parallel)u_\parallel| > \delta^{-1}$. From (6) we can conclude the integral under this condition is a small magnitude number $O(\delta)$. Thus we are able obtain some small factors in both cases. Since the integrand in $d\sigma(u, v)$ in (18) (6) still contains the variable $u_\perp, v_\perp$, we also need to apply the decomposition for these variables. The decomposition is similar and we skip the discussion here. But we point out that since the integrand for $u_\perp$ involves the first type Bessel function I_0, we need some basic estimate to verify that the integral for $u_\perp$ has the same property as $v_\parallel, u_\parallel$. We put these estimates in the appendix.

Thus our new ingredient here is that we decompose the boundary term γ_+ into the subspace

$$\gamma_+^\eta = \{u \in \gamma_+ : |n \cdot u| > \eta\delta, |u| \le (\eta\delta)^{-1}\}.$$

Here η is small number depends on the coefficient $r_\parallel$ to ensure $|u_\parallel| \ge |v_\parallel| + \delta^{-1}$ when $|v_\parallel - (1-r_\parallel)u_\parallel| < \delta^{-1}$. We comment here that due to fact that such property only holds when the coefficient for $u_\parallel$ is less than 1, we are not able to cover the specular or bounce back reflection. For these two cases, the coefficient is 1 and thus such decomposition does not help us to conclude the smallness.

During computing the trajectory formula the integral involves the variable $T_w(x)$ (the wall temperature on $x \in \partial\Omega$ in (6)). It affects the real value of the coefficient for $u_\parallel$(different to $1 - r_\parallel$). This is the reason that we need to impose some constraint on the wall temperature, which is the condition (22) in Theorem 1.1. We present the decomposition and detail in Lemma 2.4 and its proof.

The way to construct the stationary solution and the dynamical stability(Corollary 1 and Corollary 2) comes from the ideas in [8, 9]. They consider the diffuse boundary condition with a small fluctuation on the wall temperature. Thus it can be regarded as a perturbation around the diffuse boundary condition with constant temperature. For our C-L boundary condition, when $r_\perp$ and $r_\parallel$ are close to 1, it can be also regarded as a perturbation. Thus we need to restrict the accommodation coefficient to have a small fluctuation around 1. Then we need to verify the boundary condition satisfies the property as stated in Proposition 4.1 in [8] (the condition (190) in this paper). Then we can follow the standard procedure provided in [8] to prove Corollary 1 and Corollary 2.

1.4. **Outline.** In section 2 we conclude Theorem 1.1 by proving the L^∞ bound for the sequence f^m as well as the existence and L^∞ stability. In section 3, we conclude Corollary 1 and Corollary 2 by using the key propositions provided in [8]. In the appendix we prove some necessary estimates.

2. **Local well-posedness.** We start with the construction of the following iteration equation, which is positive preserving as in [13, 16]. Then equation is given by

$$\partial_t F^{m+1} + v \cdot \nabla_x F^{m+1} = Q_{\text{gain}}(F^m, F^m) - \nu(F^m)F^{m+1}, \quad F^{m+1}|_{t=0} = F_0, \tag{32}$$

with boundary condition

$$F^{m+1}(t,x,v)|n(x)\cdot v| = \int_{n(x)\cdot u>0} R(u \to v; x, t)F^m(t,x,u)\{n(x)\cdot u\}du.$$

For $m \le 0$ we set

$$F^m(t,x,v) = F_0(x,v).$$

We pose $F^{m+1} = \sqrt{\mu} f^{m+1}$ and

$$h^{m+1}(t,x,v) = e^{(\theta-t)|v|^2} f^{m+1}(t,x,v). \tag{33}$$

The equation for h^{m+1} reads

$$\partial_t h^{m+1} + v \cdot \nabla_x h^{m+1} + \nu^m h^{m+1} = e^{(\theta-t)|v|^2}\Gamma_{\text{gain}}\left(\frac{h^m}{e^{(\theta-t)|v|^2}}, \frac{h^m}{e^{(\theta-t)|v|^2}}\right), \tag{34}$$

with boundary condition

$$\begin{aligned} h^{m+1}(t,x,v) &= e^{(\theta-t)|v|^2} e^{[\frac{1}{4T_M} - \frac{1}{2T_w(x)}]|v|^2} \\ &\times \int_{n(x)\cdot u>0} h^m(t,x,u) e^{-[\frac{1}{4T_M} - \frac{1}{2T_w(x)}]|u|^2} e^{-(\theta-t)|u|^2} d\sigma(u,v). \end{aligned} \tag{35}$$

Here

$$\nu^m = |v|^2 + \nu(F^m) \ge |v|^2. \tag{36}$$

We use this section to establish the L^∞ estimate of the sequence h^{m+1} and derive the existence and uniqueness of the equation (1). The L^∞ estimate is given by the following proposition.

Proposition 1. *Assume h^{m+1} satisfies (33) with Cercignani-Lampis boundary condition. Also assume $\theta < \frac{1}{4T_M}$, $\frac{\min(T_w(x))}{T_M} > \max\left(\frac{1-r_\parallel}{2-r_\parallel}, \frac{\sqrt{1-r_\perp}-(1-r_\perp)}{r_\perp}\right)$ and*

$$\|h_0(x,v)\|_{L^\infty} < \infty, \tag{37}$$

If

$$\sup_{i\le m} \|h^i(t,x,v)\|_{L^\infty} \le C_\infty \|h_0(x,v)\|_{L^\infty}, \quad t \le t_\infty, \tag{38}$$

then we have

$$\sup_{0\le t\le t_\infty} \|h^{m+1}(t,x,v)\|_{L^\infty} \le C_\infty \|h_0(x,v)\|_{L^\infty}. \tag{39}$$

Here C_∞ is a constant defined in (166) and

$$t \le t_\infty = t_\infty(\|h_0(x,v)\|_{L^\infty}, T_M, \min\{T_w(x)\}, \theta, r_\perp, r_\parallel, \Omega) \ll 1. \tag{40}$$

Remark 6. The condition (40) is important. The smallness of the time will be used in the proof many times. And the parameters in (40) guarantee that the time only depends on the temperature, accommodation and the initial condition.

The Proposition 1 implies the uniform-in-m L^∞ estimate for $h^m(t,x,v)$,

$$\sup_m \|h^m\|_\infty < \infty \tag{41}$$

The strategy to prove Proposition 1 is to express h^{m+1} along the characteristic using the C-L boundary condition. We present the formula in Lemma 2.2. We will use Lemma 2.3 and Lemma 2.4 to bound the formula.

We represent h^{m+1} with the stochastic cycles defined as follows.

Definition 2.1. Let $\left(X^1(s;t,x,v), v\right)$ be the location and velocity along the trajectory before hitting the boundary for the first time,

$$\frac{d}{ds}\begin{pmatrix} X^1(s;t,x,v) \\ v \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix}. \tag{42}$$

Therefore, from (42), we have

$$X^1(s;t,x,v) = x - v(t-s).$$

Define the back-time cycle as

$$t_1(t,x,v) = \sup\{s < t : X^1(s;t,x,v) \in \partial\Omega\},$$
$$x_1(t,x,v) = X^1\left(t_1(t,x,v);t,x,v\right),$$
$$v_1 \in \{v_1 \in \mathbb{R}^3 : n(x_1)\cdot v_1 > 0\}.$$

Also define

$$\mathcal{V}_1 = \{v_1 : n(x_1)\cdot v_1 > 0\}, \quad x_1 \in \partial\Omega.$$

Inductively, before hitting the boundary for the k-th time, define

$$t_k(t,x,v,v_1,\cdots,v_{k-1}) = \sup\{s < t_{k-1} : X^k(s;t_{k-1},x_{k-1},v_{k-1}) \in \partial\Omega\},$$
$$x_k(t,x,v,v_1,\cdots,v_{k-1}) = X^k\left(t_k(t,x,v,v_{k-1});t_{k-1}(t,x,v),x_{k-1}(t,x,v),v_{k-1}\right),$$
$$v_k \in \{v_k \in \mathbb{R}^3 : n(x_k)\cdot v_k > 0\},$$
$$\mathcal{V}_k = \{v_k : n(x_k)\cdot v_k > 0\},$$
$$X^k(s;t_{k-1},x_{k-1},v_{k-1}) = x_{k-1} - (t_{k-1}-s)v_{k-1}.$$

Here we set

$$(t_0,x_0,v_0) = (t,x,v).$$

For simplicity, we denote

$$X^k(s) := X^k(s; t_{k-1}, x_{k-1}, v_{k-1}).$$

in the following lemmas and propositions.

Lemma 2.2. *Assume h^{m+1} satisfy (34) with the Cercignani-Lampis boundary condition (35), if $t_1 \le 0$, then*

$$|h^{m+1}(t,x,v)| \le |h_0(X^1(0),v)| + \int_0^t e^{-|v|^2(t-s)} e^{|v|^2(\theta-s)} \Gamma^m_{gain}(s) ds. \tag{43}$$

If $t_1 > 0$, for $k \ge 2$, then

$$\begin{aligned} &|h^{m+1}(t,x,v)| \\ &\le \int_{t_1}^t e^{-|v|^2(t-s)} e^{|v|^2(\theta-s)} \Gamma^m_{gain}(s) ds + e^{|v|^2(\theta-t_1)} e^{[\frac{1}{4T_M} - \frac{1}{2T_w(x_1)}]|v|^2} \int_{\prod_{j=1}^{k-1} \mathcal{V}_j} H, \end{aligned} \tag{44}$$

where H is bounded by

$$\begin{aligned} &\sum_{l=1}^{k-1} \mathbf{1}_{\{t_l>0, t_{l+1}\le 0\}} |h_0\left(X^{l+1}(0), v_l\right)| d\Sigma^k_{l,m}(0) \\ &+ \sum_{l=1}^{k-1} \int_{\max\{0,t_{l+1}\}}^{t_l} e^{|v_l|^2(\theta-s)} |\Gamma^{m-l}_{gain}(s)| d\Sigma^k_{l,m}(s) ds \\ &+ \mathbf{1}_{\{t_k>0\}} |h^{m-k+2}\left(t_k, x_k, v_{k-1}\right)| d\Sigma^k_{k-1,m}(t_k), \end{aligned} \tag{45}$$

where

$$\begin{aligned} d\Sigma^k_{l,m}(s) = &\Big\{ \prod_{j=l+1}^{k-1} d\sigma\left(v_j, v_{j-1}\right) \Big\} \\ &\times \left\{ e^{-|v_l|^2(t_l-s)} e^{-|v_l|^2(\theta-t_l)} e^{-[\frac{1}{4T_M} - \frac{1}{2T_w(x_l)}]|v_l|^2} d\sigma(v_l, v_{l-1}) \right\} \\ &\times \Big\{ \prod_{j=1}^{l-1} e^{[\frac{1}{2T_w(x_j)} - \frac{1}{2T_w(x_{j+1})}]|v_j|^2} d\sigma\left(v_j, v_{j-1}\right) \Big\}. \end{aligned} \tag{46}$$

Here we use a notation

$$\Gamma^{m-l}_{gain}(s) := \Gamma_{gain}\left(\frac{h^{m-l}(s, X^{l+1}(s), v_l)}{e^{|v_l|^2(\theta-s)}}, \frac{h^m(s, X^{l+1}(s), v_l)}{e^{|v_l|^2(\theta-s)}} \right) \text{ for } 0 \le l \le m \ . \tag{47}$$

Proof. For simplicity, we denote

$$\tilde{\mu}(t,x,v) := e^{-|v|^2(\theta-t)} e^{-[\frac{1}{4T_M} - \frac{1}{2T_w(x)}]|v|^2}. \tag{48}$$

From (34), for $0 \le s \le t$, we apply the fundamental theorem of calculus to get

$$\frac{d}{ds} \int_s^t -\nu^m d\tau = \frac{d}{ds} \int_t^s \nu^m d\tau = \nu^m.$$

Thus based on (34),

$$\frac{d}{ds} \left[e^{-\int_s^t \nu^m d\tau} h^{m+1}(s, X^1(s), v) \right] = e^{-\int_s^t \nu^m d\tau} e^{|v|^2(\theta-s)} \Gamma^m_{\text{gain}}(s). \tag{49}$$

By (36),

$$e^{-\int_s^t \nu^m d\tau} \le e^{-|v|^2(t-s)} \le 0. \tag{50}$$

Combining (49) and (50), we derive that if $t_1 \le 0$, then we have (43).

If $t_1(t,x,v) > 0$, then

$$\begin{aligned} |h^{m+1}(t,x,v)\mathbf{1}_{\{t_1>0\}}| &\le |h^{m+1}(t_1,x_1,v)|e^{-|v|^2(t-t_1)} \\ &+ \int_{t_1}^t e^{-|v|^2(t-s)} e^{|v|^2(\theta-s)} |\Gamma^m_{\text{gain}}(s)| ds. \end{aligned} \tag{51}$$

We use an induction of k to prove (44). The first term of the RHS of (51) can be expressed by the boundary condition. For $1 \le k \le m$, we rewrite the boundary condition (35) using (48) as

$$\begin{aligned} &h^{m-k+2}(t_k,x_k,v_{k-1}) \\ &= \frac{1}{\tilde{\mu}(t_k,x_k,v_{k-1})} \int_{\mathcal{V}_k} h^{m-k+1}(t_k,x_k,v_k)\tilde{\mu}(t_k,x_k,v_k) d\sigma(v_k,v_{k-1}). \end{aligned} \tag{52}$$

Directly applying (52) with $k=1$ the first term of the RHS of (51) is bounded by

$$\frac{1}{\tilde{\mu}(t_1,x_1,v)} \int_{\mathcal{V}_1} h^m(t_1,x_1,v_1)\tilde{\mu}(t_1,x_1,v_1) d\sigma(v_1,v). \tag{53}$$

Then we apply (43) and (51) to derive

$$\begin{aligned} (53) \le \frac{1}{\tilde{\mu}(t_1,x_1,v)} \Big[&\int_{\mathcal{V}_1} \mathbf{1}_{\{t_2\le 0<t_1\}} e^{-|v_1|^2 t_1} h^m(0,X^2(0),v_1)\tilde{\mu}(t_1,x_1,v_1) d\sigma(v_1,v) \\ &+ \int_{\mathcal{V}_1}\int_0^{t_1} \mathbf{1}_{\{t_2\le 0<t_1\}} e^{-|v_1|^2(t_1-s)} e^{|v_1|^2(\theta-s)} |\Gamma^{m-1}_{\text{gain}}(s)| \tilde{\mu}(t_1,x_1,v_1) d\sigma(v_1,v) ds \\ &+ \int_{\mathcal{V}_1} \mathbf{1}_{\{t_2>0\}} e^{-|v_1|^2(t_1-t_2)} |h^m(t_2,x_2,v_1)\tilde{\mu}(t_1,x_1,v_1) d\sigma(v_1,v) \\ &+ \int_{\mathcal{V}_1}\int_{t_2}^{t_1} \mathbf{1}_{\{t_2>0\}} e^{-|v_1|^2(t_1-s)} e^{|v_1|^2(\theta-s)} |\Gamma^{m-1}_{\text{gain}}(s)| \tilde{\mu}(t_1,x_1,v_1) d\sigma(v_1,v) ds \Big]. \end{aligned}$$

Therefore, the formula (44) is valid for $k=2$.

Assume (44) is valid for $k \ge 2$ (induction hypothesis). Now we prove that (44) holds for $k+1$. We express the last term in (45) using the boundary condition. In (52), since $\frac{1}{\tilde{\mu}(t_k,x_k,v_{k-1})}$ depends on v_{k-1}, we move this term to the integration over $\mathcal{V}_{k-1}$ in (44). Using the second line of (46), the integration over $\mathcal{V}_{k-1}$ is

$$\int_{\mathcal{V}_{k-1}} e^{-|v_{k-1}|^2(t_{k-1}-t_k)} \tilde{\mu}(t_{k-1},x_{k-1},v_{k-1})/\tilde{\mu}(t_k,x_k,v_{k-1}) d\sigma(v_{k-1},v_{k-2}). \tag{54}$$

We have

$$\begin{aligned} &e^{-|v_{k-1}|^2(t_{k-1}-t_k)} \tilde{\mu}(t_{k-1},x_{k-1},v_{k-1})/\tilde{\mu}(t_k,x_k,v_{k-1}) \\ &= e^{-|v_{k-1}|^2(t_{k-1}-t_k)} e^{|v_{k-1}|^2(t_{k-1}-t_k)} e^{[\frac{1}{2T_w(x_{k-1})}-\frac{1}{2T_w(x_k)}]|v_{k-1}|^2} \\ &= e^{[\frac{1}{2T_w(x_{k-1})}-\frac{1}{2T_w(x_k)}]|v_{k-1}|^2}. \end{aligned}$$

Therefore, by (54) the integration over $\mathcal{V}_{k-1}$ reads

$$\int_{\mathcal{V}_{k-1}} e^{[\frac{1}{2T_w(x_{k-1})}-\frac{1}{2T_w(x_k)}]|v_{k-1}|^2} d\sigma(v_{k-1},v_{k-2}), \tag{55}$$

which is consistent with third line in (46) with $l = k-1$.

For the remaining integration in (52), we split the integration over $\mathcal{V}_k$ into two terms as

$$\int_{\mathcal{V}_k} h^{m-k+1}(t_k,x_k,v_k)\tilde{\mu}(t_k,x_k,v_k)d\sigma(v_k,v_{k-1}) = \underbrace{\int_{\mathcal{V}_k}\mathbf{1}_{\{t_{k+1}\le 0<t_k\}}}_{(56)_1} + \underbrace{\int_{\mathcal{V}_k}\mathbf{1}_{\{t_{k+1}>0\}}}_{(56)_2}. \tag{56}$$

For the first term of the RHS of (56), we use the similar bound of (43) and derive that

$$\begin{aligned}(56)_1 \le & \int_{\mathcal{V}_k}\mathbf{1}_{\{t_{k+1}\le 0<t_k\}}e^{-|v_k|^2t_k}h^{m-k+1}(0,X^{k+1}(0),v_k)\tilde{\mu}(t_k,x_k,v_k)d\sigma(v_k,v_{k-1})\\ &+\int_{\mathcal{V}_k}\int_0^{t_k}\mathbf{1}_{\{t_{k+1}\le 0<t_k\}}e^{-|v_k|^2(t_k-s)}e^{|v_k|^2(\theta-s)}\Gamma_{\text{gain}}^{m-k}(s)\tilde{\mu}(t_k,x_k,v_k)d\sigma(v_k,v_{k-1})ds.\end{aligned} \tag{57}$$

In the first line of (57),

$$e^{-|v_k|^2t_k}\tilde{\mu}(t_k,x_k,v_k)d\sigma(v_k,v_{k-1}),$$

is consistent with the second line of (46) with $l=k$, $s=t_k$. In the second line of (57)

$$e^{-|v_k|^2(t_k-s)}\tilde{\mu}(t_k,x_k,v_k)d\sigma(v_k,v_{k-1}),$$

is consistent with the second line of (46) with $l=k$.

From the induction hypothesis((44) is valid for k) and (55), we derive the integration over $\mathcal{V}_j$ for $j\le k-1$ is consistent with the third line of (46). After taking integration $\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}$ we change $d\Sigma_{k-1,m}^k$ in (46) to $d\Sigma_{k,m}^{k+1}$. Thus the contribution of (57) is

$$\begin{aligned}&\int_{\prod_{j=1}^k\mathcal{V}_j}\mathbf{1}_{\{t_{k+1}\le 0<t_k\}}|h_0\left(X^{k+1}(0),v_k\right)|d\Sigma_{k,m}^{k+1}(0)\\ &+\int_{\prod_{j=1}^k\mathcal{V}_j}\int_0^{t_k}e^{|v_k|^2(\theta-s)}\Gamma_{\text{gain}}^{m-k}(s)d\Sigma_{k,m}^{k+1}(s)ds.\end{aligned} \tag{58}$$

For the second term of the RHS of (56), we use the same estimate as (43) and we derive

$$\begin{aligned}&(56)_2\\ &\le\int_{\mathcal{V}_k}\mathbf{1}_{\{t_{k+1}>0\}}e^{-|v_k|^2(t_k-t_{k+1})}h^{m-k+1}\left(t_{k+1},x_{k+1},v_k\right)\tilde{\mu}(t_k,x_k,v_k)d\sigma\left(v_k,v_{k-1}\right)\\ &+\int_{\mathcal{V}_k}\int_{t_{k+1}}^{t_k}\mathbf{1}_{\{t_{k+1}>0\}}e^{-|v_k|^2(t_k-s)}e^{|v_k|^2(\theta-s)}\Gamma_{\text{gain}}^{m-k}(s)\tilde{\mu}(t_k,x_k,v_k)d\sigma(v_k,v_{k-1})ds.\end{aligned} \tag{59}$$

Similar to (58), after taking integration over $\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}$ the contribution of (59) is

$$\begin{aligned}&\int_{\prod_{j=1}^k\mathcal{V}_j}\mathbf{1}_{\{t_{k+1}>0\}}|h^{m-k+1}\left(t_{k+1},x_{k+1},v_k\right)|d\Sigma_{k,m}^{k+1}(t_{k+1})\\ &+\int_{\prod_{j=1}^k\mathcal{V}_j}\int_{t_{k+1}}^{t_k}e^{|v_k|^2(\theta-s)}\Gamma_{\text{gain}}^{m-k}(s)d\Sigma_{k,m}^{k+1}(s)ds.\end{aligned} \tag{60}$$

From (60) (58), the summation in the first and second lines of (45) extends to k. And the index of the third line of (45) changes from k to $k+1$. For the rest terms, the index $l \le k-1$, we haven't done any change to them. Thus their integration are over $\prod_{l=1}^{k-1} \mathcal{V}_j$. We add $\int_{\mathcal{V}_k} d\sigma(v_k, v_{k-1}) = 1$ to all of them, so that all the integrations are over $\prod_{l=1}^{k} \mathcal{V}_j$ and we change $d\Sigma_{l,m}^{k-1}$ to $d\Sigma_{l,m}^{k}$ by

$$d\Sigma_{l,m}^{k} = d\sigma\,(v_k, v_{k-1})\, d\Sigma_{l,m}^{k-1}.$$

Therefore, the formula (45) is valid for $k+1$ and we derive the lemma. □

The next lemma is the key to prove the L^∞ bound for h^{m+1}. Below we define several notation: let

$$r_{max} := \max(r_\parallel(2-r_\parallel), r_\perp),\ r_{min} := \min(r_\parallel(2-r_\parallel), r_\perp). \tag{61}$$

Then we have

$$1 \ge r_{max} \ge r_{min} > 0. \tag{62}$$

Define

$$\xi := \frac{1}{4T_M\theta},$$

where $\theta < \frac{1}{4T_M}$ is given in (33). Then we have

$$\theta = \frac{1}{4T_M\xi}, \quad \xi > 1. \tag{63}$$

We inductively define:

$$\begin{aligned} &T_{l,l} = \frac{2\xi}{\xi+1}T_M, \quad T_{l,l-1} = r_{min}T_M + (1-r_{min})T_{l,l}, \\ &\cdots, T_{l,1} = r_{min}T_M + (1-r_{min})T_{l,2}. \end{aligned} \tag{64}$$

By a direct computation, for $1 \le i \le l$, we have

$$T_{l,i} = \frac{2\xi}{\xi+1}T_M + (T_M - \frac{2\xi}{\xi+1}T_M)[1-(1-r_{min})^{l-i}] \tag{65}$$

Moreover, let

$$\begin{aligned} d\Phi_{p,m}^{k,l}(s) :=& \{\prod_{j=l+1}^{k-1} d\sigma(v_j, v_{j-1})\} \\ &\times \{e^{-|v_l|^2(t_l-s)} e^{-|v_l|^2(\theta-t_l)} e^{-[\frac{1}{4T_M} - \frac{1}{2T_w(x_l)}]|v_l|^2} d\sigma(v_l, v_{l-1})\} \\ &\times \{\prod_{j=p}^{l-1} e^{[\frac{1}{2T_w(x_j)} - \frac{1}{2T_w(x_{j+1})}]|v_j|^2} d\sigma(v_j, v_{j-1})\}. \end{aligned} \tag{66}$$

Note that if $p=1$, $d\Phi_{1,m}^{k,l}(s) = d\Sigma_{l,m}^{k}(s)$ where $d\Sigma_{l,m}^{k}(s)$ is defined in (46). And let

$$d\Upsilon_p^{p'} := \{\prod_{j=p}^{p'} e^{[\frac{1}{2T_w(x_j)} - \frac{1}{2T_w(x_{j+1})}]|v_j|^2} d\sigma(v_j, v_{j-1})\}. \tag{67}$$

Then by the definition of (66) and (46), we have

$$d\Phi_{p,m}^{k,l}(s) = d\Phi_{p',m}^{k,l}(s) d\Upsilon_p^{p'-1}, \tag{68}$$

$$d\Sigma_{l,m}^{k}(s) = d\Phi_{p,m}^{k,l}(s) d\Upsilon_1^{p-1}. \tag{69}$$

Remark 7. We aim to bound the multiple integral in the trajectory formula in Lemma 2.2. Each integral in the formula involves the variable $T_w(x), T_M, r_\perp, r_\parallel$, thus we need to find the pattern of the upper bound for each fold integral. This is the reason we define these inductive notations.

Now we state the lemma.

Lemma 2.3. *Given the formula for h^{m+1} in (43) and (44) of lemma 2.2, there exists*

$$t^* = t^*(T_M, \xi, \mathcal{C}, k) \tag{70}$$

such that when $t \le t^*$, for any $0 \le s \le t_l$ we have

$$\int_{\prod_{j=p}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{t_l>0\}} d\Phi_{p,m}^{k,l}(s) \le (C_{T_M,\xi})^{2(l-p+1)} \mathcal{A}_{l,p}. \tag{71}$$

where we define:

$$\mathcal{A}_{l,p} = \exp\left(\Big[\frac{[T_{l,p} - T_w(x_p)][1-r_{min}]}{2T_w(x_p)[T_{l,p}(1-r_{min}) + r_{min}T_w(x_p)]} + \mathcal{C}^{l-p+1}t \Big] |v_{p-1}|^2 \right). \tag{72}$$

Here $C_{T_M,\xi}$ is a constant defined in (80) and $\mathcal{C}$ is constant defined in (83).

Moreover, for any $p < p' \le l$, we have

$$\begin{aligned} \int_{\prod_{j=p}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{t_l>0\}} d\Phi_{p,m}^{k,l}(s) &\le (C_{T_M,\xi})^{2(l-p'+1)} \int_{\prod_{j=p}^{p'-1}\mathcal{V}_j} \mathbf{1}_{\{t_l>0\}} \mathcal{A}_{l,p'} d\Upsilon_p^{p'-1} \\ &\le (C_{T_M,\xi})^{2(l-p+1)} \mathcal{A}_{l,p}. \end{aligned} \tag{73}$$

Proof. From (9) and (18), for the first bracket of the first line in (46) with $l+1 \le j \le k-1$, we have

$$\int_{\prod_{j=l+1}^{k-1}\mathcal{V}_j} \prod_{j=l+1}^{k-1} d\sigma(v_j, v_{j-1}) = 1.$$

Without loss of generality we can assume $k = l+1$. Thus $d\Phi_{p,m}^{k,l} = d\Phi_{p,m}^{l+1,l}$. We use an induction of p with $1 \le p \le l$ to prove (71).

When $p = l$, by the second line of (66), the integration over $\mathcal{V}_l$ is written as

$$\int_{\mathcal{V}_l} e^{-|v_l|^2(t_l-s)} e^{-|v_l|^2(\theta-t_l)} e^{-[\frac{1}{4T_M} - \frac{1}{2T_w(x_l)}]|v_l|^2} d\sigma(v_l, v_{l-1}). \tag{74}$$

By $\theta = \frac{1}{4T_M\xi}$ in (63) and $s \le t_l$, we bound (74) by

$$\int_{\mathcal{V}_l} e^{-[\frac{1}{2T_M}\frac{\xi+1}{2\xi} - \frac{1}{2T_w(x_l)} - t_l]|v_l|^2} d\sigma(v_l, v_{l-1}). \tag{75}$$

Expanding $d\sigma(v_l, v_{l-1})$ with (6) and (18) we rewrite (75) as

$$\begin{aligned} &\int_{\mathcal{V}_{l,\perp}} \frac{2}{r_\perp} \frac{|v_{l,\perp}|}{2T_w(x_l)} e^{-[\frac{1}{2T_M}\frac{\xi+1}{2\xi} - \frac{1}{2T_w(x_l)} - t_l]|v_{l,\perp}|^2} \\ &\times I_0\left(\frac{(1-r_\perp)^{1/2} v_{l,\perp} v_{l-1,\perp}}{T_w(x_l) r_\perp} \right) e^{-\frac{|v_{l,\perp}|^2 + (1-r_\perp)|v_{l-1,\perp}|^2}{2T_w(x_l) r_\perp}} dv_{l,\perp} \\ &\times \int_{\mathcal{V}_{l,\parallel}} \frac{1}{\pi r_\parallel (2-r_\parallel)(2T_w(x_l))} \\ &\times e^{-[\frac{1}{2T_M}\frac{\xi+1}{2\xi} - \frac{1}{2T_w(x_l)} - t_l]|v_{l,\parallel}|^2} e^{-\frac{1}{2T_w(x_l)} \frac{|v_{l,\parallel} - (1-r_\parallel)v_{l-1,\parallel}|^2}{r_\parallel(2-r_\parallel)}} dv_{l,\parallel}, \end{aligned} \tag{76}$$

where $v_{l,\|}$, $v_{l,\perp}$, $\mathcal{V}_{l,\perp}$ and $\mathcal{V}_{l,\|}$ are defined as

$$\begin{aligned} &v_{l,\perp} = v_l \cdot n(x_l),\ v_{l,\|} = v_l - v_{l,\perp} n(x_l), \\ &\mathcal{V}_{l,\perp} = \{v_{l,\perp} : v_l \in \mathcal{V}_l\},\ \mathcal{V}_{l,\|} = \{v_{l,\|} : v_l \in \mathcal{V}_l\}. \end{aligned} \tag{77}$$

$v_{l-1,\|}$ and $v_{l-1,\perp}$ are defined similarly.

First we compute the integration over $\mathcal{V}_{l,\|}$, the third, fourth line of (76). To apply (210) in Lemma 4.2, we set

$$\begin{aligned} &\varepsilon = t_l,\ w = (1-r_\|)v_{l-1,\|}\ , v = v_{l,\|}, \\ &a = -[\frac{1}{2T_M \frac{2\xi}{\xi+1}} - \frac{1}{2T_w(x_l)}],\ b = \frac{1}{2T_w(x_l) r_\| (2-r_\|)}. \end{aligned} \tag{78}$$

By $\xi > 1$ in (63), we take $t^* = t^*(\xi, T_M) \ll 1$ such that when $t_l < t \le t^*$, we have

$$b - a - \varepsilon = \frac{1}{2T_w(x_l) r_\| (2-r_\|)} - \frac{1}{2T_w(x_l)} + \frac{1}{2T_M \frac{2\xi}{\xi+1}} - t_l \ge \frac{1}{2T_M \frac{2\xi}{\xi+1}} - t \ge \frac{1}{4T_M}. \tag{79}$$

Also we take $t^* = t^*(\xi, T_M)$ to be small enough to obtain $1 + 4T_M t_l \le 1 + 4T_M t \le 2$ when $t \le t^*$. Thus the t^* we choose here is consistent with (70). Hence

$$\begin{aligned} \frac{b}{b-a-\varepsilon} &= \frac{b}{b-a}[1 + \frac{\varepsilon}{b-a-\varepsilon}] \\ &\le \frac{\frac{2\xi}{\xi+1} T_M}{\frac{2\xi}{\xi+1} T_M + [T_w(x_l) - \frac{2\xi}{\xi+1} T_M] r_\| (2-r_\|)} [1 + 4T_M t_l] \\ &\le \frac{\frac{4\xi}{\xi+1} T_M}{\frac{2\xi}{\xi+1} T_M + [\min\{T_w(x)\} - \frac{2\xi}{\xi+1} T_M] r_{max}} := C_{T_M,\xi}, \end{aligned} \tag{80}$$

where we use (61).

In regard to (210), we have

$$\frac{(a+\varepsilon)b}{b-a-\varepsilon} = \frac{ab}{b-a}[1 + \frac{\varepsilon}{b-a-\varepsilon}] + \frac{b}{b-a-\varepsilon}\varepsilon. \tag{81}$$

By (80) and $t_l < t$, we obtain

$$\frac{b}{b-a-\varepsilon}\varepsilon \le \frac{\frac{4\xi}{\xi+1} T_M}{\frac{2\xi}{\xi+1} T_M + [\min\{T_w(x)\} - \frac{2\xi}{\xi+1} T_M] r_{\max}} t.$$

By (78), we have

$$\frac{ab}{b-a} = \frac{\frac{2\xi}{\xi+1} T_M - T_w(x_l)}{2T_w(x_l)[\frac{2\xi}{\xi+1} T_M + [T_w(x_l) - \frac{2\xi}{\xi+1} T_M] r_\| (2-r_\|)]}.$$

Therefore, by (79) and (81) we obtain

$$\frac{(a+\varepsilon)b}{b-a-\varepsilon} \le \frac{\frac{2\xi}{\xi+1} T_M - T_w(x_l)}{2T_w(x_l)[\frac{2\xi}{\xi+1} T_M + [T_w(x_l) - \frac{2\xi}{\xi+1} T_M] r_\| (2-r_\|)]} + \mathcal{C}t, \tag{82}$$

where we define

$$\begin{aligned} \mathcal{C} := &\frac{4T_M\big(\frac{2\xi}{\xi+1} T_M - \min\{T_w(x)\}\big)}{2\min\{T_w(x)\}[\frac{2\xi}{\xi+1} T_M + [\min\{T_w(x)\} - \frac{2\xi}{\xi+1} T_M] r_{max}]} \\ &+ \frac{\frac{4\xi}{\xi+1} T_M}{\frac{2\xi}{\xi+1} T_M + [\min\{T_w(x)\} - \frac{2\xi}{\xi+1} T_M] r_{max}}. \end{aligned} \tag{83}$$

By (80), (82) and Lemma 4.2, using $w = (1-r_\parallel)v_{l-1,\parallel}$ we bound the third, fourth line of (76) by

$$C_{T_M,\xi}\exp\left(\Big[\frac{[\frac{2\xi}{\xi+1}T_M - T_w(x_l)]}{2T_w(x_l)[\frac{2\xi}{\xi+1}T_M(1-r_\parallel)^2 + r_\parallel(2-r_\parallel)T_w(x_l)]} + \mathcal{C}t\Big]|(1-r_\parallel)v_{l-1,\parallel}|^2\right) \tag{84}$$

$$\leq C_{T_M,\xi}\exp\left(\Big[\frac{[\frac{2\xi}{\xi+1}T_M - T_w(x_l)][1-r_{min}]}{2T_w(x_l)\big[\frac{2\xi}{\xi+1}T_M(1-r_{min}) + r_{min}T_w(x_l)\big]} + \mathcal{C}t\Big]|v_{l-1,\parallel}|^2\right). \tag{85}$$

where we use (61) and (62).

Next we compute first, second line of (76). To apply (213) in Lemma 4.3, we set

$$\varepsilon = t_l,\ w = \sqrt{1-r_\parallel}v_{l-1,\perp}\ , v = v_{l,\perp},$$

$$a = -[\frac{1}{2T_M\frac{2\xi}{\xi+1}} - \frac{1}{2T_w(x_l)}],\ b = \frac{1}{2T_w(x_l)r_\perp}.$$

Thus we can compute $\frac{b}{b-a-\varepsilon}$ and $\frac{(a+\varepsilon)b}{b-a-\varepsilon}$ using the exactly the way as (80) and (82) with replacing $r_\parallel(2-r_\parallel)$ by $r_\perp$. Hence replacing $r_\parallel(2-r_\parallel)$ by $r_\perp$ and replacing $v_{l-1,\parallel}$ by $v_{l-1,\perp}$ in (84), we bound the first, second line of (76) by

$$C_{T_M,\xi}\exp\left(\Big[\frac{[\frac{2\xi}{\xi+1}T_M - T_w(x_l)]}{2T_w(x_l)[\frac{2\xi}{\xi+1}T_M(1-r_\perp) + r_\perp T_w(x_l)]} + \mathcal{C}t_l\Big]|\sqrt{1-r_\perp}v_{l-1,\perp}|^2\right)$$

$$\leq C_{T_M,\xi}\exp\left(\Big[\frac{[\frac{2\xi}{\xi+1}T_M - T_w(x_l)][1-r_{min}]}{2T_w(x_l)\big[\frac{2\xi}{\xi+1}T_M(1-r_{min}) + r_{min}T_w(x_l)\big]} + \mathcal{C}t\Big]|v_{l-1,\perp}|^2\right). \tag{86}$$

where we use (61) and (62).

Collecting (85) (86), we derive

$$\begin{aligned}(76) &\leq (C_{T_M,\xi})^2\exp\left(\left[\frac{[\frac{2\xi}{\xi+1}T_M - T_w(x_l)][1-r_{min}]}{2T_w(x_l)\big[\frac{2\xi}{\xi+1}T_M(1-r_{min}) + r_{min}T_w(x_l)\big]} + \mathcal{C}t\right]|v_{l-1}|^2\right)\\ &= (C_{T_M,\xi})^2\mathcal{A}_{l,l},\end{aligned}$$

where $\mathcal{A}_{l,l}$ is defined in (72) and $T_{l,l} = \frac{2\xi}{\xi+1}T_M$.

Therefore, (71) is valid for $p = l$.

Suppose (71) is valid for the $p = q+1$(induction hypothesis) with $q+1 \leq l$, then

$$\int_{\prod_{j=q+1}^l \mathcal{V}_j} \mathbf{1}_{\{t_l>0\}} d\Phi_{q+1,m}^{l+1,l}(s) \leq (C_{T_M,\xi})^{2(l-q)}\mathcal{A}_{l,q+1}.$$

We want to show (71) holds for $p = q$. By the hypothesis and the third line of (66),

$$\begin{aligned}&\int_{\prod_{j=q}^l \mathcal{V}_j} \mathbf{1}_{\{t_l>0\}} d\Phi_{q,m}^{l+1,l}(s)\\ &\leq (C_{T_M,\xi})^{2(l-q)}\int_{\mathcal{V}_q}\mathcal{A}_{l,q+1}e^{[\frac{1}{2T_w(x_q)} - \frac{1}{2T_w(x_{q+1})}]|v_q|^2}d\sigma(v_q,v_{q-1}).\end{aligned} \tag{87}$$

Using the definition of $\mathcal{A}_{l,q+1}$ in (72), we obtain

$$\begin{aligned}(87) \le & (C_{T_M,\xi})^{2(l-q)} \\ & \times \int_{\mathcal{V}_q} \exp\left(\frac{(T_{l,q+1}-T_w(x_{q+1}))(1-r_{min})}{2T_w(x_{q+1})[T_{l,q+1}(1-r_{min})+r_{min}T_w(x_{q+1})]}|v_q|^2+\mathcal{C}^{l-q}t|v_q|^2\right) \\ & \times e^{[\frac{1}{2T_w(x_q)}-\frac{1}{2T_w(x_{q+1})}]|v_q|^2}d\sigma(v_q,v_{q-1}).\end{aligned} \tag{88}$$

We focus on the coefficient of $|v_q|^2$ in (88), we derive

$$\begin{aligned}&\frac{(T_{l,q+1}-T_w(x_{q+1}))(1-r_{min})}{2T_w(x_{q+1})[T_{l,q+1}(1-r_{min})+r_{min}T_w(x_{q+1})]}|v_q|^2+[\frac{1}{2T_w(x_q)}-\frac{1}{2T_w(x_{q+1})}]|v_q|^2= \\ &\frac{(T_{l,q+1}-T_w(x_{q+1}))(1-r_{min})-[T_{l,q+1}(1-r_{min})+r_{min}T_w(x_{q+1})]}{2T_w(x_{q+1})[T_{l,q+1}(1-r_{min})+r_{min}T_w(x_{q+1})]}|v_q|^2+\frac{|v_q|^2}{2T_w(x_q)} \\ &=\frac{-T_w(x_{q+1})(1-r_{min})-r_{min}T_w(x_{q+1})}{2T_w(x_{q+1})[T_{l,q+1}(1-r_{min})+r_{min}T_w(x_{q+1})]}|v_q|^2+\frac{|v_q|^2}{2T_w(x_q)} \\ &=\frac{-|v_q|^2}{2[T_{l,q+1}(1-r_{min})+r_{min}T_w(x_{q+1})]}+\frac{|v_q|^2}{2T_w(x_q)}.\end{aligned}$$

By the Definition 2.1, $x_{q+1}=x_{q+1}(t,x,v,v_1,\cdots,v_q)$, thus $T_w(x_{q+1})$ depends on v_q. In order to explicitly compute the integration over $\mathcal{V}_q$, we need to get rid of the dependence of the $T_w(x_{q+1})$ on v_q. Then we bound

$$\begin{aligned}&\exp\left(\frac{-|v_q|^2}{2[T_{l,q+1}(1-r_{min})+r_{min}T_w(x_{q+1})]}\right) \\ &\le \exp\left(\frac{-|v_q|^2}{2[T_{l,q+1}(1-r_{min})+r_{min}T_M]}\right)=\exp\left(\frac{-|v_q|^2}{2T_{l,q}}\right),\end{aligned} \tag{89}$$

where we use (64).

Hence by (18) (6) and (89), we derive

$$\begin{aligned}(88) \le & (C_{T_M,\xi})^{2(l-q)} \\ & \times \int_{\mathcal{V}_{q,\perp}} \frac{2}{r_\perp}\frac{|v_{q,\perp}|}{2T_w(x_q)}e^{-[\frac{1}{2T_{l,q}}-\frac{1}{2T_w(x_q)}-\mathcal{C}^{l-q}t]|v_{q,\perp}|^2} \\ & \times I_0\left(\frac{(1-r_\perp)^{1/2}v_{q,\perp}v_{q-1,\perp}}{T_w(x_q)r_\perp}\right)e^{-\frac{|v_{q,\perp}|^2+(1-r_\perp)|v_{q-1,\perp}|^2}{2T_w(x_q)r_\perp}}dv_{q,\perp} \\ & \times \int_{\mathcal{V}_{q,\parallel}} \frac{1}{\pi r_\parallel(2-r_\parallel)(2T_w(x_q))} \\ & \times e^{-[\frac{1}{2T_{l,q}}-\frac{1}{2T_w(x_q)}-\mathcal{C}^{l-q}t]|v_{q,\parallel}|^2}e^{-\frac{1}{2T_w(x_q)}\frac{|v_{q,\parallel}-(1-r_\parallel)v_{q-1,\parallel}|^2}{r_\parallel(2-r_\parallel)}}dv_{q,\parallel}.\end{aligned} \tag{90}$$

In the fourth, fifth line of (90), to apply (210) in Lemma 4.2, we set

$$a=-[\frac{1}{2T_{l,q}}-\frac{1}{2T_w(x_q)}],\ b=\frac{1}{2T_w(x_q)r_\parallel(2-r_\parallel)},\ \varepsilon=\mathcal{C}^{l-q}t,\ w=(1-r_\parallel)v_{q-1,\parallel}.$$

Taking (78) for comparison, we can replace $\frac{2\xi}{\xi+1}T_M$ by $T_{l,q}$ and replace t by $\mathcal{C}^{l-q}t$. Then we apply the replacement to (79) and obtain

$$b-a-\varepsilon\ge\frac{1}{2T_{l,q}}-\mathcal{C}^{l-q}t\ge\frac{1}{2T_M\frac{2\xi}{\xi+1}}-\mathcal{C}^kt\ge\frac{1}{4T_M},$$

where we take $t^* = t^*(T_M, \xi, \mathcal{C}, k)$ to be small enough and $t \le t^*$. Also we require the t satisfy

$$\frac{\varepsilon}{b-a-\varepsilon} \le 4T_M \mathcal{C}^k t \le 2.$$

We conclude the t^* only depends on the parameter in (70). Thus by the same computation as (80) we obtain

$$\frac{b}{b-a-\varepsilon} \le \frac{2T_{l,q}}{T_{l,q} + [\min\{T_w(x)\} - T_{l,q}] r_\parallel (2-r_\parallel)} \le C_{T_M,\xi},$$

where we use $T_{l,q} \le \frac{2\xi}{\xi+1} T_M$ from (64) and (61). $C_{T_M,\xi}$ is defined in (80).

By the same computation as (82), we obtain

$$\frac{(a+\varepsilon)b}{b-a-\varepsilon} = \frac{ab}{b-a} + \frac{ab}{b-a}\frac{\varepsilon}{b-a-\varepsilon} + \frac{b}{b-a-\varepsilon}\varepsilon$$

$$\le \frac{T_{l,q} - T_w(x_q)}{2T_w(x_q)[T_{l,q} + [T_w(x_q) - T_{l,q}] r_\parallel (2-r_\parallel)]} + \mathcal{C}^{l-q+1} t.$$

Here we use $T_{l,q} \le \frac{2\xi}{\xi+1} T_M$ and (61) to obtain

$$\begin{aligned}
&\frac{ab}{b-a}\frac{\varepsilon}{b-a-\varepsilon} + \frac{b\varepsilon}{b-a-\varepsilon} \\
&\le \frac{4T_M\big(T_{l,q} - \min\{T_w(x)\}\big)}{2\min\{T_w(x)\}[T_{l,q} + [\min\{T_w(x)\} - T_{l,q}] r_\parallel (2-r_\parallel)]} \mathcal{C}^{l-q} t \\
&+ \frac{2T_{l,q}}{\frac{2\xi}{\xi+1} T + [\min\{T_w(x)\} - T_{l,q}] r_\parallel (2-r_\parallel)} \mathcal{C}^{l-q} t \le \mathcal{C}^{l-q+1} t
\end{aligned}$$

with $\mathcal{C}$ defined in (83).

Thus by Lemma 4.2 with $w = (1-r_\parallel) v_{q-1,\parallel}$, the fourth, fifth line of (90) is bounded by

$$C_{T_M,\xi} \exp\left(\Big[\frac{[T_{l,q} - T_w(x_q)]}{2T_w(x_q)[T_{l,q}(1-r_\parallel)^2 + r(2-r_\parallel) T_w(x_q)]} + \mathcal{C}^{l-q+1} t \Big] |(1-r_\parallel) v_{q-1,\parallel}|^2 \right)$$

$$\le C_{T_M,\xi} \exp\left(\Big[\frac{[T_{l,q} - T_w(x_q)][1-r_{min}]}{2T_w(x_q)[T_{l,q}(1-r_{min}) + r_{min} T_w(x_q)]} + \mathcal{C}^{l-q+1} t \Big] |v_{q-1,\parallel}|^2 \right). \quad (91)$$

By the same computation the second, third line of (90) is bounded by

$$C_{T_M,\xi} \exp\left(\Big[\frac{[T_{l,q} - T_w(x_q)][1-r_{min}]}{2T_w(x_q)[T_{l,q}(1-r_{min}) + r_{min} T_w(x_q)]} + \mathcal{C}^{l-q+1} t \Big] |v_{q-1,\perp}|^2 \right). \quad (92)$$

By (91) and (92), we derive that

$$\begin{aligned}
(90) &\le (C_{T_M,\xi})^{2(l-q+1)} \\
&\exp\left(\Big[\frac{[T_{l,q} - T_w(x_q)][1-r_{min}]}{2T_w(x_q)[T_{l,q}(1-r_{min}) + r_{min} T_w(x_q)]} + \mathcal{C}^{l-q+1} t \Big] |v_{q-1}|^2 \right) \\
&= (C_{T_M,\xi})^{2(l-q+1)} \mathcal{A}_{l,q},
\end{aligned}$$

which is consistent with (71) with $p = q$. The induction is valid and we derive (71).

Now we focus on (73). The first inequality in (73) follows directly from (71) and (68). For the second inequality, by (67) we have

$$\begin{aligned}
&(C_{T_M,\xi})^{2(l-p'+1)}\int_{\prod_{j=p}^{p'-1}\mathcal{V}_j}\mathbf{1}_{\{t_l>0\}}\mathcal{A}_{l,p'}d\Upsilon_p^{p'-1}\\
&\le (C_{T_M,\xi})^{2(l-p'+1)}\int_{\prod_{j=p}^{p'-2}\mathcal{V}_j}\int_{\mathcal{V}_{p'-1}}\mathbf{1}_{\{t_l>0\}}\\
&\times\mathcal{A}_{l,p'}e^{[\frac{1}{2T_w(x_{p'-1})}-\frac{1}{2T_w(x_{p'})}]|v_{p'-1}|^2}d\sigma(v_{p'-1},v_{p'-2})d\Upsilon_p^{p'-2}. \qquad (93)
\end{aligned}$$

In the proof for (71) we have

$$(87)\le(88)\le(90)\le(C_{T_M,\xi})^{2(l-q+1)}\mathcal{A}_{l,q}.$$

Then by replacing q by $p'-1$ in the estimate $(87)\le(C_{T_M,\xi})^{2(l-q+1)}\mathcal{A}_{l,q}$ we have

$$(93)\le(C_{T_M,\xi})^{2(l-p'+2)}\int_{\prod_{j=p}^{p'-2}\mathcal{V}_j}\mathbf{1}_{\{t_l>0\}}\mathcal{A}_{l,p'-1}d\Upsilon_p^{p'-2}.$$

Keep doing this computation until integrating over $\mathcal{V}_p$ we obtain the second inequality in (73). □

The next result is the Lemma 2.4, which is the smallness of the last term of (45).

Lemma 2.4. *Assume*

$$\frac{\min(T_w(x))}{T_M}>\max\Big(\frac{1-r_\parallel}{2-r_\parallel},\frac{\sqrt{1-r_\perp}-(1-r_\perp)}{r_\perp}\Big). \qquad (94)$$

For the last term of (45), there exists

$$k_0=k_0(\Omega,C_{T_M,\xi},\mathcal{C},T_M,r_\perp,r_\parallel,\min\{T_w(x)\},\xi)\gg1, \qquad (95)$$

$$t'=t'(k_0,\xi,T_M,\min\{T_w(x)\},\mathcal{C},r_\perp,r_\parallel)\ll1 \qquad (96)$$

such that for all $t\in[0,t']$, we have

$$\int_{\prod_{j=1}^{k_0-1}\mathcal{V}_j}\mathbf{1}_{\{t_{k_0}>0\}}d\Sigma_{k_0-1,m}^{k_0}(t_{k_0})\le(\frac{1}{2})^{k_0}\mathcal{A}_{k_0-1,1}, \qquad (97)$$

where $\mathcal{A}_{k_0-1,1}$ is defined in (72).

Remark 8. The difference between this lemma and Lemma 2.3 is that we have the small term $(\frac{1}{2})^{k_0}$. This lemma implies when $k=k_0$ is large enough, the measure of the last term of (45) is small.

We need several lemmas to prove it.

Lemma 2.5. *For $1\le i\le k-1$, if*

$$|v_i\cdot n(x_i)|<\delta, \qquad (98)$$

then

$$\int_{\prod_{j=i}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{v_i\in\mathcal{V}_i:|v_i\cdot n(x_i)|<\delta\}}\mathbf{1}_{\{t_k>0\}}d\Phi_{i,m}^{k,k-1}(t_k)\le\delta(C_{T_M,\xi})^{2(k-i)}\mathcal{A}_{k-1,i}. \qquad (99)$$

If

$$|v_{i,\parallel}-\eta_{i,\parallel}v_{i-1,\parallel}|>\delta^{-1}, \qquad (100)$$

then

$$\int_{\prod_{j=i}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{t_k>0\}}\mathbf{1}_{\{|v_{i,\parallel}-\eta_{i,\parallel}v_{i-1,\parallel}|>\delta^{-1}\}}d\Phi_{i,m}^{k,k-1}(t_k)\le\delta(C_{T_M,\xi})^{2(k-i)}\mathcal{A}_{k-1,i}. \qquad (101)$$

Here $\eta_{i,\|}$ is a constant defined in (109).

If

$$|v_{i,\perp}-\eta_{i,\perp}v_{i-1,\perp}|>\delta^{-1}, \tag{102}$$

then

$$\int_{\prod_{j=i}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{t_k>0\}}\mathbf{1}_{\{|v_{i,\perp}-\eta_{i,\perp}v_{i-1,\perp}|>\delta^{-1}\}}d\Phi_{i,m}^{k,k-1}(t_k)\le\delta(C_{T_M,\xi})^{2(k-i)}\mathcal{A}_{k-1,i}. \tag{103}$$

Here $\eta_{i,\perp}$ is a constant defined in (112).

Proof. First we focus on (99). By (90) in Lemma 2.3, we can replace l by $k-1$ and replace q by i to obtain

$$\begin{aligned}
&\int_{\prod_{j=i}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{t_k>0\}}d\Phi_{i,m}^{k,k-1}(t_k)\le(C_{T_M,\xi})^{2(k-i)}\\
&\times\int_{\mathcal{V}_{i,\perp}}\frac{2}{r_\perp}\frac{|v_{i,\perp}|}{2T_w(x_i)}e^{-[\frac{1}{2T_{k-1,i}}-\frac{1}{2T_w(x_i)}-\mathcal{C}^{k-i}t]|v_{i,\perp}|^2}\\
&\times I_0\left(\frac{(1-r_\perp)^{1/2}v_{i,\perp}v_{i-1,\perp}}{T_w(x_i)r_\perp}\right)e^{-\frac{|v_{i,\perp}|^2+(1-r_\perp)|v_{i-1,\perp}|^2}{2T_w(x)r_\perp}}dv_{i,\perp}\\
&\times\int_{\mathcal{V}_{i,\|}}\frac{1}{\pi r_\|(2-r_\|)(2T_w(x_i))}\\
&\times e^{-[\frac{1}{2T_{k-1,i}}-\frac{1}{2T_w(x_i)}-\mathcal{C}^{k-i}t]|v_{i,\|}|^2}e^{-\frac{1}{2T_w(x_i)}\frac{|v_{i,\|}-(1-r_\|)v_{i-1,\|}|^2}{r_\|(2-r_\|)}}dv_{i,\|}.
\end{aligned} \tag{104}$$

Under the condition (98), we consider the second, third line of (104) with integrating over $\{v_{i,\perp}\in\mathcal{V}_{i,\perp}:|v_i\cdot n(x_i)|<\frac{1-\eta}{2(1+\eta)}\delta\}$. To apply (214) in Lemma 4.3, we set

$$a=-[\frac{1}{2T_{k-1,i}}-\frac{1}{2T_w(x_i)}],\ b=\frac{1}{2T_w(x_i)r_\perp},\ \varepsilon=\mathcal{C}^{k-i}t,\ w=\sqrt{1-r_\perp}v_{i-1,\perp}.$$

Under the condition $|v_i\cdot n(x_i)|<\frac{1-\eta}{2(1+\eta)}\delta$, applying (214) in Lemma 4.3 and using (92) with $q=i,l=k-1$, we bound the second, third line of (104) by

$$\delta C_{T_M,\xi}\exp\left([\frac{[T_{k-1,i}-T_w(x_i)][1-r_{min}]}{2T_w(x_i)[T_{k-1,i}(1-r_{min})+r_{min}T_w(x_i)]}+\mathcal{C}^{k-i}t]|v_{i-1,\perp}|^2\right). \tag{105}$$

Taking (92) for comparison, we conclude the second, third line of (104) provides one more constant term δ. The fourth, fifth line of (104) is bounded by (91) with $q=i,l=k-1$. Therefore, we derive (99).

Then we focus on (101). We consider the fourth, fifth line of (104). To apply (212) in Lemma 4.2, we set

$$a=-\frac{1}{2T_{k-1,i}}+\frac{1}{2T_w(x_i)},\quad b=\frac{1}{2T_w(x_i)r_\|(2-r_\|)},\quad \varepsilon=\mathcal{C}^{k-i}t,\ w=(1-r_\|)v_{i-1,\|}. \tag{106}$$

We define

$$B_{i,\|}:=b-a-\varepsilon. \tag{107}$$

In regard to (212),

$$\frac{b}{b-a-\varepsilon}w=\frac{b}{b-a}[1+\frac{\varepsilon}{b-a-\varepsilon}]w.$$

By (106),

$$\frac{b}{b-a} = \frac{T_{k-1,i}}{T_{k-1,i}(1-r_\parallel)^2 + T_w(x_i)r_\parallel(2-r_\parallel)}, \quad \frac{\varepsilon}{b-a-\varepsilon} = \frac{\mathcal{C}^{k-i}t}{B_{i,\parallel}}.$$

Thus we obtain

$$\frac{b}{b-a-\varepsilon}w = \eta_{i,\parallel}v_{i-1,\parallel}, \tag{108}$$

where we define

$$\eta_{i,\parallel} := \frac{T_{k-1,i}[1+\mathcal{C}^{k-i}t/B_{i,\parallel}]}{T_{k-1,i}(1-r_\parallel)^2 + T_w(x_i)r_\parallel(2-r_\parallel)}(1-r_\parallel). \tag{109}$$

Thus under the condition (100), applying (212) in Lemma 210 with $\frac{b}{b-a-\varepsilon}w = \eta_{i,\parallel}v_{i-1,\parallel}$ and using (91) with $q = i, l = k-1$, we bound the fourth, fifth line of (104) by

$$\delta C_{T_M,\xi} \exp\left(\Big[\frac{[T_{k-1,i} - T_w(x_i)][1-r_{min}]}{2T_w(x_i)[T_{k-1,i}(1-r_{min}) + r_{min}T_w(x_i)]} + \mathcal{C}^{k-i}t \Big] |v_{i-1,\parallel}|^2 \right).$$

By the same computation in Lemma 2.5, we derive (101) because of the extra constant δ.

Last we focus on (103). We consider the second, third line of (104) with integrating over $\{v_{i,\perp} : v_{i,\perp} \in \mathcal{V}_{i,\perp}, |v_{i,\perp}| > \frac{1+\eta}{1-\eta}\delta^{-1}\}$. To apply (214) in Lemma 4.4, we set

$$a = -\frac{1}{2T_{k-1,i}} + \frac{1}{2T_w(x_i)}, \quad b = \frac{1}{2T_w(x_i)r_\perp}, \quad \varepsilon = \mathcal{C}^{k-i}t, \ w = \sqrt{1-r_\perp}v_{i-1,\perp}. \tag{110}$$

Define

$$B_{i,\perp} := b - a - \varepsilon. \tag{111}$$

By the same computation as (108),

$$\frac{b}{b-a-\varepsilon}w = \eta_{i,\perp}v_{i-1,\perp},$$

where we define

$$\eta_{i,\perp} := \frac{T_{k-1,i}[1+\frac{\mathcal{C}^{k-i}t}{B_{i,\perp}}]}{T_{k-1,i}(1-r_\perp) + T_w(x_i)r_\perp}\sqrt{1-r_\perp}. \tag{112}$$

Thus under the condition (102), applying (217) in Lemma 4.4 with $\frac{b}{b-a-\varepsilon}w = \eta_{i,\perp}v_{i-1,\perp}$ and using (92) with $q = i, l = k-1$, we bound the second, third line of (104) by

$$\delta C_{T_M,\xi} \exp\left(\Big[\frac{[T_{k-1,i} - T_w(x_i)][1-r_{min}]}{2T_w(x_i)[T_{k-1,i}(1-r_{min}) + r_{min}T_w(x_i)]} + \mathcal{C}^{k-i}t \Big] |v_{i-1,\perp}|^2 \right).$$

Then we derive (101) because of the extra constant δ. □

Lemma 2.6. *For $\eta_{i,\parallel}$ and $\eta_{i,\perp}$ defined in Lemma 2.5, we suppose there exists $\eta < 1$ such that*

$$\max\{\eta_{i,\parallel}, \eta_{i,\perp}\} < \eta < 1. \tag{113}$$

Then If

$$|v_{i,\parallel}| > \frac{1+\eta}{1-\eta}\delta^{-1} \text{ and } |v_{i,\parallel} - \eta_{i,\parallel}v_{i-1,\parallel}| < \delta^{-1}, \tag{114}$$

we have

$$|v_{i-1,\parallel}| > |v_{i,\parallel}| + \delta^{-1}. \tag{115}$$

Also if

$$|v_{i,\perp}| > \frac{1+\eta}{1-\eta}\delta^{-1} \text{ and } |v_{i,\perp} - \eta_{i,\perp} v_{i-1,\perp}| < \delta^{-1}, \tag{116}$$

then we have

$$|v_{i-1,\perp}| > |v_{i,\perp}| + \delta^{-1}. \tag{117}$$

Remark 9. Lemma 2.5 includes the cases that are controllable because of the small magnitude number δ, which is the "good" factor for us to establish the Lemma 2.4. This lemma discusses those "bad" cases, which are the main difficulty since they do not directly provide δ.

Proof. Under the condition (114) we have

$$\eta_{i,\|}|v_{i-1,\|}| > |v_{i,\|}| - \delta^{-1}.$$

Thus we derive

$$\begin{aligned}
|v_{i-1,\|}| &> |v_{i,\|}| + \frac{1-\eta_{i,\|}}{\eta_{i,\|}}|v_{i,\|}| - \frac{1}{\eta_{i,\|}}\delta^{-1} \\
&> |v_{i,\|}| + \frac{1-\eta_{i,\|}}{\eta_{i,\|}}\frac{1+\eta}{1-\eta}\delta^{-1} - \frac{1}{\eta_{i,\|}}\delta^{-1} \\
&> |v_{i,\|}| + \frac{1-\eta_{i,\|}}{\eta_{i,\|}}\frac{1+\eta_{i,\|}}{1-\eta_{i,\|}}\delta^{-1} - \frac{1}{\eta_{i,\|}}\delta^{-1} \\
&> |v_{i,\|}| + \frac{1+\eta_{i,\|}}{\eta_{i,\|}}\delta^{-1} - \frac{1}{\eta_{i,\|}}\delta^{-1} > |v_{i,\|}| + \delta^{-1},
\end{aligned}$$

where we use $|v_{i,\|}| > \frac{1+\eta}{1-\eta}\delta^{-1}$ in the second line and $1 > \eta \geq \eta_{i,\|}$ in the third line. Then we obtain (115).

Under the condition (116), we apply the same computation above to obtain (117). □

Lemma 2.7. *Suppose there are n number of v_j such that*

$$|v_{j,\|} - \eta_{j,\|} v_{j-1,\|}| \geq \delta^{-1}, \tag{118}$$

and also suppose the index j in these v_j are $i_1 < i_2 < \cdots < i_n$, then

$$\begin{aligned}
&\int_{\prod_{j=i_1}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{t_k>0\}}\mathbf{1}_{\{\text{ (118) holds for } j=i_1,i_2,\cdots,i_n\}} d\Phi_{i_1,m}^{k,k-1}(t_k) \\
&\leq (\delta)^n (C_{T_M,\xi})^{2(k-i_1)} \mathcal{A}_{k-1,i_1}.
\end{aligned} \tag{119}$$

Proof. By (73) in Lemma 2 with $l = k-1$, $p = i_1$, $p' = i_n$ and using (101) with $i = i_n$, we have

$$\begin{aligned}
&\int_{\prod_{j=i_1}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{t_k>0\}}\mathbf{1}_{\{\text{ (118) holds for } j=i_1,\cdots,i_n\}} d\Phi_{i_1,m}^{k,k-1}(t_k) \\
&\leq \delta (C_{T_M,\xi})^{2(k-i_n)} \int_{\prod_{j=i_1}^{i_n-1}\mathcal{V}_j} \mathcal{A}_{k-1,i_n}\mathbf{1}_{\{t_k>0\}}\mathbf{1}_{\{\text{ (118) holds for } j=i_1,\cdots,i_{n-1}\}} d\Upsilon_{i_1}^{i_n-1} \\
&= \delta (C_{T_M,\xi})^{2(k-i_n)} \int_{\prod_{j=i_1}^{i_{n-1}-1}\mathcal{V}_j} \int_{\prod_{j=i_{n-1}}^{(i_n)-1}\mathcal{V}_j} \\
&\mathcal{A}_{k-1,i_n}\mathbf{1}_{\{t_k>0\}}\mathbf{1}_{\{\text{ (118) holds for } j=i_1,\cdots,i_{n-1}\}} d\Upsilon_{i_{n-1}}^{(i_n)-1} d\Upsilon_{i_1}^{i_{n-1}-1}.
\end{aligned} \tag{120}$$

Again by (73) and (101) with $i = i_{n-1}$ we have

$$(120) \le \delta^2 (C_{T_M,\xi})^{2(k-i_{n-1})} \int_{\prod_{j=i_1}^{i_{n-1}-1} \mathcal{V}_j} \times \mathcal{A}_{k-1,i_{n-1}} \mathbf{1}_{\{t_k>0\}} \mathbf{1}_{\{\text{ (118) holds for } j=i_1,\cdots,i_{n-2}\}} d\Upsilon_{i_1}^{i_{n-1}-1}.$$

Keep doing this computation until integrating over $\mathcal{V}_{i_1}$ we derive (119). □

Lemma 2.8. *For $0 < \delta \ll 1$, we define*

$$\mathcal{V}_j^\delta := \{v_j \in \mathcal{V}_j : |v_j \cdot n(x_j)| > \delta, |v_j| \le \delta^{-1}\}. \tag{121}$$

For the sequence $\{v_1, v_2, \cdots, v_{k-1}\}$, consider a subsequence $\{v_{l+1}, v_{l+2}, \cdots, v_{l+L}\}$ with $l+1 < l+L \le k-1$ as follows:

$$\underbrace{v_l}_{\in \mathcal{V}_l^{\frac{1-\eta}{2(1+\eta)}\delta}}, \underbrace{v_{l+1}, v_{l+2} \cdots v_{l+L}}_{all \in \mathcal{V}_{l+j} \backslash \mathcal{V}_{l+j}^{\frac{1-\eta}{2(1+\eta)}\delta}}, \underbrace{v_{l+L+1}}_{\in \mathcal{V}_{l+L+1}^{\frac{1-\eta}{2(1+\eta)}\delta}}. \tag{122}$$

In (122), if $L \ge 100\frac{1+\eta}{1-\eta}$, then we have

$$\begin{aligned} &\int_{\prod_{j=l}^{k-1} \mathcal{V}_j} \mathbf{1}_{\{t_k>0\}} \mathbf{1}_{\{v_{l+j} \in \mathcal{V}_{l+j} \backslash \mathcal{V}_{l+j}^{\frac{1-\eta}{2(1+\eta)}\delta} \text{ for } 1 \le j \le L\}} d\Phi_{l,m}^{k,k-1}(t_k) \\ &\le (3\delta)^{L/2} (C_{T_M,\xi})^{2(k-l)} \mathcal{A}_{k-1,l}. \end{aligned} \tag{123}$$

Here the η satisfies the condition (113).

Remark 10. In this lemma we combine the estimates and properties in Lemma 2.5 and Lemma 2.6. In the proof we will address the difficulty stated in Lemma 2.6 to obtain the key factor $(3\delta)^{L/2}$.

Proof. By the definition (121) we have

$$\mathcal{V}_i \backslash \mathcal{V}_i^{\frac{1-\eta}{2(1+\eta)}\delta} = \{v_i \in \mathcal{V}_i : |v_i \cdot n(x_i)| < \frac{1-\eta}{2(1+\eta)}\delta \text{ or } |v_i| \ge \frac{2(1+\eta)}{1-\eta}\delta^{-1}\}.$$

Here we summarize the result of Lemma 2.5 and Lemma 2.6. With $\frac{1-\eta}{1+\eta}\delta < \delta$, when $v_i \in \mathcal{V}_i \backslash \mathcal{V}_i^{\frac{1-\eta}{2(1+\eta)}\delta}$

1. When $|v_i \cdot n(x_i)| < \frac{1-\eta}{2(1+\eta)}\delta$, then we have (99).
2. When $|v_i| > \frac{2(1+\eta)}{1-\eta}\delta^{-1}$,
   (a) when $|v_{i,\|}| > \frac{1+\eta}{1-\eta}\delta^{-1}$, if $|v_{i,\|} - \eta_{i,\|} v_{i-1,\|}| < \delta^{-1}$, then $|v_{i-1,\|}| > |v_{i,\|}| + \delta^{-1}$.
   (b) when $|v_{i,\|}| > \frac{1+\eta}{1-\eta}\delta^{-1}$, if $|v_{i,\|} - \eta_{i,\|} v_{i-1,\|}| \ge \delta^{-1}$, then we have (101).
   (c) when $|v_{i,\perp}| > \frac{1+\eta}{1-\eta}\delta^{-1}$, if $|v_{i,\perp} - \eta_{i,\perp} v_{i-1,\perp}| < \delta^{-1}$, then $|v_{i-1,\perp}| > |v_{i,\perp}| + \delta^{-1}$.
   (d) when $|v_{i,\perp}| > \frac{1+\eta}{1-\eta}\delta^{-1}$, if $|v_{i,\perp} - \eta_{i,\perp} v_{i-1,\perp}| \ge \delta^{-1}$, then we have (103).

We define $\mathcal{W}_{i,\delta}$ as the space that provides the smallness:

$$\begin{aligned}\mathcal{W}_{i,\delta} := &\{v_i \in \mathcal{V}_i : |v_{i,\perp}| < \frac{1-\eta}{2(1+\eta)}\delta\}\\ &\bigcup\{v_i \in \mathcal{V}_i : |v_{i,\perp}| > \frac{1+\eta}{1-\eta}\delta^{-1} \text{ and } |v_{i,\perp} - \eta_{i,\perp}v_{i-1,\perp}| > \delta^{-1}\}\\ &\bigcup\bigcup\{v_i \in \mathcal{V}_i : |v_{i,\|}| > \frac{1+\eta}{1-\eta}\delta^{-1} \text{ and } |v_{i,\|} - \eta_{i,\|}v_{i-1,\|}| > \delta^{-1}\}.\end{aligned}$$

Then we have

$$\begin{aligned}&\mathcal{V}_i\backslash\mathcal{V}_i^{\frac{1-\eta}{2(1+\eta)}\delta}\\ &\subset \mathcal{W}_{i,\delta}\bigcup\{v_{i,\perp} \in \mathcal{V}_{i,\perp}|v_{i,\perp}| > \frac{1+\eta}{1-\eta}\delta^{-1} \text{ and } |v_{i,\perp} - \eta_{i,\perp}v_{i-1,\perp}| < \delta^{-1}\} \qquad (124)\\ &\bigcup\{v_{i,\|} \in \mathcal{V}_{i,\|}|v_{i,\|}| > \frac{1+\eta}{1-\eta}\delta^{-1} \text{ and } |v_{i,\|} - \eta_{i,\|}v_{i-1,\|}| < \delta^{-1}\}.\end{aligned}$$

By (99), (101) and (103) with $\frac{1-\eta}{1+\eta}\delta < \delta$, we obtain

$$\int_{\prod_{j=i}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{v_i\in\mathcal{W}_{i,\delta}\}}\mathbf{1}_{\{t_k>0\}}d\Phi_{i,m}^{k,k-1}(t_k) \le 3\delta(C_{T_M,\xi})^{2(k-i)}\mathcal{A}_{k-1,i}. \qquad (125)$$

For the subsequence $\{v_{l+1},\cdots,v_{l+L}\}$ in (122), when the number of $v_j \in \mathcal{W}_{j,\delta}$ is larger than $L/2$, by (119) in Lemma 2.7 with $n = L/2$ and replacing the condition (118) by $v_j \in \mathcal{W}_{j,\delta}$, we obtain

$$\begin{aligned}&\int_{\prod_{j=l}^{k-1}\mathcal{V}_j} \mathbf{1}_{\{\text{Number of } v_j\in\mathcal{W}_{j,\delta} \text{ is larger than } L/2\}}\mathbf{1}_{\{t_k>0\}}d\Phi_{l,m}^{k,k-1}(t_k) \qquad (126)\\ &\le (3\delta)^{L/2}(C_{T_M,\xi})^{2(k-l_i)}\mathcal{A}_{k-1,l}. \qquad (127)\end{aligned}$$

We finish the discussion with the case(1),(2b),(2d). Then we focus on the case (2a),(2c).

When the number of $v_j \notin \mathcal{W}_{j,\delta}$ is larger than $L/2$, by (124) we further consider two cases. The first case is that the number of $v_j \in \{v_j : |v_{j,\|}| > \frac{1+\eta}{1-\eta}\delta^{-1} \text{ and } |v_{j,\|} - \eta_{j,\|}v_{j-1,\|}| < \delta^{-1}\}$ is larger than $L/4$. According to the relation of $v_{j,\|}$ and $v_{j-1,\|}$, we categorize them into

Set1: $\{v_j \notin \mathcal{W}_{j,\delta} : |v_{j,\|}| > \frac{1+\eta}{1-\eta}\delta^{-1} \text{ and } |v_{j,\|} - \eta_{j,\|}v_{j-1,\|}| < \delta^{-1}\}$.

Denote $M = |\text{Set1}|$ and the corresponding index in Set1 as $j = p_1, p_2, \cdots, p_M$. Then we have

$$L/4 \le M \le L. \qquad (128)$$

By (115) in Lemma 2.6, for those v_{p_j}, we have

$$|v_{p_j,\|}| - |v_{p_j-1,\|}| < -\delta^{-1}. \qquad (129)$$

Set2: $\{v_j \in \mathcal{V}_j\backslash\mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)\delta}} : |v_{j,\|}| \ge |v_{j-1,\|}|\}$.

Denote $\mathcal{M} = |\text{Set2}|$ and the corresponding index in Set2 as $j = q_1, q_2, \cdots, q_{\mathcal{M}}$. By (128) we have

$$1 \le \mathcal{M} \le L - M \le \frac{3}{4}L. \qquad (130)$$

Then for those v_{q_j} we define

$$a_j := |v_{q_j,\|}| - |v_{q_j-1,\|}| > 0. \qquad (131)$$

Set3: $\{v_j \in \mathcal{V}_j \backslash \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)\delta}} : |v_{j,\|}| \le |v_{j-1,\|}| \le |v_{j,\|}| + \delta^{-1}\}$.

Denote $N = |\text{Set3}|$ and the corresponding index in Set3 as $j = o_1, o_2, \cdots, o_N$. Then for those o_j, we have

$$|v_{o_j,\|}| \le |v_{o_j-1,\|}| \le |v_{o_j,\|}| + \delta^{-1}. \tag{132}$$

From (122), we have $v_l \in \mathcal{V}_l^{\frac{1-\eta}{2(1+\eta)}\delta}$ and $v_{l+L+1} \in \mathcal{V}_{l+L+1}^{\frac{1-\eta}{2(1+\eta)}\delta}$, thus we can obtain

$$-\frac{2(1+\eta)}{1-\eta}\delta^{-1} < |v_{l+L+1,\|}| - |v_{l,\|}| = \sum_{j=1}^{L+1} |v_{l+j,\|}| - |v_{l+j-1,\|}|. \tag{133}$$

By (129), (131) and (132), we derive that

$$\begin{aligned}
&\frac{-2(1+\eta)}{1-\eta}\delta^{-1} \\
&< \sum_{j=1}^{M} \left(|v_{p_j,\|}| - |v_{p_j-1,\|}|\right) + \sum_{j=1}^{\mathcal{M}} \left(|v_{q_j,\|}| - |v_{q_j-1,\|}|\right) + \sum_{j=1}^{N} \left(|v_{o_j,\|}| - |v_{o_j-1,\|}|\right) \\
&\le -M\delta^{-1} + \sum_{j=1}^{\mathcal{M}} a_j.
\end{aligned}$$

Therefore, by $L \ge 100\frac{1+\eta}{1-\eta}$ and (128), we obtain

$$\frac{2(1+\eta)}{1-\eta}\delta^{-1} \le \frac{L}{10}\delta^{-1} \le \frac{M}{2}\delta^{-1}$$

and thus

$$\sum_{j=1}^{\mathcal{M}} a_j \ge M\delta^{-1} - \frac{2(1+\eta)}{1-\eta}\delta^{-1} > \frac{M\delta^{-1}}{2}. \tag{134}$$

We focus on integrating over $\mathcal{V}_{q_i}$, those index satisfy (131). Let $1 \le i \le \mathcal{M}$, we consider the third line of (104) with $i = q_i$ and with integrating over $\{v_{q_i,\|} \in \mathcal{V}_{q_i,\|} : |v_{q_i,\|}| - |v_{q_i-1,\|}| = a_i\}$. To apply (211) in Lemma 4.2, we set

$$a = -\frac{1}{2T_{k-1,q_i}} + \frac{1}{2T_w(x_{q_i})}, \quad b = \frac{1}{2T_w(x_{q_i})r_\|(2-r_\|)}, \quad \varepsilon = \mathcal{C}^{k-q_i}t.$$

By the same computation as (142), we have

$$a + \varepsilon - b = -\frac{1}{2T_{k-1,q_i}} + \frac{1}{2T_w(x_{q_i})} - \frac{1}{2T_w(x_{q_i})r_\|(2-r_\|)} + \mathcal{C}^{k-q_i}t < -\frac{1}{4T_M}. \tag{135}$$

Then we use $\eta_{q_i,\|} < 1$ to obtain

$$\mathbf{1}_{\{|v_{q_i,\|}|-|v_{q_i-1,\|}|=a_i\}} \le \mathbf{1}_{\{|v_{q_i,\|}|-\eta_{q_i,\|}|v_{q_i-1,\|}|>a_i\}} \le \mathbf{1}_{\{|v_{q_i,\|}-\eta_{q_i,\|}v_{q_i-1,\|}|>a_i\}}. \tag{136}$$

By (211) in Lemma 4.2 and (136), we apply (91) with $q = q_i$ to bound the third line of (104)(the integration over $\mathcal{V}_{q_i,\|}$) by

$$e^{-\frac{a_i^2}{4T_M}} C_{T_M,\xi} \exp\left(\left[\frac{[T_{k-1,q_i} - T_w(x_{q_i})][1-r_{min}]}{2T_w(x_{q_i})[T_{k-1,q_i}(1-r_{min}) + r_{min}T_w(x_{q_i})]} + \mathcal{C}^{k-q_i}t \right] |v_{q_i-1,\|}|^2 \right). \tag{137}$$

Hence by the constant in (137) we draw a similar conclusion as (125):

$$\int_{\prod_{j=q_i}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{t_k>0\}}\mathbf{1}_{\{|v_{q_i,\|}|-|v_{q_i-1,\|}|=a_i\}}d\Phi_{q_i,m}^{k,k-1}(t_k)\le e^{-\frac{a_i^2}{4T_M}}(C_{T_M,\xi})^{2(k-q_i)}\mathcal{A}_{k-1,q_i}. \tag{138}$$

Therefore, by Lemma 2.7, after integrating over $\mathcal{V}_{q_1,\|},\mathcal{V}_{q_2,\|},\cdots,\mathcal{V}_{q_{\mathcal{M}},\|}$ we obtain an extra constant

$$e^{-[a_i^2+a_2^2+\cdots+a_{\mathcal{M}}^2]/4T_M}\le e^{-[a_i+a_2+\cdots+a_{\mathcal{M}}]^2/(4T_M\mathcal{M})}\le e^{-[M\delta^{-1}/2]^2/(4T_M\mathcal{M})}$$
$$\le e^{-[\frac{L}{8}\delta^{-1}]^2/(4T_M\frac{3}{4}L)}\le e^{-\frac{1}{96T_M}L(\delta^{-1})^2}\le e^{-L\delta^{-1}}.$$

Here we use (134) in the last step of first line and use (128), (130) in the first step of second line and take $\delta\ll 1$ in the last step of second line. Then $e^{-L\delta^{-1}}$ is smaller than $(3\delta)^{L/2}$ in (127) and we conclude

$$\int_{\prod_{j=l}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{M=|\text{Set1}|\ge L/4\}}\mathbf{1}_{\{t_k>0\}}d\Phi_{l,m}^{k,k-1}(t_k)\le(3\delta)^{L/2}(C_{T_M,\xi})^{2(k-l_i)}\mathcal{A}_{k-1,l}. \tag{139}$$

The second case is that the number of $v_j\in\{v_j\notin\mathcal{W}_{j,\delta}:|v_{j,\perp}|>\frac{1+\eta}{1-\eta}\delta^{-1}\}$ is larger than $L/4$. We categorize $v_{j,\perp}$ into

Set4: $\{v_j\notin\mathcal{W}_{j,\delta}:|v_{j,\perp}|>\frac{1+\eta}{1-\eta}\delta^{-1}$ and $|v_{j,\perp}-\eta_{j,\perp}v_{j-1,\perp}|<\delta^{-1}\}$.

Set5: $\{v_j\in\mathcal{V}_j\backslash\mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)\delta}}:|v_{j,\perp}|>|v_{j-1,\perp}|\}$.

Set6: $\{v_j\in\mathcal{V}_j\backslash\mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)\delta}}:|v_{j,\perp}|\le|v_{j-1,\perp}|\le|v_{j,\perp}|+\delta^{-1}\}$.

Denote $|\text{Set4}|=M_1$ and the corresponding index as $p_1',p_2',\cdots,p_{M_1}'$, $|\text{Set5}|=\mathcal{M}_1$ and the corresponding index as $q_1',q_2',\cdots,q_{\mathcal{M}_1}'$, $|\text{Set6}|=N_1$ and the corresponding index as $o_1',o_2',\cdots,o_{N_1}'$. Also define $b_j:=|v_{q_j',\perp}|-|v_{q_j'-1,\perp}|$. By the same computation as (134), we have

$$\sum_{j=1}^{\mathcal{M}_1}b_j\ge M_1\delta^{-1}-\frac{2(1+\eta)}{1-\eta}\delta^{-1}>\frac{M_1\delta^{-1}}{2}.$$

We focus on the integration over $v_{q_j'}$. Let $1\le i\le\mathcal{M}_1$, we consider the second line of (104) with $i=q_i'$ and with integrating over $\{v_{q_i',\perp}\in\mathcal{V}_{q_i',\perp}:|v_{q_i',\perp}|-|v_{q_i'-1,\perp}|=b_i\}$. To apply (216) in Lemma 4.2, we set

$$a=-\frac{1}{2T_{k-1,q_i'}}+\frac{1}{2T_w(x_{q_i'})},\quad b=\frac{1}{2T_w(x_{q_i'})r_\perp},\quad \varepsilon=\mathcal{C}^{k-q_i'}t.$$

By the same computation as (142), we have

$$a+\varepsilon-b=-\frac{1}{2T_{k-1,q_i'}}+\frac{1}{2T_w(x_{q_i'})}-\frac{1}{2T_w(x_{q_i'})r_\perp}+\mathcal{C}^{k-q_i'}t<-\frac{1}{4T_M}. \tag{140}$$

Similar to (136), we have

$$\mathbf{1}_{\{|v_{q_i',\perp}|-|v_{q_i'-1,\perp}|=b_i\}}\le\mathbf{1}_{\{|v_{q_i',\perp}-\eta_{q_i',\perp}v_{q_i'-1,\perp}|>b_i\}}.$$

Hence by (216) in Lemma 4.4 and applying (92), we bound the integration over $\mathcal{V}_{q_i',\perp}$ by

$$e^{-\frac{b_i^2}{16T_M}}C_{T_M,\xi}\exp\left([\frac{[T_{k-1,q_i'}-T_w(x_{q_i'})][1-r_{min}]}{2T_w(x_{q_i'})[T_{k-1,q_i'}(1-r_{min})+r_{min}T_w(x_{q_i'})]}+\mathcal{C}^{k-q_i'}t]|v_{q_i'-1,\perp}|^2\right).$$

Therefore,

$$\int_{\prod_{j=q_i'}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{t_k>0\}}\mathbf{1}_{\{|v_{q_i',\perp}|-|v_{q_i'-1,\perp}|=b_i\}}d\Phi_{q_i',m}^{k,k-1}(t_k)\le e^{-\frac{b_i^2}{16T_M}}(C_{T_M,\xi})^{2(k-q_i')}\mathcal{A}_{k-1,q_i'}.$$

The integration over $\mathcal{V}_{q_1',\perp},\mathcal{V}_{q_2',\perp},\cdots,\mathcal{V}_{q_{\mathcal{M}_1}',\perp}$ provides an extra constant

$$e^{-[b_1^2+b_2^2+\cdots+b_{\mathcal{M}_1}^2]/16T_M}\le e^{-\frac{1}{400T_M}L(\delta^{-1})^2}\le e^{-L\delta^{-1}},$$

where we set $\delta\ll 1$ in the last step. Then $e^{-L\delta^{-1}}$ is smaller than $(3\delta)^{L/2}$ in (127) and we conclude

$$\int_{\prod_{j=l}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{M_1=|\mathrm{Set4}|\ge L/4\}}\mathbf{1}_{\{t_k>0\}}d\Phi_{l,m}^{k,k-1}(t_k)\le(3\delta)^{L/2}(C_{T_M,\xi})^{2(k-l)}\mathcal{A}_{k-1,l}. \quad (141)$$

Finally collecting (127), (139) and (141) we derive the lemma. □

Now we prove the Lemma 2.4.

Proof of Lemma 2.4. **Step 1**

To prove (97) holds for the C-L boundary condition, we mainly use the decomposition (121) done by [1] and [14] for the diffuse boundary condition. In order to apply Lemma 2.8, here we consider the space $\mathcal{V}_i^{\frac{1-\eta}{2(1+\eta)}\delta}$ and ensure η satisfy the condition (113). In this step we mainly focus on constructing the η, which is defined in (152).

First we consider $\eta_{i,\|}$, which is defined in (109). In regard to (106) and (107), we take $t'=t'(\xi,k,T_M)$(consistent with (96)) to be small enough and set $t\le t'$ to obtain

$$B_{i,\|}\ge\frac{1}{2T_{k-1,i}}-\mathcal{C}^{k-i}t\ge\frac{1}{2\frac{2\xi}{\xi+1}T_M}-\mathcal{C}^k t\ge\frac{1}{4T_M}. \quad (142)$$

By (65), $T_{k-1,i}\to T_M$ as $k-i\to\infty$. For any $\varepsilon_1>0$, there exists k_1 s.t when

$$k\ge k_1,\quad i\le k/2,\ \text{we have } T_{k-1,i}\le(1+\varepsilon_1)T_M. \quad (143)$$

Moreover, by (94), there exists ε_2 s.t

$$\frac{\min\{T_w(x)\}}{T_M}>\frac{1-r_\|}{2-r_\|}(1+\varepsilon_2). \quad (144)$$

Then we have

$$\varepsilon_2=\varepsilon_2(\min\{T_w(x)\},T_M,r_\|,r_\perp). \quad (145)$$

Thus we can bound $T_w(x_i)$ in the $\eta_{i,\|}$(defined in (109)) below as

$$T_w(x_i)=T_{k-1,i}\frac{T_w(x_i)}{T_{k-1,i}}\ge T_{k-1,i}\frac{T_w(x_i)}{T_M}\frac{1}{1+\varepsilon_1}>\frac{1-r_\|}{2-r_\|}T_{k-1,i}\frac{1+\varepsilon_2}{1+\varepsilon_1}. \quad (146)$$

Thus we obtain

$$\eta_{i,\|}<\frac{1+\frac{\mathcal{C}^{k-i}t}{B_{i,\|}}}{(1-r_\|)^2+\frac{1-r_\|}{2-r_\|}\frac{1+\varepsilon_2}{1+\varepsilon_1}r_\|(2-r_\|)}(1-r_\|)=\frac{1+\frac{\mathcal{C}^{k-i}t}{B_{i,\|}}}{1-r_\|+r_\|\frac{1+\varepsilon_2}{1+\varepsilon_1}}. \quad (147)$$

By (143), we take

$$k=k_1=k_1(\varepsilon_2,T_M,r_{\min}) \quad (148)$$

to be large enough such that $\varepsilon_1 < \varepsilon_2/4$. By (142) and (147), we derive that when $k = k_1$,

$$\sup_{i\leq k/2} \eta_{i,\|} \leq \frac{1+4T_M\mathcal{C}^k t}{1-r_\|+r_\|\frac{1+\varepsilon_2}{1+\varepsilon_2/4}} < \eta_\| < 1. \tag{149}$$

Here we define

$$\eta_\| := \frac{1}{1-r_\|+r_\|\frac{1+\varepsilon_2}{1+\varepsilon_2/2}} < 1 \tag{150}$$

and we take $t' = t'(k, T_M, \varepsilon_2, \mathcal{C}, r_\|)$ to be small enough and $t \leq t'$ such that $4T_M\mathcal{C}^k t \ll 1$ to ensure the second inequality in (149). Combining (145) and (148), we conclude the t' we choose only depends on the parameter in (96).

Then we consider $\eta_{i,\perp}$, which is defined in (112). In regard to (110) and (111), by (142) we have $B_{i,\perp} \geq \frac{1}{4T_M}$. By $\frac{\min\{T_w(x)\}}{T_M} > \frac{\sqrt{1-r_\perp}-(1-r_\perp)}{r_\perp}$ in (94) we can use the same computation as (146) to obtain

$$T_w(x_i) > \frac{\sqrt{1-r_\perp}-(1-r_\perp)}{r_\perp} T_{k-1,i}\frac{1+\varepsilon_2}{1+\varepsilon_1},$$

with $\varepsilon_1 < \varepsilon_2/4$. Thus we obtain

$$\eta_{i,\perp} < \eta_\perp < 1,$$

where we define

$$\eta_\perp := \frac{1}{\sqrt{1-r_\perp}+(1-\sqrt{1-r_\perp})\frac{1+\varepsilon_2}{1+\varepsilon_2/2}} < 1, \tag{151}$$

with $t' = t'(k, T_M, \varepsilon_2, \mathcal{C}, r_\|)$(consistent with (96)) small enough and $t \leq t'$.

Finally we define

$$\eta := \max\{\eta_\perp, \eta_\|\} < 1. \tag{152}$$

Step 2

Claim: We have

$$|t_j - t_{j+1}| \gtrsim_\Omega \Big(\frac{1-\eta}{2(1+\eta)}\delta\Big)^3, \text{ for } v_j \in \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}, 0 \leq t_j. \tag{153}$$

Proof. For $t_j \leq 1$,

$$|\int_{t_j}^{t_{j+1}} v_j ds|^2 = |x_{j+1}-x_j|^2 \gtrsim |(x_{j+1}-x_j)\cdot n(x_j)|$$

$$= |\int_{t_j}^{t_{j+1}} v_j \cdot n(x_j) ds| = |v_j \cdot n(x_j)||t_j - t_{j+1}|.$$

Here we use the fact that if $x, y \in \partial\Omega$ and $\partial\Omega$ is C^2 and Ω is bounded then $|x-y|^2 \gtrsim_\Omega |(x-y)\cdot n(x)|$(see the proof in [8]). Thus

$$|v_j \cdot n(x_j)| \lesssim \frac{1}{|t_j - t_{j+1}|}|\int_{t_j}^{t_{j+1}} v_j ds|^2 \lesssim |t_j - t_{j+1}||v_j|^2. \tag{154}$$

Since $v_j \in \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}$, $t_j \leq 0$, let $0 \leq t \leq t'$, we have

$$|v_j \cdot n(x_j)| \lesssim |t_j - t_{j+1}|\Big(\frac{1-\eta}{2(1+\eta)}\delta\Big)^{-2}. \tag{155}$$

Then we prove (153). □

In consequence, when $t_k > 0$, by (153) and $t \ll 1$, there can be at most $\{[C_\Omega(\frac{2(1+\eta)}{(1-\eta)\delta})^3] + 1\}$ numbers of $v_j \in \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}$. Equivalently there are at least $k-2-[C_\Omega(\frac{2(1+\eta)}{(1-\eta)\delta})^3]+1$ numbers of $v_j \in \mathcal{V}_j \backslash \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}$.

Step 3

In this step we combine Step 1 and Step 2 and focus on the integration over $\prod_{j=1}^{k-1}\mathcal{V}_j$.

By (153) in Step 2, we define

$$N := \Big[C_\Omega(\frac{2(1+\eta)}{\delta(1-\eta)})^3\Big] + 1. \tag{156}$$

For the sequence $\{v_1, v_2, \cdots, v_{k-1}\}$, suppose there are p number of $v_j \in \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}$ with $p \le N$, we conclude there are at most $\begin{pmatrix} k-1 \\ p \end{pmatrix}$ number of these sequences. Below we only consider a single sequence of them.

In order to get (150),(151)< 1, we need to ensure the condition (143). Thus we take $k = k_1(T_M, \xi, r_\perp, r_\parallel)$ and only use the decomposition $\mathcal{V}_j = \Big(\mathcal{V}_j \backslash \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}\Big) \cup \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}$ for $\prod_{j=1}^{k/2}\mathcal{V}_j$. Then we only consider the half sequence $\{v_1, v_2, \cdots, v_{k/2}\}$. We derive that when $t_k > 0$, there are at most N number of $v_j \in \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}$ and at least $k/2 - 1 - N$ number of $v_j \in \mathcal{V}_j \backslash \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}$ in $\prod_{j=1}^{k/2}\mathcal{V}_j$.

In this single half sequence $\{v_1, \cdots, v_{k/2}\}$, in order to apply Lemma 2.8, we only want to consider the subsequence (122) with $l+1 < l+L \le k/2$ and $L \ge 100\frac{1+\eta}{1-\eta}$. Thus we need to ignore those subsequence with $L < 100\frac{1+\eta}{1-\eta}$. By (122), we conclude that at the end of this subsequence, it is adjacent to a $v_l \in \mathcal{V}_l^{\frac{1-\eta}{2(1+\eta)}\delta}$. By (156), we conclude

$$\textit{There are at most } N \textit{ number of subsequences } (122) \textit{ with } L \le 100\frac{1+\eta}{1-\eta}. \tag{157}$$

We ignore these subsequences. Then we define the parameters for the remaining subsequence(with $L \ge 100\frac{1+\eta}{1-\eta}$) as:

$$M_1 := \text{the number of } v_j \in \mathcal{V}_j \backslash \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta} \text{ in the first subsequence starting from } v_1,$$

$$n := \text{the number of these subsequences.}$$

Similarly we can define $M_2, M_3, \cdots, M_n$ as the number in the second, third, $\cdots$, n-th subsequence. Recall that we only consider $\prod_{j=1}^{k/2}\mathcal{V}_j$, thus we have

$$100\frac{1+\eta}{1-\eta} \le M_i \le k/2, \text{ for } 1 \le i \le n. \tag{158}$$

By (157), we obtain

$$k/2 \ge M_1 + \cdots M_n \ge k/2 - 1 - 100\frac{1+\eta}{1-\eta}N > \frac{k}{2} - 101\frac{1+\eta}{1-\eta}N. \tag{159}$$

Take M_i with $1 \le i \le n$ as an example. Suppose this subsequence starts from v_{l_i+1} to $v_{l_i+M_i}$, by (123) in Lemma 2.8 with replacing l by l_i and L by M_i, we obtain

$$\begin{aligned}&\int_{\prod_{j=l_i}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{t_k>0\}}\mathbf{1}_{\{v_{l_i+j}\in\mathcal{V}_{l_i+j}\setminus\mathcal{V}_{l_i+j}^{\frac{1-\eta}{2(1+\eta)}\delta}\text{ for }1\le j\le M_i\}}d\Phi_{l_i,m}^{k,k-1}(t_k)\\&\le(3\delta)^{M_i/2}(C_{T_M,\xi})^{2(k-l)}\mathcal{A}_{k-1,l_i}.\end{aligned}\tag{160}$$

Since (160) holds for all $1 \le i \le n$, by Lemma 2.7 we can draw the conclusion for the Step 3 as follows. For a single sequence $\{v_1, v_2, \cdots, v_{k-1}\}$, when there are p number $v_j \in \mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}$, we have

$$\begin{aligned}&\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{p\text{ number }v_j\in\mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}\text{ for a single sequence}\}}\mathbf{1}_{\{t_k>0\}}d\Sigma_{k-1,m}^k(t_k)\\&\le(3\delta)^{(M_1+\cdots+M_n)/2}(C_{T_M,\xi})^{2k}\mathcal{A}_{k-1,1}.\end{aligned}\tag{161}$$

Step 4

Now we are ready to prove the lemma. By (156), we have

$$\begin{aligned}&\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{t_k>0\}}d\Sigma_{k-1,m}^k(t_k)\\&\le\sum_{p=1}^{N}\int_{\{\text{Exactly }p\text{ number of }v_j\in\mathcal{V}_j^{\frac{1-\eta}{2(1+\eta)}\delta}\}}\mathbf{1}_{\{t_k>0\}}d\Sigma_{k-1,m}^k(t_k).\end{aligned}\tag{162}$$

Since (161) holds for a single sequence, we derive

$$\begin{aligned}(162)&\le(C_{T_M,\xi})^{2k}\sum_{p=1}^{N}\binom{k-1}{p}(3\delta)^{(M_1+M_2+\cdots M_n)/2}\mathcal{A}_{k-1,1}\\&\le(C_{T_M,\xi})^{2k}N(k-1)^N(3\delta)^{k/4-101\frac{1+\eta}{1-\eta}N}\mathcal{A}_{k-1,1},\end{aligned}\tag{163}$$

where we use (159) in the second line.

Take $k = N^3$, the coefficient in (163) is bounded by

$$(C_{T_M,\xi})^{2N^3}N^{3N+1}(3\delta)^{N^3/4-101\frac{1+\eta}{1-\eta}N}\le(C_{T_M,\xi})^{2N^3}N^{4N}(3\delta)^{N^3/5},\tag{164}$$

where we choose $N = N(\eta)$ large such that $N^3/4 - 101\frac{1+\eta}{1-\eta}N \ge N^3/5$.

Using (156), we derive

$$3\delta = C(\Omega,\eta)N^{-1/3}.$$

Finally we bound (164) by

$$\begin{aligned}&(C_{T_M,\xi})^{2N^3}N^{4N}(C(\Omega,\eta)N^{-1/3})^{N^3/5}\\&\le e^{2N^3\log(C_{T_M,\xi})}e^{4N\log N}e^{(N^3/5)\log(C(\Omega,\eta)N^{-1/3})}\\&= e^{4N\log N}e^{(N^3/5)(log(C(\Omega,\eta))-\frac{1}{3}\log N)}e^{2N^3\log(C_{T_M,\xi})}\\&= e^{4N\log N-\frac{N^3}{15}(\log N-3\log C_{\Omega,\eta}-30\log C_{T_M,\xi})}\\&\le e^{4N\log N-\frac{N^3}{30}\log N}\le e^{-\frac{N^3}{50}\log N}=e^{-\frac{k}{150}\log k}\le(\frac{1}{2})^k,\end{aligned}$$

where we choose δ to be small enough in the second line such that $N = N(\Omega, \eta, C_{T_M,\xi})$ is large enough to satisfy

$$\log N - 3\log C(\Omega,\eta) - 30\log C_{T_M,\xi} \ge \frac{\log N}{2},$$

$$4N\log N-\frac{N^3}{30}\log N\le-\frac{N^3}{50}\log N.$$

And thus we choose $k=N^3=k_2=k_2(\Omega,\eta,C_{T_M,\xi})$ and we also require $\log k>150$ in the last step. Then we get (97).

Therefore, by the condition (143), we choose $k=k_0=\max\{k_1,k_2\}$. By the definition of η (152) with (150) and (151), we obtain $\eta=\eta(T_M,\mathcal{C},r_\perp,r_\parallel,\varepsilon_2)$. Thus by (145) and (148), we conclude the k_0 we choose here does not depend on t and only depends on the parameter in (95). We derive the lemma. □

Proof of Proposition 1. First we take

$$t_\infty\le t'. \tag{165}$$

with t' defined in (96). Then we let $k=k_0$ with k_0 defined in (95) so that we can apply Lemma 2.4 and Lemma 2.3. Define the constant in (38) as

$$C_\infty=3(C_{T_M,\xi})^{k_0}. \tag{166}$$

We mainly use the formula given in Lemma 2.2. We consider two cases.

Case1: $t_1\le 0$,

By (43) and using the definition of $\Gamma^m_{\text{gain}}(s)$ in (47) we have

$$\begin{aligned}
&|h^{m+1}(t,x,v)|\\
&\le|h_0(X^1(0;t,x,v),v)| &(167)\\
&+\int_0^t e^{|v|^2(\theta-t)}\int_{\mathbb{R}^3\times\mathbb{S}^2}B(v-u,w)\sqrt{\mu(u)}\\
&\times\Big|\frac{h^m(s,X^1(s),u')}{e^{|u'|^2(\theta-s)}}\Big|\Big|\frac{h^m(s,X^1(s),v')}{e^{|v'|^2(\theta-s)}}\Big|d\omega duds, &(168)
\end{aligned}$$

where $u'=u'(u,v)$ and $v'=v'(u,v)$ are defined by (3). Then we have

$$\begin{aligned}
(168)&\le(\sup_{0\le s\le t}\|h^m(s)\|_{L^\infty})^2\times\int_0^t\int_{\mathbb{R}^3\times\mathbb{S}^2}e^{|v|^2(\theta-t)}B\big(v-u,w\big)\\
&\times\sqrt{\mu(u)}e^{(|u|^2+|v|^2)(s-\theta)}d\omega duds\\
&\lesssim(\sup_{0\le s\le t}\|h^m(s)\|_{L^\infty})^2\int_0^t\int_{\mathbb{R}^3}e^{|v|^2(s-t)}|v-u|^{\mathcal{K}}\sqrt{\mu}e^{|u|^2(s-\theta)}duds\\
&\lesssim_{C_\infty}\|h_0\|^2_{L^\infty}\int_0^t e^{|v|^2(s-t)}\langle v\rangle^{\mathcal{K}+3}ds\\
&\le\|h_0\|^2_{L^\infty}\int_0^t e^{|v|^2(s-t)}\langle v\rangle^4\{\mathbf{1}_{|v|>N}+\mathbf{1}_{|v|\le N}\}ds\\
&\lesssim_{\|h_0\|_\infty}\big(\frac{1}{N^2}+Nt\big),
\end{aligned}$$

where $-3<\mathcal{K}\le 1$. Therefore, we obtain

$$(168)\le C(C_\infty,\|h_0\|_\infty)(\frac{1}{N^2}+Nt)\le\frac{1}{k_0}\|h_0\|_\infty, \tag{169}$$

where we choose

$$N=N(C_\infty,\|h_0\|_\infty,k_0)\gg 1,\quad t_\infty=t_\infty(N,C_\infty,\|h_0\|_\infty,k_0)\ll 1, \tag{170}$$

with $t\le t_\infty$ to obtain the last inequality in (169).

Finally collecting (167) and (168) we obtain

$$\|h^{m+1}(t,x,v)\mathbf{1}_{\{t_1\le 0\}}\|_\infty \le 2\|h_0\|_\infty \le C_\infty \|h_0\|_\infty, \tag{171}$$

where C_∞ is defined in (166).

Case2: $t_1 \ge 0$,

We consider (44) in Lemma 2.2. First we focus on the first line. By (169) we obtain

$$\int_{t_1}^{t} e^{|v|^2(\theta-t)}\Gamma^m_{\text{gain}}(s)ds \le \frac{1}{k_0}\|h_0\|_\infty. \tag{172}$$

Then we focus on the second line of (44). Using $\theta = \frac{1}{4T_M\xi}$ we bound the second line of (44) by

$$\exp\left(\Big[\frac{1}{2T_M\frac{2\xi}{\xi+1}} - \frac{1}{2T_w(x_1)}\Big]|v|^2\right)\int_{\prod_{j=1}^{k_0-1}\mathcal{V}_j} H. \tag{173}$$

Now we focus on $\int_{\prod_{j=1}^{k_0-1}\mathcal{V}_j} H$. We compute H term by term with the formula given in (45). First we compute the first line of (45). By Lemma 2.3 with $p=1$, for every $1\le l\le k_0-1$, we have

$$\begin{aligned}
&\int_{\prod_{j=1}^{k_0-1}\mathcal{V}_j} \mathbf{1}_{\{t_{l+1}\le 0<t_l\}}|h_0\big(X^{m-l}(0),V^{m-l}(0)\big)|d\Sigma^{k_0}_{l,m}(0)\\
&\le \|h_0\|_\infty \int_{\prod_{j=1}^{k_0-1}\mathcal{V}_j} \mathbf{1}_{\{t_{l+1}\le 0<t_l\}}d\Sigma^{k_0}_{l,m}(0)\\
&\le (C_{T_M,\xi})^l\|h_0\|_\infty\\
&\times \exp\left(\frac{(T_{l,1}-T_w(x_1))(1-r_{min})}{2T_w(x_1)[T_{l,1}(1-r_{min})+r_{min}T_w(x_1)]}|v|^2 + \mathcal{C}^l t|v|^2\right).
\end{aligned} \tag{174}$$

In regard to (173) we have

$$\exp\left(\Big[\frac{1}{2T_M\frac{2\xi}{\xi+1}} - \frac{1}{2T_w(x_1)}\Big]|v|^2\right)\times\ (174) =$$

$$(C_{T_M,\xi})^l\|h_0\|_\infty \exp\left(\Big[\frac{-1}{2\big(T_w(x_1)r_{min}+T_{l,1}(1-r_{min})\big)} + \frac{1}{2T_M\frac{2\xi}{\xi+1}}\Big]|v|^2 + (\mathcal{C})^l t|v|^2\right).$$

Using the definition (64) we have $T_w(x_1) < \frac{2\xi}{\xi+1}T_M$ and $T_{l,1} < \frac{2\xi}{\xi+1}T_M$. Then we take

$$t_\infty = t_\infty(T_M,k_0,\xi,\mathcal{C}) \tag{175}$$

to be small enough and $t\le t_\infty$ so that the coefficient for $|v|^2$ is

$$\begin{aligned}
&\frac{-1}{2\big(T_w(x_1)r_{min}+T_{l,1}(1-r_{min})\big)} + \frac{1}{2T_M\frac{2\xi}{\xi+1}} + (\mathcal{C})^l t\\
&\le \frac{-1}{2\big(T_M r_{min}+T_{l,1}(1-r_{min})\big)} + \frac{1}{2T_M\frac{2\xi}{\xi+1}} + (\mathcal{C})^{k_0} t \le 0.
\end{aligned} \tag{176}$$

Since (174) holds for all $1\le l\le k_0-1$, by (176) the contribution of the first line of (45) in (173) is bounded by

$$(C_{T_M,\xi})^{k_0}\|h_0\|_\infty. \tag{177}$$

Then we compute the second line of (45). For each $1 \le l \le k_0 - 1$ such that $\max\{0, t_{l+1}\} \le s \le t_l$, by (46), we have

$$d\Sigma^{k_0}_{l,m}(s) = e^{-|v_l|^2(t_l - s)} d\Sigma^{k_0}_{l,m}(t_l).$$

Therefore, we derive

$$\begin{aligned}
&\int_{\max\{0,t_{l+1}\}}^{t_l} \int_{\prod_{j=1}^{k_0-1}\mathcal{V}_j} e^{|v_l|^2(\theta - s)} |\Gamma^{m-l}_{\text{gain}}(s)| d\Sigma^{k_0}_{l,m}(s) ds \\
&\le \int_{\prod_{j=1}^{k_0-1}\mathcal{V}_j} \int_{\max\{0,t_{l+1}\}}^{t_l} e^{|v_l|^2(\theta - t_l)} |\Gamma^{m-l}_{\text{gain}}(s)| ds d\Sigma^{k_0}_{l,m}(t_l) \\
&\le \frac{1}{k_0} \|h_0\|_\infty \int_{\prod_{j=1}^{k_0-1}\mathcal{V}_j} \Sigma^{k_0}_{l,m}(t_l) \\
&\le \frac{1}{k_0} \|h_0\|_\infty (C_{T_M,\xi})^l \exp\left(\frac{(T_{l,1} - T_w(x_1))(1 - r_{min})}{2T_w(x_1)[T_{l,1}(1 - r_{min}) + r_{min} T_w(x_1)]} |v|^2 + (\mathcal{C})^l t |v|^2 \right),
\end{aligned} \tag{178}$$

where we apply (169) in the third line and we apply Lemma 2.3 in the last line.

In regard to (173), by (176) we obtain

$$\exp\left(\Big[\frac{1}{2T_M \frac{2\xi}{\xi+1}} - \frac{1}{2T_w(x_1)} \Big] |v|^2 \right) \times \ (178) \le \frac{1}{k_0} (C_{T_M,\xi})^l \|h_0\|_\infty.$$

Since (178) holds for all $1 \le l \le k_0 - 1$, the contribution of the second line of (45) in (173) is bounded by

$$\frac{k_0 - 1}{k_0} (C_{T_M,\xi})^{k_0} \|h_0\|_\infty. \tag{179}$$

Last we compute the third term of (45). By Lemma 2.4 and the assumption (38) we obtain

$$\begin{aligned}
&\int_{\prod_{j=1}^{k_0-1}\mathcal{V}_j} \mathbf{1}_{\{0<t_{k_0}\}} |h^{m-k_0+2}\big(t_{k_0}, x_{k_0}, V^{m-k_0+1}(t_{k_0})\big)| d\Sigma^{k_0}_{k_0-1,m}(t_{k_0}) \\
&\le \|h^{m-k_0+2}\|_\infty \int_{\prod_{j=1}^{k_0-1}\mathcal{V}_j} \mathbf{1}_{\{0<t_{k_0}\}} d\Sigma^{k_0}_{k_0-1,m}(t_{k_0}) \\
&\le 3 (C_{T_M,\xi})^{k_0} (\frac{1}{2})^{k_0} \|h_0\|_\infty \\
&\times \exp\left(\frac{(T_{l,1} - T_w(x_1))(1 - r_{min})}{2T_w(x_1)[T_{l,1}(1 - r_{min}) + r_{min} T_w(x_1)]} |v|^2 + (\mathcal{C})^l t |v|^2 \right).
\end{aligned} \tag{180}$$

In regard to (173), by (176) we have

$$\exp\left(\Big[\frac{1}{2T_M \frac{2\xi}{\xi+1}} - \frac{1}{2T_w(x_1)} \Big] |v|^2 \right) \times \ (180) \le (C_{T_M,\xi})^{k_0} \|h_0\|_\infty.$$

Thus the contribution of the third line of (45) in (173) is bounded by

$$(C_{T_M,\xi})^{k_0} \|h_0(x,v)\|_\infty. \tag{181}$$

Collecting (177) (179) (181) we conclude that the second line of (44) is bounded by

$$(C_{T_M,\xi})^{k_0} \times (2 + \frac{k_0 - 1}{k_0}) \|h_0\|_\infty. \tag{182}$$

Adding (182) to (172) we use (44) to derive

$$\|h^{m+1}(t,x,v)\mathbf{1}_{\{t_1\geq 0\}}\|_\infty \leq 3(C_{T_M,\xi})^{k_0}\|h_0\|_\infty = C_\infty\|h_0\|_\infty. \tag{183}$$

Combining (171) and (183) we derive (39).

Last we focus the parameters for t_∞ in (40). In the proof the constraints for t_∞ are (165), (170) and (175). We obtain

$$\begin{aligned} t_\infty &= t_\infty(t', N, C_\infty, \|h_0\|_\infty, T_M, k_0, \xi, \mathcal{C}) \\ &= t_\infty(k_0, \xi, T_M, \min\{T_w(x)\}, \mathcal{C}, r_\perp, r_\parallel, C_{T_M,\xi}, \|h_0\|_\infty). \end{aligned}$$

By the definition of k_0 in (95), definition of $C_{T_M,\xi}$ in (80), definition of $\mathcal{C}$ in (83), we derive (40). □

Then we can conclude the well-posedness.

Proof of Theorem 1.1. First of all we take $t < t_\infty$, where t_∞ is defined in (40) so that we can apply Proposition 1. We have

$$\sup_m \|h^m\|_\infty \lesssim \|h(0)\|_\infty.$$

- Existence

For h^m given in (33), we take the difference $h^{m+1} - h^m$ and deduce that

$$\partial_t[h^{m+1}-h^m] + v\cdot\nabla_x[h^{m+1}-h^m] + \nu^m(h^{m+1}-h^m) = e^{(\theta-t)|v|^2}\Lambda^m,$$

$$\begin{aligned}[h^{m+1}-h^m]_- = {} & e^{(\theta-t)|v|^2}e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|v|^2} \\ & \int_{n(x)\cdot u>0}[h^{m+1}(u)-h^m(u)]e^{-[\frac{1}{4T_M}-\frac{1}{2T_w(x)}]|u|^2}e^{-(\theta-t)|u|^2}d\sigma(u,v),\end{aligned}$$

where

$$\begin{aligned}\Lambda^m = {} & \Gamma_{\text{gain}}\Big(\frac{h^m-h^{m-1}}{e^{(\theta-t)|v|^2}}, \frac{h^m}{e^{(\theta-t)|v|^2}}\Big) \\ & + \Gamma_{\text{gain}}\Big(\frac{h^{m-1}}{e^{(\theta-t)|v|^2}}, \frac{h^m-h^{m-1}}{e^{(\theta-t)|v|^2}}\Big) + [\nu(F^{m-1})-\nu(F^m)]h^{m-1}.\end{aligned}$$

By the same derivation as (43) (44), when $t_1 \leq 0$, we have

$$\begin{aligned} & |h^{m+1}-h^m|(t,x,v) \\ & \leq \int_0^t e^{|v|^2(\theta-t)}\int_{\mathbb{R}^3\times\mathbb{S}^2} B(v-u,w)\sqrt{\mu} \\ & \times \Big[\Big|\frac{(h^m-h^{m-1})(s,X^1(s),u')}{e^{|u'|(\theta-s)}}\Big|\Big|\frac{h^m(s,X^1(s),v')}{e^{|v'|(\theta-s)}}\Big| \\ & + \Big|\frac{h^m(s,X^1(s),u')}{e^{|u'|(\theta-s)}}\Big|\Big|\frac{(h^m-h^{m-1})(s,X^1(s),v')}{e^{|v'|(\theta-s)}}\Big| \\ & + \Big|\frac{(h^m-h^{m-1})(s,X^1(s),u)}{e^{|u|^2(\theta-s)}}\Big|\Big|\frac{h^{m-1}(s,X^1(s),v)}{e^{|v|^2(\theta-s)}}\Big|\Big]d\omega du ds, \end{aligned}$$

where we use $h^{m+1}(0) = h^m(0)$.

Then we follow the computation for (168) to obtain

$$\begin{aligned}
&|h^{m+1}-h^m|(t,x,v)\\
&\lesssim (\|h^m-h^{m-1}\|_\infty)\|h^m\|_\infty \times \int_0^t \int_{\mathbb{R}^3\times\mathbb{S}^2} e^{|v|^2(\theta-t)}B(v-u,\omega)\\
&\times \sqrt{\mu(u)}e^{(|u|^2+|v|^2)(s-\theta)}d\omega du ds\\
&\lesssim \|h^m-h^{m-1}\|_\infty\|h^m\|_\infty(\frac{1}{N^2}+Nt)\\
&\lesssim o(1)\|h^m-h^{m-1}\|_\infty, \qquad (184)
\end{aligned}$$

where we take $N=N(\|h^m\|_\infty)$ to be large and $t<t_\infty=t_\infty(N)$ to be small as in (170).

When $t_1>0$, by the same derivation as (44), we have

$$|h^{m+1}-h^m|(t,x,v)\le \int_{t_1}^t e^{|v|^2(\theta-t)}\Lambda^m ds + e^{|v|^2(\theta-t_1)}e^{[\frac{1}{4T_M}-\frac{1}{2T_w(x_1)}]|v|^2}\int_{\prod_{j=1}^{k-1}\nu_j} H_d,$$

where H_d is bounded by

$$\begin{aligned}
&\sum_{l=1}^{k-1}\int_{\max\{0,t_{l+1}\}}^{t_l} e^{|v_l|^2(\theta-s)}|\Lambda^m(s)|d\Sigma^k_{l,m}(s)ds\\
&+\mathbf{1}_{\{t_k>0\}}|h^{m-k+2}-h^{m-k+1}|(t_k,x_k,v_{k-1})d\Sigma^k_{k-1,m}(t_k). \qquad (185)
\end{aligned}$$

By (178) and (184), the first line of (185) is bounded by

$$k_0O(t)\sup_{\ell\le m}\|h^\ell-h^{\ell-1}\|_\infty = o(1)\sup_{\ell\le m}\|h^\ell-h^{\ell-1}\|_\infty,$$

where we take $t<t_\infty=t_\infty(k_0)$ to be small.

Then we apply (180) (181) with replacing $\|h^{m-k_0+2}\|_\infty$ by $\|h^{m-k_0+2}-h^{m-k_0+1}\|_\infty$. Thus we obtain the second line of (185) is bounded by

$$\Big(\frac{1}{2}\Big)^{k_0}\sup_{\ell\le m}\|h^\ell-h^{\ell-1}\|_\infty.$$

Thus in the case $t_1>0$ we obtain

$$\|h^{m+1}-h^m\|_\infty \le o(1)\sup_{\ell\le m}\|h^\ell-h^{\ell-1}\|_\infty. \qquad (186)$$

Therefore, h^m is a Cauchy-sequence in L^∞. The existence follows by taking the limit $m\to\infty$ and the solution $h=e^{(\theta-t)|v|^2}f$ satisfies

$$\partial_t h + v\cdot\nabla_x h + |v|^2 h = e^{(\theta-t)|v|^2}\Gamma\left(\frac{h}{e^{(\theta-t)|v|^2}},\frac{h}{e^{(\theta-t)|v|^2}}\right). \qquad (187)$$

Moreover, we have

$$\|h\|_\infty \le \sup_m\|h^m\|_\infty \lesssim \|h(0)\|_\infty. \qquad (188)$$

This concludes the existence of f and (24).

- Stability

Suppose there are two solutions h_1 and h_2 satisfy (187). Also suppose there initial condition satisfy

$$\|h_1(0)\|_\infty, \|h_2\|_\infty<\infty.$$

When $t_1 \le 0$, by the same derivation as (169) and (184) we have

$$
\begin{aligned}
&|h_1 - h_2|(t,x,v) \\
&\lesssim |h_1 - h_2|(0) + (\|h_1\|_\infty + \|h_2\|_\infty)\int_0^t \|h_1 - h_2\|_\infty e^{|v|^2(s-t)}\langle v\rangle^4\{\mathbf{1}_{|v|>N} + \mathbf{1}_{|v|\le N}\}ds \\
&\lesssim \|(h_1 - h_2)(0)\|_\infty + (\|h_2\|_\infty + \|h_1\|_\infty)\Big[O(\frac{1}{N^2})\|h_1 - h_2\|_\infty + \int_0^t N\|h_1 - h_2\|_\infty ds\Big].
\end{aligned}
$$

By taking $N = N(\|h_1\|_\infty, \|h_2\|_\infty)$ to be large as in (170) so that

$$
(\|h_2\|_\infty + \|h_1\|_\infty)O(\frac{1}{N^2}) \ll 1,
$$

we derive the L^∞ stability by the Gronwall's inequality.

When $t_1 > 0$, the argument is exactly the same as the existence part and we conclude the L^∞ stability for all cases. The uniqueness follows immediately by setting $h_1(0) = h_2(0)$.

The positivity follows from the the property that iteration equation (32) is positive preserving and (186). □

3. **Steady problem with C-L boundary condition.** This section is devoted to the steady solution to the Boltzmann equation with the Cercignani-Lampis boundary condition as mention in Section 1.2.

Remark 11. The setting of the steady solution is given in Section 1.2. We remark here that in this section we no longer use notation μ. Instead we put the subscript μ_0, δ_0 only for this section in order to avoid confusion.

To prove Corollary 1 we need the following Proposition.

Proposition 2 (Proposition 4.1 of [8])**.** *Define a weight function scaled with parameter ϱ as*

$$
w_\varrho(v) = w_{\varrho,\beta,\zeta}(v) \equiv (1 + \varrho^2|v|^2)^{\frac{\beta}{2}} e^{\zeta|v|^2}. \tag{189}
$$

Assume

$$
\iint_{\Omega\times\mathbb{R}^3} g(x,v)\sqrt{\mu_0}dxdv = 0, \qquad \int_{\gamma_-} r\sqrt{\mu_0}d\gamma = 0 \tag{190}
$$

and $\beta > 4$. Then the solution f to the linear Boltzmann equation

$$
v\cdot\nabla_x f + Lf = g, \qquad f_- = P_\gamma f + r \tag{191}
$$

satisfies $\|w_\varrho f\|_\infty + |w_\varrho f|_\infty \lesssim \|w_\varrho g\|_\infty + |w_\varrho\langle v\rangle r|_\infty$.

For the purpose of applying Proposition 2, we focus on the boundary condition for the linearized equation f_s.

Lemma 3.1. *For $F_s = \mu_0 + \sqrt{\mu_0}f_s$ with F_s satisfying the boundary condition (5), (6), the boundary condition for f_s can be represented as*

$$
f_s|_-(x,v) = P_\gamma f_s + r \tag{192}
$$

such that

$$
\int_{\gamma_-} r\sqrt{\mu_0} = 0. \tag{193}
$$

Moreover,

$$
|r|_\infty \lesssim \delta_0 + \sup_{0\le s\le t}\delta_0|f(s)|_\infty. \tag{194}
$$

Before proving this lemma we need the following lemma for the C-L boundary condition.

Lemma 3.2. *In regard to the boundary condition (6), we have*

$$\frac{1}{|n(x)\cdot v|}\int_{n(x)\cdot u>0} R(u\to v;x,t)\mu_0\{n(x)\cdot u\}du=\mu_{x,r_\parallel,r_\perp}, \tag{195}$$

where

$$\begin{aligned}\mu_{x,r_\parallel,r_\perp}&=\frac{1}{2\pi[T_0(1-r_\parallel)^2+T_w(x)r_\parallel(2-r_\parallel)]}e^{-\frac{|v_\parallel|^2}{2[T_0(1-r_\parallel)^2+T_w(x)r_\parallel(2-r_\parallel)]}}\cdot\\ &\times\frac{1}{T_0(1-r_\perp)+T_w(x)r_\perp}e^{-\frac{|v_\perp|^2}{2[T_0(1-r_\perp)+T_w(x)r_\perp]}}.\end{aligned} \tag{196}$$

Moreover, for any $x\in\partial\Omega$ and $r_\parallel,r_\perp$, we have

$$\int_{n(x)\cdot v>0}\mu_{x,r_\parallel,r_\perp}\{n(x)\cdot v\}dv=1. \tag{197}$$

Proof. Using the definition of $R(u\to v;x,t)$ in (6) we can write the LHS of (195) as

$$\begin{aligned}&\int_{\mathbb{R}^+}\frac{|u_\perp|}{r_\perp T_w(x)}\exp\Big(-\frac{1}{2T_w(x)}\Big[\frac{|v_\perp|^2+(1-r_\perp)|u_\perp|^2}{r_\perp}\Big]\Big)\\ &\times I_0\Big(\frac{(1-r_\perp)^{1/2}v_\perp u_\perp}{T_w(x)r_\perp}\Big)\frac{1}{T_0}\exp\Big(-\frac{|u_\perp|^2}{2T_0}\Big)dv_\perp\\ &\times\int_{\mathbb{R}^2}\frac{1}{2T_w(x)r_\parallel(2-r_\parallel)\pi}\\ &\times\exp\Big(-\frac{1}{2T_w(x)}\frac{|v_\parallel-(1-r_\parallel)u_\parallel|^2}{r_\parallel(2-r_\parallel)}\Big)\frac{1}{2\pi T_0}\exp\Big(-\frac{|u_\parallel|^2}{2T_0}\Big)dv_\parallel.\end{aligned} \tag{198}$$

First we compute the third, fourth line of (198), in order to apply Lemma 4.2, we set

$$a=-\frac{1}{2T_0},\quad b=\frac{(1-r_\parallel)^2}{2T_w(x)r_\parallel(2-r_\parallel)},\quad v=u_\parallel,\quad w=\frac{1}{1-r_\parallel}v_\parallel,\quad \varepsilon=0,$$

$$b-a=\frac{(1-r_\parallel)^2}{2T_w(x)r_\parallel(2-r_\parallel)}+\frac{1}{2T_0}.$$

Then the third, fourth line of (198) equals to

$$\begin{aligned}&\frac{1}{(1-r_\parallel)^2}\frac{b}{b-a}\exp\Big(\frac{ab}{b-a}|\frac{v_\parallel}{1-r_\parallel}|^2\Big)\\ &=\frac{1}{2\pi}\frac{1}{T_0(1-r_\parallel)^2+T_w(x)r_\parallel(2-r_\parallel)}\exp\Big(-\frac{|v_\parallel|^2}{2\big[T_0(1-r_\parallel)^2+T_w(x)r_\parallel(2-r_\parallel)\big]}\Big).\end{aligned}$$

Then we compute the first, second line of (198), in order to apply Lemma 4.3, we set

$$a=-\frac{1}{2T_0},\quad b=\frac{1-r_\perp}{2T_w(x)r_\perp},\quad v=u_\perp,\quad w=\frac{1}{\sqrt{1-r_\perp}}v_\perp,\quad \varepsilon=0,$$

$$b-a=\frac{1-r_\perp}{2T_w(x)r_\perp}+\frac{1}{2T_0}.$$

Then the first, second line of (198) is equal to

$$\frac{1}{1-r_\perp}\frac{b}{b-a}e^{\frac{ab}{b-a}|\frac{v_\perp}{\sqrt{1-r_\perp}}|^2}$$
$$=\frac{1}{2\pi[T_0(1-r_\perp)+T_w(x)r_\perp]}\exp\Big(\frac{|v_\perp|^2}{2\pi[T_0(1-r_\perp)+T_w(x)r_\perp]}\Big).$$

Thus we conclude (195).

Then we focus on (197). The LHS of (197) can be written as

$$\begin{aligned}&\int_{\mathbb{R}_+}\frac{v_\perp}{T_0(1-r_\perp)+T_w(x)r_\perp}e^{-\frac{|v_\perp|^2}{2[T_0(1-r_\perp)+T_w(x)r_\perp]}}dv_\perp\\&\times\int_{\mathbb{R}^2}\frac{1}{2\pi[T_0(1-r_\parallel)^2+T_w(x)r_\parallel(2-r_\parallel)]}e^{-\frac{|v_\parallel|^2}{2[T_0(1-r_\parallel)^2+T_w(x)r_\parallel(2-r_\parallel)]}}dv_\parallel.\end{aligned}\tag{199}$$

Clearly (199) $=1$. □

Proof of Lemma 3.1. By plugging the linearization $F_s=\mu_0+\sqrt{\mu_0}f_s$ into the boundary condition (5) and using Lemma 3.2 we obtain

$$\mu_0+\sqrt{\mu_0}f_s=\mu_{x,r_\parallel,r_\perp}+\frac{1}{|n(x)\cdot v|}\int_{n(x)\cdot u>0}R(u\to v;x,t)\sqrt{\mu_0(u)}f_s(u)\{n(x)\cdot u\}du.$$

Thus

$$f_s(v)=\underbrace{\frac{\mu_{x,r_\parallel,r_\perp}-\mu_0}{\sqrt{\mu_0}}}_{r_1}+\underbrace{\frac{1}{\sqrt{\mu_0}}\frac{1}{|n(x)\cdot v|}\int_{n(x)\cdot u>0}R(u\to v;x,t)\sqrt{\mu_0(u)}f_s(u)\{n(x)\cdot u\}du}_{r_2(f_s)}.$$

We can rewrite the boundary condition into

$$f_s(v)=r_1+r_2(f_s)-P_\gamma f_s+P_\gamma f_s.\tag{200}$$

Clearly by (197) in Lemma 3.2 we have

$$\int_{\gamma_-}r_1\sqrt{\mu_0}=0.\tag{201}$$

To prove the Lemma we just need to focus on $r_2(f_s)-P_\gamma f_s$. By Tonelli theorem, we have

$$\begin{aligned}&\int_{\gamma_-}(r_2(f_s)-P_\gamma f_s)\sqrt{\mu_0}\\&=\int_{n(x)\cdot v<0}\Big[R(u\to v;x,t)-|n(x)\cdot v|\mu_0(v)\Big]dv\int_{n(x)\cdot u>0}\sqrt{\mu_0(u)}f_s(u)\{n(x)\cdot u\}du\\&=[1-1]\times\int_{n(x)\cdot u>0}\sqrt{\mu_0(u)}f_s(u)\{n(x)\cdot u\}du=0.\end{aligned}$$

Thus we prove (193).

Then we focus on (194). By the assumption in (29) and $\zeta<\frac{1}{\theta(4+2\delta_0)}$, for $x\in\partial\Omega$ we have

$$|w_\rho(v)r|_\infty=|w_\rho(v)\frac{\mu_{x,r_\parallel,r_\perp}-\mu_0}{\sqrt{\mu_0}}|_\infty\lesssim\delta_0.$$

Then

$$
\begin{aligned}
&|w_\rho(v)\big[r_2(f_s)-P_\gamma f_s\big]|\\
&\le |f|_\infty w_\rho(v)\frac{1}{\sqrt{\mu_0}}\int_{n(x)\cdot u>0}\Big[\frac{R(u\to v;x,t)}{|n(x)\cdot v|}-\mu_0(v)\Big]\sqrt{\mu_0(u)}f_s(u)\{n(x)\cdot u\}du\\
&\le |f|_\infty\Big|w_\rho(v)\frac{\mu_{x,r_\parallel,r_\perp}-\mu_0}{\sqrt{\mu_0}}\Big|_\infty\lesssim \delta_0|f|_\infty,
\end{aligned}
$$

where we apply Lemma 3.2 in the last line. Then we conclude the Lemma. □

Proof of Corollary 1. We consider the following iterative sequence

$$v\cdot\nabla_x f^{\ell+1}+Lf^{\ell+1}=\Gamma(f^\ell,f^\ell),\tag{202}$$

with the boundary condition given in the form (200)

$$f_-^{\ell+1}=P_\gamma f^{\ell+1}+r_1+r_2(f^\ell)-P_\gamma f^\ell.$$

We set $f^0=0$. By Lemma 3.1 we have

$$\int_{\gamma_-}\sqrt{\mu_0}\left\{r_1+r_2(f^\ell)-P_\gamma f^\ell\right\}d\gamma=0.$$

Since $\int\Gamma(f^\ell,f^\ell)\sqrt{\mu_0}=0$, we apply Proposition 2 with (194) in Lemma 3.1 to get

$$\|w_\varrho f^{\ell+1}\|_\infty+|w_\varrho f^{\ell+1}|_\infty\lesssim\left\|\frac{w_\varrho\Gamma(f^\ell,f^\ell)}{\langle v\rangle}\right\|_\infty+\delta_0|w_\varrho f^\ell|_{\infty,+}+\delta_0.$$

Since $\left\|\frac{w_\varrho\Gamma(f^\ell,f^\ell)}{\langle v\rangle}\right\|_\infty\lesssim\|w_\varrho f^\ell\|_\infty^2$, we deduce

$$\|w_\varrho f^{\ell+1}\|_\infty+|w_\varrho f^{\ell+1}|_\infty\lesssim\|w_\varrho f^\ell\|_\infty^2+\delta_0|w_\varrho f^\ell|_{\infty,+}+\delta_0,$$

so that for δ_0 small, $\|w_\varrho f^{\ell+1}\|_\infty+|w_\varrho f^{\ell+1}|_\infty\lesssim\delta_0$. Upon taking differences, we have

$$
\begin{aligned}
&[f^{\ell+1}-f^\ell]+v\cdot\nabla_x[f^{\ell+1}-f^\ell]+L[f^{\ell+1}-f^\ell]\\
&=\Gamma(f^\ell-f^{\ell-1},f^\ell)+\Gamma(f^{\ell-1},f^\ell-f^{\ell-1}),
\end{aligned}
$$

$$f_-^{\ell+1}-f_-^\ell=P_\gamma[f^{\ell+1}-f^\ell]+r_2(f^\ell)-P_\gamma f^\ell+P_\gamma f^{\ell-1}-r_2(f^{\ell-1}).$$

And by Proposition 2 again for $f^{\ell+1}-f^\ell$,

$$\|w_\varrho[f^{\ell+1}-f^\ell]\|_\infty+|w_\varrho[f^{\ell+1}-f^\ell]|_\infty\lesssim\delta_0\big\{\|w_\varrho[f^\ell-f^{\ell-1}]\|_\infty+|w_\varrho[f^\ell-f^{\ell-1}]|_\infty\big\}.$$

Hence f^ℓ is Cauchy in L^∞ and we construct our solution by taking the limit $f^\ell\to f_s$. Uniqueness follows in the standard way. □

Then we focus on the dynamical stability, which is the Corollary 2. We need this Proposition.

Proposition 3 (Proposition 7.1 from [8]). *Let $\|w_\rho f_0\|_\infty+|\langle v\rangle w_\rho r|_\infty+\|w_\rho g\|_\infty<+\infty$ and $\iint\sqrt{\mu_0}g=\int_\gamma r\sqrt{\mu_0}=\iint f_0\sqrt{\mu_0}=0$. Then the solution f*

$$\partial_t f+v\cdot\nabla_x f+Lf=g,\quad f(0)=f_0,\qquad \text{in }\Omega\times\mathbb{R}^3\times\mathbb{R}_+\tag{203}$$

satisfies

$$\|w_\rho f(t)\|_\infty+|w_\rho f(t)|_\infty\le e^{-\lambda t}\big\{\|w_\rho f_0\|_\infty+\sup e^{\lambda s}\|w_\rho g\|_\infty+\int_0^t e^{\lambda s}|\langle v\rangle w_\rho r(s)|_\infty ds\big\}.$$

Proof of Corollary 2. With the stationary solution for (26) given in Corollary 1, we set the solution to (1) as

$$F = F_s + \sqrt{\mu_0} f, \quad F_s = \sqrt{\mu_0} f + f_s.$$

Then the equation for f reads

$$\partial_t f + v \cdot \nabla_x f + Lf = L_{\sqrt{\mu_0} f_s} f + \Gamma(f, f),$$

where

$$L_{\sqrt{\mu_0} f_s} f = [Q(\sqrt{\mu_0} f_s, \sqrt{\mu_0} f) + Q(\sqrt{\mu_0} f, \sqrt{\mu_0} f_s)]/\sqrt{\mu_0}.$$

We consider the following iteration sequence

$$\partial_t f^{\ell+1} + v \cdot \nabla_x f^{\ell+1} + L f^{\ell+1} = L_{\sqrt{\mu_0} f_s} f^{\ell} + \Gamma(f^{\ell}, f^{\ell}),$$

with

$$f_-^{\ell+1} = P_\gamma f^{\ell+1} + r_1 + r_2(f^{\ell}) - P_\gamma f^{\ell}.$$

Clearly $\iint \{L_{\sqrt{\mu_0} f_s} f^{\ell} + \Gamma(f^{\ell}, f^{\ell})\} \sqrt{\mu_0} = 0$. Recall $w_\varrho(v) = (1 + \varrho^2 |v|^2)^{\frac{\beta}{2}} e^{\zeta |v|^2}$ in (189). Note that for $0 \le \zeta < \frac{1}{4}$,

$$\begin{aligned}
&\left\| e^{\frac{\lambda s}{2}} w_\varrho \left\{ \frac{1}{\langle v \rangle} [L_{\sqrt{\mu_0} f_s} f^{\ell} + \Gamma(f^{\ell}, f^{\ell})(s) \right\} \right\|_\infty \\
&\lesssim \delta_0 \sup_{0 \le s \le t} \| e^{\frac{\lambda s}{2}} w_\varrho f^{\ell}(s) \|_\infty + \left\{ \sup_{0 \le s \le t} \| e^{\frac{\lambda s}{2}} w_\varrho f^{\ell}(s) \|_\infty \right\}^2.
\end{aligned}$$

By Proposition 3 and Lemma 3.1, we deduce

$$\begin{aligned}
&\sup_{0 \le s \le t} \| e^{\frac{\lambda s}{2}} w_\varrho f^{\ell+1}(s) \|_\infty + \sup_{0 \le s \le t} |e^{\frac{\lambda s}{2}} w_\varrho f^{\ell+1}(s)|_\infty \\
&\lesssim \| w_\varrho f_0 \|_\infty + \delta_0 \sup_{0 \le s \le t} \| e^{\frac{\lambda s}{2}} w_\varrho f^{\ell}(s) \|_\infty \\
&\quad + \delta_0 \sup_{0 \le s \le t} |e^{\frac{\lambda s}{2}} w_\varrho f^{\ell}(s)|_\infty + \left\{ \sup_{0 \le s \le t} \| e^{\frac{\lambda s}{2}} w_\varrho f^{\ell}(s) \|_\infty \right\}^2.
\end{aligned}$$

For δ_0 small, there exists a ε_0 (uniform in δ_0) such that, if the initial data satisfy (30), then

$$\sup_{0 \le s \le t} \| e^{\frac{\lambda s}{2}} w_\varrho f^{\ell+1}(s) \|_\infty + \sup_{0 \le s \le t} |e^{\frac{\lambda s}{2}} w_\varrho f^{\ell+1}(s)|_\infty \lesssim \| w_\varrho f_0 \|_\infty.$$

By taking difference $f^{\ell+1} - f^{\ell}$, we deduce that

$$\begin{aligned}
&\partial_t [f^{\ell+1} - f^{\ell}] + v \cdot \nabla_x [f^{\ell+1} - f^{\ell}] + L[f^{\ell+1} - f^{\ell}] \\
&\quad = L_{\sqrt{\mu_0} f_s} [f^{\ell} - f^{\ell-1}] + \Gamma(f^{\ell} - f^{\ell-1}, f^{\ell}) + \Gamma(f^{\ell-1}, f^{\ell} - f^{\ell-1}), \\
&[f^{\ell+1} - f^{\ell}]_- = P_\gamma [f^{\ell+1} - f^{\ell}] + \frac{\mu_{x, r_\parallel, r_\perp} - \mu_0}{\sqrt{\mu_0}} \int_{\gamma_+} [f^{\ell} - f^{\ell-1}](n(x) \cdot v) dv,
\end{aligned}$$

with $f^{\ell+1} - f^{\ell} = 0$ initially. Repeating the same argument, we obtain

$$\begin{aligned}
&\sup_{0 \le s \le t} \| e^{\frac{\lambda s}{2}} w_\varrho [f^{\ell+1} - f^{\ell}](s) \|_\infty + \sup_{0 \le s \le t} |e^{\frac{\lambda s}{2}} w_\varrho [f^{\ell+1} - f^{\ell}](s)|_\infty \\
&\lesssim [\delta_0 + \sup_{0 \le s \le t} \| e^{\frac{\lambda s}{2}} w_\varrho f^{\ell}(s) \|_\infty \\
&\quad + \sup_{0 \le s \le t} \| e^{\frac{\lambda s}{2}} w_\varrho f^{\ell-1}(s) \|_\infty] \sup_{0 \le s \le t} \| e^{\frac{\lambda s}{2}} w_\varrho [f^{\ell} - f^{\ell-1}](s) \|_\infty.
\end{aligned}$$

This implies that $f^{\ell+1}$ is a Cauchy sequence. The uniqueness is standard.

To conclude the positivity, we use another sequence in [8],

$$\partial_t F^{\ell+1} + v\cdot\nabla_x F^{\ell+1} + \nu(F^\ell)F^{\ell+1} = Q_{\rm gain}(F^\ell, F^\ell).$$

We pose $F^\ell = F_s + \sqrt{\mu_0} f^\ell$, then the equation for f^ℓ reads

$$\begin{aligned}
&\partial_t f^{\ell+1} + v\cdot\nabla_x f^{\ell+1} + \nu(v) f^{\ell+1} - Kf^\ell \\
&= \Gamma_{\rm gain}(f^\ell, f^\ell) - \nu(\sqrt{\mu_0} f^\ell) f^{\ell+1} - \nu(\sqrt{\mu_0} f_s) f^{\ell+1} - \nu(\sqrt{\mu_0} f^\ell) f_s \\
&+ \frac{1}{\sqrt{\mu_0}}\Big\{ Q_{\rm gain}(\sqrt{\mu_0} f^\ell, \sqrt{\mu} f_s) + Q_{\rm gain}(\sqrt{\mu_0} f_s, \sqrt{\mu_0} f^\ell)\Big\}.
\end{aligned}$$

It is shown in [8] that f^ℓ is a Cauchy sequence. Thus by the uniqueness of the solution we conclude the positivity of F and F_s by positive preserving property of this sequence solution. □

4. **Appendix.**

Lemma 4.1. *For $R(u\to v; x, t)$ given by (6) and any u such that $u\cdot n(x) > 0$, we have*

$$\int_{n(x)\cdot v<0} R(u\to v; x, t) dv = 1. \tag{204}$$

Proof. We transform the basis from $\{\tau_1, \tau_2, n\}$ to the standard bases $\{e_1, e_2, e_3\}$. For simplicity, we assume $T_w(x) = 1$. The integration over $\mathcal{V}_\|$(defined in (77)), after the orthonormal transformation, becomes integration over $\mathbb{R}^2$. We have

$$\int_{\mathbb{R}^2} \frac{1}{r_\|(2-r_\|)} \exp\Big(\frac{|v_\| - (1-r_\|)u_\||^2}{r_\|(2-r_\|)}\Big) dv_\|,$$

which is obviously normalized.

Then we consider the integration over $\mathcal{V}_\perp$, which is $e_3 < 0$ after the transformation. We want to show

$$\frac{2}{r_\perp}\int_{-\infty}^0 -v_\perp e^{-\frac{|v_\perp|^2}{r_\perp}} e^{\frac{-(1-r_\perp)|u_\perp|^2}{r_\perp}} I_0\Big(\frac{2(1-r_\perp)^{1/2} v_\perp u_\perp}{r_\perp}\Big) dv_\perp = 1. \tag{205}$$

The Bessel function reads

$$J_0(y) = \frac{1}{\pi}\int_0^\pi e^{iy\cos\theta} d\theta = \sum_{k=0}^\infty \frac{1}{\pi}\int_0^\pi \frac{(iy\cos\theta)^k}{k!} d\theta = \sum_{k=0}^\infty \int_0^\pi \frac{(iy\cos\theta)^{2k}}{(2k)!} d\theta$$

$$\sum_{k=0}^\infty \int_0^\pi \frac{(-1)^k (y)^{2k} (\cos\theta)^{2k}}{(2k)!} d\theta = \sum_{k=0}^\infty (-1)^k \frac{(\frac{1}{4}y^2)^k}{(k!)^2},$$

where we use the Fubini's theorem and the fact that

$$\int_0^\pi \cos^{2k}\theta = \frac{\pi}{2^{2k}} \begin{pmatrix} 2k \\ k \end{pmatrix}.$$

Hence

$$I_0(y) = \frac{1}{\pi}\int_0^\pi e^{i(-iy)\cos\theta} d\theta = J_0(-iy) = \sum_{k=0}^\infty \frac{(\frac{1}{4}y^2)^k}{(k!)^2}, \quad I_0(y) = I_0(-y). \tag{206}$$

By taking the change of variable $v_\perp \to -v_\perp$, the LHS of (205) can be written as

$$\frac{2}{r_\perp}\int_0^\infty v_\perp e^{-\frac{|v_\perp|^2}{r_\perp}} e^{\frac{-(1-r_\perp)|u_\perp|^2}{r_\perp}} I_0\Big(\frac{2(1-r_\perp)^{1/2} v_\perp u_\perp}{r_\perp}\Big) dv_\perp.$$

Using (206) we rewrite the above term as

$$\sum_{k=0}^{\infty}\frac{2}{r_\perp}\int_0^\infty v_\perp e^{\frac{-|v_\perp|^2}{r_\perp}}e^{\frac{-(1-r_\perp)|u_\perp|^2}{r_\perp}}\frac{(1-r_\perp)^k v_\perp^{2k}u_\perp^{2k}}{(k!)^2 r_\perp^{2k}}dv, \tag{207}$$

where we use the Tonelli theorem. By rescaling $v_\perp=\sqrt{r_\perp}v_\perp$ we have

$$\begin{aligned}
&\frac{2}{r_\perp}\int_0^\infty v_\perp e^{\frac{-|v_\perp|^2}{r_\perp}}e^{\frac{-(1-r_\perp)|u_\perp|^2}{r_\perp}}\frac{(1-r_\perp)^k v_\perp^{2k}u_\perp^{2k}}{(k!)^2 r_\perp^{2k}}dv\\
&=2\int_0^\infty v_\perp e^{-|v_\perp|^2}e^{\frac{-(1-r_\perp)|u_\perp|^2}{r_\perp}}\frac{(1-r_\perp)^k v_\perp^{2k}u_\perp^{2k}}{(k!)^2 r_\perp^{k}}dv\\
&=2\int_0^\infty v_\perp^{2k+1} e^{-|v_\perp|^2}dv e^{\frac{-(1-r_\perp)|u_\perp|^2}{r_\perp}}\frac{(1-r_\perp)^k u_\perp^{2k}}{(k!)^2 r_\perp^{k}} \qquad (208)\\
&=2\frac{k!}{2}e^{\frac{-(1-r_\perp)|u_\perp|^2}{r_\perp}}\frac{(1-r_\perp)^k u_\perp^{2k}}{(k!)^2 r_\perp^{k}}=e^{\frac{-(1-r_\perp)|u_\perp|^2}{r_\perp}}\frac{(1-r_\perp)^k u_\perp^{2k}}{k! r_\perp^{k}}. \qquad (209)
\end{aligned}$$

Therefore, the LHS of (205) can be written as

$$e^{\frac{-(1-r_\perp)|u_\perp|^2}{r_\perp}}\sum_{k=0}^{\infty}\frac{(1-r_\perp)^k u_\perp^{2k}}{k! r_\perp^{k}}=e^{\frac{-(1-r_\perp)|u_\perp|^2}{r_\perp}}e^{\frac{(1-r_\perp)|u_\perp|^2}{r_\perp}}=1.$$

□

Lemma 4.2. *For any $a>0, b>0, \varepsilon>0$ with $a+\varepsilon<b$,*

$$\frac{b}{\pi}\int_{\mathbb{R}^2}e^{\varepsilon|v|^2}e^{a|v|^2}e^{-b|v-w|^2}dv=\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}|w|^2}. \tag{210}$$

And when $\delta\ll 1$,

$$\begin{aligned}
&\frac{b}{\pi}\int_{|v-\frac{b}{b-a-\varepsilon}w|>\delta^{-1}}e^{\varepsilon|v|^2}e^{a|v|^2}e^{-b|v-w|^2}dv\\
&\le e^{-(b-a-\varepsilon)\delta^{-2}}\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}|w|^2} \qquad (211)\\
&\le \delta\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}|w|^2}. \qquad (212)
\end{aligned}$$

Proof.

$$\begin{aligned}
&\frac{b}{\pi}\int_{\mathbb{R}^2}e^{\varepsilon|v|^2}e^{a|v|^2}e^{-b|v-w|^2}dv=\frac{b}{\pi}\int_{\mathbb{R}^2}e^{(a+\varepsilon-b)|v|^2}e^{2bv\cdot w}e^{-b|w|^2}dv\\
&=\frac{b}{\pi}\int_{\mathbb{R}^2}e^{(a+\varepsilon-b)|v+\frac{b}{a+\varepsilon-b}w|^2}e^{\frac{-b^2}{a+\varepsilon-b}|w|^2}e^{-b|w|^2}dv\\
&=\frac{b}{\pi}\int_{\mathbb{R}^2}e^{(a+\varepsilon-b)|v|^2}dv e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}|w|^2}=\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}|w|^2},
\end{aligned}$$

where we apply change of variable $v+\frac{b}{a+\varepsilon-b}w\to v$ in the first step of the last line, then we obtain (210).

Following the same derivation

$$\begin{aligned}
&\frac{b}{\pi}\int_{|v-\frac{b}{b-a-\varepsilon}w|>\delta^{-1}} e^{\varepsilon|v|^2}e^{a|v|^2}e^{-b|v-w|^2}dv\\
&=\frac{b}{\pi}\int_{|v-\frac{b}{b-a-\varepsilon}w|>\delta^{-1}} e^{(a+\varepsilon-b)|v-\frac{b}{b-a-\varepsilon}w|^2}dve^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}|w|^2}\\
&\le e^{-(b-a-\varepsilon)\delta^{-2}}\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}|w|^2}\le\delta\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}|w|^2},
\end{aligned}$$

thus we obtain (212). □

Lemma 4.3. *For any $a>0, b>0, \varepsilon>0$ with $a+\varepsilon<b$,*

$$2b\int_{\mathbb{R}^+} ve^{\varepsilon v^2}e^{av^2}e^{-bv^2}e^{-bw^2}I_0(2bvw)dv=\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}w^2}. \tag{213}$$

And when $\delta\ll 1$,

$$2b\int_{0<v<\delta} ve^{\varepsilon v^2}e^{av^2}e^{-bv^2}e^{-bw^2}I_0(2bvw)dv\le\delta\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}w^2}. \tag{214}$$

Proof.

$$\begin{aligned}
&2b\int_{\mathbb{R}^+} ve^{\varepsilon v^2}e^{av^2}e^{-bv^2}e^{-bw^2}I_0(2bvw)dv\\
&=2b\int_{\mathbb{R}^+} ve^{(a+\varepsilon-b)v^2}I_0(2bvw)e^{\frac{b^2}{a+\varepsilon-b}w^2}e^{\frac{b^2}{b-a-\varepsilon}w^2}dve^{-bw^2}\\
&=2(b-a-\varepsilon)\int_{\mathbb{R}^+} ve^{(a+\varepsilon-b)v^2}I_0(2bvw)e^{\frac{(bw)^2}{a+\varepsilon-b}}dv\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}w^2}\\
&=\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}w^2},
\end{aligned}$$

where we use (205) in Lemma 4.1 in the last line, then we obtain (213).

Following the same derivation we have

$$\begin{aligned}
&2b\int_{0<v<\delta} ve^{\varepsilon v^2}e^{av^2}e^{-bv^2}e^{-bw^2}I_0(2bvw)dv\\
&=2(b-a-\varepsilon)\int_{0<v<\delta} ve^{(a+\varepsilon-b)v^2}I_0(2bvw)e^{\frac{(bw)^2}{a+\varepsilon-b}}dv\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}w^2}.
\end{aligned}$$

Using the definition of I_0 we have

$$I_0(y)=\frac{1}{\pi}\int_0^\pi e^{y\cos\phi}d\phi\le e^y.$$

Thus when $a-b+\varepsilon<0$,

$$\begin{aligned}
&2(b-a-\varepsilon)\int_{0<v<\delta} ve^{(a+\varepsilon-b)v^2}I_0(2bvw)e^{\frac{(bw)^2}{a+\varepsilon-b}}dv\\
&\le 2(b-a-\varepsilon)\int_{0<v<\delta} ve^{(a-b+\varepsilon)v^2}e^{2vbw}e^{\frac{(bw)^2}{a-b+\varepsilon}}\\
&=2(b-a-\varepsilon)\int_{0<v<\delta} ve^{(a-b+\varepsilon)(v+\frac{bw}{a-b+\varepsilon})^2}dv\\
&\le 2(b-a-\varepsilon)\int_{0<v<\delta} vdv<\delta,
\end{aligned}$$

where we use $\delta\ll 1$ in the last step, then we obtain (214). Then we derive (217). □

Lemma 4.4. *For any $m, n > 0$, when $\delta \ll 1$, we have*

$$2m^2 \int_{\frac{n}{m}u_\perp + \delta^{-1}}^{\infty} v_\perp e^{-m^2 v_\perp^2} I_0(2mnv_\perp u_\perp) e^{-n^2 u_\perp^2} dv_\perp \lesssim e^{-\frac{m^2}{4\delta^2}}. \tag{215}$$

In consequence, for any $a > 0, b > 0, \varepsilon > 0$ with $a + \varepsilon < b$,

$$\begin{aligned} &2b \int_{\frac{b}{b-a-\varepsilon}w + \delta^{-1}}^{\infty} v e^{\varepsilon v^2} e^{av^2} e^{-bv^2} e^{-bw^2} I_0(2bvw) dv \\ &\leq e^{\frac{-(b-a-\varepsilon)}{4\delta^2}} \frac{b}{b-a-\varepsilon} e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}w^2} \qquad (216) \\ &\leq \delta \frac{b}{b-a-\varepsilon} e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}w^2}. \qquad (217) \end{aligned}$$

Proof. We discuss two cases. The first case is $v_\perp > 2\frac{n}{m}u_\perp$. We bound I_0 as

$$I_0(2mnv_\perp u_\perp) \leq \frac{1}{\pi} \int_0^\pi \exp\Big(2mnv_\perp u_\perp\Big) d\theta = \exp\Big(2mnv_\perp u_\perp\Big).$$

The LHS of (215) is bounded by

$$2m^2 \int_{\max\{2\frac{n}{m}u_\perp, \frac{n}{m}u_\perp + \delta^{-1}\}}^{\infty} v e^{-m^2(v_\perp - \frac{n}{m}u_\perp)^2} dv.$$

Using $v_\perp > 2\frac{n}{m}u_\perp$ we have

$$(v_\perp - \frac{n}{m}u_\perp)^2 \geq (\frac{v_\perp}{2} + \frac{v_\perp}{2} - \frac{n}{m}u_\perp)^2 \geq \frac{v_\perp^2}{4}.$$

Thus we can further bound LHS of (215) by

$$2m^2 \int_{\max\{2\frac{n}{m}u_\perp, \frac{n}{m}u_\perp + \delta^{-1}\}}^{\infty} v_\perp e^{-\frac{m^2 v_\perp^2}{4}} dv_\perp \lesssim e^{-\frac{m^2}{4\delta^2}}.$$

The second case is $0 \leq v_\perp \leq 2\frac{n}{m}u_\perp$. Since $\frac{n}{m}u_\perp + \delta^{-1} < v_\perp$, without loss of generality, we can assume $u_\perp > \delta^{-1}$. We compare the Taylor series of $v_\perp I_0(2mnv_\perp u_\perp)$ and $\exp\Big(2mnv_\perp u_\perp\Big)$. We have

$$v_\perp I_0(2mnv_\perp u_\perp) = \sum_{k=0}^{\infty} \frac{m^{2k} n^{2k} v_\perp^{2k+1} u_\perp^{2k}}{(k!)^2}, \tag{218}$$

and

$$\exp\Big(2mnv_\perp u_\perp\Big) = \sum_{k=0}^{\infty} \frac{2^k m^k n^k v_\perp^k u_\perp^k}{k!}. \tag{219}$$

We choose k_1 such that when $k > k_1$, we can apply the Sterling formula such that

$$\frac{1}{2} \leq \Big|\frac{k!}{k^k e^{-k}\sqrt{2\pi k}}\Big| \leq 2.$$

Then we observe the quotient of the k-th term of (218) and the $2k+1$-th term of (219),

$$\begin{aligned}
&\frac{m^{2k}n^{2k}v_\perp^{2k+1}u_\perp^{2k}}{(k!)^2}\Big/\Big(\frac{2^{2k+1}m^{2k+1}n^{2k+1}v_\perp^{2k+1}u_\perp^{2k+1}}{(2k+1)!}\Big)\\
&\le \frac{4}{k^{2k}e^{-2k}2\pi k}\Big/\Big(\frac{2^{2k+1}mnu_\perp}{(2k+1)^{2k+1}e^{-(2k+1)}\sqrt{2\pi(2k+1)}}\Big)\\
&= \frac{4e}{2\pi mn}\Big(\frac{k+1/2}{k}\Big)^{2k+1}\frac{\sqrt{2\pi(2k+1)}}{u_\perp}\\
&= \frac{4e}{2\pi mn}\Big(\frac{2k+1}{2k}\Big)^{2k+1}\frac{\sqrt{2\pi(2k+1)}}{u_\perp} \le \frac{4e^2}{\sqrt{\pi}mn}\frac{\sqrt{k}}{u_\perp}.
\end{aligned}$$

Thus we can take $k_u = u_\perp^2$ such that when $k \le k_u$,

$$\sum_{k=k_1}^{k_u}\frac{m^{2k}n^{2k}v_\perp^{2k+1}u_\perp^{2k}}{(k!)^2} \le \frac{4e^2}{\sqrt{\pi}mn}\sum_{k=k_1}^{k_u}\frac{2^{2k+1}m^{2k+1}n^{2k+1}v_\perp^{2k+1}u_\perp^{2k+1}}{(2k+1)!}. \tag{220}$$

Similarly we observe the quotient of the k-th term of (218) and the $2k$-th term of (219),

$$\begin{aligned}
&\frac{m^{2k}n^{2k}v_\perp^{2k+1}u_\perp^{2k}}{(k!)^2}\Big/\Big(\frac{2^{2k}m^{2k}n^{2k}v_\perp^{2k}u_\perp^{2k}}{(2k)!}\Big)\\
&\le \frac{4v_\perp}{k^{2k}e^{-2k}2\pi k}\Big/\Big(\frac{2^{2k}}{(2k)^{2k}e^{-2k}\sqrt{4\pi k}}\Big) = \frac{4v_\perp}{\sqrt{\pi}\sqrt{k}}.
\end{aligned}$$

When $k > k_u = u_\perp^2$, by $u_\perp > \delta^{-1}$ and $v_\perp < 2\frac{n}{m}u_\perp$ we have

$$\frac{4v_\perp}{\sqrt{\pi}\sqrt{k}} \le \frac{4v_\perp}{\sqrt{\pi}u_\perp} \le \frac{8n}{m\sqrt{\pi}}.$$

Thus we have

$$\sum_{k=k_u}^{\infty}\frac{m^{2k}n^{2k}v_\perp^{2k+1}u_\perp^{2k}}{(k!)^2} \le \frac{8n}{m\sqrt{\pi}}\sum_{k=k_u}^{\infty}\frac{2^{2k}m^{2k}n^{2k}v_\perp^{2k}u_\perp^{2k}}{(2k)!}. \tag{221}$$

Collecting (221) (220), when $v_\perp < 2\frac{n}{m}u_\perp$, we obtain

$$v_\perp I_0(2mnv_\perp u_\perp) \lesssim \exp\Big(\frac{2(1-r_\perp)^{1/2}v_\perp u_\perp}{r_\perp}\Big). \tag{222}$$

By (222), we have

$$\begin{aligned}
&\int_{\frac{n}{m}u_\perp+\delta^{-1}}^{2\frac{n}{m}u_\perp} v_\perp I_0(2mnv_\perp u_\perp))e^{-m^2v_\perp^2}e^{n^2v_\perp^2}dv\\
&\lesssim \int_{\frac{n}{m}u_\perp+\delta^{-1}}^{2\frac{n}{m}u_\perp} e^{-m^2(v_\perp-\frac{n}{m}u_\perp)^2}dv \le e^{-m^2\delta^{-2}}.
\end{aligned} \tag{223}$$

Collecting (219) and (223) we prove (215).

Then following the same derivation as (213),

$$
\begin{aligned}
&2b\int_{\frac{b}{b-a-\varepsilon}w+\delta^{-1}}^{\infty} ve^{\varepsilon v^2}e^{av^2}e^{-bv^2}e^{-bw^2}I_0(2bvw)dv\\
&=2(b-a-\varepsilon)\int_{\frac{b}{b-a-\varepsilon}w+\delta^{-1}}^{\infty} ve^{(a+\varepsilon-b)v^2}I_0(2bvw)e^{\frac{(bw)^2}{a+\varepsilon-b}}dv\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}w^2}\\
&\le e^{\frac{-(b-a-\varepsilon)}{4\delta^2}}\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}w^2}\le\delta\frac{b}{b-a-\varepsilon}e^{\frac{(a+\varepsilon)b}{b-a-\varepsilon}w^2},
\end{aligned}
$$

where we apply (215) in the first step in the third line and take $\delta \ll 1$ in the last step of the third line. □

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