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A NOTE ON TWO SPECIES COLLISIONAL PLASMA IN BOUNDED DOMAINS

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Abstract. We construct a unique global-in-time solution to the two species Vlasov-Poisson-Boltzmann system in convex domains with the diffuse boundary condition, which can be viewed as one of the ideal scattering boundary model. The construction follows a new L^2-L^∞ framework in [3]. In our knowledge this result is the first construction of strong solutions for *two species* plasma models with *self-consistent field* in general bounded domains.

1. **Introduction.** One of the fundamental models for dynamics of dilute charged particles (e.g., electrons and ions) is the Vlasov-Maxwell-Boltzmann (VMB) system, in which particles interact with themselves through collisions and with their self-consistent electromagnetic field:

$$\begin{aligned}\partial_t F_+ + v\cdot\nabla_x F_+ + \frac{e_+}{m_+}(E+\frac{v}{c}\times B)\cdot\nabla_v F_+ &= Q(F_+,F_+)+Q(F_+,F_-),\\ \partial_t F_- + v\cdot\nabla_x F_- - \frac{e_-}{m_-}(E+\frac{v}{c}\times B)\cdot\nabla_v F_- &= Q(F_-,F_+)+Q(F_-,F_-).\end{aligned} \tag{1.1}$$

Here $F_\pm(t,x,v)\geq 0$ are the density functions for the ions $(+)$ and electrons $(-)$ respectively, and $e_\pm$, $m_\pm$ the magnitude of their charges and masses, c the speed of light. The self-consistent electromagnetic field $E(t,x)$, $B(t,x)$ in 1.1 is coupled with $F(t,x,v)$ through the Maxwell system (see [15]). Previous studies for the VMB system, for example the existence of global in time classical solution, uniqueness, and asymptotic behavior without boundaries, can be found in [15], [6].

Now formally as the speed of light $c\to\infty$, one can derive the so-called two species Vlasov-Poisson-Boltzmann (VPB) system, where $B(t,x)=0$. And the field E, that we are interested in, is associated with an electrostatic potential ϕ as

$$E(t,x) := -\nabla_x\phi(t,x), \tag{1.2}$$

where the potential is determined by the Poisson equation:

$$-\Delta_x\phi(t,x) = \int_{\mathbb{R}^3}(F_+ - F_-)dv := \rho. \tag{1.3}$$

In this paper we consider the zero Neumann boundary condition for ϕ:

$$\frac{\partial\phi}{\partial n} = 0 \text{ for } x\in\partial\Omega. \tag{1.4}$$

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It turns out that the presence of all the physical constants does not create essential mathematical difficulties. Therefore, for simplicity we normalize all constants in 1.1 to be one, and the VPB system takes the form:

$$\begin{aligned}\partial_t F_+ + v\cdot\nabla_x F_+ + E\cdot\nabla_v F_+ &= Q(F_+,F_+)+Q(F_+,F_-),\\ \partial_t F_- + v\cdot\nabla_x F_- - E\cdot\nabla_v F_+ &= Q(F_-,F_+)+Q(F_-,F_-).\end{aligned}\tag{1.5}$$

The collision operator between particles measures "the change rate" in binary hard sphere collisions and takes the form of

$$\begin{aligned}Q(F_1,F_2)(v) &:= Q_{\text{gain}}(F_1,F_2)-Q_{\text{loss}}(F_1,F_2)\\ &:= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|(v-u)\cdot\omega|[F_1(v')F_2(u')-F_1(v)F_2(u)]\mathrm{d}\omega\mathrm{d}u,\end{aligned}\tag{1.6}$$

where $u' = u-[(u-v)\cdot\omega]\omega$ and $v' = v+[(u-v)\cdot\omega]\omega$. The collision operator enjoys a collision invariance: for any measurable G_1, G_2,

$$\int_{\mathbb{R}^3}\begin{bmatrix}1 & v & \frac{|v|^2-3}{2}\end{bmatrix}Q(G_1,G_1)\mathrm{d}v = \begin{bmatrix}0 & 0 & 0\end{bmatrix},\quad \int_{\mathbb{R}^3}Q(G_1,G_2)=0.\tag{1.7}$$

It is well-known that a global Maxwellian μ satisfies $Q(\cdot,\cdot)=0$ where

$$\mu(v) := \frac{1}{(2\pi)^{3/2}}\exp\left(-\frac{|v|^2}{2}\right).\tag{1.8}$$

Throughout this paper, let's use the notation

$$\iota = + \text{ or } -,\ \text{and denote } -\iota = \begin{cases}- & ,\ \text{if } \iota = +\\ + & ,\ \text{if } \iota = -.\end{cases}\tag{1.9}$$

Being an important equation in both theoretic and application aspects, the Boltzmann equation has drawn attentions and there have been a lot of research activities in analytic study of the equation. Notably the nonlinear energy method has led to solutions of many open problems [14, 15] including global strong solution of both the VMB system and the VPB system, when the initial data are close to the Maxwellian μ. One thing to note is that these results deal with idealized periodic domains or whole space, in which the solutions can remain bounded in H^k for large k.

In many important physical applications, e.g. semiconductor and tokamak, the charged dilute gas is confined within a container, and its interaction with the boundary often plays a crucial role both in physics and mathematics. So it's natural to consider the equation 1.5 in a bounded domain Ω, and the interaction of the gas with the boundary is described by suitable boundary conditions [4, 24]. In this paper we consider one of the physical conditions, a so-called diffuse boundary condition:

$$F_\iota(t,x,v) = c_\mu\mu(v)\int_{n(x)\cdot u>0}F_\iota(t,x,u)(n(x)\cdot u)du \text{ for } (x,v)\in\gamma_-.\tag{1.10}$$

Here, $\gamma_- := \{(x,v)\in\partial\Omega\times\mathbb{R}^3 : n(x)\cdot v<0\}$ and $n(x)$ is the outward unit normal at a boundary point x. A number c_μ is chosen to be $\sqrt{2\pi}$ so that $c_\mu\int_{n(x)\cdot u>0}\mu(u)(n(x)\cdot u)du = 1$. Due to this normalization the distrubution of 1.10 enjoys a null flux condition at the boundary:

$$\int_{\mathbb{R}^3}F_\iota(t,x,v)(n(x)\cdot v)du = 0 \text{ for } x\in\partial\Omega.\tag{1.11}$$

One can view this boundary condition as one of the ideal scattering model.

However, in general, higher regularity may not be expected for solutions of the Boltzmann equation in physical bounded domains. Such a drastic difference of

solutions with boundaries had been demonstrated as the formation and propagation of discontinuity in non-convex domains [23, 7], and a non-existence of some second order derivatives at the boundary in convex domains [16]. Evidently the nonlinear energy method is not generally available to the boundary problems. In order to overcome such critical difficulty, Guo developed a L^2-L^∞ framework in [13] to study global solutions of the Boltzmann equation with various boundary conditions. The core of the method lays in a direct approach (without taking derivatives) to achieve a pointwise bound using trajectory of the transport operator, which leads substantial development in various directions including [8, 7, 16, 17]. There are also studies on different type of collisional plasma models such as a Fokker-Planck equation with some boundary conditions (for example, see [19] and reference therein).

The main goal of the paper is to study the 2 species VPB system coupled of 1.5 with 1.2 and 1.3, which describes the dynamics of electrons in the absence of a magnetic field. From 1.7 and 1.11, a smooth solution of VPB with the diffuse BC 1.10preserves total mass:

$$\iint_{\Omega\times\mathbb{R}^3} F_\iota(t,x,v)\mathrm{d}v\mathrm{d}x \equiv \iint_{\Omega\times\mathbb{R}^3} F_\iota(0,x,v)\mathrm{d}v\mathrm{d}x \quad \text{for all} \quad t\geq 0. \tag{1.12}$$

We assume that initially $F_0(x,v)$ satisfies

$$\iint_{\Omega\times\mathbb{R}^3} (F_+(0,x,v)-F_-(0,x,v))\mathrm{d}v\mathrm{d}x = 0. \quad \text{(a neutral condition)} \tag{1.13}$$

Then $\int_\Omega \left\{\int_{\mathbb{R}^3}(F_+(t,x,v)-F_-(t,x,v))\mathrm{d}v\right\}\mathrm{d}x = 0$ for all $t>0$ from 1.12. This zero-mean condition guarantees a solvability of the Poisson equation 1.3 with the Neumann boundary condition 1.4.

There are some previous studies for the one-species VPB system (which is obtained by letting $F_- = 0$) with physical boundary conditions. For example the time asymptotics of a solution to the VPB system is studied [5] under some a priori assumption on the solutions. In [25] renormalized solutions (no uniqueness) were constructed for the VPB system with diffuse boundary condition. Recently in [3] the authors constructed a unique global strong solution to the VPB system with diffuse boundary condition. They also had a weighted $W^{1,p}$, $3<p<6$ estimate for the solution of such system. This regularity result was later improved in [2] where the author obtained a weighted $W^{1,\infty}$ estimate for the solution under the appearance of an external field with a favorable sign condition $E\cdot n>0$ on the boundary which will be explained later.

We consider a perturbation around μ:

$$F_\iota = \mu + \sqrt{\mu} f_\iota. \tag{1.14}$$

Then the corresponding problem is given by

$$\begin{aligned}
&\partial_t f_+ + v\cdot\nabla_v f_+ - \nabla\phi\cdot\nabla_v f_+ + \frac{v}{2}\cdot\nabla\phi f_+ \\
&-\frac{2}{\sqrt{\mu}}Q(\sqrt{\mu}f_+,\,u) - \frac{1}{\sqrt{\mu}}Q(\mu,\sqrt{\mu}(f_++f_-)) = \Gamma(f_+,f_++f_-) - v\cdot\nabla\phi\sqrt{\mu}, \\
&\partial_t f_- + v\cdot\nabla_v f_- + \nabla\phi\cdot\nabla_v f_- - \frac{v}{2}\cdot\nabla\phi f_- \\
&-\frac{2}{\sqrt{\mu}}Q(\sqrt{\mu}f_-,\,u) - \frac{1}{\sqrt{\mu}}Q(\mu,\sqrt{\mu}(f_-+f_+)) = \Gamma(f_-,f_++f_-) + v\cdot\nabla\phi\sqrt{\mu}, \\
&f(0,x,v) = f_0(x,v),
\end{aligned} \tag{1.15}$$

$$-\Delta_x\phi(t,x)=\int_{\mathbb{R}^3}\sqrt{\mu}(f_+-f_-)dv,\ \frac{\partial\phi}{\partial n}=0 \text{ for } x\in\partial\Omega, \tag{1.16}$$

$$f_\iota(t,x,v)=c_\mu\sqrt{\mu}(v)\int_{n(x)\cdot u>0}\sqrt{\mu}(u)f_\iota(t,x,u)(n(x)\cdot u)du \text{ for } (x,v)\in\gamma_-. \tag{1.17}$$

For $g=\begin{bmatrix}g_1\\g_2\end{bmatrix}, h=\begin{bmatrix}h_1\\h_2\end{bmatrix}$, let

$$Lg:=-\frac{1}{\sqrt{\mu}}\begin{bmatrix}2Q(\sqrt{\mu}g_1,\mu)+Q(\mu,\sqrt{\mu}(g_1+g_2))\\2Q(\sqrt{\mu}g_2,\mu)+Q(\mu,\sqrt{\mu}(g_1+g_2))\end{bmatrix}:=\nu(v)g-Kg. \tag{1.18}$$

Here the collision frequency is defined as

$$\nu(v):=\frac{2}{\sqrt{\mu}}Q_{\text{loss}}(\sqrt{\mu},\mu):=2\int_{\mathbb{S}^2}\int_{\mathbb{R}^3}|(v-u)\cdot\omega|\mu(u)\mathrm{d}u\mathrm{d}\omega\sim\langle v\rangle, \tag{1.19}$$

It is well-known that for hard-sphere case,

$$\begin{aligned}&\frac{1}{\sqrt{\mu}(v)}Q_{\text{gain}}(\sqrt{\mu}g_1,\mu)=\frac{1}{\sqrt{\mu}(v)}Q_{\text{gain}}(\mu,\sqrt{\mu}g_1)=\int_{\mathbb{R}^3}\mathbf{k}_2(v,u)g_1(u)du,\\&\frac{1}{\sqrt{\mu}(v)}Q_{\text{loss}}(\mu,\sqrt{\mu}g_1)=\int_{\mathbb{R}^3}\mathbf{k}_1(v,u)g_1(u)du,\end{aligned}$$

with

$$\begin{aligned}\mathbf{k}_1(v,u)=&\pi|v-u|e^{-\frac{|v|^2+|u|^2}{4}},\\\mathbf{k}_2(v,u)=&\pi|v-u|^{-1}e^{-\frac{|v-u|^2}{8}}e^{-\frac{||v|^2-|u|^2|^2}{8|v-u|^2}}.\end{aligned} \tag{1.20}$$

Thus

$$\begin{aligned}Kg:=&\begin{bmatrix}\frac{2}{\sqrt{\mu}}Q_{\text{gain}}(\sqrt{\mu}g_1,\mu)+Q(\mu,\sqrt{\mu}(g_1+g_2))\\\frac{2}{\sqrt{\mu}}Q_{\text{gain}}(\sqrt{\mu}g_2,\mu)+Q(\mu,\sqrt{\mu}(g_1+g_2))\end{bmatrix}\\:=&\begin{bmatrix}\int_{\mathbb{R}^3}\mathbf{k}_2(v,u)(3g_1(u)+g_2(u))du-\int_{\mathbb{R}^3}\mathbf{k}_1(v,u)(g_1(u)+g_2(u))du\\\int_{\mathbb{R}^3}\mathbf{k}_2(v,u)(3g_2(u)+g_1(u))du-\int_{\mathbb{R}^3}\mathbf{k}_1(v,u)(g_1(u)+g_2(u))du\end{bmatrix}.\end{aligned} \tag{1.21}$$

The nonlinear operator is defined as

$$\begin{aligned}\Gamma(g,h):=&\Gamma_{\text{gain}}(g,h)-\Gamma_{\text{loss}}(g,h)\\:=&\frac{1}{\sqrt{\mu}}\begin{bmatrix}Q_{\text{gain}}(\sqrt{\mu}g_1,\sqrt{\mu}(h_1+h_2)-Q_{\text{loss}}(\sqrt{\mu}g_1,\sqrt{\mu}(h_1+h_2))\\Q_{\text{gain}}(\sqrt{\mu}g_2,\sqrt{\mu}(h_1+h_2)-Q_{\text{loss}}(\sqrt{\mu}g_2,\sqrt{\mu}(h_1+h_2))\end{bmatrix}.\end{aligned} \tag{1.22}$$

Then for $f=\begin{bmatrix}f_+\\f_-\end{bmatrix}$, 1.15 becomes

$$\partial_t f+v\cdot\nabla_x f-q\nabla\phi\cdot\nabla_v f+q\frac{v}{2}\cdot\nabla\phi f+Lf=\Gamma(f,f)-q_1v\cdot\nabla\phi\sqrt{\mu}, \tag{1.23}$$

where $q=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$, and $q_1=\begin{bmatrix}1\\-1\end{bmatrix}$.

Let's clarify some notations. We denote

$$w_\vartheta(v)=e^{\vartheta|v|^2}. \tag{1.24}$$

The boundary of the phase space $\gamma:=\{(x,v)\in\partial\Omega\times\mathbb{R}^3\}$ can be decomposed as

$$\begin{aligned}\gamma_-&=\{(x,v)\in\partial\Omega\times\mathbb{R}^3:n(x)\cdot v<0\},\quad\text{(the incoming set)},\\\gamma_+&=\{(x,v)\in\partial\Omega\times\mathbb{R}^3:n(x)\cdot v>0\},\quad\text{(the outcoming set)},\\\gamma_0&=\{(x,v)\in\partial\Omega\times\mathbb{R}^3:n(x)\cdot v=0\},\quad\text{(the grazing set)}.\end{aligned} \tag{1.25}$$

For any function $z(x,v):\bar{\Omega}\times\mathbb{R}^3\to\mathbb{R}$, denote

$$|z|_{2,+}^2=\int_{\gamma_+}z^2d\gamma,\quad |z|_{2,-}^2=\int_{\gamma_-}z^2d\gamma,\quad |z|_{\gamma,2}^2=\iint_{\partial\Omega\times\mathbb{R}^3}z^2|n(x)\cdot v|dvdx$$

Now for any vector-valued function $f,g:\Omega\times\mathbb{R}^3\to\mathbb{R}^2$, with $f=\begin{bmatrix}f_+\\ f_-\end{bmatrix}$, and $g=\begin{bmatrix}g_+\\ g_-\end{bmatrix}$, let's clarify the following notations:

$$\begin{aligned}
|f|&:=|f_+|+|f_-|,\ f\cdot g:=f_+g_++f_-g_-,\\
\langle f,g\rangle&:=\iint_{\Omega\times\mathbb{R}^3}f\cdot g\,dvdx=\iint_{\Omega\times\mathbb{R}^3}(f_+g_++f_-g_-)\,dvdx,\\
f^p&:=\begin{bmatrix}f_+^p\\ f_-^p\end{bmatrix},\ \int f:=\begin{bmatrix}\int f_+\\ \int f_-\end{bmatrix},\ \partial f:=\begin{bmatrix}\partial f_+\\ \partial f_-\end{bmatrix},\\
|f|_{p,+}^p&:=\int_{\gamma_+}|f|^pd\gamma\sim\int_{\gamma_+}(|f_+|^p+|f_-|^p)d\gamma,\\
|f|_{p,-}^p&:=\int_{\gamma_-}|f|^pd\gamma\sim\int_{\gamma_-}(|f_+|^p+|f_-|^p)d\gamma,|f|_{\gamma,p}^p:=\iint_{\partial\Omega\times\mathbb{R}^3}|f|^p|n(x)\cdot v|dvdx,\\
\|f(t)\|_p^p&:=\iint_{\Omega\times\mathbb{R}^3}|f|^pdvdx\sim\iint_{\Omega\times\mathbb{R}^3}(|f_+(t)|^p+|f_-(t)|^p)dvdx,\\
\|f(t)\|_\infty&:=\sup_{(x,v)\in\Omega\times\mathbb{R}^3}|f_+(t)|+|f_-(t)|.
\end{aligned}$$

1.1. **A new distance function.** Throughout this paper we extend ϕ_f for a *negative time*. Let

$$\phi_f(s,x,v):=\phi_{f_0}(x,v)\ \text{ for }\ -\infty<s<0,\tag{1.26}$$

where $\phi_{f_0}(x,v)$ satisfies $-\Delta\phi_{f_0}(x,v)=\int_{\mathbb{R}}^3(f_{0,+}-f_{0,-})\sqrt{\mu}dv$.

The *characteristics (trajectory)* is determined by the Hamilton ODEs for f_+ and f_- separately

$$\frac{d}{ds}\begin{bmatrix}X_\iota^f(s;t,x,v)\\ V_\iota^f(s;t,x,v)\end{bmatrix}=\begin{bmatrix}V_\iota^f(s;t,x,v)\\ -\iota\nabla_x\phi_f(s,X_\iota^f(s;t,x,v))\end{bmatrix}\quad\text{for}-\infty<s,t<\infty,\tag{1.27}$$

with $(X_\iota^f(t;t,x,v),V_\iota^f(t;t,x,v))=(x,v)$.

For $(t,x,v)\in\mathbb{R}\times\Omega\times\mathbb{R}^3$, we define *the backward exit time* $t_{\mathbf{b},\iota}^f(t,x,v)$ as

$$t_{\mathbf{b},\iota}^f(t,x,v):=\sup\{s\geq0:X_\iota^f(\tau;t,x,v)\in\Omega\ \text{ for all }\tau\in(t-s,t)\}.\tag{1.28}$$

Furthermore, we define $x_{\mathbf{b},\iota}^f(t,x,v):=X_\iota^f(t-t_{\mathbf{b},\iota}(t,x,v);t,x,v)$ and $v_{\mathbf{b},\iota}^f(t,x,v):=V_\iota^f(t-t_{\mathbf{b},\iota}(t,x,v);t,x,v)$.

Definition 1.1 (Distance Function)**.** For $\varepsilon>0$, for $\iota=+$ or $-$ as in 1.9, define

$$\begin{aligned}\alpha_{f,\varepsilon,\iota}(t,x,v):=&\chi\Big(\frac{t-t_{\mathbf{b},\iota}^f(t,x,v)+\varepsilon}{\varepsilon}\Big)|n(x_{\mathbf{b},\iota}^f(t,x,v))\cdot v_{\mathbf{b},\iota}^f(t,x,v)|\\&+\Big[1-\chi\Big(\frac{t-t_{\mathbf{b},\iota}^f(t,x,v)+\varepsilon}{\varepsilon}\Big)\Big].\end{aligned}\tag{1.29}$$

Here we use a smooth function $\chi : \mathbb{R} \to [0,1]$ satisfying

$$\chi(\tau) = 0, \ \tau \le 0, \ \text{and} \ \ \chi(\tau) = 1, \ \tau \ge 1, \\ \frac{d}{d\tau}\chi(\tau) \in [0,4] \ \ \text{for all } \tau \in \mathbb{R}. \tag{1.30}$$

Note that $\alpha_{f,\varepsilon,\iota}(0,x,v) \equiv \alpha_{f_0,\varepsilon,\iota}(0,x,v)$ is determined by f_0 and its extension 1.26. For the sake of simplicity, we could drop the superscription ^{f} in X_ι^f, V_ι^f, $t_{\mathbf{b},\iota}^f$, $x_{\mathbf{b},\iota}^f$, $v_{\mathbf{b},\iota}^f$ unless they could cause any confusion.

Also, denote

$$\alpha_{f,\varepsilon}(t,x,v) := \begin{bmatrix} \alpha_{f,\varepsilon,+}(t,x,v) & 0 \\ 0 & \alpha_{f,\varepsilon,-}(t,x,v) \end{bmatrix}, \tag{1.31}$$

and let $|\alpha_{f,\varepsilon}(t,x,v)| := |\alpha_{f,\varepsilon,+}(t,x,v)| + |\alpha_{f,\varepsilon,-}(t,x,v)|$.

One of the crucial properties of the new distance function in 1.29 is an invariance under the Vlasov operator:

$$\big[\partial_t + v\cdot\nabla_x - \iota\nabla_x\phi_f\cdot\nabla_v\big]\alpha_{f,\varepsilon,\iota}(t,x,v) = 0. \tag{1.32}$$

This is due to the fact that the characteristics solves a deterministic system 1.27 (See the proof in the appendix). This crucial invariant property under the Vlasov operator is one of the key points in our approach.

It is important to note that a different version of the distance function which has been used in the author's previous paper [2] to establish the regularity of the one specie VPB system is not applicable here. In [2], the weight $\tilde{\alpha}$ took the form

$$\tilde{\alpha}(t,x,v) = \Big[|v\cdot\nabla\xi(x)|^2 + \xi(x)^2 - 2(v\cdot\nabla^2\xi(x)\cdot v)\xi(x) - 2(E(t,\overline{x})\cdot\nabla\xi(\overline{x}))\xi(x)\Big]^{1/2} \tag{1.33}$$

for $x \in \Omega$ close to boundary, where $\overline{x} := \{\bar{x} \in \partial\Omega : d(x,\bar{x}) = d(x,\partial\Omega)\}$ is uniquely defined. And ξ was assumed to be a C^3 function $\xi : \mathbb{R}^3 \to \mathbb{R}$ such that $\Omega = \{x \in \mathbb{R}^3 : \xi(x) < 0\}$, $\partial\Omega = \{x \in \mathbb{R}^3 : \xi(x) = 0\}$, and $\nabla\xi(x) \neq 0$ when $|\xi(x)| \ll 1$. And the domain was assumed to be strictly convex:

$$\sum_{i,j} \partial_{ij}\xi(x)\zeta_i\zeta_j \ge C_\xi|\zeta|^2 \ \ \text{for all} \ \ \zeta \in \mathbb{R}^3 \ \text{and for all} \ x \in \bar{\Omega} = \Omega \cup \partial\Omega.$$

One of the crucial property this $\tilde{\alpha}$ enjoys is the velocity lemma:

$$|\{\partial_t + v\cdot\nabla_x + E\cdot\nabla_v\}\tilde{\alpha}(t,x,v)| \lesssim |v|\tilde{\alpha}, \tag{1.34}$$

when under the sign condition

$$E\cdot n > \delta > 0, \ \text{on } \partial\Omega, \tag{1.35}$$

where n is the outward normal vector. This can be seen by direct computation:

$$|\{\partial_t + v\cdot\nabla_x + E\cdot\nabla_v\}\tilde{\alpha}^2(t,x,v)| \sim |v|\tilde{\alpha}^2 + C_\xi(E,\nabla_x E,\partial_t E)|v|\xi(x), \tag{1.36}$$

for some bounded function C_ξ. Now under 1.35, we get an extra stronger control for $\xi(x)$ from $\tilde{\alpha}^2$, and therefore the second term on the right-hand side of 1.36 can be bounded by:

$$C_\xi|v|\xi(x) \le \frac{C_\xi}{\inf_{y\in\partial\Omega} E(t,y)\cdot\nabla\xi(x)}|v|(E(t,\overline{x})\cdot\nabla\xi(\overline{x}))\xi(x) \le \frac{C_\xi}{\delta}\tilde{\alpha}^2(t,x,v). \tag{1.37}$$

Thus combing 1.36 and 1.37 we obtain 1.34. This means $\tilde{\alpha}(t,x,v)$ retains its full power under the transport operator, which is crucially used for establishing the theories in [2].

Thus it's clear that without the last term in 1.33, i.e. in the case $E\cdot\nabla\xi=0$ on $\partial\Omega$, in order to have the $\xi(x)$ control from the second term on the right hand side of 1.36, we can only obtain

$$|\{\partial_t+v\cdot\nabla_x+E\cdot\nabla_v\}\tilde{\alpha}^2(t,x,v)|\lesssim|v|\tilde{\alpha}(t,x,v). \tag{1.38}$$

Therefore $\tilde{\alpha}(t,x,v)$ suffers a loss of power under the transport operator, and would result it's been inapplicable for the situation here.

Therefore the previous distance function $\tilde{\alpha}$ would work only under a crucial favorable sign condition 1.35. But for the two species VPB system, it's clear from the equation 1.5 that if one requires the sign condition for the field for F_+, i.e. $-\nabla\phi\cdot n>0$, then inevitably one would have $+\nabla\phi\cdot n<0$, so the field for F_- would fail to satisfy the sign condition. We note that the similar $\tilde{\alpha}$ has also been used by [12], [18] in the study of one-species problem of Vlasov equation.

Thus one of the major benefit for this new distance function α is that it only requires the zero-Neuuman boundary condition $E\cdot n=0$ (see Lemma 2.1, Proposition 4), and therefore with $\pm\nabla\phi\cdot n=0$ from 1.4, we can apply this distance function to the two species VPB system 1.5.

1.2. **Main Theorem.** The main goal of this paper is the construction of a unique global *strong* solution of the two species VPB system with the diffuse boundary condition when the domain is C^3 and *convex*. Moreover an asymptotic stability of the global Maxwellian μ is studied.

Here a C^3 domain means that for any $p\in\partial\Omega$, there exists sufficiently small $\delta_1>0,\delta_2>0$, and an one-to-one and onto C^3-map

$$\begin{aligned}\eta_p:\{x_\parallel\in\mathbb{R}^2:|x_\parallel|<\delta_1\}&\ \to\ \partial\Omega\cap B(p,\delta_2),\\ x_\parallel=(x_{\parallel,1},x_{\parallel,2})&\ \mapsto\ \eta_p(x_{\parallel,1},x_{\parallel,2}).\end{aligned} \tag{1.39}$$

A *convex* domain means that there exists $C_\Omega>0$ such that for all $p\in\partial\Omega$ and η_p and for all $x_\parallel$ in 1.39

$$\sum_{i,j=1}^{2}\zeta_i\zeta_j\partial_i\partial_j\eta_p(x_\parallel)\cdot n(x_\parallel)\le-C_\Omega|\zeta|^2\quad\text{for all }\zeta\in\mathbb{R}^2. \tag{1.40}$$

Theorem 1.2. *Assume a bounded open C^3 domain $\Omega\subset\mathbb{R}^3$ is convex 1.40. Let $0<\tilde{\vartheta}<\vartheta\ll1$. Assume the neutral condition 1.13 and the compatibility condition*

$$f_{0,\iota}(x,v)=c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0}f_{0,\iota}(x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u\quad\text{on }\gamma_-. \tag{1.41}$$

Then there exists a small constant $0<\varepsilon_0\ll1$ such that for all $0<\varepsilon\le\varepsilon_0$ if an initial datum $F_0=\mu+\sqrt{\mu}f_0\ge0$ satisfies

$$\|w_\vartheta f_0\|_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)}<\varepsilon, \tag{1.42}$$

and, recall the matrix definition of α in 1.31,

$$\|w_{\tilde{\vartheta}}\alpha^\beta_{f_0,\varepsilon}\nabla_{x,v}f_0\|_{L^p(\Omega\times\mathbb{R}^3)}<\varepsilon\quad\text{for}\quad3<p<6,\quad1-\frac{2}{p}<\beta<\frac{2}{3}, \tag{1.43}$$

and

$$\|w_{\tilde{\vartheta}}\nabla_v f_0\|_{L^3(\Omega\times\mathbb{R}^3)}<\infty, \tag{1.44}$$

then there exists a unique global-in-time solution (f,ϕ_f) *to 1.15, 1.16, 1.17 such that* $F(t)=\mu+\sqrt{\mu}f(t)\geq 0$. *Moreover there exists* $\lambda_\infty>0$ *such that*

$$\sup_{t\geq 0}e^{\lambda_\infty t}\|w_\vartheta f(t)\|_{L^\infty(\bar\Omega\times\mathbb{R}^3)}+\sup_{t\geq 0}e^{\lambda_\infty t}\|\phi_f(t)\|_{C^2(\Omega)}\lesssim 1, \tag{1.45}$$

and, for some $C>0$,

$$\|w_{\tilde\vartheta}\alpha^\beta_{f,\varepsilon}\nabla_{x,v}f(t)\|_{L^p(\Omega\times\mathbb{R}^3)}\lesssim e^{Ct}\quad \textit{for all } t\geq 0, \tag{1.46}$$

and, for $0<\delta=\delta(p,\beta)\ll 1$,

$$\|\nabla_v f(t)\|_{L^3_x(\Omega)L^{1+\delta}_v(\mathbb{R}^3)}\lesssim_t 1\quad \textit{for all}\quad t\geq 0. \tag{1.47}$$

Furthermore, if (f,ϕ_f) *and* (g,ϕ_g) *are both solutions to 1.15, 1.16, 1.17 then*

$$\|f(t)-g(t)\|_{L^{1+\delta}(\Omega\times\mathbb{R}^3)}\lesssim_t \|f(0)-g(0)\|_{L^{1+\delta}(\Omega\times\mathbb{R}^3)}\quad \textit{for all}\quad t\geq 0. \tag{1.48}$$

The proof of Theorem 1.2 devotes a nontrivial extension of the argument of [3] now for the two species VPB system. One of the major difference here is the L^2 coercivity estimate.

We now illustrate the main ideas in the proof of Theorem 1.2 which largely follows the framework in [3]. In the energy-type estimate of $\nabla_{x,v}f$ in $\alpha^\beta_{f,\varepsilon}$-weighted L^p-norm, the operator $v\cdot\nabla_x$ causes a boundary term to be controlled:

$$\int_0^t\int_{\partial\Omega}\int_{n\cdot v\leq 0}|\alpha^\beta_{f,\varepsilon}\nabla_{x,v}f|^p|n\cdot v|\mathrm{d}v\mathrm{d}S_x\mathrm{d}s. \tag{1.49}$$

It turns out this integrand is integrable if

$$\beta>\frac{p-2}{p}\quad\text{so that}\quad |n\cdot v|^{p\beta-p+1}\in L^1_{loc}(\mathbb{R}^3). \tag{1.50}$$

On the other hand to control the terms in the bulk we need *a bound of* $\phi_f(t)$ in C^2_x. A key observation is that

$$\left\|\int_{\mathbb{R}^3}\nabla_x f\sqrt{\mu}\mathrm{d}v\right\|_{L^p_x(\Omega)}\lesssim\sup_x\sum_{\iota=\pm}\left\|\frac{\sqrt{\mu}}{\alpha^\beta_{f,\varepsilon,\iota}}\right\|_{L^{p^*}(\mathbb{R}^3)}\left\|\alpha^\beta_{f,\varepsilon}\nabla_x f\right\|_{L^p(\Omega\times\mathbb{R}^3)},\quad \frac{1}{p}+\frac{1}{p^*}=1, \tag{1.51}$$

which leads $C^{2,0+}$-bound of ϕ_f by the Morrey inequality for $p>3$ as long as

$$\alpha^{-\beta p^*}_{f,\varepsilon,\iota}\in L^1_{loc}(\mathbb{R}^3)\ \text{ for some } \beta p^*>\frac{p-2}{p-1}. \tag{1.52}$$

The proof of 1.52 can be found in [3], where the authors employ a change of variables $v\mapsto(x^f_{\mathbf{b}}(t,x,v),t^f_{\mathbf{b}}(t,x,v))$, and carefully compute and bound the determinant of the Jacobian matrix to get

$$\int_{|v|\lesssim 1}\alpha^{-\beta p^*}_{f,\varepsilon}\mathrm{d}v\lesssim\int_{\text{boundary}}\frac{|(x-x^f_{\mathbf{b}})\cdot n(x^f_{\mathbf{b}})|^{1-\beta p^*}}{|x-x^f_{\mathbf{b}}|^{3-\beta p^*}}\mathrm{d}x^f_{\mathbf{b}}+\text{good terms}<\infty, \tag{1.53}$$

which turns to be bounded as long as $\beta p^*<1$.

In order to run the L^2-L^∞ bootstrap argument we need to prove the L^2 coercivity property of the solution f (Proposition 6). This is one of the major difference from [3], as here for the two species VPB system, the null space of the linear operator L in 1.18 is a six-dimensional subspace of $L^2_v(\mathbb{R}^3;\mathbb{R}^2)$ spanned by orthonormal vectors

$$\left\{\begin{bmatrix}\sqrt{\mu}\\0\end{bmatrix},\begin{bmatrix}0\\\sqrt{\mu}\end{bmatrix},\begin{bmatrix}\frac{v_i}{\sqrt{2}}\sqrt{\mu}\\\frac{v_i}{\sqrt{2}}\sqrt{\mu}\end{bmatrix},\begin{bmatrix}\frac{|v|^2-3}{2\sqrt{2}}\sqrt{\mu}\\\frac{|v|^2-3}{2\sqrt{2}}\sqrt{\mu}\end{bmatrix}\right\},\ i=1,2,3, \tag{1.54}$$

(see Lemma 1 from [15] for the proof). And the projection of f onto the null space $N(L)$ can be denoted by

$$\begin{aligned}&\mathbf{P}f(t,x,v)\\&:=\left\{a_+(t,x)\begin{bmatrix}\sqrt{\mu}\\0\end{bmatrix}+a_-(t,x)\begin{bmatrix}0\\\sqrt{\mu}\end{bmatrix}+b(t,x)\cdot\frac{v}{\sqrt{2}}\begin{bmatrix}\sqrt{\mu}\\\sqrt{\mu}\end{bmatrix}+c(t,x)\frac{|v|^2-3}{2\sqrt{2}}\begin{bmatrix}\sqrt{\mu}\\\sqrt{\mu}\end{bmatrix}\right\}.\end{aligned}\tag{1.55}$$

Using the standard L^2 energy estimate of the equation, it is well-known (See [15]) that L is degenerate: $\langle Lf,f\rangle\gtrsim\|\nu^{1/2}(I-\mathbf{P})f\|^2_{L^2_{(\Omega\times\mathbb{R}^3)}}$. Thus it's clear that in order to control the L^2 norm of $f(t)$, we need a way to bound the missing $\|\mathbf{P}(t)\|_{L^2}$ term.

From there we adopt the ideas from [7] and apply it to our setting (two species system). By using weak formulation of the equation 1.23, we properly choose a set of test functions:

$$\begin{aligned}\psi_a&\equiv\begin{bmatrix}-(|v|^2-\beta_a)\sqrt{\mu}v\cdot\nabla_x\varphi_{a_+}\\-(|v|^2-\beta_a)\sqrt{\mu}v\cdot\nabla_x\varphi_{a_-}\end{bmatrix},\\\psi^{i,j}_{b,1}&\equiv\begin{bmatrix}(v_i^2-\beta_b)\sqrt{\mu}\partial_j\varphi^j_b\\(v_i^2-\beta_b)\sqrt{\mu}\partial_j\varphi^j_b\end{bmatrix},\quad i,j=1,2,3,\\\psi^{i,j}_{b,2}&\equiv\begin{bmatrix}|v|^2v_iv_j\sqrt{\mu}\partial_j\varphi^i_b(x)\\|v|^2v_iv_j\sqrt{\mu}\partial_j\varphi^i_b(x)\end{bmatrix},\quad i\neq j,\\\psi_c&\equiv\begin{bmatrix}(|v|^2-\beta_c)\sqrt{\mu}v\cdot\nabla_x\varphi_c\\(|v|^2-\beta_c)\sqrt{\mu}v\cdot\nabla_x\varphi_c\end{bmatrix},\end{aligned}\tag{1.56}$$

where $\varphi_{a_\pm}(t,x)$, $\varphi_b(t,x)$, and $\varphi_c(t,x)$ solve

$$\begin{aligned}-\Delta\varphi_{a_\pm}=a_\pm(t,x),\quad\partial_n\varphi_{a_\pm}|_{\partial\Omega}=0,\\-\Delta\varphi^j_b=b_j(t,x),\quad\varphi^j_b|_{\partial\Omega}=0,\ \text{and}\ -\Delta\varphi_c=c(t,x),\quad\varphi_c|_{\partial\Omega}=0,\end{aligned}\tag{1.57}$$

and carefully choose $\beta_a=10$, $\beta_b=1$, and $\beta_c=5$ to satisfy 7.11. Integrating against those test functions $\int_0^t\langle\phi,1.23\rangle$, we can nicely extract the L^2 norms of the $N(L)$ projections of f: $\|a_\pm(t)\|^2_{L^2},\|b(t)\|^2_{L^2},\|c(t)\|^2_{L^2}$ through the term $\langle v\cdot\nabla_x f,\phi\rangle$. And therefore we recover the bound for the missing $\|\mathbf{P}f(t)\|^2_{L^2}$ term from the L^2 energy estimate of f.

Finally we use L^2-L^∞ bootstrap argument to derive an exponential decay of f in L^∞. The main idea here is to control f_+ and f_- separately along their trajectories $(X_+(s),V_+(s))$ and $(X_-(s),V_-(s))$ by using the double Duhamel expansion, and then use change of variables to get the L^2 bound. But here as we are working with the two species system, it's important to note that in the process of the double Duhamel expansion, a mix of trajectories would occur 8.25. That is if we start with either $\iota=+$ or $-$, both the f_+ and f_- terms would appear in the first Duhamel expansion of f_ι. From there we perform the second Duhamel expansion by expanding f_+ along $(X_+(s),V_+(s))$, and expanding f_- along $(X_-(s),V_-(s))$. And then we treat them using two different change of variables

$$u\mapsto X_+(s';s,X_\iota(s;t,x,v),u),\quad u\mapsto X_-(s';s,X_\iota(s;t,x,v),u)\tag{1.58}$$

accordingly to get the bound with $\|f_+\|_{L^2}+\|f_-\|_{L^2}$ in the bulk. But thanks to the L^2 coercivity (Proposition 6) which gives control to the whole $\|f\|_{L^2}$, we can take the sum $\sum_{\iota=\pm}|f_\iota|$ and close the estimates.

2. **Preliminary.** In this section, we give some basic estimates of initial-boundary problems of the transport equation in the presence of a time-dependent field $E(t,x)$, and f here is assumed to be a scalar valued function $f(t,x,v):[0,\infty)\times\Omega\times\mathbb{R}^3\to\mathbb{R}$ satisfies

$$\partial_t f + v\cdot\nabla_x f + E\cdot\nabla_v f + \psi f = H, \tag{2.1}$$

where $H=H(t,x,v)$ and $\psi=\psi(t,x,v)\geq 0$. We assume that E is defined for all $t\in\mathbb{R}$. Throughout this section $(X(s;t,x,v),V(s;t,x,v))$ denotes the characteristic which is determined by 1.27 with replacing $-\iota\nabla_x\phi_f$ by E.

Lemma 2.1. *Assume that Ω is convex 1.40. Suppose that $\sup_t\|E(t)\|_{C^1_x}<\infty$ and*

$$n(x)\cdot E(t,x)=0 \quad \textit{for } x\in\partial\Omega \textit{ and for all } t. \tag{2.2}$$

Assume $(t,x,v)\in\mathbb{R}_+\times\bar{\Omega}\times\mathbb{R}^3$ and $t+1\geq t_{\mathbf{b}}(t,x,v)$. If $x\in\partial\Omega$ then we further assume that $n(x)\cdot v>0$. Then we have

$$n(x_{\mathbf{b}}(t,x,v))\cdot v_{\mathbf{b}}(t,x,v)<0. \tag{2.3}$$

Proof. The proof is the same as that of Lemma 1 in [3]. But since we are going to use some of the argument for later purpose, let's present the proof here.

Step 1. Note that locally we can parametrize the trajectory (see Lemma 15 in [16] or [22] for details). We consider local parametrization 1.39. We drop the subscript p for the sake of simplicity. If $X(s;t,x,v)$ is near the boundary then we can define $(X_n,X_\parallel)$ to satisfy

$$X(s;t,x,v)=\eta(X_\parallel(s;t,x,v))+X_n(s;t,x,v)[-n(X_\parallel(s;t,x,v))]. \tag{2.4}$$

For the normal velocity we define

$$V_n(s;t,x,v):=V(s;t,x,v)\cdot[-n(X_\parallel(s;t,x,v))]. \tag{2.5}$$

We define $V_\parallel$ tangential to the level set $\big(\eta(X_\parallel)+X_n(-n(X_\parallel))\big)$ for fixed X_n. Note that

$$\frac{\partial\big(\eta(x_\parallel)+x_n(-n(x_\parallel))\big)}{\partial x_{\parallel,i}}\perp n(x_\parallel) \quad \text{for } i=1,2.$$

We define $(V_{\parallel,1},V_{\parallel,2})$ as

$$V_{\parallel,i}:=\Big(V-V_n[-n(X_\parallel)]\Big)\cdot\Big(\frac{\partial\eta(X_\parallel)}{\partial x_{\parallel,i}}+X_n\Big[-\frac{\partial n(X_\parallel)}{\partial x_{\parallel,i}}\Big]\Big). \tag{2.6}$$

Therefore we obtain

$$V(s;t,x,u)=V_n[-n(X_\parallel)]+V_\parallel\cdot\nabla_{x_\parallel}\eta(X_\parallel)-X_nV_\parallel\cdot\nabla_{x_\parallel}n(X_\parallel). \tag{2.7}$$

Directly we have

$$\dot{X}(s;t,x,u)=\dot{X}_\parallel\cdot\nabla_{x_\parallel}\eta(X_\parallel)+\dot{X}_n[-n(X_\parallel)]-X_n\dot{X}_\parallel\cdot\nabla_{x_\parallel}n(X_\parallel).$$

Comparing coefficients of normal and tangential components, we obtain that

$$\dot{X}_n(s;t,x,v)=V_n(s;t,x,v),\quad \dot{X}_\parallel(s;t,x,v)=V_\parallel(s;t,x,v). \tag{2.8}$$

On the other hand, from 2.7,

$$\begin{aligned}\dot{V}(s)=\dot{V}_n[-n(X_\parallel)]-V_n\nabla_{x_\parallel}n(X_\parallel)\dot{X}_\parallel+V_\parallel\cdot\nabla^2_{x_\parallel}\eta(X_\parallel)\dot{X}_\parallel+\dot{V}_\parallel\cdot\nabla_{x_\parallel}\eta(X_\parallel)\\ -\dot{X}_n\nabla_{x_\parallel}n(X_\parallel)V_\parallel-X_n\nabla_{x_\parallel}n(X_\parallel)\dot{V}_\parallel-X_nV_\parallel\cdot\nabla^2_{x_\parallel}n(X_\parallel)\dot{X}_\parallel.\end{aligned} \tag{2.9}$$

From $2.9\cdot[-n(X_\parallel)]$, 2.8, and $\dot V = E$, we obtain that

$$\begin{aligned}\dot V_n(s) = {} & [V_\parallel(s)\cdot\nabla^2\eta(X_\parallel(s))\cdot V_\parallel(s)]\cdot n(X_\parallel(s)) + E(s,X(s))\cdot[-n(X_\parallel(s))]\\ & - X_n(s)[V_\parallel(s)\cdot\nabla^2 n(X_\parallel(s))\cdot V_\parallel(s)]\cdot n(X_\parallel(s)).\end{aligned} \tag{2.10}$$

Step 2. We prove 2.3 by the contradiction argument. Assume we choose (t,x,v) satisfying the assumptions of Lemma 2.1. Let us assume

$$X_n(t-t_{\mathbf b};t,x,v) + V_n(t-t_{\mathbf b};t,x,v) = 0. \tag{2.11}$$

First we choose $0<\varepsilon\ll 1$ such that $X_n(s;t,x,v)\ll 1$ and

$$V_n(s;t,x,v)\ge 0 \quad \text{for } t-t_{\mathbf b}(t,x,v) < s < t-t_{\mathbf b}(t,x,v)+\varepsilon. \tag{2.12}$$

The sole case that we cannot choose such $\varepsilon>0$ is when there exists $0<\delta\ll 1$ such that $V_n(s;t,x,v)<0$ for all $s\in(t-t_{\mathbf b}(t,x,v), t-t_{\mathbf b}(t,x,v)+\delta)$. But from 2.8 for $s\in(t-t_{\mathbf b}(t,x,v), t-t_{\mathbf b}(t,x,v)+\delta)$,

$$0\le X_n(s;t,x,v) = X_n(t-t_{\mathbf b}(t,x,v);t,x,v) + \int_{t-t_{\mathbf b}(t,x,v)}^{s} V_n(\tau;t,x,v)\mathrm{d}\tau < 0.$$

Now with $\varepsilon>0$ in 2.12, temporarily we define that $t_* := t-t_{\mathbf b}(t,x,v)+\varepsilon$, $x_* = X(t-t_{\mathbf b}(t,x,v)+\varepsilon;t,x,v)$, and $v_* = V(t-t_{\mathbf b}(t,x,v)+\varepsilon;t,x,v)$. Then $(X_n(s;t,x,v), X_\parallel(s;t,x,v)) = (X_n(s;t_*,x_*,v_*), X_\parallel(s;t_*,x_*,v_*))$ and $(V_n(s;t,x,v), V_\parallel(s;t,x,v)) = (V_n(s;t_*,x_*,v_*), V_\parallel(s;t_*,x_*,v_*))$.

Now we consider the RHS of 2.10. From 1.40, the first term $[V_\parallel(s)\cdot\nabla^2\eta(X_\parallel(s))\cdot V_\parallel(s)]\cdot n(X_\parallel(s))\le 0$. By an expansion and 2.2 we can bound the second term

$$\begin{aligned} & E(s,X(s))\cdot n(X_\parallel(s))\\ = {} & E(s,X_n(s),X_\parallel(s))\cdot n(X_\parallel(s))\\ = {} & E(s,0,X_\parallel(s))\cdot n(X_\parallel(s)) + \|E(s)\|_{C^1_x}O(|X_n(s)|)\\ = {} & \|E(s)\|_{C^1_x}O(|X_n(s)|).\end{aligned} \tag{2.13}$$

From 1.27 and assumptions of Lemma 2.1,

$$|V_\parallel(s;t,x,v)|\le |v| + t_{\mathbf b}(t,x,v)\|E\|_\infty \le |v| + (1+t)\|E\|_\infty.$$

Combining the above results with 2.10, we conclude that

$$\dot V_n(s;t_*,x_*,v_*) \lesssim (|v|+(1+t)\|E\|_\infty)^2 X_n(s;t_*,x_*,v_*),$$

and hence from 2.8 for $t-t_{\mathbf b}(t,x,v)\le s\le t_*$,

$$\begin{aligned} & \frac{d}{ds}[X_n(s;t_*,x_*,v_*) + V_n(s;t_*,x_*,v_*)]\\ \lesssim {} & (|v|+(1+t)\|E\|_\infty)^2[X_n(s;t_*,x_*,v_*) + V_n(s;t_*,x_*,v_*)].\end{aligned} \tag{2.14}$$

By the Gronwall inequality and 2.11, for $t-t_{\mathbf b}(t,x,v)\le s\le t_*$,

$$\begin{aligned} & [X_n(s;t_*,x_*,v_*) + V_n(s;t_*,x_*,v_*)]\\ \lesssim {} & [X_n(t-t_{\mathbf b}(t,x,u)) + V_n(t-t_{\mathbf b}(t,x,u))]e^{C\varepsilon(|v|+(1+t)\|E\|_\infty)^2)}\\ = {} & 0.\end{aligned}$$

From 2.12 we conclude that $X_n(s;t,x,v)\equiv 0$ and $V_n(s;t,x,v)\equiv 0$ for all $s\in[t-t_{\mathbf b}(t,x,u), t-t_{\mathbf b}(t,x,u)+\varepsilon]$. We can continue this argument successively to deduce that $X_n(s;t,x,v)\equiv 0$ and $V_n(s;t,x,v)\equiv 0$ for all $s\in[t-t_{\mathbf b}(t,x,v),t]$. Therefore $x_n = 0 = v_n$ which implies $x\in\partial\Omega$ and $n(x)\cdot v = 0$. This is a contradiction since we chose $n(x)\cdot v>0$ if $x\in\partial\Omega$. □

Lemma 2.2. *Assume that, for $\Lambda_1 > 0$, $\delta_1 > 0$,*

$$\sup_{t\geq 0} e^{\Lambda_1 t}\|E(t)\|_\infty \leq \delta_1 \ll 1. \tag{2.15}$$

We also assume $\frac{1}{C}\langle v\rangle \leq \psi(t,x,v) \leq C\langle v\rangle$ for some $C > 0$. For ε satisfying

$$\varepsilon > \frac{2\delta_1}{\Lambda_1} > 0, \tag{2.16}$$

there exists a constant $C_{\delta_1,\Lambda_1,\Omega} > 0$ such that, for all $t \geq 0$,

$$\begin{aligned}&\int_0^t\int_{\gamma_+\setminus\gamma_+^\varepsilon} |h|\mathrm{d}\gamma\mathrm{d}s\\ &\leq C_{\delta_1,\Lambda_1,\Omega}\left\{||h_0||_1 + \int_0^t \|h(s)\|_1 + \left\|[\partial_t + v\cdot\nabla_x + E\cdot\nabla_v + \psi]h(s)\right\|_1\mathrm{d}s\right\}.\end{aligned} \tag{2.17}$$

If $E \in L^\infty$ does not decay but

$$\|E(t)\|_\infty \leq \delta, \tag{2.18}$$

then for $\varepsilon > 0$,

$$\begin{aligned}&\int_0^t\int_{\gamma_+\setminus\gamma_+^\varepsilon} |h|\mathrm{d}\gamma\mathrm{d}s\\ &\leq C_{\delta,t,\varepsilon,\Omega}\left\{\ \|h_0\|_1 + \int_0^t \|h(s)\|_1 + \left\|[\partial_t + v\cdot\nabla_x + E\cdot\nabla_v + \psi]h(s)\right\|_1\mathrm{d}s\ \right\},\end{aligned} \tag{2.19}$$

where we have time-dependent constant $C_{\delta,t,\varepsilon,\Omega} > 0$.

Proof. See the proof of Lemma 6 in [3]. □

Lemma 2.3 (Green's identity)**.** *For $p \in [1,\infty)$, we assume $f \in L^p_{loc}(\mathbb{R}_+\times\Omega\times\mathbb{R}^3)$ satisfies*

$$\partial_t f + v\cdot\nabla_x f + E\cdot\nabla_v f \in L^p_{loc}(\mathbb{R}_+; L^p(\Omega\times\mathbb{R}^3)), \quad f \in L^p_{loc}(\mathbb{R}_+; L^p(\gamma_+)).$$

Then $f \in C^0_{loc}(\mathbb{R}_+; L^p(\Omega\times\mathbb{R}^3))$ and $f \in L^p_{loc}(\mathbb{R}_+; L^p(\gamma_-))$.

Moreover

$$\begin{aligned}\|f(T)\|_p^p + \int_0^T |f|_{p,+}^p &= \|f(0)\|_p^p + \int_0^T |f|_{p,-}^p\\ &\quad + p\int_0^T\iint_{\Omega\times\mathbb{R}^3}\{\partial_t + v\cdot\nabla_x f + E\cdot\nabla_v f\}|f|^{p-2}f.\end{aligned} \tag{2.20}$$

Proof. See the proof of Lemma 5 in [3]. □

Proposition 1. *Assume the compatibility condition*

$$f_0(x,v) = g(0,x,v) \quad \text{for} \quad (x,v)\in\gamma_-. \tag{2.21}$$

Let $p \in [1,\infty)$ and $0 < \vartheta < 1/4$. Assume

$$\begin{aligned}\nabla_x f_0, \nabla_v f_0 &\in L^p(\Omega\times\mathbb{R}^3),\\ \nabla_{x,v} t_{\mathbf{b}}\partial_t g, \nabla_{x,v} v_{\mathbf{b}}\nabla_v g, \nabla_{x,v} x_{\mathbf{b}}\partial_{x_{\mathbf{b}}} g, \nabla_{x,v} t_{\mathbf{b}}\psi g &\in L^p([0,T]\times\gamma_-),\\ \nabla_x H, \nabla_v H &\in L^p([0,T]\times\Omega\times\mathbb{R}^3),\\ e^{-\vartheta|v|^2}\nabla_x\psi, e^{-\vartheta|v|^2}\nabla_v\psi &\in L^p([0,T]\times\Omega\times\mathbb{R}^3),\\ e^{\vartheta|v|^2} f_0 \in L^\infty(\Omega\times\mathbb{R}^3), e^{\vartheta|v|^2} g &\in L^\infty([0,T]\times\gamma_-),\\ e^{\vartheta|v|^2} H &\in L^\infty([0,T]\times\Omega\times\mathbb{R}^3).\end{aligned} \tag{2.22}$$

Then for any $T > 0$, there exists a unique solution f to 2.1 such that $\nabla_{x,v} f \in C^0([0,T]; L^p(\Omega \times \mathbb{R}^3)) \cap L^1((0,T); L^p(\gamma))$.

Proof. See the proof of Proposition 2 in [3]. □

Lemma 2.4. *Assume $E(t,x) \in C^1_x$ is given and 2.15 and*

$$\sup_{t\geq 0} e^{\Lambda_2 t} \|\nabla_x E(t)\|_\infty \leq \delta_2 \ll 1, \tag{2.23}$$

with $\Lambda_2 + \delta_2 + \varepsilon \leq 1$. Then there exists $C > 0$ such that

$$|\nabla_v X(s;t,x,v)| \leq C e^{C\delta_2 (\Lambda_2)^{-2}} |t-s|, \quad \text{for all } \max(t - t_{\mathbf{b}}(t,x,v), -\varepsilon) \leq s \leq t. \tag{2.24}$$

Proof. See the proof of Lemma 9 in [3]. □

3. L^∞ **estimate.** Let $\iota = +$ or $-$ as in 1.9. We set $F^0_\iota(t,x,v) \equiv \mu$ and $\phi^0 \equiv 0$. We then apply proposition 1 for $\ell = 0, 1, 2...$ to get a sequence F^ℓ such that

$$\begin{aligned} &\partial_t F^{\ell+1}_\iota + v \cdot \nabla_x F^{\ell+1}_\iota - \iota \nabla \phi^\ell \cdot \nabla_v F^{\ell+1}_\iota \\ &\qquad = Q_{\text{gain}}(F^\ell_\iota, F^\ell_\iota + F^\ell_{-\iota}) - Q_{\text{loss}}(F^{\ell+1}_\iota, F^\ell_\iota + F^\ell_{-\iota}), \\ &-\Delta \phi^\ell = \int_{\mathbb{R}^3} F^\ell_+ - F^\ell_- \mathrm{d}v, \quad \int_\Omega \phi^\ell \mathrm{d}x = 0, \quad \frac{\partial \phi^\ell}{\partial n}\Big|_{\partial\Omega} = 0, \end{aligned} \tag{3.1}$$

and, on $(x,v) \in \gamma_-$,

$$F^{\ell+1}_\iota(t,x,v) = c_\mu \mu \int_{n(x)\cdot v > 0} F^\ell_\iota(t,x,u)\{n(x)\cdot u\} \mathrm{d}u, \tag{3.2}$$

and $F^{\ell+1}_\iota(0,x,v) = F_{0\iota}(x,v)$.

Then $f^{\ell+1}$ solves

$$\begin{aligned} &[\partial_t + v\cdot\nabla_x - q\nabla_x \phi^\ell \cdot \nabla_v + \nu + q\frac{v}{2}\cdot\nabla\phi^\ell] f^{\ell+1} \\ &= K f^\ell - q_1 v \cdot \nabla \phi^\ell \sqrt{\mu} + \Gamma_{\text{gain}}(f^\ell, f^\ell) - \Gamma_{\text{loss}}(f^{\ell+1}, f^\ell), \\ &-\Delta \phi^\ell = \int_{\mathbb{R}^3} (f^\ell_+ - f^\ell_-)\sqrt{\mu} \mathrm{d}v, \quad \int_\Omega \phi^\ell \mathrm{d}x = 0, \quad \frac{\partial \phi^\ell}{\partial n}\Big|_{\partial\Omega} = 0, \end{aligned} \tag{3.3}$$

Denote the characteristics $(X^\ell_\iota, V^\ell_\iota)$ which solves

$$\begin{aligned} &\frac{d}{ds} X^\ell_\iota(s;t,x,v) = V^\ell_\iota(s;t,x,v), \\ &\frac{d}{ds} V^\ell_\iota(s;t,x,v) = -\iota \nabla \phi^\ell(s, X^\ell_\iota(s;t,x,v)). \end{aligned} \tag{3.4}$$

$$\begin{aligned} t^\ell_{1,\iota}(t,x,v) &:= \sup\{s < t : X^\ell_\iota(s;t,x,v) \in \partial\Omega\}, \\ x^\ell_{1,\iota}(t,x,v) &:= X^\ell_\iota(t^\ell_{1,\iota}(t,x,v);t,x,v), \\ t^{\ell-1}_{2,\iota}(t,x,v,v_1) &:= \sup\{s < t^\ell_{1,\iota} : X^{\ell-1}_\iota(s;t^\ell_{1,\iota}(t,x,v), x^\ell_{1,\iota}(t,x,v), v_1) \in \partial\Omega\}, \\ x^{\ell-1}_{2,\iota}(t,x,v,v_1) &:= X^{\ell-1}_\iota(t^{\ell-1}_{2,\iota}(t,x,v,v_1); t^\ell_{1,\iota}(t,x,v), x^\ell_{1,\iota}(t,x,v), v_1), \end{aligned} \tag{3.5}$$

and inductively

$$
\begin{aligned}
&t_{k,\iota}^{\ell-(k-1)}(t,x,v,v_1,\cdots,v_{k-1})\\
&:=\sup\big\{s<t_{k-1,\iota}^{\ell-(k-2)}:X_{\iota}^{\ell-(k-1)}(s;t_{k-1,\iota}^{\ell-(k-2)},x_{k-1,\iota}^{\ell-(k-2)},v_{k-1})\in\partial\Omega\big\},\\
&x_{k,\iota}^{\ell-(k-1)}(t,x,v,v_1,\cdots,v_{k-1})\\
&:=X_{\iota}^{\ell-(k-1)}(t_{k,\iota}^{\ell-(k-1)};t_{k-1,\iota}^{\ell-(k-2)},x_{k-1,\iota}^{\ell-(k-2)},v_{k-1}).
\end{aligned}
\tag{3.6}
$$

Here,

$$
\begin{aligned}
t_{i,\iota}^{\ell-(i-1)}&:=t_{i,\iota}^{\ell-(i-1)}(t,x,v,v_1,\cdots,v_{i-1}),\\
x_{i,\iota}^{\ell-(i-1)}&:=x_{i,\iota}^{\ell-(i-1)}(t,x,v,v_1,\cdots,v_{i-1}).
\end{aligned}
$$

Proposition 2. *Assume that for sufficiently small $M>0$, such that*

$$
\|w_\vartheta f_0\|_\infty<\frac{M}{2},
\tag{3.7}
$$

then there exits $T^*(M)>0$ such that

$$
\sup_{0\le t\le T^*}\max_{\ell}\|w_\vartheta f^\ell(t)\|_\infty\le M.
\tag{3.8}
$$

Proof. We define

$$
h^\ell(t,x,v):=w_\vartheta(v)f^\ell(t,x,v).
\tag{3.9}
$$

By an induction hypothesis we assume

$$
\sup_{0\le t\le T^*}\|h^\ell(t)\|_\infty\le M.
\tag{3.10}
$$

Then $h^{\ell+1}$ solves

$$
\begin{aligned}
&[\partial_t+v\cdot\nabla_x-q\nabla_x\phi^\ell\cdot\nabla_v+\nu+q\frac{v}{2}\cdot\nabla\phi^\ell-q\frac{\nabla_x\phi^\ell\cdot\nabla_v w_\vartheta}{w_\vartheta}]h^{\ell+1}\\
&=K_{w_\vartheta}h^\ell-q_1v\cdot\nabla\phi^\ell w_\vartheta\sqrt{\mu}+w_\vartheta\Gamma_{\text{gain}}(\frac{h^\ell}{w_\vartheta},\frac{h^\ell}{w_\vartheta})-w_\vartheta\Gamma_{\text{loss}}(\frac{h^{\ell+1}}{w_\vartheta},\frac{h^\ell}{w_\vartheta}),\\
&-\Delta\phi^\ell=\int_{\mathbb{R}^3}(f_+^\ell-f_-^\ell)\sqrt{\mu}\mathrm{d}v,\quad\int_\Omega\phi^\ell\mathrm{d}x=0,\quad\frac{\partial\phi^\ell}{\partial n}\Big|_{\partial\Omega}=0,
\end{aligned}
\tag{3.11}
$$

where $K_{w_\vartheta}(\,\cdot\,)=w_\vartheta K(\frac{1}{w_\vartheta}\,\cdot)$. The boundary condition is

$$
h_\iota^{\ell+1}|_{\gamma_-}=c_\mu w_\vartheta\sqrt{\mu}\int_{n\cdot u>0}h_\iota^\ell w_\vartheta^{-1}\sqrt{\mu}\{n\cdot u\}\mathrm{d}u.
\tag{3.12}
$$

We define

$$
\begin{aligned}
\nu^\ell(t,x,v)&:=\begin{bmatrix}\nu_+^\ell(t,x,v)&0\\0&\nu_-^\ell(t,x,v)\end{bmatrix}\\
&:=\begin{bmatrix}\nu(v)+\frac{v}{2}\cdot\nabla\phi^\ell-\frac{\nabla_x\phi^\ell\cdot\nabla_v w_\vartheta}{w_\vartheta}&0\\0&\nu(v)-\frac{v}{2}\cdot\nabla\phi^\ell+\frac{\nabla_x\phi^\ell\cdot\nabla_v w_\vartheta}{w_\vartheta})\end{bmatrix}.
\end{aligned}
\tag{3.13}
$$

From 3.10, for $M\ll1$, $\|\nabla\phi^\ell\|_\infty\ll1$ and hence

$$
\nu_\iota^\ell(t,x,v)\ge\frac{4\nu_0}{5}\langle v\rangle.
\tag{3.14}
$$

Let

$$
g^\ell:=-q_1v\cdot\nabla\phi^\ell\sqrt{\mu}+\Gamma_{\text{gain}}(\frac{h^\ell}{w_\vartheta},\frac{h^\ell}{w_\vartheta}):=\begin{bmatrix}g_+^\ell\\g_-^\ell\end{bmatrix}.
\tag{3.15}
$$

Note that

$$|w_\vartheta g^\ell| \lesssim \|h^\ell\|_\infty + \langle v\rangle \|h^\ell\|_\infty^2, \tag{3.16}$$

where we have used

$$|w_\vartheta \Gamma(\frac{h}{w_\vartheta}, \frac{h}{w_\vartheta})| \lesssim \langle v\rangle \|h\|_\infty^2. \tag{3.17}$$

Consider the trajectories of $h_+^{\ell+1}$ and $h_-^{\ell+1}$ separately from 3.11,

$$\begin{aligned}
&\frac{d}{ds}\Big\{e^{-\int_s^t \nu_\iota^\ell(\tau, X_\iota^\ell(\tau), V_\iota^\ell(\tau))\mathrm{d}\tau} h_\iota^{\ell+1}(s, X_\iota^\ell(s;t,x,v), V_\iota^\ell(s;t,x,v))\Big\} \\
&= e^{-\int_s^t \nu_\iota^\ell(\tau, X_\iota^\ell(\tau), V_\iota^\ell(\tau))\mathrm{d}\tau}\Big\{K_{w_\vartheta,\iota} h^\ell(s, X_\iota^\ell(s), V_\iota^\ell(s)) + w_\vartheta g_\iota^\ell(s, X_\iota^\ell(s), V_\iota^\ell(s))\Big\}.
\end{aligned} \tag{3.18}$$

From 3.18 and 3.12, we have

$$\begin{aligned}
&h_\iota^{\ell+1}(t,x,v) \\
&= \mathbf{1}_{t_{1,\iota}^\ell \le 0} e^{-\int_0^t \nu_\iota^\ell} h_\iota^{\ell+1}(0, X_\iota^\ell(0), V_\iota^\ell(0)) \\
&\quad + \int_{\max\{t_{1,\iota}^\ell, 0\}}^t e^{-\int_s^t \nu_\iota^\ell} [K_{w_\vartheta,\iota} h^\ell + w_\vartheta g_\iota^\ell](s, X_\iota^\ell(s;t,x,v), V_\iota^\ell(s;t,x,v))\mathrm{d}s \\
&\quad + \mathbf{1}_{t_{1,\iota}^\ell \ge 0} e^{-\int_{t_{1,\iota}^\ell}^t \nu_\iota^\ell} h^{\ell+1}(t_{1,\iota}^\ell, X_\iota^\ell(t_{1,\iota}^\ell;t,x,v), V_\iota^\ell(t_{1,\iota}^\ell;t,x,v)).
\end{aligned} \tag{3.19}$$

We define

$$\tilde{w}_\vartheta(v) \equiv \frac{1}{w_\vartheta(v)\sqrt{\mu(v)}}. \tag{3.20}$$

From 3.12,

$$\begin{aligned}
&\textit{the last line of } 3.19 \\
&= \mathbf{1}_{t_{1,\iota}^\ell \ge 0} e^{-\int_{t_{1,\iota}^\ell}^t \nu_\iota^\ell} \frac{1}{\tilde{w}_\vartheta(V_\iota^\ell(t_{1,\iota}^\ell))} \\
&\qquad \int_{n(x_{1,\iota}^\ell)\cdot v_1 > 0} h_\iota^\ell(t_{1,\iota}^\ell, x_{1,\iota}^\ell, v_1)\tilde{w}_\vartheta(v_1) c_\mu \mu\{n(x_{1,\iota}^\ell)\cdot v_1\}\mathrm{d}v_1.
\end{aligned}$$

We define $\mathcal{V}(x) = \{v \in \mathbb{R}^3 : n(x)\cdot v > 0\}$ with a probability measure $\mathrm{d}\sigma = \mathrm{d}\sigma(x)$ on $\mathcal{V}(x)$ which is given by

$$\mathrm{d}\sigma \equiv c_\mu \mu(v)\{n(x)\cdot v\}\mathrm{d}v. \tag{3.21}$$

Let

$$\mathcal{V}_{j,\iota} := \{v_j \in \mathbb{R}^3 : n(x_{j,\iota}^{\ell-(j-1)})\cdot v_j > 0\}. \tag{3.22}$$

Then inductively we obtain from 3.19, 3.18 and 3.12,

$$\begin{aligned}
&|h_\iota^{\ell+1}(t,x,v)| \\
&\le \mathbf{1}_{t_{1,\iota}^\ell \le 0} e^{-\int_0^t \nu_\iota^\ell} |h^{\ell+1}(0, X_\iota^\ell(0), V_\iota^\ell(0))| \\
&\quad + \int_{\max\{t_{1,\iota}^\ell, 0\}}^t e^{-\int_s^t \nu_\iota^\ell} |[K_{w_\vartheta} h^\ell + w_\vartheta g^\ell](s, X_\iota^\ell(s;t,x,v), V_\iota^\ell(s;t,x,v))|\mathrm{d}s \\
&\quad + \mathbf{1}_{t_{1,\iota}^\ell > 0} \frac{e^{-\int_{t_{1,\iota}^\ell}^t \nu_\iota^\ell}}{\tilde{w}_{\vartheta\varrho}(V_\iota^\ell(t_{1,\iota}^\ell))} \int_{\prod_{j=1}^{k-1}\mathcal{V}_{j,\iota}} |H|,
\end{aligned} \tag{3.23}$$

where $|H|$ is bounded by

$$\begin{aligned}
&\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1,\iota}^{\ell-l}\le 0<t_{l,\iota}^{\ell-(l-1)}\}}|h^{\ell-l}(0,X_\iota^{\ell-l}(0;v_l),V_\iota^{\ell-l}(0;v_l))|\mathrm{d}\Sigma_{l,\iota}(0)\\
+&\sum_{l=1}^{k-1}\int_{\max\{t_{l+1,\iota}^{\ell-l},0\}}^{t_{l,\iota}^{\ell-(l-1)}}\mathbf{1}_{\{t_{l+1,\iota}^{\ell-l}\le 0<t_{l,\iota}^{\ell-(l-1)}\}}\\
&\quad\times|[K_{w_\vartheta}h^{\ell-l}+w_\vartheta g^{\ell-l}](s,X_\iota^{\ell-l}(s;v_l),V_\iota^{\ell-l}(s;v_l)|\mathrm{d}\Sigma_{l,\iota}(s)\mathrm{d}s\\
+&\mathbf{1}_{\{0<t_{k,\iota}^{\ell-(k-1)}\}}|h^{\ell-(k-1)}(t_{k,\iota}^{\ell-(k-1)},x_{k,\iota}^{\ell-(k-1)},v_{k-1})|\mathrm{d}\Sigma_{k-1,\iota}(t_{k,\iota}^{\ell-(k-1)}),
\end{aligned}$$

where

$$\begin{aligned}
d\Sigma_{l,\iota}^{k-1}(s)=\{\Pi_{j=l+1}^{k-1}\mathrm{d}\sigma_{j,\iota}\}\times\{e^{-\int_s^{t_{l,\iota}}\nu_\iota^\ell\tilde{w}_\vartheta(v_l)}\mathrm{d}\sigma_{l,\iota}\}\\
\times\Pi_{j=1}^{l-1}\{e^{-\int_{t_{j+1,\iota}}^{t_{j,\iota}}\nu_\iota^j}\frac{w_\vartheta(v_{j,\mathbf{b}})\sqrt{\mu(v_{j,\mathbf{b}})}}{w_\vartheta(v_j)\sqrt{\mu(v_j)}}\mathrm{d}\sigma_{j,\iota}\},
\end{aligned}\tag{3.24}$$

and

$$\begin{aligned}
X_\iota^{\ell-l}(s;v_l)&:=X_\iota^{\ell-l}(s;t_{l,\iota}^{\ell-(l-1)},x_{l,\iota}^{\ell-(l-1)},v_l),\\
V_\iota^{\ell-l}(s;v_l)&:=V_\iota^{\ell-l}(s;t_{l,\iota}^{\ell-(l-1)},x_{l,\iota}^{\ell-(l-1)},v_l),\\
v_{j,\mathbf{b}}&:=V_\iota^{\ell-j}(t_{j,\iota}^{\ell-(j-1)}-t_{\mathbf{b}};t_{j,\iota}^{\ell-(j-1)},x_{j,\iota}^{\ell-(j-1)},v_j).
\end{aligned}\tag{3.25}$$

Step 2-2. We claim that there exist $T>0$ and $k_0>0$ such that for all $k\ge k_0$ and for all $(t,x,v)\in[0,T]\times\bar{\Omega}\times\mathbb{R}^3$, we have

$$\int_{\Pi_{j=1}^{k-1}\mathcal{V}_{j,\iota}}\mathbf{1}_{\{t_{k,\iota}^{\ell-(k-1)}(t,x,v,v^1,\cdots,v^{k-1})>0\}}\mathrm{d}\Sigma_{k-1,\iota}^{k-1}\lesssim_\Omega\left\{\frac{1}{2}\right\}^{k/5}.\tag{3.26}$$

The proof of the claim is a modification of a proof of Lemma 14 of [16].

For $0<\delta\ll 1$ we define

$$\mathcal{V}_{j,\iota}^\delta:=\{v_j\in\mathcal{V}_{j,\iota}:|v_j\cdot n(x_{j,\iota}^{\ell-(j-1)})|>\delta,\ |v_j|\le\delta^{-1}\}.\tag{3.27}$$

Choose

$$T=\frac{2}{\delta^{2/3}(1+\|\nabla\phi\|_\infty)^{2/3}}.\tag{3.28}$$

We claim that

$$|t_{j,\iota}^{\ell-(j-1)}-t_{j+1,\iota}^{\ell-j}|\gtrsim\delta^3,\quad\text{for } v_j\in\mathcal{V}_{j,\iota}^\delta,\ 0\le t\le T,\ 0\le t_{j,\iota}^{\ell-(j-1)}.\tag{3.29}$$

For $j\ge 1$,

$$\begin{aligned}
&\Big|\int_{t_{j,\iota}^{\ell-(j-1)}}^{t_{j+1,\iota}^{\ell-j}}V_\iota^{\ell-j}(s;t_{j,\iota}^{\ell-(j-1)},x_{j,\iota}^{\ell-(j-1)},v_j)\mathrm{d}s\Big|^2\\
=\ &|x_{j+1,\iota}^{\ell-j}-x_{j,\iota}^{\ell-(j-1)}|^2\\
\gtrsim\ &|(x_{j+1,\iota}^{\ell-j}-x_{j,\iota}^{\ell-(j-1)})\cdot n(x_{j,\iota}^{\ell-(j-1)})|\\
=\ &\Big|\int_{t_{j,\iota}^{\ell-(j-1)}}^{t_{j+1,\iota}^{\ell-j}}V_\iota^{\ell-j}(s;t_{j,\iota}^{\ell-(j-1)},x_{j,\iota}^{\ell-(j-1)},v_j)\cdot n(x_{j,\iota}^{\ell-(j-1)})\mathrm{d}s\Big|\\
=\ &\Big|\int_{t_{j,\iota}^{\ell-(j-1)}}^{t_{j+1,\iota}^{\ell-j}}\Big(v_j-\int_{t_{j,\iota}^{\ell-(j-1)}}^s\nabla\phi^{\ell-j}(\tau,X_\iota^{\ell-j}(\tau))\mathrm{d}\tau\Big)\cdot n(x_{j,\iota}^{\ell-(j-1)})\mathrm{d}s\Big|
\end{aligned}$$

$$\begin{aligned}\geq\quad & |v_j\cdot n(x_{j,\iota}^{\ell-(j-1)})||t_{j,\iota}^{\ell-(j-1)}-t_{j+1,\iota}^{\ell-j}|\\ & -\Big|\int_{t_{j,\iota}^{\ell-(j-1)}}^{t_{j+1,\iota}^{\ell-j}}\int_{t_{j,\iota}^{\ell-(j-1)}}^{s}\nabla\phi^{\ell-j}(\tau,X_\iota^{\ell-j}(\tau))\cdot n(x_{j,\iota}^{\ell-(j-1)})\mathrm{d}\tau\mathrm{d}s\Big|,\end{aligned}$$

where $X_\iota^{\ell-j}(\tau)=X_\iota^{\ell-j}(\tau;t_{j,\iota}^{\ell-(j-1)},x_{j,\iota}^{\ell-(j-1)},v_j)$.

Here we have used the fact if $x,y\in\partial\Omega$ and $\partial\Omega$ is C^2 and Ω is bounded then $|x-y|^2\gtrsim_\Omega|(x-y)\cdot n(x)|$. Hence

$$\begin{aligned}&|v_j\cdot n(x_{j,\iota}^{\ell-(j-1)})|\\ &\lesssim\frac{1}{|t_{j,\iota}^{\ell-(j-1)}-t_{j+1,\iota}^{\ell-j}|}\Big|\int_{t_{j,\iota}^{\ell-(j-1)}}^{t_{j+1,\iota}^{\ell-j}}V_\iota^{\ell-j}(s;t_{j,\iota}^{\ell-(j-1)},x_{j,\iota}^{\ell-(j-1)},v_j)\mathrm{d}s\Big|^2\\ &\quad+\frac{1}{|t_{j,\iota}^{\ell-(j-1)}-t_{j+1,\iota}^{\ell-j}|}\\ &\times\Big|\int_{t_{j,\iota}^{\ell-(j-1)}}^{t_{j+1,\iota}^{\ell-j}}\int_{t_{j,\iota}^{\ell-(j-1)}}^{s}\nabla\phi^{\ell-j}(\tau,X_\iota^{\ell-j}(\tau;t_{j,\iota}^{\ell-(j-1)},x_{j,\iota}^{\ell-(j-1)},v_j))\cdot n(x_{j,\iota}^{\ell-(j-1)})\mathrm{d}\tau\mathrm{d}s\Big|\\ &\lesssim|t_{j,\iota}^{\ell-(j-1)}-t_{j+1,\iota}^{\ell-j}|\big\{|v_j|^2+|t_{j,\iota}^{\ell-(j-1)}-t_{j+1,\iota}^{\ell-j}|^3\|\nabla\phi\|_\infty^2\\ &\quad+\frac{1}{2}\sup_{t_{j+1,\iota}^{\ell-j}\leq\tau\leq t_{j,\iota}^{\ell-(j-1)}}|\nabla\phi^{\ell-j}(\tau,X_\iota^{\ell-j}(\tau;t_{j,\iota}^{\ell-(j-1)},x_{j,\iota}^{\ell-(j-1)},v_j))\cdot n(x_{j,\iota}^{\ell-(j-1)})|\big\}.\end{aligned}\tag{3.30}$$

For $v_j\in\mathcal{V}_{j,\iota}^\delta$, $0\leq t\leq T$, and $t_{j,\iota}^{\ell-(j-1)}\geq0$,

$$|v_j\cdot n(x_{j,\iota}^{\ell-(j-1)})|\lesssim|t_{j,\iota}^{\ell-(j-1)}-t_{j+1,\iota}^{\ell-j}|\{\delta^{-2}+T^3\|\nabla\phi^{\ell-j}\|_\infty^2+\|\nabla\phi^{\ell-j}\|_\infty\}.$$

We choose T as 3.28 then prove 3.29.

Therefore if $t_{k,\iota}^{\ell-(k-1)}\geq0$ then there can be at most $\big\{\big[\frac{C_\Omega}{\delta^3}\big]+1\big\}$ numbers of $v_m\in\mathcal{V}_{m,\iota}^\delta$ for $1\leq m\leq k-1$. Equivalently there are at least $k-2-\big[\frac{C_\Omega}{\delta^3}\big]$ numbers of $v_i\in\mathcal{V}_{i,\iota}\backslash\mathcal{V}_{i,\iota}^\delta$ for $0\leq i\leq m$.

Let us choose $k=N\times\big(\big[\frac{C_\Omega}{\delta^3}\big]+1\big)$ and $N=\big(\big[\frac{C_\Omega}{\delta^3}\big]+1\big)\gg C>1$. Then we have

$$\begin{aligned}&\int_{\prod_{j=1}^{k-1}\mathcal{V}_{j,\iota}}1_{\{t_{k,\iota}^{\ell-(k-1)}(t,x,v,v^1,\cdots,v^{k-1})>0\}}d\Sigma_{k-1}^{k-1}\\ \leq\quad&\sum_{m=1}^{\big[\frac{C_\Omega}{\delta^3}\big]+1}\int_{\left\{\begin{array}{c}\text{there are exactly } m \text{ of } v_i\in\mathcal{V}_{i,\iota}^\delta\\ \text{and } \ k-1-m \ of \ v_i\in\mathcal{V}_{i,\iota}\backslash\mathcal{V}_{i,\iota}^\delta\end{array}\right\}}\prod_{j=1}^{k-1}C_0\mu(v_j)^{1/4}\mathrm{d}v_j\\ \leq\quad&\sum_{m=1}^{\big[\frac{C_\Omega}{\delta^3}\big]+1}\left(\begin{array}{c}k-1\\ m\end{array}\right)\Big\{\int_{\mathcal{V}}C_0\mu(v)^{1/4}\mathrm{d}v\Big\}^m\Big\{\int_{\mathcal{V}\backslash\mathcal{V}^\delta}C_0\mu(v)^{1/4}\mathrm{d}v\Big\}^{k-1-m}\\ \leq\quad&\Big(\Big[\frac{C_\Omega}{\delta^3}\Big]+1\Big)\{k-1\}^{\big[\frac{C_\Omega}{\delta^3}\big]+1}\{\delta\}^{k-2-\big[\frac{C_\Omega}{\delta^3}\big]}\Big\{\int_{\mathcal{V}}C_0\mu(v)^{1/4}dv\Big\}^{\big[\frac{C_\Omega}{\delta^3}\big]+1}\\ \leq\quad&\{CN\}^{\frac{k}{N}}\Big\{\frac{k}{N}\Big\}^{\frac{k}{N}}\Big\{\frac{k}{N}\Big\}^{-\frac{k}{N}\frac{N^2}{20}}\leq\Big\{\frac{k}{N}\Big\}^{\frac{k}{N}\left(-\frac{N^2}{20}+3\right)}\leq\Big\{\frac{1}{2}\Big\}^k,\end{aligned}$$

where we have chosen $k=N\times\big(\big[\frac{C_\Omega}{\delta^3}\big]+1\big)$ and $N=\big(\big[\frac{C_\Omega}{\delta^3}\big]+1\big)\gg C>1$.

Step 2-3. We define a notation

$$\mathbf{k}_{\varrho}(v,u) := \frac{1}{|v-u|}\exp\left\{-\varrho|v-u|^2 - \varrho\frac{||v|^2-|u|^2|^2}{|v-u|^2}\right\}. \tag{3.31}$$

For $0 < \frac{\vartheta}{4} < \varrho$, if $0 < \tilde{\varrho} < \varrho - \frac{\vartheta}{4}$ then

$$\mathbf{k}_{\varrho}(v,u)\frac{e^{\vartheta|v|^2}}{e^{\vartheta|u|^2}} \lesssim \mathbf{k}_{\tilde{\varrho}}(v,u). \tag{3.32}$$

See the proof in the appendix.

Moreover, for $0 < \frac{\vartheta}{4} < \varrho$, (see the proof of Lemma 7 in [13])

$$\int_{\mathbb{R}^3} \mathbf{k}_{\varrho}(v,u)\frac{e^{\vartheta|v|^2}}{e^{\vartheta|u|^2}}\mathrm{d}u \lesssim \langle v\rangle^{-1}. \tag{3.33}$$

Also, we have

$$\prod_{j=1}^{l-1}\sup_{v_j}\frac{w_{\vartheta}(v_{j,\mathbf{b}})\sqrt{\mu(v_{j,\mathbf{b}})}}{w_{\vartheta}(v_j)\sqrt{\mu(v_j)}} \lesssim \prod_{j=1}^{l-1} e^{\int_{t_{j+1,\iota}}^{t_{j,\iota}}\|\nabla\phi^{\ell-j}(s)\|_{\infty}^2} \lesssim \prod_{j=1}^{l-1} e^{M^2(t_{j,\iota}-t_{j+1,\iota})} \lesssim e^{M^2 t}, \tag{3.34}$$

and

$$\sum_{l=1}^{k-1}\mathbf{1}_{\{t_{l+1,\iota}^{\ell-l}\le 0<t_{l,\iota}^{\ell-(l-1)}\}} = \mathbf{1}_{\{t_k^{\ell-(k-1)}\le 0\}}. \tag{3.35}$$

Then from 3.16, 3.14, and 3.23, 3.24, 3.26, 3.34, and 3.35, if we choose $\ell \ge k_0$ and $0 \le t \le T$ where k_0 and T in 3.26, and let $M^2 \ll \nu_0$, we have

$$\begin{aligned}
&|h_{\iota}^{\ell+1}(t,x,v)| \\
&\le \|e^{-\frac{3}{4}\nu_0 t}h_0\|_{\infty} \\
&\quad + \int_{\max\{t_{1,\iota}^{\ell},0\}}^{t} e^{-\frac{3}{4}\nu_0(t-s)}\int_{\mathbb{R}^3}\mathbf{k}_{\varrho}(V_{\iota}^{\ell}(s;t,x,v),u)|h^{\ell}(s,X_{\iota}^{\ell}(s;t,x,v),u)|\mathrm{d}u\mathrm{d}s \\
&\quad + C_k\sup_{l}\int_{\max\{t_{l+1,\iota}^{\ell-l},0\}}^{t_{l,\iota}^{\ell-(l-1)}} e^{-\frac{3}{4}\nu_0(t-s)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathbf{k}_{\varrho}(V_{\iota}^{\ell-l}(s;v_l),u) \\
&\qquad\qquad \times |h^{\ell-l}(s,X_{\iota}^{\ell-l}(s;v_l),u)|\{n(x_l)\cdot v_l\}\frac{\sqrt{\mu(v_l)}}{w_{\vartheta}(v_l)}\mathrm{d}v_l\mathrm{d}u\mathrm{d}s \\
&\quad + \int_{\max\{t_{1,\iota}^{\ell},0\}}^{t}\langle V_{\iota}^{\ell}(s;t,x,v)\rangle e^{-\int_s^t\frac{\nu_{\iota}^{\ell}(\tau)}{2}\mathrm{d}\tau}\|e^{-\frac{3}{4}\nu_0(t-s)}h^{\ell}(s)\|_{\infty}^2\mathrm{d}s \\
&\quad + C_k\sup_{l}\int_{\max\{t_{l+1,\iota}^{\ell-l},0\}}^{t_{l,\iota}^{\ell-(l-1)}}\langle V_{\iota}^{\ell-l}(s;v_l)\rangle \\
&\qquad\qquad \times e^{-\int_s^{t_{l,\iota}^{\ell-(l-1)}}\frac{\nu_{\iota}^{\ell-l}(\tau)}{2}\mathrm{d}\tau}\|e^{-\frac{3}{4}\nu_0(t-s)}h^{\ell-l}(s)\|_{\infty}^2\mathrm{d}s \\
&\quad + \int_{\max\{t_{1,\iota}^{\ell},0\}}^{t}\|e^{-\frac{3}{4}\nu_0(t-s)}\nabla\phi^{\ell}(s)\|_{\infty}\mathrm{d}s \\
&\quad + C_k\sup_{l}\int_{\max\{t_{l+1,\iota}^{\ell-l},0\}}^{t_{l,\iota}^{\ell-(l-1)}}\|e^{-\frac{3}{4}\nu_0(t-s)}\nabla\phi^{\ell-l}(s)\|_{\infty}\mathrm{d}s
\end{aligned} \tag{3.36}$$

$$+\left\{\tfrac{1}{2}\right\}^{k/5}\|e^{-\frac{3}{4}\nu_0(t-t_{k,\iota}^{\ell-(k-1)})}h(t_{k,\iota}^{\ell-(k-1)})\|_\infty,$$

where we used the abbreviation of 3.25.

From $\int_0^t\langle V_\iota^{\ell-l}(s;v_l)\rangle e^{-\int_s^{t_{l,\iota}^{\ell-(l-1)}}\frac{\nu_\iota^{\ell-l}(\tau)}{2}\mathrm{d}\tau}\mathrm{d}s\lesssim 1$ and 3.33, we derive that

$$\begin{aligned}\|h^{\ell+1}(t)\|_\infty&\lesssim\|h_+^{\ell+1}(t)\|_\infty+\|h_-^{\ell+1}(t)\|_\infty\\&\lesssim_k\|h(0)\|_\infty+o(1)\|h(t_k^{\ell-(k-1)})\|_\infty\\&\quad+t\max_{l\geq0}\sup_{0\leq s\leq t}\|h^{\ell-l}(s)\|_\infty+\max_{l\geq0}\sup_{0\leq s\leq t}\|h^{\ell-l}(s)\|_\infty^2.\end{aligned}\tag{3.37}$$

By taking supremum in ℓ and choosing $M\ll1$ and $0\leq t\leq T^*\leq T$ with $T^*\ll1$, we conclude 3.8. □

4. Weighted $W^{1,p}$ estimates.

Proposition 3. *The main goal of this section is to prove the following weighted $W^{1,p}$ estimate for the sequence f^ℓ in 3.3 Let us choose $0<\tilde\vartheta<\vartheta\ll1$ and*

$$\frac{p-2}{p}<\beta<\frac{2}{3},\quad \text{for}\quad 3<p<6.\tag{4.1}$$

Assume f^l solves 3.3, and for some $T>0$

$$\sup_{\ell\geq0}\sup_{0\leq t\leq T}\|w_\vartheta f^\ell(t)\|_\infty\ll1,\tag{4.2}$$

$$\sup_{\ell\geq0}\sup_{0\leq t\leq T}e^{\Lambda_1 t}\|\nabla_x\phi^\ell(t)\|_\infty<\delta_1,\tag{4.3}$$

with

$$0<\frac{\delta_1}{\Lambda_1}\ll_\Omega1.\tag{4.4}$$

Then there exists $T^{**}\ll1$ and $C>0$ such that the sequence 3.3 satisfies

$$\max_{\ell\geq0}\sup_{0\leq t\leq T^{**}}\mathcal{E}^\ell(t)\leq C\{\|w_\vartheta f_0\|_\infty+\|w_{\tilde\vartheta}f_0\|_p^p+\|w_{\tilde\vartheta}\alpha_{f_0,\varepsilon}^\beta\nabla_{x,v}f_0\|_p^p+|\nabla_{\tau,v}f_0|_{p,+}^p\}<\infty.\tag{4.5}$$

where we define, for $0<\varepsilon\ll1$,

$$\begin{aligned}\mathcal{E}^{\ell+1}(t):=&\|w_\vartheta f^{\ell+1}(t)\|_\infty+\|w_{\tilde\vartheta}f^{\ell+1}(t)\|_p^p+\|w_{\tilde\vartheta}\alpha_{f^\ell,\varepsilon}^\beta\nabla_{x,v}f^{\ell+1}(t)\|_p^p\\&+\int_0^t|w_{\tilde\vartheta}\alpha_{f^\ell,\varepsilon}^\beta\nabla_{x,v}f^{\ell+1}(t)|_{p,+}^p+\int_0^t|w_{\tilde\vartheta}f^{\ell+1}(t)|_{p,+}^p.\end{aligned}\tag{4.6}$$

To prove this, we need the following results:

Proposition 4. *Assume $\phi_f(t,x)$ obtained from 1.16 with $\nabla\phi_f$ satisfies 2.15 and*

$$\sup_{t\geq0}e^{\Lambda_2 t}\|\nabla^2\phi_f(t)\|_\infty\leq\delta_2\ll1.\tag{4.7}$$

Then for $\iota=+$ or $-$ as in 1.9, for all $0<\sigma<1$ and $N>1$ and for all $s\geq0$, $x\in\bar\Omega$,

$$\int_{|u|\leq N}\frac{\mathrm{d}u}{\alpha_{f,\varepsilon,\iota}(s,x,u)^\sigma}\lesssim_{\sigma,\Omega,\Lambda_1,\delta_1,\Lambda_2,\delta_2,N}1,\tag{4.8}$$

and, for any $0<\kappa\leq2$,

$$\int_{|u|\geq N}\frac{e^{-C|v-u|^2}}{|v-u|^{2-\kappa}}\frac{1}{\alpha_{f,\varepsilon,\iota}(s,x,u)^\sigma}\mathrm{d}u\lesssim_{\sigma,\Omega,\Lambda_1,\delta_1,\Lambda_2,\delta_2,N,\kappa}1.\tag{4.9}$$

Proof. It is important to note that from 2.15 we have $n(x)\cdot\nabla\phi_f=0$ for all $x\in\Omega$. Thus for both trajectories $(X_\pm(s;t,x,v),V_\pm(s;t,x,v))$, their corresponding fields $\mp\nabla_x\phi_f$ satisfy $\mp\nabla_x\phi_f\cdot n(x)=0$. Therefore we can apply Proposition 3 from [3] to $\alpha_{f,\varepsilon,+}$ and $\alpha_{f,\varepsilon,-}$ separately to conclude 4.8 and 4.9. □

Lemma 4.1. *For any $0<\delta<1$, we claim that if (f,ϕ_f) solves 1.16 then*

$$\|\phi_f(t)\|_{C^{1,1-\delta}(\bar\Omega)}\lesssim_{\delta,\Omega}\|w_\vartheta f(t)\|_\infty\quad\textit{for all}\quad t\geq 0. \tag{4.10}$$

Proof. We have, for any $p>1$,

$$\Big\|\int_{\mathbb{R}^3}(f_+-f_-)\sqrt{\mu(v)}\mathrm{d}v\Big\|_{L^p(\Omega)}\leq|\Omega|^{1/p}\left(\int_{\mathbb{R}^3}w_\vartheta(v)^{-1}\sqrt{\mu(v)}\mathrm{d}v\right)\|w_\vartheta f(t)\|_\infty.$$

Then we apply the standard elliptic estimate to 1.16 and deduce that

$$\|\phi_f(t)\|_{W^{2,p}(\Omega)}\lesssim\|w_\vartheta f(t)\|_\infty.$$

On the other hand, from the Morrey inequality, we have, for $p>3$ and $\Omega\subset\mathbb{R}^3$,

$$\|\phi_f(t)\|_{C^{1,1-3/p}(\Omega)}\lesssim_{p,\Omega}\|\phi(t)\|_{W^{2,p}(\Omega)}.$$

Now we choose $p=3/\delta$ for $0<\delta<1$. Then we can obtain 4.10. □

To close the estimate, we use the following lemma crucially.

Lemma 4.2. *Assume 4.1. If ϕ_f solves 1.16 then*

$$\|\phi_f(t)\|_{C^{2,1-\frac{3}{p}}(\bar\Omega)}\leq(C_1)^{1/p}\big\{\|f(t)\|_p+\|\alpha^\beta_{f,\varepsilon}\nabla_x f(t)\|_p\big\}\quad\textit{for}\quad p>3. \tag{4.11}$$

Proof. Applying the Schauder estimate to 1.16, we deduce

$$\|\phi_f(t)\|_{C^{2,1-\frac{3}{p}}(\bar\Omega)}\lesssim_{p,\Omega}\Big\|\int_{\mathbb{R}^3}(f_+(t)-f_-(t))\sqrt{\mu}\mathrm{d}v\Big\|_{C^{0,1-\frac{3}{p}}(\bar\Omega)}\quad\text{for}\quad p>3. \tag{4.12}$$

By the Morrey inequality, $W^{1,p}\subset C^{0,1-\frac{3}{p}}$ with $p>3$ for a domain $\Omega\subset\mathbb{R}^3$ with a smooth boundary $\partial\Omega$, we derive

$$\begin{aligned}&\Big\|\int_{\mathbb{R}^3}(f_+(t)-f_-(t))\sqrt{\mu}\mathrm{d}v\Big\|_{C^{0,1-\frac{n}{p}}(\bar\Omega)}\\&\lesssim\Big\|\int_{\mathbb{R}^3}(f_+(t)-f_-(t))\sqrt{\mu}\mathrm{d}v\Big\|_{W^{1,p}(\Omega)}\\&\lesssim\left(\int_{\mathbb{R}^3}\mu^{q/2}\mathrm{d}v\right)^{1/q}\|f(t)\|_{L^p(\Omega\times\mathbb{R}^3)}+\Big\|\int_{\mathbb{R}^3}\nabla_x(f_+(t)-f_-(t))\sqrt{\mu}\mathrm{d}v\Big\|_{L^p(\Omega)}.\end{aligned} \tag{4.13}$$

By the Hölder inequality, for $\iota=+$ or $-$ as in 1.9,

$$\begin{aligned}&\Big|\int_{\mathbb{R}^3}\nabla_x f_\iota(t,x,v)\sqrt{\mu(v)}\mathrm{d}v\Big|\\&\leq\Big\|\frac{\sqrt{\mu(\cdot)}}{\alpha_{f,\varepsilon,\iota}(t,x,\cdot)^\beta}\Big\|_{L^{\frac{p}{p-1}}(\mathbb{R}^3)}\Big\|\alpha_{f,\varepsilon,\iota}(t,x,\cdot)^\beta\nabla_x f_\iota(t,x,\cdot)\Big\|_{L^p(\mathbb{R}^3)}\\&=\underbrace{\left(\int_{\mathbb{R}^3}\frac{\mu(v)^{\frac{p}{2(p-1)}}}{\alpha_{f,\varepsilon,\iota}(t,x,v)^{\frac{\beta p}{p-1}}}\mathrm{d}v\right)^{\frac{p-1}{p}}}_{4.14_1}\|\alpha_{f,\varepsilon,\iota}(t,x,\cdot)^\beta\nabla_x f_\iota(t,x,\cdot)\|_{L^p(\mathbb{R}^3)}.\end{aligned} \tag{4.14}$$

Note that $\frac{p-2}{p-1}<\frac{\beta p}{p-1}<\frac{2}{3}\frac{p}{p-1}<1$ from 4.1. We apply Proposition 4 and conclude that $4.14_1\lesssim 1$. Taking $L^p(\Omega)$-norm on 4.14 and from 4.13, we conclude 4.11. □

We need some basic estimates to prove Proposition 3. Recall the decomposition of L in 1.18. From 1.19

$$|\nabla_v \nu(v)| \leq \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |\omega|\mu(u)\mathrm{d}\omega\mathrm{d}u \lesssim 1. \tag{4.15}$$

Recall the definition of $\mathbf{k}_\varrho(v,u)$ from 3.31. From 1.20 and a direction computation, for $0<\varrho<\frac{1}{8}$,

$$|\partial_{v_i}\mathbf{k}_1(v,u+v)| = C_{\mathbf{k}_1}\partial_{v_i}\Big(|u|e^{-\frac{|v|^2+|u+v|^2}{4}}\Big) \lesssim \mathbf{k}_\varrho(v,u+v), \tag{4.16}$$

and

$$\begin{aligned}
\partial_{v_i}\mathbf{k}_2(v,u+v) &= C_{\mathbf{k}_2}\partial_{v_i}\Big(\frac{1}{|u|}e^{-\frac{|u|^2}{8}}e^{-\frac{||v|^2-|u+v|^2|^2}{8|u|^2}}\Big)\\
&= -\frac{C_{\mathbf{k}_2}}{|u|}e^{-\frac{|u|^2}{8}}e^{-\frac{||v|^2-|u+v|^2|^2}{8|u|^2}}\frac{||v|^2-|u+v|^2|}{4|u|}\frac{u_i}{|u|}\\
&\lesssim \frac{e^{-\frac{|u|^2}{8}}}{|u|}e^{-\frac{||v|^2-|u+v|^2|^2}{16|u|^2}}\\
&\lesssim \mathbf{k}_\varrho(v,u+v).
\end{aligned} \tag{4.17}$$

For $g_1,g_2:\mathbb{R}^3\to\mathbb{R}$, $g=\begin{bmatrix}g_1\\g_2\end{bmatrix}$, we define

$$K_v g(v) := \begin{bmatrix}\int_{\mathbb{R}^3}\nabla_v\mathbf{k}_2(v,u)(3g_1(u)+g_2(u))du - \int_{\mathbb{R}^3}\nabla_v\mathbf{k}_1(v,u)(g_1(u)+g_2(u))du\\ \int_{\mathbb{R}^3}\nabla_v\mathbf{k}_2(v,u)(3g_2(u)+g_1(u))du - \int_{\mathbb{R}^3}\nabla_v\mathbf{k}_1(v,u)(g_1(u)+g_2(u))du\end{bmatrix}. \tag{4.18}$$

From 3.32, 4.16, and 4.17,

$$\begin{aligned}
|w_{\tilde\vartheta}K\nabla_v g(v)| &\lesssim \sum_i\int_{\mathbb{R}^3}|\mathbf{k}_i(v,u+v)|\frac{w_{\tilde\vartheta}(v)}{w_{\tilde\vartheta}(u+v)}\\
&\qquad\times(|w_{\tilde\vartheta}\nabla_v g_1(u+v)|+|w_{\tilde\vartheta}\nabla_v g_2(u+v)|)\mathrm{d}u\\
&\lesssim \int_{\mathbb{R}^3}\mathbf{k}_{\tilde\varrho}(v,u)|w_{\tilde\vartheta}\nabla_v g(u)|\mathrm{d}u,\\
|w_{\tilde\vartheta}K_v g(v)|&\\
&\lesssim \sum_i\int_{\mathbb{R}^3}|\nabla_v\mathbf{k}_i(v,u+v)|\frac{w_{\tilde\vartheta}(v)}{w_\vartheta(u+v)}(|w_\vartheta g_1(u+v)|+|w_\vartheta g_2(u+v)|)\mathrm{d}u\\
&\lesssim \int_{\mathbb{R}^3}\mathbf{k}_\varrho(v,u)\frac{w_{\tilde\vartheta}(v)}{w_\vartheta(u)}|w_\vartheta g(u)|\mathrm{d}u\\
&\lesssim \|w_\vartheta g\|_\infty.
\end{aligned} \tag{4.19}$$

For $g=\begin{bmatrix}g_1\\g_2\end{bmatrix}$ and $h=\begin{bmatrix}h_1\\h_2\end{bmatrix}$, the nonlinear Boltzmann operator $\Gamma(g,h)$ in 1.22 equals

$$\begin{aligned}
\Gamma(g,h) = &\begin{bmatrix}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|u\cdot\omega|(h_1+h_2)(v+u_\perp)g_1(v+u_\parallel)\sqrt{\mu(v+u)}\mathrm{d}\omega\mathrm{d}u\\ \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|u\cdot\omega|(h_1+h_2)(v+u_\perp)g_2(v+u_\parallel)\sqrt{\mu(v+u)}\mathrm{d}\omega\mathrm{d}u\end{bmatrix}\\
&-\begin{bmatrix}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|u\cdot\omega|(h_1+h_2)(v+u)g_1(v)\sqrt{\mu(v+u)}\mathrm{d}\omega\mathrm{d}u\\ \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|u\cdot\omega|(h_1+h_2)(v+u)g_2(v)\sqrt{\mu(v+u)}\mathrm{d}\omega\mathrm{d}u\end{bmatrix},
\end{aligned} \tag{4.20}$$

where $u_{\|}=(u\cdot\omega)\omega$ and $u_{\perp}=u-u_{\|}$. Following the derivation of 1.20 in Chapter 3 of [10], by exchanging the role of $\sqrt{\mu}$ and w^{-1}, we have

$$\begin{aligned}|w_{\vartheta}\Gamma(g,h)|&\lesssim\|w_{\vartheta}g\|_{\infty}\int_{\mathbb{R}^3}\mathbf{k}_{\tilde{\varrho}}(v,u)|w_{\vartheta}h(u)|\mathrm{d}u,\\|w_{\vartheta}\Gamma(g,h)|&\lesssim\|w_{\vartheta}h\|_{\infty}\left(\int_{\mathbb{R}^3}\mathbf{k}_{\tilde{\varrho}}(v,u)|w_{\vartheta}g(u)|\mathrm{d}u+\langle v\rangle|w_{\vartheta}g(v)|\right).\end{aligned}\tag{4.21}$$

By direct computations

$$\begin{aligned}&\nabla_v\Gamma(g,h)(v)\\&=\nabla_v\Gamma_{\mathrm{gain}}(g,h)-\nabla_v\Gamma_{\mathrm{loss}}(g,h)\\&=\Gamma_{\mathrm{gain}}(\nabla_v g,h)+\Gamma_{\mathrm{gain}}(g,\nabla_v h)-\Gamma_{\mathrm{loss}}(\nabla_v g,h)-\Gamma_{\mathrm{loss}}(g,\nabla_v h)+\Gamma_v(g,h).\end{aligned}\tag{4.22}$$

Here we have defined

$$\begin{aligned}\Gamma_v(g,h)(v):=&\Gamma_{v,\mathrm{gain}}-\Gamma_{v,\mathrm{loss}}\\=&\begin{bmatrix}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|u\cdot\omega|(h_1+h_2)(v+u_{\perp})g_1(v+u_{\|})\nabla_v\sqrt{\mu(v+u)}\mathrm{d}\omega\mathrm{d}u\\\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|u\cdot\omega|(h_1+h_2)(v+u_{\perp})g_2(v+u_{\|})\nabla_v\sqrt{\mu(v+u)}\mathrm{d}\omega\mathrm{d}u\end{bmatrix}\\&-\begin{bmatrix}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|u\cdot\omega|(h_1+h_2)(v+u)g_1(v)\nabla_v\sqrt{\mu(v+u)}\mathrm{d}\omega\mathrm{d}u\\\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|u\cdot\omega|(h_1+h_2)(v+u)g_2(v)\nabla_v\sqrt{\mu(v+u)}\mathrm{d}\omega\mathrm{d}u\end{bmatrix}.\end{aligned}\tag{4.23}$$

Note that

$$\begin{aligned}&|w_{\tilde{\vartheta}}\Gamma_{\mathrm{gain}}(\nabla_v g,h)|+|w_{\tilde{\vartheta}}\Gamma_{\mathrm{gain}}(g,\nabla_v h)|\\&\lesssim(\|w_{\vartheta}g\|_{\infty}+\|w_{\vartheta}h\|_{\infty})\big\{|w_{\tilde{\vartheta}}\Gamma_{\mathrm{gain}}(|\nabla_v g|,w_{\vartheta}^{-1})|+|w_{\tilde{\vartheta}}\Gamma_{\mathrm{gain}}(w_{\vartheta}^{-1},|\nabla_v h|)|\big\}\\&\lesssim\ (\|w_{\vartheta}g\|_{\infty}+\|w_{\vartheta}h\|_{\infty})\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|(v-u)\cdot\omega|\frac{w_{\tilde{\vartheta}}(v)}{w_{\vartheta}(u)}\Big\{\frac{|\nabla_v h(u')|}{w_{\vartheta}(v')}+\frac{|\nabla_v g(v')|}{w_{\vartheta}(u')}\Big\}\mathrm{d}\omega\mathrm{d}u.\end{aligned}$$

Then following the derivation of 1.20 in Chapter 3 of [10], by exchanging the role of $\sqrt{\mu}$ and w_{ϑ}^{-1}, we can obtain a bound of

$$\begin{aligned}&|w_{\tilde{\vartheta}}\Gamma_{\mathrm{gain}}(\nabla_v g,h)|+|w_{\tilde{\vartheta}}\Gamma_{\mathrm{gain}}(g,\nabla_v h)|\\&\lesssim(\|w_{\vartheta}g\|_{\infty}+\|w_{\vartheta}h\|_{\infty})\int_{\mathbb{R}^3}\mathbf{k}_{\varrho}(v,u)\frac{w_{\tilde{\vartheta}}(v)}{w_{\tilde{\vartheta}}(u)}(|w_{\tilde{\vartheta}}\nabla_v g(u)|+|w_{\tilde{\vartheta}}\nabla_v h(u)|)\mathrm{d}u\\&\lesssim(\|w_{\vartheta}g\|_{\infty}+\|w_{\vartheta}h\|_{\infty})\int_{\mathbb{R}^3}\mathbf{k}_{\tilde{\varrho}}(v,u)(|w_{\tilde{\vartheta}}\nabla_v g(u)|+|w_{\tilde{\vartheta}}\nabla_v h(u)|)\mathrm{d}u.\end{aligned}\tag{4.24}$$

Clearly

$$\begin{aligned}|w_{\tilde{\vartheta}}\Gamma_{\mathrm{loss}}(g,\nabla_v h)|&\lesssim\|w_{\vartheta}g\|_{\infty}\int_{\mathbb{R}^3}\frac{w_{\tilde{\vartheta}}(v)}{w_{\vartheta}(v)w_{\tilde{\vartheta}}(u)}|w_{\tilde{\vartheta}}\nabla_v h(u)|\mu(u)^{\frac{1}{2}}\mathrm{d}u\\&\lesssim\|w_{\vartheta}g\|_{\infty}\int_{\mathbb{R}^3}\mathbf{k}_{\tilde{\varrho}}(v,u)|w_{\tilde{\vartheta}}\nabla_v h(u)|\mathrm{d}u,\\|w_{\tilde{\vartheta}}\Gamma_{\mathrm{loss}}(\nabla_v g,h)|&\lesssim\langle v\rangle\|w_{\vartheta}h\|_{\infty}|w_{\tilde{\vartheta}}\nabla_v g(v)|.\end{aligned}\tag{4.25}$$

For $\Gamma_{v,\mathrm{loss}}(g,h)$ defined in 4.23,

$$\begin{aligned}&|w_{\tilde{\vartheta}}\Gamma_{v,\mathrm{loss}}(g,h)|\\&\lesssim\frac{w_{\tilde{\vartheta}}(v)}{w_{\vartheta}(v)}|w_{\vartheta}g|\iint_{\mathbb{R}^3\times\mathbb{S}^2}|(u-v)\cdot\omega|\frac{1}{w_{\vartheta}(u)}|w_{\vartheta}h(u)|\nabla_v\sqrt{\mu(u)}\mathrm{d}u\mathrm{d}\omega\\&\lesssim\langle v\rangle|w_{\vartheta}g|\|w_{\vartheta}h\|_{\infty}.\end{aligned}\tag{4.26}$$

For $\Gamma_{v,\text{gain}}(g,h)$, following the derivation of 1.20 in Chapter 3 of [10], by exchanging the role of $\sqrt{\mu}$ and w_ϑ^{-1}

$$\begin{aligned}|w_{\tilde\vartheta}\Gamma_{v,\text{gain}}(g,h)| &\lesssim \|w_\vartheta h\|_\infty \iint_{\mathbb{R}^3\times\mathbb{S}^2} |(u-v)\cdot\omega| \frac{w_{\tilde\vartheta}(v)}{w_\vartheta(v')}\frac{w_\vartheta g(v')}{w_\vartheta(u')}\nabla_v\sqrt{\mu(u)}\mathrm{d}u\mathrm{d}\omega \\ &\lesssim \|w_\vartheta h\|_\infty \int_{\mathbb{R}^3} \mathbf{k}_{\tilde\varrho}(v,u)|w_\vartheta g(u)|\mathrm{d}u.\end{aligned} \tag{4.27}$$

The next result is about estimates of derivatives on the boundary. Assume 3.2 and 3.3. We claim that for $(x,v)\in\gamma_-$,

$$|\nabla_{x,v}f^{\ell+1}(t,x,v)| \lesssim \langle v\rangle\sqrt{\mu(v)}\Big(1+\frac{1}{|n(x)\cdot v|}\Big)\times 4.29. \tag{4.28}$$

with

$$\begin{aligned}\int_{n(x)\cdot u>0}\Big\{&(\langle u\rangle + |\nabla_x\phi^\ell| + |\nabla_x\phi^{\ell-1}|)|\nabla_{x,v}f^\ell(t,x,u)| + \langle u\rangle(|f^{\ell+1}|+|f^\ell|) \\ &+ (1+\|w_\vartheta f^\ell\|_\infty + \|w_\vartheta f^{\ell-1}\|_\infty)\int_{\mathbb{R}^3}\mathbf{k}_\varrho(u,u')(|f^\ell(u')|+|f^{\ell-1}(u')|)\mathrm{d}u' \\ &+ (\langle u\rangle(|f^{\ell+1}|+|f^\ell|) + \mu(u)^{\frac14})(|\nabla_x\phi^\ell|+|\nabla_x\phi^{\ell-1}|)\Big\}\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u.\end{aligned} \tag{4.29}$$

From 3.3,

$$\begin{aligned}&\partial_n f^{\ell+1}(t,x,v) \\ =&\frac{-1}{n(x)\cdot v}\Big\{\partial_t f^{\ell+1} + \sum_{i=1}^2 (v\cdot\tau_i)\partial_{\tau_i}f^{\ell+1} - q\nabla_x\phi^l\cdot\nabla_v f^{\ell+1} + q\frac{v}{2}\cdot\nabla_x\phi^l f^{\ell+1} \\ &+\nu f^{\ell+1} - Kf^\ell - \Gamma_{\text{gain}}(f^\ell,f^\ell) + \Gamma_{\text{loss}}(f^{\ell+1},f^\ell) + q_1 v\cdot\nabla_x\phi^l\sqrt{\mu}\Big\}.\end{aligned} \tag{4.30}$$

Let $\tau_1(x)$ and $\tau_2(x)$ be unit tangential vectors to $\partial\Omega$ satisfying $\tau_1(x)\cdot n(x) = 0 = \tau_2(x)\cdot n(x)$ and $\tau_1(x)\times\tau_2(x) = n(x)$. Define the orthonormal transformation from $\{n,\tau_1,\tau_2\}$ to the standard basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, i.e. $\mathcal{T}(x)n(x) = \mathbf{e}_1$, $\mathcal{T}(x)\tau_1(x) = \mathbf{e}_2$, $\mathcal{T}(x)\tau_2(x) = \mathbf{e}_3$, and $\mathcal{T}^{-1} = \mathcal{T}^T$. Upon a change of variable: $u' = \mathcal{T}(x)u$, we have

$$n(x)\cdot u = n(x)\cdot\mathcal{T}^t(x)u' = n(x)^t\mathcal{T}^t(x)u' = [\mathcal{T}(x)n(x)]^t u' = \mathbf{e}_1\cdot u' = u_1',$$

then the RHS of the diffuse BC 3.2 equals

$$c_\mu\sqrt{\mu(v)}\int_{u_1'>0} f^\ell(t,x,\mathcal{T}^t(x)u')\sqrt{\mu(u')}\{u_1'\}\mathrm{d}u'.$$

Then we can further take tangential derivatives ∂_{τ_i} as, for $(x,v)\in\gamma_-$,

$$\begin{aligned}&\partial_{\tau_i}f^{\ell+1}(t,x,v) \\ &= c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\partial_{\tau_i}f^\ell(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u \\ &+ c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\nabla_v f^\ell(t,x,u)\frac{\partial\mathcal{T}^t(x)}{\partial\tau_i}\mathcal{T}(x)u\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u.\end{aligned} \tag{4.31}$$

We can take velocity derivatives directly to 1.17 and obtain that for $(x,v)\in\gamma_-$,

$$\begin{aligned}\nabla_v f^{\ell+1}(t,x,v) &= c_\mu \nabla_v \sqrt{\mu(v)}\int_{n(x)\cdot u>0} f^\ell(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u,\\ \partial_t f^{\ell+1}(t,x,v) &= c_\mu \sqrt{\mu(v)}\int_{n(x)\cdot u>0} \partial_t f^\ell(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u.\end{aligned} \tag{4.32}$$

For the temporal derivative, we use 1.23 again to deduce that

$$\begin{aligned}&\partial_t f^{\ell+1}(t,x,v)\\ =&c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\Big\{-u\cdot\nabla_x f^\ell+q\nabla_x\phi^{\ell-1}\cdot\nabla_v f^\ell-q\frac{u}{2}\cdot\nabla_x\phi^{l-1}f^\ell+\nu f^\ell\\ &-Kf^{l-1}+\Gamma_{\text{gain}}(f^{\ell-1},f^{\ell-1})-\Gamma_{\text{loss}}(f^\ell,f^{\ell-1})\\ &-q_1u\cdot\nabla_x\phi^{\ell-1}\sqrt{\mu}\Big\}\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u.\end{aligned} \tag{4.33}$$

From 4.30-4.33, 1.20, and 4.21, we conclude 4.28.

Proof of Proposition 3. **Step 1.** Note that by our choice of f^1, we have $\partial_t f^1(t,x,v)|_{\gamma_-}=0$. Therefore combing 4.31, 4.30, and 4.32 and the assumption that $|\nabla_{\tau,v}f_0|^p_{p,+}<\infty$, we get 4.5 is valid for $\ell\le 1$.

Thus it suffices to prove the following induction statement: there exist $T^{**}\ll 1$ (and $T^{**}<T^*(M)$) and $C>0$ such that

$$\begin{aligned}&\text{if }\max_{0\le m\le\ell}\sup_{0\le t\le T^{**}}\mathcal{E}^m(t)\\ &\le C\{\|w_\vartheta f_0\|_\infty+\|w_{\tilde\vartheta}f_0\|_p^p+\|w_{\tilde\vartheta}\alpha^\beta_{f_0,\varepsilon}\nabla_{x,v}f_0\|_p^p+|\nabla_{\tau,v}f_0|^p_{p,+}\}<\infty,\\ &\text{then }\sup_{0\le t\le T^{**}}\mathcal{E}^{\ell+1}(t)\\ &\le C\{\|w_\vartheta f_0\|_\infty+\|w_{\tilde\vartheta}f_0\|_p^p+\|w_{\tilde\vartheta}\alpha^\beta_{f_0,\varepsilon}\nabla_{x,v}f_0\|_p^p+|\nabla_{\tau,v}f_0|^p_{p,+}\}.\end{aligned} \tag{4.34}$$

Define

$$\nu_{\phi^\ell}(t,x,v):=\begin{bmatrix}\nu(v)+\frac{v}{2}\cdot\nabla_x\phi^\ell & 0\\ 0 & \nu(v)-\frac{v}{2}\cdot\nabla_x\phi^\ell\end{bmatrix}. \tag{4.35}$$

From the assumption 4.2, we have that $\nu(v)+\frac{v}{2}\cdot\nabla_x\phi^\ell\gtrsim\frac{\nu(v)}{2}$, and $\nu(v)-\frac{v}{2}\cdot\nabla_x\phi^\ell\gtrsim\frac{\nu(v)}{2}$.

From 1.16, 1.20, and 4.21, we can easily obtain that, for $0<\varrho\ll 1$

$$\begin{aligned}&\|w_{\tilde\vartheta}f^{\ell+1}(t)\|_p^p+\int_0^t\|\nu_{\phi^\ell}^{1/p}w_{\tilde\vartheta}f^{\ell+1}\|_p^p+\int_0^t|w_{\tilde\vartheta}f^{\ell+1}|^p_{p,+}\\ \lesssim&\|w_{\tilde\vartheta}f(0)\|_p^p+\|w_{\tilde\vartheta}f^\ell\|_\infty\int_0^t\int_{\Omega\times\mathbb{R}^3}\langle v\rangle|w_{\tilde\vartheta}f^{\ell+1}|^p\\ &+(1+\|w_\vartheta f^\ell\|_\infty)\int_0^t\iint_{\Omega\times\mathbb{R}^3}|w_{\tilde\vartheta}f^{\ell+1}(v)|^{p-1}\int_{\mathbb{R}^3}\mathbf{k}_\varrho(v,u)\frac{w_{\tilde\vartheta}(v)}{w_{\tilde\vartheta}(u)}|w_{\tilde\vartheta}f^\ell(u)|\mathrm{d}u\\ &+o(1)\int_0^t\|w_{\tilde\vartheta}f^{\ell+1}\|_p^p+\int_0^t\|\nabla\phi^\ell\|_p^p+\int_0^t|w_{\tilde\vartheta}f^{\ell+1}|^p_{p,-}.\end{aligned} \tag{4.36}$$

Note that by the Hölder inequality, 3.33, and 3.32,

$$\begin{aligned}
&\int_{\mathbb{R}^3} |w_{\tilde{\vartheta}} f^{\ell+1}(v)|^{p-1} \int_{\mathbb{R}^3} \mathbf{k}_{\tilde{\varrho}}(v,u)|w_{\tilde{\vartheta}} f^{\ell}(u)|\mathrm{d}u\mathrm{d}v \\
&\lesssim \|w_{\tilde{\vartheta}} f^{\ell+1}\|_{L^p_v}^{\frac{1}{p-1}} \left\| \int_{\mathbb{R}^3} \mathbf{k}_{\tilde{\varrho}}(v,u)^{1/q} \mathbf{k}_{\tilde{\varrho}}(v,u)^{1/p} |w_{\tilde{\vartheta}} f^{\ell}(u)|\mathrm{d}u \right\|_{L^p_v} \\
&\lesssim \|w_{\tilde{\vartheta}} f^{\ell+1}\|_{L^p_v}^{\frac{1}{p-1}} \left(\int_{\mathbb{R}^3} \mathbf{k}_{\tilde{\varrho}}(v,u)\mathrm{d}u \right)^{1/q} \left\| \left(\int_{\mathbb{R}^3} \mathbf{k}_{\tilde{\varrho}}(v,u)|w_{\tilde{\vartheta}} f^{\ell}(u)|^p \mathrm{d}u \right)^{1/p} \right\|_{L^p_v} \\
&\lesssim o(1)\|w_{\tilde{\vartheta}} f^{\ell+1}\|_{L^p_v}^p + \|w_{\tilde{\vartheta}} f^{\ell}\|_{L^p_v}^p.
\end{aligned} \tag{4.37}$$

From a standard elliptic theorem and 1.16, we have

$$\int_0^t \|\nabla \phi^{\ell}\|_p^p \lesssim \int_0^t \|w_{\tilde{\vartheta}} f^{\ell}\|_p^p. \tag{4.38}$$

Now we focus on $\int_0^t |w_{\tilde{\vartheta}} f^{\ell+1}|_{p,-}^p$ in 4.36. We plug in 3.2 and then decompose $\gamma_+^{\varepsilon} \cup \gamma_+ \backslash \gamma_+^{\varepsilon}$ where ε is small but satisfies 2.16. This leads

$$\begin{aligned}
&\int_0^t |w_{\tilde{\vartheta}} f^{\ell+1}|_{p,-}^p \\
&\lesssim \int_0^t \int_{\partial\Omega} \left(\int_{\gamma_+^{\varepsilon}(x)} w_{\tilde{\vartheta}} |f^{\ell}| \sqrt{\mu}\{n\cdot u\}\mathrm{d}u \right)^p \\
&\quad + \int_0^t \int_{\partial\Omega} \left(\int_{\gamma_+(x)\backslash\gamma_+^{\varepsilon}(x)} w_{\tilde{\vartheta}} |f^{\ell}| \sqrt{\mu}\{n\cdot u\}\mathrm{d}u \right)^p \\
&\lesssim \Big(\int_{\gamma_+^{\varepsilon}} \sqrt{\mu}\{n\cdot u\}\mathrm{d}u \Big)^{p/q} \int_0^t |w_{\tilde{\vartheta}} f^{\ell}|_{p,+}^p + \int_0^t \int_{\gamma_+\backslash\gamma_+^{\varepsilon}} |w_{\tilde{\vartheta}} f^{\ell}\sqrt{\mu}|^p \\
&\lesssim o(1) \int_0^t |w_{\tilde{\vartheta}} f^{\ell}|_{p,+}^p + \int_0^t \int_{\gamma_+\backslash\gamma_+^{\varepsilon}} |w_{\tilde{\vartheta}} f^{\ell}\sqrt{\mu}|^p.
\end{aligned}$$

From 3.3, Lemma 2.2, 4.21, and 4.37

$$\begin{aligned}
&\int_0^t |w_{\tilde{\vartheta}} f^{\ell+1}|_{p,-}^p \\
&\lesssim \|w_{\tilde{\vartheta}} f(0)\|_p^p + o(1) \int_0^t |w_{\tilde{\vartheta}} f^{\ell}|_{p,+}^p + (1+\|w_{\vartheta} f^{\ell-1}\|_{\infty}) \int_0^t \|(w_{\tilde{\vartheta}} f^{\ell}\|_p^p + w_{\tilde{\vartheta}} f^{\ell-1}\|_p^p).
\end{aligned} \tag{4.39}$$

Collecting terms from 4.36, 4.37, 4.38, and 4.39, we conclude that for $\sup_{\ell\geq 0} \|w_{\tilde{\vartheta}} f^{\ell}\|_{\infty} \ll 1$,

$$\begin{aligned}
&\|w_{\tilde{\vartheta}} f^{\ell+1}(t)\|_p^p + \int_0^t \|\nu_{\phi^{\ell}}^{1/p} w_{\tilde{\vartheta}} f^{\ell+1}\|_p^p + \int_0^t |w_{\tilde{\vartheta}} f^{\ell+1}|_{p,+}^p \\
&\lesssim \|w_{\tilde{\vartheta}} f(0)\|_p^p + (1+\|w_{\vartheta} f^{\ell}\|_{\infty}) \int_0^t (\|w_{\tilde{\vartheta}} f^{\ell}\|_p^p + \|w_{\tilde{\vartheta}} f^{\ell-1}\|_p^p + \int_0^t |w_{\tilde{\vartheta}} f^{\ell}|_{p,+}^p) \\
&\lesssim w_{\tilde{\vartheta}} f(0)\|_p^p + \big(o(1) + t(1+\|w_{\vartheta} f^{\ell}\|_{\infty})\big) \max_{m=\ell,\ell-1} \sup_{0\leq t\leq T^{**}} \mathcal{E}^m.
\end{aligned} \tag{4.40}$$

Step 2. By taking derivatives $\partial \in \{\nabla_{x_i}, \nabla_{v_i}\}$ to 3.3,

$$[\partial_t + v\cdot\nabla_x - q\nabla_x\phi^\ell\cdot\nabla_v + \nu_{\phi^\ell, w_{\tilde\vartheta}}](w_{\tilde\vartheta}\partial f^{\ell+1}) = w_{\tilde\vartheta}\mathcal{G}^{\ell+1}, \tag{4.41}$$

where

$$\begin{aligned}\mathcal{G}^{\ell+1} =& -\partial v\cdot\nabla_x f^{\ell+1} + q\partial\nabla\phi^\ell\cdot\nabla_v f^{\ell+1} + \partial\Gamma_{\text{gain}}(f^\ell, f^\ell) - \partial\Gamma_{\text{loss}}(f^{\ell+1}, f^\ell)\\ &- \partial\big[\nu(v) + q\frac{v}{2}\cdot\nabla\phi^\ell(t,x)\big]f^{\ell+1} - \partial K f^\ell - q_1\partial(v\cdot\nabla_x\phi^\ell\sqrt{\mu}).\end{aligned} \tag{4.42}$$

Here we have used

$$\begin{aligned}\nu_{\phi^\ell, w_{\tilde\vartheta}} &= \nu_{\phi^\ell, w_{\tilde\vartheta}}(t,x,v)\\ &:= \begin{bmatrix} \nu(v) + \frac{v}{2}\cdot\nabla\phi^\ell(t,x) + \frac{\nabla_x\phi^\ell\cdot\nabla_v w_{\tilde\vartheta}}{w_{\tilde\vartheta}} & 0\\ 0 & \nu(v) - \frac{v}{2}\cdot\nabla\phi^\ell(t,x) + \frac{\nabla_x\phi^\ell\cdot\nabla_v w_{\tilde\vartheta}}{w_{\tilde\vartheta}}\end{bmatrix}.\end{aligned} \tag{4.43}$$

Denote

$$\begin{aligned}\nu_{\phi^\ell, w_{\tilde\vartheta}+} &= \nu(v) + \frac{v}{2}\cdot\nabla\phi^\ell(t,x) + \frac{\nabla_x\phi^\ell\cdot\nabla_v w_{\tilde\vartheta}}{w_{\tilde\vartheta}},\ \nu_{\phi^\ell, w_{\tilde\vartheta}-}\\ &= \nu(v) - \frac{v}{2}\cdot\nabla\phi^\ell(t,x) + \frac{\nabla_x\phi^\ell\cdot\nabla_v w_{\tilde\vartheta}}{w_{\tilde\vartheta}}.\end{aligned}$$

From 1.32 and 4.41, for $\iota = +$ or $-$ we have

$$\begin{aligned}&\frac{1}{p}|w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon,\iota}\partial f^{\ell+1}_\iota|^{p-1}\big[\partial_t + v\cdot\nabla_x - \iota\nabla_x\phi^\ell\cdot\nabla_v + \nu_{\phi^\ell, w_{\tilde\vartheta}\iota}\big]|w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon,\iota}\partial f^{\ell+1}_\iota|\\ &= \alpha^{\beta p}_{f^\ell,\varepsilon,\iota}|w_{\tilde\vartheta}\partial f^{\ell+1}_\iota|^{p-1}\big[\partial_t + v\cdot\nabla_x - \iota\nabla_x\phi^\ell\cdot\nabla_v + \nu_{\phi^\ell, w_{\tilde\vartheta}\iota}\big]|w_{\tilde\vartheta}\partial f^{\ell+1}_\iota|\\ &= w^p_{\tilde\vartheta}\alpha^{\beta p}_{f^\ell,\varepsilon,\iota}|\partial f^{\ell+1}_\iota|^{p-1}\mathcal{G}^{\ell+1}_\iota.\end{aligned} \tag{4.44}$$

From 4.15, 4.43, 4.18, and 4.23

$$\begin{aligned}|\mathcal{G}| \lesssim\ & |\nabla_x f^{\ell+1}| + |\nabla^2\phi^\ell||\nabla_v f^{\ell+1}|\\ &+ |\Gamma_{\text{gain}}(\partial f^\ell, f^\ell)| + |\Gamma_{\text{gain}}(f^\ell, \partial f^\ell)| + |\Gamma_{\text{loss}}(f^{\ell+1}, \partial f^\ell)|\\ &+ |\Gamma_{\text{loss}}(\partial f^{\ell+1}, f^\ell)| + |K\partial f^\ell| + |f^{\ell+1}| + |\Gamma_{v,\text{gain}}(f^\ell, f^\ell)| + |\Gamma_{v,\text{loss}}(f^{\ell+1}, f^\ell)|\\ &+ |K_v f^\ell| + w_\vartheta(v)^{-1/2}(|\nabla\phi^\ell| + |\nabla^2\phi^\ell|)(1 + \|w_\vartheta f^{\ell+1}\|_\infty).\end{aligned} \tag{4.45}$$

Now we apply Lemma 2.3 to 4.44 to both $f^{\ell+1}_+$ and $f^{\ell+1}_-$ separately and add them together to obtain

$$\begin{aligned}&\|w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\partial f^{\ell+1}(t)\|^p_p + \int_0^t \|\nu^{1/p}_{\phi^\ell, w_{\tilde\vartheta}} w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\partial f^{\ell+1}\|^p_p + \int_0^t |w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\partial f^{\ell+1}|^p_{p,+}\\ &\le \|w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\partial f(0)\|^p_p\\ &\quad + \underbrace{\int_0^t |w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\partial f^{\ell+1}|^p_{p,-}}_{4.46_{\gamma_-}} + \underbrace{\int_0^t\iint_{\Omega\times\mathbb{R}^3} p|\alpha^{\beta p}_{f^\ell,\varepsilon} w^p_{\tilde\vartheta}(\partial f^{\ell+1})^{p-1}|\mathcal{G}^{\ell+1}|}_{4.46_{\mathcal{G}}}.\end{aligned} \tag{4.46}$$

First we consider 4.46$_{\mathcal{G}}$. Directly, the contribution of $|\nabla_x f^{\ell+1}| + |\nabla^2\phi^\ell||\nabla_v f^{\ell+1}|$ of 4.45 in 4.46$_{\mathcal{G}}$ is bounded by

$$(1+\sup_{0\le s\le t}\|\nabla^2\phi^\ell\|_\infty)\int_0^t \|w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\partial f^{\ell+1}\|_p^p. \tag{4.47}$$

From 4.19, 4.24, and 4.25, the contribution of $|\Gamma_{\text{gain}}(f^\ell,\partial f^\ell)| + |\Gamma_{\text{gain}}(\partial f^\ell, f^\ell)| + |\Gamma_{\text{loss}}(f^{\ell+1},\partial f^l)| + |K\partial f^\ell|$ of 4.45 in 4.46$_{\mathcal{G}}$ is bounded by

$$\begin{aligned}&(1+\sup_{0\le s\le t}\|w_\vartheta f^\ell(s)\|_\infty + \sup_{0\le s\le t}\|w_\vartheta f^{\ell+1}(s)\|_\infty)\\ &\times\int_0^t\iint_{\Omega\times\mathbb{R}^3}|\alpha^\beta_{f^\ell,\varepsilon}w_{\tilde\vartheta}\partial f^{\ell+1}(v)|^{p-1}\int_{\mathbb{R}^3}|\alpha_{f^\ell,\varepsilon}(v)^\beta\mathbf{k}_\varrho(v,u)w_{\tilde\vartheta}(v)\partial f^\ell(u)|\mathrm{d}u\mathrm{d}v\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{4.48}$$

The estimate of 4.48 is carried out in *Step 3*.

From 4.25, the contribution of $|\Gamma_{\text{loss}}(\partial f^{\ell+1}, f^\ell)|$ of 4.45 in 4.46$_{\mathcal{G}}$ is bounded by

$$\sup_{0\le s\le t}\|w_\vartheta f^\ell(s)\|_\infty\int_0^t\|\nu^{1/p}_{\phi_f,w_{\tilde\vartheta}}w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\partial f^{\ell+1}\|_p^p. \tag{4.49}$$

For the $|f^{\ell+1}|$ contribution of 4.45 in 4.46$_{\mathcal{G}}$, we bound

$$\begin{aligned}&\int_0^t\iint_{\Omega\times\mathbb{R}^3}p|w^p_{\tilde\vartheta}\alpha^{\beta p}_{f^\ell,\varepsilon}(\partial f^{\ell+1})^{p-1}||f^{\ell+1}|\mathrm{d}x\mathrm{d}v\mathrm{d}s\\ \lesssim&\int_0^t\iint_{\Omega\times\mathbb{R}^3}|\nu^{1/p}_{\phi_f,w_{\tilde\vartheta}}w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\partial f^{\ell+1}|^{p-1}|w_{\tilde\vartheta}f^{\ell+1}|\frac{|\alpha_{f^\ell,\varepsilon}(s,x,v)|^\beta}{\langle v\rangle^{(p-1)/p}}\mathrm{d}x\mathrm{d}v\mathrm{d}s\\ \lesssim& o(1)\int_0^t\iint_{\Omega\times\mathbb{R}^3}|\nu^{1/p}_{\phi_f,w_{\tilde\vartheta}}w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\partial f^{\ell+1}|^p+(1+\delta_1/\Lambda_1)\int_0^t\iint_{\Omega\times\mathbb{R}^3}|w_{\tilde\vartheta}f^{\ell+1}|^p.\end{aligned} \tag{4.50}$$

Here we have used the fact that, from 1.24 and 4.3

$$\begin{aligned}&|\alpha_{f^\ell,\varepsilon}(s,x,v)|\\ \le&\, 2(\mathbf{1}_{s+1\ge t_{\mathbf{b}}(s,x,v)}|v_{\mathbf{b}}(s,x,v)|+\mathbf{1}_{s\le t_{\mathbf{b}}(s,x,v)+1})\\ \lesssim&\, 1+|v|+\int_{-1}^0|\nabla\phi^\ell(\tau,X(\tau;s,x,v))|\mathrm{d}\tau+\int_0^s|\nabla\phi^\ell(\tau,X(\tau;s,x,v))|\mathrm{d}\tau\\ \lesssim&\,(1+\|w_\vartheta f_0\|_\infty+\delta_1/\Lambda_1)\langle v\rangle,\end{aligned} \tag{4.51}$$

and from 4.1, $\frac{|\alpha_{f^\ell,\varepsilon}(s,x,v)|^\beta}{\langle v\rangle^{(p-1)/p}}\lesssim(1+\delta_1/\Lambda_1)\times\frac{\langle v\rangle^\beta}{\langle v\rangle^{(p-1)/p}}\lesssim(1+\delta_1/\Lambda_1)$.

From 4.26, the contribution of $|\Gamma_{v,\text{loss}}(f^{\ell+1},f^\ell)|$ of 4.45 in 4.46$_{\mathcal{G}}$ is bounded by

$$\begin{aligned}&\|w_\vartheta f^{\ell+1}\|_\infty\int_0^t\iint_{\Omega\times\mathbb{R}^3}p|w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\partial f^{\ell+1}|^{p-1}\\ &\qquad\times|\alpha_{f^\ell,\varepsilon}(v)|^\beta\langle v\rangle w_{\tilde\vartheta}(v)w_\vartheta(v)^{-1}\|w_\vartheta f^\ell(s,x,\cdot)\|_{L^p(\mathbb{R}^3)}\\ \lesssim&\,\|w_\vartheta f^{\ell+1}\|_\infty\left\{\int_0^t\iint_{\Omega\times\mathbb{R}^3}|\alpha^\beta_{f^\ell,\varepsilon}\partial f^{\ell+1}|^p+\int_0^t\iint_{\Omega\times\mathbb{R}^3}|w_\vartheta f^\ell|^p\right\},\end{aligned} \tag{4.52}$$

where we have used, from 4.51, $|\alpha_{f^\ell,\varepsilon}(v)|^\beta\langle v\rangle w_{\tilde\vartheta}(v)w_\vartheta(v)^{-1}\lesssim 1$.

From 4.19 and 4.27, the contribution of $|\Gamma_{v,\mathrm{gain}}|$ and $|K_v f|$ in $4.46_{\mathcal{G}}$ is bounded by

$$
\begin{aligned}
&(1+\sup_{0\le s\le t}\|w_\vartheta f^\ell\|_\infty)\int_0^t\iint_{\Omega\times\mathbb{R}^3}|\alpha^{\beta p}_{f^\ell,\varepsilon}(w_{\tilde\vartheta}\partial f^{\ell+1}(v))^{p-1}|\int_{\mathbb{R}^3}\mathbf{k}_\varrho(v,u)\frac{w_{\tilde\vartheta}(v)}{w_{\tilde\vartheta}(u)}|f^\ell(u)|\\
\lesssim\ & o(1)\int_0^t\iint_{\Omega\times\mathbb{R}^3}|\nu_{\phi^\ell}^{1/p}\alpha^\beta_{f^\ell,\varepsilon}\partial f^{\ell+1}|^p+(1+\sup_{0\le s\le t}\|w_\vartheta f^\ell(s)\|_\infty)\int_0^t\iint_{\Omega\times\mathbb{R}^3}|w_{\tilde\vartheta}f^\ell|^p,
\end{aligned}
\tag{4.53}
$$

where we have used, for $1/p+1/p^*=1$ and $0<\tilde\varrho\ll\varrho$, from 3.32, 3.33,

$$
\begin{aligned}
&\int_{\mathbb{R}^3}|\alpha_{f^\ell,\varepsilon}(v)^{\beta p}(w_{\tilde\vartheta}\partial f^{\ell+1}(v))^{p-1}|\int_{\mathbb{R}^3}\mathbf{k}_{\tilde\varrho}(v,u)w_{\tilde\vartheta}(u)|f^\ell(u)|\mathrm{d}u\mathrm{d}v\\
\lesssim\ &\int_{\mathbb{R}^3}|\alpha_{f^\ell,\varepsilon}(v)^{\beta p}(w_{\tilde\vartheta}\partial f^{\ell+1}(v))^{p-1}|\int_{\mathbb{R}^3}\mathbf{k}_{\tilde\varrho}(v,u)^{1/p^*}\mathbf{k}_{\tilde\varrho}(v,u)^{1/p}|w_{\tilde\vartheta}f^\ell(u)|\mathrm{d}u\mathrm{d}v\\
\lesssim\ &\int_{\mathbb{R}^3}\frac{|\alpha_{f^\ell,\varepsilon}(v)|^\beta}{\langle v\rangle^{\frac{p-1}{p}}}|\langle v\rangle^{1/p}w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\partial f^{\ell+1}(v)|^{p-1}\\
&\quad\times\left(\int_{\mathbb{R}^3}\mathbf{k}_{\tilde\varrho}(v,u)\mathrm{d}u\right)^{1/p^*}\left(\int_{\mathbb{R}^3}\mathbf{k}_{\tilde\varrho}(v,u)|w_{\tilde\vartheta}f^\ell(u)|^p\mathrm{d}u\right)^{1/p}\mathrm{d}v\\
\lesssim\ &\left(\int_{\mathbb{R}^3}|\langle v\rangle^{1/p}w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\partial f^{\ell+1}(v)|^p\mathrm{d}v\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathbf{k}_{\tilde\varrho}(v,u)|w_{\tilde\vartheta}f^\ell(u)|^p\mathrm{d}u\mathrm{d}v\right)^{\frac1p}\\
\lesssim\ &\left(\int_{\mathbb{R}^3}|\nu_{\phi^\ell}^{1/p}w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\partial f^{\ell+1}|^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^3}|w_{\tilde\vartheta}f^\ell|^p\right)^{\frac1p}.
\end{aligned}
$$

Note that from the standard elliptic estimate and 1.16,

$$
\|\phi^\ell(t)\|_{W^{2,p}(\Omega)}\lesssim\left\|\int_{\mathbb{R}^3}(f^\ell_+-f^\ell_-)(t,x,v)\sqrt{\mu(v)}\mathrm{d}v\right\|_{L^p(\Omega)}\lesssim\|f^\ell(t)\|_{L^p(\Omega\times\mathbb{R}^3)}. \tag{4.54}
$$

Then from 4.54 we bound the part of $w_{\tilde\vartheta}(v)^{-1/2}(|\nabla\phi^\ell|+|\nabla^2\phi^\ell|)(1+\|w_\vartheta f^{\ell+1}\|_\infty)$ of 4.45 in 4.44 by

$$
\begin{aligned}
&(1+\|w_\vartheta f^{\ell+1}\|_\infty)\int_0^t\iint_{\Omega\times\mathbb{R}^3}p|w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\partial f^{\ell+1}|^{p-1}\frac{|\alpha_{f^\ell,\varepsilon}(v)|^\beta}{w_{\tilde\vartheta}(v)^{1/2}}(|\nabla\phi^\ell|+|\nabla^2\phi^\ell|)\\
\lesssim\ &(1+\|w_\vartheta f^{\ell+1}\|_\infty)\int_0^t\iint_{\Omega\times\mathbb{R}^3}|w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\partial f^{\ell+1}|^{p-1}w_{\tilde\vartheta}^{-1/4}(|\nabla\phi^\ell|+|\nabla^2\phi^\ell|)\\
\lesssim\ &(1+\|w_\vartheta f^{\ell+1}\|_\infty)\Bigg\{o(1)\int_0^t\iint_{\Omega\times\mathbb{R}^3}|w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\partial f^{\ell+1}|^p\\
&\qquad+\int_0^t\iint_{\Omega\times\mathbb{R}^3}|w_{\tilde\vartheta}\alpha^\beta_{f^{\ell-1},\varepsilon}\partial f^\ell|^p+\int_0^t\int_\Omega\|\phi^\ell\|^p_{W^{2,p}}\int_{\mathbb{R}^3}w_{\tilde\vartheta}^{-p/4}\Bigg\}\\
\lesssim\ &o(1)(1+\|w_\vartheta f^{\ell+1}\|_\infty)\int_0^t\iint_{\Omega\times\mathbb{R}^3}|w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\partial f^{\ell+1}|^p\\
&+\int_0^t\iint_{\Omega\times\mathbb{R}^3}|w_{\tilde\vartheta}\alpha^\beta_{f^{\ell-1},\varepsilon}\partial f^\ell|^p+\int_0^t\iint_{\Omega\times\mathbb{R}^3}|f^\ell|^p.
\end{aligned}
\tag{4.55}
$$

where we have used, from 4.51, $\alpha_{f^\ell,\varepsilon}(v)^\beta w_\vartheta(v)^{-1/2}\lesssim w_\vartheta(v)^{-1/4}$.

Step 3. We focus on 4.48. With $N>0$, we split the u-integration of 4.48 into the integrations over $\{|u|\le N\}$ and $\{|u|\ge N\}$.

For $\{|u|\ge N\}$ and $0<\tilde{\varrho}\ll\varrho$, by Holder inequality with $\frac{1}{p}+\frac{1}{p^*}=1$

$$\begin{aligned}
&\int_{|u|\ge N}|\alpha^{\beta}_{f^{\ell},\varepsilon}(v)\mathbf{k}_{\tilde{\varrho}}(v,u)\partial f^{\ell}(u)|\\
&\le|\alpha^{\beta}_{f^{\ell},\varepsilon}(v)|\Big(\sum_{\iota=\pm}\int_{|u|\ge N}\mathbf{k}_{\tilde{\varrho}}(v,u)\frac{1}{\alpha_{f^{\ell-1},\varepsilon,\iota}(u)^{\beta p^*}}\Big)^{1/p^*}\\
&\qquad\times\Big(\int_{|u|\ge N}\mathbf{k}_{\tilde{\varrho}}(v,u)|\alpha^{\beta}_{f^{\ell-1},\varepsilon}\partial f^{\ell}(u)|^p\Big)^{1/p}\\
&\lesssim\alpha^{\beta}_{f^{\ell},\varepsilon}(v)\Big(\int_{|u|\ge N}\mathbf{k}_{\tilde{\varrho}}(v,u)|\alpha^{\beta}_{f^{\ell-1},\varepsilon}\partial f^{\ell}(u)|^p\mathrm{d}u\Big)^{1/p},
\end{aligned}\tag{4.56}$$

where have used Proposition 4 with $\beta q<\frac{p-1}{p}\frac{p}{p-1}=1$ from 4.1.

Then the contribution of $\{|u|\ge N\}$ in 4.48 is bounded by

$$\begin{aligned}
&\int_0^t\int_\Omega\int_{v\in\mathbb{R}^3}|\nu_{\phi_f}^{1/p}w_{\tilde{\vartheta}}\alpha^{\beta}_{f^{\ell},\varepsilon}\partial f^{\ell+1}(v)|^{p-1}\frac{|\alpha_{f^{\ell},\varepsilon}(v)|^{\beta}}{\langle v\rangle^{\frac{p-1}{p}}}\\
&\qquad\times\int_{|u|\ge N}\mathbf{k}_{\varrho}(v,u)\frac{w_{\tilde{\vartheta}}(v)}{w_{\tilde{\vartheta}}(u)}|w_{\tilde{\vartheta}}\partial f^{\ell}(u)|\mathrm{d}u\mathrm{d}v\mathrm{d}x\mathrm{d}s\\
&\le\int_0^t\int_\Omega\Big(\int_v|\nu_{\phi_f}^{1/p}w_{\tilde{\vartheta}}\alpha^{\beta}_{f^{\ell},\varepsilon}\partial f^{\ell+1}(v)|^p\Big)^{1/q}\\
&\qquad\times\Big(\int_{|u|\ge N}|w_{\tilde{\vartheta}}\alpha^{\beta}_{f^{\ell-1},\varepsilon}\partial f^{\ell}(u)|^p\int_v\mathbf{k}_{\tilde{\varrho}}(v,u)\Big)^{1/p}\\
&\lesssim o(1)\int_0^t\|\nu_{\phi_f}^{1/p}w_{\tilde{\vartheta}}\alpha^{\beta}_{f^{\ell},\varepsilon}\partial f^{\ell+1}(s)\|_p^p\mathrm{d}s+\int_0^t\|w_{\tilde{\vartheta}}\alpha^{\beta}_{f^{\ell-1},\varepsilon}\partial f^{\ell}(s)\|_p^p\mathrm{d}s,
\end{aligned}\tag{4.57}$$

where we have used, from 4.51, $\frac{|\alpha_{f^{\ell},\varepsilon}(v)|^{\beta}}{\langle v\rangle^{\frac{p-1}{p}}}\lesssim 1$ for β in 4.1, 3.32, and 3.33.

The contribution of $\{|u|\le N\}$ in 4.48 is bounded by, from the Hölder inequality,

$$\begin{aligned}
&\int_0^t\int_\Omega\int_{\mathbb{R}^3}|\nu_{\phi_f}^{1/p}w_{\tilde{\vartheta}}\alpha^{\beta}_{f^{\ell},\varepsilon}\partial f^{\ell+1}(v)|^{p-1}\\
&\qquad\times\int_{|u|\le N}\sum_{\iota=\pm}\mathbf{k}_{\varrho}(v,u)\frac{w_{\tilde{\vartheta}}(v)}{w_{\tilde{\vartheta}}(u)}\frac{|\alpha_{f^{\ell},\varepsilon}(v)|^{\beta}|w_{\tilde{\vartheta}}\alpha^{\beta}_{f^{\ell-1},\varepsilon}\partial f^{\ell}(u)|}{\langle v\rangle^{(p-1)/p}\alpha_{f^{\ell-1},\varepsilon,\iota}(u)^{\beta}}\mathrm{d}u\mathrm{d}v\mathrm{d}x\mathrm{d}s\\
&\le\int_0^t\|\nu_{\phi_f}^{1/p}w_{\tilde{\vartheta}}\alpha^{\beta}_{f^{\ell},\varepsilon}\partial f^{\ell+1}(s)\|_p^{p-1}\\
&\qquad\times\Big[\int_\Omega\int_{\mathbb{R}^3}\Big(\int_{|u|\le N}\sum_{\iota=\pm}\mathbf{k}_{\tilde{\varrho}}(v,u)\frac{|w_{\tilde{\vartheta}}\alpha^{\beta}_{f^{\ell-1},\varepsilon}\partial f^{\ell}(u)|}{\alpha_{f^{\ell-1},\varepsilon,\iota}(u)^{\beta}}\mathrm{d}u\Big)^p\mathrm{d}v\mathrm{d}x\Big]^{1/p}\mathrm{d}s.
\end{aligned}\tag{4.58}$$

where we have used 3.32 and the fact $|\alpha_{f^{\ell},\varepsilon}|^{\beta}/\langle v\rangle^{\frac{p-1}{p}}\lesssim 1$ from 4.51 and 4.1.

By the Hölder inequality, we bound an underlined u-integration inside 4.58 as

$$\|w_{\tilde{\vartheta}}\alpha^{\beta}_{f^{\ell-1},\varepsilon}\partial f^{\ell}(\cdot)\|_{L^p(\mathbb{R}^3)}\times\left(\sum_{\iota=\pm}\int_{\mathbb{R}^3}\frac{e^{-p^*\tilde{\varrho}|v-u|^2}}{|v-u|^{p^*}}\frac{\mathbf{1}_{|u|\le N}}{\alpha_{f^{\ell-1},\varepsilon,\iota}(u)^{\beta p^*}}\mathrm{d}u\right)^{1/q},\tag{4.59}$$

where $1/p+1/p^*=1$.

It is important to note that for $\iota=+$ or $-$,

$$\left(\int_{\mathbb{R}^3}\frac{e^{-p^*\tilde{\varrho}|v-u|^2}}{|v-u|^{p^*}}\frac{\mathbf{1}_{|u|\le N}}{\alpha_{f^{\ell-1},\varepsilon,\iota}(u)^{\beta p^*}}\mathrm{d}u\right)^{1/p^*}\le\left|\frac{1}{|\cdot|^{p^*}}*\frac{\mathbf{1}_{|\cdot|\le N}}{\alpha_{f^{\ell-1},\varepsilon,\iota}(\cdot)^{p^*\beta}}\right|^{1/p^*}.\tag{4.60}$$

By the Hardy-Littlewood-Sobolev inequality with

$$1+\frac{1}{p/p^*}=\frac{1}{3/p^*}+\frac{1}{\frac{3}{2}\frac{p-1}{p}},$$

we have

$$\begin{aligned}\left\|\left|\frac{1}{|\cdot|^{p^*}}*\frac{\mathbf{1}_{|\cdot|\le N}}{\alpha_{f^{\ell-1},\varepsilon,\iota}(\cdot)^{p^*\beta}}\right|^{1/p^*}\right\|_{L^p(\mathbb{R}^3)}&=\left\|\frac{1}{|\cdot|^{p^*}}*\frac{\mathbf{1}_{|\cdot|\le N}}{\alpha_{f^{\ell-1},\varepsilon,\iota}(\cdot)^{p^*\beta}}\right\|^{1/p^*}_{L^{p/p^*}(\mathbb{R}^3)}\\&\lesssim\left\|\frac{\mathbf{1}_{|\cdot|\le N}}{\alpha_{f^{\ell-1},\varepsilon,\iota}(\cdot)^{p^*\beta}}\right\|^{1/p^*}_{L^{\frac{3(p-1)}{2p}}(\mathbb{R}^3)}\\&\lesssim\left(\int_{\mathbb{R}^3}\frac{\mathbf{1}_{|v|\le N}}{\alpha_{f^{\ell-1},\varepsilon,\iota}(v)^{\frac{p}{p-1}\beta\frac{3(p-1)}{2p}}}\mathrm{d}v\right)^{\frac{2p}{3(p-1)}\frac{p-1}{p}}\\&=\left(\int_{\mathbb{R}^3}\frac{\mathbf{1}_{|v|\le N}}{\alpha_{f^{\ell-1},\varepsilon,\iota}(v)^{3\beta/2}}\mathrm{d}v\right)^{2/3}.\end{aligned}\tag{4.61}$$

For $3<p<6$, we have $\frac{3}{2}\frac{p-2}{p}<1$ and $\frac{2}{3}<\frac{p-1}{p}$. Importantly from 4.1 we have $\frac{3\beta}{2}<1$. Now we apply 4.8 in Proposition 4 to conclude that

$$\left(\int_{\mathbb{R}^3}\frac{\mathbf{1}_{|v|\le M}}{\alpha_{f^{\ell-1},\varepsilon,\iota}(v)^{3\beta/2}}\mathrm{d}v\right)^{2/3}\lesssim_{p,\beta,M,\Omega}1.$$

Finally from 4.58, 4.59, 4.60, 4.61, and 4.57 we bound

$$\begin{aligned}(4.48)\lesssim&o(1)\int_0^t\|\nu_\phi^{1/p}w_{\tilde{\vartheta}}\alpha^{\beta}_{f^{\ell},\varepsilon}\partial f^{\ell+1}\|_p^p\\&+(1+\sup_{0\le s\le t}\|w_\vartheta f^{\ell}(s)\|_\infty+\sup_{0\le s\le t}\|w_\vartheta f^{\ell+1}(s)\|_\infty)\int_0^t\|w_{\tilde{\vartheta}}\alpha^{\beta}_{f^{\ell-1},\varepsilon}\partial f^{\ell}\|_p^p.\end{aligned}\tag{4.62}$$

Collecting terms from 4.47, 4.48, 4.49, 4.50, 4.52, 4.53, 4.55, 4.57, and 4.62 we have

$$\begin{aligned}4.46_{\mathcal{G}}\lesssim&o(1)\int_0^t\|\nu_{\phi^\ell}^{1/p}w_{\tilde{\vartheta}}\alpha^{\beta}_{f^{\ell},\varepsilon}\partial f^{\ell+1}\|_p^p\\&+(1+\sup_{0\le s\le t}\|\nabla^2\phi^\ell(s)\|_\infty)\int_0^t\|w_{\tilde{\vartheta}}\alpha^{\beta}_{f^{\ell},\varepsilon}\partial f^{\ell+1}\|_p^p\\&+(1+\sup_{0\le s\le t}\|w_\vartheta f^{\ell}(s)\|_\infty+\delta_1/\Lambda_1)\int_0^t\|w_{\tilde{\vartheta}}f^{\ell+1}\|_p^p\end{aligned}\tag{4.63}$$

$$+(1+\sup_{0\le s\le t}\|w_\vartheta f^\ell(s)\|_\infty+\sup_{0\le s\le t}\|w_\vartheta f^{\ell+1}(s)\|_\infty)$$
$$\times\int_0^t(\|w_{\tilde\vartheta}f^\ell\|_p^p+\|w_{\tilde\vartheta}\alpha^\beta_{f^{\ell-1},\varepsilon}\partial f^\ell\|_p^p).$$

Step 4. We focus on 4.46_{γ_-}. From 4.28 and 4.29,

$$\begin{aligned}&\int_{n(x)\cdot v<0}|n(x)\cdot v|^{\beta p}|w_{\tilde\vartheta}\nabla_{x,v}f^{\ell+1}(t,x,v)|^p|n(x)\cdot v|\mathrm{d}v\\&\lesssim\int_{n(x)\cdot v<0}\langle v\rangle^p\mu(v)^{\frac{p}{2}}w_{\tilde\vartheta}^p\Big(|n(x)\cdot v|^{\beta p+1}+|n(x)\cdot v|^{(\beta-1)p+1}\Big)\times|4.29|^p\mathrm{d}v.\end{aligned}\tag{4.64}$$

Note that for $0<\tilde\vartheta\ll_p 1$ we have $\mu(v)^{\frac{p}{2}}w_{\tilde\vartheta}^p\lesssim e^{C|v|^2}$ for some $C>0$ when $|v|\gg 1$.

On the other hand, from 4.1, we have

$$(\beta-1)p+1>\frac{p-2}{p}p-p+1=-1,\quad |n(x)\cdot v|^{(\beta-1)p+1}\in L^1_{loc}(\mathbb{R}^3).\tag{4.65}$$

Now we bound $|4.29|^p$. For the first line of 4.29, we split the u-integration into $\gamma_+^\varepsilon(x)\cup\gamma_+(x)\backslash\gamma_+^\varepsilon(x)$ where ε is small but satisfies 2.16. By the Hölder inequality

$$\begin{aligned}&\Big\{\sum_{\iota=\pm}\int_{n(x)\cdot u>0}|w_{\tilde\vartheta}\alpha^\beta_{f^{\ell-1},\varepsilon}\nabla_{x,v}f^\ell(s,x,u)|\{w_{\tilde\vartheta}\alpha_{f^\ell,\varepsilon,\iota}(u)\}^{-\beta}\langle u\rangle\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u\Big\}^p\\&\lesssim\Big\{\int_{\gamma_+^\varepsilon(x)}|w_{\tilde\vartheta}\alpha^\beta_{f^{\ell-1},\varepsilon}\nabla_{x,v}f^\ell(s,x,u)|^p\{n(x)\cdot u\}\mathrm{d}u\Big\}\\&\quad\times\underbrace{\Big\{\sum_{\iota=\pm}\int_{\gamma_+^\varepsilon(x)}\{w_{\tilde\vartheta}\alpha_{f^{\ell-1},\varepsilon,\iota}(u)\}^{-\beta p^*}|n(x)\cdot u|\mu^{\frac{q}{4}}\mathrm{d}u\Big\}^{p/p^*}}\\&+\Big\{\int_{\gamma_+(x)\backslash\gamma_+^\varepsilon(x)}|w_{\tilde\vartheta}\alpha^\beta_{f^{\ell-1},\varepsilon}\nabla_{x,v}f^\ell(s,x,u)|^p\mu^{\frac{p}{8}}\{n(x)\cdot u\}\mathrm{d}u\Big\}\\&\quad\times\underbrace{\Big\{\sum_{\iota=\pm}\int_{\gamma_+(x)\backslash\gamma_+^\varepsilon(x)}\{w_{\tilde\vartheta}\alpha_{f^{\ell-1},\varepsilon,\iota}(s,x,u)\}^{-\beta p^*}|n(x)\cdot u|\mu^{\frac{p^*}{8}}\mathrm{d}u\Big\}^{p/p^*}}\quad,\ p^*:=\frac{p}{p-1}.\end{aligned}\tag{4.66}$$

Note that $\alpha_{f^\ell,\varepsilon,\iota}(s,x,u)\neq|n(x)\cdot u|$ for $(x,u)\in\gamma_+$ in general. From 4.1, $\beta p^*<1$. From 4.8 and 4.9 with $v=0$, we have $\alpha^{-\beta p^*}_{f^{\ell-1},\varepsilon,\iota}|n(x)\cdot u|\lesssim\alpha^{-\beta p^*}_{f^{\ell-1},\varepsilon,\iota}\in L^1_{loc}(\{u\in\mathbb{R}^3\})$. Since $\mathbf{1}_{\gamma_+^\varepsilon(x)}(v)\downarrow 0$ almost everywhere in $\mathbb{R}^3$ as $\varepsilon\downarrow 0$, by the dominant convergence theorem, for 4.4, we choose $\varepsilon:=\frac{2\delta_1}{\Lambda_1}\ll_\Omega 1$

$$\begin{aligned}4.66\lesssim o(1)&\int_{\gamma_+^\varepsilon(x)}|w_{\tilde\vartheta}\alpha^\beta_{f^{\ell-1},\varepsilon}\nabla_{x,v}f^\ell(s,x,u)|^p\{n(x)\cdot u\}\mathrm{d}u\\&+\int_{\gamma_+(x)\backslash\gamma_+^\varepsilon(x)}|w_{\tilde\vartheta}\alpha^\beta_{f^{\ell-1},\varepsilon}\nabla_{x,v}f^\ell(s,x,u)|^p\mu(u)^{p/8}\{n(x)\cdot u\}\mathrm{d}u.\end{aligned}\tag{4.67}$$

Now applying Lemma 2.2 and 4.44 to f_+^ℓ and f_-^ℓ separately and adding them together, the last term of 4.67 has a bound as

$$\int_0^t\int_{\partial\Omega}\int_{\gamma_+(x)\setminus\gamma_+^\varepsilon(x)}|w_{\tilde\vartheta}\alpha^\beta_{f^{\ell-1},\varepsilon}\nabla_{x,v}f^\ell(s,x,u)|^p\mu(u)^{p/8}\{n(x)\cdot u\}\mathrm{d}u\mathrm{d}S_x\mathrm{d}s \tag{4.68}$$
$$\lesssim \|w_{\tilde\vartheta}\alpha^\beta_{f_0,\varepsilon}\nabla_{x,v}f(0)\mu^{1/8}\|_p^p+\int_0^t\|w_{\tilde\vartheta}\alpha^\beta_{f^{\ell-1},\varepsilon}\nabla_{x,v}f^\ell\|_p^p+4.69,$$

where, from 4.41, 4.42,

$$\int_0^t\iint_{\Omega\times\mathbb{R}^3}|[\partial_t+v\cdot\nabla_x-q\nabla_x\phi^{\ell-1}\cdot\nabla_v+\nu_{\phi^\ell,w_{\tilde\vartheta}}](w_{\tilde\vartheta}\mu^{1/8}\alpha^\beta_{f^{\ell-1},\varepsilon}\nabla_{x,v}f^\ell)^p| \tag{4.69}$$
$$\le\int_0^t\iint_{\Omega\times\mathbb{R}^3}p|\alpha^{\beta p}_{f^{\ell-1},\varepsilon}(\nabla_{x,v}f^\ell)^{p-1}||w_{\tilde\vartheta}\mu^{1/8}|^p\big|\mathcal{G}^\ell\big| \tag{4.70}$$
$$+\int_0^t\iint_{\Omega\times\mathbb{R}^3}|\nabla_x\phi^\ell|\mu^{0+}|\alpha^\beta_{f^{\ell-1},\varepsilon}\nabla_{x,v}f^\ell|^p. \tag{4.71}$$

Clearly $4.70\lesssim 4.63|_{\ell\longleftrightarrow\ell-1}$. And, from 4.3,

$$4.71\lesssim\delta_1\int_0^t\|w_{\tilde\vartheta}\alpha^\beta_{f^{\ell-1},\varepsilon}\nabla_{x,v}f^\ell\|_p^p.$$

Now we consider the third term of 4.29. From the trace theorem $W^{1,p}(\Omega)\to W^{1-\frac1p,p}(\partial\Omega)$ and 4.54

$$\|\nabla\phi^m\|_{L^p(\partial\Omega)}\lesssim\|\nabla\phi^m\|_{W^{1-\frac1p,p}(\partial\Omega)}\lesssim\|\nabla\phi^m\|_{W^{1,p}(\Omega)}\lesssim\|w_{\tilde\vartheta}f^m\|_{L^p(\Omega\times\mathbb{R}^3)}. \tag{4.72}$$

Then

$$\begin{aligned}&\int_{\partial\Omega}\Big\{\int_{n(x)\cdot u>0}(\langle u\rangle(|f^{\ell+1}|+|f^\ell|)+\mu(u)^{\frac14})\\&\qquad\times(|\nabla_x\phi^\ell|+|\nabla_x\phi^{\ell-1}|)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u\Big\}^p\mathrm{d}S_x\\&\lesssim(1+\|w_\vartheta f^\ell\|_\infty+\|w_\vartheta f^{\ell+1}\|_\infty)\left(\sum_{m=\ell}^{\ell+1}\|\nabla\phi^m\|^p_{L^p(\partial\Omega)}\right)\\&\lesssim(1+\|w_\vartheta f^\ell\|_\infty+\|w_\vartheta f^{\ell+1}\|_\infty)(\|w_{\tilde\vartheta}f^\ell\|^p_{L^p(\Omega\times\mathbb{R}^3)}+\|w_{\tilde\vartheta}f^{\ell+1}\|^p_{L^p(\Omega\times\mathbb{R}^3)}).\end{aligned} \tag{4.73}$$

For the second term of 4.29, by the Hölder inequality with $\frac1p+\frac1q=1$ for $3<p<6$,

$$\begin{aligned}&\Big\{\int_{n\cdot u>0}\Big(\langle u\rangle(|f|^\ell+|f|^{\ell+1})+(1+\|w_\vartheta f^\ell\|_\infty+\|w_\vartheta f^{\ell-1}\|_\infty)\\&\times\int_{\mathbb{R}^3}\frac{\mathbf{k}_\varrho(u,u')^{1/q}}{|n\cdot u'|^{1/p}}\mathbf{k}_\varrho(u,u')^{1/p}(|f^\ell(u')|+f^{\ell-1}(u'))|n\cdot u'|^{1/p}\mathrm{d}u'\Big)\sqrt{\mu}\{n\cdot u\}\mathrm{d}u\Big\}^p\\\lesssim&\left(\int_{n\cdot u>0}(|f^{\ell+1}|^p+|f^\ell|^p)\{n\cdot u\}\sqrt{\mu}\mathrm{d}u\right)^p+(1+\|w_\vartheta f^\ell\|_\infty+\|w_\vartheta f^{\ell+1}\|_\infty)\\&\times\int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}\mathbf{k}_\varrho(u,u')|n\cdot u'|^{-q/p}\mathrm{d}u'\Big)^{p/q}\\&\times\int_{\mathbb{R}^3}\mathbf{k}_\varrho(u,u')(|f^\ell(u')|^p+|f^{\ell+1}(u')|^p)|n\cdot u'|\mathrm{d}u'\mathrm{d}u\\\lesssim&(1+\|w_\vartheta f^\ell\|_\infty+\|w_\vartheta f^{\ell+1}\|_\infty)\int_{n\cdot u>0}(|f^{\ell+1}|^p+|f^\ell|^p)\{n\cdot u\}\mathrm{d}u\end{aligned} \tag{4.74}$$

$$\lesssim \quad (1+\|w_\vartheta f^\ell\|_\infty+\|w_\vartheta f^{\ell+1}\|_\infty)(|w_\vartheta f^\ell\|_\infty+\|w_\vartheta f^{\ell+1}\|_\infty).$$

Collecting terms from 4.64, 4.67, 4.69, 4.73, and 4.74 we derive that

$$\begin{aligned}
&4.46_{\gamma_-}\\
\lesssim\ & \|w_{\tilde\vartheta}\alpha^\beta_{f_0,\varepsilon}\nabla_{x,v}f(0)\mu(u)^{1/8}\|_p^p\\
&+o(1)\left(\int_0^t |w_{\tilde\vartheta}\alpha^\beta_{f^{\ell-1},\varepsilon}\partial f^\ell|^p_{p,+}\right)+\int_0^t(1+\|w_\vartheta f^\ell\|_\infty+\|w_\vartheta f^{\ell+1}\|_\infty)\\
&+\left(o(1)+\sup_{0\le s\le t}\|w_\vartheta f^{\ell-1}(s)\|_\infty+\sup_{0\le s\le t}\|w_\vartheta f^\ell(s)\|_\infty+\sup_{0\le s\le t}\|w_\vartheta f^{\ell+1}(s)\|_\infty\right)\\
&\qquad\times\int_0^t\sum_{m=\ell-1}^{\ell}\|\nu_{\phi_f}^{1/p}w_{\tilde\vartheta}\alpha^\beta_{f^{m-1},\varepsilon}\partial f^m\|_p^p\\
&+\sum_{m=\ell}^{\ell+1}(1+\sup_{0\le s\le t}\|w_\vartheta f^m(s)\|_\infty+\sup_{0\le s\le t}\|\nabla^2\phi^{m-1}(s)\|_\infty)\int_0^t\|\alpha^\beta_{f^{m-1},\varepsilon}w_{\tilde\vartheta}\partial f^m\|_p^p\\
&+(1+\sum_{m=\ell}^{\ell+1}\sup_{0\le s\le t}\|w_\vartheta f^m(s)\|_\infty)\int_0^t\sum_{m=\ell}^{\ell+1}\left(\|w_{\tilde\vartheta}f^m\|_p^p+|w_{\tilde\vartheta}f^m|^p_{p,+}\right).
\end{aligned}\tag{4.75}$$

Step 5. From 4.40, 4.46, 4.63, 4.75 we have

$$\begin{aligned}
&\sup_{0\le s\le t}\mathcal{E}^{\ell+1}(s)\\
\le& C_0\max_{0\le m\le\ell+1}\sup_{0\le s\le t}\mathcal{E}^m(s)\Bigg\{\|w_{\tilde\vartheta}\alpha^\beta_{f_0,\varepsilon}\nabla_{x,v}f_0\|_p^p\\
&+t\left(1+\|w_\vartheta f^{\ell-1}\|_\infty+\sum_{m=\ell-1}^{\ell+1}\|w_\vartheta f^m l\|_\infty+\|\nabla^2\phi^{\ell-1}\|_\infty+\|\nabla^2\phi^\ell\|_\infty\right)\Bigg\}\\
&+o(1)\max_{0\le m\le\ell}\sup_{0\le s\le t}\mathcal{E}^m(s).
\end{aligned}\tag{4.76}$$

On the other hand, from Lemma 4.2,

$$\|\nabla^2\phi^\ell(t)\|_\infty+\|\nabla^2\phi^{\ell-1}(t)\|_\infty\lesssim[\mathcal{E}^\ell(t)+\mathcal{E}^{\ell-1}(t)]^{1/p}.\tag{4.77}$$

Therefore from 4.76, 4.77, and the induction hypothesis in 4.34, we first choose a small $o(1)$, then large $C\gg C_0$, and finally small $0<T^{**}\ll 1$ to conclude

$$\begin{aligned}
\sup_{0\le s\le t}\mathcal{E}^{\ell+1}(s)\le&\frac{C}{10}\|w_{\tilde\vartheta}\alpha^\beta_{f_0,\varepsilon}\nabla_{x,v}f_0\|_p^p+\frac{1}{10}\sup_{0\le s\le t}\sup_{m\le\ell}\mathcal{E}^m(s)\\
\le&C\{\|w_\vartheta f_0\|_\infty+\|w_{\tilde\vartheta}f_0\|_p^p+\|w_{\tilde\vartheta}\alpha^\beta_{f_0,\varepsilon}\nabla_{x,v}f_0\|_p^p\}.
\end{aligned}$$

This proves 4.34. □

5. $L^3_xL^{1+}_v$ bound of $\nabla_v f^\ell$.

Proposition 5. *Assume the inital condition satisfies 3.7, 4.5, and*

$$\|w_{\tilde\vartheta}\nabla_v f_0\|_{L^3_{x,v}}<\infty.\tag{5.1}$$

Then for $T^{**}\ll 1$, the sequence 3.3 satisfies

$$\sup_\ell\sup_{0\le t\le T^{**}}\|\nabla_v f^\ell(t)\|_{L^3_x(\Omega)L^{1+\delta}_v(\mathbb{R}^3)}\lesssim 1\quad\text{for all}\quad t\ge 0.\tag{5.2}$$

Proof. **Step 1.** Note that from 3.3 and 4.32, we have

$$\begin{aligned}&[\partial_t + v\cdot\nabla_x - q\nabla_x\phi^\ell\cdot\nabla_v + \nu(v) + q\frac{v}{2}\cdot\nabla_x\phi^\ell]\partial_v f^{\ell+1}\\ =&-\partial_x f^{\ell+1} - q\frac{1}{2}\partial_x\phi^\ell f^{\ell+1} - \partial_v\nu f^{\ell+1} + \partial_v(Kf^\ell) + \partial_v(\Gamma_{\text{gain}}(f^\ell,f^\ell))\\ &-\partial_v(\Gamma_{\text{loss}}(f^{\ell+1},f^\ell)) - q_1(\partial_x\phi^\ell\sqrt{\mu} - \frac{v_i^2}{2}\partial_x\phi^\ell\sqrt{\mu})\end{aligned}\tag{5.3}$$

with the boundary bound for $(x,v)\in\gamma_-$

$$\left|\partial_v f^{\ell+1}\right| \lesssim |v|\sqrt{\mu}\int_{n\cdot u>0}|f^\ell|\sqrt{\mu}\{n\cdot u\}\mathrm{d}u \quad \text{on} \quad \gamma_-.\tag{5.4}$$

From 4.15, 4.19, 4.24, 4.25, 4.26, and 4.27, we obtain the following bound along the characteristics for f_+ and f_- seperately. For $\iota = +$ or $-$ as in 1.9,

$$\begin{aligned}&|\partial_v f_\iota^{\ell+1}(t,x,v)|\\ \le\ & \mathbf{1}_{t_{1,\iota}^\ell(t,x,v)>t}|\partial_v f^{\ell+1}(0,X_\iota^\ell(0;t,x,v),V_\iota^\ell(0;t,x,v))| &(5.5)\\ +\ & \mathbf{1}_{t_{1,\iota}^\ell(t,x,v)<t}\mu(v_{\mathbf{b}})^{\frac14}\int_{n(x_{\mathbf{b}})\cdot u>0}|f^{\ell+1}(t-t_{1,\iota}^\ell,x_{\mathbf{b}},u)|\sqrt{\mu}\{n(x_{\mathbf{b}})\cdot u\}\mathrm{d}u &(5.6)\\ +\ & \int_{\max\{t-t_{1,\iota}^\ell,0\}}^t|\partial_x f^\ell(s,X_\iota^\ell(s;t,x,v),V_\iota^\ell(s;t,x,v))|\mathrm{d}s &(5.7)\\ +\ & \int_{\max\{t-t_{1,\iota}^\ell,0\}}^t(1+\|w_\vartheta f^\ell\|_\infty+\|w_\vartheta f^{\ell+1}\|_\infty) &(5.8)\\ &\qquad\times\int_{\mathbb{R}^3}\mathbf{k}_\varrho(V_\iota^\ell(s),u)|\partial_v f^\ell(s,X_\iota^\ell(s),u)|\mathrm{d}u\mathrm{d}s\\ +\ & \int_{\max\{t-t_{1,\iota}^\ell,0\}}^t\|f^{\ell+1}(s)\|_\infty|\nabla_x\phi^\ell(s,X_\iota^\ell(s;t,x,v))|\mu^{1/4}\mathrm{d}s, &(5.9)\end{aligned}$$

where δ_1 is in 4.3. Here we used that from 4.25, on the RHS of 5.3, $|\Gamma_{\text{loss}}(\partial_v f^{\ell+1},f^\ell)|$ $\lesssim\langle v\rangle\|w_\theta f^\ell\|_\infty|\partial_v f^{\ell+1}|\le\frac{\nu(v)}{8}|\partial_v f^{\ell+1}|$, and thus can be absorbed to the LHS.

Note that if $|v|>2\frac{\delta_1}{\Lambda_1}$, then from 4.3 and 4.4, for $0\le s\le t$,

$$\begin{aligned}|V_\iota^\ell(s;t,x,v)| &\ge |v| - \int_0^t|\nabla_x\phi^\ell(\tau;t,x,v)|\mathrm{d}\tau\\ &\ge |v| - \delta_1/\Lambda_1\\ &\ge \frac{|v|}{2}.\end{aligned}\tag{5.10}$$

Therefore

$$\sup_{s,t,x}\left\|\frac{1}{w_{\tilde\vartheta}(V_\iota^\ell(s;t,x,v))}\right\|_{L_v^r}\lesssim 1 \quad \text{for any } 1\le r\le\infty.\tag{5.11}$$

We derive

$$\begin{aligned}&\|5.5\|_{L_x^3L_v^{1+\delta}}\\ \lesssim&\left(\int_\Omega\left(\int_{\mathbb{R}^3}|w_{\tilde\vartheta}\partial_v f^{\ell+1}(0,X_\iota^\ell(0),V_\iota^\ell(0))|^3\right)\left(\int_{\mathbb{R}^3}\frac{\mathrm{d}v}{|w_{\tilde\vartheta}(V_\iota^\ell(0))|^{(1+\delta)\frac{3}{2-\delta}}}\right)^{\frac{2-\delta}{1+\delta}}\right)^{1/3}\end{aligned}\tag{5.12}$$

$$\lesssim \left(\iint_{\Omega\times\mathbb{R}^3} |w_{\tilde{\vartheta}}(V_\iota^\ell(0;t,x,v))\partial_v f^{\ell+1}(0,X_\iota^\ell(0;t,x,v),V_\iota^\ell(0;t,x,v))|^3 \mathrm{d}v\mathrm{d}x\right)^{1/3}$$
$$\lesssim |w_{\tilde{\vartheta}}\partial_v f(0)\|_{L^3_{x,v}},$$

where we have used a change of variables $(x,v)\mapsto (X_\iota^\ell(0;t,x,v),V_\iota^\ell(0;t,x,v))$ and 5.11.

Clearly

$$\|5.6\|_{L^3_x L^{1+\delta}_v} \lesssim \sup_{0\le s\le t}\|w_\vartheta f^{\ell+1}(s)\|_\infty. \tag{5.13}$$

From $W^{1,2}(\Omega)\subset L^6(\Omega)\subset L^2(\Omega)$ for a bounded $\Omega\subset\mathbb{R}^3$, and the change of variables $(x,v)\mapsto (X_\iota^\ell(s;t,x,v),V_\iota^\ell(s;t,x,v))$ for fixed $s\in(\max\{t-t_{\mathbf{b}},0\},t)$,

$$\begin{aligned}
\|5.9\|_{L^3_x L^{1+\delta}_v} &\lesssim \|w_\vartheta f^{\ell+1}\|_\infty \int_0^t \|\mu^{1/8}\nabla_x\phi^\ell(s,X_\iota^\ell(s;t,x,v))\|_{L^3_{x,v}}\|\mu^{1/8}\|_{L_v^{\frac{3(1+\delta)}{2-\delta}}}\\
&\lesssim \|w_\vartheta f^{\ell+1}\|_\infty \int_0^t \|\nabla_x\phi^\ell(s)\|_{L^3_x}\\
&\lesssim \|w_\vartheta f^{\ell+1}\|_\infty \int_{\max\{t-t_{\mathbf{b}},0\}}^t \|\phi^\ell(s)\|_{W^{2,2}_x}\\
&\lesssim \|w_\vartheta f^{\ell+1}\|_\infty \int_0^t \|w_{\tilde{\vartheta}} f^\ell(s)\|_2.
\end{aligned} \tag{5.14}$$

Step 2. We claim

$$\|5.7\|_{L^3_x L^{1+\delta}_v} \lesssim \int_0^t \|w_{\tilde{\vartheta}}\alpha^\beta_{f^{\ell-1},\varepsilon}\partial_x f^\ell(s)\|_{L^p_{x,v}}. \tag{5.15}$$

Now we have for $3<p<6$, by the Hölder inequality $\frac{1}{1+\delta}=\frac{1}{\frac{p+p\delta}{p-1-\delta}}+\frac{1}{p}$,

$$\begin{aligned}
&\left\|\left\|\int_{\max\{t-t^\ell_{1,\iota},0\}}^t |\partial_x f^\ell(s,X_\iota^\ell(s;t,x,v),V_\iota^\ell(s;t,x,v))|\mathrm{d}s\right\|_{L^{1+\delta}_v(\mathbb{R}^3)}\right\|_{L^3_x}\\
\lesssim &\left\|\left\|\int_{\max\{t-t^\ell_{1,\iota},0\}}^t \frac{|w_{\tilde{\vartheta}}\alpha^\beta_{f^{\ell-1},\varepsilon}\partial_x f^\ell(s,X_\iota^\ell(s;t,x,v),V_\iota^\ell(s;t,x,v))|}{w_{\tilde{\vartheta}}\alpha_{f^{\ell-1},\varepsilon,\iota}(s,X_\iota^\ell(s;t,x,v),V_\iota^\ell(s;t,x,v))^\beta}\mathrm{d}s\right\|_{L^{1+\delta}_v(\mathbb{R}^3)}\right\|_{L^3_x}\\
\lesssim &\left\|\frac{w_{\tilde{\vartheta}}(v)^{-1}}{\alpha_{f^{\ell-1},\varepsilon,\iota}(t,x,v)^\beta}\right\|_{L^{\frac{p+p\delta}{p-1-\delta}}_v(\mathbb{R}^3)}\\
&\times\left\|\left\|\int_0^t |w_{\tilde{\vartheta}}\alpha^\beta_{f^{\ell-1},\varepsilon}\partial_x f^\ell(s,X_\iota^\ell(s;t,x,v),V_\iota^\ell(s;t,x,v))|\mathrm{d}s\right\|_{L^p_v(\mathbb{R}^3)}\right\|_{L^3_x}\\
\lesssim &\left\|\frac{w_{\tilde{\vartheta}}(v)^{-1}}{\alpha_{f^{\ell-1},\varepsilon,\iota}(t,x,v)^\beta}\right\|_{L^{\frac{p+p\delta}{p-1-\delta}}_v(\mathbb{R}^3)}\times\int_0^t \|w_{\tilde{\vartheta}}\alpha^\beta_{f^{\ell-1},\varepsilon}\partial_x f^\ell(s)\|_{L^p_{x,v}}\mathrm{d}s,
\end{aligned} \tag{5.16}$$

where we have used $\alpha_{f^{\ell-1},\varepsilon,\iota}(t,x,v)=\alpha_{f^{\ell-1},\varepsilon,\iota}(s,X_\iota^\ell(s;t,x,v),V_\iota^\ell(s;t,x,v))$ for $t-t_{\mathbf{b}}(t,x,v)\le s\le t$ and the change of variables $(x,v)\mapsto(X_\iota^\ell(s;t,x,v),V_\iota^\ell(s;t,x,v))$ and the Minkowski inequality.

For β in 1.43, we have $\beta\frac{p}{p-1} < 1$ since $\frac{2}{3} < \frac{p-1}{p}$ for $3 < p$. Therefore, we can choose $0 < \delta \ll 1$ so that β in 1.43 satisfies

$$\beta \times \frac{p+p\delta}{p-1-\delta} < 1. \tag{5.17}$$

We apply Proposition 4 to conclude that

$$\sup_{t,x} \left\| \frac{w_{\tilde{\vartheta}}(v)^{-1}}{\alpha_{f^{\ell-1},\varepsilon,\iota}(t,x,v)^{\beta}} \right\|^{\frac{p+p\delta}{p-1-\delta}}_{L_v^{\frac{p+p\delta}{p-1-\delta}}(\mathbb{R}^3)} = \sup_{t,x} \int_{\mathbb{R}^3} \frac{e^{-\tilde{\vartheta}\frac{p+p\delta}{p-1-\delta}|v|^2}}{\alpha_{f^{\ell-1},\varepsilon,\iota}(t,x,v)^{\beta\frac{p+p\delta}{p-1-\delta}}} \mathrm{d}v \lesssim 1. \tag{5.18}$$

Finally, from 5.16, 5.18, and 1.46, we conclude the claim 5.15.

Step 3. We consider 5.9. We split the u-integration of 5.9 into two parts with $N \gg 1$ as

$$\int_{|u|\le N} \mathbf{k}_{\varrho}(V_\iota^\ell(s),u)|\nabla_v f^\ell(s,X_\iota^\ell(s),u)|\mathrm{d}u \tag{5.19}$$

$$+ \int_{|u|\ge N} \mathbf{k}_{\varrho}(V_\iota^\ell(s),u)|\nabla_v f^\ell(s,X_\iota^\ell(s),u)|\mathrm{d}u. \tag{5.20}$$

First we bound 5.19. From the change of variables

$$\begin{aligned} &(x,v) \mapsto (X_\iota^\ell(s;t,x,v), V_\iota^\ell(s;t,x,v)) \text{ for } t - t_{1,\iota}^\ell \le s \le t, \\ &\left\| \int_{|u|\le N} \mathbf{k}_{\varrho}(V_\iota^\ell(s;t,x,v),u)|\nabla_v f^\ell(s,X_\iota^\ell(s;t,x,v),u)|\mathrm{d}u \right\|_{L_x^3 L_v^3} \\ &= \left\| \int_{|u|\le N} \mathbf{k}_{\varrho}(v,u)|\nabla_v f^\ell(s,x,u)|\mathrm{d}u \right\|_{L_x^3 L_v^3}. \end{aligned} \tag{5.21}$$

If $|v| \ge 2N$ then $|v-u|^2 \gtrsim |v|^2$ and $\mathbf{k}_{\varrho}(v,u) \lesssim \frac{e^{-C|v|^2}}{|v-u|^2}$ for $|v| \ge 2N$ and $|u| \le N$. For $0 < \delta \ll 1$ with $\frac{3(1+\delta)}{1-2\delta} > 3$,

$$\begin{aligned} &5.21 \\ &\lesssim C_N \left\| \left\| \int_{|u|\le N} \mathbf{k}_{\varrho}(v,u)|\nabla_v f^\ell(s,x,u)|\mathrm{d}u \right\|_{L_v^{\frac{3(1+\delta)}{1-2\delta}}(\{|v|\le 2N\})} \right\|_{L_x^3} \\ &+ \left\| \left\| e^{-C|v|^2} \right\|_{L_v^{3/2}} \left\| \int_{|u|\le N} \frac{1}{|v-u|} |\nabla_v f^\ell(s,x,u)|\mathrm{d}u \right\|_{L_v^{\frac{3(1+\delta)}{1-2\delta}}(\{|v|\ge 2N\})} \right\|_{L_x^3} \\ &\lesssim \left\| \left\| \frac{1}{|v-\cdot|} * |\nabla_v f^\ell(s,x,\cdot)| \right\|_{L_v^{\frac{3(1+\delta)}{1-2\delta}}} \right\|_{L_x^3}. \end{aligned} \tag{5.22}$$

Then by the Hardy-Littlewood-Sobolev inequality with $1 + \frac{1}{\frac{3(1+\delta)}{1-2\delta}} = \frac{1}{3} + \frac{1}{1+\delta}$, we derive that

$$5.22 \lesssim \left\| \|\nabla_v f^\ell(s,x,v)\|_{L_v^{1+\delta}} \right\|_{L_x^3} = \|\nabla_v f^\ell(s)\|_{L_x^3 L_v^{1+\delta}}.$$

Combining the last estimate with 5.21, 5.22, we prove that

$$\|5.19\|_{L_x^3 L_v^{1+\delta}} \lesssim \|\nabla_v f^\ell(s)\|_{L_x^3 L_v^{1+\delta}}. \tag{5.23}$$

Now we consider 5.20. Choose $0<\delta'\ll 1$. We have

5.20

$$\leq \int_{|u|\geq N}\frac{1}{w_{\tilde{\vartheta}}(V_\iota^\ell(s;t,x,v))^{1-\delta'}}\frac{w_{\tilde{\vartheta}}(V_\iota^\ell(s;t,x,v))}{w_{\tilde{\vartheta}}(u)}\frac{\mathbf{k}_\varrho(V_\iota^\ell(s;t,x,v),u)}{\alpha_{f^{\ell-1},\varepsilon,\iota}(s,X_\iota^\ell(s;t,x,v),u)^\beta}$$
$$\times\frac{1}{w_{\tilde{\vartheta}}(V_\iota^\ell(s;t,x,v))^{\delta'}}w_{\tilde{\vartheta}}(u)|\alpha_{f^{\ell-1},\varepsilon}(s,X_\iota^\ell(s;t,x,v),u)^\beta\nabla_v f^\ell(s,X_\iota^\ell(s;t,x,v),u)|\mathrm{d}u.$$

By the Hölder inequality with $\frac{1}{p}+\frac{1}{p^*}=1$ with $3<p<6$

$$|5.20|$$
$$\lesssim \frac{1}{w_{\tilde{\vartheta}}(V_\iota^\ell(s;t,x,v))^{1-\delta'}}$$
$$\times\left\|\frac{w_{\tilde{\vartheta}}(V_\iota^\ell(s;t,x,v))}{w_{\tilde{\vartheta}}(u)}\frac{\mathbf{k}_\varrho(V_\iota^\ell(s;t,x,v),u)}{\alpha_{f^{\ell-1},\varepsilon,\iota}(s,X_\iota^\ell(s;t,x,v),u)^\beta}\right\|_{L^{p^*}(\{|u|\geq N\})}$$
$$\times\left\|\frac{w_{\tilde{\vartheta}}(u)}{w_{\tilde{\vartheta}}(V_\iota^\ell(s;t,x,v))^{\delta'}}\alpha_{f^{\ell-1},\varepsilon}(s,X_\iota^\ell(s;t,x,v),u)^\beta\nabla_v f^\ell(s,X_\iota^\ell(s;t,x,v),u)\right\|_{L^p_u(\mathbb{R}^3)}. \tag{5.24}$$

Then by the Hölder inequality with $\frac{1}{1+\delta}=\frac{1}{p}+\frac{1}{\frac{(1+\delta)p}{p-(1+\delta)}}$,

$$\|5.20\|_{L^{1+\delta}_v}$$
$$\lesssim \left\|\frac{1}{w_{\tilde{\vartheta}}(V_\iota^\ell(s;t,x,v))^{1-\delta'}}\right\|_{L^{\frac{(1+\delta)p}{p-(1+\delta)}}_v}$$
$$\times\sup_v\left\|\frac{w_{\tilde{\vartheta}}(V_\iota^\ell(s;t,x,v))}{w_{\tilde{\vartheta}}(u)}\frac{\mathbf{k}_\varrho(V_\iota^\ell(s;t,x,v),u)}{\alpha_{f^{\ell-1},\varepsilon,\iota}(s,X_\iota^\ell(s;t,x,v),u)^\beta}\right\|_{L^{p^*}(\{|u|\geq N\})}$$
$$\times\left\|\left\|\frac{w_{\tilde{\vartheta}}(u)}{w_{\tilde{\vartheta}}(V_\iota^\ell(s;t,x,v))^{\delta'}}\alpha_{f^{\ell-1},\varepsilon}(s,X_\iota^\ell(s;t,x,v),u)^\beta\nabla_v f^\ell(s,X_\iota^\ell(s;t,x,v),u)\right\|_{L^p_u}\right\|_{L^p_v}.$$

Note that, from 3.32, $\mathbf{k}_\varrho(v,u)\frac{e^{\tilde{\vartheta}|v|^2}}{e^{\tilde{\vartheta}|u|^2}}\lesssim\mathbf{k}_{\tilde{\varrho}}(v,u)$ for some $0<\tilde{\varrho}<\varrho$. Hence we derive, using 5.11

$$\left\|\|5.20\|_{L^{1+\delta}_v}\right\|_{L^3_x}$$
$$\lesssim_\Omega \sup_{X_\iota^\ell,V_\iota^\ell}\left\|\frac{e^{-\frac{\tilde{\vartheta}}{10}|V_\iota^\ell-u|^2}}{|V_\iota^\ell-u|}\frac{1}{\alpha_{f^{\ell-1},\varepsilon,\iota}(s,X_\iota^\ell,u)^\beta}\right\|_{L^{p^*}(\{|u|\geq N\})}$$
$$\times\left\|\frac{\tilde{w}(u)}{\tilde{w}(V_\iota^\ell(s;t,x,v))^{\delta'}}\alpha_{f^{\ell-1},\varepsilon}(s,X_\iota^\ell(s;t,x,v),u)^\beta\nabla_v f^\ell(s,X_\iota^\ell(s;t,x,v),u)\right\|_{L^p_{u,v,x}}.$$

Finally using 4.9 in Proposition 4 with $\frac{p-2}{p-1}<\beta p^*<1$ from 1.43 and applying the change of variables $(x,v)\mapsto(X_\iota^\ell(s;t,x,v),V_\iota^\ell(s;t,x,v))$, we derive that

$$\left\|\|5.20\|_{L^{1+\delta}_v}\right\|_{L^3_x}$$
$$\lesssim_\Omega\left\|\left\|\frac{1}{w_{\tilde{\vartheta}}(v)^{\delta'}}w_{\tilde{\vartheta}}(u)\alpha_{f^{\ell-1},\varepsilon}(s,x,u)^\beta\nabla_v f^\ell(s,x,u)\right\|_{L^p_v}\right\|_{L^p_{u,x}} \tag{5.25}$$

$$\begin{aligned}\lesssim\quad & \left\|\tfrac{1}{w_{\tilde\vartheta}(v)^{\delta'}}\right\|_{L^p_v}\left\|w_{\tilde\vartheta}(u)\alpha_{f^{\ell-1},\varepsilon}(s,x,u)^\beta\nabla_v f^\ell(s,x,u)\right\|_{L^p_{u,x}}\\ \lesssim\quad & \left\|w_{\tilde\vartheta}\alpha^\beta_{f^{\ell-1},\varepsilon}\nabla_v f^\ell(s)\right\|_{L^p}.\end{aligned}$$

Combining 5.24 and 5.25 we conclude that

$$\|5.20\|_{L^3_xL^{1+\delta}_v}\lesssim\|w_{\tilde\vartheta}\alpha^\beta_{f^{\ell-1},\varepsilon}\nabla_v f^\ell(s)\|_{L^p_{x,v}}. \tag{5.26}$$

Finally from 5.23 and 5.26, and using the Minkowski inequality, we conclude that

$$\begin{aligned}&\|5.9\|_{L^3_vL^{1+\delta}_x}\\ &\lesssim(1+\|w_\vartheta f^\ell\|_\infty+\|w_\vartheta f^{\ell+1}\|_\infty)\\ &\quad\times\int_0^t\left[\|\nabla_v f^\ell(s)\|_{L^3_xL^{1+\delta}_v}+\|w_{\tilde\vartheta}\alpha^\beta_{f^{\ell-1},\varepsilon}\nabla_v f^\ell(s)\|_{L^p_{x,v}}\right]\mathrm{d}s.\end{aligned} \tag{5.27}$$

Collecting terms from 5.5-5.9, and 5.12, 5.6, 5.14, 5.15, 5.27, we derive

$$\begin{aligned}&\sup_{0\le s\le t}\|\nabla_v f^{\ell+1}(s)\|_{L^3_xL^{1+\delta}_v}\\ &\lesssim\sup_{0\le s\le t}\|\nabla_v f^{\ell+1}_+(s)\|_{L^3_xL^{1+\delta}_v}+\sup_{0\le s\le t}\|\nabla_v f^{\ell+1}_-(s)\|_{L^3_xL^{1+\delta}_v}\\ &\lesssim\|w_{\tilde\vartheta}\nabla_v f(0)\|_{L^3_{x,v}}+\sup_{0\le s\le t}\|w_\vartheta f^{\ell+1}(s)\|_\infty+\sup_{0\le s\le t}\|w_\vartheta f^\ell(s)\|_\infty\\ &\quad+t(1+\sup_{0\le s\le t}\|w_\vartheta f^{\ell+1}(s)\|_\infty+\sup_{0\le s\le t}\|w_\vartheta f^\ell(s)\|_\infty)\\ &\qquad\times(\sup_{0\le s\le t}\|w_{\tilde\vartheta}\alpha^\beta_{f,\varepsilon}\nabla_{x,v}f^\ell(s)\|_p+\|\nabla_v f^\ell(s)\|_{L^3_xL^{1+\delta}_v}).\end{aligned} \tag{5.28}$$

Therefore from 3.8 and 4.5, we can choose $T^{**}\ll 1$ and conclude 5.2. □

6. **Local existence.**

Theorem 6.1. *Let $0<\tilde\vartheta<\vartheta\ll 1$. Assume that for sufficiently small $M>0$, $F_0=\mu+\sqrt{\mu}f_0\ge 0$ satisfying 3.7, 4.5, 5.1 and the compatibility condition 1.41.*

Then there exists $T^*(M)>0$ and a unique solution $F(t,x,v)=\mu+\sqrt{\mu}f(t,x,v)\ge 0$ to 1.15, 1.16, and 1.17 in $[0,T^*(M))\times\Omega\times\mathbb{R}^3$ such that

$$\sup_{0\le t\le T*}\|w_\vartheta f(t)\|_\infty\le M. \tag{6.1}$$

Moreover

$$\sup_{0\le t\le T^*}\|\nabla_v f(t)\|_{L^3_xL^{1+\delta}_v}<\infty\quad\text{for } 0<\delta\ll 1, \tag{6.2}$$

and

$$\sup_{0\le t\le T*}\left\{\|w_{\tilde\vartheta}\alpha^\beta_{f,\varepsilon}\nabla_{x,v}f(t)\|^p_p+\int_0^t|w_{\tilde\vartheta}\alpha^\beta_{f,\varepsilon}\nabla_{x,v}f(t)|^p_{p,+}\right\}<\infty. \tag{6.3}$$

Furthermore, $\|w_\vartheta f(t)\|_\infty$, $\|\nabla_v f(t)\|_{L^3_xL^{1+\delta}_v}$ and $\|w_{\tilde\vartheta}\alpha^\beta_{f,\varepsilon}\nabla_{x,v}f(t)\|^p_p+\int_0^t|w_{\tilde\vartheta}\alpha^\beta_{f,\varepsilon}\nabla_{x,v}f(t)|^p_{p,+}$ are continuous in t.

Proof. **Step 1.** We claim that for $T^{**}\ll 1$, the whole sequence 3.3 satisfies

$$f^\ell\to f\text{ strongly in }L^\infty((0,T);L^{1+}(\Omega\times\mathbb{R}^3)). \tag{6.4}$$

Note that $f^{\ell+1}-f^{\ell}$ satisfies $(f^{\ell+1}-f^{\ell})|_{t=0}=0$, so

$$\begin{aligned}
&\partial_t[f^{\ell+1}-f^{\ell}]+v\cdot\nabla_x[f^{\ell+1}-f^{\ell}]-q\nabla_x\phi^{\ell}\cdot\nabla_v[f^{\ell+1}-f^{\ell}]\\
&+q\frac{v}{2}\cdot\nabla_x\phi^{\ell}[f^{\ell+1}-f^{\ell}]+\nu[f^{\ell+1}-f^{\ell}]\\
=&q\nabla_x\phi_{f^{\ell}-f^{\ell-1}}\cdot\nabla_v f^{\ell-1}\\
&+\Gamma_{\text{gain}}(f^{\ell},f^{\ell})-\Gamma_{\text{loss}}(f^{\ell+1},f^{\ell})-\Gamma_{\text{gain}}(f^{\ell-1},f^{\ell-1})+\Gamma_{\text{loss}}(f^{\ell},f^{\ell-1})\\
&+K[f^{\ell}-f^{\ell-1}]-q\frac{v}{2}\cdot\nabla_x\phi_{f^{\ell}-f^{\ell-1}}f^{\ell-1}-q_1v\cdot\nabla_x\phi_{f^{\ell}-f^{\ell-1}}\sqrt{\mu}.
\end{aligned}\tag{6.5}$$

By Lemma 2.3 for $L^{1+\delta}$-space with $0<\delta\ll 1$, we obtain

$$\begin{aligned}
&\|[f^{\ell+1}-f^{\ell}](t)\|_{1+\delta}^{1+\delta}+\int_0^t\|\nu_{\phi^{\ell}}^{1/1+\delta}[f^{\ell+1}-f^{\ell}]\|_{1+\delta}^{1+\delta}+\int_0^t|[f^{\ell+1}-f^{\ell}]|_{1+\delta,+}^{1+\delta}\\
\le&\|[f^{\ell+1}-f^{\ell}](0)\|_{1+\delta}^{1+\delta}+\int_0^t\iint_{\Omega\times\mathbb{R}^3}|\text{RHS of 6.5}||f^{\ell+1}-f^{\ell}|^{\delta}\\
&+\int_0^t|[f^{\ell+1}-f^{\ell}]|_{1+\delta,-}^{1+\delta},
\end{aligned}\tag{6.6}$$

where $\nu_{\phi^{\ell}}$ is defined as 4.35.

Now for $0<\delta\ll 1$, by the Hölder inequality with $1=\frac{1}{\frac{3(1+\delta)}{2-\delta}}+\frac{1}{3}+\frac{1}{\frac{1+\delta}{\delta}}$ and the Sobolev embedding $W^{1,1+\delta}(\Omega)\subset L^{\frac{3(1+\delta)}{2-\delta}}(\Omega)$ when $\Omega\subset\mathbb{R}^3$,

$$\begin{aligned}
&\int_0^t\iint_{\Omega\times\mathbb{R}^3}|\nabla_x\phi_{f^{\ell}-f^{\ell-1}}\cdot\nabla_v f^{\ell-1}||f^{\ell+1}-f^{\ell}|^{\delta}\\
\lesssim&\int_0^t\|\nabla_x\phi_{f^{\ell}-f^{\ell-1}}\|_{L_x^{\frac{3(1+\delta)}{2-\delta}}}\|\nabla_v f^{\ell-1}\|_{L_x^3L_v^{1+\delta}}\left\||f^{\ell+1}-f^{\ell}|^{\delta}\right\|_{L_{x,v}^{\frac{1+\delta}{\delta}}}\\
\lesssim&\sup_{0\le s\le t}\|\nabla_v f^{\ell-1}(s)\|_{L_x^3L_v^{1+\delta}}\times\int_0^t\|[f^{\ell+1}-f^{\ell}](s)\|_{1+\delta}^{1+\delta}ds.
\end{aligned}\tag{6.7}$$

A simple modification of 4.37 and 4.38 as

$$\begin{aligned}
&\int_0^t\int_x\int_v\int_u\mathbf{k}_{\varrho}(v,u)|f^{\ell}(u)-f^{\ell-1}(u)||f^{\ell+1}(v)-f^{\ell}(v)|^{\delta}\\
\lesssim&\int_0^t\int_x\int_v\int_u\mathbf{k}_{\varrho}(v,u)^{\frac{1}{1+\delta}}|f^{\ell}(u)-f^{\ell-1}(u)|\mathbf{k}_{\varrho}(v,u)^{\frac{\delta}{1+\delta}}|f^{\ell+1}(v)-f^{\ell}(v)|^{\delta}\\
\lesssim&\int_0^t\int_x\int_v(|f^{\ell}(v)-f^{\ell-1}(v)|^{1+\delta}+|f^{\ell+1}(v)-f^{\ell}(v)|^{1+\delta})\int_u\mathbf{k}_{\varrho}(v,u)\\
\lesssim&\int_0^t\|f^{\ell}-f^{\ell-1}\|_{1+\delta}^{1+\delta}+\int_0^t\|f^{\ell+1}-f^{\ell}\|_{1+\delta}^{1+\delta},
\end{aligned}$$

leads to

$$\begin{aligned}
&\int_0^t\iint_{\Omega\times\mathbb{R}^3}|\text{the }2^{\text{nd}}\text{and }3^{\text{rd}}\text{ line of RHS of 6.5}||f^{\ell+1}-f^{\ell}|^{\delta}\\
\lesssim&\sup_{0\le s\le t}\left\{1+\|w_{\vartheta}f^{\ell+1}(s)\|_{\infty}+\|w_{\vartheta}f^{\ell}(s)\|_{\infty}\right\}\\
&\times\left(\int_0^t\|f^{\ell}-f^{\ell-1}\|_{1+\delta}^{1+\delta}+\int_0^t\|f^{\ell+1}-f^{\ell}\|_{1+\delta}^{1+\delta}\right).
\end{aligned}\tag{6.8}$$

Then following the proof of 4.39 and applying 6.7 to 6.8, we can obtain

$$\begin{aligned}
&\int_0^t |[f^{\ell+1}-f^\ell]|^{1+\delta}_{1+\delta,-} \\
&\lesssim o(1)\int_0^t |[f^\ell-f^{\ell-1}]|^{1+\delta}_{1+\delta,+} + \|[f^\ell-f^{\ell-1}](0)\|^{1+\delta}_{1+\delta} \\
&+ \sup_{0\le s\le t}\left\{1+\|\nabla_v f^{\ell-2}(s)\|_{L^3_x L^{1+\delta}_v} + \|w_\vartheta f^\ell(s)\|_\infty + \|w_\vartheta f^{\ell-1}(s)\|_\infty\right\} \\
&\qquad \times\left(\int_0^t \|f^\ell-f^{\ell-1}\|^{1+\delta}_{1+\delta} + \int_0^t \|f^{\ell-1}-f^{\ell-2}\|^{1+\delta}_{1+\delta}\right).
\end{aligned} \tag{6.9}$$

Using 3.8, 5.2, 6.6, 6.7, 6.8, 6.9 and $[f^{\ell+1}-f^\ell]|_{t=0}=0$ we get

$$\begin{aligned}
&\sup_{0\le s\le t}\|f^{\ell+1}(s)-f^\ell(s)\|^{1+\delta}_{1+\delta} + \int_0^t |f^{\ell+1}-f^\ell|^{1+\delta}_{1+\delta,+} \\
&\le [O(t)+o(1)] \\
&\left(\sup_{0\le s\le t}\|f^\ell-f^{\ell-1}\|^{1+\delta}_{1+\delta} + \int_0^t |f^\ell-f^{\ell-1}|^{1+\delta}_{1+\delta,+} + \sup_{0\le s\le t}\|f^{\ell-1}-f^{\ell-2}\|^{1+\delta}_{1+\delta}\right).
\end{aligned} \tag{6.10}$$

Thus adding 6.10 with the same estimate $6.10|_{f^{\ell+2}-f^{\ell+1}}$ we get

$$\begin{aligned}
&\sup_{0\le s\le t}\|f^{\ell+1}(s)-f^\ell(s)\|^{1+\delta}_{1+\delta} + \int_0^t |f^{\ell+1}-f^\ell|^{1+\delta}_{1+\delta,+} \\
&\quad + \sup_{0\le s\le t}\|f^{\ell+2}(s)-f^{\ell+1}(s)\|^{1+\delta}_{1+\delta} + \int_0^t |f^{\ell+2}-f^{\ell+1}|^{1+\delta}_{1+\delta,+} \\
&\le [O(t)+o(1)]\left(\sup_{0\le s\le t}\|f^\ell-f^{\ell-1}\|^{1+\delta}_{1+\delta} + \int_0^t |f^\ell-f^{\ell-1}|^{1+\delta}_{1+\delta,+}\right. \\
&\qquad\qquad \left. + \sup_{0\le s\le t}\|f^{\ell-1}-f^{\ell-2}\|^{1+\delta}_{1+\delta}\right).
\end{aligned}$$

Therefore, inductively we have

$$\sup_{0\le s\le t}\|f^{\ell+1}(s)-f^\ell(s)\|^{1+\delta}_{1+\delta} + \int_0^t |f^{\ell+1}-f^\ell|^{1+\delta}_{1+\delta,+} \le [O(t)+o(1)]^m.$$

Hence we derive stability

$$\sup_{0\le s\le t}\|f^\ell(s)-f^m(s)\|^{1+\delta}_{1+\delta} \le [O(t)+o(1)]^{\min\{m,\ell\}}, \tag{6.11}$$

and this concludes 6.4.

Step 2. We combine 3.8 and 6.4 to get unique weak-$*$ convergence (up to subsequence if necessary), $(w_\vartheta f^\ell, w_\vartheta f^{\ell+1}) \overset{*}{\rightharpoonup} (w_\vartheta f, w_\vartheta f)$ weakly$-*$ in $L^\infty(\mathbb{R}\times\Omega\times\mathbb{R}^3;\mathbb{R}^2)\cap L^\infty(\mathbb{R}\times\gamma;\mathbb{R}^2)$. For $\varphi=\begin{bmatrix}\varphi_+\\ \varphi_-\end{bmatrix}\in C^\infty_c(\mathbb{R}\times\bar\Omega\times\mathbb{R}^3;\mathbb{R}^2)$,

$$\int_0^T \langle f^{\ell+1}, [-\partial_t - v\cdot\nabla_x+\nu]\varphi\rangle + \underbrace{\langle q f^{\ell+1}, \nabla_x\phi^\ell\cdot\nabla_v\varphi + \frac{v}{2}\cdot\nabla_x\phi^\ell\varphi\rangle}_{6.12_\phi} \tag{6.12}$$

$$= \int_0^T \langle Kf^\ell, \varphi\rangle - \langle q_1 v\cdot\nabla_x\phi^\ell\sqrt{\mu}, \varphi\rangle + \underbrace{\langle \Gamma_{\text{gain}}(f^\ell, f^\ell), \varphi\rangle}_{6.12_{\text{gain}}} - \underbrace{\langle \Gamma_{\text{loss}}(f^{\ell+1}, f^\ell), \varphi\rangle}_{6.12_{\text{loss}}}$$
$$+ \int_0^T \langle f^{\ell+1}, \varphi\rangle_{\gamma_+} - \int_0^T \langle c_\mu\sqrt{\mu}\int_{n\cdot u>0} f^\ell\sqrt{\mu}\{n\cdot u\}\mathrm{d}u, \varphi\rangle_{\gamma_-}.$$

Except the underbraced terms in 6.12 all terms converges to limits with f instead of $f^{\ell+1}$ or f^ℓ.

We define, for $(t,x,v)\in\mathbb{R}\times\bar\Omega\times\mathbb{R}^3$ and for $0<\delta\ll 1$,

$$\begin{aligned} f^\ell_\delta(t,x,v) &:= \kappa_\delta(x,v) f^\ell(t,x,v) \\ &:= \chi\Big(\frac{|n(x)\cdot v|}{\delta} - 1\Big)\Big[1-\chi(\delta|v|)\Big]\chi\Big(\frac{|v|}{\delta}-1\Big) f^\ell(t,x,v). \end{aligned} \tag{6.13}$$

Note that $f_\delta(t,x,v)=0$ if either $|n(x)\cdot v|\le\delta$, $|v|\ge\frac{1}{\delta}$, or $|v|\le\delta$.

From 3.8

$$\Big|\int_0^T 6.12_{\text{loss}} - \int_0^T \langle\Gamma_{\text{loss}}(f,f),\varphi\rangle\Big|$$
$$\le \Big|\int_0^T \langle\int_{\mathbb{R}^3}|v-u|\{f^\ell_+(u)-f_+(u)+f^\ell_-(u)-f_-(u)\}\sqrt{\mu(u)}\mathrm{d}u f^{\ell+1}(v), \varphi(t,x,v)\rangle\mathrm{d}t\Big|$$
$$+ \Big|\int_0^T \langle\int_{\mathbb{R}^3}|v-u|(f_+(u)+f_-(u))\sqrt{\mu(u)}\mathrm{d}u\{f^{\ell+1}(v)-f(v)\}, \varphi(t,x,v)\rangle\mathrm{d}t\Big|.$$

The second term converges to zero from the weak$-*$ convergence in L^∞ and 3.8. The first term is bounded by, from 3.8,

$$\Big[\int_0^T \Big\|\int_{\mathbb{R}^3}\kappa_\delta(x,u)(f^\ell(t,x,u)-f(t,x,u))\langle u\rangle\sqrt{\mu(u)}\mathrm{d}u\Big\|^2_{L^2(\Omega\times\mathbb{R}^3)}\Big]^{1/2} \tag{6.14}$$
$$\times \sup_{0\le t\le T}\|w_\vartheta f^{\ell+1}(t)\|_\infty + O(\delta).$$

On the other hand, from Lemma A.1, we have an extension $\bar f^\ell(t,x,v)$ of $\kappa_\delta(x,u)$ $f^\ell(t,x,u)$. We apply the average lemma (see Theorem 7.2.1 in page 187 of [10], for example) to $\bar f^\ell(t,x,v)$. From 3.3 and 3.8

$$\sup_\ell\Big\|\int_{\mathbb{R}^3}\bar f^\ell(t,x,u)\langle u\rangle\sqrt{\mu(u)}\mathrm{d}u\Big\|_{H^{1/4}_{t,x}(\mathbb{R}\times\mathbb{R}^3)} < \infty. \tag{6.15}$$

Then by $H^{1/4}\subset\subset L^2$, up to subsequence, we conclude that

$$\int_{\mathbb{R}^3}\kappa_\delta(x,u)f^\ell(t,x,u)\langle u\rangle\sqrt{\mu(u)}\mathrm{d}u \to \int_{\mathbb{R}^3}\kappa_\delta(x,u)f(t,x,u)\langle u\rangle\sqrt{\mu(u)}\mathrm{d}u$$
strongly in $L^2_{t,x}$.

So we conclude that 6.14 $\to 0$ as $\ell\to\infty$.

For 6.12_{gain} let us use a test function $\varphi_1(v)\varphi_2(t,x)$. From the density argument, it suffices to prove a limit by testing with $\varphi(t,x,v)$.

We use a standard change of variables $(v,u)\mapsto(v',u')$ and $(v,u)\mapsto(u',v')$ (for example see page 10 of [10]) to get

$$
\begin{aligned}
&\int_0^T 6.12_{\text{gain}} - \int_0^T \langle \Gamma_{\text{gain}}(f,f),\varphi\rangle \\
=\;& \int_0^T \langle \Gamma_{\text{gain}}(f^\ell, f^\ell - f),\varphi\rangle + \int_0^T \langle \Gamma_{\text{gain}}(f^\ell - f, f),\varphi\rangle \\
=\;& \sum_{\iota=\pm}\int_0^T \iint_{\Omega\times\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}(f_+^\ell(u)-f_+(u)+f_-^\ell(u)-f_-(u)) \qquad (6.16)\\
&\quad\times\sqrt{\mu(u')}|(v-u)\cdot\omega|\varphi_{1,\iota}(v')\mathrm{d}\omega\mathrm{d}u\Big)f_\iota^\ell(v)\varphi_{2,\iota}(t,x)\mathrm{d}v\mathrm{d}x\mathrm{d}t \\
+\;& \sum_{\iota=\pm}\int_0^T \iint_{\Omega\times\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}(f_\iota^\ell(u)-f_\iota(u)) \qquad (6.17)\\
&\quad\times\sqrt{\mu(v')}|(v-u)\cdot\omega|\varphi_{1,\iota}(u')\mathrm{d}\omega\mathrm{d}u\Big)(f_+(v)+f_-(v))\varphi_{2,\iota}(t,x)\mathrm{d}v\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

For $N\gg 1$ we decompose the integration of 6.17 and 6.18 using

$$
\begin{aligned}
1 =&\{1-\chi(|u|-N)\}\{1-\chi(|v|-N)\}\\
&+\chi(|u|-N)+\chi(|v|-N)-\chi(|u|-N)\chi(|v|-N).
\end{aligned}
\qquad (6.18)
$$

Note that $\{1-\chi(|u|-N)\}\{1-\chi(|v|-N)\}\neq 0$ if $|v|\le N+1$ and $|u|\le N+1$, and if $\chi(|u|-N)+\chi(|v|-N)-\chi(|u|-N)\chi(|v|-N)\neq 0$ then either $|v|\ge N$ or $|u|\ge N$. From 3.8, the second part of 6.17 and 6.18 from 6.18 are bounded by

$$
\begin{aligned}
&\int_0^T\iint_{\Omega\times\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}[\cdots]\times\{\chi(|u|-N)+\chi(|v|-N)-\chi(|u|-N)\chi(|v|-N)\}\\
\le\;& \sup_\ell \|w_\vartheta f^\ell\|_\infty\|w_\vartheta f\|_\infty\times\{e^{-\frac{\vartheta}{2}|v|^2}e^{-\frac{\vartheta}{2}|u|^2}\}\{\mathbf{1}_{|v|\ge N}+\mathbf{1}_{|u|\ge N}\}\\
\le\;& O(\frac{1}{N}).
\end{aligned}
$$

Now we only need to consider the parts with $\{1-\chi(|u|-N)\}\{1-\chi(|v|-N)\}$. Then

$$
\begin{aligned}
&6.17\\
=&\sum_{\iota=\pm}\int_0^T\iint_{\Omega\times\mathbb{R}^3}\int_{\mathbb{R}^3}(f_+^\ell(t,x,u)-f_+(t,x,u)+f_-^\ell(t,x,u)-f_-(t,x,u))\\
&\qquad\times\{1-\chi(|u|-N)\}\Big(\int_{\mathbb{S}^2}\sqrt{\mu(u')}|(v-u)\cdot\omega|\varphi_{1,\iota}(v')\mathrm{d}\omega\Big)\,\mathrm{d}u\\
&\qquad\times\{1-\chi(|v|-N)\}f_\iota^\ell(t,x,v)\varphi_{2,\iota}(t,x)\mathrm{d}v\mathrm{d}x\mathrm{d}t.
\end{aligned}
\qquad (6.19)
$$

Now, let us define

$$
\Phi_{v,\iota}(u):=\{1-\chi(|u|-N)\}\int_{\mathbb{S}^2}\sqrt{\mu(u')}|(v-u)\cdot\omega|\varphi_{1,\iota}(v')\mathrm{d}\omega\ \text{ for } |v|\le N+1. \qquad (6.20)
$$

For $0<\delta\ll 1$ we have $O(\frac{N^3}{\delta^3})$ number of $v_i\in\mathbb{R}^3$ such that $\{v\in\mathbb{R}^3:|v|\le N+1\}\subset\bigcup_{i=1}^{O(\frac{N^3}{\delta^3})}B(v_i,\delta)$. Since 6.20 is smooth in u and v and compactly supported, for $0<\varepsilon\ll 1$ we can always choose $\delta>0$ such that

$$
|\Phi_{v,\iota}(u)-\Phi_{v_i,\iota}(u)|<\varepsilon\ \text{ if } v\in B(v_i,\delta). \qquad (6.21)
$$

Now we replace $\Phi_{v,\iota}(u)$ in the second line of 6.19 by $\Phi_{v_i,\iota}(u)$ whenever $v \in B(v_i,\delta)$. Moreover we use κ_δ-cut off in 6.13. If v is included in several balls then we choose the smallest i. From 6.21 and 3.8 the difference of 6.19 and the one with $\Phi_{v_i}(u)$ can be controlled and we conclude that

$$\begin{aligned}
6.19 = \{O(\varepsilon)+O(\delta)\}\sup_\ell \|w_\vartheta f^\ell\|_\infty^2 \\
+\sum_{\iota=\pm}\int_0^T\int_\Omega\sum_i\int_{\mathbb{R}^3}\mathbf{1}_{v\in B(v_i,\delta)} \\
\times\int_{\mathbb{R}^3}\kappa_\delta(x,u)(f_+^\ell(t,x,u)-f_+(t,x,u)+f_-^\ell(t,x,u)-f_-(t,x,u))\Phi_{v_i,\iota}(u)\mathrm{d}u \\
\times\{1-\chi(|v|-N)\}f_\iota^\ell(t,x,v)\varphi_{2,\iota}(t,x)\mathrm{d}v\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{6.22}$$

From Lemma A.1 and the average lemma

$$\max_{1\le i\le O(\frac{N^3}{\delta^3})}\sup_\ell\left\|\int_{\mathbb{R}^3}\kappa_\delta(x,u)f^\ell(t,x,u)\Phi_{v_i,\iota}(u)\mathrm{d}u\right\|_{H^{1/4}_{t,x}(\mathbb{R}\times\mathbb{R}^3)}<\infty. \tag{6.23}$$

For $i=1$ we extract a subsequence $\ell_1\subset\mathcal{I}_1$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\kappa_\delta(x,u)f^{\ell_1}(t,x,u)\Phi_{v_i,\iota}(u)\mathrm{d}u\to\int_{\mathbb{R}^3}\kappa_\delta(x,u)f(t,x,u)\Phi_{v_i,\iota}(u)\mathrm{d}u \\
&\text{strongly in } L^2_{t,x}.
\end{aligned} \tag{6.24}$$

Successively we extract subsequences $\mathcal{I}_{O(\frac{N^3}{\delta^3})}\subset\cdots\subset\mathcal{I}_2\subset\mathcal{I}_1$. Now we use the last subsequence $\ell\in\mathcal{I}_{O(\frac{N^3}{\delta^3})}$ and redefine f^ℓ with it. Clearly we have 6.24 for all i. Finally we bound the last term of 6.22 by

$$C_{\varphi_2,N}\max_i\int_0^T\sum_{\iota=\pm}\left\|\int_{\mathbb{R}^3}\kappa_\delta(x,u)(f^\ell(t,x,u)-f(t,x,u))\Phi_{v_i,\iota}(u)\mathrm{d}u\right\|_{L^2_{t,x}}\sup_\ell\|w_\vartheta f^\ell\|_\infty$$
$$\to 0 \quad\text{as}\quad \ell\to\infty.$$

Together with 6.22 we prove $6.17\to 0$. Similarly we can prove $6.18\to 0$.

Now we consider 6.12_ϕ. From

$$-(\Delta\phi^\ell-\Delta\phi)=\int\kappa_\delta(f_+^\ell-f_++f_-^\ell-f_-)\sqrt{\mu}+\int(1-\kappa_\delta)(f_+^\ell-f_++f_-^\ell-f_-)\sqrt{\mu},$$

we have

$$\|\nabla_x\phi^\ell-\nabla_x\phi\|_{L^2_{t,x}}\le\left\|\int\kappa_\delta(f^\ell-f)\sqrt{\mu}\right\|_{L^2_{t,x}}+O(\delta)\sup_\ell\|w_\vartheta f^\ell\|_\infty. \tag{6.25}$$

Then following the previous argument, we prove $\nabla_x\phi^\ell\to\nabla_x\phi$ strongly in $L^2_{t,x}$ as $\ell\to\infty$. Combining with $w_\vartheta f^\ell\overset{*}{\rightharpoonup}w_\vartheta f$ in L^∞, we prove $\int_0^T 6.12_\phi$ converges to $\int_0^T\langle qf,\{\nabla_x\phi\cdot\nabla_v\varphi+\frac{v}{2}\cdot\nabla_x\phi\varphi\}\rangle$. This proves the existence of a (weak) solution $f\in L^\infty$.

Step 7. We claim 6.3. By the weak lower-semicontinuity of L^p we know that (if necessary we further extract a subsequence out of the subsequence of *Step 6*)

$$w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\nabla_{x,v}f^{\ell+1}\rightharpoonup\mathcal{F},\qquad\sup_{0\le t\le T^{**}}\|\mathcal{F}(t)\|_p^p\le\liminf\sup_{0\le t\le T^{**}}\|w_{\tilde\vartheta}\alpha^\beta_{f^\ell,\varepsilon}\nabla_{x,v}f^{\ell+1}(t)\|_p^p,$$

and

$$\int_0^{T^{**}} |\mathcal{F}|_{p,+}^p \le \liminf \int_0^{T^{**}} |w_{\tilde{\vartheta}} \alpha_{f^{\ell},\varepsilon}^{\beta} \nabla_{x,v} f^{\ell+1}(t)|_{p,+}^p.$$

We need to prove that

$$\mathcal{F} = w_{\tilde{\vartheta}} \alpha_{f,\varepsilon}^{\beta} \nabla_{x,v} f \quad \text{almost everywhere except } \gamma_0. \tag{6.26}$$

We claim that, up to some subsequence, for any given smooth test function $\psi \in C_c^{\infty}(\bar{\Omega} \times \mathbb{R}^3 \backslash \gamma_0; \mathbb{R}^2)$

$$\lim_{\ell \to \infty} \int_0^t \iint_{\Omega \times \mathbb{R}^3} w_{\tilde{\vartheta}} \alpha_{f^{\ell},\varepsilon}^{\beta} \nabla_{x,v} f^{\ell+1} \psi \mathrm{d}x\mathrm{d}v = \int_0^t \iint_{\Omega \times \mathbb{R}^3} w_{\tilde{\vartheta}} \alpha_{f,\varepsilon}^{\beta} \nabla_{x,v} f \psi \mathrm{d}x\mathrm{d}v. \tag{6.27}$$

We note that we need to extract a single subsequence, let say $\{\ell_*\} \subset \{\ell\}$, satisfying 6.27 for all test functions in $C_c^{\infty}(\bar{\Omega} \times \mathbb{R}^3 \backslash \gamma_0; \mathbb{R}^2)$. Of course the convergent rate needs not to be uniform and it could vary with test functions.

For each $N \in \mathbb{N}$ we define a set

$$\mathcal{S}_N := \Big\{ (x,v) \in \bar{\Omega} \times \mathbb{R}^3 : \mathrm{dist}(x, \partial\Omega) \le \frac{1}{N} \text{ and } |n(x) \cdot v| \le \frac{1}{N} \Big\} \cup \{|v| > N\}. \tag{6.28}$$

For a given test function we can always find $N \gg 1$ such that

$$supp(\psi) \subset (\mathcal{S}_N)^c := \bar{\Omega} \times \mathbb{R}^3 \backslash \mathcal{S}_N. \tag{6.29}$$

We will exam 6.27 by the identity obtained from the integration by parts

$$\begin{aligned} & \int_0^t \langle w_{\tilde{\vartheta}} \alpha_{f^{\ell},\varepsilon}^{\beta}, \nabla_{x,v} f^{\ell+1} \psi \rangle \\ = & -\int_0^t \langle \alpha_{f^{\ell},\varepsilon}^{\beta} f^{\ell+1}, \nabla_{x,v} (w_{\tilde{\vartheta}} \psi) \rangle & (6.30) \\ & + \sum_{\iota = \pm} \int_0^t \iint_{\gamma} n \alpha_{f^{\ell},\varepsilon,\iota}^{\beta} f_{\iota}^{\ell+1} (w_{\tilde{\vartheta}} \psi) & (6.31) \\ & - \int_0^t \langle \nabla_{x,v} \alpha_{f^{\ell},\varepsilon}^{\beta}, f^{\ell+1} (w_{\tilde{\vartheta}} \psi) \rangle. & (6.32) \end{aligned}$$

We finish this step by proving the convergence of 6.30 and 6.31. From 1.29 and 3.8, if $(x,v) \in (\mathcal{S}_N)^c$ then

$$\begin{aligned} \sup_{\ell \ge 0} |\alpha_{f^{\ell},\varepsilon,\iota}^{\beta}(t,x,v)| & \lesssim |v|^{\beta} + (t+\varepsilon)^{\beta} \sup_{\ell \ge 0} \|\nabla \phi_{f^{\ell}}\|_{\infty}^{\beta} \\ & \lesssim N^{\beta} + (T^{**} + \varepsilon)^{\beta} \sup_{\ell \ge 0} \|w_{\vartheta} f^{\ell}\|_{\infty}^{\beta} \le C_N < +\infty. \end{aligned}$$

Hence we extract a subsequence (let say $\{\ell_N\}$) out of subsequence in *Step 6* such that $\alpha_{f^{\ell_N},\varepsilon,\iota}^{\beta} \overset{*}{\rightharpoonup} A_{\iota} \in L^{\infty}$ *weakly* $-*$ *in* $L^{\infty}((0,T^{**}) \times (\mathcal{S}_N)^c) \cap L^{\infty}((0,T^{**}) \times (\gamma \cap (\mathcal{S}_N)^c))$. Note that $\alpha_{f^{\ell_N},\varepsilon,\iota}^{\beta}$ satisfies $[\partial_t + v \cdot \nabla_x - \iota \nabla_x \phi^{\ell_N} \cdot \nabla_v] \alpha_{f^{\ell_N},\varepsilon,\iota}^{\beta} = 0$ and $\alpha_{f^{\ell_N},\varepsilon,\iota}^{\beta}|_{\gamma_-} = |n \cdot v|^{\beta}$. By passing a limit in the weak formulation we conclude that $[\partial_t + v \cdot \nabla_x - \iota \nabla_x \phi_f \cdot \nabla_v] A_{\iota} = 0$ and $A_{\iota}|_{\gamma_-} = |n \cdot v|^{\beta}$. By the uniqueness of the Vlasov equation ($\nabla \phi_f \in W^{1,p}$ for any $p < \infty$) we derive $A_{\iota} = \alpha_{f,\varepsilon,\iota}^{\beta}$ almost everywhere and hence conclude that

$$\alpha_{f^{\ell_N},\varepsilon,\iota}^{\beta} \overset{*}{\rightharpoonup} \alpha_{f,\varepsilon,\iota}^{\beta} \text{ weakly} - * \text{ in } L^{\infty}((0,T^{**}) \times (\mathcal{S}_N)^c) \cap L^{\infty}((0,T^{**}) \times (\gamma \cap (\mathcal{S}_N)^c)). \tag{6.33}$$

Now the convergence of 6.30 and 6.31 is a direct consequence of strong convergence of 6.4 and the weak$-*$ convergence of 6.33.

Step 8. We devote the entire *Step 8* to prove the convergence of 6.32.

Step 8-a. Let us choose $(x,v)\in(\mathcal{S}_N)^c$. From 1.24

$$If\quad t^{f^\ell}_{\mathbf{b},\iota}\geq t+\varepsilon\quad then\quad \alpha_{f^\ell,\varepsilon,\iota}(t,x,v)=1. \tag{6.34}$$

From now we only consider that case

$$t^{f^\ell}_{\mathbf{b},\iota}(t,x,v)\leq \varepsilon+t. \tag{6.35}$$

If $|v|\geq 2(\varepsilon+T^{**})\sup_\ell\|\nabla\phi^\ell\|_\infty$ then

$$\begin{aligned}|V_\iota^{f^\ell}(s;t,x,v)|&\geq |v|-\int_s^t\|\nabla\phi^\ell(\tau)\|_\infty \mathrm{d}\tau\\&\geq(\varepsilon+T^{**})\sup_\ell\|\nabla\phi^\ell\|_\infty\quad for\ all\ \ell\ and\ s\in[-\varepsilon,T^{**}].\end{aligned} \tag{6.36}$$

For this case we need a version of velocity lemma of $\tilde{\alpha}$ in 1.33, which shows up in the author's previous paper [2], but this time with neutral boundary condition $\pm\nabla\phi^\ell\cdot n=0$ on $\partial\Omega$. So $\tilde{\alpha}$ now takes the form

$$\tilde{\alpha}(t,x,v):=\sqrt{\xi(x)^2+|\nabla\xi(x)\cdot u|^2-2(u\cdot\nabla_x^2\xi(x)\cdot u)\xi(x)}. \tag{6.37}$$

From a direct computation,

$$\begin{aligned}&[\partial_t+u\cdot\nabla_x-\iota\nabla\phi^\ell(t,x)\cdot\nabla_u]\{\xi(x)^2+|\nabla\xi(x)\cdot u|^2-2(u\cdot\nabla_x^2\xi(x)\cdot u)\xi(x)\}\\&=2\{u\cdot\nabla\xi\}\xi+\cancel{2\{u\cdot\nabla^2\xi\cdot u\}\{u\cdot\nabla_x\xi\}}-2u\cdot(u\cdot\nabla\nabla^2\xi\cdot u)\xi\\&\quad\cancel{-2\{u\cdot\nabla^2\xi\cdot u\}\{u\cdot\nabla\xi\}}+2\{-\iota\nabla\phi^\ell\cdot\nabla\xi\}\{\nabla\xi\cdot u\}-4\{-\iota\nabla\phi^\ell\cdot\nabla^2\xi\cdot u\}\xi\\&\lesssim|u\cdot\nabla\xi|^2+|\xi|^2+\{|u|+\frac{1}{|u|}\}(-2(u\cdot\nabla_x^2\xi(x)\cdot u)\xi(x))+|\nabla\phi^\ell\cdot\nabla\xi||\nabla\xi\cdot u|.\end{aligned} \tag{6.38}$$

From the Neumann BC $(n(x)\cdot E(t,x)=0$ on $x\in\partial\Omega)$, we have

$$\begin{aligned}&|\nabla\phi^\ell(t,x)\cdot\nabla\xi(x)|\\&\leq|\nabla\phi^\ell(t,x_*)\cdot\nabla\xi(x_*)|+\|\nabla\phi^\ell(t)\|_{C^1(\bar{\Omega})}\|\xi\|_{C^2(\bar{\Omega})}|x-x_*|\\&\lesssim_\Omega\|\nabla\phi^\ell(t)\|_{C^1(\bar{\Omega})}|\xi(x)|,\end{aligned} \tag{6.39}$$

where $x_*\in\partial\Omega$ such that $|x-x_*|=\inf_{y\in\partial\Omega}|x-y|$.

By controlling the last term of 6.38 by 6.39 and using 6.36, we conclude that

$$\begin{aligned}&\frac{d}{ds}\tilde{\alpha}(s,X_\iota^{f^\ell}(s;t,x,u),V_\iota^{f^\ell}(s;t,x,u))^2\\&\lesssim_\Omega\Big(1+|V_\iota^{f^\ell}(s;t,x,u)|+\frac{1}{|V_\iota^{f^\ell}(s;t,x,u)|}\Big)\tilde{\alpha}(s,X_\iota^{f^\ell}(s;t,x,u),V_\iota^{f^\ell}(s;t,x,u))^2\\&\lesssim_{\Omega,R,N}\Big(1+|V_\iota^{f^\ell}(s;t,x,u)|\Big)\tilde{\alpha}(s,X_\iota^{f^\ell}(s;t,x,u),V_\iota^{f^\ell}(s;t,x,u))^2,\end{aligned}$$

so

$$\begin{aligned}&|\tilde{\alpha}(s, X_\iota^{f^\ell}(s;t,x,v), V_\iota^{f^\ell}(s;t,x,v))|\\ &\geq \frac{1}{C_\Omega}\tilde{\alpha}(t,x,v)e^{-C_\Omega\frac{|t-s|}{(\varepsilon+T^{**})\sup_\ell\|\nabla\phi^\ell\|_\infty}}\\ &\geq \frac{e^{-\frac{C_\Omega}{\sup_\ell\|\nabla\phi^\ell\|_\infty}}}{C_\Omega}\times\frac{1}{N}\quad \textit{for all } \ell \textit{ and } s\in[-\varepsilon,T^{**}].\end{aligned}$$

Especially at $s = t - t_{\mathbf{b},\iota}^{f^\ell}(t,x,v)$, from 6.37,

$$|n(x_{\mathbf{b},\iota}^{f^\ell})\cdot v_{\mathbf{b},\iota}^{f^\ell}| \geq \frac{e^{-\frac{C_\Omega}{\sup_\ell\|\nabla\phi^\ell\|_\infty}}}{C_\Omega}\times\frac{1}{N}\quad \textit{for all } \ell. \tag{6.40}$$

Step 8-b. From now on we assume 6.35 and

$$\begin{aligned}&|v|\leq 2(\varepsilon+T^{**})\sup_\ell\|\nabla\phi^\ell\|_\infty,\\ &\textit{or, from 1.27, } |V_\iota^{f^\ell}(s;t,x,v)|\leq 3(\varepsilon+T^{**})\sup_\ell\|\nabla\phi^\ell\|_\infty \quad \textit{for} \quad s\in[-\varepsilon,T^{**}].\end{aligned} \tag{6.41}$$

Let $(X_{n,\iota}^{f^\ell}, X_{\|,\iota}^{f^\ell}, V_{n,\iota}^{f^\ell}, V_{\|,\iota}^{f^\ell})$ satisfy 2.8, 2.6, and 2.10 with $E = -\iota\nabla\phi^\ell$.

Let us define

$$\tau_1 := \sup\big\{\tau\geq 0 : V_{n,\iota}^{f^\ell}(s;t,x,v)\geq 0 \textit{ for all } s\in[t-t_{\mathbf{b},\iota}^{f^\ell}(t,x,v),\tau]\big\}. \tag{6.42}$$

Since $(X_\iota^{f^\ell}(s;t,x,v), V_\iota^{f^\ell}(s;t,x,v))$ is C^1 (note that $\nabla\phi^\ell\in C^1_{t,x}$) in s we have $V_{n,\iota}^{f^\ell}(\tau_1;t,x,v) = 0$.

We claim that, there exists some constant $\delta_{**} = O_{\varepsilon,T^{**},\sup_\ell\|\nabla\phi^\ell\|_{C^1}}(\frac{1}{N})$ in 6.48 which does not depend on ℓ such that

$$\begin{aligned}&\textit{If}\quad 0\leq V_{n,\iota}^{f^\ell}(t-t_{\mathbf{b},\iota}^{f^\ell}(t,x,v);t,x,v)<\delta_{**} \textit{and 6.41}, \quad \textit{then}\\ &V_{n,\iota}^{f^\ell}(s;t,x,v)\leq e^{C|s-(t-t_{\mathbf{b},\iota}^{f^\ell}(t,x,v))|^2}V_{n,\iota}^{f^\ell}(t-t_{\mathbf{b},\iota}^{f^\ell}(t,x,v);t,x,v)\\ &\quad \textit{for } s\in[t-t_{\mathbf{b},\iota}^{f^\ell},\tau_1].\end{aligned} \tag{6.43}$$

For the proof we regard the equations 2.8, 2.6, and 2.10 as the forward-in-time problem with an initial datum at $s = t - t_{\mathbf{b},\iota}^{f^\ell}(t,x,v)$. Clearly we have $X_{n,\iota}^{f^\ell}(t - t_{\mathbf{b},\iota}^{f^\ell}(t,x,v);t,x,v) = 0$ and $V_{n,\iota}^{f^\ell}(t - t_{\mathbf{b},\iota}^{f^\ell}(t,x,v);t,x,v)\geq 0$ from Lemma 2.1. Again from Lemma 2.1, if $V_{n,\iota}^{f^\ell}(t - t_{\mathbf{b},\iota}^{f^\ell}(t,x,v);t,x,v) = 0$ then $X_{n,\iota}^{f^\ell}(s;t,x,v) = 0$ for all $s\geq t - t_{\mathbf{b},\iota}^{f^\ell}(t,x,v)$. From now on we assume $V_{n,\iota}^{f^\ell}(t - t_{\mathbf{b},\iota}^{f^\ell}(t,x,v);t,x,v)] > 0$. From 2.10, as long as $t - t_{\mathbf{b},\iota}^{f^\ell}(t,x,v)\leq s\leq T^{**}$ and

$$V_{n,\iota}^{f^\ell}(s;t,x,v)\geq 0 \quad \textit{and} \quad X_{\perp,\iota}^{f^\ell}(s;t,x,v)\leq\frac{1}{N}\ll 1, \tag{6.44}$$

then we have

$$\dot{V}_n^{f^\ell}(s) = \underbrace{[V_{\|,\iota}^{f^\ell}(s)\cdot\nabla^2\eta(X_{\|,\iota}^{f^\ell}(s))\cdot V_{\|,\iota}^{f^\ell}(s)]\cdot n(X_{\|,\iota}^{f^\ell}(s))}_{\leq 0 \textit{ from 1.40}}$$

$$
\begin{aligned}
&- \underbrace{\nabla\phi^\ell(s, X^{f^\ell}(s)) \cdot [-n(X^{f^\ell}_{\|,\iota}(s))]}_{=O(1)\sup_\ell \|\nabla\phi^\ell\|_{C^1}\times X^{f^\ell}_{n,\iota}(s) \ \ from\ 2.13} \\
&- \underbrace{X^{f^\ell}_{n,\iota}(s)[V^{f^\ell}_{\|,\iota}(s)\cdot\nabla^2 n(X^{f^\ell}_{\|,\iota}(s))\cdot V^{f^\ell}_{\|,\iota}(s)]\cdot n(X^{f^\ell}_{\|,\iota}(s))}_{=O(1)\{3(\varepsilon+T^{**})\sup_\ell\|\nabla\phi^\ell\|_\infty\}^2\times X^{f^\ell}_{n,\iota}(s) \ \ from\ 6.41} \qquad (6.45)\\
&\le C(1+\varepsilon+T^{**})^2(\sup_\ell \|\nabla\phi^\ell\|_{C^1}\sup_\ell\|\nabla\phi^\ell\|_\infty)\times X^{f^\ell}_{n,\iota}(s).
\end{aligned}
$$

Let us consider 6.45 together with $\dot{X}^{f^\ell}_n(s;t,x,v) = V^{f^\ell}_n(s;t,x,v)$. Then, as long as s satisfies 6.44,

$$
\begin{aligned}
&V^{f^\ell}_{n,\iota}(s)\\
&= V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota}) + \int_{t-t^{f^\ell}_{\mathbf{b},\iota}}^s \dot{V}^{f^\ell}_n(\tau)\mathrm{d}\tau\\
&\le V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota}) + \int_{t-t^{f^\ell}_{\mathbf{b},\iota}}^s C(1+\varepsilon+T^{**})^2(\sup_\ell\|\nabla\phi^\ell\|_{C^1}\sup_\ell\|\nabla\phi^\ell\|_\infty)\times X^{f^\ell}_{n,\iota}(\tau)\mathrm{d}\tau\\
&= V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota})\\
&\quad + \int_{t-t^{f^\ell}_{\mathbf{b},\iota}}^s C(1+\varepsilon+T^{**})^2(\sup_\ell\|\nabla\phi^\ell\|_{C^1}\sup_\ell\|\nabla\phi^\ell\|_\infty)\int_{t-t^{f^\ell}_{\mathbf{b},\iota}}^\tau V^{f^\ell}_{n,\iota}(\tau')\mathrm{d}\tau'\mathrm{d}\tau.
\end{aligned}
$$

Following the same argument of the proof of Lemma 2.4, we derive that

$$
\begin{aligned}
&V^{f^\ell}_{n,\iota}(s) \le\ V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota})\\
&+ C(1+\varepsilon+T^{**})^2(\sup_\ell\|\nabla\phi^\ell\|_{C^1}\sup_\ell\|\nabla\phi^\ell\|_\infty)\int_{t-t^{f^\ell}_{\mathbf{b},\iota}}^s |s-(t-t^{f^\ell}_{\mathbf{b},\iota})|V^{f^\ell}_{n,\iota}(\tau')\mathrm{d}\tau'.
\end{aligned}
$$

From the Gronwall's inequality, we derive that, as long as 6.44 holds,

$$
\begin{aligned}
&V^{f^\ell}_{n,\iota}(s;t,x,v)\\
&\le V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota}(t,x,v))e^{C(1+\varepsilon+T^{**})^2(\sup_\ell\|\nabla\phi^\ell\|_{C^1}\sup_\ell\|\nabla\phi^\ell\|_\infty)\times|s-(t-t^{f^\ell}_{\mathbf{b},\iota}(t,x,v))|^2}.
\end{aligned} \qquad (6.46)
$$

Now we verify the conditions of 6.44 for all $-\varepsilon \le t-t^{f^\ell}_{\mathbf{b},\iota}(t,x,v) \le s \le T^{**}$. Note that we are only interested in the case of $V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota}(t,x,v);t,x,v) < \delta_{**}$. From the argument of 6.45, ignoring negative curvature term,

$$
\begin{aligned}
&|X^{f^\ell}_{n,\iota}(s;t,x,v)|\\
&\le (\varepsilon+T^{**})|V^{f^\ell}_{n,\iota}(t^{f^\ell}_{\mathbf{b},\iota};t,x,v)|\\
&\quad + C[1+(\varepsilon+T^{**})^2\sup_\ell\|\nabla\phi^\ell\|_\infty]\sup_\ell\|\nabla\phi^\ell\|_{C^1}\int_{t-t^{f^\ell}_{\mathbf{b},\iota}}^s\int_{t-t^{f^\ell}_{\mathbf{b},\iota}}^\tau |X^{f^\ell}_{n,\iota}(\tau;t,x,v)|\mathrm{d}\tau\mathrm{d}s\\
&\le (\varepsilon+T^{**})|V^{f^\ell}_{n,\iota}(t^{f^\ell}_{\mathbf{b},\iota};t,x,v)| + C\int_{t-t^{f^\ell}_{\mathbf{b},\iota}}^s |\tau-(t-t^{f^\ell}_{\mathbf{b},\iota})||X^{f^\ell}_{n,\iota}(\tau;t,x,v)|\mathrm{d}\tau.
\end{aligned}
$$

Then by the Gronwall's inequality we derive that, in case of 6.35,

$$
|X^{f^\ell}_{n,\iota}(s;t,x,v)| \le C_{\varepsilon+T^{**}}|V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota};t,x,v)| \ \ for\ all\ -\varepsilon \le t-t^{f^\ell}_{\mathbf{b},\iota} \le s \le t \le T^{**}. \qquad (6.47)
$$

If we choose

$$\delta_{**} = \frac{o(1)}{|T^{**}+\varepsilon|} \times \frac{1}{N}, \tag{6.48}$$

then 6.46 holds for $-\varepsilon \le t - t^{f^\ell}_{\mathbf{b},\iota}(t,x,v) \le s \le T^{**}$. Hence we complete the proof of 6.43.

Step 8-c. Suppose that 6.41 holds and $0 \le V^{f^\ell}_{n,\iota}(t - t^{f^\ell}_{\mathbf{b},\iota}(t,x,v);t,x,v) < \delta_{**}$ with δ_{**} of 6.48. Recall the definition of τ_1 in 6.42. Inductively we define $\tau_2 := \sup\big\{\tau \ge 0 : V^{f^\ell}_{n,\iota}(s;t,x,v) \le 0 \ \textit{for all}\ s \in [\tau_1,\tau]\big\}$ and $\tau_3,\tau_4,\cdots$. Clearly such points can be countably many at most in an interval of $[t-t^{f^\ell}_{\mathbf{b},\iota},t]$. Suppose $\lim_{k\to\infty}\tau_k = t$. Then choose $k_0 \gg 1$ such that $|\tau_{k_0} - t| \ll |V^{f^\ell}_{n,\iota}(t - t^{f^\ell}_{\mathbf{b},\iota};t,x,v)|$. Then, for $s \in [\tau_{k_0},t]$, from 6.45 and 6.41,

$$|V^{f^\ell}_{n,\iota}(t;t,x,v)| \lesssim |V^{f^\ell}_{n,\iota}(t - t^{f^\ell}_{\mathbf{b},\iota};t,x,v)|. \tag{6.49}$$

Now we assume that $\tau_{k_0} < t \le \tau_{k_0+1}$. From the definition of τ_i in 6.42 we split the case in two.

Case 1: Suppose $V^{f^\ell}_{n,\iota}(s;t,x,v) > 0$ for $s \in (\tau_{k_0},t)$.

From 6.45 and 6.47

$$V^{f^\ell}_{n,\iota}(t;t,x,v) \lesssim \int_{\tau_{k_0}}^{T^{**}} X^{f^\ell}_{n,\iota}(s) \lesssim |V^{f^\ell}_{n,\iota}(t - t^{f^\ell}_{\mathbf{b},\iota};t,x,v)|. \tag{6.50}$$

Case 2: Suppose $V^{f^\ell}_{n,\iota}(s;t,x,v) < 0$ for $s \in (\tau_{k_0},t)$.

Suppose

$$-V^{f^\ell}_{n,\iota}(t;t,x,v) = |V^{f^\ell}_{n,\iota}(t;t,x,v)| \ge |V^{f^\ell}_{n,\iota}(t - t^{f^\ell}_{\mathbf{b},\iota};t,x,v)|^A \quad \textit{for any}\ 0 < A < \frac{1}{2}. \tag{6.51}$$

From 6.45, now taking account of the curvature term this time, we derive that

$$\begin{aligned} -V^{f^\ell}_{n,\iota}(t;t,x,v) \le & \int_{\tau_{k_0}}^{t} (-1)[V^{f^\ell}_{\|,\iota}(s)\cdot\nabla^2\eta(X^{f^\ell}_{\|,\iota}(s))\cdot V^{f^\ell}_{\|,\iota}(s)]\cdot n(X^{f^\ell}_{\|,\iota}(s))\mathrm{d}s \\ & + C|V^{f^\ell}_{n,\iota}(t - t^{f^\ell}_{\mathbf{b},\iota}(t,x,v);t,x,v)|, \end{aligned}$$

where we have used 6.41 and 6.47. From 6.51 the above inequality implies that, for $|V^{f^\ell}_{n,\iota}(t - t^{f^\ell}_{\mathbf{b},\iota}(t,x,v);t,x,v)| \ll 1$,

$$|V^{f^\ell}_{n,\iota}(t - t^{f^\ell}_{\mathbf{b},\iota};t,x,v)|^A \lesssim \int_{\tau_{k_0}}^{t} (-1)[V^{f^\ell}_{\|,\iota}(s)\cdot\nabla^2\eta(X^{f^\ell}_{\|,\iota}(s))\cdot V^{f^\ell}_{\|,\iota}(s)]\cdot n(X^{f^\ell}_{\|,\iota}(s))\mathrm{d}s.$$

Note that $|\frac{d}{ds}V^{f^\ell}_{\|,\iota}(s)|$ and $|\frac{d}{ds}X^{f^\ell}_{\|,\iota}(s)|$ are all bound from $\nabla\phi^\ell \in C^1$, 6.41, and 6.47. Hence we have

$$\begin{aligned} & \frac{1}{2}|V^{f^\ell}_{n,\iota}(t - t^{f^\ell}_{\mathbf{b},\iota};t,x,v)|^A \\ & \lesssim \int_{\tau_{k_0}}^{t-|V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota};t,x,v)|^A} (-1)[V^{f^\ell}_{\|,\iota}(s)\cdot\nabla^2\eta(X^{f^\ell}_{\|,\iota}(s))\cdot V^{f^\ell}_{\|,\iota}(s)]\cdot n(X^{f^\ell}_{\|,\iota}(s))\mathrm{d}s. \end{aligned} \tag{6.52}$$

On the other hand, if $t - |V^{f^\ell}_{n,\iota}(t - t^{f^\ell}_{\mathbf{b},\iota};t,x,v)|^A \le \tau_{k_0}$ then $|t-\tau_{k_0}| \le |V^{f^\ell}_{n,\iota}(t - t^{f^\ell}_{\mathbf{b},\iota};t,x,v)|^A$, which implies that, from 6.45, 6.41, and 6.47,

$$|V^{f^\ell}_{n,\iota}(t;t,x,v)| \lesssim |V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota};t,x,v)|^A \quad \textit{for any } 0<A<\frac{1}{2}. \tag{6.53}$$

Now we consider $X^{f^\ell}_{n,\iota}(t;t,x,v)$. From 6.45 and $\dot{X}^{f^\ell}_n(s;t,x,v) = V^{f^\ell}_n(s;t,x,v)$ together with 6.47 and 6.41

$$\begin{aligned}
&X^{f^\ell}_{n,\iota}(t;t,x,v)\\
&\lesssim |V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota};t,x,v)| + \int_{\tau_{k_0}}^{t}\int_{\tau_{k_0}}^{\tau} \underbrace{[V^{f^\ell}_{\|,\iota}(s)\cdot\nabla^2\eta(X^{f^\ell}_{\|,\iota}(s))\cdot V^{f^\ell}_{\|,\iota}(s)]\cdot n(X^{f^\ell}_{\|,\iota}(s))}_{\leq 0}\,\mathrm{d}s\mathrm{d}\tau\\
&\lesssim |V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota};t,x,v)| + |V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota};t,x,v)|^A\\
&\quad\times\int_{\tau_{k_0}}^{t-|V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota};t,x,v)|^A} [V^{f^\ell}_{\|,\iota}(s)\cdot\nabla^2\eta(X^{f^\ell}_{\|,\iota}(s))\cdot V^{f^\ell}_{\|,\iota}(s)]\cdot n(X^{f^\ell}_{\|,\iota}(s))\mathrm{d}s\\
&\lesssim |V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota};t,x,v)| - |V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota};t,x,v)|^{2A} \quad \textit{from 6.52}\\
&\lesssim |V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota};t,x,v)| - |V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota};t,x,v)|^{1-}\\
&< 0,
\end{aligned} \tag{6.54}$$

for $|V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota};t,x,v)| \ll 1$. Clearly this cannot happen since $x\in\bar{\Omega}$ and $x_n\geq 0$. Therefore our assumption 6.51 was wrong and we conclude 6.53.

Step 8-d. From 6.43, 6.49, 6.50, and 6.53 in *Step 8-a* and *Step 8-b*, we conclude that the same estimate 6.53 for $|V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota};t,x,v)| \ll 1$ in the case of 6.35 and 6.41. Finally from 6.34, 6.40, 6.43, and 6.53 we conclude that

$$|V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota}(t,x,v);t,x,v)| \geq \left(\frac{1}{N}\right)^{1/A} \quad (t,x,v)\in[0,T^{**}]\times(\mathcal{S}_N)^c. \tag{6.55}$$

From (2.36), (2.37), (2.40), and (2.41) in Lemma 2.4 in [21], and combing with 6.41 we have

$$\begin{aligned}
&\nabla_{x,v}t^{f^\ell}_{\mathbf{b},\iota} \lesssim_\Omega \frac{1}{|v_n(x^{f^\ell}_{\mathbf{b},\iota})|}|v^{f^\ell}_{\mathbf{b},\iota}||t^{f^\ell}_{\mathbf{b},\iota}|^2 e^{\|\nabla^2\phi^\ell\|_\infty (t^{f^\ell}_{\mathbf{b},\iota})^2/2} \lesssim_\Omega |V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota}(t,x,v);t,x,v)|,\\
&\nabla_{x,v}v_n(x^{f^\ell}_{\mathbf{b},\iota})\\
&\lesssim_\Omega \frac{1}{|v_n(x^{f^\ell}_{\mathbf{b},\iota})|}\left(|v||t^{f^\ell}_{\mathbf{b},\iota}|^2 e^{\|\nabla^2\phi^\ell\|_\infty (t^{f^\ell}_{\mathbf{b},\iota})^2/2} + |v^{f^\ell}_{\mathbf{b},\iota}||t^{f^\ell}_{\mathbf{b},\iota}|^3(1+t^{f^\ell}_{\mathbf{b},\iota})e^{\|\nabla^2\phi^\ell\|_\infty (t^{f^\ell}_{\mathbf{b},\iota})^2/2}\right)\\
&\lesssim_\Omega |V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota}(t,x,v);t,x,v)|.
\end{aligned} \tag{6.56}$$

Therefore from above we have

$$\begin{aligned}
|\nabla_{x,v}\alpha^\beta_{f^\ell,\varepsilon,\iota}(t,x,v)| &\lesssim_\chi \beta|\alpha^\beta_{f^\ell,\varepsilon,\iota}(t,x,v)|^{\beta-1}|\nabla_{x,v}t^{f^\ell}_{\mathbf{b},\iota} + \nabla_{x,v}v_n(x^{f^\ell}_{\mathbf{b},\iota})|\\
&\lesssim_\chi |V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota};t,x,v)|^{\beta-1}|V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota};t,x,v)|^{-1}.
\end{aligned} \tag{6.57}$$

Combing 6.55 and 6.57 we achieve

$$\sup_{\substack{\ell\in\mathbb{N},\ (x,v)\in(\mathcal{S}_N)^c,\\ -\varepsilon\leq t-t^{f^\ell}_{\mathbf{b},\iota}(t,x,v)\leq t\leq T^{**}}} |\nabla_{x,v}\alpha^\beta_{f^\ell,\varepsilon,\iota}(t,x,v)| \lesssim \frac{1}{|V^{f^\ell}_{n,\iota}(t-t^{f^\ell}_{\mathbf{b},\iota};t,x,v)|^{2-\beta}} \lesssim_{\varepsilon,N,T^{**}} 1.$$

Hence we extract another subsequence out of all previous steps for $\iota = +$ first, and then from that subsequence extract further another subsequence for $\iota = -$ (and redefine this as $\{\ell_N\}$) such that

$$\nabla_{x,v}\alpha^{\beta}_{f^{\ell_N},\varepsilon,\iota} \overset{*}{\rightharpoonup} \nabla_{x,v}\alpha^{\beta}_{f,\varepsilon,\iota} \ \textit{weakly} - * \ \textit{in}\ L^\infty((-\varepsilon, T^{**}) \times (\mathcal{S}_N)^c). \tag{6.58}$$

Note that the limiting function is identified from 6.33. Clearly the convergence of 6.32 is an easy consequence of strong convergence of 6.4 and the weak$-*$ convergence of 6.58.

Step 8-c. Inductively we extract $\{\ell_N\} \supseteq \{\ell_{N+1}\} \supseteq \{\ell_{N+2}\} \supseteq \cdots$ by following all the process in *Step 7* to *Step 8-b*. Then finally we define the subsequence, by the Cantor's diagonal argument,

$$\ell_* = \ell_\ell. \tag{6.59}$$

Then clearly 6.27 holds with this subsequence for any test function ψ. For any $\psi \in C_c^\infty(\bar{\Omega} \times \mathbb{R}^3 \backslash \gamma_0; \mathbb{R}^2)$ there exists $N_\psi \in \mathbb{N}$ such that $supp(\psi) \subset (\mathcal{S}_{N_\psi})^c$. Then all the proofs work.

This implies 6.26 from 6.30, 6.31, 6.32. Positivity $F = \mu + \sqrt{\mu} f \geq 0$ comes from *Step 1* and *Step 6*.

Step 5. Choose $t > t' \geq 0$. Instead of expanding $h_\iota(t, x, v)$ at $t = 0$ as 3.19, we expand at t'. Then by the iteration we have 3.23 replacing 0 by t'. Collecting all terms at time t', we have

$$\begin{aligned}\|h_\iota(t)\|_\infty \leq & \|h_\iota(t')\|_\infty \Big\{ \mathbf{1}_{t_{1,\iota} \leq t'} e^{-\int_{t'}^t \nu_\iota} \\ & + \mathbf{1}_{t_{1,\iota} \geq t'} \frac{e^{-\int_{t_{1,\iota}}^t \nu_\iota}}{\widetilde{w}_\vartheta(V_\iota(t_{1,\iota}))} \int_{\prod_{j=1}^{k-1} \mathcal{V}_{j,\iota}} \sum_{l=1}^{k-1} \mathbf{1}_{\{t_{l+1,\iota}^{\ell-l} \leq t' < t_{l,\iota}^{\ell-(l-1)}\}} \mathrm{d}\Sigma_{l_\iota}(t') \Big\}.\end{aligned} \tag{6.60}$$

Since 3.21 is a probability measure and $|e^{-\int_{t'}^t \nu_\iota} - 1| \ll |t - t'|$ for $|t - t'| \ll 1$,

$$|6.60 - \|h_\iota(t')\|_\infty| \leq O(|t - t'|) + \int_{\prod_{j=1}^{k-1} \mathcal{V}_{j,\iota}} \mathbf{1}_{\{t_\iota^k(t,x,v,u^1,\cdots,u^{k-1})>0\}} \mathrm{d}\Sigma_{k-1,\iota}^{k-1}.$$

Then by 3.23 we have $\|h(t)\|_\infty - \|h(t')\|_\infty < \frac{1}{2^k} + O_k(|t - t'|)$. For large k, choosing $|t - t'| \ll 1$, we can prove $\|h(t)\|_\infty - \|h(t')\|_\infty \ll 1$ as $|t - t'| \ll 1$.

Now we can expand $h(t', x, v)$ at t by 3.19. Following the same argument we have $\|h(t')\|_\infty - \|h(t)\|_\infty \ll 1$ as $|t - t'| \ll 1$. Hence $\|w_\vartheta f(t)\|_\infty$ is continuous in t.

The continuity of $\|\nabla_v f(t)\|_{L^3_x L^{1+\delta}_v}$ and $\|w_{\tilde{\vartheta}} \alpha^{\beta}_{f,\varepsilon} \nabla_{x,v} f(t)\|_p^p + \int_0^t |w_{\tilde{\vartheta}} \alpha^{\beta}_{f,\varepsilon} \nabla_{x,v} f(t)|_{p,+}^p$ is an easy consequence of 5.5-5.9, and 4.46, 4.75, 4.63 as well. □

7. L^2 coercivity.

Proposition 6. *Suppose* (f, ϕ) *solves 1.15, 1.16, and 1.17. Then there is* $0 < \lambda_2 \ll 1$ *such that for* $0 \leq s \leq t$,

$$\begin{aligned} & \|e^{\lambda_2 t} f(t)\|_2^2 + \|e^{\lambda_2 t} \nabla \phi(t)\|_2^2 \\ & + \int_s^t \|e^{\lambda_2 \tau} f(\tau)\|_\nu^2 + \|e^{\lambda_2 \tau} \nabla \phi_f(\tau)\|_2^2 \mathrm{d}\tau + \int_s^t |e^{\lambda_2 \tau} f|_{2,+}^2 \\ \lesssim & \|e^{\lambda_2 s} f(s)\|_2^2 + \|e^{\lambda_2 s} \nabla \phi_f(s)\|_2^2 \\ & + \sup_{s \leq \tau \leq t} \|w_\vartheta f(\tau)\|_\infty \int_s^t \|e^{\lambda_2 \tau} f(\tau)\|_\nu^2. \end{aligned} \tag{7.1}$$

The null space of linear operator L is a six-dimensional subspace of $L^2_v(\mathbb{R}^3;\mathbb{R}^2)$ spanned by orthonormal vectors

$$\left\{\begin{bmatrix}\sqrt{\mu}\\0\end{bmatrix},\begin{bmatrix}0\\\sqrt{\mu}\end{bmatrix},\begin{bmatrix}\frac{v_i}{\sqrt{2}}\sqrt{\mu}\\\frac{v_i}{\sqrt{2}}\sqrt{\mu}\end{bmatrix},\begin{bmatrix}\frac{|v|^2-3}{2\sqrt{2}}\sqrt{\mu}\\\frac{|v|^2-3}{2\sqrt{2}}\sqrt{\mu}\end{bmatrix}\right\},\ i=1,2,3, \tag{7.2}$$

and the projection of f onto the null space $N(L)$ is denoted by

$$\begin{aligned}&\mathbf{P}f(t,x,v)\\&:=\left\{a_+(t,x)\begin{bmatrix}\sqrt{\mu}\\0\end{bmatrix}+a_-(t,x)\begin{bmatrix}0\\\sqrt{\mu}\end{bmatrix}+b(t,x)\cdot\frac{v}{\sqrt{2}}\begin{bmatrix}\sqrt{\mu}\\\sqrt{\mu}\end{bmatrix}+c(t,x)\frac{|v|^2-3}{2\sqrt{2}}\begin{bmatrix}\sqrt{\mu}\\\sqrt{\mu}\end{bmatrix}\right\}.\end{aligned} \tag{7.3}$$

In order to prove the proposition we need the following:

Lemma 7.1. *There exists a function $G(t)$ such that, for all $0\le s\le t$, $G(s)\lesssim\|f(s)\|_2^2$ and*

$$\begin{aligned}&\int_s^t\|\mathbf{P}f(\tau)\|_\nu^2+\int_s^t\|\nabla\phi_f\|_2^2\\&\lesssim G(t)-G(s)+\int_s^t\|(\mathbf{I}-\mathbf{P})f(\tau)\|_\nu^2+\int_s^t|(1-P_\gamma)f(\tau)|_{2,+}^2\\&\quad+\int_s^t\|\nu^{-1/2}\Gamma(f,f)\|_2^2+\int_s^t\|w_\vartheta f(\tau)\|_\infty\|\mathbf{P}f(\tau)\|_2^2.\end{aligned} \tag{7.4}$$

Proof of Proposition 6. **Step 1.** Without loss of generality we prove the result with $s=0$. We have an L^2-estimate from $\int_0^t\langle 2e^{\lambda_2 t}f, 1.23\rangle$

$$\begin{aligned}&\|e^{\lambda_2 t}f(t)\|_2^2-\|f(0)\|_2^2+\int_0^t|e^{\lambda_2\tau}(1-P_\gamma)f^j|_{2,+}^2\\&+\int_0^t\iint_{\Omega\times\mathbb{R}^3}v\cdot\nabla\phi_f e^{2\lambda_2\tau}(f_+^2-f_-^2)+2\int_0^t e^{\lambda_2\tau}\langle f,Lf\rangle\\&=2\int_0^t e^{2\lambda_2\tau}\langle f,\Gamma(f,f)\rangle-2\int_0^t e^{2\lambda_2\tau}\int_\Omega\nabla\phi_f\cdot\int_{\mathbb{R}^3}v\sqrt{\mu}(f_+-f_-)\\&+2\lambda_2\int_0^t\|e^{\lambda_2\tau}f(\tau)\|_2^2,\end{aligned}$$

where

$$P_\gamma f:=\begin{bmatrix}P_\gamma f_+\\P_\gamma f_-\end{bmatrix}:=\begin{bmatrix}c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0}f_+(u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u\\c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0}f_-(u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u\end{bmatrix}. \tag{7.5}$$

On the other hand multiplying $\sqrt{\mu(v)}\phi_f(t,x)$ with a test function $\psi(t,x)$ to 1.23 and applying the Green's identity, (from the charge conservation) we obtain

$$\begin{aligned}&\int_\Omega\nabla\phi_f(t,x)\cdot\int_{\mathbb{R}^3}v\sqrt{\mu}(f_+-f_-)\mathrm{d}v\mathrm{d}x\\&=-\int_\Omega\phi_f(t,x)\left(\int_{\mathbb{R}^3}v\cdot\nabla_x\sqrt{\mu}(f_+-f_-)\mathrm{d}v\right)\mathrm{d}x\\&\quad+\iint_{\partial\Omega\times\mathbb{R}^3}\phi_f(t,x)(f_+-f_-)\sqrt{\mu}\{n\cdot v\}\mathrm{d}v\mathrm{d}S_x\\&=\int_\Omega\phi_f(t,x)\partial_\tau\left(\int_{\mathbb{R}^3}(f_+-f_-)\sqrt{\mu}\mathrm{d}v\right)\mathrm{d}x\end{aligned}$$

$$+\iint_{\partial\Omega\times\mathbb{R}^3}\phi_f(t,x)(f_+-f_-)\sqrt{\mu}\{n\cdot v\}\mathrm{d}v\mathrm{d}S_x.$$

From 1.11, the last boundary contribution equals zero. Now we use 1.16 and deduce that

$$\begin{aligned}
&\int_0^t e^{2\lambda_2\tau}\int_\Omega \phi_f(t,x)\partial_\tau\left(\int_{\mathbb{R}^3}(f_+-f_-)(\tau)\sqrt{\mu}\mathrm{d}v\right)\mathrm{d}x\mathrm{d}\tau\\
=&-\int_0^t e^{2\lambda_2\tau}\int_\Omega \phi_f(t,x)\partial_\tau\Delta_x\phi_f(\tau,x)\mathrm{d}x\mathrm{d}\tau\\
=&\frac{1}{2}\int_0^t e^{2\lambda_2\tau}\int_\Omega \partial_\tau|\nabla_x\phi_f(\tau,x)|^2\mathrm{d}x\mathrm{d}\tau\\
=&\frac{1}{2}\left(\int_\Omega e^{2\lambda_2 t}|\nabla_x\phi_f(t,x)|^2\mathrm{d}x\right)-\frac{1}{2}\left(\int_\Omega|\nabla_x\phi_f(0,x)|^2\mathrm{d}x\right)\\
&-\lambda_2\int_0^t e^{2\lambda_2\tau}\int_\Omega|\nabla_x\phi_f(\tau,x)|^2\mathrm{d}x\mathrm{d}\tau.
\end{aligned}$$

Hence we derive

$$\begin{aligned}
&\|e^{\lambda_2 t}f(t)\|_2^2+\|e^{\lambda_2 t}\nabla\phi_f(t)\|_2^2+\int_0^t\iint_{\Omega\times\mathbb{R}^3}e^{2\lambda_2\tau}v\cdot\nabla\phi_f(f_+^2-f_-^2)\\
&+2C\int_0^t\iint_{\Omega\times\mathbb{R}^3}\|e^{\lambda_2\tau}(\mathbf{I}-\mathbf{P})f\|_\nu^2+\int_0^t|e^{\lambda_2\tau}(1-P_\gamma)f^j|_{2,+}^2\\
\lesssim&\|f(0)\|_2^2+\|\nabla\phi_f(0)\|_2^2+\int_0^t\|e^{\lambda_2\tau}\nu^{-1/2}\Gamma(f,f)\|_2^2\\
&+\{\lambda_2+o(1)\}\int_0^t\|e^{\lambda_2\tau}f\|_\nu^2+\lambda_2\int_0^t\|e^{\lambda_2\tau}\nabla_x\phi_f\|_2^2.
\end{aligned}$$

Now we apply Lemma 7.1 and add $o(1)\times$ 7.4 to the above inequality and choose $0<\lambda_2\ll 1$ to conclude 7.1 except the full boundary control.

Step 2. Note that from 7.5, $P_\gamma f_\pm=z_\pm(t,x)\sqrt{\mu(v)}$ for a suitable functions $z_\pm(t,x)$ on the boundary. Then for $\iota=+$ or $-$, for $0<\varepsilon\ll 1$

$$\begin{aligned}
|P_\gamma f_\iota|_{\gamma,2}^2=&\int_{\partial\Omega}|z_\iota(t,x)|^2\mathrm{d}x\times\int_{\mathbb{R}^3}\mu(v)|n(x)\cdot v|\mathrm{d}v\\
\lesssim&\int_{\partial\Omega}|z_\iota(t,x)|^2\mathrm{d}x\times\int_{\gamma_+(x)\setminus\gamma_+^\varepsilon(x)}\mu(v)^{3/2}|n(x)\cdot v|\mathrm{d}v\\
=&|\mathbf{1}_{\gamma_+\setminus\gamma_+^\varepsilon}\mu^{1/4}P_\gamma f_\iota|_{2,+}^2.
\end{aligned}$$

Since $P_\gamma f=f-(1-P_\gamma)f$ on γ_+ we have $|\mathbf{1}_{\gamma_+\setminus\gamma_+^\varepsilon}\mu^{1/4}P_\gamma f|_{2,+}^2\lesssim|\mathbf{1}_{\gamma_+\setminus\gamma_+^\varepsilon}\mu^{1/4}f|_{2,+}^2+|(1-P_\gamma)f|_{2,+}^2$. Therefore

$$\int_0^t|P_\gamma f|_{\gamma,2}^2\lesssim\int_0^t|\mathbf{1}_{\gamma_+\setminus\gamma_+^\varepsilon}\mu^{1/4}f|_{2,+}^2+\int_0^t|(1-P_\gamma)f|_{2,+}^2.\tag{7.6}$$

Note that

$$\begin{aligned}
&\left|[\partial_t+v\cdot\nabla_x-q\nabla\phi\cdot\nabla_v](\mu^{1/4}f)\right|\\
\lesssim&\mu^{1/4}\{|v||\nabla_x\phi|f+|v||\nabla_x\phi|+|Lf|+|\Gamma(f,f)|\}.
\end{aligned}$$

By the trace theorem Lemma 2.2,

$$\begin{aligned}&\int_0^t |\mathbf{1}_{\gamma_+\setminus\gamma_+^\varepsilon}\mu^{1/4}f|_{2,+}^2\\ &\lesssim \|f_0\|_2 + (1+\|wf\|_\infty)\int_0^t \|f\|_2^2 + \int_0^t \|\nabla\phi\|_2^2.\end{aligned} \tag{7.7}$$

Adding $o(1)\times$ 7.6 to the result of Step 1 and using 7.7 we conclude 7.1. □

Proof of Lemma 7.1. From the Green's identity, a solution f of 1.23 satisfies

$$\begin{aligned}&\langle f(t),\psi(t)\rangle - \langle f(s),\psi(s)\rangle - \underbrace{\int_s^t \langle f,\partial_t\psi\rangle}_{7.8_T} + \underbrace{\int_s^t\int_\gamma (\psi\cdot f)(v\cdot n(x))}_{7.8_B}\\ &- \underbrace{\int_s^t \langle \mathbf{P}f, v\cdot\nabla_x\psi\rangle}_{7.8_C} - \int_s^t \langle (\mathbf{I}-\mathbf{P})f, v\cdot\nabla_x\psi\rangle + \underbrace{\int_s^t \langle q\sqrt{\mu}f, \nabla_x\phi_f\cdot\nabla_v(\frac{1}{\sqrt{\mu}}\psi)\rangle}_{7.8_P}\\ &= \int_s^t \langle \psi, \{-L(\mathbf{I}-\mathbf{P})f + \Gamma(f,f)\}\rangle - \underbrace{\int_s^t \langle \psi, q_1 v\cdot\nabla_x\phi_f\sqrt{\mu}\rangle}_{7.8_{\phi_f}}.\end{aligned} \tag{7.8}$$

We use a set of test functions:

$$\begin{aligned}\psi_a &\equiv \begin{bmatrix}-(|v|^2-\beta_a)\sqrt{\mu}v\cdot\nabla_x\varphi_{a_+}\\ -(|v|^2-\beta_a)\sqrt{\mu}v\cdot\nabla_x\varphi_{a_-}\end{bmatrix},\\ \psi_{b,1}^{i,j} &\equiv \begin{bmatrix}(v_i^2-\beta_b)\sqrt{\mu}\partial_j\varphi_b^j\\ (v_i^2-\beta_b)\sqrt{\mu}\partial_j\varphi_b^j\end{bmatrix}, \quad i,j=1,2,3,\\ \psi_{b,2}^{i,j} &\equiv \begin{bmatrix}|v|^2v_iv_j\sqrt{\mu}\partial_j\varphi_b^i(x)\\ |v|^2v_iv_j\sqrt{\mu}\partial_j\varphi_b^i(x)\end{bmatrix}, \quad i\neq j,\\ \psi_c &\equiv \begin{bmatrix}(|v|^2-\beta_c)\sqrt{\mu}v\cdot\nabla_x\varphi_c\\ (|v|^2-\beta_c)\sqrt{\mu}v\cdot\nabla_x\varphi_c\end{bmatrix},\end{aligned} \tag{7.9}$$

where $\varphi_{a_\pm}(t,x)$, $\varphi_b(t,x)$, and $\varphi_c(t,x)$ solve

$$\begin{aligned}-\Delta\varphi_{a_\pm} = a_\pm(t,x), \quad \partial_n\varphi_{a_\pm}|_{\partial\Omega}=0,\\ -\Delta\varphi_b^j = b_j(t,x), \quad \varphi_b^j|_{\partial\Omega}=0, \text{ and } -\Delta\varphi_c = c(t,x), \quad \varphi_c|_{\partial\Omega}=0,\end{aligned} \tag{7.10}$$

and $\beta_a=10$, $\beta_b=1$, and $\beta_c=5$ such that for all $i=1,2,3$,

$$\begin{aligned}&\int_{\mathbb{R}^3}(|v|^2-\beta_a)(\frac{|v|^2-3}{2\sqrt{2}})v_i^2\mu(v)\mathrm{d}v=0,\\ &\int_{\mathbb{R}^3}(v_i^2-\beta_b)\mu(v)\mathrm{d}v=0,\\ &\int_{\mathbb{R}^3}(|v|^2-\beta_c)v_i^2\mu(v)\mathrm{d}v=0.\end{aligned} \tag{7.11}$$

Step 1. Estimate of 7.8_{ϕ_f}: From 7.9 and 7.11, we have $7.8_{\phi_f}\equiv 0$ for $\psi_{b,1}^{i,j}$, $\psi_{b,2}^{i,j}$, and ψ_c. For $\psi=\psi_a$, because from definition $\phi=\varphi_{a_+}-\varphi_{a_-}$, 7.8_{ϕ_f} equals

$$\begin{aligned}7.8_{\phi_f}\big|_{\psi=\psi_a} &= \int_{\mathbb{R}^3} -(|v|^2-\beta_a)(v_1)^2\mu\mathrm{d}v\int_s^t\int_\Omega(\nabla\varphi_{a_+}-\nabla\varphi_{a_-})\cdot\nabla\phi_f\\ &= C_1\int_s^t\|\nabla\phi_f\|_2^2,\end{aligned} \tag{7.12}$$

where

$$C_1 = \int_{\mathbb{R}^3} -(|v|^2 - \beta_a)(v_1)^2 \mu \mathrm{d}v = 5. \tag{7.13}$$

Now we look at 7.8_C. For $\psi = \psi_c$, from oddness in velocity integration and 7.11, 7.8_C becomes

$$\int_s^t \langle \mathbf{P}f, v \cdot \nabla_x \psi_c \rangle = -C_2 \int_s^t \|c(\tau)\|_2^2, \tag{7.14}$$

where $C_2 = 2\int_{\mathbf{R}^3}(|v|^2 - \beta_c)v_i^2(\frac{|v|^2-3}{2\sqrt{2}})\mu(v)dv = 20\pi^{3/2}$.

For $\psi = \psi_a$, from oddness in velocity integration and 7.11, 7.8_C becomes

$$\int_s^t \langle \mathbf{P}f, v \cdot \nabla_x \psi_a \rangle = -C_1 \int_s^t \|a_+(\tau)\|_2^2 + \|a_-(\tau)\|_2^2, \tag{7.15}$$

where $C_1 = 5$ as in 7.13.

For fixed i, j, we choose test function $\psi = \psi_{b,1}^{i,j}$ in 7.9 where β_b and φ_b are defined in 7.11 and 7.10. From oddness in velocity integration and definition of β_b, 7.8_C in 7.8 yields

$$7.8_C|_{\psi_{b,1}^{i,j}} := \int_s^t \langle \mathbf{P}f, v \cdot \nabla \psi_{b,1}^{i,j} \rangle = -C_3 \int_s^t \int_\Omega b_i(\partial_{ij}\Delta^{-1}b_j), \tag{7.16}$$

where $C_3 := 2\int_{\mathbb{R}^3}(v_i^2 - \beta_b)\frac{v_i^2}{\sqrt{2}}\mu dv = 4\sqrt{\pi}$.

Next, we try test function $\psi_{b,2}^{i,j}$ with $i \neq j$ to obtain

$$\begin{aligned} 7.8_C|_{\psi_{b,2}^{i,j}} &:= \int_s^t \langle \mathbf{P}f, \psi_{b,2}^{i,j} \rangle = 2\int_s^t \iint_{\Omega\times\mathbb{R}^3} (b \cdot \frac{v}{\sqrt{2}})\sqrt{\mu} v \cdot \nabla \psi_{b,2}^{i,j} \\ &= -C_4 \int_s^t \int_\Omega \big(b_j(\partial_{ij}\Delta^{-1}b_i) + b_i(\partial_{jj}\Delta^{-1}b_i)\big). \end{aligned} \tag{7.17}$$

by oddness in velocity integral where $C_4 := 14\sqrt{\pi}$. Note that the RHS of 7.16 cancel out with the first term in the RHS of 7.17, therefore combining them we get

$$\begin{aligned} \left(\sum_{i,j} -\frac{C_4}{C_3} \times 7.8_C|_{\psi_{b,1}^{i,j}}\right) + \left(\sum_{i\neq j} 7.8_C|_{\psi_{b,2}^{i,j}}\right) &= -C_4 \sum_{i,j} \int_s^t \int_\Omega b_i(\partial_{jj}\Delta^{-1}b_i) \\ &= -C_4 \int_s^t \|b(\tau)\|_2^2. \end{aligned}$$

Estimate of 7.8_P: From 7.9,

$$\begin{aligned} 7.8_P &= \int_s^t \langle q\sqrt{\mu}f, \nabla_x\phi_f \cdot \nabla_v(\frac{1}{\sqrt{\mu}}\psi)\rangle, \quad \psi = \psi_{a_\pm,b,c}, \\ &\lesssim \int_s^t \|w_\vartheta f\|_\infty \int_\Omega \nabla_x\phi_f \cdot \nabla_x\varphi_{a_\pm,b,c} \lesssim \int_s^t \|w_\vartheta f(\tau)\|_\infty \|\mathbf{P}f(\tau)\|_2^2, \end{aligned} \tag{7.18}$$

by elliptic estimate $\|\nabla_x\varphi_{a_\pm,b,c}\|_2 \lesssim \|\varphi_{a_\pm,b,c}\|_{H^2} \lesssim \|\mathbf{P}f\|_2$.

Step 2. Estimate of c : For boundary integral 7.8_B, we decompose $f_\gamma = P_\gamma f + \mathbf{1}_{\gamma_+}(1 - P_\gamma)f$. Then from 7.11 and trace theorem $|\nabla\varphi_c|_2 \lesssim \|\varphi_c\|_{H^2} \lesssim \|c\|_2$,

$$\begin{aligned}\int_s^t\int_\gamma \psi_c\cdot f(v\cdot n(x)) &= \int_s^t\int_\gamma \psi_c\cdot \mathbf{1}_{\gamma_+}(1-P_\gamma)f d\gamma\\ &\lesssim \varepsilon\int_s^t \|c(\tau)\|_2^2 + C_\varepsilon\int_s^t |(1-P_\gamma)f(\tau)|_{2,+}^2,\quad \varepsilon\ll 1.\end{aligned}\tag{7.19}$$

If we define

$$Re := \int_s^t \langle \psi, \{L(\mathbf{I}-\mathbf{P})f - \Gamma(f,f)\}\rangle + \int_s^t \langle (\mathbf{I}-\mathbf{P})f, v\cdot\nabla_x\psi\rangle,\tag{7.20}$$

then from 7.12, 7.18, elliptic estimate and Young's inequality we have

$$Re|_{\psi_c} \lesssim \varepsilon\int_s^t \|c\|_2^2 + \int_s^t \|(\mathbf{I}-\mathbf{P})f(\tau)\|_\nu^2 + \int_s^t \|\nu^{-1/2}\Gamma(f,f)(\tau)\|_2^2,\tag{7.21}$$

We also use even/oddness in velocity integration, 7.11, and Young's inequality to estimate,

$$\begin{aligned}7.8_T|_{\psi_c} &= \int_s^t \langle f, \partial_t\psi_c\rangle = \int_s^t \langle (\mathbf{I}-\mathbf{P})f, \partial_t\psi_c\rangle\\ &\lesssim \varepsilon\int_s^t \|\nabla\Delta^{-1}\partial_t c(\tau)\|_2^2 + \int_s^t \|(\mathbf{I}-\mathbf{P})f(\tau)\|_\nu^2.\end{aligned}\tag{7.22}$$

Now, we choose a new test function $\psi_c^t := \begin{bmatrix} (\frac{|v|^2-3}{2\sqrt{2}})\sqrt{\mu}\partial_t\varphi_c(t,x)\\ (\frac{|v|^2-3}{2\sqrt{2}})\sqrt{\mu}\partial_t\varphi_c(t,x)\end{bmatrix}$. Note that $\partial_t\varphi_c$ solves $-\Delta\partial_t\varphi_c = \partial_t c(t,x)$ with $\partial_t\varphi_c(t,x)|_{\partial\Omega}=0$. We taking difference quotient for $\partial_t f$ and it replace first three terms in the LHS of 7.8. With help of Poincaré inequality $\|\partial_t\varphi_c\|_2 \lesssim \|\nabla\partial_t\varphi_c\|_2$, we can also compute $7.8_{\phi_f}\big|_{\psi=\psi_c^t}=0$, and

$$\begin{aligned}7.8_P\big|_{\psi=\psi_c^t} &= \int_s^t \langle q\sqrt{\mu}f, \begin{bmatrix}\nabla_x\phi_f\cdot\frac{v}{\sqrt{2}}\partial_t\varphi_c\\ \nabla_x\phi_f\cdot\frac{v}{\sqrt{2}}\partial_t\varphi_c\end{bmatrix}\rangle\\ &\lesssim \int_s^t \|w_\vartheta f\|_\infty\left(\varepsilon\|\nabla\Delta^{-1}\partial_t c(\tau)\|_2^2 + \left(\|a_+(\tau)\|_2^2 + \|a_-(\tau)\|_2^2\right)\right),\end{aligned}\tag{7.23}$$

$$\begin{aligned}&\int_s^t \langle \mathbf{P}f, v\cdot\nabla\psi_c^t\rangle + \int_s^t \langle (\mathbf{I}-\mathbf{P})f, v\cdot\nabla\psi_c^t\rangle\\ &\lesssim \varepsilon\int_s^t \|\nabla\Delta^{-1}\partial_t c(\tau)\|_2^2 + \int_s^t \|b(\tau)\|_2^2 + \int_s^t \|(\mathbf{I}-\mathbf{P})f\|_\nu^2.\end{aligned}\tag{7.24}$$

Since ψ_c^t vanishes when it acts with Lf and $\Gamma(f,f)$, and boundary integral 7.8_B vanishes by Dirichlet boundary condition of φ_c , from 7.23, 7.24, and 7.8, we obtain

$$\begin{aligned}&\int_s^t \langle \partial_t f, \psi_c^t\rangle = \int_s^t\int_\Omega \partial_t\varphi_c(\tau,x)\partial_t c(\tau,x)dx = \int_s^t \|\nabla\Delta^{-1}\partial_t c(\tau)\|_2^2\\ &\lesssim \varepsilon\int_s^t \|\nabla\Delta^{-1}\partial_t c(\tau)\|_2^2\\ &\quad + \int_s^t\left(\|a_+(\tau)\|_2^2 + \|a_-(\tau)\|_2^2 + \int_s^t \|b(\tau)\|_2^2 + \int_s^t \|(\mathbf{I}-\mathbf{P})f\|_\nu^2\right).\end{aligned}\tag{7.25}$$

We combine 7.8, 7.12, 7.18, 7.19, 7.21, 7.22, and 7.25 with $\varepsilon \ll 1$ to obtain

$$\begin{aligned}\int_s^t \|c(\tau)\|_2^2 \lesssim G_c(t) - G_c(s) + \int_s^t \|(\mathbf{I}-\mathbf{P})f(\tau)\|_\nu^2 + \int_s^t |(1-P_\gamma)f(\tau)|_{2,+}^2 \\ + \int_s^t \|\nu^{-1/2}\Gamma(f,f)(\tau)\|_2^2 + \int_s^t \|w_\vartheta f(\tau)\|_\infty \|\mathbf{P}f(\tau)\|_2^2 \\ + \varepsilon \int_s^t \big(\|a_+(\tau)\|_2^2 + \|a_-(\tau)\|_2^2 + \|b(\tau)\|_2^2\big),\end{aligned} \tag{7.26}$$

for $\varepsilon \ll 1$ where $G_c(t) := \langle f(t), \psi_c(t)\rangle \lesssim \|f(t)\|_2^2$.

Step 3. Estimate of a : From mass conservation $\int_\Omega a_\pm(t,x)dv = 0$, $\varphi_{a_\pm}$ in 7.10 is well-defined. Moreover, we choose $\varphi_{a_\pm}$ so that has mean zero, $\int_\Omega \varphi_{a_\pm}(t,x)dx = 0$. Therefore, Poincaré inequality $\|\varphi_{a_\pm}\|_2 \lesssim \|\nabla\varphi_{a_\pm}\|_2$ holds and these are also true for $\partial_t\varphi_{a_\pm}$ which solves same elliptic equation with Neumann boundary condition.

For boundary integral 7.8_B, we decompose $f_\gamma = P_\gamma f + \mathbf{1}_{\gamma_+}(1-P_\gamma)f$. From Neumann boundary condition $\partial_n\varphi_a = 0$ and oddness in velocity integral, $\int_\gamma \psi_a \cdot P_\gamma f(v\cdot n(s)) = 0$ and we obtain similar esimate as 7.19,

$$\begin{aligned}\int_s^t\int_\gamma \psi_a \cdot f(v\cdot n(x)) = \int_s^t\int_\gamma \psi_a \cdot \mathbf{1}_{\gamma_+}(1-P_\gamma)f d\gamma \\ \lesssim \varepsilon \int_s^t \|a(\tau)\|_2^2 + C_\varepsilon \int_s^t |(1-P_\gamma)f(\tau)|_{2,+}^2, \quad \varepsilon \ll 1.\end{aligned} \tag{7.27}$$

For 7.8_T, from oddness,

$$\begin{aligned}7.8_T|_{\psi_a} = \int_s^t \langle f, \partial_t\psi_a\rangle = \int_s^t \iint_{\Omega\times\mathbb{R}^3} ((b\cdot v)\begin{bmatrix}\sqrt{\mu}\\ \sqrt{\mu}\end{bmatrix} + (\mathbf{I}-\mathbf{P})f)\cdot\partial_t\psi_a \\ \lesssim \varepsilon \int_s^t (\|\nabla\Delta^{-1}\partial_t a_+(\tau)\|_2^2 + \|\nabla\Delta^{-1}\partial_t a_-(\tau)\|_2^2) \\ + \int_s^t \|b(\tau)\|_2^2 + \int_s^t \|(\mathbf{I}-\mathbf{P})f(\tau)\|_\nu^2.\end{aligned} \tag{7.28}$$

Now let us estimate $\int_s^t \|\nabla\Delta^{-1}\partial_t a_+(\tau)\|_2^2 + \|\nabla\Delta^{-1}\partial_t a_-(\tau)\|_2^2$ which appear in 7.22 type estimate. We use new test function $\psi_a^t = \begin{bmatrix}\partial_t\varphi_{a_+}(x)\sqrt{\mu}\\ \partial_t\varphi_{a_-}(x)\sqrt{\mu}\end{bmatrix}$. It's easy to check

$$\begin{aligned}7.8_{\phi_f}\big|_{\psi=\psi_a^t} = \int_s^t \iint_{\Omega\times\mathbb{R}^3} q_1\sqrt{\mu}v\cdot\nabla_x\phi_f\cdot\psi_a^t = 0, \\ 7.8_P\big|_{\psi=\psi_a^t} = \int_s^t \iint_{\Omega\times\mathbb{R}^3} q\sqrt{\mu}f\cdot\nabla_x\phi_f\cdot\nabla_v\begin{bmatrix}\partial_t\varphi_{a_+}\\ \partial_t\varphi_{a_-}\end{bmatrix} = 0,\end{aligned} \tag{7.29}$$

and from the null condition on boundary 1.11, we have $7.8_B|_{\psi=\psi_a^t} = 0$. Moreover,

$$\begin{aligned}\int_s^t \langle \mathbf{P}f, v\cdot\nabla_x\psi_a^t\rangle + \int_s^t \langle \mathbf{I}-\mathbf{P}f, v\cdot\nabla_x\psi_a^t\rangle \\ \lesssim \varepsilon \int_s^t (\|\nabla\Delta^{-1}\partial_t a_+(\tau)\|_2^2 + \|\nabla\Delta^{-1}\partial_t a_-(\tau)\|_2^2) \\ + \int_s^t \|b(\tau)\|_2^2 + \int_s^t \|\mathbf{I}-\mathbf{P}f(\tau)\|_\nu^2,\end{aligned} \tag{7.30}$$

and from 1.7

$$\int_s^t \langle \psi_a^t, \Gamma(f,f)\rangle = 0. \tag{7.31}$$

Now taking difference quotient, we obtain from 7.29, 7.30, and 7.30, for almost t,

$$\begin{aligned}\int_s^t \langle \partial_t f, \psi_a^t\rangle &= \int_s^t \int_\Omega \partial_t \varphi_{a_+} \partial_t a_+ + \partial_t \varphi_{a_-} \partial_t a_- dx \\ &= \int_s^t (\|\nabla \Delta^{-1} \partial_t a_+(\tau)\|_2^2 + \|\nabla \Delta^{-1} \partial_t a_-(\tau)\|_2^2) \\ &\lesssim \int_s^t \|b(\tau)\|_2^2 + \int_s^t \|\mathbf{I}-\mathbf{P} f(\tau)\|_\nu^2 + \int_s^t \|\nu^{-1/2}\Gamma(f,f)\|_2^2.\end{aligned} \tag{7.32}$$

Finally we change c into a in 7.21 and combine with 7.8, 7.12) 7.15, 7.18, 7.27, 7.28, and 7.32 with $\varepsilon \ll 1$ to obtain

$$\begin{aligned}&\int_s^t \|a(\tau)\|_2^2 + \int_s^t \|\nabla \phi_f(\tau)\|_2^2 \\ &\lesssim G_a(t) - G_a(s) + \int_s^t \|(\mathbf{I}-\mathbf{P})f(\tau)\|_\nu^2 + \int_s^t |(1-P_\gamma)f(\tau)|_{2,+}^2 \\ &\quad + \int_s^t \|\nu^{-1/2}\Gamma(f,f)(\tau)\|_2^2 + \int_s^t \|w_\vartheta f(\tau)\|_\infty \|\mathbf{P} f(\tau)\|_2^2 + \int_s^t \|b(\tau)\|_2^2,\end{aligned} \tag{7.33}$$

for $\varepsilon \ll 1$ where $G_a(t) := \iint_{\Omega\times\mathbb{R}^3} f(t)\psi_a(t) \lesssim \|f(t)\|_2^2$.

Step 4. Estimate of b : For fixed i,j, we choose test function $\psi = \psi_{b,1}^{i,j}$ in 7.9 where β_b and φ_b are defined in 7.11 and 7.10. For boundary integration, contribution of $P_\gamma f$ vanishes by oddness.

$$7.8_B|_{\psi_{b,1}^{i,j}} := \int_s^t \int_\gamma \psi_{b,1}^{i,j} \cdot \mathbf{1}_{\gamma_+}(1-P_\gamma) f(v\cdot n(x)) \lesssim \varepsilon \int_s^t \|b(\tau)\|_2^2 + \int_s^t |(1-P_\gamma)f|_{2,+}^2, \tag{7.34}$$

and similar as 7.22 and 7.21, we use oddness and definition of β_b to vanish contribution of a and b. We obtain

$$7.8_T|_{\psi_{b,1}^{i,j}} \lesssim \varepsilon \int_s^t \|\nabla \Delta^{-1}\partial_t b_j(\tau)\|_2^2 + \int_s^t \|c(\tau)\|_2^2 + \int_s^t \|(\mathbf{I}-\mathbf{P})f(\tau)\|_\nu^2, \tag{7.35}$$

$$Re|_{\psi_{b,1}^{i,j}} \lesssim \varepsilon \int_s^t \|b(\tau)\|_2^2 + \int_s^t \|(\mathbf{I}-\mathbf{P})f(\tau)\|_\nu^2 + \int_s^t \|\nu^{-1/2}\Gamma(f,f)(\tau)\|_2^2. \tag{7.36}$$

Next, we try test function $\psi_{b,2}^{i,j}$ with $i \neq j$. We also have the following three estimates using oddness of velocity integral,

$$\begin{aligned}&7.8_B|_{\psi_{b,2}^{i,j}} := \int_s^t \int_\gamma \psi_{b,1}^{i,j} \cdot \mathbf{1}_{\gamma_+}(1-P_\gamma) f(v\cdot n(x)) \lesssim \varepsilon \int_s^t \|b(\tau)\|_2^2 + \int_s^t |(1-P_\gamma)f|_{2,+}^2, \\ &7.8_T|_{\psi_{b,2}^{i,j}} \lesssim \varepsilon \int_s^t \|\nabla \Delta^{-1}\partial_t b_i(\tau)\|_2^2 + \int_s^t \|(\mathbf{I}-\mathbf{P})f(\tau)\|_\nu^2, \\ &Re|_{\psi_{b,2}^{i,j}} \lesssim \varepsilon \int_s^t \|b(\tau)\|_2^2 + \int_s^t \|(\mathbf{I}-\mathbf{P})f(\tau)\|_\nu^2 + \int_s^t \|\nu^{-1/2}\Gamma(f,f)(\tau)\|_2^2.\end{aligned} \tag{7.37}$$

To obtain estimate for $\|\nabla\Delta^{-1}\partial_t b_j\|_2$, we use a test function

$$\psi^t_{b,j} := \begin{bmatrix} \frac{v_j}{\sqrt{2}}\sqrt{\mu}\partial_t\varphi^j_b(t,x) \\ \frac{v_j}{\sqrt{2}}\sqrt{\mu}\partial_t\varphi^j_b(t,x) \end{bmatrix}.$$

Note that $\partial_t\varphi^j_b$ solves $-\Delta\partial_t\varphi^j_b = \partial_t b_j(t,x)$ with $\partial_t\varphi^j_b(t,x)|_{\partial\Omega} = 0$. We taking difference quotient for $\partial_t f$ in 1.16 and with help of Poincaré inequality, we get

$$\begin{aligned} 7.8_{\phi_f}\big|_{\psi=\psi^t_{b,j}} &= 0, \\ 7.8_P\big|_{\psi=\psi^t_{b,j}} &= \int_s^t \iint_{\Omega\times\mathbb{R}^3} \sqrt{\mu} f\cdot \begin{bmatrix} \partial_j\phi_f\cdot\partial_t\varphi^j_b \\ \partial_j\phi_f\cdot\partial_t\varphi^j_b \end{bmatrix} \\ &\lesssim \int_s^t \|w_\vartheta f\|_\infty(\varepsilon\|\nabla\Delta^{-1}\partial_t b_j(\tau)\|_2^2 + \|a_+(\tau)\|_2^2 + a_-(\tau)\|_2^2). \end{aligned} \tag{7.38}$$

Moreover,

$$\begin{aligned} &\int_s^t \langle \mathbf{P}f, v\cdot\nabla\psi^t_{b,j}\rangle + \int_s^t \langle (\mathbf{I}-\mathbf{P})f, v\cdot\nabla\psi^t_{b,j}\rangle \\ &\quad\lesssim \varepsilon\int_s^t \|\nabla\Delta^{-1}\partial_t b_j(\tau)\|_2^2 + \int_s^t (\|a_+(\tau)\|_2^2 + \|a_-(\tau)\|_2^2 + \|c(\tau)\|_2^2) \\ &\qquad + \int_s^t \|(\mathbf{I}-\mathbf{P})f\|_\nu^2. \end{aligned} \tag{7.39}$$

Since $\psi^t_{b,j}$ vanishes when it acts with Lf and $\Gamma(f,f)$, and boundary integral 7.8_B vanishes by Dirichlet boundary condition of $\partial_t\varphi^j_b$, from 7.38, 7.39, and 7.8, we obtain

$$\begin{aligned} &\int_s^t\int_\Omega \partial_t\varphi^j_b(\tau,x)\partial_t b_j(\tau,x)dx = \int_s^t \|\nabla\Delta^{-1}\partial_t b_j(\tau)\|_2^2 \\ &\lesssim \varepsilon\int_s^t \|\nabla\Delta^{-1}\partial_t b_j(\tau)\|_2^2 + \int_s^t (\|a_+(\tau)\|_2^2 + \|a_-(\tau)\|_2^2 + \|c(\tau)\|_2^2) \\ &\quad + \int_s^t \|(\mathbf{I}-\mathbf{P})f(\tau)\|_\nu^2. \end{aligned} \tag{7.40}$$

Now we combine 7.8, 7.12, 7.18, 7.16, 7.34, 7.35, 7.36, 7.17, and 7.37 for all i,j with proper constant weights. In particular, we note that RHS of 7.16 is cancelled by the first term on the RHS of 7.17. Therefore,

$$\begin{aligned} &\int_s^t \|b(\tau)\|_2^2 = -\sum_{i,j}\int_s^t\int_\Omega b_i(\partial_{jj}\Delta^{-1}b_i) \\ &\lesssim G_b(t) - G_b(s) + \int_s^t \|(\mathbf{I}-\mathbf{P})f(\tau)\|_\nu^2 + \int_s^t |(1-P_\gamma)f(\tau)|_{2,+}^2 \\ &\quad + \int_s^t \|\nu^{-1/2}\Gamma(f,f)(\tau)\|_2^2 + \int_s^t \|w_\vartheta f(\tau)\|_\infty^2\|\mathbf{P}f(\tau)\|_2^2 \\ &\quad + \int_s^t \|c(\tau)\|_2^2 + \varepsilon\int_s^t (\|a_+(\tau)\|_2^2 + \|a_-(\tau)\|_2^2), \quad G_b(t)\lesssim \|f(t)\|_2^2, \quad \varepsilon\ll 1. \end{aligned} \tag{7.41}$$

Finally we combine 7.26, 7.33, and 7.41 with $\varepsilon\ll 1$ to conclude 7.4. □

8. **Global existence and exponential decay.** The following time-dependent interpolation estimate is crucial in the proof of Theorem 1.2.

Lemma 8.1. *Assume $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$. For $0 < D_1 < 1$, $0 < D_2 < 1$, $\Lambda_0 > 0$, and for all $t \geq 0$,*

$$\|\nabla_x^2 \phi(t)\|_{L^\infty(\Omega)} \lesssim_{\Omega, D_1, D_2} e^{D_1 \Lambda_0 t} \|\phi(t)\|_{C^{1,1-D_1}(\Omega)} + e^{-D_2 \Lambda_0 t} \|\phi(t)\|_{C^{2,D_2}(\Omega)} \tag{8.1}$$

Proof. Let Ω_1 be an open bounded subset of $\mathbb{R}^3$ containing the closure $\bar{\Omega}$. Suppose $\phi(t) \in C^{2,D_2}(\Omega)$. From a standard extension theorem (e.g. see Lemma 6.37 of [9] in page 136) there exists a function $\bar{\phi}(t) \in C^{2,D_2}(\Omega_1)$ and $\bar{\phi}(t) \equiv 0$ in $\mathbb{R}^3 \backslash \Omega_1$ such that $\phi(t) \equiv \bar{\phi}(t)$ in Ω and

$$\begin{aligned} \|\bar{\phi}(t)\|_{C^{1,1-D_1}(\Omega_1)} &\leq C_{\Omega,\Omega_1,D_1,D_2} \|\phi(t)\|_{C^{1,1-D_1}(\Omega)}, \\ \|\bar{\phi}(t)\|_{C^{2,D_2}(\Omega_1)} &\leq C_{\Omega,\Omega_1,D_1,D_2} \|\phi(t)\|_{C^{2,D_2}(\Omega)}, \end{aligned} \tag{8.2}$$

where $C_{\Omega,\Omega_1,D_1,D_2}$ does not depend on $\phi(t)$ and t.

Choose arbitrary points x, y in $\mathbb{R}^3$. For $0 \leq s \leq 1$, $(1-s)x + sy \in \overline{xy}$. Note that

$$\begin{aligned} &[(y-x)\cdot\nabla]\nabla\bar{\phi}(t,(1-s)x+sy) \\ &= \frac{[(y-x)\cdot\nabla]\nabla\bar{\phi}(t,(1-s)x+sy) - [(y-x)\cdot\nabla]\nabla\bar{\phi}(t,x)}{|(1-s)x+sy-x|^{D_2}} |(1-s)x+sy-x|^{D_2} \\ &\quad + \Big(\frac{y-x}{|y-x|}\cdot\nabla\Big)\nabla\bar{\phi}(t,x)|y-x| \\ &= O(|x-y|^{1+D_2}) s^{D_2} [\nabla^2\bar{\phi}(t)]_{C^{0,D_2}} + \Big(\frac{y-x}{|y-x|}\cdot\nabla\Big)\nabla\bar{\phi}(t,x)|y-x|. \end{aligned}$$

Taking an integration on $s \in [0,1]$, we obtain that

$$\begin{aligned} &\left|\Big(\frac{y-x}{|y-x|}\cdot\nabla\Big)\nabla\bar{\phi}(t,x)\right| \\ &\leq \frac{1}{|y-x|}\left|\int_0^1 [(y-x)\cdot\nabla]\nabla\bar{\phi}(t,(1-s)x+sy)\mathrm{d}s\right| + \frac{1}{1+D_2}|x-y|^{D_2}[\nabla^2\bar{\phi}(t)]_{C^{0,D_2}}. \end{aligned} \tag{8.3}$$

On the other hand, from an expansion along s,

$$\nabla\bar{\phi}(t,y) - \nabla\bar{\phi}(t,x) = \int_0^1 [(y-x)\cdot\nabla]\nabla\bar{\phi}(t,(1-s)x+sy)\mathrm{d}s.$$

We plug this identity into 8.3 and deduce that for $0 < D_1 < 1$

$$\begin{aligned} &\left|\left(\frac{x-y}{|x-y|}\cdot\nabla\right)\nabla\bar{\phi}(t,x)\right| \\ &\leq \frac{|\nabla\bar{\phi}(t,x) - \nabla\bar{\phi}(t,y)|}{|x-y|} + \frac{1}{1+D_2}|x-y|^{D_2}[\nabla^2\bar{\phi}(t)]_{C_x^{0,D_2}} \\ &\leq \frac{1}{|x-y|^{D_1}}[\nabla\bar{\phi}(t)]_{C^{0,1-D_1}} + \frac{1}{1+D_2}|x-y|^{D_2}[\nabla^2\bar{\phi}(t)]_{C_x^{0,D_2}}. \end{aligned} \tag{8.4}$$

Now let us choose

$$|x-y| = e^{-\Lambda_0 t}, \quad \hat{\omega} := \frac{x-y}{|x-y|} \in \mathbb{S}^2.$$

From 8.4

$$|(\hat{\omega}\cdot\nabla)\nabla\bar{\phi}(t,x)| \leq e^{D_1\Lambda_0 t}[\nabla\bar{\phi}(t)]_{C^{0,1-D_1}} + \frac{1}{1+D_2}e^{-D_2\Lambda_0 t}[\nabla^2\bar{\phi}(t)]_{C_x^{0,D_2}}.$$

Taking supremum in x and $\hat{\omega}$ to the above inequality and using $\|\nabla_x^2\bar{\phi}(t)\|_{L_x^\infty} = \sup_x \sup_{\hat{\omega}\in\mathbb{S}^2} |(\hat{\omega}\cdot\nabla)\nabla\bar{\phi}(t,x)|$, we get

$$\|\nabla_x^2\bar{\phi}(t)\|_{L^\infty(\Omega_1)} \le e^{D_1\Lambda_0 t}[\nabla_x\bar{\phi}(t)]_{C^{0,1-D_1}(\Omega_1)} + e^{-D_2\Lambda_0 t}[\nabla^2\bar{\phi}(t)]_{C^{0,D_2}(\Omega_1)}.$$

Finally from 8.2 and the above estimate we conclude 8.1. □

Now we are ready to prove the global-in-time result.

Proof of Theorem 1.2. **Step 1.** For $0 < M \ll 1$ and $0 < \delta_* \ll 1$, we first assume that an initial datum satisfies

$$\begin{aligned} \|w_\vartheta f_0\|_\infty + \|w_{\tilde{\vartheta}} f_0\|_p + \|w_{\tilde{\vartheta}}\alpha_{f_0,\varepsilon}^\beta \nabla_{x,v} f_0\|_p \le \delta_* M, \\ \|w_{\tilde{\vartheta}}\nabla_v f_0\|_{L^3(\Omega\times\mathbb{R}^3)} + \|\nabla_x^2\phi_f(0)\|_\infty < \infty. \end{aligned} \tag{8.5}$$

We will choose M, δ_* later. For the sake of convenience we choose a large constant $L \gg \max\big(M, \|\nabla_x^2\phi_f(0)\|_\infty\big)$. In order to use the continuation argument along the lines of the local existence theorem, Theorem 6.1, we set

$$\begin{aligned} T = \sup_t \Big\{ t \ge 0 : \|e^{\lambda_\infty s} w_\vartheta f(t)\|_\infty + \|w_{\tilde{\vartheta}} f(t)\|_p \le M, \\ \text{and } \|w_{\tilde{\vartheta}}\alpha_{f,\varepsilon}^\beta \nabla_{x,v} f(t)\|_p^p + \int_0^t |w_{\tilde{\vartheta}}\alpha_{f,\varepsilon}^\beta \nabla_{x,v} f(t)|_{p,+}^p < \infty, \\ \text{and } \|\nabla_v f(t)\|_{L_x^3(\Omega)L_v^{1+\delta}(\mathbb{R}^3)} < \infty, \\ \text{and } \|\nabla_x^2\phi_f(t)\|_\infty \le L \Big\}. \end{aligned} \tag{8.6}$$

Here for fixed $\delta \ll 1$, we choose λ_∞ such that

$$20\sqrt{CC_2 M} \le \lambda_\infty \le \min\left(\frac{\lambda_2}{2}, \frac{\nu_0}{4}\right), \quad \textit{for } M \ll 1, \tag{8.7}$$

where λ_2 is obtained in Proposition 6. Note that from 4.10 the condition 4.4 holds for $M \ll 1$.

Step 2. We claim that

$$\sup_{0\le t\le T} e^{\frac{\lambda_\infty}{2}t}\|\nabla_x^2\phi_f(t)\|_\infty \le C_2 M, \quad \textit{with} \quad C_2 := C_\Omega + (C_1C_p)^{1/p}\delta_*. \tag{8.8}$$

Here C_Ω appears in 4.10, and C_1 in 4.11, and C_p in Proposition 3.

From 4.10 and 8.6, for $0 \le t \le T$, for all $D_1 > 0$

$$\|\phi_f(t)\|_{C^{1,1-D_1}(\bar{\Omega})} \le C_\Omega\|w_\vartheta f(t)\|_\infty \le C_\Omega M e^{-\lambda_\infty t}. \tag{8.9}$$

On the other hand, from Proposition 3, replacing f^ℓ and $f^{\ell+1}$ by f in 4.63, 4.75, and by Gronwall's inequality and 8.5, we derive that for $0 \le t \le T$

$$\begin{aligned} \|f(t)\|_p^p + \|w_{\tilde{\vartheta}}\alpha_{f,\varepsilon}^\beta\nabla_{x,v} f(t)\|_p^p + \int_0^t |w_{\tilde{\vartheta}}\alpha_{f,\varepsilon}^\beta\nabla_{x,v} f(s)|_{p,+}^p \\ \le C_p e^{C_p(1+L)t} \times (\delta_* M)^p. \end{aligned} \tag{8.10}$$

Now we use Lemma 4.2, from 4.11, for $p > 3$ and $0 \le t \le T$,

$$\|\phi_f(t)\|_{C^{2,1-\frac{3}{p}}(\bar{\Omega})} \le (C_1C_p)^{1/p} e^{\frac{1}{p}C_p(1+L)t} \times \delta_* M. \tag{8.11}$$

Finally we use an interpolation between $C^{1,1-D_1}(\bar{\Omega})$ and $C^{2,1-\frac{3}{p}}(\bar{\Omega})$ and derive an estimate of $C^2(\bar{\Omega})$: Applying Lemma 8.1 and 8.1 with $D_2 = 1 - \frac{3}{p}$, from 8.10 and 8.9, we derive that for all $0 < D_1 < 1$, $3 < p < 6$, $\Lambda_0 > 0$, and $0 \le t \le T$,

$$\|\nabla_x^2 \phi_f(t)\|_\infty \le e^{-[\lambda_\infty - D_1\Lambda_0]t} C_\Omega M + e^{-[(1-\frac{3}{p})\Lambda_0 - \frac{1}{p}C_p(1+L)]t}(C_1 C_p)^{1/p}\delta_* M. \quad (8.12)$$

Then we choose

$$\Lambda_0 = \frac{\frac{\lambda_\infty}{2} + \frac{C_p}{p}(1+L)}{1-\frac{3}{p}} \quad \textit{and then } D_1 = \frac{\lambda_\infty}{2\Lambda_0}. \quad (8.13)$$

In conclusion we have, for all $0 \le t \le T$,

$$\|\nabla_x^2 \phi_f(t)\|_\infty \le e^{-\frac{\lambda_\infty}{2}t}[C_\Omega + (C_1 C_p)^{1/p}\delta_*]M.$$

As long as $M \ll L$ then $\|\nabla_x^2 \phi_f(t)\|_\infty \le L$ for all $0 \le t \le T$ and hence the claim 8.8 holds.

Step 3. We claim that there exists $T_\infty \gg 1$ such that, for $N \in \mathbb{N}$, $t \in [NT_\infty, (N+1)T_\infty]$, and $(N+1)T_\infty \le T$,

$$\begin{aligned}
&\|w_\vartheta f(t)\|_\infty \\
&\le (t - NT_\infty)e^{-\frac{3}{4}\nu_0(t-NT_\infty)}\|w_\vartheta f(NT_\infty)\|_\infty + o(1)\sup_{NT_\infty \le s \le t} e^{-\frac{3}{4}\nu_0(t-s)}\|w_\vartheta f(s)\|_\infty \\
&+ C_{T_\infty}\int_{NT_\infty}^t e^{-\frac{3}{4}\nu_0(t-s)}\|f(s)\|_{L^2_{x,v}}\mathrm{d}s \\
&+ C_{T_\infty}\int_{NT_\infty}^t e^{-\frac{3}{4}\nu_0(t-s)}\|\nabla\phi_f(s)\|_\infty \mathrm{d}s.
\end{aligned} \quad (8.14)$$

For the sake of simplicity we present a proof of 8.14 for $N = 0$. The proof for $N > 0$ can be easily obtained by considering $f(NT_\infty)$ as an initial datum.

As 3.9 we define $h(t,x,v) := w_\vartheta f(t,x,v)$. Then h solves 3.3 and 3.12 with exchanging all $(h^\ell, h^{\ell+1}, \phi^\ell)$ to (h, h, ϕ_f). We define

$$\begin{aligned}
\nu_{\phi_f, w_\vartheta}(t,x,v) &:= \begin{bmatrix} \nu_{\phi_f,w_\vartheta,+} & 0 \\ 0 & \nu_{\phi_f,w_\vartheta,-} \end{bmatrix} \\
&= \begin{bmatrix} \nu(v) + \frac{v}{2}\cdot\nabla\phi_f - \frac{\nabla_x\phi_f\cdot\nabla_v w_\vartheta}{w_\vartheta} & 0 \\ 0 & \nu(v) - \frac{v}{2}\cdot\nabla\phi_f + \frac{\nabla_x\phi_f\cdot\nabla_v w_\vartheta}{w_\vartheta} \end{bmatrix}.
\end{aligned} \quad (8.15)$$

From 8.6 and 4.10, for $0 \le t \le T$

$$\begin{aligned}
\nu_{\phi_f,w_\vartheta,\pm} &\ge \{\nu_0 - \frac{\|\nabla\phi_f\|_\infty}{2} - 2\vartheta\|\nabla\phi_f\|_\infty\}\langle v\rangle \\
&\ge \{\nu_0 - (\frac{1}{2} - 2\vartheta)M\}\langle v\rangle \\
&\ge \frac{4\nu_0}{5}\langle v\rangle.
\end{aligned} \quad (8.16)$$

Then h solves 3.18 along the trajectory with deleting all superscriptions of ℓ and $\ell+1$ and exchanging ν^ℓ to ν_{ϕ_f,w_ϑ} and with new g

$$g := -q_1 v\cdot\nabla\phi_f\sqrt{\mu} + \Gamma(\frac{h}{w_\vartheta}, \frac{h}{w_\vartheta}). \quad (8.17)$$

We define a stochastic cycles for $\iota = +$ or $-$ as in 1.9,

$$(t_{l,\iota}(t,x,v,v_1,\cdots,v_{l-1}), x_{l,\iota}(t,x,v,v_1,\cdots,v_{l-1})),$$

by deleting all superscriptions in 3.5 and 3.6. Then by deleting all superscriptions of ℓ and $\ell+1$ from 3.36, we obtain the bound for h_ι:

$$\begin{aligned}
&|h_\iota(t,x,v)| \\
&\le \|e^{-\frac{3}{4}\nu_0 t}h_0\|_\infty + O(k)\sup_{0\le s\le t}\|e^{-\frac{3}{4}\nu_0(t-s)}h(s)\|_\infty^2 \\
&\quad + O(k)\int_0^t \|e^{-\frac{3}{4}\nu_0(t-s)}\nabla\phi_{\frac{h}{w_\vartheta}}(s)\|_\infty \mathrm{d}s + \Big\{\frac{1}{2}\Big\}^{k/5}\sup_{0\le s\le t}\|e^{-\frac{3}{4}\nu_0(t-s)}h(s)\|_\infty \\
&\quad + \underbrace{\int_{\max\{t_{1,\iota},0\}}^t e^{-\frac{3}{4}\nu_0(t-s)}\int_{\mathbb{R}^3}\mathbf{k}_\varrho(V_\iota(s;t,x,v),u)|h(s,X_\iota(s;t,x,v),u)|\mathrm{d}u\mathrm{d}s}_{8.18_1} \\
&\quad + O(k)\sup_l \int_{\max\{t_{l+1,\iota},0\}}^{t_{l,\iota}} e^{-\frac{3}{4}\nu_0(t-s)} \\
&\qquad \times \underbrace{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathbf{k}_\varrho(V_\iota(s;t_l,x_l,v_l),u)|h(s,X_\iota(s;t_l,x_l,v_l),u)||n(x_l)\cdot v_l|\sqrt{\mu(v_l)}\mathrm{d}u\mathrm{d}v_l\mathrm{d}s}_{8.18_2}.
\end{aligned} \tag{8.18}$$

For any large $m \gg 1$ we define

$$\mathbf{k}_{\varrho,m}(v,u) = \mathbf{1}_{|v-u|\ge\frac{1}{m},|v|\le m}\mathbf{k}_\varrho(v,u), \tag{8.19}$$

such that $\sup_v\int_{\mathbb{R}^3}|\mathbf{k}_{\varrho,m}(v,u)-\mathbf{k}_\varrho(v,u)|\mathrm{d}u \lesssim \frac{1}{m}$, and $|\mathbf{k}_{\varrho,m}(v,u)| \lesssim_m 1$.

Furthermore we split the time interval as, for each ℓ, l

$$\{\max\{t_{l+1,\iota},0\}\le s\le t_{l,\iota}\} = \{\max\{t_{l+1,\iota},0\}\le s\le t_{l,\iota}-\delta\}\cup\{t_{l,\iota}-\delta\le s\le t_{l,\iota}\}, \tag{8.20}$$

where we choose a small constant $0<\delta\ll_k 1$ later in 8.26.

For 8.18_1, we have

$$\begin{aligned}
8.18_1 &\le \int_{\max\{t_{1,\iota},0\}}^{t-\delta} + \int_{t-\delta}^t \\
&\lesssim \int_{\max\{t_{1,\iota},0\}}^{t-\delta} e^{-\frac{3}{4}\nu_0(t-s)}\int_{\mathbb{R}^3}\mathbf{k}_\varrho(V_\iota(s;t,x,v),u)|h(s,X_\iota(s;t,x,v),u)|\mathrm{d}u\mathrm{d}s \\
&\quad + \delta\sup_{0\le s\le t}\|e^{-\frac{3}{4}\nu_0(t-s)}h(s)\|_\infty \\
&\lesssim \int_{\max\{t_{1,\iota},0\}}^{t-\delta} e^{-\frac{3}{4}\nu_0(t-s)}\int_{\mathbb{R}^3}\mathbf{k}_{\varrho,m}(V_\iota(s;t,x,v),u)|h(s,X_\iota(s;t,x,v),u)|\mathrm{d}u\mathrm{d}s \\
&\quad + (\frac{1}{m}+\delta)\sup_{0\le s\le t}\|e^{-\frac{3}{4}\nu_0(t-s)}h(s)\|_\infty.
\end{aligned} \tag{8.21}$$

Now for 8.18_2 we separate into several cases:

Case 1: For $|v_l| > \frac{m}{2}$, we have $|V_\iota(s;t_l,x_l,v_l)| > \frac{m}{4}$, so

$$\int_{|v_l|>\frac{m}{2}}\int_{\mathbb{R}^3} \mathbf{k}_\varrho(V_\iota(s;t_l,x_l,v_l),u)|(n(x_l)\cdot v_l|\sqrt{\mu(v_l)}\mathrm{d}u\mathrm{d}v_l \lesssim \frac{1}{m}.$$

Thus

$$\begin{aligned}&\int_{\max\{t_{l+1,\iota},0\}}^{t_{l,\iota}} e^{-\frac{3}{4}\nu_0(t-s)}\int_{|v_l|>\frac{m}{2}}\int_{\mathbb{R}^3}\\ &\quad\times \mathbf{k}_\varrho(V_\iota(s;t_l,x_l,v_l),u)|h(s,X_\iota(s;t_l,x_l,v_l),u)||n(x_l)\cdot v_l|\sqrt{\mu(v_l)}\mathrm{d}u\mathrm{d}v_l\mathrm{d}s\\ \lesssim&\frac{1}{m}\sup_{0\le s\le t}\|e^{-\frac{3}{4}\nu_0(t-s)}h(s)\|_\infty.\end{aligned}$$

Case 2: For $|v_l| \le \frac{m}{2}$, $|u| > m$, we have $|V_\iota(s;t_l,x_l,v_l) - u| > \frac{m}{4}$ so

$$\sup_{|v_l|\le\frac{m}{2}}\int_{|u|>m} \mathbf{k}_\varrho(V_\iota(s;t_l,x_l,v_l),u)|(n(x_l)\cdot v_l|\sqrt{\mu(v_l)}\mathrm{d}u \lesssim \frac{1}{m}.$$

Thus

$$\begin{aligned}&\int_{\max\{t_{l+1,\iota},0\}}^{t_{l,\iota}} e^{-\frac{3}{4}\nu_0(t-s)}\int_{|v_l|\le\frac{m}{2}}\int_{|u|>m}\\ &\quad\times \mathbf{k}_\varrho(V_\iota(s;t_l,x_l,v_l),u)|h(s,X_\iota(s;t_l,x_l,v_l),u)||n(x_l)\cdot v_l|\sqrt{\mu(v_l)}\mathrm{d}u\mathrm{d}v_l\mathrm{d}s\\ \lesssim&\frac{1}{m}\sup_{0\le s\le t}\|e^{-\frac{3}{4}\nu_0(t-s)}h(s)\|_\infty.\end{aligned}$$

Case 3: For $|v_l| \le \frac{m}{2}$ and $|u| \le m$, we split the time integration as $\int_{\max\{t_{l+1,\iota},0\}}^{t_{l,\iota}} = \int_{\max\{t_{l+1,\iota},0\}}^{t_{l,\iota}-\delta} + \int_{t_{l,\iota}-\delta}^{t_{l,\iota}}$ and use 8.19 to conclude that

$$\begin{aligned}&8.18_2\\ \lesssim&O(k)\sup_l\int_{\max\{t_{l+1,\iota},0\}}^{t_{l,\iota}-\delta} e^{-\frac{3}{4}\nu_0(t-s)}\int_{|v_l|\le m}\int_{|u|\le m}\\ &\quad\times \mathbf{k}_{\varrho,m}(V_\iota(s;t_l,x_l,v_l),u)|h(s,X_\iota(s;t_l,x_l,v_l),u)||n(x_l)\cdot v_l|\sqrt{\mu(v_l)}\mathrm{d}u\mathrm{d}v_l\mathrm{d}s\\ &+O(k)(\frac{1}{m}+\delta)\sup_{0\le s\le t}\|e^{-\frac{3}{4}\nu_0(t-s)}h(s)\|_\infty\\ \lesssim&O_m(k)\sup_l\int_{\max\{t_{l+1,\iota},0\}}^{t_{l,\iota}-\delta} e^{-\frac{3}{4}\nu_0(t-s)}\\ &\times\int_{|v_l|\le m}\int_{|u|\le m}|h(s,X_\iota(s;t_l,x_l,v_l),u)||n(x_l)\cdot v_l|\sqrt{\mu(v_l)}\mathrm{d}u\mathrm{d}v_l\mathrm{d}s\\ &+O(k)(\frac{1}{m}+\delta)\sup_{0\le s\le t}\|e^{-\frac{3}{4}\nu_0(t-s)}h(s)\|_\infty.\end{aligned}\tag{8.22}$$

Combining 8.18, 8.21, and 8.22 we get

$$\begin{aligned}&|h_\iota(t,x,v)|\\ \le\quad&\|e^{-\frac{3}{4}\nu_0 t}h_0\|_\infty + O(k)\sup_{0\le s\le t}\|e^{-\frac{3}{4}\nu_0(t-s)}h(s)\|_\infty^2\end{aligned}$$

$$
\begin{aligned}
&+O(k)\int_0^t \|e^{-\frac{3}{4}\nu_0(t-s)}\nabla\phi_{\frac{h}{w_\vartheta}}(s)\|_\infty \mathrm{d}s\\
&+\left\{O(k)(\delta+\frac{1}{m})+\left\{\frac{1}{2}\right\}^{k/5}\right\}\sup_{0\le s\le t}\|e^{-\frac{3}{4}\nu_0(t-s)}h(s)\|_\infty\\
&+\int_{\max\{t_{1,\iota},0\}}^{t-\delta} e^{-\frac{3}{4}\nu_0(t-s)}\int_{\mathbb{R}^3}\mathbf{k}_{\varrho,m}(V_\iota(s;t,x,v),u)|h(s,X_\iota(s;t,x,v),u)|\mathrm{d}u\mathrm{d}s\\
&+O_m(k)\sup_l\int_{\max\{t_{l+1,\iota},0\}}^{t_{l,\iota}-\delta} e^{-\frac{3}{4}\nu_0(t-s)}\\
&\qquad\times\int_{|u|\le m}\int_{|v_l|\le m}|h(s,X_\iota(s;t_l,x_l,v_l),u)|\mathrm{d}v_l\mathrm{d}u\mathrm{d}s.
\end{aligned}
\tag{8.23}
$$

Now for $|h(s,X_\iota(s;t,x,v),u)|$ we use similar bounds for $|h_+(s,X_\iota(s;t,x,v),u)|$ and $|h_-(s,X_\iota(s;t,x,v),u)|$ separately and add them together to get

$$
\begin{aligned}
&|h(s,X_\iota(s;t,x,v),u)|\\
&=|h_+(s,X_\iota(s;t,x,v),u)|+|h_-(s,X_\iota(s;t,x,v),u)|\\
&\le \|e^{-\frac{3}{4}\nu_0 s}h_0\|_\infty+O(k)\sup_{0\le s'\le s}\|e^{-\frac{3}{4}\nu_0(s-s')}h(s')\|_\infty^2\\
&\quad+O(k)\int_0^s\|e^{-\frac{3}{4}\nu_0(s-s')}\nabla\phi_{\frac{h}{w_\vartheta}}(s')\|_\infty\mathrm{d}s'\\
&\quad+\left\{O(k)\left(\delta+\frac{1}{m}\right)+\left\{\frac{1}{2}\right\}^{k/5}\right\}\sup_{0\le s'\le s}\|e^{-\frac{3}{4}\nu_0(s-s')}h(s')\|_\infty\\
&\quad+O(m)\int_{\max\{t'_{1,+},0\}}^{s-\delta}e^{-\frac{3}{4}\nu_0(s-s')}\int_{|u'|\le m}\\
&\qquad\qquad\times|h_+(s',X_+(s';s,X_\iota(s;t,x,v),u),u')|\mathrm{d}u'\mathrm{d}s'\\
&\quad+O(m)\int_{\max\{t'_{1,-},0\}}^{s-\delta}e^{-\frac{3}{4}\nu_0(s-s')}\int_{|u'|\le m}\\
&\qquad\qquad\times|h_-(s',X_-(s';s,X_\iota(s;t,x,v),u),u')|\mathrm{d}u'\mathrm{d}s'\\
&\quad+O_m(k)\sup_{l,l'}\int_{\max\{t'_{l'+1,+},0\}}^{t'_{l',+}-\delta}e^{-\frac{3}{4}\nu_0(s-s')}\\
&\qquad\qquad\times\int_{|u'|\le m}\int_{|v'_{l'}|\le m}|h_+(s',X_\iota(s';t'_{l'},x'_{l'},v'_{l'}),u')|\mathrm{d}v'_{l'}\mathrm{d}u'\mathrm{d}s'\\
&\quad+O_m(k)\sup_{l,l'}\int_{\max\{t'_{l'+1,-},0\}}^{t'_{l',-}-\delta}e^{-\frac{3}{4}\nu_0(s-s')}\\
&\qquad\qquad\times\int_{|u'|\le m}\int_{|v'_{l'}|\le m}|h_-(s',X_\iota(s';t'_{l'},x'_{l'},v'_{l'}),u')|\mathrm{d}v'_{l'}\mathrm{d}u'\mathrm{d}s',
\end{aligned}
\tag{8.24}
$$

where

$$
\begin{aligned}
t'_{l',+}&=t_{l',+}(s,X_\iota(s;t,x,v),u,v'_1,\cdots,v'_{l'-1}),\\
t'_{l',-}&=t_{l',-}(s,X_\iota(s;t,x,v),u,v'_1,\cdots,v'_{l'-1}),\\
x'_{l',+}&=x_{l',+}(s,X_\iota(s;t,x,v),u,v'_1,\cdots,v'_{l'-1})\\
x'_{l',-}&=x_{l',-}(s,X_\iota(s;t,x,v),u,v'_1,\cdots,v'_{l'-1}).
\end{aligned}
$$

Plugging 8.24 into 8.23 we conclude that

$$
\begin{aligned}
&|h_\iota(t,x,v)|\\
&\le \|(1+t)e^{-\frac{3}{4}\nu_0 t}h_0\|_\infty + O_m(k)\sup_{0\le s\le t}\|e^{-\frac{3}{4}\nu_0(t-s)}h(s)\|_\infty^2\\
&\quad + O_m(k)\int_0^t \|e^{-\frac{3}{4}\nu_0(t-s)}\nabla\phi_{\frac{h}{w_\vartheta}}(s)\|_\infty \mathrm{d}s\\
&\quad + \left\{O_m(k)\delta + O(k)\frac{1}{m} + O_m(1)\left\{\frac{1}{2}\right\}^{k/5}\right\}\sup_{0\le s\le t}\|e^{-\frac{3}{4}\nu_0(t-s)}h(s)\|_\infty\\
&\quad + O(m^2)\int_0^t\int_{|u'|\le m}\int_0^{s-\delta} e^{-\frac{3}{4}\nu_0(t-s')}\\
&\qquad\qquad \times\int_{|u|\le m}|h_+(s',X_+(s';s,X_\iota(s;t,x,v),u),u')|\mathrm{d}u\mathrm{d}s'\mathrm{d}u'\mathrm{d}s\\
&\quad + O(m^2)\int_0^t\int_{|u'|\le m}\int_0^{s-\delta} e^{-\frac{3}{4}\nu_0(t-s')}\\
&\qquad\qquad \times\int_{|u|\le m}|h_-(s',X_-(s';s,X_\iota(s;t,x,v),u),u')|\mathrm{d}u\mathrm{d}s'\mathrm{d}u'\mathrm{d}s\\
&\quad + O(k)\sup_l\int_0^{t_{l,\iota}-\delta} e^{-\frac{3}{4}\nu_0(t-s)}\int_{|u|\le m}\int_{|v_l|\le m}|h_\iota(s,X_\iota(s;t_l,x_l,v_l),u)|\mathrm{d}v_l\mathrm{d}u\mathrm{d}s\\
&\quad + O(k)\sup_{l,l'}\int_0^t\int_0^{t'_{l',+}-\delta} e^{-\frac{3}{4}\nu_0(t-s')}\\
&\qquad\qquad \times\int_{|u'|\le m}\int_{|v'_{l'}|\le m}|h_+(s',X_\iota(s';t'_{l'},x'_{l'},v'_{l'}),u')|\mathrm{d}v'_{l'}\mathrm{d}u'\mathrm{d}s'\mathrm{d}s\\
&\quad + O(k)\sup_{l,l'}\int_0^t\int_0^{t'_{l',-}-\delta} e^{-\frac{3}{4}\nu_0(t-s')}\\
&\qquad\qquad \times\int_{|u'|\le m}\int_{|v'_{l'}|\le m}|h_-(s',X_\iota(s';t'_{l'},x'_{l'},v'_{l'}),u')|\mathrm{d}v'_{l'}\mathrm{d}u'\mathrm{d}s'\mathrm{d}s.
\end{aligned}
\tag{8.25}
$$

Choose $T_\infty \gg 1$ and $k \gg 1$ in 3.26 and 3.28. Then we choose

$$
m = k^2 \quad \text{and} \quad \delta = \frac{1}{m^3 k}, \tag{8.26}
$$

so that $O_m(k)\delta + O(k)\frac{1}{m} + O_m(1)\{\frac{1}{2}\}^{k/5} \ll 1$.

Note that

$$
\begin{aligned}
&\frac{\partial X_\pm(s;t_l,x_l,v_l)}{\partial v_l}\\
&= -(t_l-s)\mathrm{Id}_{3\times 3}\\
&\quad \mp\int_{t_l}^s\int_{t_l}^\tau\left(\frac{\partial X_\pm(\tau';t_l,x_l,v_l)}{\partial v_l}\cdot\nabla_x\right)\left(\nabla_x\phi_{\frac{h}{w_\vartheta}}(\tau',X_\pm(\tau';t_l,x_l,v_l))\right)\mathrm{d}\tau'\mathrm{d}\tau,
\end{aligned}
\tag{8.27}
$$

Now we use Lemma 2.4. Note that from 8.8, the condition 4.7 of Lemma 2.4 is satisfied with $\Lambda_2 = \frac{\lambda_\infty}{2}$ and $\delta_2 = C_2 M$. From Lemma 2.4 and 2.24 we have for

$\iota = +$ or $-$,

$$\left|\frac{\partial X_\iota(\tau';t_l,x_l,v_l)}{\partial v_l}\right| \le Ce^{\frac{4CC_2M}{(\lambda_\infty)^2}}|t_l-\tau'|. \tag{8.28}$$

From 8.28 and 8.8, the second term of RHS in 8.27 is bounded by

$$\begin{aligned} &CC_2Me^{\frac{4CC_2M}{(\lambda_\infty)^2}}\int_s^{t_l}\int_\tau^{t_l}(t_l-\tau')e^{-\frac{\lambda_\infty}{2}\tau'}\mathrm{d}\tau'\mathrm{d}\tau \\ \le\ &\frac{4CC_2M}{(\lambda_\infty)^2}e^{\frac{4CC_2M}{(\lambda_\infty)^2}}|t_l-s|. \end{aligned} \tag{8.29}$$

From our choice of λ_∞ in 8.7, we have

$$\frac{4CC_2M}{(\lambda_\infty)^2}e^{\frac{4CC_2M}{(\lambda_\infty)^2}} < \frac{1}{10}.$$

Therefore from 8.27, for $0 \le s \le t_l - \delta$

$$\begin{aligned} &\det\left(\frac{\partial X_\iota(s;t_l,x_l,v_l)}{\partial v_l}\right) \\ =\ &\det\left(-(t_l-s)\mathrm{Id}_{3\times 3}+o(1)\right) \\ \gtrsim\ &|t_l-s|^3 \\ \gtrsim\ &\delta. \end{aligned} \tag{8.30}$$

We can obtain the exactly same lower bound of $\det\left(\frac{\partial X_\iota(s;t'_{l'},x'_{l'},v'_{l'})}{\partial v'_{l'}}\right)$, $\det\left(\frac{\partial X_+(s';s,X_\iota(s;t,x,v),u)}{\partial u}\right)$, and $\det\left(\frac{\partial X_-(s';s,X_\iota(s;t,x,v),u)}{\partial u}\right)$ for $0 \le s' \le s-\delta$ and $0 \le s' \le t'_{l'}-\delta$.

Now we apply the change of variables

$$\begin{aligned} v_l &\mapsto X_\iota(s;t_l,x_l,v_l), \\ v'_{l'} &\mapsto X_\iota(s;t'_{l'},x'_{l'},v'_{l'}), \\ u &\mapsto X_+(s';s,X_\iota(s;t,x,v),u), \\ u &\mapsto X_-(s';s,X_\iota(s;t,x,v),u), \end{aligned}$$

and conclude 8.14 from 8.25 and 8.26.

By choosing $T_\infty \gg 1$ so that $(1+T_\infty)e^{-\frac{3}{4}\nu_0 T_\infty} \le e^{-\frac{1}{2}\nu_0 T_\infty}$, and applying 8.14 successively, we achieve that

$$\begin{aligned} &\|w_\vartheta f(t)\|_\infty \\ \le\ &C_{T_\infty}e^{-\frac{\nu_0}{2}t}\|w_\vartheta f(0)\|_\infty + o(1)\frac{e^{\nu_0 T_\infty}}{1-e^{-\nu_0 T_\infty}}\sup_{0\le s\le t}e^{-\nu_0(t-s)}\|w_\vartheta f(s)\|_\infty \\ &+\underbrace{C_{T_\infty}e^{\frac{\nu_0}{2}}\int_0^t e^{-\frac{\nu_0}{2}(t-s)}\|f(s)\|_2\mathrm{d}s}_{8.31_{L^2}}+\underbrace{C_{T_\infty}e^{\frac{\nu_0}{2}}\int_0^t e^{-\frac{\nu_0}{2}(t-s)}\|\nabla\phi_f(s)\|_\infty\mathrm{d}s}_{8.31_{\phi_f}}, \end{aligned} \tag{8.31}$$

where we have used

$$e^{\nu_0 T_\infty}\{1+e^{-\nu_0 T_\infty}+\cdots+e^{-\nu_0 NT_\infty}\} = \frac{e^{\nu_0 T_\infty}}{1-e^{-\nu_0 T_\infty}}.$$

Step 4. From Proposition 6 and 7.1 we have

$$\begin{aligned}&\|e^{\lambda_2 t}f(t)\|_2^2+\|e^{\lambda_2 t}\nabla\phi(t)\|_2^2\\&+\int_0^t\|e^{\lambda_2\tau}f(\tau)\|_\nu^2+\|e^{\lambda_2\tau}\nabla\phi_f(\tau)\|_2^2\mathrm{d}\tau+\int_0^t|e^{\lambda_2\tau}f|_{2,+}^2\\&\lesssim\|f_0\|_2^2+\|\nabla\phi_{f_0}\|_2^2.\end{aligned}\tag{8.32}$$

Hence

$$\begin{aligned}8.31_{L^2}&\lesssim te^{-\min(\frac{\nu_0}{2},\lambda_2)\times t}\{\|f_0\|_2+\|\nabla\phi_{f_0}\|_2\}\\&\lesssim e^{-\min(\frac{\nu_0}{4},\frac{\lambda_2}{2})\times t}\{\|f_0\|_2+\|\nabla\phi_{f_0}\|_2\}.\end{aligned}\tag{8.33}$$

Now we consider 8.31_{ϕ_f}. In order to close the estimate in 8.31 we need to improve the decay rate of $\|\nabla\phi_f(s)\|_\infty$. We claim that, for $\theta_{2,r,p}>0$ (which is specified in 8.38,

$$\|\nabla_x\phi_f(s)\|_\infty\lesssim e^{-(1+\theta_{2,r,p})\lambda_\infty s}\{\sup_{t\geq0}\|e^{\lambda_2 t}f(s)\|_2+\sup_{t\geq0}\|e^{\lambda_\infty t}f(s)\|_\infty\}.\tag{8.34}$$

By Morrey's inequality for $\Omega\subset\mathbb{R}^3$ and $r>3$

$$\|\nabla_x\phi_f\|_\infty\lesssim\|\nabla_x\phi_f\|_{C^{0,1-3/r}(\Omega)}\lesssim\|\nabla_x\phi_f\|_{W^{1,r}(\Omega)}.\tag{8.35}$$

Then applying the standard elliptic estimate to 1.16, we get

$$\|\nabla_x\phi_f(t)\|_{W^{1,2}(\Omega)}\lesssim\left\|\int_{\mathbb{R}^3}(f_+(t)-f_-(t))\sqrt{\mu}\mathrm{d}v\right\|_{L^2(\Omega)}\lesssim e^{-\lambda_2 t}\sup_{t\geq0}\|e^{\lambda_2 t}f(t)\|_2,\tag{8.36}$$

$$\|\nabla_x\phi_f(t)\|_{W^{1,p}(\Omega)}\lesssim\left\|\int_{\mathbb{R}^3}(f_+(t)-f_-(t))\sqrt{\mu}\mathrm{d}v\right\|_{L^p(\Omega)}\lesssim e^{-\lambda_\infty t}\sup_{t\geq0}\|e^{\lambda_\infty t}f(t)\|_\infty.\tag{8.37}$$

Now we use the standard interpolation: For $p>r>3$,

$$\|\nabla_x\phi_f\|_{W^{1,r}(\Omega)}\lesssim\|\nabla_x\phi_f(t)\|_{W^{1,2}(\Omega)}^{\theta_{2,r,p}}\|\nabla_x\phi_f(t)\|_{W^{1,p}(\Omega)}^{1-\theta_{2,r,p}},$$

for

$$\theta_{2,r,p}:=\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{2}-\frac{1}{p}}>\frac{2}{3}\cdot\frac{p-3}{p-2}.\tag{8.38}$$

Then we derive

$$\begin{aligned}&\sup_{t\geq0}\|e^{[\theta_{2,r,p}\lambda_2+(1-\theta_{2,r,p})\lambda_\infty]t}\nabla_x\phi_f(t)\|_\infty\\&\lesssim\left(\sup_{t\geq0}\|e^{\lambda_2 t}f(t)\|_2\right)^{\theta_{2,r,p}}\left(\sup_{t\geq0}\|e^{\lambda_\infty t}f(t)\|_\infty\right)^{1-\theta_{2,r,p}}\\&\lesssim\sup_{t\geq0}\|e^{\lambda_2 t}f(t)\|_2+o(1)\sup_{t\geq0}\|e^{\lambda_\infty t}f(t)\|_\infty.\end{aligned}\tag{8.39}$$

From our choice 8.7 and $0<p-3\ll1$,

$$\theta_{2,r,p}\lambda_2+(1-\theta_{2,r,p})\lambda_\infty\geq(1+\theta_{2,r,p})\lambda_\infty.\tag{8.40}$$

From 8.39

$$
\begin{aligned}
8.31_{\phi_f} \lesssim & \int_0^t e^{-\frac{\nu_0}{2}(t-s)} e^{-(1+\theta_{2,r,p})\lambda_\infty s} \|e^{\lambda_2 s} f(s)\|_2 \mathrm{d}s \\
& + o(1) \int_0^t e^{-\frac{\nu_0}{2}(t-s)} e^{-(1+\theta_{2,r,p})\lambda_\infty s} \|e^{\lambda_\infty s} w_\vartheta f(s)\|_\infty \mathrm{d}s \\
\lesssim & e^{-\min(\frac{\nu_0}{4},\lambda_\infty)\times t}\{\|f_0\|_2 + \|\nabla\phi_{f_0}\|_2\} \quad \textit{from 8.32} \\
& + o(1) e^{-\min(\frac{\nu_0}{4},\lambda_\infty)\times t} \sup_{0\le s\le t} \|e^{\lambda_\infty s} w_\vartheta f(s)\|_\infty.
\end{aligned} \tag{8.41}
$$

Multiplying $e^{\lambda_\infty t}$ and taking $\sup_{t\ge 0}$ to 8.31 with $\lambda_\infty \le \min\left(\frac{\nu_0}{4}, \frac{\lambda_2}{2}\right)$, and from 8.33 and 8.41, we obtain that

$$
\sup_{t\ge 0} e^{\lambda_\infty t}\|w_\vartheta f(t)\|_\infty \lesssim \|w_\vartheta f(0)\|_\infty + \|f_0\|_2 + \|\nabla\phi_{f_0}\|_2 + o(1) \sup_{0\le s\le t} e^{\lambda_\infty s}\|w_\vartheta f(s)\|_\infty. \tag{8.42}
$$

By absorbing the last (small) term, we conclude that

$$
\sup_{0\le t\le T} e^{\lambda_\infty t}\|w_\vartheta f(t)\|_\infty \le \mathfrak{C}\delta_* M. \tag{8.43}
$$

If we choose $\delta_* \ll 1/\mathfrak{C}$ then by the local existence theorem (Theorem 6.1) and continuity of $\|w_\vartheta f(t)\|_\infty$, $\|w_{\tilde\vartheta} f(t)\|_p^p + \|w_{\tilde\vartheta}\alpha_{f,\varepsilon}^\beta \nabla_{x,v} f(s)\|_p^p + \int_0^t |w_{\tilde\vartheta}\alpha_{f,\varepsilon}^\beta \nabla_{x,v} f(s)|_{p,+}^p$, and $\|\nabla_v f(t)\|_{L^3_x(\Omega)L^{1+\delta}_v(\mathbb{R}^3)}$, we conclude that $T = \infty$.

Then the estimates of 1.46 and 1.47 are direct consequence of Proposition 3, Lemma 4.2, and Proposition 5.

And 1.48 can be derived from 6.5-6.11 by replacing f^ℓ, $f^{\ell+1}$ with f, g. □

Appendix A. Auxiliary results and proofs.

Proof of 1.32. Let $\iota = +$ or $-$, from 1.27, for $t - t_{\mathbf{b},\iota}(t,x,v) < s \le t$,

$$
\begin{aligned}
& x_{\mathbf{b},\iota}(s, X_\pm(s;t,x,v), V_\pm(s;t,x,v)) = x_{\mathbf{b},\iota}(t,x,v), \\
& v_{\mathbf{b},\iota}(s, X_\pm(s;t,x,v), V_\pm(s;t,x,v)) = v_{\mathbf{b},\iota}(t,x,v).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& [\partial_t + v\cdot\nabla_x \mp \nabla_x\phi_f\cdot\nabla_v]\alpha_{f,\varepsilon,\pm}(t,x,v) \\
= & \frac{d}{ds}\alpha_{f,\varepsilon,\pm}(s, X_\pm(s;t,x,v), V_\pm(s;t,x,v))\big|_{s=t} \\
= & \frac{d}{ds}\alpha_{f,\varepsilon,\pm}(t,x,v) = 0.
\end{aligned}
$$

From 1.28 and 1.27,

$$
t_{\mathbf{b},\iota}(s, X_\pm(s;t,x,v), V_\pm(s;t,x,v)) = t_{\mathbf{b},\iota}(t,x,v) - (t-s).
$$

Therefore

$$
\begin{aligned}
& [\partial_t + v\cdot\nabla_x \mp \nabla_x\phi_f\cdot\nabla_v](t - t_{\mathbf{b},\iota}(t,x,v)) \\
= & \frac{d}{ds}[s - t_{\mathbf{b},\iota}(s, X_\pm(s;t,x,v), V_\pm(s;t,x,v))]\big|_{s=t} \\
= & \frac{d}{ds}[t - t_{\mathbf{b},\iota}(t,x,v)] = 0.
\end{aligned}
$$

These prove 1.32. □

Proof of 3.32. The proof follows the argument of Lemma 7 in [13]. Note

$$\mathbf{k}_\varrho(v,u)\frac{e^{\vartheta|v|^2}}{e^{\vartheta|u|^2}}=\frac{1}{|v-u|}\exp\left\{-\varrho|v-u|^2-\varrho\frac{||v|^2-|u|^2|^2}{|v-u|^2}+\vartheta|v|^2-\vartheta|u|^2\right\}.$$

Let $v-u=\eta$ and $u=v-\eta$. Then the exponent equals

$$\begin{aligned}&-\varrho|\eta|^2-\varrho\frac{||\eta|^2-2v\cdot\eta|^2}{|\eta|^2}-\vartheta\{|v-\eta|^2-|v|^2\}\\ =\;&-2\varrho|\eta|^2+4\varrho v\cdot\eta-4\varrho\frac{|v\cdot\eta|^2}{|\eta|^2}-\vartheta\{|\eta|^2-2v\cdot\eta\}\\ =\;&(-2\varrho-\vartheta)|\eta|^2+(4\varrho+2\vartheta)v\cdot\eta-4\varrho\frac{\{v\cdot\eta\}^2}{|\eta|^2}.\end{aligned}$$

If $0<\vartheta<4\varrho$ then the discriminant of the above quadratic form of $|\eta|$ and $\frac{v\cdot\eta}{|\eta|}$ is

$$(4\varrho+2\vartheta)^2-4(-2\varrho-\vartheta)(-4\varrho)=4\vartheta^2-16\varrho\vartheta<0.$$

Hence, the quadratic form is negative definite. We thus have, for $0<\tilde{\varrho}<\varrho-\frac{\vartheta}{4}$, the following perturbed quadratic form is still negative definite

$$-(\varrho-\tilde{\varrho})|\eta|^2-(\varrho-\tilde{\varrho})\frac{||\eta|^2-2v\cdot\eta|^2}{|\eta|^2}-\vartheta\{|\eta|^2-2v\cdot\eta\}\le 0.$$

Therefore we conclude 3.32. □

Recall $\kappa_\delta(x,v)$ in 6.13. Let us denote $f_\delta(t,x,v):=\kappa_\delta(x,v)f(t,x,v)$. We assume that $f(s,x,v)=e^s f_0(x,v)$ for $s<0$. Then $\|f_\delta\|_{L^2(\mathbb{R}\times\Omega\times\mathbb{R}^3)}\lesssim\|f\|_{L^2(\mathbb{R}_+\times\Omega\times\mathbb{R}^3)}+\|f_0\|_{L^2(\Omega\times\mathbb{R}^3)}$, $\|f_\delta\|_{L^2(\mathbb{R}\times\gamma)}\lesssim\|f_\gamma\|_{L^2(\mathbb{R}_+\times\gamma)}+\|f_0\|_{L^2(\gamma)}$.

Lemma A.1. *Assume Ω is convex in 1.40 and $\sup_{0\le t\le T}\|E(t)\|_{L^\infty(\Omega)}<\infty$. Let $\bar{E}(t,x)=\mathbf{1}_\Omega(x)E(t,x)$ for $x\in\mathbb{R}^3$. There exists $\bar{f}(t,x,v)\in L^2(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3;\mathbb{R}^2)$, an extension of f_δ, such that*

$$\bar{f}|_{\Omega\times\mathbb{R}^3}\equiv f_\delta\quad\text{and}\quad\bar{f}|_\gamma\equiv f_\delta|_\gamma\quad\text{and}\quad\bar{f}|_{t=0}\equiv f_\delta|_{t=0}.$$

Moreover, in the sense of distributions on $\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^2$,

$$[\partial_t+v\cdot\nabla_x+q\bar{E}\cdot\nabla_v]\bar{f}=h,\tag{A.1}$$

where

$$\begin{aligned}h(t,x,v)=\;&\kappa_\delta(x,v)\mathbf{1}_{t\in[0,\infty)}[\partial_t+v\cdot\nabla_x+qE\cdot\nabla_v]f\\&+\kappa_\delta(x,v)\mathbf{1}_{t\in(-\infty,0]}e^t[1+v\cdot\nabla_x+qE\cdot\nabla_v]f_0\kappa_\delta(x,v)\\&+f(t,x,v)[v\cdot\nabla_x+q_1E\cdot\nabla_v]\kappa_\delta(x,v),\end{aligned}\tag{A.2}$$

where $t^{EX}_{\mathbf{b}},x^{EX}_{\mathbf{b}},t^{EX}_{\mathbf{f}},x^{EX}_{\mathbf{f}}$ are defined in A.5.

Moreover,

$$\begin{aligned}\|h\|_{L^2(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)}\lesssim\;&\|[\partial_t+v\cdot\nabla_x+qE\cdot\nabla_v]f\|_{L^2(\mathbb{R}_+\times\Omega\times\mathbb{R}^3)}+\|f\|_{L^2(\mathbb{R}\times\Omega\times\mathbb{R}^3)}\\&+\|[v\cdot\nabla_x+qE\cdot\nabla_v]f_0\|_{L^2(\Omega\times\mathbb{R}^3)}.\end{aligned}\tag{A.3}$$

Proof. In the sense of distributions

$$\partial_t f_\delta+v\cdot\nabla_x f_\delta+qE\cdot\nabla_v f_\delta=h\text{ in }A.2.\tag{A.4}$$

Clearly $|[v\cdot\nabla_x+q_1E\cdot\nabla_v]\kappa_\delta(x,v)|\lesssim_\delta 1$.

For $x \in \mathbb{R}^3\backslash\bar{\Omega}$ we define

$$\begin{aligned} t_{\mathbf{b}}^{EX}(x,v) &:= \sup\{s \geq 0 : x - \tau v \in \mathbb{R}^3\backslash\bar{\Omega} \ \text{ for all } \ \tau \in (0,s)\}, \\ t_{\mathbf{f}}^{EX}(x,v) &:= \sup\{s \geq 0 : x + \tau v \in \mathbb{R}^3\backslash\bar{\Omega} \ \text{ for all } \ \tau \in (0,s)\}, \end{aligned} \tag{A.5}$$

and $x_{\mathbf{b}}^{EX}(x,v) = x - t_{\mathbf{b}}^{EX}(t,x,v))v$, $x_{\mathbf{f}}^{EX}(x,v) = x + t_{\mathbf{f}}^{EX}(t,x,v))v$.

We define, for $x \in \mathbb{R}^3\backslash\bar{\Omega}$,

$$\begin{aligned} f_E(t,x,v) =& \mathbf{1}_{x_{\mathbf{b}}^{EX}(t,x,v)\in\partial\Omega} f_\delta(t - t_{\mathbf{b}}^{EX}(x,v), x_{\mathbf{b}}^{EX}(x,v), v) \\ &+ \mathbf{1}_{x_{\mathbf{f}}^{EX}(t,x,v)\in\partial\Omega} f_\delta(t + t_{\mathbf{f}}^{EX}(x,v), x_{\mathbf{f}}^{EX}(x,v), v). \end{aligned} \tag{A.6}$$

Recall that, from 6.13, $f_\delta \equiv 0$ when $n(x)\cdot v = 0$, and hence $f_E \equiv 0$ for $n(x)\cdot v = 0$. Since Ω is convex if $v \neq 0$ then $\{x_{\mathbf{b}}^{EX}(x,v) \in \partial\Omega\} \cap \{x_{\mathbf{f}}^{EX}(x,v) \in \partial\Omega\} = \emptyset$. Note that

$$f_E(t,x,v) = f_\gamma(t,x,v) = f_\delta(t,x,v) \ \text{ for } x \in \partial\Omega. \tag{A.7}$$

And since for any $s > 0$,

$$\begin{aligned} (t + s - t_{\mathbf{b}}^{EX}(x+sv,v), x_{\mathbf{b}}^{EX}(x+sv,v), v) &= (t - t_{\mathbf{b}}^{EX}(x,v), x_{\mathbf{b}}^{EX}(x,v), v), \\ (t + s + t_{\mathbf{f}}^{EX}(x+sv,v), x_{\mathbf{f}}^{EX}(x+sv,v), v) &= (t - t_{\mathbf{f}}^{EX}(x,v), x_{\mathbf{f}}^{EX}(x,v), v), \end{aligned}$$

so in the sense of distribution, in $\mathbb{R}\times[\mathbb{R}^3\backslash\bar{\Omega}]\times\mathbb{R}^3$

$$\partial_t f_E + v\cdot\nabla_x f_E = 0. \tag{A.8}$$

We define

$$\bar{f}(t,x,v) := \mathbf{1}_\Omega(x) f_\delta(t,x,v) + \mathbf{1}_{\mathbb{R}^3\backslash\bar{\Omega}}(x) f_E(t,x,v). \tag{A.9}$$

From A.4, A.7, and A.8 we prove A.1. The estimates of A.3 are direct consequence of Lemma 2.2. □

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