Genericity of pseudo-Anosov mapping classes, when seen as mapping classes

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Abstract. We prove that pseudo-Anosov mapping classes are generic with respect to certain notions of genericity reflecting that we are dealing with mapping classes.

1. Introduction

Throughout this paper let Σ be a complete orientable hyperbolic surface of finite area, with genus g and r punctures, where $(g, r) \neq (0, 3)$.

Thurston's classification asserts that elements in the mapping class group $\mathrm{Map}(\Sigma)$ fall into three categories: finite order, reducible, and pseudo-Anosov. However, it seems that from any reasonable point of view most elements are pseudo-Anosov. For example, Maher [14] proved that, with few assumptions, random walks on the mapping class group give rise to pseudo-Anosov elements with asymptotic probability one. This result was later enhanced and generalized by Maher himself and others [16, 17, 26, 28, 29].

We will however care about another notion of genericity: if $\rho : \mathrm{Map}(\Sigma) \to \mathbb{R}_{\geq 0}$ is a proper positive function, then we say that a set $X \subset \mathrm{Map}(\Sigma)$ is *generic* with respect to ρ, or *ρ-generic* for short, if we have

$$\lim_{R\to\infty} \frac{|B^\rho(R) \cap X|}{|B^\rho(R)|} = 1$$

where $B^\rho(R) = \{\phi \in \mathrm{Map}(\Sigma) \text{ with } \rho(\phi) \leq R\}$. Here properness of ρ just means that $B^\rho(R)$ is a finite set for all R. A *negligible* set is one whose complement is generic.

Maybe the first function that comes to mind is the word length with respect to a finite generating set $\mathcal{G}$ of $\mathrm{Map}(\Sigma)$, and Cumplido and Wiest [7] proved that indeed the set of pseudo-Anosov elements is not negligible in this sense. For braid groups equipped with the Garside's generating set, Caruso and Wiest [5] showed that it is also generic. But beyond this case, genericity of pseudo-Anosov elements remain open for word lengths.

However, one can make the case that the word length, while being related to the group theory of the mapping class group, has little to do with the fact that the mapping class group consists of mapping classes. To illustrate this point identify $\mathrm{SL}_2\,\mathbb{Z}$ with the mapping class group of the once punctured torus and note that the two matrices

$$A = \begin{pmatrix} 5904283700961130691 & 4322235651404355330 \\ 2161117825702177665 & 1582048049556775361 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 99 \\ 0 & 1 \end{pmatrix}$$

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have the same word length, namely 99, with respect to the standard generating set of $\mathrm{SL}_2\,\mathbb{Z}$. Arguably, it would be more natural to say that A is farther from the identity than B. Not only because the coefficients of A are much larger than those of B but, more importantly, because the map induced by A on the torus distorts both the metric and conformal structure much more dramatically than the map induced by B.

Our goal is to prove that pseudo-Anosov mapping classes are ρ-generic with respect to a number of functions on $\mathrm{Map}(\Sigma)$ measuring the complexity of mapping classes when seen as mapping classes:

Theorem 1.1. *The set of pseudo-Anosov mapping classes is generic with respect to any one of the functions*

$$\rho_K(\phi) = \inf\{K(f) \text{ where } f \in \mathrm{Diff}(\Sigma) \text{ represents } \phi\}$$

where $K(f)$ is the quasi-conformal distortion of f,

$$\rho_{\mathrm{Lip}}(\phi) = \inf\{\mathrm{Lip}(f) \text{ where } f \in \mathrm{Diff}(\Sigma) \text{ represents } \phi\}$$

where $\mathrm{Lip}(f)$ is the Lipschitz constant for f, and

$$\rho_{\sigma,\eta}(\phi) = \iota(\phi(\sigma), \eta)$$

where σ and η are filling multicurves and $\iota(\cdot,\cdot)$ is the geometric intersection number.

Remark. Note that, although amazingly it is not formally stated in the paper, the claim for $\rho_K(\phi)$ in Theorem 1.1 was obtained by Maher in [15]. Unfortunately, we were unaware of this fact until we finished writing our paper. Both the argument in [15] and ours have the same starting point, namely an earlier, again not formally stated, result from [14]. However, after that starting point, the arguments use different methods and techniques. We will return to this at the end of the introduction.

We sketch now the proof of Theorem 1.1. We begin by addressing the reason why we are including $\rho_{\sigma,\eta}$ at all among the functions in Theorem 1.1. There are a few reasons. First, both quantities $\rho_K(\phi)$ and $\rho_{\mathrm{Lip}}(\phi)$ can be estimated in terms of $\rho_{\sigma,\eta}$. Second, there is the maybe not very important observation that, after identifying $\mathrm{SL}_2\,\mathbb{Z}$ with the mapping class group of a punctured torus, the ℓ_1-norm on $\mathrm{SL}_2\,\mathbb{Z}$ agrees with $\rho_{\sigma,\sigma}$ where σ is the union of the two simple curves representing the standard generators of homology. However, the main reason to consider $\rho_{\sigma,\eta}$ is that it is the more natural quantity from the point of view of proofs.

In fact, if we denote by $\mathcal{C}(\Sigma)$ the space of geodesic currents on Σ endowed with the weak-* topology, and consider multicurves as currents, then what we will actually prove is the following theorem:

Theorem 1.2. *Let $\mathcal{R} \subset \mathrm{Map}(\Sigma)$ be the set of non-pseudo-Anosov mapping classes and let $\gamma_0 \subset \Sigma$ be a filling multicurve. Then we have*

$$\lim_{L\to\infty} \frac{|\{\phi \in \mathcal{R} \text{ with } F(\phi(\gamma_0)) \leq L\}|}{L^{6g-6+2r}} = 0$$

for every continuous homogenous function $F : \mathcal{C}(\Sigma) \to \mathbb{R}_{\geq 0}$ which, for every compact $K \subset \Sigma$, is proper when restricted on the set $\mathcal{C}_K(\Sigma)$ of currents supported by K.

Recall that a function $F : \mathcal{C}(\Sigma) \to \mathbb{R}$ is *homogenous* if $F(t \cdot \lambda) = t \cdot F(\lambda)$ for every $t \geq 0$ and $\lambda \in \mathcal{C}(\Sigma)$. Note also that for Σ open, the properness condition we impose on F is much weaker it being proper on $\mathcal{C}(\Sigma)$. For example, if η is a filling multicurve then $F(\cdot) = \iota(\cdot, \eta)$ is not proper on $\mathcal{C}(\Sigma)$ but is proper on $\mathcal{C}_K(\Sigma)$ for any K. Theorem 1.1 follows when we apply Theorem 1.2 to the corresponding functions combined with the fact, see [8, 27], that

$$\liminf_{L\to\infty} \frac{|\{\phi \in \mathrm{Map}(\Sigma) \text{ with } F(\phi(\gamma_0)) \leq L\}|}{L^{6g-6+2r}} > 0 \tag{1.1}$$

for any F as in Theorem 1.2.

The starting point of the proof of Theorem 1.2 is a result of Maher [14] asserting that the set $\mathcal{R} \subset \mathrm{Map}(S)$ of non-pseudo-Anosov mapping classes is the union, for each k, of k-isolated points (that is, points which are at distance at least k from any other element of $\mathcal{R}$) together with the union of finitely many sets, each one of which consists of mapping classes at *relative distance* $L(k)$ around the centralizer of some mapping class. Here the relative distance is the semi-distance on $\mathrm{Map}(S)$ arising, with the help of a base point, form its action on the curve complex. It follows that proving that $\mathcal{R}$ is negligible boils down to proving (1) that the set $\mathcal{R}_k \subset \mathcal{R}$ of k-isolated points has low density and (2) that sets of mapping classes with small relative distance of centralizers of elements are negligible. Rephrasing this in terms of measures (on the space of currents) it suffices to prove (1) that

$$\lim_{k\to\infty} \lim_{L\to\infty} \frac{1}{L^{6g-6+2r}} \sum_{\phi\in\mathcal{I}_k} \delta_{\frac{1}{L}\phi(\gamma_0)} = 0, \tag{1.2}$$

and (2) that

$$\lim_{L\to\infty} \frac{1}{L^{6g-6+2r}} \sum_{\phi\in\mathcal{N}_{\mathrm{rel}}(C(\phi_0),R)} \delta_{\frac{1}{L}\phi(\gamma_0)} = 0 \tag{1.3}$$

for $\phi_0 \in \mathrm{Map}(\Sigma)$ non-central. Here δ_x is the Dirac measure centred on x and the convergence takes place with respect to the weak-*-topology. We get (1.3) from the fact that any limit is absolutely continuous to the Thurston measure — an immediate consequence of for example Proposition 4.1 in [8] — and of the fact that the set of limits of sequences of the form $(\phi_i(\gamma_0))$ with $\phi \in \mathcal{N}_{\mathrm{rel}}(C(\phi_0), R)$ has vanishing Thurston measure. To establish (1.2) we use again that any limit is absolutely continuous with respect to the Thurston measure, but this time we have to use Masur's result [19] on the ergodicity of the Thurston measure with respect to the action of the mapping class group.

Remark. Maher's proof in [15] of Theorem 1.1 also relies on the decomposition of $\mathcal{R}$ as the union of $\mathcal{I}_k$ and finitely many sets consisting of mapping classes at bounded relative distance from the centralizer of some mapping class. At this point the two arguments diverge. While we rely on the fact that every limit of (1.2) and (1.3) is absolutely continuous with respect to the Thurston measure, Maher makes use of a rather sophisticated lattice counting result of Athreya-Bufetov-Eskin-Mirzakhani [2]. Similarly, while we rely on the ergodicity of the Thurston measure, that is the ergodicity of the Teichmüller flow, Maher relies on the mixing property of that flow. We might be partial, but we believe that our argument is not only different but also simpler than that of Maher.

Remark. As it is the case for Maher's argument, all the results here hold with unchanged proofs if we replace the set $\mathcal{R}$ of non-pseudo Anosov elements by any set of elements for which there is a uniform upper bound for the translation length in the curve complex.

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2. Maher's theorem

As we already did in the introduction, we denote by $\mathcal{R}$ the set of all non-pseudo-Anosov mapping classes of $\mathrm{Map}(\Sigma)$. We also fix an arbitrary finite generating set $\mathcal{G}$ for $\mathrm{Map}(\Sigma)$ and let $d_{\mathcal{G}}$ be the induced left-invariant distance:

$$d_{\mathcal{G}}(\phi, \psi) = \text{word length with respect to } \mathcal{G} \text{ of } \psi^{-1}\phi.$$

Given $k > 0$ let

$$\mathcal{I}_k = \{\phi \in \mathcal{R} \text{ with } d_{\mathcal{G}}(\phi, \phi') \geq k \text{ for all } \phi' \in \mathcal{R} \setminus \{\phi\}\}$$

be the set of elements in $\mathcal{R}$ which do not have any other elements in $\mathcal{R}$ within distance less than k. We denote the complement of $\mathcal{I}_k$ by

$$\mathcal{D}_k = \mathcal{R} \setminus \mathcal{I}_k.$$

The notations are chosen to suggest that $\mathcal{I}_k$ consists of *k-isolated points* and that $\mathcal{D}_k$ consists of *k-dense points*.

Recall that distances in the definition of $\mathcal{I}_k$ (and thus in that of $\mathcal{D}_k$ as well) are measured with respect to the distance $d_{\mathcal{G}}$. We stress that this is the case because we will also be working with another distance, or rather a semi-distance, namely the *relative distance*

$$d_{\mathrm{rel}}(\phi, \psi) = d_{C(\Sigma)}(\phi(\alpha_0), \psi(\alpha_0))$$

where $d_{C(\Sigma)}(\cdot, \cdot)$ denotes the distance in the curve complex $C(\Sigma)$, and where α_0 is a fixed but otherwise arbitrary simple essential curve in Σ.

Armed with this notation we can state Maher's theorem:

Theorem 2.1 (Maher)**.** *For every k, there is a finite set of non-central mapping classes $\mathcal{F} \subset \mathrm{Map}(\Sigma) \setminus C(\mathrm{Map}(\Sigma))$ and some $L > 0$ such that*

$$\mathcal{D}_k \subset \bigcup_{\phi \in \mathcal{F}} \{\psi \in \mathrm{Map}(\Sigma) \text{ with } d_{\mathrm{rel}}(\psi, C(\phi)) \leq L\},$$

where $C(\phi)$ is the centralizer of ϕ in $\mathrm{Map}(S)$ and $C(\mathrm{Map}(\Sigma))$ is the center of $\mathrm{Map}(\Sigma)$.

Although it is proved and used in [14] (see the discussion at the beginning of section 5 in said paper), Theorem 2.1 is not explicitly stated therein. Hence we discuss how to deduce it from the stated results here:

Proof. First, suppose that $\mathrm{Map}(\Sigma)$ is center free. Then, from the very definition of $\mathcal{D}_k$, we get that there is a finite subset $\mathcal{F} \subset \mathrm{Map}(\Sigma)$ with

$$\mathcal{D}_k \subset \bigcup_{\phi \in \mathcal{F}} (\mathcal{R} \cap \mathcal{R}\phi). \tag{2.4}$$

To see this, note that one can take $\mathcal{F}$ to be all non-trivial elements in the ball of radius k around the identity with respect to $d_{\mathcal{G}}$.

Now, Theorem 4.1 in [14] implies that for each $\phi \in \mathcal{F}$ there is some L such that

$$\mathcal{R} \cap \mathcal{R}\phi \subset \{\psi \in \mathrm{Map}(\Sigma) \text{ with } d_{\mathrm{rel}}(\psi, C(\phi)) \leq L\}.$$

This theorem applies because the mapping class group is weakly relatively hyperbolic with relative conjugacy bounds [14, Theorem 3.1] and because $\mathcal{R}$ consists of elements conjugated to elements of bounded relative length [14, Lemma 5.5]. This concludes the discussion of Theorem 2.1 if $\mathrm{Map}(\Sigma)$ is center free.

In the presence of a non-trivial center the argument is almost the same: Note that $\mathcal{R} = \mathcal{R}\phi$ for every central element and hence the only change to the above argument is that one has to take $\mathcal{F}$ to be the set of all non-central elements in the ball of radius k around the identity with respect to $d_{\mathcal{G}}$. □

3. Currents

In this section we recall a few facts about the space of geodesic currents on Σ. We then describe the (projective) accumulation points of sequences of the form $(\phi_i(\gamma_0))$ where γ_0 is an essential multicurve and where (ϕ_i) is a sequence of mapping classes at bounded relative distance of the centralizer of some $\phi \in \mathrm{Map}(\Sigma)$. Recall that a multicurve is a finite union of (disjoint or not) of (simple or not) primitive essential curves in Σ. We say that a multicurve is filling if its geodesic representative cuts the surface into a collection of disks and once-punctured disks.

Properties of the space of currents. Let $\overline{\Sigma}$ be a compact surface with interior $\Sigma = \overline{\Sigma} \setminus \partial\overline{\Sigma}$, endowed with an arbitrary hyperbolic metric with totally geodesic boundary. We suggest the reader to think, in a first reading, that $\overline{\Sigma} = \Sigma$; that is, Σ is closed.

Geodesic currents on Σ are fundamental group invariant Radon measures on the space of geodesics on the universal cover of $\overline{\Sigma}$. However, that they are such measures will not really be relevant here—what is more important for our purposes are the properties the space $\mathcal{C}(\Sigma)$ of currents have (when endowed with the weak-*-topology). We list the facts about $\mathcal{C}(\Sigma)$ that we will use:

(1) $\mathcal{C}(\Sigma)$ is a locally compact metrizable topological space.
(2) $\mathcal{C}(\Sigma)$ is a cone as a topological vector space, meaning in particular that there are continuous maps

$$\mathcal{C}(\Sigma) \times \mathcal{C}(\Sigma) \to \mathcal{C}(\Sigma),\ (\lambda, \mu) \mapsto \lambda + \mu$$
$$\mathbb{R}_{\geq 0} \times \mathcal{C}(\Sigma) \to \mathcal{C}(\Sigma),\ (t, \lambda) \mapsto t\lambda$$

satisfying the usual associativity, commutativity and distributivity properties as in vector spaces.
(3) The set $\{\gamma \text{ closed geodesic in } \Sigma\}$ is a subset of $\mathcal{C}(\Sigma)$ and in fact the set

$$\mathbb{R}_+ \cdot \{\gamma \text{ closed geodesic in } \Sigma\}$$

of weighted closed geodesics is dense in $\mathcal{C}(\Sigma)$.
(4) The inclusion of the set of weighted simple geodesics into $\mathcal{C}(\Sigma)$ extends to a continuous embedding of the space $\mathcal{ML}(\Sigma)$ of measured laminations into $\mathcal{C}(\Sigma)$.
(5) There is a continuous bilinear map

$$\iota : \mathcal{C}(\Sigma) \times \mathcal{C}(\Sigma) \to \mathbb{R}_{\geq 0}$$

such that $\iota(\gamma, \gamma')$ is nothing other than the geometric intersection number for all closed geodesics γ, γ'.

(6) The mapping class group acts continuously on $\mathcal{C}(\Sigma)$ by linear automorphisms. Moreover, the inclusion of the set of closed geodesics into $\mathcal{C}(\Sigma)$ is equivariant with respect to this action.

Moreover, for every compact $K \subset \Sigma$, let $\mathcal{C}_K(\Sigma) \subset \mathcal{C}(\Sigma)$ be the subcone consisting of the currents supported by K. Then the following holds:

(7) The set $\{\lambda \in \mathcal{C}_K(\Sigma) \text{ with } \iota(\lambda, \eta) \leq L\}$ is compact for every $L \geq 0$ and every filling multicurve η. In particular, the image $\mathrm{P}\mathcal{C}_K(\Sigma)$ of $\mathcal{C}_K(\Sigma)$ in the space

$$\mathrm{P}\mathcal{C}(\Sigma) = (\mathcal{C}(\Sigma) \setminus \{0\})/\mathbb{R}_{>0}$$

of projective currents is compact.

(8) For every multicurve γ_0 there is a compact $K \subset \Sigma$ such that

$$\mathrm{Map}(\Sigma) \cdot \gamma_0 \subset \mathcal{C}_K(\Sigma).$$

In particular, every sequence (ϕ_i) in $\mathrm{Map}(\Sigma)$ contains a subsequence (ϕ_{i_j}) such that the limit $\lim_{j\to\infty} \phi_{i_j}(\gamma_0)$ exists in $\mathrm{P}\mathcal{C}(\Sigma)$.

Currents were introduced by Bonahon in [3, 4] and all the facts here can be found in a more or less transparent way in these papers. In the case of closed surfaces, [1] is a very readable account of currents, measured laminations, and the relation between them. Finally, we hope that the presentation of currents, for both open and closed surfaces, in the forthcoming book [11] will also be similarly readable.

Accumulation points of thickened centralizers. It will be important later on to know that projective accumulation points, in the space of currents, of sequences of the form $(\phi_i(\gamma_0))$ where γ_0 is a multicurve and with

$$\phi_i \in \mathcal{N}_{\mathrm{rel}}(C(\phi), L) = \{\psi \in \mathrm{Map}(S) \text{ with } d_{\mathrm{rel}}(\psi, C(\phi)) \leq L\}$$

are very particular:

Proposition 3.1. *Let $\phi \in \mathrm{Map}(\Sigma) \setminus C(\mathrm{Map}(\Sigma))$ be a non-central mapping class, let (ϕ_n) be a sequence of pairwise distinct elements in $\mathcal{N}_{\mathrm{rel}}(C(\phi), L)$, and let γ_0 be a filling multicurve. If the sequence $(\phi_n(\gamma_0))$ converges projectively to a uniquely ergodic measured lamination λ, then $\phi(\lambda)$ is a multiple of λ.*

Recall that a measure lamination λ is *uniquely ergodic* if every measured lamination μ with $\iota(\lambda, \mu) = 0$ is a multiple of λ.

We start with the following observation:

Lemma 3.2. *Let $\gamma_0 \subset \Sigma$ be a filling multicurve and (ϕ_n) and (ψ_n) be sequences of mapping classes with $d_{\mathrm{rel}}(\phi_n, \psi_n) \leq L$. Given any simple multicurve α, suppose that the sequences $(\phi_n(\gamma_0))$ and $(\psi_n(\alpha))$ converge projectively to $\lambda, \lambda' \in P\mathcal{C}(\Sigma)$, respectively. If (ϕ_n) consists of pairwise distinct elements, then there is a chain*

$$\lambda = \lambda_0, \lambda_1, \ldots, \lambda_k = \lambda'$$

of measured laminations with $\iota(\lambda_i, \lambda_{i+1}) = 0$ for all $i = 0, \ldots, k-1$.

Proof. We first prove the statement for $\alpha = \alpha_0$, where α_0 is the base point in $C(\Sigma)$ used to define d_{rel}. Assume that $(\phi_n(\gamma_0))$ and $(\psi_n(\alpha_0))$ converge projectively to $\lambda, \lambda' \in P\mathcal{C}(\Sigma)$. Abusing notation consider λ and λ' not only as projective currents but also as actual currents. The assumption that the sequences $(\phi_n(\gamma_0))$ and $(\psi_n(\alpha_0))$ converge projectively to $\lambda, \lambda' \in P\mathcal{C}(\Sigma)$ implies that there are bounded sequences (ϵ_n) and (ϵ'_n) consisting of positive numbers and such that

$$\lambda = \lim_n \epsilon_n \phi_n(\gamma_0), \ \ \lambda' = \lim_n \epsilon'_n \psi_n(\alpha_0).$$

The assumptions that (ϕ_n) consists of pairwise distinct elements and that γ_0 is filling implies that the sequence $(\phi_n(\gamma_0))$ is not eventually constant, and thus that $\epsilon_n \to 0$.

Also, the assumption that $d_{\mathrm{rel}}(\phi_n, \psi_n) \leq L$ implies that for all n there is a chain of simple curves

$$\phi_n(\alpha_0) = \beta_n^1, \beta_n^2, \dots, \beta_n^{L+1} = \psi_n(\alpha_0)$$

with $\iota(\beta_n^i, \beta_n^{i+1}) = 0$ for all $i = 1, \dots, L$ and all n. Projective compactness of the space of currents (or rather of measured laminations) implies that passing to a subsequence we might assume that there are bounded positive sequences $(\epsilon_n^1), \dots, (\epsilon_n^{L+1})$ such that

$$\lim_{n\to\infty} \epsilon_n^i \beta_n^i = \lambda_i \neq 0$$

exists in the space $\mathcal{ML}(\Sigma)$ of measured lamination. We might also assume without loss of generality that $\epsilon_n^{L+1} = \epsilon'_n$ and thus that $\lambda_{L+1} = \lambda'$.

The claim will follow when we show that

$$\iota(\lambda, \lambda_1) = \iota(\lambda_1, \lambda_2) = \iota(\lambda_2, \lambda_3) = \cdots = \iota(\lambda_L, \lambda_{L+1}) = 0.$$

To do so, first note that

$$\begin{aligned} \iota(\lambda, \lambda_1) &= \lim_n \epsilon_n \cdot \epsilon_n^1 \cdot \iota(\phi_n(\gamma_0), \beta_n^1) \\ &= \lim_n \epsilon_n \cdot \epsilon_n^1 \cdot \iota(\phi_n(\gamma_0), \phi_n(\alpha_0)) \\ &= \lim_n \epsilon_n \cdot \epsilon_n^1 \cdot \iota(\gamma_0, \alpha_0) = 0, \end{aligned}$$

where the last equality follows from the fact that the sequence (ϵ_n^1) is bounded while (ϵ_n) tends to 0. The proof of the other equalities is even simpler: since the curves β_n^i and β_n^{i+1} are disjoint for all n and i we have

$$\iota(\lambda_i, \lambda_{i+1}) = \lim_n \epsilon_n^i \cdot \epsilon_n^{i+1} \cdot \iota(\beta_n^i, \beta_n^{i+1}) = 0.$$

Now suppose α is an arbitrary simple multicurve with $(\psi_n(\alpha))$ converging to $\lambda'' \in P\mathcal{C}(\Sigma)$. There is a sequence $\alpha_0, \alpha_1, \dots, \alpha_m = \alpha$ of simple multicurves, with $\iota(\alpha_i, \alpha_{i+1}) = 0$ for all $i = 0, \dots, m-1$. By passing to subsequences of $(\psi_n(\alpha_i))$ and taking limits as $n \to \infty$, we get a sequence of measured laminations

$$\lambda' = \lambda'_0, \dots, \lambda'_m = \lambda''$$

with $\iota(\lambda'_i, \lambda'_{i+1}) = 0$ for all $i = 0, \dots, m-1$. This chain extends the one from λ to λ' to a chain from λ to λ''. This finishes the proof of the lemma. □

We are ready to prove the proposition:

Proof of Proposition 3.1. Take for all n some $\psi_n \in C(\phi)$ with $d_{\mathrm{rel}}(\phi_n, \psi_n) \leq L$. Let α be any simple multicurve and let $\beta = \phi(\alpha)$. Compactness of $\mathrm{P}\mathcal{ML}(\Sigma)$ implies that, up to passing to a subsequence, we might assume that the limits

$$\lambda' = \lim_n \psi_n(\alpha) \quad \text{and} \quad \lambda'' = \lim_n \psi_n(\beta)$$

exist in $\mathrm{P}\mathcal{C}(\Sigma)$.

From Lemma 3.2, there is a chain of measure laminations

$$\lambda = \lambda_0, \lambda_1, \ldots, \lambda_m = \lambda'$$

with $\iota(\lambda_i, \lambda_{i+1}) = 0$ for $i = 1, \ldots, m-1$. There is a similar chain from λ to λ''.

Recall now that $\lambda_0 = \lambda$ is uniquely ergodic. Since $\iota(\lambda_0, \lambda_1) = 0$, we get that λ_1 is a multiple of λ and thus uniquely ergodic. Then, since $\iota(\lambda_1, \lambda_2) = 0$, we get that λ_2 is a multiple of λ_1 and thus of λ and uniquely ergodic and so on. Iteratively we get that λ' is a multiple of λ. Using the chain from λ to λ'', we also get that λ'' is a multiple of λ.

Finally, since $\beta = \phi(\alpha)$ and $\psi_n \in C(\phi)$, we have that, projectively,

$$\phi(\lambda') = \lim_{n\to\infty} \phi(\psi_n(\alpha)) = \lim_{n\to\infty} \psi_n(\phi(\alpha)) = \lim_{n\to\infty} \psi_n(\beta) = \lambda''.$$

This implies that λ is projectively fixed by ϕ, so $\phi(\lambda)$ is a multiple of λ as claimed. □

4. A technical result

The reason why we stressed earlier that $\mathcal{C}(\Sigma)$ is metrizable and locally compact is that these are the properties needed to work as customary with the weak-*-topology on the space of measures[1] on $\mathcal{C}(\Sigma)$. In fact, to establish Theorem 1.2 we will prove that the measures

$$m^{\mathcal{R}}_{\gamma_0, L} = \frac{1}{L^{6g-6+2r}} \sum_{\phi \in \mathcal{R}} \delta_{\frac{1}{L}\phi(\gamma_0)} \tag{4.5}$$

converge when $L \to \infty$ to the trivial measure. Here we consider the weighted multicurve $\frac{1}{L}\phi(\gamma_0)$ as a current and denote by $\delta_{\frac{1}{L}\phi(\gamma_0)}$ the Dirac measure on $\mathcal{C}(\Sigma)$ centered therein.

In [8, 9, 12, 25] we considered a closely related family of measures and proved that the limit

$$C \cdot \mathfrak{m}_{\mathrm{Thu}} = \lim_{L\to\infty} \frac{1}{L^{6g-6+2r}} \sum_{\phi \in \mathrm{Map}(S)} \delta_{\frac{1}{L}\phi(\gamma_0)}. \tag{4.6}$$

exists (see also [11]). Here $C = C(\gamma_0)$ is a positive real number and $\mathfrak{m}_{\mathrm{Thu}}$ is the Thurston measure on $\mathcal{C}(\Sigma)$. Recall that the Thurston measure is a Radon measure supported on the space $\mathcal{ML}(\Sigma)$ of measured laminations. The Thurston measure can be constructed either as a scaling limit [21, 11] or using the symplectic structure on $\mathcal{ML}(\Sigma)$. See [23] for a discussion of both points of view.

[1] This is also the reason why we didn't encourage the reader to think of currents as measures, because it is a well-established fact that thinking of "the weak-*-topology on the space of measures on the space of measures endowed with the weak-*-topology" leads the unprepared reader to tremors, shaking and cold sweats.

The only facts about the Thurston measure we will need are that it is preserved by the mapping class group, that the action

$$\mathrm{Map}(\Sigma) \curvearrowright (\mathcal{ML}(\Sigma), \mathfrak{m}_{\mathrm{Thu}})$$

is *almost free* in the sense that the fixed point set of every non-central element in $\mathrm{Map}(\Sigma)$ has vanishing Thurston measure—central elements act trivially on $\mathcal{ML}(\Sigma)$—and that it is ergodic with respect to $\mathrm{Map}(\Sigma)$ [19].

In this section we prove:

Proposition 4.1. *Let $\gamma_0 \subset \Sigma$ be a filling multicurve. The family of measures $\left(m^{\mathcal{R}}_{\gamma_0,L}\right)_{L\geq 1}$ is precompact with respect to the weak-*-topology on the space of Radon measures on $\mathcal{C}(\Sigma)$. Moreover for any sequence $L_n \to \infty$ such that the limit*

$$\mathfrak{m} = \lim_{n\to\infty} m^{\mathcal{R}}_{\gamma_0,L_n}$$

exists, one has that

$$\sum_{\phi\in\mathrm{Map}(\Sigma)} \phi_*\mathfrak{m} \leq C \cdot \mathfrak{m}_{\mathrm{Thu}}$$

where C is as in (4.6).

We start by proving that the family of measures in Proposition 4.1 is precompact and that any limit must be uniformly continuous with respect to $\mathfrak{m}_{\mathrm{Thu}}$.

Lemma 4.2. *The family of measures $\left(m^{\mathcal{R}}_{\gamma_0,L}\right)_{L\geq 1}$ is precompact with respect to the weak-*-topology on the space Radon measures on $\mathcal{C}(\Sigma)$. Moreover, any accumulation point is absolutely continuous with respect to the Thurston measure.*

Proof. The measure $m^{\mathcal{R}}_{\gamma_0,L}$ is bounded from above for all L by the measure

$$m_{\gamma_0,L} = \frac{1}{L^{6g-6+2r}} \sum_{\phi\in\mathrm{Map}(S)} \delta_{\frac{1}{L}\phi(\gamma_0)}. \tag{4.7}$$

From the existence of the limit (4.6) we get

$$\limsup \int f dm^{\mathcal{R}}_{\gamma_0,L} \leq \limsup \int f dm_{\gamma_0,L} = C \cdot \int f d\mathfrak{m}_{\mathrm{Thu}} < \infty \tag{4.8}$$

for every continuous function $f : \mathcal{C}(\Sigma) \to \mathbb{R}$ with compact support. This implies that the family $(m^{\mathcal{R}}_{\gamma_0,L})_{L\geq 1}$ is bounded and thus precompact in the weak-*-topology. Moreover, (4.8) implies that any accumulation point of $m^{\mathcal{R}}_{\gamma_0,L}$ is bounded from above by $C \cdot \mathfrak{m}_{\mathrm{Thu}}$ and hence is absolutely continuous to the Thurston measure, as we had claimed. □

Note that the same argument also proves that both families

$$m^{\mathcal{I}_k}_{\gamma_0,L} = \frac{1}{L^{6g-6+2r}} \sum_{\phi\in\mathcal{I}_k} \delta_{\frac{1}{L}\phi(\gamma_0)} \quad \text{and} \quad m^{\mathcal{D}_k}_{\gamma_0,L} = \frac{1}{L^{6g-6+2r}} \sum_{\phi\in\mathcal{D}_k} \delta_{\frac{1}{L}\phi(\gamma_0)}$$

are precompact and that any limit when $L \to \infty$ is absolutely continuous with respect to the Thurston measure. Here $\mathcal{I}_k$ and $\mathcal{D}_k$ are, as before, the subsets of $\mathcal{R}$ consisting of k-isolated points and k-dense points, respectively.

We can from now on fix a sequence (L_n) with $L_n \to \infty$ such that the following limits all exist:

$$\begin{aligned} \mathfrak{m}_{\gamma_0} &= \lim_{n\to\infty} m^{\mathcal{R}}_{\gamma_0,L_n}, \\ \mathfrak{m}^{\mathcal{I}_k}_{\gamma_0} &= \lim_{n\to\infty} m^{\mathcal{I}_k}_{\gamma_0,L_n}, \text{ and} \\ \mathfrak{m}^{\mathcal{D}_k}_{\gamma_0} &= \lim_{n\to\infty} m^{\mathcal{D}_k}_{\gamma_0,L_n}. \end{aligned} \tag{4.9}$$

Since $\mathcal{R}$ is the disjoint union of $\mathcal{I}_k$ and $\mathcal{D}_k$ they automatically satisfy that

$$\mathfrak{m}_{\gamma_0} = \mathfrak{m}^{\mathcal{I}_k}_{\gamma_0} + \mathfrak{m}^{\mathcal{D}_k}_{\gamma_0}.$$

Our next goal is to prove that the second of these limits is 0:

Lemma 4.3. *We have* $\mathfrak{m}^{\mathcal{D}_k}_{\gamma_0} = 0$.

Proof. By Maher's Theorem 2.1 it is enough to prove that, for any non-central $\phi_0 \in \mathrm{Map}(S)$ and any $R \geq 0$, the trivial measure is the only accumulation point when $L \to \infty$ of the family of measures

$$m^{\mathcal{N}}_{\gamma_0,L} = \frac{1}{L^{6g-6+2r}} \sum_{\phi\in\mathcal{N}_{\mathrm{rel}}(C(\phi_0),R)} \delta_{\frac{1}{L}\phi(\gamma_0)}.$$

Well, each $m^{\mathcal{N}}_{\gamma_0,L}$ is bounded by the measure $m_{\gamma_0,L}$ given by (4.7) and hence any such accumulation point $\mathfrak{m}' = \lim_{n\to\infty} m^{\mathcal{N}}_{\gamma_0,L_n}$ is bounded by $C \cdot \mathfrak{m}_{\mathrm{Thu}}$ by (4.6). The claim will then follow when we say that the support of $\mathfrak{m}'$ is contained in a set of vanishing Thurston measure.

First, the support of the limiting measure $\mathfrak{m}'$ is contained in the set of accumulation points of sequences (x_n) where x_n is in the support of $m^{\mathcal{N}}_{\gamma_0,L_n}$, that is, a multiple of $\phi_n(\gamma_0)$ for some $\phi_n \in \mathcal{N}_{\mathrm{rel}}(C(\phi_0), R)$. On the other hand, since the set of uniquely ergodic lamination has full $\mathfrak{m}_{\mathrm{Thu}}$-measure [18], we also get that $\mathfrak{m}'$ is supported by uniquely ergodic laminations. It thus follows from Proposition 3.1 that $\mathfrak{m}'$ is supported by the set of measured laminations projectively fixed by ϕ_0. Since this set has vanishing $\mathfrak{m}_{\mathrm{Thu}}$-measure we get that $\mathfrak{m}'$ is trivial, as we needed to prove. □

As a final step towards the proof of Proposition 4.1 we establish an equivariance property for the limits of the measures $m^{\mathcal{R}}_{\gamma_0,L}$:

Lemma 4.4. *We have* $\phi_*(\mathfrak{m}_{\gamma_0}) = \lim_{n\to\infty} m^{\mathcal{R}}_{\phi(\gamma_0),L_n}$ *for all* $\phi \in \mathrm{Map}(\Sigma)$.

Proof. Noting that the set $\mathcal{R}$ is closed under conjugation we get that $\mathcal{R}\phi = \phi\mathcal{R}$. This means that

$$\begin{aligned} m^{\mathcal{R}}_{\phi(\gamma_0),L_n} &= \frac{1}{L^{6g-6+2r}} \sum_{\psi\in\mathcal{R}} \delta_{\frac{1}{L}\psi\phi(\gamma_0)} = \frac{1}{L^{6g-6+2r}} \sum_{\psi\in\mathcal{R}\phi} \delta_{\frac{1}{L}\psi(\gamma_0)} \\ &= \frac{1}{L^{6g-6+2r}} \sum_{\psi\in\phi\mathcal{R}} \delta_{\frac{1}{L}\psi(\gamma_0)} = \frac{1}{L^{6g-6+2r}} \sum_{\psi\in\mathcal{R}} \delta_{\frac{1}{L}\phi\psi(\gamma_0)} \\ &= \frac{1}{L^{6g-6+2r}} \sum_{\psi\in\mathcal{R}} \phi_*\left(\delta_{\frac{1}{L}\psi(\gamma_0)}\right) = \phi_*\left(\frac{1}{L^{6g-6+2r}} \sum_{\psi\in\mathcal{R}} \delta_{\frac{1}{L}\psi(\gamma_0)}\right) \\ &= \phi_*\left(m^{\mathcal{R}}_{\gamma_0,L_n}\right) \end{aligned}$$

The claim follows now from (4.9) and the continuity of the action of $\mathrm{Map}(\Sigma)$ on the space of currents. □

We are ready to prove the proposition:

Proof of Proposition 4.1. Recall that Lemma 4.2 asserts that the given family of measures is precompact and hence we can assume that we are given a sequence (L_n) with $L_n \to \infty$ such that the limit

$$\mathfrak{m}_{\gamma_0} = \lim_{n\to\infty} m^{\mathcal{R}}_{\gamma_0,L_n}$$

exists. To prove Proposition 4.1 it will suffice to show, with C as in (4.6), that for every finite set $\mathcal{Z} \subset \mathrm{Map}(\Sigma)$ we have

$$\sum_{\phi\in\mathcal{Z}} \phi_*(\mathfrak{m}_{\gamma_0}) \leq C \cdot \mathfrak{m}_{\mathrm{Thu}}.$$

Fixing such a finite set $\mathcal{Z}$ choose

$$k > 2 \cdot \max\{d_{\mathcal{G}}(\mathrm{id}, \phi) \text{ where } \phi \in \mathcal{Z}\}.$$

Lemma 4.4 and Lemma 4.3 imply, respectively, the first and last of the following equalities:

$$\sum_{\phi\in\mathcal{Z}} \phi_*(\mathfrak{m}_{\gamma_0}) = \sum_{\phi\in\mathcal{Z}} \lim_{n\to\infty} m^{\mathcal{R}}_{\phi(\gamma_0),L_n} = \lim_{n\to\infty} \sum_{\phi\in\mathcal{Z}} m^{\mathcal{R}}_{\phi(\gamma_0),L_n} = \lim_{n\to\infty} \sum_{\phi\in\mathcal{Z}} m^{\mathcal{I}_k}_{\phi(\gamma_0),L_n}.$$

Moreover, from the choice of k we get that $\mathcal{I}_k\phi \cap \mathcal{I}_k\phi' = \emptyset$ for any two distinct $\phi, \phi' \in \mathcal{Z}$ and we can thus rewrite

$$\begin{aligned} \sum_{\phi\in\mathcal{Z}} m^{\mathcal{I}_k}_{\phi(\gamma_0),L_n} &= \frac{1}{L_n^{6g-6+2r}} \sum_{\phi\in\mathcal{Z}} \sum_{\psi\in\mathcal{I}_k} \delta_{\frac{1}{L_n}\psi\phi(\gamma_0)} = \frac{1}{L_n^{6g-6+2r}} \sum_{\phi\in\mathcal{Z}} \sum_{\psi\in\mathcal{I}_k\phi} \delta_{\frac{1}{L_n}\psi(\gamma_0)} \\ &= \frac{1}{L_n^{6g-6+2r}} \sum_{\psi\in\bigcup_{\phi\in\mathcal{Z}}\mathcal{I}_k\phi} \delta_{\frac{1}{L_n}\psi(\gamma_0)}. \end{aligned}$$

It thus follows that

$$\sum_{\phi\in\mathcal{Z}} m^{\mathcal{I}_k}_{\phi(\gamma_0),L_n} \leq \frac{1}{L_n^{6g-6+2r}} \sum_{\psi\in\mathrm{Map}(S)} \delta_{\frac{1}{L_n}\psi(\gamma_0)}$$

and hence that

$$\sum_{\phi\in\mathcal{Z}} \phi_*(\mathfrak{m}_{\gamma_0}) \leq \lim_{n\to\infty} \frac{1}{L_n^{6g-6+2r}} \sum_{\phi\in \mathrm{Map}(S)} \delta_{\frac{1}{L_n}\phi(\gamma_0)} = C\cdot \mathfrak{m}_{\mathrm{Thu}}.$$

We are done. □

5. Proofs of the theorems

We are now ready to prove the main results.

Theorem 1.2. *Let $\mathcal{R}\subset \mathrm{Map}(\Sigma)$ be the set of non pseudo-Anosov mapping classes and let $\gamma_0\subset\Sigma$ be a filling multicurve. Then we have*

$$\lim_{L\to\infty} \frac{|\{\phi\in\mathcal{R} \text{ with } F(\phi(\gamma_0))\leq L\}|}{L^{6g-6+2r}} = 0$$

for every continuous homogenous function $F:\mathcal{C}(\Sigma)\to\mathbb{R}_{\geq 0}$ which, for every $K\subset\Sigma$ compact, is proper when restricted on the set $\mathcal{C}_K(\Sigma)$ of currents supported by K.

Proof. The claim will follow easily once we prove that

$$\lim_{L\to\infty} m^{\mathcal{R}}_{\gamma_0,L} = 0 \tag{5.10}$$

with $m^{\mathcal{R}}_{\gamma_0,L}$ as in (4.5). Since this family of measures is precompact by Proposition 4.1, it suffices to prove that 0 is the only accumulation point when $L\to\infty$. So let (L_n) be a sequence tending to ∞ and such that the limit

$$\mathfrak{m}_{\gamma_0} = \lim_{n\to\infty} m^{\mathcal{R}}_{\gamma_0,L_n}$$

exists. By Lemma 4.2 $\mathfrak{m}_{\gamma_0}$ is absolutely continuous with respect to $\mathfrak{m}_{\mathrm{Thu}}$. This means that there is a function (the Radon-Nikodym derivative) $\kappa:\mathcal{C}(\Sigma)\to\mathbb{R}_{\geq 0}$ with the property that

$$\int_{\mathcal{C}(\Sigma)} f(\zeta) d\mathfrak{m}_{\gamma_0}(\zeta) = \int_{\mathcal{C}(\Sigma)} f(\zeta)\cdot\kappa(\zeta) d\mathfrak{m}_{\mathrm{Thu}}(\zeta)$$

for any continuous compactly supported function f on the space of currents.

Proposition 4.1 asserts that the measure $\sum_{\phi\in\mathrm{Map}(S)}\phi_*\mathfrak{m}_{\gamma_0}$ is not only finite, but actually bounded by a multiple $C\cdot\mathfrak{m}_{\mathrm{Thu}}$ of the Thurston measure. In terms of the function κ, this implies that

$$\sum_{\phi\in\mathrm{Map}(S)} \kappa(\phi(\zeta)) \leq C \text{ for } \mathfrak{m}_{\mathrm{Thu}}\text{-almost every } \zeta\in\mathcal{C}(\Sigma). \tag{5.11}$$

We claim that this implies that $\kappa(\zeta)=0$ almost surely:

Claim. *$\kappa(\zeta)=0$ almost surely with respect to the Thurston measure.*

In a nutshell, the claim follows from the fact that ergodic actions of discrete groups on non-atomic measure spaces are recurrent (the condition on the measure being non-atomic is just there to rule out actions with only one orbit). In any case, we give a direct argument to prove the claim:

Proof of the Claim. If the claim fails to be true, then there is a positive $\mathfrak{m}_{\mathrm{Thu}}$-measure set $U \subset \mathcal{C}(\Sigma)$ with $\kappa(\zeta) \geq \epsilon > 0$ for every $\zeta \in U$. Noting that the action $\mathrm{Map}(\Sigma) \curvearrowright \mathcal{C}(\Sigma)$ is almost free we get from (5.11) that, for almost every $\zeta \in U$,

$$\#\{\phi \in \mathrm{Map}(\Sigma) \text{ with } \phi(\zeta) \in U\} \leq \frac{C}{\epsilon}.$$

It follows that there is a set $V \subset U$ of positive Thurston measure such that the set

$$\mathcal{Z} = \{\phi \in \mathrm{Map}(\Sigma) \text{ with } \phi(V) \cap U \neq \emptyset\}$$

is finite. Now, since the action is essentially free we can in fact find $W \subset V$ of positive Thurston measure with $W \cap \phi(W) = \emptyset$ for all $\phi \notin C(\mathrm{Map}(\Sigma))$. This contradicts the ergodicity of the action of the mapping class group on $(\mathcal{ML}(\Sigma), \mathfrak{m}_{\mathrm{Thu}})$. □

The claim implies that the limiting measure vanishes, that is $\mathfrak{m}_{\gamma_0} = 0$, establishing (5.10). We can now conclude the proof: let $F : \mathcal{C}(\Sigma) \to \mathbb{R}_{\geq 0}$ be as in the statement and note that

$$\begin{aligned}\frac{|\{\gamma \in \mathcal{R} \cdot \gamma_0 \text{ with } F(\gamma) \leq L\}|}{L^{6g-6+2r}} &\leq \frac{|\{\phi \in \mathcal{R} \text{ with } F\left(\frac{1}{L}\phi(\gamma_0)\right) \leq 1\}|}{L^{6g-6+2r}} \\ &= m^{\mathcal{R}}_{\gamma_0,L}(\{F(\cdot) \leq 1\})\end{aligned}$$

and by (5.10) together with the fact that $\{F(\cdot) \leq 1\}$ is compact we have that

$$\lim_{L\to\infty} m^{\mathcal{R}}_{\gamma_0,L}(\{F(\cdot) \leq 1\}) = 0. \quad \square$$

Finally, we prove Theorem 1.1:

Theorem 1.1. *The set of pseudo-Anosov mapping classes is generic with respect to either one of the functions*

$$\rho_K(\phi) = \inf\{K(f) \text{ where } f \in \mathrm{Diff}(\Sigma) \text{ represents } \phi\}$$

where $K(f)$ is the quasi-conformal distortion,

$$\rho_{\mathrm{Lip}}(\phi) = \inf\{\mathrm{Lip}(f) \text{ where } f \in \mathrm{Diff}(\Sigma) \text{ represents } \phi\}$$

where $\mathrm{Lip}(f)$ is the lipschitz constant, and

$$\rho_{\sigma,\eta}(\phi) = \iota(\phi(\sigma), \eta)$$

where σ and η are filling multicurves and $\iota(\cdot,\cdot)$ is the geometric intersection number.

Proof. We start by proving that the set of pseudo-Anosov mapping classes is $\rho_{\sigma,\eta}$-generic for filling multicurves σ and η. Well, the function

$$\mathcal{C}(\Sigma) \to \mathbb{R}_{\geq 0}, \ \lambda \mapsto \iota(\lambda, \eta)$$

is continuous and proper on the set $\mathcal{C}_K(\Sigma)$ of currents supported by compact sets $K \subset \Sigma$. We thus get from Theorem 1.2 that

$$\lim_{L\to\infty} \frac{|\{\phi \in \mathcal{R} \text{ with } \iota(\phi(\sigma), \eta) \leq L\}|}{L^{6g-6+2r}} = 0 \tag{5.12}$$

On the other hand we get from [27] or [22] (see also [10, 11]) that

$$\liminf_{L\to\infty} \frac{|\{\phi \in \mathrm{Map}(\Sigma) \text{ with } \iota(\phi(\sigma), \eta) \leq L\}|}{L^{6g-6+2r}} = \mathrm{const}(\sigma, \eta) > 0. \tag{5.13}$$

Since $\rho_{\sigma,\eta}(\phi) = \iota(\phi(\sigma), \eta)$ we get from (5.12) and (5.13) that

$$\lim_{L\to\infty} \frac{|\{\phi \in \mathcal{R} \text{ with } \rho_{\sigma,\eta}(\phi) \leq L\}|}{|\{\phi \in \mathrm{Map}(\Sigma) \text{ with } \rho_{\sigma,\eta}(\phi) \leq L\}|} = 0.$$

This shows the set of pseudo-Anosov mapping classes is generic with respect to $\rho_{\sigma,\eta}$.

We consider now genericity with respect to ρ_{Lip}. Fix once and for all a filling multicurve σ. Although it does not really matter, we could for example assume that σ is a marking in the sense of [20]. We need the following fact:

Fact 1. *There is $C = C(\Sigma, \sigma) \geq 1$ with*

$$\frac{1}{C} \cdot \rho_{\mathrm{Lip}}(\phi) \leq \rho_{\sigma,\sigma}(\phi) \leq C \cdot \rho_{\mathrm{Lip}}(\phi)$$

for all $\phi \in \mathrm{Map}(\Sigma)$.

Fact 1 is well known but, for the convenience of the reader, we will comment on its proof once we are done with Theorem 1.1. From Fact 1 we get that

$$|\{\phi \in \mathcal{R} \text{ with } \rho_{\mathrm{Lip}}(\phi) \leq L\}| \leq |\{\phi \in \mathcal{R} \text{ with } \rho_{\sigma,\sigma}(\phi) \leq CL\}|$$

$$|\{\phi \in \mathrm{Map}(\Sigma) \text{ with } \rho_{\mathrm{Lip}}(\phi) \leq L\}| \geq \left|\left\{\phi \in \mathrm{Map}(\Sigma) \text{ with } \rho_{\sigma,\sigma}(\phi) \leq \frac{L}{C}\right\}\right|.$$

We thus get from (5.12) and (5.13) that

$$\lim_{L\to\infty} \frac{|\{\phi \in \mathcal{R} \text{ with } \rho_{\mathrm{Lip}}(\phi) \leq L\}|}{|\{\phi \in \mathrm{Map}(\Sigma) \text{ with } \rho_{\mathrm{Lip}}(\phi) \leq L\}|} = 0,$$

as we had claimed.

The genericity with respect to ρ_K follows by the same argument when we replace Fact 1 by the following also well-known fact:

Fact 2. *There is $C = C(\Sigma) \geq 1$ with*

$$\frac{1}{C} \cdot \rho_{\mathrm{Lip}}(\phi)^2 \leq \rho_K(\phi) \leq C \cdot \rho_{\mathrm{Lip}}(\phi)^2$$

for all $\phi \in \mathrm{Map}(\Sigma)$.

We have proved Theorem 1.1. □

We comment now on the proofs of the two facts used in the proof above. By properties (7) and (8) of the space of currents, we have that for any other filling multicurve σ' there is a constant $C_1 = C_1(\Sigma, \sigma, \sigma')$ with

$$\frac{1}{C_1}\iota(\sigma, \phi(\sigma)) \leq \ell_\Sigma(\phi(\sigma')) \leq C_1\iota(\sigma, \phi(\sigma)), \tag{5.14}$$

where $\ell_\Sigma(\cdot)$ is the hyperbolic length function. Choosing σ' to be a *short marking* in the sense of [13], we get from Theorem 4.1 in that paper that there is a constant $C_2 = C_2(\Sigma, \sigma')$ such that

$$\frac{1}{C_2}\ell_\Sigma(\sigma') \leq \rho_{\mathrm{Lip}}(\phi) \leq C_2\ell_\Sigma(\sigma') \tag{5.15}$$

Fact 1 follows, with $C = C_1 \cdot C_2$, from these two inequalities.

A similar argument, replacing results from [13] by results from [24], yields Fact 2. Alternatively one can directly refer to Theorem B in [6].

For the reader who feels cheated by a proof which only consists of a sequence of references, we sketch a more direct proof of Fact 1 and Fact 2. Suppose Σ is closed. By the Arzelá-Ascoli theorem, there is a Lipschitz map f on Σ representing ϕ with $L_f = \rho_{\mathrm{Lip}}(\phi)$. By Teichmüller's theorem, there is a unit-area quadratic differential q on Σ and a map g representing ϕ, such that $\rho_K(\phi) = L_g^2$, where L_g is the Lipschitz constant of g with respect to the singular Euclidean metric induced by q. Moreover, L_g is the minimal Lipschitz constant of all maps on q representing ϕ. By compactness of Σ, the q–metric and the hyperbolic metric on Σ are bilipschitz equivalent. By compactness of the space of unit-area quadratic differentials, this bilipschitz equivalence is uniform. Therefore, there is a constant B depending only on Σ such that

$$\frac{1}{B} L_f \leq L_g \leq B L_f.$$

This obtains Fact 2 with $C = B^2$.

Let σ be a filling multicurve which we realize by a q–geodesic. Because σ is filling, it cannot be entirely q–vertical. Compactness of the space of unit-area quadratic differentials implies that in fact the horizontal length of σ is a definite proportion of its total length. Under the map g, the q–horizontal direction gets stretched by the factor L_g, so the q–length of $\phi(\sigma)$ grows proportionally to L_g. By comparing to the hyperbolic metric and using compactness of Σ again, we get Equation (5.15) with $\sigma = \sigma'$. We still have (5.14) (with $\sigma = \sigma'$). This shows Fact 1.

For the general case, losing compactness of Σ means losing bilipschitz equivalence between the q–metric and the hyperbolic metric. However, the argument we just sketched can be modified to take care of this issue and we refer to the above listed references for the details.

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