Knot Theory and Complex Curves

Matthew Hedden

Topology is often intertwined with analysis and geometry in unexpected and beautiful ways. A familiar example of this phenomenon is provided by the Riemann mapping theorem: if the topology of a domain in the complex plane is simple enough, then it is analytically simple. In particular, if it is simply connected and not all of $\mathbb{C}$, then it is holomorphically equivalent to the unit disk. A variant of the mapping theorem can be stated as "a loop in the plane without self-intersections bounds a region that is holomorphically equivalent to the unit disk."

From a topological perspective, loops in $\mathbb{C} \cong \mathbb{R}^2$ without self-intersections are quite simple. Indeed, any such loop can be deformed to any other through a family. If one allows loops to extend into $\mathbb{R}^3$, the situation becomes far more complicated. Knot theory studies loops in $\mathbb{R}^3$ without self-intersections, up to an equivalence defined by smooth deformation. It is convenient to regard loops in $\mathbb{R}^3$ as living instead in the 3-dimensional sphere $\mathbb{S}^3$ that arises from compactification. This sphere can also be viewed as the set of unit vectors in $\mathbb{C}^2$. From this perspective one can ask about an analogue of the Riemann mapping theorem: which knots in the unit sphere $\mathbb{S}^3 \subset \mathbb{C}^2$ bound holomorphically embedded disks in the interior of the unit 4-ball?

It turns out that an answer comes from a solution to a closely related question that will frame my talk at the AMS Spring Central Sectional Meeting. Recall that a complex polynomial in two variables $f(z,w) \in \mathbb{C}[z,w]$ specifies an *algebraic curve* $V_f \subset \mathbb{C}^2$ through its zero locus. My talk will address the following question.

Question. Which knots arise as the intersection of an algebraic curve with the unit 3-sphere in $\mathbb{C}^2$?

I'll call such knots $\mathbb{C}$-*knots*. The connection between the question above and holomorphically embedded disks comes from the fact that the part of an algebraic curve in the interior of the unit 4-ball can be parametrized by a holomorphic map from a Riemann surface. Because I work with knots, which are loops up to deformation, I can smoothly vary the loop in its equivalence class so that the

Matthew Hedden is a professor of mathematics at Michigan State University. His email address is `mhedden@math.msu.edu`.

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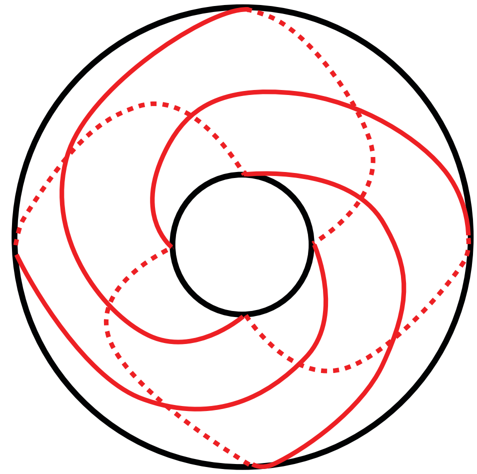


Figure 1. The red curve represents the intersection of the unit 3-sphere $\mathbb{S}^3 \subset \mathbb{C}^2$ with the zero locus of the polynomial $f(z,w) = z^3 - w^4$.

distinction between a holomorphic map and an algebraic curve disappears.

Example. A quintessential example of a $\mathbb{C}$-knot is provided by the polynomial $f(z,w) = z^3 - w^4$. The zero locus of f can be parametrized by the holomorphic map $\gamma : \mathbb{C} \to \mathbb{C}^2$ given by

$$t \xrightarrow{\gamma} (t^4, t^3).$$

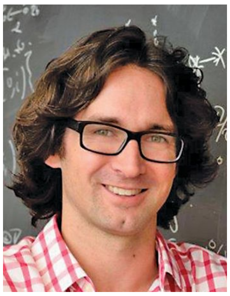
Considering the intersection of the curve with the 3-sphere of vectors (z,w) satisfying $|z|^2 + |w|^2 = 1$, we obtain the equation $|t|^8 + |t|^6 = 1$. As the left-hand side is an increasing continuous function of $|t| \geq 0$, we find there is a unique solution λ. This shows that the intersection between the image of γ and the 3-sphere is an embedded circle, parametrized by

$$\lambda e^{i\theta} \xrightarrow{\gamma} (\lambda^4 e^{4i\theta}, \lambda^3 e^{3i\theta}).$$

Geometrically, this is a curve that lies on the torus $\lambda^4\mathbb{S}^1 \times \lambda^3\mathbb{S}^1 \subset \mathbb{C}\times\mathbb{C}$ and that wraps 4 times around the first circular coordinate while wrapping 3 times around the second. The knot represented by this curve is called the *torus knot of type* $(4,3)$ and is illustrated in Figure 1. Note that the knot bounds an embedded disk, holomorphically parametrized by the restriction of γ to the disk of radius λ, but that the mapping is singular at the origin where the gradient of γ vanishes.

The subject of $\mathbb{C}$-knots lies at the crossroads of a number of branches of mathematics. These knots play an important role in *low-dimensional topology*, where they have been the subject of guiding conjectures. In *algebraic geometry* they figure prominently in the study of singular points of algebraic curves (like the origin in the example above). They are tightly connected to certain geometric structures on 3-manifolds called *contact structures* and to analogous *symplectic structures* on smooth 4-manifolds.

They are closely related to *braid groups* and *mapping class groups*, wherein $\mathbb{C}$-knots admit group theoretic characterizations. A rich interaction has also been witnessed between $\mathbb{C}$-knots and topological invariants stemming from the fields of *gauge theory, Floer homology*, and *categorification*. In my talk I will survey the history of $\mathbb{C}$-knots, discuss a number of the interactions mentioned above, and highlight some recent advances and generalizations in their study.



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Credits

Figure 1 was created by the author.
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