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# Floer homology and fractional Dehn twists

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ABSTRACT

We establish a relationship between Heegaard Floer homology and the fractional Dehn twist coefficient of surface automorphisms. Specifically, we show that the rank of the Heegaard Floer homology of a 3-manifold bounds the absolute value of the fractional Dehn twist coefficient of the monodromy of any of its open book decompositions with connected binding. We prove this by showing that the rank of Floer homology gives bounds for the number of boundary parallel right or left Dehn twists necessary to add to a surface automorphism to guarantee that the associated contact manifold is tight or overtwisted, respectively. By examining branched double covers, we also show that the rank of the Khovanov homology of a link bounds the fractional Dehn twist coefficient of its odd-stranded braid representatives.



## 1. Introduction

Let $S$ be a compact oriented 2-manifold with a single boundary component, and $\phi$ a homeomorphism of $S$ fixing its boundary pointwise. The fractional Dehn twist coefficient of $\phi$ is a rational number $\tau(\phi) \in \mathbb{Q}$ that depends only on the isotopy class of $\phi$ rel boundary, and can be understood as a measure of the amount of twisting around the boundary effected by $\phi$ compared to a "canonical"—e.g., pseudo-Anosov—representative

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of its (free) isotopy class. More precisely, consider the image of $\phi$ under the natural map $\mathrm{Aut}(S, \partial S) \to \mathrm{Aut}(S)$ which drops the requirement that an isotopy fixes the boundary pointwise. In this latter group, $\phi$ is isotopic to its Nielsen–Thurston representative; that is, there is an isotopy $\Phi : S \times [0, 1] \to S$ such that $\Phi_0 = \phi$ and $\Phi_1$ is either periodic, reducible, or pseudo-Anosov.[1] Considering the restriction of $\Phi$ to the boundary, we obtain a homeomorphism:

$$\Phi_\partial : \partial S \times [0, 1] \to \partial S \times [0, 1]$$

defined by $\Phi_\partial(x, t) = (\Phi_t(x), t)$. The fractional Dehn twist coefficient $\tau(\phi)$ can be defined as the winding number of the arc $\Phi(\theta \times [0, 1])$ where $\theta \in \partial S$ is a basepoint.[2] This would appear only to associate a real number to $\phi$, which could depend on the choice of basepoint and isotopy. The Nielsen–Thurston classification, however, shows that this winding number is a well-defined rational-valued invariant $\tau(\phi) \in \mathbb{Q}$. The definition extends easily to surfaces with several boundary circles, in which case there is a corresponding twist coefficient for each component of the boundary. Here we will be concerned only with the case of connected boundary.

The study of fractional Dehn twist coefficients dates at least from the work of Gabai and Oertel [7] in the context of essential laminations of 3-manifolds, where, with different conventions than those used here, it appeared as the slope of the "degenerate curve" [7, pg. 62]. Honda, Kazez, and Matic [13,14] observed a connection with contact topology through open book decompositions, which has been explored by various authors [3,18, 16]. The following proposition summarizes a few key properties of the fractional Dehn twist coefficient.

**Proposition** *([21,16]). Let $\tau : \mathrm{Aut}(S, \partial S) \to \mathbb{Q}$ be the fractional Dehn twist coefficient, and let $t_\partial$ denote the mapping class of a right-handed Dehn twist around a curve parallel to $\partial S$. Then for all $\phi, \psi \in \mathrm{Aut}(S, \partial S)$, we have:*

(1) (Quasimorphism) $|\tau(\phi \circ \psi) - \tau(\phi) - \tau(\psi)| \leqslant 1$.
(2) (Homogeneity) $\tau(\phi^n) = n\tau(\phi)$.
(3) (Boundary Twisting) $\tau(\phi \circ t_\partial) = \tau(\phi) + 1$.

The first two properties easily imply that the fractional Dehn twist is invariant under conjugation (see e.g., [8, Proposition 5.3]), and the third implies that it can be arbitrarily large, either positively or negatively. There are constraints, however, on the possible denominators of $\tau(\phi)$ based on the topology of $S$; cf. [6, Theorem 8.8], [18, Theorem 4.4], [36].

[1] As in [18], such a map is called reducible only if is not periodic. Moreover, in the reducible case, after an isotopy rel $\partial S$ we get a subsurface of $S$ to which $\phi$ restricts as a map with periodic or pseudo-Anosov representative: we apply the definition of fractional Dehn twist coefficient to the restriction of $\phi$ to that subsurface.

[2] $\tau(\phi)$ can be defined without Nielsen–Thurston theory by lifting $\phi$ to the universal cover and using the translation number of an associated action on a line at infinity [21].

Surface homeomorphisms of the sort we consider arise naturally as monodromies of fibered knots in 3-manifolds or, equivalently, open book decompositions of 3-manifolds with connected binding. Indeed, if $K \subset Y$ is a fibered knot then the complement of a neighborhood of $K$ is a bundle over $S^1$ with fiber a compact surface $S$ with one boundary component. This bundle is described by a monodromy homeomorphism $\phi_K : S \to S$ that is the identity on the boundary and well-defined up to isotopy and conjugation. Hence we can think of the twist coefficient as giving rise to an invariant of fibered knots in 3-manifolds, $K \mapsto \tau(\phi_K)$, where we suppress the choice of fibration from our notation. Our main result shows that if the 3-manifold is fixed, then there is an *a priori* bound on the value of the twist coefficient for any fibered knot in that manifold.

**Theorem 1.** *Let $Y$ be a closed oriented 3-manifold. Then there exists a real number $M \geqslant 0$ with the following property: Let $K$ be any fibered knot in $Y$ and let $\phi_K$ denote its monodromy. Then*

$$|\tau(\phi_K)| \leqslant M.$$

In the case that a knot fibers in many distinct ways, the bound is to be interpreted as stated: regardless of the choice of fiber, the twist coefficient of the resulting monodromy is bounded by a number depending only on $Y$.

Given $Y$, we let $M_Y$ denote the smallest number satisfying the conclusion of Theorem 1.

To the best of our knowledge, the only situation prior to our theorem in which the bound $M_Y$ was known to exist is for knots in the 3-sphere, in which case work of Gabai [6] and Kazez–Roberts [18] shows that $|\tau(\phi_K)| \leq 1/2$. Their proof relies on the application of thin position, among other things, and does not extend to other manifolds in an obvious way. Our proof exploits the connection between twist coefficients and contact topology, and a connection between contact topology and Heegaard Floer homology. Recall that by a construction of Thurston–Winkelnkemper [37], a fibered knot $K \subset Y$, regarded as an open book decomposition, uniquely determines a contact structure $\xi_K$ on $Y$ (see [38] for uniqueness). It was shown by Honda, Kazez, and Matic that if $\xi_K$ is tight, then $\tau(\phi_K) \geqslant 0$ [13, Theorem 1.1 and Propositions 3.1, 3.2]. Using property (3) of $\tau$, we see that to obtain a lower bound on $\tau(\phi_K)$ it suffices to show that there is an integer $N$ depending only on $Y$ such that the monodromy $\phi_K \circ t_\partial^n$ describes a tight contact structure (on a different 3-manifold) for any $n > N$. Therefore Theorem 1 is implied by the following.

**Theorem 2.** *For a closed oriented 3-manifold $Y$, there is an integer $N \geqslant 0$ with the following property: Let $\xi$ be a contact structure on $Y$, and choose any open book decomposition $(S, \phi)$ that supports $\xi$ and has connected binding. Then for any $n > N$, the open book $(S, \phi \circ t_{\partial S}^n)$ determines a tight contact structure.*

As before, we write $N_Y$ for the smallest integer $N$ satisfying the conclusion of Theorem 2. Theorem 1 follows from Theorem 2 by observing that open book decompositions for $Y$ are in bijection with those for $-Y$ under a correspondence induced by inverting monodromies. Thus $M = 1 + \max\{N_Y, N_{-Y}\}$ satisfies Theorem 1.

Theorem 2 was first observed by Ozsváth and Szabó in the case that $Y$ is an $L$-space, in which case $N_Y = N_{-Y} = 0$ [30, Theorem 1.6]. Indeed, in that paper they ask the following question:

**Question 3** *([30, pg. 43]). Given an open book decomposition $(S, \phi)$ for $Y$, what is the minimum $n$ such that $(S, \phi \circ t^n_{\partial S})$ specifies a tight contact structure?*

Theorem 2 is proved by a generalization of Ozsváth and Szabó's argument, and we obtain a bound depending on the Heegaard Floer homology of $Y$. Indeed, our proof shows

$$N_Y \leqslant \frac{1}{2}(\dim_{\mathbb{F}} \widehat{HF}(Y) - |\mathrm{Tor}\ H_1(Y;\mathbb{Z})|) \tag{1}$$

where $\widehat{HF}(Y)$ denotes the Heegaard Floer groups of $Y$ with coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ and $|\mathrm{Tor}\ H_1(Y;\mathbb{Z})|$ is the number of elements in the torsion submodule of first singular homology. Since the right side of (1) does not depend on the orientation, we immediately obtain a similar estimate for the number $M_Y$ bounding twist coefficients. Theorem 2 can be viewed as an answer to Ozsváth and Szabó's question, and Theorem 1 as a geometric interpretation of the rank of the Heegaard Floer homology groups of a 3-manifold $Y$: it is a "speed limit" for fibered knots in $Y$ with respect to the twist coefficient. Such an interpretation raises the natural question:

**Question 4.** *Does every 3-manifold that is not an $L$-space contain a "fast" knot? That is, a fibered knot for which the absolute value of the twist coefficient is at least 1?*

This question is closely tied to the conjecture that L-spaces are exactly those 3-manifolds without taut foliations. Indeed, an affirmative answer to Question 4 would imply this conjecture, by recent work of Kazez and Roberts [17]. In a related direction, is perhaps worth pointing out the following corollary, stated in terms of the *reduced* Heegaard Floer homology groups:

**Corollary 5.** *Let $K \subset Y$ be a fibered knot, and let $\Sigma_n(K)$ denote its $n$-fold branched cyclic cover. Then*

$$\dim_{\mathbb{F}} HF^{red}(\Sigma_n(K)) \geq n \cdot |\tau(\phi_K)| - 1.$$

*In particular, if $K$ has right- (or left-)veering monodromy then all cyclic branched covers over $K$ with sufficiently large order are not L-spaces.*

Note that work of Kazez and Roberts could be used to show that high order branched cyclic covers of fibered knots with right- or left-veering monodromy are not L-spaces, but that their work wouldn't produce the quantitative growth rate of the corollary. We expect that for most knots the rank of the reduced Floer homology of branched cyclic covers will have at least positive linear growth in the order of the cover. There are examples, however (such as the figure eight knot [2]), for which all branched cyclic covers are L-spaces; the corollary of course implies such knots have vanishing twist coefficient.

We should remark that there are many more refined estimates of $N_Y$ made possible by taking into account further structure on the Floer groups (see the remarks after the proof of Theorem 2.5). For example, since our argument depends only on one spin$^c$ structure at a time, we can show

$$N_Y \leq \max_{\mathfrak{s}\in \mathrm{spin}^c(Y)} \frac{1}{2}(\dim_{\mathbb{F}} \widehat{HF}(Y, \mathfrak{s}) - 1) \tag{2}$$

(here we assume $Y$ is a rational homology sphere for convenience, cf. Remark 2.6). In general (2) is a much better estimate than (1), though both recover $N_Y = 0$ in the case that $Y$ is an $L$-space.

If we are given more data about the knot our bound for the twist coefficient can be sharpened further. To state one such result, recall that an oriented plane field distribution on a closed oriented 3-manifold is determined up to homotopy by two pieces of data: its associated spin$^c$ structure, together with a "3-dimensional invariant," as described by Gompf [10] (ultimately this classification goes back to Pontryagin). Supposing $\xi$ to be a plane field on $Y$ whose spin$^c$ structure $\mathfrak{s}_\xi$ has torsion first Chern class, the 3-dimensional invariant is a rational number called the Hopf invariant $h(\xi)$ (see Equation (7) in Section 4 below). Strictly this quantity also depends on the orientation of the ambient 3-manifold; when necessary we will write $h(\xi_Y)$ or $h(\xi_{-Y})$ to indicate that $\xi$ is to be considered on the oriented manifold $Y$ or $-Y$ (meaning $Y$ with the reversed orientation), respectively. Now whenever a spin$^c$ structure has torsion Chern class, the associated Heegaard Floer homology group carries a rational-valued grading, and in fact for a spin$^c$ structure $\mathfrak{s}_\xi$ the grading takes values in $\mathbb{Z}+h(\xi)$. The reduced Floer homology groups are finite-dimensional and, in particular, can be nonzero in at most finitely many degrees. Keeping this in mind, the following theorem provides a more precise bound on the twist coefficient of a fibered knot, given the homotopy data of its associated contact structure.

**Theorem 6.** *Let $\xi$ be a contact structure on $Y$ whose associated* spin$^c$ *structure $\mathfrak{s}_\xi$ is torsion, and let $(S, \phi)$ be an open book supporting $\xi$ with genus $g$ and connected binding. Then the twist coefficient of $\phi$ satisfies*

$$-1 - \sum_{\substack{d \equiv -h(\xi_Y)+1 \\ \bmod\ 2g-2}} \dim_{\mathbb{F}} HF_d^{red}(-Y, \mathfrak{s}_\xi) \leqslant \tau(\phi) \leqslant 1 + \sum_{\substack{d \equiv -h(\xi_Y) \\ \bmod\ 2g-2}} \dim_{\mathbb{F}} HF_d^{red}(-Y, \mathfrak{s}_\xi).$$

(The change in sign of $h(\xi)$ arises since we consider Floer homology for $-Y$ instead of $Y$; note the shift in degree between the two sides.)

A slightly sharper version is given in Corollary 4.8 below. As before, the bounds on twist number come from estimating the number of boundary twists which, when added to the monodromy, is sufficient to obtain a tight contact structure. Concretely, Theorem 6 follows from:

**Theorem 7.** *Let $\xi$ be a contact structure on $Y$ with torsion Chern class, and for a rational number $d$ let*

$$N(d) = \dim_{\mathbb{F}} HF_d^{red}(-Y, \mathfrak{s}_\xi).$$

*Then for an open book $(S, \phi)$ compatible with $\xi$, having genus $g$ and connected binding, any open book obtained by composing $\phi$ with at least $1 + \sum_d N(d)$ boundary Dehn twists describes a tight contact structure, where the sum is over degrees $d$ congruent to $-h(\xi_Y) + 1$ modulo $2g - 2$.*

We remark that in both Theorem 6 and Theorem 7, if the page genus is 1 then the sums consist only of a single term in the appropriate degree.

The first inequality in Theorem 6 follows as before; the other inequality follows similarly by inverting the monodromy, though note this inversion does introduce a shift in gradings. This issue is discussed more thoroughly in Section 4; see the proof of Corollary 4.8. Theorem 7 yields a surprising corollary: it shows that for "most" contact structures $\xi$, "most" open books which support $\xi$ yield a tight structure after adding a single right-handed Dehn twist along the boundary.

**Corollary 8.** *Let $(Y, \mathfrak{s})$ be a* spin$^c$ *3-manifold with $c_1(\mathfrak{s})$ torsion. Let $\mathcal{S} \cong \mathbb{Z}$ be the set of homotopy classes of contact structures on $Y$ whose induced* spin$^c$ *structure is $\mathfrak{s}$. Then there is a finite subset $\mathcal{S}_0 \subset \mathcal{S}$ such any $\xi$ whose homotopy class is in $\mathcal{S} - \mathcal{S}_0$ has the following property. There exists an integer $g_0 \geqslant 0$ such that for any open book decomposition which supports $\xi$ (with connected binding) and has genus $g \geqslant g_0$, adding a single right-handed, boundary-parallel Dehn twist to the monodromy produces a tight contact structure.*

Indeed, we take $\mathcal{S}_0$ to be the set of homotopy classes of $\xi$ such that the group $HF^{red}_{-h(\xi)+1}(-Y, \mathfrak{s})$ is nontrivial (see section 4 for a discussion of homotopy classification of plane fields). Then for $\xi \in \mathcal{S} - \mathcal{S}_0$ and $g$ sufficiently large it is clear that $HF_d^{red}(-Y, \mathfrak{s}_\xi) = 0$ when $d = -h(\xi_Y) + 1$ modulo $2g - 2$.

In a different direction, our results readily imply a connection between the "twist number" of a closed braid in $S^3$ and the reduced Khovanov homology of the link obtained as its closure. We thank John Baldwin and Liam Watson for bringing this to our attention.

**Theorem 9.** *Let $L$ be a link in $S^3$, and let $\hat{\beta}$ be any closed braid isotopic to $L$ and having an odd number of strands. Then*

$$|\tau(\hat{\beta})| \leqslant \dim_{\mathbb{F}} \widetilde{Kh}(-L) - |\det(L)| + 2.$$

*Here $\tau(\hat{\beta})$ is the twist coefficient of $\beta$, viewed as an element in the mapping class group of the disk with $n$ marked points, and $\widetilde{Kh}$ denotes reduced Khovanov homology.*

The organization of this article is as follows. In the next section we give a proof of our main results, Theorems 1 and 2, based on a surgery exact triangle for Heegaard Floer homology with twisted coefficients. Section 2 also contains the proof of Corollary 5. In Section 3, we spell out the connection between twist numbers and Khovanov homology. Then in Section 4 we revisit our proof of Theorems 1 and 2 to refine our estimates on $N_Y$ and give the proof of Theorems 6 and 7, making use of an absolute grading on Heegaard Floer homology by homotopy classes of oriented plane fields on $Y$ due to Huang and Ramos [15]. In the final section we provide more details on the construction of the twisted surgery triangle that plays a primary role in the proof of our main theorems.

## 2. Proof of Theorems 1 and 2

We work in characteristic two throughout, and let $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. This is for simplicity, and all our arguments could be made with $\mathbb{Z}$ in place of $\mathbb{F}$.

In this section we prove Theorem 2, from which Theorem 1 will follow easily. More precisely, we show that adding

$$N(Y) = \frac{1}{2}(\dim_{\mathbb{F}} \widehat{HF}(Y) - |\mathrm{Tor}_{\mathbb{Z}}\, H_1(Y;\mathbb{Z})|)$$

right-handed Dehn twists to the boundary of any open book decomposition $(S, \phi)$ of $Y$ will produce an open book decomposition for a tight contact structure. The key observation is that the manifold specified by $(S, \phi \circ t_\partial^n)$ is homeomorphic to $Y_{-1/n}(K)$, where $K = \partial S$ is the binding of the open book, viewed as a knot in $Y$. Let $\xi_n$ denote the contact structure on $Y_{-1/n}(K)$ induced by $(S, \phi \circ t_\partial^n)$. Our strategy is as follows

(1) Observe that to show $\xi_n$ is tight, it suffices by [30, Theorem 1.4] to show that its contact invariant $c(\xi_n) \in HF^+(-(Y_{-1/n}(K)))$ is not zero.
(2) Fit $HF^+(-(Y_{-1/n}(K)))$ into an exact triangle of modules over $\mathbb{F}[U]$

$$\begin{array}{ccc} HF^+(-(Y_{-1/n}(K));\mathbb{F}) & \longrightarrow & HF^+(-Y;\mathbb{F}) \\ & {\nwarrow}^{F} \qquad {\swarrow}_{G} & \\ & \underline{HF}^+(-Y_0(K);\mathbb{F}[C_n]) & \end{array}$$

where the bottom term is a twisted version of the Floer homology for zero surgery on the binding, with coefficients in the group algebra of the cyclic group $C_n = \mathbb{Z}/n\mathbb{Z}$.

(3) Show that non-triviality of $F$, restricted to a particular subgroup

$$\underline{HF}^+(-Y_0(K), \mathfrak{s}_{1-g}; \mathbb{F}[C_n]),$$

implies $c(\xi_n) \neq 0$. Here $\mathfrak{s}_{1-g}$ is the spin$^c$ structure on the fibered 3-manifold $Y_0(K)$ whose Chern class evaluates to $2-2g$ on the fiber and which is cobordant through the surgery cobordism to the spin$^c$ structure associated to $\xi$.

(4) Show that the subgroup from Step (3) is isomorphic to $\mathbb{F}[C_n]$, as an $\mathbb{F}[U]$-module where $U$ acts as zero. In particular, this group is a vector space of dimension $n$ over $\mathbb{F}$.

(5) Conclude, by Step (3) and exactness at $-Y_0$, that $c(\xi_n) \neq 0$ provided that

$$n > \dim_{\mathbb{F}} \text{ coker } U : HF^+(-Y) \to HF^+(-Y),$$

and relate $\dim_{\mathbb{F}} \text{coker } U$ to $N(Y)$.

There are two main technical issues involved in implementing this strategy. The first pertains to Steps (2) and (3). The issue is that while the surgery exact triangle used for Step (2) appears in various places in the literature, neither the definition nor the geometric content of the maps in the triangle as required in Step (3) is totally clear. We resolve this issue by first relating the maps in the exact triangle to maps on twisted Floer homology groups associated to 2-handle cobordisms, and then relying on a naturality result for the contact submodule in twisted Floer homology under these latter maps. In order to achieve this, we establish a general exact triangle satisfied by the (twisted) Floer homologies of certain triples of Dehn filled manifolds using a well-known "exact triangle detection lemma". The above surgery triangle, and indeed all previously known exact triangles satisfied by Heegaard Floer modules of closed three manifolds, can be viewed as specializations. So as not to disrupt the flow of the argument, this discussion is postponed to Section 5.

The other technical issue is that Step (4) fails when the fiber surface $S$ has genus one; the relevant summand of $\underline{HF}^+(-Y_0(K), \mathbb{F}[C_n])$ is infinite dimensional in this case. To account for this, we alter our coefficients through the discussion, replacing $\mathbb{F}$ with a certain Novikov field $\Lambda$, which is the coefficient module for Floer homology perturbed by a 2-form. Using Floer homology perturbed by a 2-form Poincaré dual to a meridian of the binding, the case of genus one proceeds exactly as above.

## *2.1. Essentials of the proof*

With the general outline of our proof in place, we turn to the details of the argument. Suppose $W : Z_1 \to Z_2$ is a compact oriented cobordism between closed connected oriented 3-manifolds $Z_1$ and $Z_2$. For each spin$^c$ structure $\mathfrak{s}$ on $W$ there is an induced homomorphism between the Heegaard Floer homology groups of $Z_1$ and $Z_2$. More generally, if $\mathbb{A}$ is a module for the group algebra $\mathbb{F}[H^1(Z_1;\mathbb{Z})]$, there is a homomorphism in Floer homology with twisted coefficients,

$$F^{\mathbb{A}}_{W,\mathfrak{s}} : \underline{HF}^+(Z_1,\mathfrak{s}_1;\mathbb{A}) \to \underline{HF}^+(Z_2,\mathfrak{s}_2;\mathbb{A}\otimes_{\mathbb{F}[H^1(Z_1)]}\mathbb{K}(W)),$$

where $\mathfrak{s}_i = \mathfrak{s}|_{Z_i}$, and $\mathbb{K}(W) = \mathbb{F}[\mathrm{Im}(H^1(\partial W) \to H^2(W,\partial W))]$ (cf. [32, Theorem 3.8]).

In the case that $W$ consists of a single 2-handle addition along a knot, the induced homomorphism is defined by counting holomorphic triangles in a suitable Heegaard triple-diagram. Explicitly, suppose that $Z_i$ are described by pointed Heegaard diagrams $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\gamma}^i,w)$ such that $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\gamma}^1,\boldsymbol{\gamma}^2,w)$ is an admissible triple diagram describing $W$ and adapted to the knot in the standard way. Then the Floer chain groups for $Z_i$ are generated over the appropriate coefficient modules by intersection points in $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}^i}$, and $F^{\mathbb{A}}_W$ is the map induced in homology by the chain map

$$F^{\mathbb{A}}_W(U^{-j}\cdot\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\gamma}^2}} \sum_{\psi\in\pi_2(\mathbf{x},\Theta,\mathbf{y})} \#\mathcal{M}(\psi)\,\mathcal{A}(\psi)\,U^{n_w(\psi)-j}\cdot\mathbf{y},$$

where the sum is over homotopy classes of triangles $\psi$ whose associated moduli space $\mathcal{M}(\psi)$ has dimension 0, and $\Theta \in \mathbb{T}_{\boldsymbol{\gamma}^1} \cap \mathbb{T}_{\boldsymbol{\gamma}^2}$ is a canonical intersection point. Here $\mathcal{A}$ : $\pi_2(\mathbf{x},\Theta,\mathbf{y}) \to \mathbb{K}(W)$ is an "additive assignment" that we now describe in the situations relevant for us; namely, in the case of a 2-handle cobordism associated to a "zero surgery" (see [32] for more details, or Section 5 below).

Assume that $Z_0$ is the 3-manifold resulting from 0-framed surgery along a null-homologous knot in a 3-manifold, $Z$, and that $W$ is the associated cobordism. The oriented boundary of $W$ is given as

$$\partial W = -Z \cup Z_0 = -(-Z_0) \cup -Z,$$

indicating that we can view $W$ as a cobordism either from $Z$ to $Z_0$, or from $-Z_0$ to $-Z$. The latter viewpoint will be more relevant for our purposes. Note that $\mathbb{K}(W) = \mathbb{F}[H^1(Z_0)]$, so that any choice of coefficient module $\mathbb{A}$ chosen for $-Z_0$ will induce the module $\mathbb{A}\otimes_{\mathbb{F}[H^1(Z_0)]}\mathbb{K}(W) = \mathbb{A}$ for the Floer homology of $-Z$.

We will primarily specialize to the case where $\mathbb{A} = \mathbb{F}[C_n]$ is the group algebra over $\mathbb{F}$ on the cyclic group $C_n$, though we will also use coefficients in the group algebra on $C_n$ over the Novikov field $\Lambda$. For both, suppose that we are given a Heegaard triple diagram compatible with the cobordism as above, so that it contains a curve representing the 0-framed longitude. On this curve we place a basepoint $p$. Then for $\psi \in \pi_2(\mathbf{x},\Theta,\mathbf{y})$ let $n_p(\partial\psi)$ be the algebraic number of times the boundary of $\psi$ meets the codimension-one submanifold (of the Lagrangian torus) determined by $p$. Taking coefficients in the module $\mathbb{F}[C_n]$, where $C_n$ is the cyclic group of order $n$ with fixed generator $\zeta$, the map induced by $W$ can be written

$$F_W^{\mathbb{F}[C_n]}(\zeta^k \cdot U^{-j} \cdot \mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_{\boldsymbol\alpha}\cap\mathbb{T}_{\boldsymbol\gamma^2}} \sum_{\psi\in\pi_2(\mathbf{x},\Theta,\mathbf{y})} \#\mathcal{M}(\psi)\, \zeta^{n_p(\partial\psi)+k} \cdot U^{n_w(\psi)-j} \cdot \mathbf{y} \tag{3}$$

yielding a map on homology

$$F_W^{\mathbb{F}[C_n]} : \underline{HF}^+(-Z_0; \mathbb{F}[C_n]) \to \underline{HF}^+(-Z; \mathbb{F}[C_n]).$$

Note that the factor involving $\zeta$ on the right side of (3) is the additive assignment.

When we view $W$ as a cobordism from $Z$ to $Z_0$, then for any coefficient module $\mathbb{A}$ over $\mathbb{F}[H^1(Z)]$, the induced module over $\mathbb{F}[H^1(Z_0)]$ is

$$\mathbb{A} \otimes_{\mathbb{F}[H^1(Z)]} \mathbb{K}(W) \cong \mathbb{A} \otimes_{\mathbb{F}[H^1(Z)]} \mathbb{F}[H^1(Z_0)] \cong \mathbb{A}[T, T^{-1}],$$

with isomorphisms induced by the splitting $H^1(Z_0) \cong H^1(Z) \oplus \mathbb{Z}$, and where the additional variable $T$ corresponds to a generator of the $\mathbb{Z}$ summand. For $\mathbb{F}$ coefficients, we thus have a chain map

$$F_W^{\mathbb{F}}(U^{-j} \cdot \mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{T}_{\boldsymbol\alpha}\cap\mathbb{T}_{\boldsymbol\gamma^2}} \sum_{\psi\in\pi_2(\mathbf{x},\Theta,\mathbf{y})} \#\mathcal{M}(\psi)\, T^{n_p(\partial\psi)} \cdot U^{n_w(\psi)-j} \cdot \mathbf{y} \tag{4}$$

which induces a map

$$F_W^{\mathbb{F}} : HF^+(Z) \to \underline{HF}^+(Z_0; \mathbb{F}[T, T^{-1}]).$$

Note that in equations (3) and (4) we use the same symbols $\mathbf{x}, \mathbf{y}, \boldsymbol\alpha, \ldots$ to correspond to generators and Heegaard circles that play corresponding roles in the equations, though they denote generators in different groups, and circles in different Heegaard triples, in the two equations. In particular, in (3) the pair $(\boldsymbol\alpha, \boldsymbol\gamma^1)$ describe $-Z_0$ and $(\boldsymbol\alpha, \boldsymbol\gamma^2)$ gives $-Z$, while in (4), $(\boldsymbol\alpha, \boldsymbol\gamma^1)$ corresponds to $Z$ and $(\boldsymbol\alpha, \boldsymbol\gamma^2)$ describes $Z_0$.

We will ultimately need to use the map (3) in the case that $-Z_0 = -Y_0(K)$ and $-Z = -(Y_{-1/n}(K)) = (-Y)_{1/n}(K)$, where $K$ is the connected binding of an open book in a 3-manifold $Y$ supporting a contact structure $\xi$ as above. Note that $Z_0$ is indeed obtained by zero surgery on a knot in $Z$; namely, the core of the surgery solid torus used to obtain $Z$ as $-1/n$ surgery on $K \subset Y$. Moreover, this knot is fibered in $Z$ with monodromy differing from that of $K$ by $n$ right-handed boundary Dehn twists, and thus it induces the contact structure we called $\xi_n$ on $Z = Y_{-1/n}(K)$. While this is the application we have in mind (cf. Steps (2) and (3) of the outline given at the beginning of this section), for the moment we suppress the auxiliary 3-manifold, fibered knot, and contact structure, $(Y, \xi, K)$, and simply consider the general case of a fibered knot $L \subset Z$ inducing a contact structure which we abusively denote by $\xi$, and the associated zero surgery $Z_0$.

We will need a generalization of the Heegaard Floer contact invariant introduced in [30] to the situation of twisted coefficients. This generalization is alluded to in [30, Remark 4.5] and further developed in [25, Section 4]. The construction associates to a contact structure $\xi$ on $Z$ and any module $\mathbb{A}$ over $\mathbb{F}[H^1(Z)]$, a distinguished submodule:

$$\underline{c}(\xi;\mathbb{A}) := \iota(\mathbb{A}) \subset \underline{HF}^+(-Z;\mathbb{A}).$$

This *contact submodule* is generated by the inclusion $\iota$ of the homology of the "bottom-most" non-trivial filtered submodule of the knot Floer homology of a fibered knot $L$ supporting $\xi$ (which is isomorphic to $\mathbb{A}$ by [30, proof of Theorem 1.1]) into the Floer homology of $-Z$. Strictly speaking, the literature only refers to the contact *element* in twisted Floer homology, but this does not make sense with coefficients in a general module.

The contact submodule behaves well with respect to 2-handle cobordisms like the one described above, corresponding to 0-surgery on the binding $L$, a fact which we now make precise. To state the result, note that there is a canonical spin$^c$ structure $\mathfrak{s}_{1-g}$ on $Z_0$ determined by

- $\mathfrak{s}_{1-g}$ is cobordant through the surgery cobordism to the spin$^c$ structure on $Z$ determined by the contact structure, and
- if $\widehat{S}$ denotes the fiber of the open book, capped off in $Z_0$, then we have:

$$\langle c_1(\mathfrak{s}_{1-g}), [\widehat{S}]\rangle = 2-2g. \tag{5}$$

**Lemma 2.1.** *Let $L \subset Z$ be a fibered knot with induced contact structure $\xi_L$. If the fiber of $L$ has genus greater than one, then for any module $\mathbb{A}$ over $\mathbb{F}[H^1(Z_0(L))]$ there is an identification*

$$\underline{HF}^+(-Z_0(L),\mathfrak{s}_{1-g};\mathbb{A}) \cong \mathbb{A}$$

*as a trivial $\mathbb{F}[U]$-module, i.e. $U$ acts as zero. Moreover, $\mathfrak{s}_{1-g}$ is the unique* spin$^c$*-structure satisfying (5) that supports non-zero Floer homology. The image of the map*

$$\underline{F}^{\mathbb{A}}_W : \underline{HF}^+(-Z_0(L),\mathfrak{s}_{1-g};\mathbb{A}) \to \underline{HF}^+(-Z,\mathfrak{s}_\xi;\mathbb{A})$$

*induced by the 0-surgery cobordism is the contact submodule $\underline{c}(\xi_L;\mathbb{A})$.*

*All of the above remains true if the genus of the fiber is one, provided that we take coefficients in an algebra over $\Lambda_\omega$, where $\Lambda_\omega$ is the Novikov field viewed as a module over $\mathbb{F}[H^1(Z_0)]$ via a choice of closed 2-form $\omega$ which evaluates non-trivially on the capped-off fiber.*

In the Novikov twisted case, the primary ground algebra for our purposes is $\Lambda_\omega[C_n]$, where $\omega$ is Poincaré dual to the class of the fiber. We refer to Section 5 for details regarding Novikov ring coefficients.

**Proof.** In [30, Proposition 3.1], Ozsváth-Szabó construct a Heegaard triple diagram for the surgery cobordism $W : -Z_0(L) \to -Z$ with the following properties:

- The diagram is weakly admissible for the unique spin$^c$ structure on $W$ extending $\mathfrak{s}_{1-g}$.
- There are precisely two intersection points $\mathbf{u}$, $\mathbf{v}$ providing generators for the chain complex $CF^+(-Z_0(L), \mathfrak{s}_{1-g})$, and the only nontrivial differential is $\partial^+(U^{-j}\mathbf{u}) = U^{-j+1} \cdot \mathbf{v}$. Thus $HF^+(-Z_0(L), \mathfrak{s}_{1-g})$ is generated by the homology class of $\mathbf{u}$. Moreover, for any other intersection point, $\mathbf{x}$, the quantity $\langle c_1(\mathfrak{s}_w(\mathbf{x})), [\widehat{S}]\rangle$ is strictly greater than $2 - 2g$. In particular there are no intersection points corresponding to any other spin$^c$ structures satisfying (5).
- There is a unique holomorphic triangle $\psi$ contributing to the image of $\mathbf{u}$ under the chain map $F_W$.
- The image of $\mathbf{u}$ is a cycle representing the contact invariant $c(\xi) \in HF^+(-Z)$. More precisely, the image of $\mathbf{u}$ is the unique generator for the knot Floer chain complex for $L$, in filtration level $-g$.

Using the same diagram for the chain complexes and chain maps with twisted coefficients gives the desired result. Indeed, all of the statements above remain true with $\mathbb{A}$ replacing the implicit $\mathbb{F}$ coefficients. Note that the second item in this case establishes an isomorphism $HF^+(-Z_0(L), \mathfrak{s}_{1-g}; \mathbb{A}) \cong \mathbb{A}$ as a trivial $\mathbb{F}[U]$-module, but generation by $\mathbf{u}$ is ambiguous; in particular, it does *not* mean generation as an $\mathbb{F}[H^1(Z_0)]$-module since $\mathbb{A}$ may not even be finitely generated over $\mathbb{F}[H^1(Z_0)]$. This is, in essence, why one needs to talk about the contact *submodule* rather than the contact *element* in the most general case.

When $g = 1$, the key difference is that $c_1(\mathfrak{s}_{1-g})$ is torsion and the diagram fails to be admissible. However, it fails to be admissible only because of positivity of the periodic domain corresponding to the homology class of the fiber. If we take coefficients in an algebra over $\Lambda_\omega$, where $\omega$ evaluates non-trivially on this class, then no admissibility is required for this periodic domain. □

Just as with contact element in untwisted Floer homology, non-vanishing of the contact submodule implies tightness (cf. [30, Theorem 1.4]).

**Lemma 2.2.** *Suppose a contact structure $\xi$ is overtwisted. Then the contact submodule is trivial, i.e. $\underline{c}(\xi; \mathbb{A}) \equiv 0$ for any ground module $\mathbb{A}$.*

**Proof.** This follows in exactly the same manner as [30, Proof of Theorem 1.4], noting only that the Künneth theorem for the knot Floer filtration of the connected sums of knots holds with arbitrary coefficient modules and that the contact submodule associated to the overtwisted contact structure induced by the left-handed trefoil is trivial over any ground module (this latter fact can be calculated directly or deduced from the universal coefficients theorem and the fact that $\mathbb{F}[H^1(S^3)] = \mathbb{F}$). □

The preceding two lemmas yield an immediate corollary.

**Corollary 2.3.** *In the situation of Lemma 2.1, and for $g \geqslant 2$, if the map*

$$F_W^{\mathbb{F}[C_n]} : \underline{HF}^+(-Z_0(K), \mathfrak{s}_{1-g}; \mathbb{F}[C_n]) \to \underline{HF}^+(-Z, \mathfrak{s}_\xi; \mathbb{F}[C_n])$$

*is nonzero, then $\xi_L$ is tight. If $g = 1$ and the map*

$$F_W^{\Lambda_\omega[C_n]} : \underline{HF}^+(-Z_0(K), \mathfrak{s}_{1-g}; \Lambda_\omega[C_n]) \to \underline{HF}^+(-Z, \mathfrak{s}_\xi; \Lambda_\omega[C_n])$$

*is nonzero, then $\xi_L$ is tight.* □

We now return to the surgery exact triangle. In this case, the manifold $Z$ above becomes $Y_{-1/n}(K)$, while $Z_0 = Y_0(K)$. We have:

**Proposition 2.4.** *Let $n > 0$, and suppose the genus of the fiber is greater than one. If, for the map*

$$F : \underline{HF}^+(-Y_0(K), \mathbb{F}[C_n]) \to HF^+(-Y_{-1/n}(K); \mathbb{F})$$

*appearing in the surgery triangle, the restriction to the summand corresponding to $\mathfrak{s}_{1-g}$ is nontrivial, then the contact structure $\xi_n$ on $Y_{-1/n}$ is tight. The same is true if the genus of the fiber is one, provided we consider the surgery triangle with coefficients in the Novikov module $\Lambda_\omega$ associated to a 2-form evaluating non-trivially on the fiber.*

**Proof.** In Section 5 the map $F$ appearing in the surgery triangle is defined on the chain level by

$$F(U^{-j}\zeta^k \mathbf{x}) := \sum_{\substack{\psi \in \pi_2(\mathbf{x}, \Theta, \mathbf{y}) \\ \mu(\psi)=0 \\ n_p(\partial\psi) = -k \bmod n}} \#\mathcal{M}(\psi) \cdot t^{\omega(\psi)} \cdot U^{n_w(\psi)-j} \cdot \mathbf{y}$$

(cf. equation (11) below). Here $t$ is the variable appearing in the Novikov ring $\Lambda_\omega$, which we set equal to 1 if the genus of the fiber is at least two. Comparing this to the definition of the cobordism-induced homomorphism $F_W^{\mathbb{F}[C_n]}$ in (3) above, it is easy to check that there is a commutative diagram

$$\begin{array}{ccc} \underline{CF}^+(-Y_0;\mathbb{F}[C_n]) & \xrightarrow{F_W^{\mathbb{F}[C_n]}} & \underline{CF}^+(-Y_{-1/n};\mathbb{F}[C_n]) \\ \mathcal{N}\uparrow & \overset{F}{\searrow} & \downarrow\Pi \\ CF^+(-Y_0;\mathbb{F}) & \xrightarrow{F_W} & CF^+(-Y_{-1/n};\mathbb{F}) \end{array}$$

where $\mathcal{N}$ and $\Pi$ are chain maps induced by the coefficient $\mathbb{F}$-homomorphisms $\mathcal{N}:\mathbb{F}\to\mathbb{F}[C_n]$ and $\Pi:\mathbb{F}[C_n]\to\mathbb{F}$ given by

$$\mathcal{N}(1)=\sum_{k=0}^{n-1}\zeta^k \quad \text{and} \quad \Pi(p(\zeta))=p(0),$$

where $p(\zeta)$ denotes a polynomial in $\zeta$. In particular, we see that if $F$ induces a nontrivial map in homology (in a particular spin$^c$ structure), then so does $F_W^{\mathbb{F}[C_n]}$. An analogous diagram exists with $\Lambda_\omega$ replacing $\mathbb{F}$ throughout. The result follows now from the previous corollary. □

Our main results (Theorems 1 and 2) now follow easily. We give a combined restatement.

**Theorem 2.5.** *Let $Y$ be a closed oriented 3-manifold. Then there exists a constant $N(Y)\geqslant 0$ with the following property. Let $K\subset Y$ be a fibered knot with monodromy $\phi_K$, and let $\tau_K=\tau(\phi_K)$ be the fractional Dehn twist coefficient of $\phi_K$. Then*

(1) *For all $n>N(Y)$, the contact structure $\xi_n$ supported by the open book with monodromy obtained from $\phi_K$ by composition with $n$ boundary-parallel Dehn twists is tight.*
(2) *We have the bound*

$$|\tau_K|\leqslant N(Y)+1.$$

*Moreover, we can take*

$$N(Y)=\frac{1}{2}(\dim_{\mathbb{F}}\widehat{HF}(Y)-|\mathrm{Tor}_{\mathbb{Z}}\ H_1(Y;\mathbb{Z})|). \tag{6}$$

Here $\widehat{HF}(Y)$ indicates the sum of Heegaard Floer groups over all spin$^c$ structures on $Y$, while $|\mathrm{Tor}_{\mathbb{Z}}\ H_1(Y;\mathbb{Z})|$ is the order of the torsion subgroup of the first homology.

**Proof.** We treat the case that the fiber genus is at least two explicitly; the genus one case is exactly the same, using coefficients in $\Lambda_\omega$, where $\omega$ is Poincaré dual to the meridian of the binding. Strictly speaking, this latter argument produces (6) with

$\dim_\Lambda \widehat{HF}(Y;\Lambda_\omega)$ in place of $\dim_{\mathbb{F}} \widehat{HF}(Y)$. These dimensions are equal, however, since $\widehat{HF}(Y;\Lambda_\omega) \cong \widehat{HF}(Y) \otimes_{\mathbb{F}} \Lambda$ by the universal coefficient theorem and the fact that $[\omega] = 0 \in H^2(Y;\mathbb{R})$.

Proceeding, then, consider the surgery exact triangle:

$$\begin{array}{ccc} HF^+(-Y_{-1/n}(K);\mathbb{F}) & \longrightarrow & HF^+(-Y;\mathbb{F}), \\ {\scriptstyle F}\nwarrow & & \swarrow{\scriptstyle G} \\ & \underline{HF}^+(-Y_0(K);\mathbb{F}[C_n]) & \end{array}$$

The summand of the bottom module corresponding to $\mathfrak{s}_{1-g}$ is, according to Lemma 2.1, isomorphic to $\mathbb{F}[C_n]$ and therefore of dimension $n$ over $\mathbb{F}$. By Lemma 2.1 again and $U$-equivariance of the sequence, the component of $G$ mapping into the $\mathfrak{s}_{1-g}$ summand factors through the cokernel of the action of $U$ on $HF^+(-Y;\mathbb{F})$, which is finite-dimensional and independent of $n$. Hence for $n$ sufficiently large, we conclude $F$ is nonzero and therefore the contact structure $\xi_n$ on $Y_{-1/n}$ is tight by Proposition 2.4.

To estimate the size of $n$ required, observe that it suffices that $n$ be larger than the dimension of the cokernel of $U$, acting on $HF^+(-Y)$. In a given spin$^c$ structure it is easy to see that $\dim \widehat{HF}(Y,\mathfrak{s}) = \dim\ker U + \dim \mathrm{coker} U = 2\dim\mathrm{coker} U + k_{\mathfrak{s}}$, where $k_{\mathfrak{s}}$ is the rank of $HF^\infty(Y,\mathfrak{s})$ as a module over $\mathbb{F}[U,U^{-1}]$. Note that $k_{\mathfrak{s}}$ is 0 if $\mathfrak{s}$ is non-torsion, and at least 1 in the torsion case (cf. [27, Theorem 10.1] and [29, Lemma 2.3]). Adding over all spin$^c$ structures, if we set $N(Y) = \frac{1}{2}(\dim \widehat{HF}(Y) - |\mathrm{Tor}_{\mathbb{Z}} H_1(Y)|)$ it follows that adding at least $N(Y)+1$ right twists to the monodromy of any open book with connected binding will produce a tight contact structure.

According to Honda–Kazez–Matic [13, Theorem 1.1 and Propositions 3.1, 3.2], the fractional Dehn twist coefficient of the monodromy of an open book supporting a tight contact structure is nonnegative. Since we added $N(Y)+1$ right twists, the new monodromy is $\phi_K \circ t_\partial^{N(Y)+1}$. Hence

$$0 \leqslant \tau(\phi_K \circ t_\partial^{N(Y)+1}) = \tau_K + N(Y) + 1,$$

which gives half the desired inequality. For the other half, replace $\phi_K$ by $\phi_K^{-1}$. This amounts to reversing the orientation on $Y$, giving a lower bound on $\tau(\phi_K^{-1}) = -\tau(\phi_K)$ in terms of $N_{-Y}$. But since (6) is insensitive to the orientation of $Y$ (cf. [27, Proposition 2.5]), the result follows. □

**Remark 2.6.** We could, by the argument in the proof, take

$$N(Y) = \frac{1}{2}\max_{\mathfrak{s}}(\dim \widehat{HF}(Y,\mathfrak{s}) - k_{\mathfrak{s}}),$$

which gives (2) in the case that $Y$ is a rational homology sphere. Indeed, note that the homomorphism $G$ is a sum of homogeneous terms corresponding to spin$^c$ structures on the surgery cobordism $-Y \to -Y_0(K)$, hence just one term of $G$ maps to the spin$^c$ structure $\mathfrak{s}_{1-g}$ on $-Y_0(K)$. Therefore we may consider one spin$^c$ structure on $-Y$ at a time.

**Remark 2.7.** It follows from the main result of [23] together with work of Lidman [20] that if $\mathfrak{s}$ is a torsion spin$^c$ structure on a 3-manifold with positive first Betti number, there is an estimate

$$k_{\mathfrak{s}} \geqslant L_Y := \begin{cases} 2 \cdot 3^{(b_1(Y)-1)/2} & \text{if } b_1(Y) \text{ is odd} \\ 4 \cdot 3^{b_1(Y)/2-1} & \text{if } b_1(Y) \text{ is even.} \end{cases}$$

Thus, letting $L(\mathfrak{s}) = L_Y$ if $\mathfrak{s}$ is torsion and $L(\mathfrak{s}) = 0$ otherwise, we can take

$$N(Y) = \frac{1}{2} \max_{\mathfrak{s}} (\dim \widehat{HF}(Y, \mathfrak{s}) - L(\mathfrak{s})).$$

**Remark 2.8.** If $Y$ is an $L$-space, meaning that $Y$ is a rational homology sphere with $\dim \widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|$, then the theorem says $|\tau(\phi_K)| \leqslant 1$ for any fibered knot $K$ in $Y$. In fact, we must have

$$|\tau(\phi_K)| < 1 \quad \text{for any fibered } K \text{ in an } L\text{-space.}$$

Indeed, if $K \subset Y$ has $|\tau(\phi_K)| \geqslant 1$ then $Y$ admits a taut foliation, according to [14, Theorem 1.2]. On the other hand, $L$-spaces do not admit taut foliations by [25, Theorem 1.4]. Note that while $|\tau(K)| \leqslant 1/2$ for knots in $S^3$, this is not true for knots in arbitrary $L$-spaces.

To illustrate the last statement, if $K \subset S^3$ is the right trefoil, then $K$ is fibered with $\tau_K(\phi) = 1/6$. The result of $+1$ surgery on $K$ is the Poincaré sphere $Y = -\Sigma(2, 3, 5)$, an $L$-space, and the induced knot $K \subset Y$ (the core of the surgery) is fibered with twist coefficient $-5/6$, since $+1$ surgery corresponds to addition of a left Dehn twist.

In a similar spirit, we turn to the proof of Corollary 5:

**Proof of Corollary 5.** Suppose that $K$ is a fibered knot in a 3-manifold $Y$ with monodromy $\phi_K$. Then the $n$-fold cyclic branched cover $\Sigma_n(K)$ is well-defined (in general, it depends on a homomorphism of the knot group to $\mathbb{Z}/n\mathbb{Z}$, but this is specified by counting intersections with the fiber) and has an open book decomposition with the same fiber and monodromy $\phi_K^n$. The proof of Theorem 2.5 shows that

$$|\tau(\phi_K^n)| \leq 1 + \dim_{\mathbb{F}} \operatorname{coker}(U : HF^+(\Sigma_n) \to HF^+(\Sigma_n)).$$

Homogeneity of the twist coefficient shows that the left-hand side equals $n \cdot |\tau(\phi_K))|$ whereas $1 + \dim HF^{red}(\Sigma_n)$ is at least as large as the right-hand side, since reduced Floer homology is defined as the limit (see [28, Definition 4.7])

$$HF^{red} := \lim_{k\to\infty} \operatorname{coker} U^k. \quad \square$$

## 3. Application to braids

Fractional Dehn twist coefficients can be defined in various contexts; for example Malyutin [21] gave a definition of a "twist number" for closed braids by considering a braid as an element of the mapping class group of a punctured disk (see also Ito–Kawamuro [16]). For a braid $\beta$ write $\tau(\beta) \in \mathbb{Q}$ for the twist number; note that while $\tau(\beta)$ is conjugation-invariant and so depends only on the closure $\hat{\beta}$, it is not an invariant of the link type of $\hat{\beta}$. That is to say, different closed braid representatives of a given link may have different twist numbers (for example, while it is easy to construct braids with arbitrarily large twist number, it follows from [21, Proposition 13.1] that if $\beta$ is a Markov stabilization of another braid, then $|\tau(\beta)| \leqslant 1$).

The following application of Theorem 2.5 was pointed out to us by John Baldwin and Liam Watson.

**Theorem 3.1.** *Let $L$ be a link in $S^3$, and let $\hat{\beta}$ be any closed braid isotopic to $L$ and having an odd number of strands. Then*

$$|\tau(\beta)| \leqslant \dim_{\mathbb{F}} \widetilde{Kh}(-L) - |\det(L)| + 2.$$

Here $\widetilde{Kh}(-L)$ denotes the reduced Khovanov homology of the mirror of $L$, with coefficients in $\mathbb{F}$.

**Proof.** If $\hat{\beta}$ is a closed braid with axis the unknot $U$ and representing the link type $L$, then forming the double cover of $S^3$ branched along $\hat{\beta}$ gives rise to a 3-manifold $\Sigma_2(L)$ equipped with an open book decomposition lifting the decomposition of $S^3$ with disk pages and binding $U$. Since $\beta$ has an odd number of strands, this open book structure on $\Sigma_2(L)$ has connected binding.

If $\phi$ denotes the monodromy of the lifted open book, then it is not hard to check that $\tau(\phi) = \frac{1}{2}\tau(\beta)$. From Theorem 1 and using (1), we have

$$|\tau(\phi)| \leqslant \frac{1}{2}(\dim \widehat{HF}(\Sigma_2(L)) - |H_1(\Sigma_2(L))|) + 1.$$

Now recall that there is a spectral sequence whose $E_2$ page is the reduced Khovanov homology $\widetilde{Kh}(-L)$, and which converges to $\widehat{HF}(\Sigma_2(L))$ (see [31]). Hence

$$\dim \widehat{HF}(\Sigma_2(L)) \leqslant \dim \widetilde{Kh}(-L).$$

Moreover, $|H_1(\Sigma_2(L))| = |\det(L)|$ unless the latter quantity is 0. Combining these observations gives the desired result. □

Suppose that $\beta'$ is an alternating braid on $n \geqslant 3$ strands. Using Corollary 5.5 of [21], for example, it is easy to see that $\tau(\beta') = 0$. However, it is not the case that any braid Markov equivalent to $\beta'$ has vanishing twist number. The following shows that nevertheless, there is an upper bound on the twist number of a braid representing the same link type as $\hat{\beta}'$.

**Corollary 3.2.** *Let $L$ be an alternating link, and $\beta$ any braid on an odd number of strands with the property that the closure $\hat{\beta}$ is isotopic to $L$. Then*

$$|\tau(\beta)| < 2.$$

**Proof.** Since $\hat{\beta}$ is the alternating link $L$, the double branched cover $\Sigma_2(L)$ is an $L$-space, and moreover the spectral sequence from $\widetilde{Kh}(-L)$ to $\widehat{HF}(\Sigma_2(L))$ collapses (cf. [31]). The proof of Theorem 3.1 then gives $|\tau(\hat{\beta}')| \leqslant 2$, and the strict inequality follows from the remark at the end of Section 2. □

The corollary applies equally, of course, to braids whose closures are quasi-alternating in the sense of Ozsváth-Szabó [31]: the double cover of $S^3$ branched along such a link is also an $L$-space.

## 4. Graded refinement

Here we provide the proof of Theorem 6. To do so, we make use of absolute gradings constructed on Heegaard Floer homology by Ozsváth and Szabó [32] (in the case of torsion spin$^c$ structures) and Huang and Ramos [15] (in general). We also clarify some properties of the general grading by homotopy classes of plane fields in the latter case.

### *4.1. Plane field grading on Floer homology*

Recall that Heegaard Floer homology carries a relative cyclic grading in each spin$^c$ structure. Huang and Ramos [15] proved that this can be lifted to an absolute grading on $HF^\circ(Y)$ by the set of homotopy classes of oriented 2-plane fields on $Y$, which we will denote by $\mathcal{P}(Y)$ (here $HF^\circ$ indicates any of the versions of Heegaard Floer homology). Our goal in this subsection is to calculate the absolute grading on the Floer homology of a fibered 3-manifold, in a canonical spin$^c$ structure.

To begin, recall the homotopy classification of oriented plane fields on a closed oriented 3-manifold (for a much more detailed discussion, see [10]). The set $\mathcal{P}(Y)$ is a $\mathbb{Z}$-set whose orbits correspond bijectively to spin$^c$ structures; in particular if $\xi$ is an oriented plane field, then there is an associated spin$^c$ structure $\mathfrak{s}_\xi$. The $\mathbb{Z}$-orbit in $\mathcal{P}(Y)$ corresponding to a given spin$^c$ structure $\mathfrak{s}$ is isomorphic to $\mathbb{Z}/d(\mathfrak{s})\mathbb{Z}$, where $d(\mathfrak{s})$ is the divisibility of $c_1(\mathfrak{s})$, see e.g., [10, Section 4] or [19, Lemma 2.3 and Section 5(i)]. In particular, the orbit in $\mathcal{P}(Y)$ corresponding to a spin$^c$ structure with torsion first Chern class is free. We remark that while the set $\mathcal{P}(Y)$ is independent of the orientation on $Y$, the action of $\mathbb{Z}$ on $\mathcal{P}(Y)$ does depend on this orientation: specifically, this action is negated under orientation reversal.

The homotopy class of a plane field $\xi$ for which $c_1(\mathfrak{s}_\xi)$ is a torsion class is specified by $\mathfrak{s}_\xi$ together with a rational number called the *Hopf invariant*. This quantity is defined as follows: choose a compact almost-complex 4-manifold $(W, J)$ with $\partial W = Y$ as oriented manifolds, and $TY \cap J(TY)$ homotopic to $\xi$. Then

$$h(\xi) = h(\xi_Y) = \frac{1}{4}(c_1^2(W, J) - 3\sigma(W) - 2\chi(W) + 2), \tag{7}$$

where $\sigma$ is the signature of $W$, $\chi(W)$ is the Euler characteristic, and $c_1^2$ denotes the rational-valued square of the Chern class.

It is not hard to determine the effect of reversing orientation of $Y$ on the Hopf invariant: since the quantity $c_1^2 - 3\sigma - 2\chi$ vanishes for closed almost-complex 4-manifolds, we have

$$h(\xi_Y) + h(\xi_{-Y}) = 1. \tag{8}$$

Huang and Ramos proved certain properties of the grading on Floer homology by $\mathcal{P}(Y)$, notably:

- For any plane field $\xi \in \mathcal{P}(Y)$, we have $HF^+_{[\xi]}(Y) \subset HF^+(Y, \mathfrak{s}_\xi)$.
- The grading by plane fields lifts the relative grading on Heegaard Floer homology defined by Ozsváth and Szabó. Moreover, if $c_1(\mathfrak{s}_\xi)$ is torsion, the summand $HF^\circ_{[\xi]}(Y)$ coincides with the $\mathbb{Q}$-graded summand $HF^\circ_{h(\xi)}(Y, \mathfrak{s}_\xi)$.

To describe the final property, let $W : Y_1 \to Y_2$ be a cobordism, and let $p_i \in \mathcal{P}(Y_i)$ for $i = 1, 2$. We say plane fields $p_1$ and $p_2$ are *related by* $W$ if there is an almost-complex structure $J$ on $W$ such that the fields of complex tangencies $TY_i \cap J(TY_i)$ represent $p_i$, for $i = 1, 2$.

- If $x \in HF^+_{p_1}(Y_1)$ is a homogeneous element such that $F_W(x)$ has a nonzero component in $HF^+_{p_2}(Y_2)$, then $p_1$ and $p_2$ are related by $W$.

To expand on this point, we recall some of the homotopy classification of almost-complex structures on 4-manifolds. First, observe that an almost-complex structure on an oriented 4-manifold $W$ is the same as a lift of the classifying map for $TW$ from $BSO(4)$ to $BU(2)$. Since the fiber of the bundle $BU(2) \to BSO(4)$ is $SO(4)/U(2) \cong S^2$, we are interested in obstruction theory for a bundle over $W$ with fiber $S^2$. (An alternate perspective is given by choosing a metric on $W$, after which a choice of almost-complex structure is the same as a non-vanishing section of the rank-3 vector bundle $\Lambda^+$ of self-dual 2-forms on $W$.)

**Proposition 4.1.** *Let $W : Y_1 \to Y_2$ be a cobordism between connected oriented 3-manifolds. For a* spin$^c$ *structure $\mathfrak{s} \in \mathrm{spin}^c(W)$, let $\mathcal{J}_\mathfrak{s}(W)$ denote the set of homotopy classes of* *almost-complex structures on* $W$ *whose associated* spin$^c$ *structure is* $\mathfrak{s}$*, and assume* $\mathcal{J}_{\mathfrak{s}}(W)$ *is nonempty. Then there is a transitive* $\mathbb{Z}$*-action on* $\mathcal{J}_{\mathfrak{s}}(W)$*, such that the restriction* $\mathcal{J}_{\mathfrak{s}}(W) \to \mathcal{P}(Y_i)$*,* $J \mapsto TY_i \cap JTY_i$*, is a map of* $\mathbb{Z}$*-sets.*

**Proof.** Choose a nicely embedded path $[0,1] \to W$ connecting the two boundary components; identify its neighborhood with $[0,1] \times B^3$. Given an almost-complex structure $J$, regarded as a section of the unit sphere bundle of $\Lambda^+(W)$, we trivialize the latter over $[0,1] \times B^3$ and construct $J'$ such that $J' = J$ away from the arc, and over each $t \times B^3$ the two sections glue over $\partial B^3$ to give the Hopf map $S^3 \to S^2$. This is easily seen to correspond to generators of the $\mathbb{Z}$ actions on $\mathcal{P}(Y_i)$. Transitivity of the action follows from the fact that the obstruction to homotopy between two elements of $\mathcal{J}_{\mathfrak{s}}(W)$ lies in $H^3(W; \pi_3(S^2)) = \mathbb{Z}$, and there is no further obstruction. $\square$

The following provides the calculation we need:

**Lemma 4.2.** *Let* $M$ *be an oriented, fibered 3-manifold with oriented fiber* $\widehat{S}$ *having genus* $g > 1$*. Write* $[T\widehat{S}]$ *for the plane field of oriented tangents to the fibers. Then* $HF^+(M, \mathfrak{s}_{1-g})$ *is supported in absolute grading* $[T\widehat{S}] - 1$*.*

**Proof.** Construct a Lefschetz fibration $X$ over $D^2$ with oriented boundary $\partial X = -M$, whose singular fibration extends the surface bundle structure on the boundary. Removing a 4-ball from $X$, Ozsváth and Szabó show in [29, Theorem 5.3] that there is a unique spin$^c$ structure $\mathfrak{t}_{can}$ on $X$ having $\langle c_1(\mathfrak{t}_{can}), \widehat{S}\rangle = 2 - 2g$ and inducing a nontrivial map (in fact an isomorphism) $HF^+(M, \mathfrak{s}_{1-g}) \to HF^+_0(-S^3)$, where the subscript in the latter refers to the absolute $\mathbb{Q}$-grading, and we take the natural orientation on $S^3$ to be that induced by $B^4$. According to the third bullet point above, there is an almost-complex structure on $X - B^4$ relating the plane fields giving the gradings on these groups; we can identify one such as follows. Note that $X$ admits a canonical symplectic structure [12, Theorem 10.2.18] (cf. [11]) to which is associated a natural homotopy class of compatible almost-complex structure, from which an element $J_0$ can be chosen so that the fibers of the Lefschetz fibration $X \to D^2$ are $J_0$-holomorphic. In particular the adjunction formula implies that $c_1(J_0)$ pairs with the fiber $\widehat{S}$ to give $2 - 2g$, and from [29] the spin$^c$ structure associated to $J_0$ is $\mathfrak{t}_{can}$. Now $\mathcal{P}(S^3)$ is identified with the integers via the Hopf invariant (7), and the standard tight contact structure on $S^3$ has Hopf invariant 0. We may suppose that the chosen $B^4 \subset X$ is a standard Darboux ball, so that the plane field $\xi_0 = TS^3 \cap J_0(TS^3)$ is isotopic to the standard contact structure on $S^3$. However, we are considering $\xi_0$ as a plane field on $-S^3$, and from (8), we have that $h([\xi_0]_{-S^3})$ is $+1$. Therefore a member of $\mathcal{J}_{\widehat{S}}(X - B^4)$ relates the tangent field $[T\widehat{S}]$ (and no other plane field on $M$, by Proposition 4.1) to grading level 1 in $HF^+(-S^3)$. Hence only $[T\widehat{S}] - 1$ is related to the grading of $HF^+_0(-S^3) = HF^+_0(S^3)$. The result follows from the third bullet point above. $\square$

Given an oriented 3-manifold $Y$ with positive co-oriented (and hence oriented) contact structure $\xi$, Giroux's theorem [9] implies that we can find an open book decomposition for $Y$ that supports $\xi$, has connected binding $K$, and has pages $S$ of genus $g > 1$. Let $W : -Y_0 \to -Y$ be the surgery cobordism obtained by attaching a 2-handle along $K$ with framing determined by the pages, "turned around" and equipped with the unique spin$^c$ structure extending $\mathfrak{s}_{1-g}$ on $-Y_0$. Then Lemma 2.1 implies that the contact invariant of $\xi$ is equal to the image under $F^+_W$ of the nonzero element of $HF^+(-Y_0, \mathfrak{s}_{1-g}) \cong \mathbb{F}$ (in fact, in this untwisted situation Lemma 2.1 is nothing but [30, Proposition 3.1]).

Now construct a Lefschetz fibration $X$ with oriented boundary $-Y_0$ and whose singular fibration extends the surface bundle structure on the boundary, as in the proof above. The spin$^c$ structure on $W$ glues with $\mathfrak{t}_{can}$ on $X$ to give a spin$^c$ structure on the cobordism $X \cup W$, with the property that the contact invariant $c(\xi)$ is equal to the "mixed invariant" $F^{mix}_{X\cup W}(\Theta^-)$ (cf. Plamenevskaya [35, Lemma 1] for more details). In the case that $c_1(\mathfrak{s}_\xi)$ is torsion, it follows quickly from the formula for the shift in rational grading induced by cobordisms that $c(\xi)$ lies in rational grading $h(\xi_{-Y}) - 1 = -h(\xi_Y)$. This corrects [30, Prop. 4.6], though the correction has been made in the literature long ago, e.g. [35, Section 4].

We remark that for general $\mathfrak{s}_\xi$, Huang and Ramos claim that the contact invariant lies in the absolute grading of Floer homology corresponding to the plane field $[\xi]$. In light of the above and the second bullet point previously, this should be interpreted as saying that $c(\xi)$ lies in the graded summand of $\widehat{HF}(-Y)$ that is dual to the summand of $\widehat{HF}(Y)$ in grading $[\xi]$; in terms of plane fields this says $c(\xi) \in \widehat{HF}_{[\xi]-1}(-Y)$ (where in the subscript the action of $-1$ is taken using the orientation on $-Y$). However, since in our application we will focus on the case that $c_1(\mathfrak{s}_\xi)$ is torsion, we do not pursue this discussion.

Observe that the set $\mathcal{P}(Y)$ carries a natural involution $[\xi] \mapsto [\xi]^*$ induced by reversing the orientation of the plane field $\xi$. It can be seen that this involution respects the $\mathbb{Z}$ action, i.e., $([\xi] + n)^* = [\xi]^* + n$. In fact we have:

**Lemma 4.3.** *For any oriented 3-manifold $Y$ with oriented plane field $[\xi]$, there is an isomorphism*

$$HF^+_{[\xi]}(Y, \mathfrak{s}_\xi) \cong HF^+_{[\xi]^*}(Y, \mathfrak{s}_{\xi^*}).$$

This can be seen as a refinement of the conjugation invariance of Floer homology, since the spin$^c$ structure associated to $[\xi]^*$ is conjugate to $\mathfrak{s}_\xi$. Given Huang and Ramos's construction, the proof is routine and based on the observation that if $\mathbf{x}$ is a Heegaard Floer generator coming from the diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ and with corresponding spin$^c$ structure $\mathfrak{s}$, then the same intersection point interpreted in $(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ corresponds to the conjugate of $\mathfrak{s}$ and gives rise to the negative of the gradient-like vector field originally determined by $\mathbf{x}$ (cf. [15], [27, Theorem 2.4]).

### 4.2. Adding twists to open books

With these preliminaries in place, we return to the homomorphism

$$G : HF^+(-Y, \mathbb{F}) \to \underline{HF}^+(-Y_0(K), \mathbb{F}[C_n])$$

appearing in the surgery triangle. We assume for the rest of the section that the spin$^c$ structure corresponding to the contact structure $\xi$ has torsion first Chern class. Here, as before, our arguments are given for the case $g > 1$ but carry over directly to the genus 1 case by replacing $\mathbb{F}$ by the Novikov field $\Lambda$. We leave the attendant adjustments of the following proofs to the reader.

Comparing the definition of $G$, given in (10) below, with (4) defining cobordism maps in twisted Floer homology, we observe that $G$ is the map on Floer homology induced by a 2-handle cobordism connecting $-Y$ to $-Y_0(K)$, followed by the change of coefficients homomorphism induced by the projection $\mathbb{F}[T, T^{-1}] \to \mathbb{F}[T, T^{-1}]/\langle T^n - 1\rangle \cong \mathbb{F}[C_n]$. Note that if $W : Y \to Y_0(K)$ is the standard 2-handle cobordism such as the one considered above, then the cobordism under consideration here is $-W$. For the present purposes it is convenient to consider $-W$ as a cobordism from $Y_0(K)$ to $Y$; recall that $Y$ carries a contact structure we denote by $\xi$, supported by an open book with connected binding $K$ and page genus $g$.

**Proposition 4.4.** *The surgery cobordism $-W : Y_0(K) \to Y$ admits an almost-complex structure $J$ with the following properties:*

- *The tangents to the fiber surfaces in $Y_0(K)$ are positively $J$-invariant.*
- *If $\eta = TY \cap J(TY)$ is the plane field on $Y$ induced by $J$, then the homotopy classes $[\eta]$ and $[\xi]$ are related by*

$$h([\xi]_Y) - h([\eta]_Y) = 2g - 1.$$

**Proof.** We have seen that an oriented, fibered knot $L$ in an oriented 3-manifold $M$ gives rise to a positive, oriented contact structure via the Thurston–Winkelnkemper construction, whose homotopy class we denote $TW(M, L)$. Implicitly, the fiber surface is oriented using the orientation on $L$. Recall that Eliashberg [4] has constructed a symplectic structure on the surgery cobordism $Z(M, L) : M \to M_0(L)$ for which the fibration on $M_0(L)$ is symplectic and such that the contact planes on $M$ are symplectically positive. It follows as before that there is an almost-complex structure (compatible with the symplectic structure) on $Z(M, L)$ relating the contact field $TW(M, L)$ on $M$ to the plane field tangent to the fibers in $M_0(L)$.

Applying this construction to the oriented knot $K$ lying in $-Y$ gives a plane field $TW(-Y, K)$ and an almost-complex structure on the corresponding surgery cobordism, which is $Z(-Y, K) = -Z(Y, K) = -W$. It remains to compare the Hopf invariants of $[\eta] = TW(-Y, K)$ and $[\xi] = TW(Y, K)$, bearing in mind that we must reverse the ambient orientation to do so.

For this, recall that (by the determination above of the grading of the contact invariant, together with symmetries of knot Floer homology) the Hopf invariant of $TW(M, L)$ is equal to the grading in which the highest nontrivial filtered summand of knot Floer homology $\widehat{HFK}(M, L, g)$ is supported, which we write as $\mathrm{gr}(\widehat{HFK}(M, L, g))$. Therefore, $h(TW(-Y, K)_{-Y})$ is equal to $\mathrm{gr}(\widehat{HFK}(-Y, K, g))$. Using symmetries of knot Floer homology ([26, Proposition 3.7] and [26, Proposition 3.10] in particular) we obtain

$$\begin{aligned} h(TW(-Y, K)_{-Y}) &= \mathrm{gr}(\widehat{HFK}(-Y, K, g)) \\ &= -\,\mathrm{gr}(\widehat{HFK}(Y, K, -g)) \\ &= -(\mathrm{gr}(\widehat{HFK}(Y, K, g)) - 2g) \\ &= -h(TW(Y, K)_Y) + 2g. \end{aligned}$$

The result now follows from (8). □

**Corollary 4.5.** *The only elements of $HF^+(-Y)$ that map nontrivially to $\underline{HF}^+(-Y_0(K), \mathfrak{s}_{1-g}; \mathbb{F}[C_n])$ under the surgery cobordism lie in degree congruent to $-h([\xi]_Y) + 1$ modulo $2g - 2$.*

**Proof.** The target group is supported in degree $[T\widehat{S}]_{-Y_0} - 1$, and we have just seen that $[T\widehat{S}]_{Y_0}$ is related through $-W$ to $[\eta]_Y$. By Proposition 4.1, and since the divisibility of $\mathfrak{s}_{1-g}$ is $2g - 2$, we have that $[T\widehat{S}]_{Y_0}$ is related only to plane fields congruent to $[\eta]_Y$ modulo the action of $2g - 2 \in \mathbb{Z}$. Turning the cobordism around, $-W$ relates $[\eta]_{-Y}$ to $[T\widehat{S}]_{-Y_0}$, and likewise any plane field obtained from $[\eta]_{-Y}$ by the action of $2g - 2$ (where now the $\mathbb{Z}$ action uses the orientation $-Y$). Therefore, the only plane fields on $-Y$ that are related to $[T\widehat{S}]_{-Y_0} - 1$ through $-W$ have Hopf invariant congruent modulo $2g - 2$ to

$$h([\eta]_{-Y} - 1) = h([\eta]_{-Y}) - 1 = -h([\eta]_Y) = -h([\xi]_Y) + 2g - 1. \quad \square$$

**Theorem 4.6.** *Suppose that for some $n > 0$ the restriction to the canonical grading $[T\widehat{S}] - 1$ of the map in the surgery triangle,*

$$F : \underline{HF}^+_{[T\widehat{S}]-1}(-Y_0(K); \mathbb{F}[C_n]) \to HF^+(-Y_{-1/n}(K); \mathbb{F}),$$

*vanishes. Then*

$$\sum_{k \in \mathbb{Z}} \dim_{\mathbb{F}}(HF^+_{-h(\xi_Y)+1+(2g-2)k}(-Y, \mathfrak{s}_\xi; \mathbb{F}) / \operatorname{Im}(U)) \geqslant n.$$

*If $\mathfrak{s}_\xi$ is self-conjugate and $g > 1$, then*

$$\sum_{k\in\mathbb{Z}} \dim_{\mathbb{F}}(HF^+_{-h(\xi_Y)+1+(2g-2)k}(-Y, \mathfrak{s}_\xi; \mathbb{F})/\operatorname{Im}(U)) \geqslant 2n.$$

**Proof.** The hypothesis holds if and only if the map $G : HF^+(-Y, \mathbb{F}) \to \underline{HF}^+(-Y_0(K); \mathbb{F}[C_n])$ maps onto the summand in degree $[T\widehat{S}] - 1$. Corollary 4.5 shows that the only contribution to the image of the map $G$ in that degree comes from its restriction to gradings congruent to $-h(\xi_Y) + 1$ modulo $2g - 2$. By Lemma 2.1 and Corollary 4.2 the twisted Floer homology of $-Y_0(K)$ in degree $[T\widehat{S}] - 1$ has dimension $n$ over $\mathbb{F}$ and has trivial $U$-action. Since cobordism maps are $U$-equivariant, the first statement follows.

For the strengthened conclusion in the self-conjugate case, begin by noting that by conjugation invariance, our hypotheses imply that the restriction of $F$ to the summand corresponding to $\mathfrak{s}^*_{1-g}$ also vanishes, and hence the map $G$ surjects to the group

$$\underline{HF}^+(-Y_0(K), \mathfrak{s}_{1-g}; \mathbb{F}[C_n]) \oplus \underline{HF}^+(-Y_0(K), \mathfrak{s}^*_{1-g}; \mathbb{F}[C_n]) \cong \mathbb{F}^{2n}.$$

Since $\mathfrak{s}_{1-g}$ and $\mathfrak{s}^*_{1-g}$ are cobordant through $-W$ only to $\mathfrak{s}_\xi$ and $\mathfrak{s}^*_\xi$, respectively, when $\mathfrak{s}_\xi$ is self-conjugate the group $HF^+(-Y, \mathfrak{s}_\xi; \mathbb{F})$ maps onto the group above. We claim only the indicated degrees can map nontrivially. We have seen this already for the component of $G$ mapping into the first factor; for the second, note that from Lemma 4.3, if $M$ is fibered then $HF^+(M, \mathfrak{s}^*_{1-g})$ is supported in degree $[T\widehat{S}]^* - 1$. Observe that if $J$ is an almost-complex structure on $-W : Y_0(K) \to Y$ relating $[T\widehat{S}]_{Y_0}$ to $[\eta]_Y$, then $-J$ relates $[T\widehat{S}]^*$ to $[\eta]^*$. Since $h(\eta) = h(\eta^*)$, the same argument as above then gives that the degrees that can map nontrivially to the second factor above are also congruent to $-h(\xi_Y) + 1$ modulo $2g - 2$. Hence the sum of groups in these degrees must map onto $\mathbb{F}^{2n}$, which gives the result. □

For the next results it is convenient to introduce the notation

$$\mathcal{K}_d(M, \mathfrak{s}) = \dim_{\mathbb{F}}(\ker(U) \cap HF^{+,red}_d(M, \mathfrak{s}; \mathbb{F})) \in \mathbb{Z}$$

and

$$\mathcal{K}^*_d(M, \mathfrak{s}) = \dim_{\mathbb{F}}(HF^+_d(M, \mathfrak{s}; \mathbb{F})/\operatorname{Im}(U)) \in \mathbb{Z}$$

**Corollary 4.7.** *Let $\xi$ be a contact structure on a 3-manifold $Y$, having torsion first Chern class and supported by a genus $g$ open book with connected binding $K$. Let $\xi_n$ denote the contact structure obtained by adding $n$ right Dehn twists along the boundary of the page. Then*

$$n > \sum_{\substack{d=-h(\xi_Y)+1 \\ \bmod\ 2g-2}} \mathcal{K}^*_d(-Y, \mathfrak{s}_\xi) \quad \Longrightarrow \quad \xi_n \text{ is tight.}$$

*If $\mathfrak{s}_\xi$ is self-conjugate, then*

$$2n > \sum_{\substack{d=-h(\xi_Y)+1 \\ \bmod 2g-2}} \mathcal{K}^*_d(-Y, \mathfrak{s}_\xi) \quad \Longrightarrow \quad \xi_n \text{ is tight.}$$

**Proof.** Combine Theorem 4.6 and Proposition 2.4. □

**Proof of Theorem 7.** The statement follows from Corollary 4.7 since the group $HF^+_d(-Y, \mathfrak{s}_\xi; \mathbb{F})/\operatorname{Im}(U)$ appearing in the corollary is a quotient of $HF^{red}_d(-Y, \mathfrak{s}_\xi)$. □

**Corollary 4.8.** *Let $K \subset Y$ be a fibered knot, and let $\tau_K$ be the fractional Dehn twist coefficient of the monodromy of $K$. Then we have an inequality*

$$-1 - \sum_{\substack{d=-h(\xi_Y)+1 \\ \bmod 2g-2}} \mathcal{K}^*_d(-Y, \mathfrak{s}_\xi) \leqslant \tau_K \leqslant 1 + \sum_{\substack{d=-h(\xi_Y) \\ \bmod 2g-2}} \mathcal{K}_d(-Y, \mathfrak{s}_\xi),$$

*where $\xi$ is the contact structure associated to $K$ by the Thurston–Winkelnkemper construction, and we assume $c_1(\mathfrak{s}_\xi)$ is a torsion class.*

*If the* spin$^c$ *structure associated to this contact structure is self-conjugate, then in fact*

$$-1 - \left\lfloor \frac{1}{2} \sum_{\substack{d=-h(\xi_Y)+1 \\ \bmod 2g-2}} \mathcal{K}^*_d(-Y, \mathfrak{s}_\xi) \right\rfloor \leqslant \tau_K \leqslant 1 + \left\lfloor \frac{1}{2} \sum_{\substack{d=-h(\xi_Y) \\ \bmod 2g-2}} \mathcal{K}_d(-Y, \mathfrak{s}_\xi) \right\rfloor.$$

**Proof.** For the first inequality of the corollary, observe that adding $1+\sum \mathcal{K}^*_d(-Y, \mathfrak{s}_\xi)$ right twists to the monodromy of $K$ produces an open book supporting a tight contact structure by the previous corollary, where the sum is over degrees $d$ congruent to $-h(\xi_Y)+1$ modulo $2g-2$. The new monodromy has twist coefficient $\tau_K + 1 + \sum \mathcal{K}^*_d(-Y, \mathfrak{s}_\xi)$, which must be nonnegative since the supported contact structure is tight [14].

For the upper bound on $\tau_K$, first observe that for any 3-manifold $M$ with spin$^c$ structure $\mathfrak{s}$, there is the relation

$$\mathcal{K}_d(M, \mathfrak{s}) = \mathcal{K}^*_{-d-1}(-M, \mathfrak{s}),$$

which follows quickly from the isomorphism $CF^\pm_d(M, \mathfrak{s}) \cong CF^{-d-2}_\mp(-M, \mathfrak{s})$, where the superscript index indicates cohomology, proved by Ozsváth and Szabó [24], together with the long exact sequence relating the different versions of Floer homology.

Now the fibered knot $K$ induces an open book on $-Y$ with oriented fiber $S$ and monodromy $\phi_K^{-1}$. Letting $\bar{\xi}$ denote the associated positive contact structure on $-Y$, the result just obtained says

$$-\tau_K = \tau(\phi_K^{-1}) \geqslant -1 - \sum \mathcal{K}^*_d(-Y, \mathfrak{s}_{\bar{\xi}}),$$

the sum over degrees congruent modulo $2g-2$ to $-h(\bar{\xi}_{-Y})+1 = h(\bar{\xi}_Y)$. Using the observation above, this gives

$$\tau_K \leqslant 1 + \sum_{d \equiv h(\bar{\xi}_Y)} \mathcal{K}_{-d-1}(-Y, \mathfrak{s}_{\bar{\xi}}) = 1 + \sum_{d \equiv -h(\bar{\xi}_Y)-1} \mathcal{K}_d(-Y, \mathfrak{s}_{\bar{\xi}}).$$

As plane fields on $Y$, the positive contact structure $\xi$ and the now-negative contact structure $\bar{\xi}$ stand in the same relationship as $\xi$ and $\eta$ in Proposition 4.4, so that $h(\bar{\xi}_Y) = h(\xi_Y) - 2g + 1 \equiv h(\xi_Y) - 1$ modulo $2g-2$. The desired upper bound follows from the fact that $\mathfrak{s}_\xi = \mathfrak{s}_{\bar{\xi}}$.

The bound on $\tau_K$ when $\mathfrak{s}_\xi$ is self-conjugate follows similarly, using the stronger conclusion in Corollary 4.7 for this case. □

**Proof of Theorem 6.** By definition, $\mathcal{K}^*_d(-Y, \mathfrak{s}_\xi)$ and $\mathcal{K}_d(-Y, \mathfrak{s}_\xi)$ are the dimensions of quotient- and sub-spaces of $HF^{red}_d(-Y, \mathfrak{s}_\xi)$, respectively. Hence the statement of the theorem follows immediately from the first inequality in Corollary 4.8. □

## 5. Twisted Floer homology and the surgery exact triangle

In this subsection we state and sketch a proof of a general surgery exact triangle relating the (twisted) Floer homology of three 3-manifolds obtained by Dehn filling a single manifold $M$ with torus boundary. The discussion can be viewed as a synthesis and clarification of the literature.

Before stating the theorem, we briefly recall that Heegaard Floer homology of a 3-manifold $Y$ can be defined with coefficients in any $\mathbb{F}[H^1(Y;\mathbb{Z})]$-module, by appealing to standard constructions of homology with twisted coefficients (imported to the setting of Morse homology) and noting that the fundamental group of the configuration space of paths between the Heegaard tori is given by

$$\pi_1(\mathcal{P}(\mathbb{T}_{\boldsymbol{\alpha}}, \mathbb{T}_{\boldsymbol{\beta}}), \mathbf{x}) \cong \pi_2(\mathbf{x}, \mathbf{x}) \cong \mathbb{Z} \oplus H^1(Y;\mathbb{Z}).$$

The totally twisted Floer complex $\underline{CF}^\infty(Y)$ is thus freely generated over $\mathbb{F}[\mathbb{Z} \oplus H^1(Y;\mathbb{Z})] \cong \mathbb{F}[U, U^{-1}] \otimes \mathbb{F}[H^1(Y;\mathbb{Z})]$ by $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. See [27, Section 8]. The $\mathbb{F}[U, U^{-1}]$-module structure coming from the $\mathbb{Z}$ summand in $\pi_2(\mathbf{x}, \mathbf{x})$ gives rise to a filtration (by complexes of $\mathbb{F}[U]$ submodules) of $\underline{CF}^\infty$ with sub-, quotient-, and subquotient complexes $\underline{CF}^-, \underline{CF}^+, \underline{\widehat{CF}}$. We denote the collection of complexes by $\underline{CF}^\circ$. Now let $\Lambda$ denote the Novikov ring (which is a field, in this case)

$$\Lambda := \left\{ \sum_{r \in \mathbb{R}} a_r \cdot t^r \mid a_r \in \mathbb{F}, \text{ and } \{a_r | a_r \neq 0, r < \lambda\} \text{ is finite for all } \lambda \in \mathbb{R} \right\},$$

with multiplication defined on monomials by $a_r t^r \cdot b_s t^s = a_r b_s t^{r+s}$ and extended linearly. A choice of two form $\omega \in \Omega^2(Y;\mathbb{R})$ defines an $\mathbb{F}[H^1(Y)]$-module structure on $\Lambda$, where $\eta \in H^1(Y) \subset H^1(Y;\mathbb{R})$ acts by:

$$\eta(\sum_{r\in\mathbb{R}} a_r t^r) := \sum_{r\in\mathbb{R}} a_r t^{(r+\int_Y \eta\wedge\omega)}.$$

Viewed as an $\mathbb{F}[H^1(Y)]$-module in this way, we denote the Novikov ring by $\Lambda_\omega$. See [1] for a nice discussion. Similarly, given a closed curve $\gamma \subset Y$ we can define an $\mathbb{F}[H^1(Y)]$-module structure on the group algebra of the cyclic group $C_n = \mathbb{Z}/n\mathbb{Z}$ by:

$$\eta(a_i \cdot \zeta^i) := a_i \cdot \zeta^{i+\eta([\gamma])}, \text{where } \zeta = e^{2\pi i/n} \in \mathbb{F}[C_n],$$

and $[\gamma] \in H_1(Y)$ is the homology class of the curve. When we view $\mathbb{F}[C_n]$ as a module in this way, we may refer to it as $\mathbb{F}[C_n]_\gamma$. These definitions thus allow us to speak of Floer homology with coefficients in $\Lambda_\omega$ or $\mathbb{F}[C_n]_\gamma$:

$$\underline{CF}^\circ(Y;\Lambda_\omega) := \underline{CF}^\circ(Y)\otimes_{\mathbb{F}[H^1]}\Lambda_\omega, \quad \underline{CF}^\circ(Y;\mathbb{F}[C_n]_\gamma) := \underline{CF}^\circ(Y)\otimes_{\mathbb{F}[H^1]}\mathbb{F}[C_n]_\gamma$$

Given $\omega \in \Omega^2(Y;\mathbb{R})$, or a curve $\gamma \subset Y$, we can also amalgamate the actions above to consider $\Lambda[C_n]$ as an $\mathbb{F}[H^1(Y)]$-module, where the action takes place on $\Lambda$ and $C_n$ independently, as defined above. It will often be more convenient to use concrete models for these chain complexes, which will be described in the course of the proof of the following theorem.

**Theorem 5.1.** *Let $M$ be an oriented 3-manifold with oriented boundary $\partial M = T^2$, and let $\sigma_0, \sigma_1, \sigma_2 \subset T^2$ be a triple of simple closed curves, whose algebraic intersection numbers satisfy (for some choice of orientations)*

$$\#\{\sigma_0 \cap \sigma_1\} = -n, \quad \#\{\sigma_1 \cap \sigma_2\} = \#\{\sigma_2 \cap \sigma_0\} = -1,$$

*where $n > 0$. Then for any 2-form $\omega$ which vanishes on $\partial M$, and for $\mathcal{R} = \mathbb{F}$ or $\Lambda_\omega$, there is a long exact sequence*

$$\begin{array}{ccc} HF^+(M_0;\mathcal{R}) & \longrightarrow & HF^+(M_1;\mathcal{R}), \\ {\scriptstyle F}\nwarrow & & \swarrow{\scriptstyle G} \\ & \underline{HF}^+(M_2;\mathcal{R}[C_n]) & \end{array}$$

*where $M_j$ is the 3-manifold obtained by Dehn filling $M$ with slope $\sigma_j$. The module structure on $\mathcal{R}[C_n]$ is defined by the curve $\sigma_2^*$ obtained as the core of the filling torus and, for $n = 1$, is isomorphic to $\mathcal{R}$. In each case $\Lambda_\omega$ should be interpreted as the module associated with the extension of $\omega$ by zero to a 2-form over the filling solid torus.*

*The maps $G$ and $F$ are related to the maps on twisted Floer homology groups induced by the canonical 2-handle cobordisms between the filled 3-manifolds, and are defined by chain maps in Equations (10) and (11) below.*

**Remark 5.2.** The assumption on the intersection numbers is equivalent to the condition that the slopes satisfy

$$[\sigma_0 + \sigma_1 + n\sigma_2] = 0 \in H_1(\partial M).$$

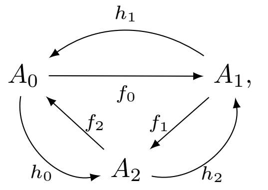
**Remark 5.3.** The theorem also holds with $\mathbb{Z}$ replacing $\mathbb{F}$ throughout, and for the other versions of Floer homology provided that we complete the coefficients with respect to $U$ in the case of minus and infinity.

Before proving the theorem, we discuss a collection of closely related results in the literature. To begin, the theorem with $\mathcal{R} = \mathbb{Z}$ was first proved in [27, Theorem 9.14], in the (not-so) special case that $M = Y \setminus \mathrm{nbd}(K)$ is the complement of a null-homologous knot in a homology sphere, $\sigma_1$ its meridian, and $\sigma_2$ its Seifert longitude. This yields an exact triangle for the Floer homologies of the triple $Y_{1/n}(K), Y, Y_0(K)$ with twisting on the zero surgery term. In the same paper, the case where $\sigma_2$ is the meridian of a null-homologous knot and $\sigma_0$ its Seifert longitude was also addressed, yielding an exact triangle for $Y_0(K), Y_n(K), Y$, with trivially twisted coefficients for the Floer homology of $Y$ (groups which are isomorphic to the direct sum of $n$ copies of the untwisted Floer homology). In both cases, the proof relied on an adaptation of Floer's argument for an exact triangle in instanton homology [5]. In particular, the long exact sequences came from short exact sequences on the chain level. This left the geometric meaning of the connecting homomorphisms unclear. This was remedied for the fractional surgery exact triangle in [24, Section 3.1], where the maps starting and terminating on the twisted term were interpreted in terms of holomorphic triangle counts in a cover of the symmetric product of a Heegaard diagram (the third map, too, was identified with triangle counts, but this fact was already explicit in [27]).

In [31, Theorem 4.5], the exact sequence with $\mathcal{R} = \mathbb{Z}/2\mathbb{Z}$ and $n = 1$ was reproved in such a way to put all of the maps on equal footing. In particular, each map was defined using the same holomorphic triangle counts involved in the definition of the theory's 2-handle cobordism maps; indeed, in the case $n = 1$, consecutive pairs of 3-manifolds in the triangle are manifestly cobordant through a single 2-handle attachment. Each map in the exact sequence is a sum of the maps on Floer homology induced by this cobordism. The key for this approach was a break from Floer's proof of the exact triangle, and the implementation of an "exact triangle detection lemma" [31, Lemma 4.2] (see proof below a statement). In [1, Theorem 3.1], this approach was extended to the case $n = 1$ and $\mathcal{R} = \Lambda_\omega$.

In [34, Theorem 3.1], the case with $\mathcal{R} = \mathbb{Z}$, $M = Y \setminus \mathrm{nbd}(K)$ a null-homologous knot complement, $\sigma_2$ its meridian, and $\sigma_0$ an $m$-framed longitude was treated. This yields an exact triangle between the Floer homology of $Y_m(K)$, $Y_{m+n}(K)$, and $n$ copies of that of $Y$. There, the treatment was again via the exact triangle detection lemma, but the discussion left ambiguous the precise definition of certain maps relevant in the application of the lemma. A theorem equivalent to Theorem 5.1 in the case $\mathcal{R} = \mathbb{Z}$ was stated for the completed minus version of Floer homology [22, Proposition 9.5]. We turn to the proof.

**Proof.** Since many elements of the proof appear in the literature, we will outsource various details to specific references, and focus on issues that are either absent or ambiguous elsewhere. An easily stated version of the exact triangle detection lemma says that if $A_i$ are chain complexes, $f_i$ chain maps, and $h_i$ chain homotopies arranged as:




which satisfy, for each $i \in \{0, 1, 2\}$ (regarded cyclically):

(1) (Null-homotopy) $f_{i+1} \circ f_i = \partial_{i+2} \circ h_i + h_i \circ \partial_i$
(2) (Quasi-isomorphism) $f_{i+2} \circ h_i + h_{i+1} \circ f_i$ induces an isomorphism on homology,

then the maps induced by $f_i$ form a long exact sequence on homology. Chain complexes, maps, and homotopies satisfying these assumptions, and which induce the desired exact triangle are produced from a specific Heegaard quadruple diagram:

$$(\Sigma; \{\boldsymbol{\alpha}, \boldsymbol{\gamma}^0, \boldsymbol{\gamma}^1, \boldsymbol{\gamma}^2\}; \{w, p\}),$$

where

(1) $\Sigma$ is a closed oriented surface of genus $g$.
(2) $\boldsymbol{\gamma}^i = \{\gamma_1^i, \ldots, \gamma_g^i\}$, $i = 0, 1, 2$, are $g$-tuples of simple closed curves in $\Sigma$, arranged so that the first $g - 1$ curves are all small Hamiltonian translates of each other, and so that $\gamma_g^i$ live in a torus connect summand of $\Sigma$ and intersect minimally in the same way as the filling slopes.
(3) $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_g\}$ is a $g$-tuple of homologically independent, pairwise disjoint, simple closed curves in $\Sigma$, transverse to the union of $\boldsymbol{\gamma}^j$.
(4) For each $i = 0, 1, 2$, the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}^i)$ specifies $M_i$.
(5) $w$ is a basepoint in the complement of all curves, and $p$ is a basepoint in $\gamma_g^2$.
(6) The diagram is admissible, in the sense that any multi-periodic domain satisfying $n_w(\mathcal{P}) = 0$ either has at least one negative coefficient or satisfies $\omega([\mathcal{P}]) > 0$, where $\omega$ is the perturbation 2-form.

In terms of this diagram we define chain groups

$$A_i = \begin{cases} \displaystyle\bigoplus_{\mathbf{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}^i}} \mathcal{R}^+ \cdot \mathbf{x} & i = 0, 1, \\ \displaystyle\bigoplus_{\mathbf{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}^i}} \mathcal{R}^+[C_n] \cdot \mathbf{x} & i = 2. \end{cases}$$

Here $\mathcal{R}^+$ is the $\mathcal{R}[U]$ module $\mathcal{R}[U,U^{-1}]/U\cdot\mathcal{R}[U]$. The boundary operators $\partial_i : A_i \to A_i$ are given by:

$$\partial_i(U^{-j}\cdot\mathbf{x}) = \sum_{\substack{\phi\in\pi_2(\mathbf{x},\mathbf{y})\\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}(\phi)\cdot t^{\omega(\phi)}\cdot U^{n_w(\phi)-j}\cdot\mathbf{y}, \quad \text{for } i=0,1,$$

and

$$\partial_2(\zeta^k\cdot U^{-j}\cdot\mathbf{x}) = \sum_{\substack{\phi\in\pi_2(\mathbf{x},\mathbf{y})\\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}(\phi)\cdot t^{\omega(\phi)}\cdot \zeta^{k+n_p(\partial\phi)}\cdot U^{n_w(\phi)-j}\cdot\mathbf{y}.$$

In the case that $\mathcal{R}=\mathbb{F}$, the coefficients can be obtained from the above formulae by setting $t=1$. Here, the exponent of $t$ is given by the evaluation of $\omega$ on the two-chain in $Y$ arising from the domain of the Whitney disk (viewed as a two-chain on $\Sigma$), together with two-chains that cone off its boundary with gradient flowlines to the index one and two critical points of a Morse function on $Y$ specifying the Heegaard diagram. In the $\mathcal{R}[C_n]$ twisted case, we further multiply by the $n$-th root of unity, raised to the algebraic number of times the boundary of the domain crosses the $p$ basepoint. Tracing through the definitions, one see that these complexes compute the Heegaard Floer groups in the theorem:

$$H_*(A_i,\partial_i) \cong HF^+(M_i,\mathcal{R}), \quad i=0,1,$$
$$H_*(A_2,\partial_2) \cong HF^+(M_2,\mathcal{R}[C_n]).$$

(For the $\mathcal{R}[C_n]$ twisted complexes, the key point is that the multiplicity $n_p(\partial\phi)$ equals the intersection number of $\partial\mathcal{D}(\phi)$ with a curve that intersects $\gamma_g^2$ exactly once and no other curves; such a curve is isotopic to the core of the filling solid torus.)

The hypotheses required by the triangle detection lemma will follow from the $A_\infty$ structure present in the Fukaya category of the symmetric product of $\Sigma$, together with the standard nature of the tori $\mathbb{T}_{\boldsymbol{\gamma}^j}$ coming from the collections $\boldsymbol{\gamma}^j$, $j=0,1,2$. Most of the gross features of the argument appear in the aforementioned references (see especially [34, Proof of Theorem 3.1] and [33]). The new technical challenges reside primarily in understanding exactly how the twisting should be incorporated in the definition of the chain maps $f_i$ and homotopies $h_i$, and how these definitions affect algebraic and geometric aspects of the argument. Since these details are particularly relevant to the proof of our main theorem, we will try to provide a thorough treatment.

To begin, we must consider the (twisted) completed minus Floer complexes $\mathbf{CF}^-(\mathbb{T}_{\boldsymbol{\gamma}^i},\mathbb{T}_{\boldsymbol{\gamma}^{i+1}})$ for $i=0,1,2$. This notation seems to be dominant in the literature, but we should note that it differs from [33, Section 2.5] where the complexes are denoted $CF^{--}$. In each case, the complex is freely generated by $\mathbf{x}\in\mathbb{T}_{\boldsymbol{\gamma}^i}\cap\mathbb{T}_{\boldsymbol{\gamma}^{i+1}}$. For $i=0$ the ground ring is $\mathcal{R}[[U]]$ and for $i=1,2$ we use $\mathcal{R}[[U]][C_n]$. The boundary operators are defined as above, with the cases $i=1,2$ accounting for the multiplicity of domains of

Whitney disks at $p \in \gamma_g^2$. The reason to consider power series in $U$ is that there may be infinitely many homotopy classes of Whitney polygons defined by the Heegaard diagram which admit holomorphic representatives. The admissibility conditions placed on our diagram ensure, however, that there are only finitely many such homotopy classes with fixed $n_w(\psi)$. It follows that the polygon counts can be used to define maps between the completed minus (or infinity) groups.

Observe that the 3-manifold specified by $(\Sigma, \mathbb{T}_{\boldsymbol{\gamma}^i}, \mathbb{T}_{\boldsymbol{\gamma}^{i+1}})$ is homeomorphic to $\#^{g-1} S^1 \times S^2$ when $i = 1, 2$, while $(\Sigma, \mathbb{T}_{\boldsymbol{\gamma}^0}, \mathbb{T}_{\boldsymbol{\gamma}^1})$ specifies the connected sum $L(n,1)\#^{g-1} S^1 \times S^2$. Their Floer homologies are given as follows:

$$\mathbf{HF}^-(\mathbb{T}_{\boldsymbol{\gamma}^0}, \mathbb{T}_{\boldsymbol{\gamma}^1}) \cong \mathcal{R}^n \otimes \Lambda^*(\mathcal{R}^{g-1}) \otimes \mathcal{R}[[U]]$$
$$\underline{\mathbf{HF}}^-(\mathbb{T}_{\boldsymbol{\gamma}^i}, \mathbb{T}_{\boldsymbol{\gamma}^{i+1}}) \cong \Lambda^*(\mathcal{R}^{g-1}) \otimes \mathcal{R}[[U]][C_n], \quad \text{for } i = 1, 2.$$

One can compute this directly, or apply the Künneth theorem for the (completed) Floer homology of a connected sum of 3-manifolds. For $i = 1, 2$, the highest graded summand of the Floer group is rank one over $\mathcal{R}[C_n]$, and we denote a generator by $\Theta_{i,i+1}$. For the $i = 0$ case, the $n$ summands correspond to the $n$ different spin$^c$-structures on $L(n,1)$. Picking a particular spin$^c$-structure we obtain a top-dimensional generator for its summand, which we denote $\Theta_{0,1}$. Our choice appears to be specified instead by the particular generator, but could be described more intrinsically in terms of the Chern class of a spin$^c$ structure on the 4-manifold with 3-boundary components determined by the pointed Heegaard triple diagram $(\Sigma; \{\boldsymbol{\alpha}, \boldsymbol{\gamma}^0, \boldsymbol{\gamma}^1, w\})$. From either perspective, we have made a choice of spin$^c$-structure on $L(n,1)$; however, each such choice would produce a (presumably different, in general) exact triangle. Indeed, our particular choice of $\Theta_{0,1}$ can be viewed, even less intrinsically, as the unique one for which the maps that we are about to define satisfy the conditions required by the exact triangle detection lemma. Picking a different spin$^c$ structure on $L(n,1)$ (or different top-dimensional generator) would still result in an exact triangle, but would necessitate modification of the mod $n$ congruences demanded of the $n_p$ multiplicities for its maps.

With all this in mind, we can now define the chain maps and homotopies which serve as input for the exact triangle detection lemma. The chain maps are given as follows:

$$f_0(U^{-j}\mathbf{x}) := \sum_{\substack{\psi \in \pi_2(\mathbf{x}, \Theta_{0,1}, \mathbf{y}) \\ \mu(\psi)=0}} \#\mathcal{M}(\psi) \cdot t^{\omega(\psi)} \cdot U^{n_w(\psi)-j} \cdot \mathbf{y} \tag{9}$$

$$f_1(U^{-j}\mathbf{x}) := \sum_{\substack{\psi \in \pi_2(\mathbf{x}, \Theta_{1,2}, \mathbf{y}) \\ \mu(\psi)=0}} \#\mathcal{M}(\psi) \cdot t^{\omega(\psi)} \cdot \zeta^{n_p(\partial\psi)} \cdot U^{n_w(\psi)-j} \cdot \mathbf{y} \tag{10}$$

$$f_2(U^{-j}\zeta^k\mathbf{x}) := \sum_{\substack{\psi \in \pi_2(\mathbf{x}, \Theta_{2,0}, \mathbf{y}) \\ \mu(\psi)=0 \\ n_p(\partial\psi) = -k \bmod n}} \#\mathcal{M}(\psi) \cdot t^{\omega(\psi)} \cdot U^{n_w(\psi)-j} \cdot \mathbf{y} \tag{11}$$

Note that the map $f_i$ is defined by counting holomorphic triangles with boundary mapping to $\mathbb{T}_{\boldsymbol{\alpha}}, \mathbb{T}_{\boldsymbol{\gamma}^i}, \mathbb{T}_{\boldsymbol{\gamma}^{i+1}}$, and with the vertex that maps into $\mathbb{T}_{\boldsymbol{\gamma}^i} \cap \mathbb{T}_{\boldsymbol{\gamma}^{i+1}}$ sent to our distinguished generator $\Theta_{i,i+1}$. In each case the Novikov ring enters as with the definition of the boundary operators: we simply measure the $\omega$ area of the coned-off domains of the Whitney triangles. The two chains arising from coning Whitney triangles are contained within the four-manifold $X_{\alpha,\gamma^i,\gamma^{i+1}}$ specified by the Heegaard triple diagram via the construction of [28, Section 8], and $\omega$ canonically extends to this four-manifold by our assumption that $\omega|_{\partial M} = 0$. The only difference between the maps, then, is how they incorporate the $C_n$ twisting: $f_0$ makes no use of it; $f_1$ uses it similarly to the boundary operator on $\underline{CF}^+(M_2)$, via the signed crossing number of the boundary of a triangle at the twisting basepoint $p$; $f_2$ incorporates the twisting by requiring triangles counted in the expansion of $f_2(\zeta^k\mathbf{x})$ to have boundary which crosses $p$ *negative* $k$ times (modulo $n$). Verification that these define chain maps is, as usual, a consequence of Gromov compactness together with a homotopy conservation principle; namely, that intersection numbers (in the case of the $U$ action and $C_n$ twisting) and $\omega$ areas (in the case of the Novikov twisting) are homotopy invariants of a class $\psi \in \pi_2(a,b,c)$ which are additive under decomposition of such a class into the juxtaposition of a triangle with a disk.

Similarly, we define homotopy operators using pseudo-holomorphic quadrilateral counts:

$$h_0(U^{-j}\mathbf{x}) := \sum_{\substack{\psi\in\pi_2(\mathbf{x},\Theta_{0,1},\Theta_{1,2},\mathbf{y}) \\ \mu(\psi)=-1}} \#\mathcal{M}(\psi)\cdot t^{\omega(\psi)}\cdot \zeta^{n_p(\partial\psi)}\cdot U^{n_w(\psi)-j}\cdot \mathbf{y} \tag{12}$$

$$h_1(U^{-j}\mathbf{x}) := \sum_{\substack{\psi\in\pi_2(\mathbf{x},\Theta_{1,2},\Theta_{2,0},\mathbf{y}) \\ \mu(\psi)=-1 \\ n_p(\partial\psi)=0 \bmod n}} \#\mathcal{M}(\psi)\cdot t^{\omega(\psi)}\cdot U^{n_w(\psi)-j}\cdot \mathbf{y} \tag{13}$$

$$h_2(U^{-j}\zeta^k\mathbf{x}) := \sum_{\substack{\psi\in\pi_2(\mathbf{x},\Theta_{2,0},\Theta_{0,1},\mathbf{y}) \\ \mu(\psi)=-1 \\ n_p(\partial\psi)=-k \bmod n}} \#\mathcal{M}(\psi)\cdot t^{\omega(\psi)}\cdot U^{n_w(\psi)-j}\cdot \mathbf{y} \tag{14}$$

If we consider one dimensional families of pseudo-holomorphic quadrilateral (arising from $\mu = 0$ homotopy classes) then Gromov compactness, together with additivity of $\omega(\psi), n_w(\psi), n_p(\partial\psi)$ under juxtaposition, implies that $h_i$ provides a homotopy between $f_{i+1} \circ f_i$ and the operator:

$$f_{\alpha,i,i+2}(- \otimes f_{i,i+1,i+2}(\Theta_{i,i+1} \otimes \Theta_{i+1,i+2})),$$

where

$$f_{\alpha,i,i+2} : CF^+(M_i;\mathcal{R}) \otimes \mathbf{CF}^-(\mathbb{T}_{\boldsymbol{\gamma}^i}, \mathbb{T}_{\boldsymbol{\gamma}^{i+2}}) \to CF^+(M_{i+2};\mathcal{R})$$
$$f_{i,i+1,i+2} : \mathbf{CF}^-(\mathbb{T}_{\boldsymbol{\gamma}^i}, \mathbb{T}_{\boldsymbol{\gamma}^{i+1}}) \otimes \mathbf{CF}^-(\mathbb{T}_{\boldsymbol{\gamma}^{i+1}}, \mathbb{T}_{\boldsymbol{\gamma}^{i+2}}) \to \mathbf{CF}^-(\mathbb{T}_{\boldsymbol{\gamma}^i}, \mathbb{T}_{\boldsymbol{\gamma}^{i+2}})$$

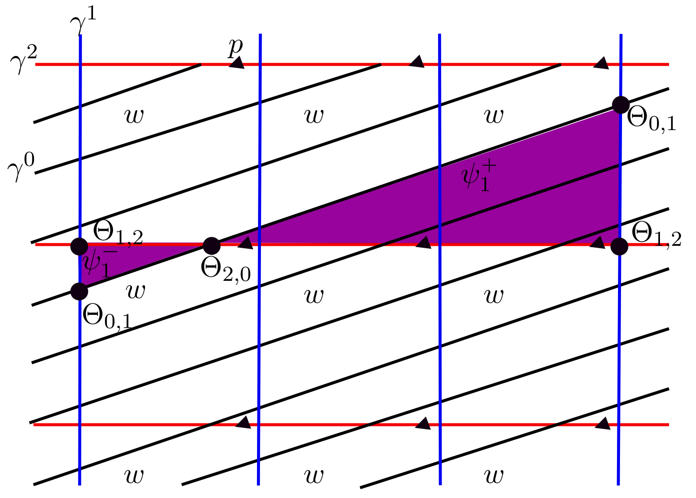



**Fig. 1.** The universal cover of the torus summand of the Heegaard diagram where the filling slopes lie in the case $n = 3$. The black lines of slope 1/3 represent lifts of $\gamma_g^0$, and the blue vertical lines are lifts of $\gamma_g^1$. The red horizontal lines are lifts of $\gamma_g^2$, and contain lifts of the basepoint $p$ which defines the $C_3$ twisting (we represent lifts of $p$ by small black triangles). Shown are the triangles $\psi_1^{\pm}$ with vertices on $\Theta_{i,i+1}$ and $\Theta_{i+1,i+2}$. They satisfy $n_w(\psi_1^{\pm}) = 0$, $n_p(\partial\psi_1^{\pm}) = 0$ mod 3. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

are chain maps defined by counting holomorphic triangles with appropriate boundary conditions (for these latter maps, we have suppressed notation indicating which complexes are $C_n$ twisted, but remind the reader that the complexes for $M_j$ are twisted only when $j = 2$, and the complexes for pairs $\mathbb{T}_{\gamma^i}, \mathbb{T}_{\gamma^j}$ are twisted unless $\{i, j\} = \{0, 1\}$). For all of the maps, homotopies, etc. involved, the key idea to keep in mind is that if the map emanates from a $C_n$ twisted complex, then the holomorphic polygons counted must cross the twisting point $p$ a number of times equal to negative the exponent of the $\zeta$ power appearing in front of the intersection point. Another notable feature is the requirement by $h_1$ that the $\mathbb{T}_{\gamma^2}$ boundary of the rectangles should cross $p$ zero times, modulo $n$. This is actually a convention which is tied to our choice of spin$^c$-structure on $L(n, 1)$ used to determine $\Theta_{0,1}$. Choosing a different spin$^c$-structure would force us to require $n_p(\partial\psi) = m$ mod $n$ for some other value of $m$.

To verify that $h_i$, so defined, is a null-homotopy for $f_{i+1} \circ f_i$, it suffices to show that $f_{i,i+1,i+2}(\Theta_{i,i+1} \otimes \Theta_{i+1,i+2}) = 0$. This is essentially a local calculation in the torus summand of the Heegaard surface where the filling slopes lie, together with a neck stretching argument and similar local considerations for the torus summands where the other $\gamma$ curves lie. See [33, Proposition 2.10] for details on the argument, as applied to the hat theory, and [33, Section 2.5] for its extension to plus. For us, the only difference will be in the torus connect summand of the Heegaard surface where the filling slopes lie and the added bells and whistles that our twisting(s) incorporate. The universal cover of this torus, together with the lifts of the filling slopes, is shown in Fig. 1 in the case where $n = 3$.

The key fact about this region for this part of the argument is that triangles with two vertices on the $g$-th component of $\Theta_{i,i+1}$ and $\Theta_{i+1,i+2}$ and fixed values of $n_w(\psi), \omega(\psi)$ and $n_p(\partial\psi)$ mod $n$ come in canceling pairs. More precisely, for each $k > 0$, there are exactly two triangles, $\psi_k^{\pm}$ with two vertices on the $g$-th component of $\Theta_{i,i+1}$ and $\Theta_{i+1,i+2}$, and these triangles satisfy $n_w(\psi_k^{\pm}) = n \cdot \frac{k(k-1)}{2}$ and $n_p(\partial\psi_k^{\pm}) = 0 \bmod n$. That the triangles cancel comes from the facts that our base rings have characteristic two and that $\omega(\psi_k^+) = \omega(\psi_k^-)$ for all $k$. To see this latter fact, it suffices to observe that the cohomology class determined by $\omega$ on the four-manifold $X_{\gamma^i,\gamma^{i+2},\gamma^{i+2}}$ is trivial, cf. [1, Theorem 3.1, last paragraph of proof].

We now turn to the quasi-isomorphism condition in the triangle detection lemma. For this we consider an augmented Heegaard diagram which, in addition to the four sets of attaching curves previously mentioned, contains an additional $g$-tuple of curves $\tilde{\boldsymbol{\gamma}}^i$ each of which arises via small Hamiltonian perturbation from a corresponding curve in $\boldsymbol{\gamma}^i$ (in particular, the 3-manifold specified by $(\Sigma, \alpha, \tilde{\boldsymbol{\gamma}}^i)$ is homeomorphic to $M_i$). We further require that each curve in $\tilde{\boldsymbol{\gamma}}^i$ intersects the corresponding curve in $\boldsymbol{\gamma}^i$ in exactly two points. There are corresponding complexes, denoted $\tilde{A}_i$, and we consider maps $g_i : A_i \to \tilde{A}_i$, defined by counting pseudo-holomorphic pentagons:

$$g_0(U^{-j}\mathbf{x}) := \sum_{\substack{\psi\in\pi_2(\mathbf{x},\Theta_{0,1},\Theta_{1,2},\widetilde{\Theta}_{2,0},\mathbf{y}) \\ \mu(\psi)=-2 \\ n_p(\partial\psi)=0 \bmod n}} \#\mathcal{M}(\psi)\cdot t^{\omega(\psi)}\cdot U^{n_w(\psi)-j}\cdot\mathbf{y} \tag{15}$$

$$g_1(U^{-j}\mathbf{x}) := \sum_{\substack{\psi\in\pi_2(\mathbf{x},\Theta_{1,2},\Theta_{2,0},\widetilde{\Theta}_{0,1},\mathbf{y}) \\ \mu(\psi)=-2 \\ n_p(\partial\psi)=0 \bmod n}} \#\mathcal{M}(\psi)\cdot t^{\omega(\psi)}\cdot U^{n_w(\psi)-j}\cdot\mathbf{y} \tag{16}$$

$$g_2(U^{-j}\zeta^k\mathbf{x}) := \sum_{\substack{\psi\in\pi_2(\mathbf{x},\Theta_{2,0},\Theta_{0,1},\widetilde{\Theta}_{1,2},\mathbf{y}) \\ \mu(\psi)=-2 \\ n_p(\partial\psi)=-k \bmod n}} \#\mathcal{M}(\psi)\cdot t^{\omega(\psi)}\cdot \zeta^{n_{\tilde{p}}(\partial\psi)}\cdot U^{n_w(\psi)-j}\cdot\mathbf{y}, \tag{17}$$

where $\widetilde{\Theta}_{i,i+1}$ is a top-dimensional generator for the complex associated to the Lagrangians coming from $\boldsymbol{\gamma}^i$ and $\tilde{\boldsymbol{\gamma}}^{i+1}$. Note the appearance of $\tilde{p}$ in the last equation: this is a basepoint on $\tilde{\gamma}_g^2$ which is the image of $p$ under the Hamiltonian isotopy defining $\gamma_g^2$.

Gromov compactness for one dimensional families of pseudo-holomorphic pentagons, applied in this context, implies that such a family will have ten types of ends. Five arise from the non-compactness of the domain coming from the vertices (boundary punctures) of the pentagon. Using the fact that the $\Theta$ intersection points are cycles rules out three of the these ends, and the remaining two give rise to terms of the form $g_i \circ \partial_i + \tilde{\partial}_i \circ g_i$. The other five ends correspond to ends of the moduli space of conformal structures on a pentagon, over which the moduli spaces $\mathcal{M}(\psi)$ fiber. Each of these comes from a conformal degeneration of a pentagon into a rectangle and triangle joined at a vertex. Of these, two give rise to the terms in the sum of compositions $\tilde{f}_{i+2} \circ h_i + \tilde{h}_{i+1} \circ f_i$, where

$\tilde{f}_{i+2}$ and $\tilde{h}_{i+1}$ are defined exactly as in Equations (9)–(14), but with the $\tilde{\gamma}^i$ curves used in place of $\gamma^i$ in the range of the map. Two of the remaining ends involve triangles which contribute to the maps $f_{i,i+1,i+2}(\Theta_{i,i+1} \otimes \Theta_{i+1,i+2})$ and $f_{i+1,i+2,\tilde{i}}(\Theta_{i+1,i+2} \otimes \widetilde{\Theta}_{i+2,i})$, which were previously shown to be zero. The remaining ends contribute to the map

$$q_i(-) := f_{\alpha,i,\tilde{i}}(- \otimes h_{i,i+1,i+2,\tilde{i}}(\Theta_{i,i+1} \otimes \Theta_{i+1,i+2} \otimes \widetilde{\Theta}_{i+2,i})) \tag{18}$$

where $h_{i,i+1,i+2,\tilde{i}}$ is an operator defined by counting holomorphic quadrilaterals. Thus the pentagon operators provide a chain homotopy between $\tilde{f}_{i+2} \circ h_i + \tilde{h}_{i+1} \circ f_i$ and $q_i$. We claim that $q_i$ induces an isomorphism on homology (in fact, it is an isomorphism of chain complexes, but we will not need this). Granting this, we have essentially proved the theorem. The one caveat is that $q$ is not a map from $A_i$ to itself, but to a (quasi-)isomorphic complex $\tilde{A}_i$. The easiest way around this technicality is to tweak the detection lemma to address a family of chain complexes which have three-periodic homology. This is the route taken by [31] and subsequent incarnations. We follow suit, so that our $f, h$, and $g$ maps increase the index (by 1, 2, and 3, respectively) in the family of complexes $\{A_i\}_{i\in\mathbb{Z}}$ which we will show have three-periodic homology via $q_i : A_i \to A_{i+3}$, with $A_{i+3} := \tilde{A}_i$.

Working with this setup, it only remains to show that $q_i$ induces an isomorphism on homology. When $i \neq 2 \bmod 3$ it will suffice to show that

$$\widehat{h}_{i,i+1,i+2,\tilde{i}}(\Theta_{i,i+1} \otimes \Theta_{i+1,i+2} \otimes \Theta_{i+2,\tilde{i}}) = t^\lambda \cdot \Theta_{i,\tilde{i}},$$

for some $\lambda$ (since $t^\lambda$ is a unit in $\Lambda$), and that

$$\widehat{f}_{\alpha,i,\tilde{i}}(- \otimes \Theta_{i,\tilde{i}})$$

induces an isomorphism on homology, where in both cases the "hat" refers to the induced map on the corresponding hat Floer complex (that verification of isomorphism for the hat complex implies it for the plus complex is a consequence of [33, Exercise 1.4]). Verifying the former is essentially the same argument found in [33, Discussion surrounding Equation 15, Figures 8 and 9], the only real difference being the local calculation in the torus region where the filling slopes lie and the implicit Novikov twisting. Indeed, we obtain the factor of $t^\lambda$ in front of $\Theta_{i,\tilde{i}}$, where $\lambda$ is the $\omega$ area of the coned off domain of the unique pseudo-holomorphic quadrilateral with $n_p(\partial\psi) = 0$ modulo $n$ and $n_w(\psi) = 0$.

For the latter, when $i \neq 2 \bmod 3$, one can easily show that that $f_{\alpha,i,\tilde{i}}(- \otimes \Theta_{i,\tilde{i}})$ is an isomorphism by arguing that it agrees, up to higher order terms with respect to the area filtration, with the "closest point" map $\iota$ discussed in [33, Proof of Lemma 2.17]. When Novikov coefficients are used, one needs to be careful if a non-admissible Heegaard diagram is employed: in that case one cannot find an area form which vanishes on all periodic domains, hence the area filtration is not well-defined. The argument still works, however, if the Heegaard diagram is admissible in the weaker sense that $\omega$ evaluates positively on positive multi-periodic domain. For then one can filter the complex using a combination of area and the natural filtration of $\Lambda_\omega$ by powers of $t$.

The cases with $i = 2 \bmod 3$ are somewhat different than the other two. Here the chain maps considered in Equation (18) are defined by holomorphic triangle counts with twisting on both the input and output complexes:

$$\underline{CF}^+(M_2; \mathcal{R}[C_n]) \underset{\mathcal{R}[C_n][[U]]}{\otimes} \underline{\mathbf{CF}}^-(\mathbb{T}_{\boldsymbol{\gamma}^2}, \widetilde{\mathbb{T}_{\boldsymbol{\gamma}^2}}) \xrightarrow{f_{\alpha,2,\tilde{2}}} \underline{CF}^+(M_{\tilde{2}}; \mathcal{R}[C_n])$$

where the twisting on the input is induced by the basepoint $p \subset \gamma_g^2$, and on the output by $\tilde{p} \subset \tilde{\gamma}_g^2$. Note that the complex associated to the pair $\mathbb{T}_{\boldsymbol{\gamma}^2}, \widetilde{\mathbb{T}_{\boldsymbol{\gamma}^2}}$ is twisted by both basepoints, and thus is freely generated over

$$\mathcal{R}[[U]][C_n] \otimes_{\mathcal{R}[[U]]} \mathcal{R}[[U]][C_n].$$

Equivalently, we can think of it as a complex of $\mathcal{R}[[U]][C_n] - \mathcal{R}[[U]][C_n]$ bimodules. The boundary operator is given by

$$\partial(\zeta^i \mathbf{x} \tilde{\zeta}^j) = \sum_{\substack{\phi \in \pi_2(\mathbf{x},\mathbf{y}) \\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}(\phi) \cdot t^{\omega(\phi)} \cdot U^{n_w(\phi)} \cdot \zeta^{i+n_p(\partial\phi)} \mathbf{y} \tilde{\zeta}^{j+n_{\tilde{p}}(\partial\phi)},$$

where we use $\zeta$ (resp. $\tilde{\zeta}$) to record the twisting induced by $p$ (resp. $\tilde{p}$). Its homology, viewed as either a right or left module over $\mathcal{R}[[U]][C_n]$ can easily be computed:

$$\underline{\mathbf{HF}}^-(\mathbb{T}_{\boldsymbol{\gamma}^2}, \widetilde{\mathbb{T}_{\boldsymbol{\gamma}^2}}) \cong (\mathcal{R}[[U]][C_n] \oplus \mathcal{R}[[U]][C_n]) \otimes_{\mathcal{R}} \Lambda^*(\mathcal{R}^{g-1}),$$

where a bimodule generator for the top dimensional summand is given by

$$\Theta_{2,\tilde{2}} = \sum_{i=1}^{n} \zeta^{-i} \theta_+ \tilde{\zeta}^i.$$

Here, $\theta_+$ is the explicit $g$-tuple of intersection points representing the top-graded generator of the chain complex for $\#^g S^1 \times S^2$ coming from the Heegaard diagram $(\Sigma, \boldsymbol{\gamma}^2, \tilde{\boldsymbol{\gamma}}^2)$. Now the map $f_{\alpha,2,\tilde{2}}$ is defined on generators by (we suppress the role of $U$):

$$f_{\alpha,2,\tilde{2}}(\mathbf{x}\zeta^i \otimes \zeta^j \mathbf{y} \tilde{\zeta}^k) := \sum_{\substack{\psi \in \pi_2(\mathbf{x},\mathbf{y},\mathbf{r}) \\ \mu(\psi)=0 \\ n_p(\partial\psi) = -i-j \bmod n}} \#\mathcal{M}(\psi) \cdot t^{\omega(\psi)} \cdot \tilde{\zeta}^{k+n_{\tilde{p}}(\partial\psi)} \cdot \mathbf{r} \qquad (19)$$

We wish to show that the map $q_2$ defined in (18) induces an isomorphism on homology. To do this, we observe

$$\widehat{h}_{2,0,1,\tilde{2}}(\Theta_{2,0} \otimes \Theta_{0,1} \otimes \Theta_{1,\tilde{2}}) = \sum_{i=1}^{n} \zeta^{n-i} \theta_+ \tilde{\zeta}^i = \Theta_{2,\tilde{2}}, \qquad (20)$$

Fig. 2 and its caption explain the first equality, and for the second we use the fact $\zeta^{n-i} = \zeta^{-i}$.

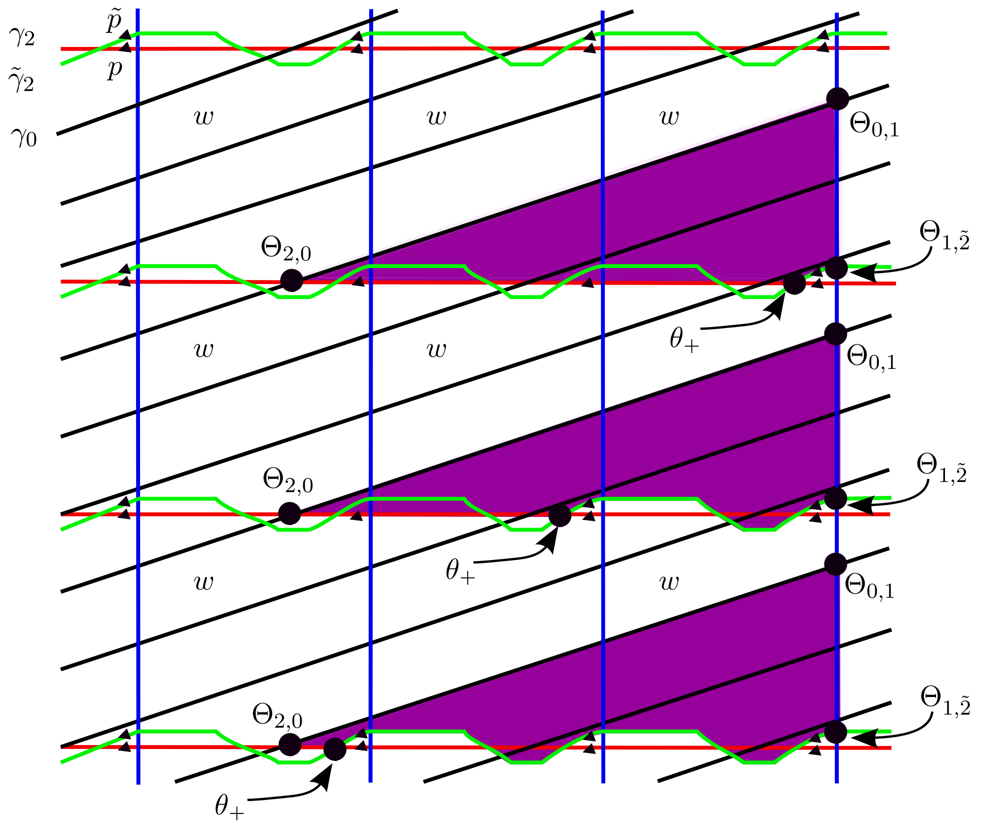



**Fig. 2.** The figure shows the domains of $n = 3$ holomorphic quadrilaterals embedded in the universal cover of the torus summand where the filling slopes lie. These account for the terms in the sum (20). Each can be viewed as a slight perturbation of the triangle $\psi_1^+$ from Fig. 1, and they differ only in which lift of the $g$-th component of $\theta_+$ the boundary of the quadrilateral "jumps" from $\tilde{\gamma}^2$ to $\gamma^2$. This difference affects the values of $n_p(\psi)$ and $n_{\tilde{p}}(\psi)$, giving rise to the different terms in Equation (20). The top, middle, and bottom quadrilaterals give rise to the terms $\zeta^2\theta_+\tilde{\zeta}^1$, $\zeta^1\theta_+\tilde{\zeta}^2$, and $\zeta^0\theta_+\tilde{\zeta}^3$, respectively.

Next we note that

$$\widehat{f}_{\alpha,2,\tilde{2}}(\mathbf{x} \otimes 1\theta_+\tilde{\zeta}^j) = \tilde{\zeta}^j \iota(\mathbf{x}) + \text{lower order terms},$$

where $\iota$ is the closest point map on generators, and lower order is with respect to the area filtration. This follows from the existence of small triangles connecting $\mathbf{x}$ to $\iota(\mathbf{x})$ with third vertex mapping to $\theta_+$ whose boundaries do not cross the basepoints $p, \tilde{p}$. Now consider the restriction of $q_2$ to the hat complex. We have

$$\begin{aligned} \widehat{q}_2(\zeta^j\mathbf{x}) &:= \widehat{f}_{\alpha,2,\tilde{2}}(\mathbf{x}\zeta^j \otimes \Theta_{2,\tilde{2}}) \\ &= \textstyle\sum_{i=1}^n \widehat{f}_{\alpha,2,\tilde{2}}(\mathbf{x}\zeta^j \otimes \zeta^{-i}\theta_+\tilde{\zeta}^i) \\ &= \textstyle\sum_{i=1}^n \widehat{f}_{\alpha,2,\tilde{2}}(\mathbf{x} \quad \otimes \zeta^{j-i}\theta_+\tilde{\zeta}^i) \\ &= \widehat{f}_{\alpha,2,\tilde{2}}(\mathbf{x} \ \otimes 1\theta_+\tilde{\zeta}^j) + \text{lower order terms} \\ &= \tilde{\zeta}^j\iota(\mathbf{x}) + \text{lower order terms}. \end{aligned}$$

Thus $\widehat{q}_2$ is an isomorphism up to lower order terms which implies that it, and $q_2$, induce isomorphisms on homology. □

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