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Noncommutative Polynomials Describing Convex Sets

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Abstract

The free closed semialgebraic set $\mathcal{D}_f$ determined by a hermitian noncommutative polynomial $f \in \mathrm{M}_\delta(\mathbb{C}{<}x, x^*{>})$ is the closure of the connected component of $\{(X, X^*) \mid f(X, X^*) \succ 0\}$ containing the origin. When L is a hermitian monic linear pencil, the free closed semialgebraic set $\mathcal{D}_L$ is the feasible set of the linear matrix inequality $L(X, X^*) \succeq 0$ and is known as a free spectrahedron. Evidently these are convex and it is well known that a free closed semialgebraic set is convex if and only it is a free spectrahedron. The main result of this paper solves the basic problem of determining those f for which $\mathcal{D}_f$ is convex. The solution leads to an efficient algorithm that not only determines if $\mathcal{D}_f$ is convex, but if so, produces a minimal hermitian monic pencil L such that $\mathcal{D}_f = \mathcal{D}_L$. Of independent interest is a subalgorithm based on a Nichtsingulärstellensatz presented here: given a linear pencil $\widetilde{L}$ and a hermitian monic pencil L, it determines if $\widetilde{L}$ takes invertible values on the interior of $\mathcal{D}_L$. Finally, it is shown that if $\mathcal{D}_f$ is convex for an irreducible hermitian $f \in \mathbb{C}{<}x, x^*{>}$, then f has degree at most two, and arises as the Schur complement of an L such that $\mathcal{D}_f = \mathcal{D}_L$.

Keywords Linear matrix inequality (LMI) · Semialgebraic set · Convex set · Spectrahedron · Noncommutative rational function · Free locus

Mathematics Subject Classification Primary 13J30 · 47A56 · 52A05; Secondary 14P10 · 15A22 · 16W10

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1 Introduction

Semidefinite programming (SDP) [52,69] is the main branch of convex optimization to emerge in the last 25 years. Feasibility sets of semidefinite programs are given by linear matrix inequalities (LMIs) and are called spectrahedra. We refer to the book [8] for an overview of the substantial theory of LMIs and spectrahedra and the connection to real algebraic geometry. Spectrahedra are now basic objects in a number of areas of mathematics. They figure prominently in determinantal representations [10,29,54,58,63], in the solution of the Kadison–Singer paving conjecture [51] and the solution of the Lax conjecture [38,50].

One of the main applications of SDP lies in linear systems and control theory [61]. From both empirical observation and the textbook classics, one sees that many problems in this subject are described by signal flow diagrams and naturally convert to inequalities involving polynomials in matrices. These polynomials depend only upon the signal flow diagram and are otherwise independent of either the matrices or their sizes. Thus, many problems in systems and control naturally lead to noncommutative polynomials, or more generally rational functions and matrix inequality conditions. This paper solves the basic problem of identifying those noncommutative rational matrix inequalities that give rise to convex feasibility sets. For expository purposes, the body of the paper presents a detailed proof of this fact for noncommutative polynomials. The modifications needed to handle the more technically challenging case of matrix rational functions are indicated in "Appendix A."

The main results of the article are stated in this introduction. Following a review of basic definitions including that of a free spectrahedron and free semialgebraic set in Sect. 1.1, the three main results are presented in Sect. 1.2 followed by a guide to the paper in Sect. 1.3.

1.1 Definitions

Let $x = (x_1, \dots, x_g)$ denote freely noncommuting variables and $x^* = (x_1^*, \dots, x_g^*)$ their formal adjoints. Let $<x, x^*>$ denote the set of words in x and x^* and $\mathbb{C}<x, x^*>$ the free polynomials in (x, x^*) equal the finite $\mathbb{C}$-linear combinations from $<x, x^*>$. For a positive integer δ, the set of free polynomials with coefficients in $M_\delta(\mathbb{C})$ is denoted $M_\delta(\mathbb{C}<x, x^*>)$ and is naturally identified with the tensor product $M_\delta(\mathbb{C}) \otimes \mathbb{C}<x, x^*>$. The ring $\mathbb{C}<x, x^*>$ has a natural involution $*$ determined by sending the variables x_j to x_j^*, and vice versa, sending scalars to their complex conjugates and $(fg)^* = g^* f^*$ for $f, g \in \mathbb{C}<x, x^*>$. An element $f \in \mathbb{C}<x, x^*>$ is hermitian if $f = f^*$. This involution, and the notion of a hermitian polynomial, naturally extends to $M_\delta(\mathbb{C}<x, x^*>)$.

An element $f \in M_\delta(\mathbb{C}<x, x^*>)$ is a finite sum

$$f = \sum_{w \in <x, x^*>} f_w w \in M_\delta(\mathbb{C}) \otimes \mathbb{C}<x, x^*> = M_\delta(\mathbb{C}<x, x^*>), \tag{1}$$

where $f_w \in \mathrm{M}_\delta(\mathbb{C})$. Given a g-tuple $X = (X_1, \dots, X_g) \in \mathrm{M}_n(\mathbb{C})^g$, a word $w \in <x, x^*>$ is evaluated at (X, X^*) in the natural way, resulting in an $n \times n$ matrix $w(X, X^*)$. The polynomial f of equation (1) is then evaluated at X as

$$f(X, X^*) = \sum_{w \in <x,x^*>} f_w \otimes w(X, X^*) \in \mathrm{M}_\delta(\mathbb{C}) \otimes \mathrm{M}_n(\mathbb{C}) = \mathrm{M}_{n\delta}(\mathbb{C}).$$

It is a standard fact that f is hermitian if and only if $f^*(X, X^*) = f(X, X^*)^*$ for each n and $X \in \mathrm{M}_n(\mathbb{C})^g$.

Affine linear polynomials play a special role. A monic (linear) pencil of size δ is an element L of $\mathrm{M}_\delta(\mathbb{C}<x, x^*>)$ of the form

$$L(x, x^*) = I_\delta - A \odot x - B \odot x^* = I_\delta - \sum_{j=1}^{g} A_j x_j - \sum_{j=1}^{g} B_j x_j^*. \tag{2}$$

In the case $B = A^*$, the pencil L is a hermitian monic (linear) pencil. The associated spectrahedron[1]

$$\mathcal{D}_L(n) = \{(X, X^*) \in \mathrm{M}_n(\mathbb{C})^{2g} : L(X, X^*) \succeq 0\}$$

is a convex semialgebraic set and is the closure of the connected set $\{(X, X^*) \in \mathrm{M}_n(\mathbb{C})^{2g} : L(X, X^*) \succ 0\}$. The union, over n, of $\mathcal{D}_L(n)$ is a free spectrahedron, denoted $\mathcal{D}_L$.

Given $f \in \mathrm{M}_\delta(\mathbb{C}<x, x^*>)$ with $\det f(0) \neq 0$ and a positive integer n, let $\mathcal{K}_f(n)$ denote the closure of the connected component of 0 of

$$\{(X, X^*) \in \mathrm{M}_n(\mathbb{C})^{2g} : \det f(X, X^*) \neq 0\}.$$

The free invertibility set $\mathcal{K}_f$ associated to f is then the union, over n, of the $\mathcal{K}_f(n)$. By replacing f by $f(0)^{-1} f$ we may, and usually do, assume that $f(0) = I$. A free invertibility set $\mathcal{K}_f$ is convex if each $\mathcal{K}_f(n)$ is. If $f = f^*$ is hermitian, then $\mathcal{K}_f$ is a free semialgebraic set that we denote here by $\mathcal{D}_f$. (Letting $g = f^* f$, we see that g is hermitian with $g(0) = I$, and $\mathcal{K}_f = \mathcal{K}_g = \mathcal{D}_g$.) In particular, if L is a hermitian monic pencil, then $\mathcal{D}_L$ is a convex free semialgebraic set. Questions surrounding convexity of free semialgebraic sets arise in applications such as systems engineering and are natural from the point of view of the theories of completely positive maps, operator systems and matrix convex sets [19,56], and quantum information theory [9,33]. It is known, [37,47], that $\mathcal{K}_f$ is convex if and only if there is an hermitian monic pencil L such that $\mathcal{K}_f = \mathcal{D}_L$.

1.2 Main Results

We are now ready to exposit our main results. Using the theory of realizations for noncommutative rational functions [3,7,27,40,65], in Theorem 1.1 we explicitly and

[1] For a square matrix T, the notation $T \succeq 0$ indicates that T is positive semidefinite.

constructively describe the structure of noncommutative matrix polynomials f whose invertibility set $\mathcal{K}_f$ is convex. Each $\delta \times \delta$ noncommutative polynomial or noncommutative rational function r with $r(0) = I$ has a noncommutative Fornasini–Marchesini (FM) realization. Namely, there exists a positive integer d (the size of the realization), a monic linear pencil with coefficients from $\mathrm{M}_d(\mathbb{C})$, and $c, b_1, \ldots, b_{2g} \in \mathrm{M}_{d\times\delta}(\mathbb{C})$ such that

$$r(x, x^*) = I_\delta + c^* L(x, x^*)^{-1}\mathbf{b}, \tag{3}$$

where $\mathbf{b} := \sum_{j=1}^{g}(b_j x_j + b_{g+j}x_j^*)$. A $d \times d$ linear pencil L as in (2) is irreducible if $A_1, \ldots, A_g, B_1, \ldots, B_g$ generate $\mathrm{M}_d(\mathbb{C})$ as a $\mathbb{C}$-algebra. For non-constant r, the FM realization (3) is minimal if L has minimal size amongst all FM realizations of r.[2] Since any two minimal realizations are equivalent up to change of basis (see also Remark 2.6 for details), Theorem 1.1 does not depend upon the choice of minimal realization.

Theorem 1.1 *Let* $f \in \mathrm{M}_\delta(\mathbb{C}<x, x^*>)$ *with* $f(0) = I$. *Let* $f^{-1} = I + c^* L^{-1}\mathbf{b}$ *be a minimal FM realization. After a basis change, we can assume that*

$$L = \begin{pmatrix} L^1 & \star & \star \\ & \ddots & \star \\ & & L^\ell \end{pmatrix}, \tag{4}$$

with each L^i *either irreducible or an identity matrix.*

Let $\widehat{L}$ *be the direct sum of those irreducible blocks* L^i *of* L *that are similar to a hermitian monic pencil, and let* $\widecheck{L}$ *be the direct sum of the remaining* L^j. *Then, the following are equivalent:*

(i) $\mathcal{K}_f$ *is convex;*
(ii) $\mathcal{K}_f$ *is a free spectrahedron;*
(iii) $\mathcal{K}_f = \mathcal{K}_{\widehat{L}}$;
(iv) $\widecheck{L}$ *is invertible on the interior of* $\mathcal{K}_{\widehat{L}}$.

Proof If $\mathcal{K}_f$ is convex, then it is a free spectrahedron (by [37]). Hence, (i) implies (ii). The converse is immediate. The equivalence of items (iii) and (iv) is straightforward. Evidently item (iii) implies (ii). The converse is proved in Sect. 4.1. □

Theorem 1.1 implies that, for a monic linear pencil L, the invertibility set $\mathcal{K}_L$ is convex if and only if the semisimple part of a minimal size pencil L describing $\mathcal{K}_L$ is similar to a hermitian pencil.

A non-invertible element $f \in \mathrm{M}_\delta(\mathbb{C}<x, x^*>)$ with $\det f(0) \neq 0$ is an atom [14, Section 3.2] if it does not factor; that is, it cannot be written as $f_1 f_2$ for non-invertible $f_j \in \mathrm{M}_\delta(\mathbb{C}<x, x^*>)$. Given $f_j \in \mathrm{M}_{\delta_j}(\mathbb{C}<x, x^*>)$ for $1 \le j \le t$, the intersection $\mathcal{K} := \bigcap_j \mathcal{K}_{f_j}$ is irredundant if $\mathcal{K}_{f_j} \not\subseteq \bigcap_{k\neq j}\mathcal{K}_{f_k}$ for all j. Theorem 1.1 yields the following striking result providing further evidence of the rigid nature of convexity for free semialgebraic sets.

[2] It is convenient to declare the size of a minimal realization for constant r to be 0.

Corollary 1.2 *Suppose* $f_j \in \mathrm{M}_{\delta_j \times \delta_j}(\mathbb{C}{<}x, x^*{>})$ *are atoms with* $f_j(0) = I$. *If* $\mathcal{K} := \bigcap_j \mathcal{K}_{f_j}$ *is irredundant, then* $\mathcal{K}$ *is convex if and only if each* $\mathcal{K}_{f_j}$ *is convex.*

Proof See Sect. 4.1. □

Theorem 1.1 leads to algorithms based on semidefinite programming. Note that Part (2) of Corollary 1.3 asserts the existence of an effective version of the main result of [37].

Corollary 1.3 *Let* $f \in \mathrm{M}_\delta(\mathbb{C}{<}x, x^*{>})$ *with* $\det f(0) \neq 0$ *be given. There is an efficient deterministic algorithm based on linear algebra and semidefinite programming (SDP) to:*

(1) *check whether* $\mathcal{K}_f$ *is convex;*
(2) *(in the case* $\mathcal{K}_f$ *is convex) compute a linear matrix inequality (LMI) representation for* $\mathcal{K}_f$*; that is, a hermitian monic pencil* L *(of minimal size) with* $\mathcal{K}_f = \mathcal{D}_L$.

An SDP can be solved up to a given arbitrary precision in polynomial time [53, Section 6.4]. Thus in practice, our algorithm runs in polynomial time. The proof of (2) is based on Theorem 1.1 (see Sect. 4.2), while the proof of (1) in Sect. 4.3 uses (2) and new, of independent interest, (recursive) certificates for invertibility of linear pencils on interiors of free spectrahedra.

Theorem 1.4 (Nichtsingulärstellensatz) *Let* L *be a hermitian monic pencil, and let* $\widetilde{L}$ *be a not necessarily square affine linear matrix polynomial. Consider the set of all matrices* D, C_k, P_0 *such that* $P_0 \succeq 0$ *and*

$$D\widetilde{L} + \widetilde{L}^* D^* = P_0 + \sum_k C_k^* L C_k. \tag{5}$$

(Such certificates can be searched for using semidefinite programming.)

(1) *If the only solutions of (5) have* $P_0 = 0 = C_k$*, then for some* (X, X^*) *in the interior of* $\mathcal{D}_L$*, the matrix* $\widetilde{L}(X, X^*)$ *is rank deficient;*
(2) *Otherwise let* $V = \ker P_0 \cap \bigcap_k \ker C_k$.
    (a) *If* $V = \{0\}$*, then* $\widetilde{L}$ *is full rank on* $\operatorname{int} \mathcal{D}_L$.
    (b) *If* $V \neq \{0\}$*, then* $\widetilde{L}$ *is full rank on* $\operatorname{int} \mathcal{D}_L$ *if and only if* $\widetilde{L}|_V$ *is full rank on* $\operatorname{int} \mathcal{D}_L$ *and the theorem now applies with* $\widetilde{L}$ *replaced by the smaller pencil* $\widetilde{L}|_V$.

Proof See Proposition 4.3, Corollary 4.6 and its proof. □

For the special case of hermitian atoms with $\delta = 1$, the conclusion of Theorem 1.1 can be significantly strengthened as the final main result shows.

Theorem 1.5 *Suppose* $f \in \mathbb{C}{<}x, x^*{>}$ *is a hermitian atom and* $f(0) \succ 0$. *If* $\mathcal{D}_f$ *is proper and convex, then* f *is of degree at most two, is concave and is the Schur complement of any minimal size hermitian monic pencil* L *satisfying* $\mathcal{D}_f = \mathcal{D}_L$.

Proof See Sect. 3. □

Theorem 1.5 settles [17, Conjecture 1.4]. In [68, Theorem 5.4], this result is further extended to the case when f is not necessarily an atom, but $\mathcal{D}_{f+\varepsilon}$ is proper and convex for all small $\varepsilon > 0$.

Noncommutative, or more accurately, freely noncommutative analysis has implications in the commutative setting, particularly for LMIs. Given a hermitian monic pencil L the set $\mathcal{D}_L(1)$, level 1 of the free spectrahedron $\mathcal{D}_L$, consisting of $\xi \in \mathbb{C}^g$ such that $L(\xi, \overline{\xi}) \succeq 0$ is a spectrahedron [64]. Spectrahedra are currently of intense interest in a number of areas; e.g., real algebraic geometry [8,48,62], optimization [20,52,69] and quantum information theory [49,57]. Problems involving free spectrahedra are typically tractable semidefinite programming problems. Thus, elevating a problem involving spectrahedra to its free analog often produces a tractable relaxation. The matrix cube problem of [6,52] is a notable example of these phenomena [32,34]. See also [15,42]. Theorem 1.4 provides another example as it gives a computationally tractable relaxation for the problem of determining whether a polynomial is of constant sign on the interior of a spectrahedron.

1.3 Reader's Guide

Section 2 contains background and some preliminary results on linear pencils, free spectrahedra and realizations of noncommutative rational functions needed in the sequel. The proof of Theorem 1.5 is given in Sect. 3, followed by the proof of Theorem 1.1 and its corollary, Corollary 1.2, in Sect. 4.1. Corollary 1.3 and Theorem 1.4 are proved in the remainder of Sect. 4. Section 4.2 contains an algorithm that, for a given noncommutative polynomial f with convex $\mathcal{K}_f$, constructs a hermitian monic pencil $\widehat{L}$ with $\mathcal{D}_{\widehat{L}} = \mathcal{K}_f$. Indeed, up to similarity, $\widehat{L}$ is extracted from the monic linear pencil L appearing in a minimal FM realization of f^{-1}. Section 4.3 presents an efficient algorithm for checking whether $\mathcal{K}_f$ is convex. It is based on (the proof of) Theorem 1.1 and representation theory and produces a finite sequence of semidefinite programs of decreasing size whose feasibility determines if $\mathcal{K}_f$ is convex. Section 5 presents several illustrative examples establishing optimality of our main results. Further, Sect. 5.3 settles a conjecture from [17] on the degrees of atoms f with convex $\mathcal{K}_f$ in the negative. In Sect. 6, we characterize hermitian monic pencils that can arise in a minimal realization of a noncommutative polynomial; these pencils underpin our constructions in Sect. 5. Finally, Sect. 7 provides a detailed analysis of factorizations of hereditary noncommutative polynomials. As a consequence, an hereditary minimal degree defining polynomial for a free spectrahedron is an atom and hence has degree at most two, see Corollary 7.2.

2 Preliminaries

Let $z = (z_1, \dots, z_g, z_{g+1}, \dots, z_{2g}) = (x_1, \dots, x_g, y_1, \dots, y_g)$ denote $2g$ freely noncommuting variables. Replacing $z_{g+j} = y_j$ with x_j^* identifies $\mathbb{C}\langle z\rangle$ with $\mathbb{C}\langle x, x^*\rangle$. On the other hand, elements $f \in \mathbb{C}\langle z\rangle$ are naturally evaluated at tuples $Z = (X, Y) \in \mathrm{M}_n(\mathbb{C})^g \times \mathrm{M}_n(\mathbb{C})^g = \mathrm{M}_n(\mathbb{C})^{2g}$, whereas we evaluate $f \in \mathbb{C}\langle x, x^*\rangle$

at $(X, X^*) \in \mathrm{M}_n(\mathbb{C})^{2g}$. The use of $\mathbb{C}\langle z\rangle$ versus $\mathbb{C}\langle x, x^*\rangle$ only signals our intent on viewing the domain of f as either $\mathrm{M}_n(\mathbb{C})^{2g}$ or $\{(X, X^*) : X \in \mathrm{M}_n(\mathbb{C})^g\} \subset \mathrm{M}_n(\mathbb{C})^{2g}$ respectively. Indeed, we can identify $\mathbb{C}\langle z\rangle$ with $\mathbb{C}\langle x, x^*\rangle$ whenever we work with attributes of free polynomials that are *per se* independent of evaluations. For example, ring-theoretically there is no difference in using symbols z_{g+j} instead of x_j^* when talking about atomicity of polynomials. Therefore, the results and definitions for matrix polynomials in $z = (z_1, \dots, z_h)$, whose assumptions refer only to the structure, and not to evaluations, of polynomials, directly apply to matrix polynomials in $x_1, \dots, x_g, x_1^*, \dots, x_g^*$.

The free locus $\mathcal{Z}_f$ of $f \in \mathbb{C}\langle z\rangle^{\delta\times\delta}$ is the union, over $n \in \mathbb{N}$, of

$$\mathcal{Z}_f(n) = \left\{(X, Y) \in \mathrm{M}_n(\mathbb{C})^{2g} :\ \det f(X, Y) = 0\right\}.$$

Assuming $\det f(0) \neq 0$, as in the introduction, let $\mathcal{K}_f = \bigcup_n \mathcal{K}_f(n)$, where $\mathcal{K}_f(n)$ is the closure of the connected component of

$$\left\{(X, X^*) \in \mathrm{M}_n(\mathbb{C})^{2g} :\ \det f(X, X^*) \neq 0\right\}$$

containing the origin.

For $A = (A_1, \dots, A_g) \in \mathrm{M}_{d\times e}(\mathbb{C})^g$ and $P \in \mathrm{M}_{e\times\delta}(\mathbb{C})$, we write

$$\begin{aligned} A^* &:= (A_1^*, \dots, A_g^*), \qquad & A \odot x &:= \sum_j^g A_j x_j, \\ AP &:= (A_1 P, \dots, A_g P), \qquad & \ker A &:= \bigcap_j^g \ker A_j. \end{aligned}$$

For a hermitian monic pencil $L = I - A \odot x - A^* \odot x^*$ set $\partial\mathcal{D}_L(n) = \mathcal{Z}_L(n) \cap \mathcal{D}_L(n)$ and

$$\partial\mathcal{D}_L = \bigcup_{n\in\mathbb{N}} \partial\mathcal{D}_L(n).$$

Observe that since $L(0) \succ 0$, it is easy to see that $\partial\mathcal{D}_L(n)$ is precisely the topological boundary of $\mathcal{D}_L(n)$. Furthermore, $\mathcal{D}_L(n)$ is the closure of its interior because of convexity. A non-constant hermitian monic pencil L is minimal if it is of minimal size among hermitian monic pencils L' satisfying $\mathcal{D}_{L'} = \mathcal{D}_L$. If L and M are minimal and $\mathcal{D}_L = \mathcal{D}_M$, then L and M are unitarily equivalent. (See Proposition 2.2.) It is convenient to declare that the minimal pencil for the largest free spectrahedron $\mathcal{D}_I = \{(X, X^*) \colon X \in \mathrm{M}_n(\mathbb{C})^n, n \in \mathbb{N}\}$ is of size 0. Every free semialgebraic set strictly contained in $\mathcal{D}_I$ is called **proper**.

2.1 Free Loci and Spectrahedra

For $h, n \in \mathbb{N}$, let $\Omega^{(n)} = (\Omega_1^{(n)}, \ldots, \Omega_h^{(n)})$ be an h-tuple of $n \times n$ generic matrices, that is,

$$\Omega_j^{(n)} = (\omega_{j\imath\jmath})_{\imath\jmath},$$

where $\omega_{j\imath\jmath}$ for $1 \leq j \leq h$ and $1 \leq \imath, \jmath \leq n$ are commuting indeterminates.

Lemma 2.1 *A linear pencil $L = I - A \odot z$ is irreducible if and only if*

(1) $\ker A = \{0\}$ *and* $\ker A^* = \{0\}$*; and*
(2) $\det L(\Omega^{(n)})$ *is an irreducible polynomial for all n large enough.*

Proof Assume L is irreducible. Thus, the A_j have no common invariant subspace. In particular, $\ker A = \{0\}$ and $\ker A^* = \{0\}$. Thus, (1) holds. The fact that (2) holds is contained in [36, Theorem 3.4].

For the converse implication, assume L is not irreducible. So the A_j have an invariant subspace, and L can be written in block form as

$$L = \begin{pmatrix} L_1 & \star \\ 0 & L_2 \end{pmatrix}.$$

If the coefficients of L_1 are jointly nilpotent, then $\ker A \neq \{0\}$. If the coefficients of L_2 are jointly nilpotent, then $\ker A^* \neq \{0\}$. Otherwise $\det L_i(\Omega^{(n)})$ are non-constant for all large n (cf. Remark 2.6(5) below), and hence

$$\det L(\Omega^{(n)}) = \det L_1(\Omega^{(n)}) \det L_2(\Omega^{(n)})$$

is not irreducible for large n. □

Note that every irreducible hermitian monic pencil is minimal. Two hermitian monic pencils $I_\delta - \sum_{j=1}^g A_j x_j - \sum_{j=1}^g A_j^* x_j^*$ and $I_\delta - \sum_{j=1}^g B_j x_j - \sum_{j=1}^g B_j^* x_j^*$ are unitarily equivalent if there is a unitary matrix U such that $UA_j = B_j U$ for $1 \leq j \leq g$.

Proposition 2.2 *A minimal hermitian monic pencil is an orthogonal direct sum of irredundant irreducible hermitian monic pencils. If L_1 and L_2 are minimal hermitian monic pencils with $\mathcal{D}_{L_1} = \mathcal{D}_{L_2}$, then L_1 and L_2 are unitarily equivalent.*

Proof Let $L = I_\delta - \sum_{j=1}^g A_j x_j - \sum_{j=1}^g A_j^* x_j^*$ be a given hermitian monic pencil. By an invariant subspace for L, we mean an invariant subspace for $\{A_1, \ldots, A_g, A_1^*, \ldots, A_g^*\}$. Since L is hermitian, any invariant subspace for L is in fact reducing. Hence, $L = \oplus_i L^i$, where each L^i is a hermitian monic pencil with no nontrivial invariant (equivalently reducing) subspaces. Thus, each L^i is an irreducible hermitian monic pencil.

If there is an i such that $\mathcal{D}_{L^i} \subseteq \bigcap_{j \neq i} \mathcal{D}_{L^j}$, then, setting $M = \oplus_{j \neq i} L^j$ it follows that $\mathcal{D}_M = \mathcal{D}_L$ and M has smaller size than L. Hence, if L is minimal, then L is irredundant.

The last statement is [32, Theorem 1.2]. See also [15, Section 6]. □

Proposition 2.3 [36, Proposition 8.3] *If L is a minimal hermitian monic pencil, then $\partial\mathcal{D}_L(n)$ is Zariski dense in $\mathcal{Z}_L(n)$ for all n large enough.*

In particular, if f is a polynomial and $\partial\mathcal{D}_L \subseteq \mathcal{Z}_f$, then $\mathcal{Z}_L \subseteq \mathcal{Z}_f$.

Proposition 2.4 *If $f \in \mathrm{M}_\delta(\mathbb{C}{<}z{>})$ and $\det f(0) \neq 0$, then f is an atom if and only if $\det f(\Omega^{(n)})$ is an irreducible polynomial for all n large enough.*

Proof The forward implication is [36, Theorem 4.3(1)]. For the converse, suppose f factors as $f = f_1 f_2$, where the f_i are non-invertible. By Remark 2.6(5), $\det f_i(\Omega^{(n)})$ is non-constant for large n. But then $\det f(\Omega^{(n)})$ is not irreducible for large n. □

Proposition 2.5 *Let $f \in \mathrm{M}_\delta(\mathbb{C}{<}x, x^*{>})$ satisfy $\det f(0) \neq 0$, and let L be a hermitian monic pencil.*

(1) *If $\mathcal{Z}_f = \mathcal{Z}_L$, then $\mathcal{K}_f = \mathcal{D}_L$.*
(2) *If L is minimal and $\mathcal{K}_f = \mathcal{D}_L$, then $\mathcal{Z}_f \supseteq \mathcal{Z}_L$.*
(3) *If f is an atom and L is minimal, then $\mathcal{K}_f = \mathcal{D}_L$ implies $\mathcal{Z}_f = \mathcal{Z}_L$.*

Proof To prove item (1) let (X, X^*) be a point in the connected component $\mathcal{O}$ of

$$\left\{(X, X^*) \in \mathrm{M}_n(\mathbb{C})^{2g} : \det f(X, X^*) \neq 0\right\}$$

containing the origin. Thus, there exists a path γ in $\mathcal{O}$ with $\gamma(0) = 0$ and $\gamma(1) = (X, X^*)$. If $L(X, X^*) \not\succ 0$, then there exists $t \in (0, 1)$ such that $\det L(\gamma(t)) = 0$, contradicting $\mathcal{Z}_f = \mathcal{Z}_L$. Therefore, $L(X, X^*) \succ 0$. A similar argument shows $L(X, X^*) \succ 0$ implies $(X, X^*) \in \mathcal{O}$. Taking closures obtains $\mathcal{K}_f = \mathcal{D}_L$.

Taking up items (2) and (3), suppose L is minimal. If $\mathcal{K}_f = \mathcal{D}_L$, then they have the same topological boundary. Since the topological boundary of $\mathcal{K}_f(n)$ is contained in $\mathcal{Z}_f(n)$ and $\partial\mathcal{D}_L(n)$ is Zariski dense in $\mathcal{Z}_L(n)$ for large n by Proposition 2.3, $\mathcal{Z}_f \supseteq \mathcal{Z}_L$. If also f is atom, then $\mathcal{Z}_f(n)$ is irreducible for large n by Proposition 2.4, and thus, $\mathcal{Z}_f = \mathcal{Z}_L$. □

2.2 Realization Theory

Let $\mathrm{M}_\delta(\mathbb{C}\langle\!\langle z\rangle\!\rangle)$ denote the $\delta \times \delta$ noncommutative (nc) rational functions in $z_1, \ldots, z_h$ [14,41,66]. Evaluations and the involution for polynomials naturally extend to $\mathrm{M}_\delta(\mathbb{C}\langle\!\langle z\rangle\!\rangle)$ and $\mathrm{M}_\delta(\mathbb{C}\langle\!\langle x, x^*\rangle\!\rangle)$, respectively. Both operations are entirely transparent for FM realizations (Eq. (3)), the realizations we use in this paper.

Remark 2.6 Realization theory in general has roots in automata theory [21,22,59,60] and can be traced back further to [44], while FM realizations in the commutative setting arise from control theory [23]. For later use we recall the following fundamental results about minimal FM realizations. Each is an embodiment of a well-understood general principle of realization theory for matrix functions in one (commutative) variable [4]. For the original statements and proofs, see [3,40,65].

(1) An FM realization $I + c^*(I - A \odot z)^{-1}(b \odot z)$ of size d with $c, b_j \in M_{d\times\delta}(\mathbb{C})$ is controllable if

$$\operatorname{span}\{A^w b_j u \colon w \in <z>, 1 \le j \le g, u \in \mathbb{C}^\delta\} = \mathbb{C}^d,$$

and observable if

$$\operatorname{span}\{(A^*)^w c u \colon w \in <z>, u \in \mathbb{C}^\delta\} = \mathbb{C}^d.$$

It is a fundamental result that a realization is minimal if and only if it is observable and controllable. An immediate consequence, and one that is used here, is, for a minimal realization,

$$c^* v = 0 \text{ and } v \in \ker A \quad \Rightarrow \quad v = 0$$

and

$$v^* b = 0 \text{ and } v \in \ker A^* \quad \Rightarrow \quad v = 0.$$

(2) The state space isomorphism theorem says minimal FM realizations are unique up to an isomorphism (change of basis) between their state spaces. That is, if $I + c^*(I - A \odot z)^{-1}(b \odot z)$ and $I + \gamma^*(I - B \odot z)^{-1}(\beta \odot z)$ are two minimal FM realizations for the same rational function, then they have the same size, say d, and there is $d \times d$ invertible matrix S such that $SA = BS$, $Sb = \beta$ and $c^* S^{-1} = \gamma$.
(3) Given a realization $I + c^*(I - A \odot z)^{-1}(b \odot z)$ there is a linear algebra algorithm—an extension of the Kalman decomposition—that produces a minimal realization.
(4) In the classical (commutative) one-variable setting, if $\mathbbm{r}(\zeta) = I + \zeta c^*(I - A\zeta)^{-1} b$ is a minimal FM realization, then the domain of $\mathbbm{r}$ is precisely the set of ζ for which $I - \zeta A$ is invertible. In the present several variable noncommutative setting, while there are some subtleties in the statement of the analogous result found in [40,65], these results do justify calling the complement of $\mathscr{Z}_L$ the domain of regularity of the rational function with minimal realization $\mathbbm{r} = I + c^* L^{-1}\mathbf{b}$.
(5) If $\mathbbm{r} = I + c^* L^{-1}\mathbf{b}$ is a minimal realization, then $\mathbbm{r}$ is a polynomial if and only if the coefficients of L are jointly nilpotent. Indeed, if $\mathbbm{r}$ is a polynomial, then by item (4) , $\mathscr{Z}_L = \varnothing$. By [46, Proposition 3.3], $\mathscr{Z}_L = \varnothing$ if and only the coefficients of L are jointly nilpotent. The converse is immediate.
(6) Lastly, if $\mathbbm{r} = I + c^* L^{-1}\mathbf{b}$ is an FM realization, then

$$\mathbbm{r}^{-1} = I - c^*\big(I - (A - bc^*) \odot z\big)^{-1}\mathbf{b} \tag{6}$$

is an FM realization of $\mathbbm{r}^{-1}$ by [3, Theorem 4.3]. Because the realizations (3) and (6) are of the same size, (3) is minimal for $\mathbbm{r}$ if and only if (6) is minimal for $\mathbbm{r}^{-1}$.

Proposition 2.7 *Let $f \in \mathrm{M}_\delta(\mathbb{C}<z>)$ be non-constant with $f(0) = I$. If $I + c^* L^{-1}\mathbf{b}$ is a minimal FM realization of f^{-1} with $L = I - A \odot z$, then*

(1) $\det f(\Omega^{(n)}) = \det L(\Omega^{(n)})$ *for all n.*

If moreover $\delta = 1$, then

(2) $\ker A^* = \{0\}$ *and* $\ker A = \{0\}$*;*
(3) *L is irreducible if and only if f is an atom.*

Proof (1) By the well-known determinantal identity $\det(M + uv^*) = \det(I + v^* M^{-1} u) \det M$ for an invertible M,

$$\det L(Z)\ \det f(Z)^{-1} = \det\big((L + \mathbf{b}c^*)(Z)\big)$$

for every Z with $\det f(Z) \neq 0$. By Remark 2.6(5) , $N_j := A_j - b_j c^*$, the coefficients of $L + \mathbf{b}c^*$, are the coefficients in a minimal realization of the polynomial f. By Remark 2.6(5) , the N_j are jointly nilpotent. Hence, $\det f(\Omega^{(n)}) = \det L(\Omega^{(n)})$ for all n.

(2) If $0 \neq v \in \ker A$, then

$$N_j v = -(c^* v) b_j,$$

and $c^* v \in \mathbb{C} \setminus \{0\}$ by Remark 2.6(1) . Hence, $b_j \in \operatorname{ran} N_j$. Since the N_j are jointly nilpotent, there exists a nonzero vector u such that $u^* N_j = 0$. Hence, $u^* b_j = 0$. By Remark 2.6(1) , the FM realization Eq. (6) is not minimal, contradicting Remark 2.6(6) .

A similar line of reasoning shows that $\ker A^* = \{0\}$. If $v^* A_j = 0$ and $N_j u = 0$, then $-v^* b_j c^* u = 0$. By minimality, there is a k such that $v^* b_k \neq 0$. Hence, $c^* u = 0$, and thus, $A_j u = 0$, contradicting minimality.

(3) Let f be an atom. By Proposition 2.4, $\det L(\Omega^{(n)}) = \det f(\Omega^{(n)})$ is an irreducible polynomial for all n large enough. Hence, L is irreducible by Lemma 2.1 and (2). Conversely, if L is irreducible, then $\det f(\Omega^{(n)}) = \det L(\Omega^{(n)})$ is an irreducible polynomial for all n large enough by Lemma 2.1. Therefore, f is an atom by Proposition 2.4. □

3 Proof of Theorem 1.5

We start the proof of Theorem 1.5 with a lemma.

Lemma 3.1 *Suppose $\mathbb{r} \in \mathbb{C}\plarr x, x^* \prarr \setminus \mathbb{C}$ is defined at the origin and $\mathbb{r}(0) = 1$. Assume that $\mathbb{r}$ is hermitian and $\mathbb{r} = 1 + c^* L^{-1}\mathbf{b}$ is a minimal FM realization, where $\mathbf{b} = \sum_j \breve{b}_j x_j + \sum_j \widehat{b}_j x_j^*$. If L is irreducible and monic hermitian, say $L = I - A \odot x - A^* \odot x^*$, then there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that*

$$\breve{b}_j = \lambda A_j c \ \ \text{and}\ \ \widehat{b}_j = \lambda A_j^* c \quad \text{for all } j = 1, \ldots, g.$$

Proof Since $\mathbb{r}$ is hermitian, the comparison of formal power series expansions of $1 + c^* L^{-1}\mathbf{b}$ and $1 + \mathbf{b}^* L^{-1} c$ yields

$$c^* A_k\, w(A, A^*)\breve{b}_j = \widehat{b}_k^*\, w(A, A^*)\, A_j c \tag{7}$$

$$c^* A_k^*\, w(A, A^*)\breve{b}_j = \breve{b}_k^*\, w(A, A^*)\, A_j c \tag{8}$$

$$c^* A_k\, w(A, A^*)\,\widehat{b}_j = \widehat{b}_k^*\, w(A, A^*)\, A_j^* c \tag{9}$$

for all $w \in <x, x^*>$ and $1 \le j, k \le g$. Since L is irreducible, the matrices $w(A, A^*)$, for $w \in <x, x^*>$, span $\mathrm{M}_d(\mathbb{C})$. It is easy to see that if $v_1, v_2, v_3, v_4 \in \mathbb{C}^d$ satisfy

$$v_1^* M v_2 = v_3^* M v_4 \qquad \text{for all } M \in \mathrm{M}_d(\mathbb{C}),$$

then v_1 and v_3 are collinear, and v_2 and v_4 are collinear. Hence by (7),(8),(9) and the fact that $w(A, A^*)$ span $\mathrm{M}_d(\mathbb{C})$, there exist $\lambda^1_{jk}, \lambda^2_{jk}, \lambda^3_{jk} \in \mathbb{C}$ such that

$$\breve{b}_j = \lambda^1_{jk} A_j c, \qquad \widehat{b}_k = \overline{\lambda^1_{jk}} A_k^* c$$

$$\breve{b}_j = \lambda^2_{jk} A_j c, \qquad \breve{b}_k = \overline{\lambda^2_{jk}} A_k c \tag{10}$$

$$\widehat{b}_j = \lambda^3_{jk} A_j^* c, \qquad \widehat{b}_k = \overline{\lambda^3_{jk}} A_k^* c \tag{11}$$

for all j, k. By minimality, there exists ℓ such that $\breve{b}_\ell \neq 0$ or $\widehat{b}_\ell \neq 0$. By symmetry, we may assume $\widehat{b}_\ell \neq 0$.

Since $\widehat{b}_\ell \neq 0$, Eq. (10) implies $\lambda := \lambda^1_{j\ell} \neq 0$ is independent of j. It also implies $A_\ell^* c \neq 0$. Likewise, by Eq. (11), $\lambda^3_{j\ell}$ is independent of j and $\lambda^3_{j\ell} = \lambda^1_{j\ell} = \lambda$. By Eq. (11), $\widehat{b}_\ell = \lambda A_\ell^* c = \overline{\lambda} A_\ell^* c$. Thus $\lambda \in \mathbb{R} \setminus \{0\}$. Finally, from Eq. (10), $\breve{b}_j = \lambda A_j c$ and $\widehat{b}_j = \lambda A_j^* c$ as desired. □

Proposition 3.2 *Suppose $f \in \mathbb{C}<x, x^*>$ is a hermitian atom and $f^{-1} = 1 + c^* L^{-1}\mathbf{b}$ is a minimal FM realization. If L is hermitian, then f is concave, has degree at most two and is a Schur complement of L. Further, $f(X, X^*) \succ 0$ if and only if $L(X, X^*) \succ 0$.*

Proof Since L is hermitian, it has the form $L = I - A \odot x - A^* \odot x^*$. Since f is an atom and the realization $f^{-1} = I + c^* L^{-1}\mathbf{b}$ is minimal, L is irreducible by Proposition 2.7(3) . Since f is hermitian, so is f^{-1}. Thus, by Lemma 3.1 we may assume that

$$\mathbf{b} = \varepsilon(A \odot x + A^* \odot x^*)c$$

for $\varepsilon \in \{-1, 1\}$. By Remark 2.6(6) , f admits a minimal realization

$$f = 1 - \varepsilon c^* \Big(I - A(I - \varepsilon cc^*) \odot x - A^*(I - \varepsilon cc^*) \odot x^*\Big)^{-1} (A \odot x + A^* \odot x^*)c. \tag{12}$$

Since f is a polynomial, the $A_j(I-\varepsilon cc^*)$, $A_j^*(I-\varepsilon cc^*)$ are jointly nilpotent by Remark 2.6(5) . In particular, they have a nontrivial common kernel. Since A_j, A_j^* generate $\mathrm{M}_d(\mathbb{C})$, it follows that $P = I-\varepsilon cc^*$ is singular, so in particular $\varepsilon = 1$. Since also P is hermitian and a rank-one perturbation of the identity, it is an orthogonal projection. After a unitary change of basis, we assume that $P = 0 \oplus I_{d-1}$. Let

$$A = \begin{pmatrix} \alpha & v^* \\ u & \widetilde{A} \end{pmatrix}$$

be the decomposition of A with respect to this new basis. Then,

$$AP = \begin{pmatrix} 0 & v^* \\ 0 & \widetilde{A} \end{pmatrix}, \qquad A^*P = \begin{pmatrix} 0 & u^* \\ 0 & \widetilde{A}^* \end{pmatrix}$$

are jointly nilpotent, so $\widetilde{A}, \widetilde{A}^*$ are jointly nilpotent. Hence, $\widetilde{A}_j^*\widetilde{A}_j$ is nilpotent, and thus, $\widetilde{A} = 0$. It follows that AP, A^*P are jointly nilpotent of order at most two and

$$\begin{aligned} &\Big(I - A(I-cc^*)\odot x - A^*(I-cc^*)\odot x^*\Big)^{-1} \\ &= I + A(I-cc^*)\odot x + A^*(I-cc^*)\odot x^*. \end{aligned}$$

Now (12) gives

$$\begin{aligned} f &= 1 - c^*\Big(I + A(I-cc^*)\odot x + A^*(I-cc^*)\odot x^*\Big)\big(A\odot x + A^*\odot x^*\big)c \\ &= 1 - c^*(A\odot x + A^*\odot x^*)c - c^* \\ &\quad (A\odot x + A^*\odot x^*)(I-cc^*)(A\odot x + A^*\odot x^*)c. \end{aligned}$$

Therefore, f has the form

$$f = 1 - (\alpha\odot x + \bar{\alpha}\odot x^*) - (u\odot x + v\odot x^*)^*(u\odot x + v\odot x^*),$$

which is a Schur complement of

$$L = I - \begin{pmatrix} \alpha & v^* \\ u & 0 \end{pmatrix}\odot x - \begin{pmatrix} \bar{\alpha} & u^* \\ v & 0 \end{pmatrix}\odot x^*.$$

In particular, f is concave, has degree at most two and $f(X, X^*) \succ 0$ if and only if $L(X, X^*) \succ 0$. $\square$

Proposition 3.3 *Suppose $f \in \mathbb{C}$<x, x^*> is a hermitian atom with $f(0) = 1$ and L is a minimal hermitian monic pencil of size $d \geq 1$. If $\mathcal{D}_f = \mathcal{D}_L$, then L is irreducible and there exists $b_j, c \in \mathbb{C}^d$ such that $f^{-1} = I + c^*L^{-1}\mathbf{b}$ is a minimal FM realization.*

Proof Write $L = I - A\odot x - A^*\odot x^*$. By Proposition 2.5(3) , $\mathcal{Z}_f = \mathcal{Z}_L$. After a unitary change of basis, we can assume that L equals a direct sum of irreducible hermitian monic pencils $L^1, \dots, L^\ell$. Since L is minimal, the pencils $L^1, \dots, L^\ell$ are pairwise unitarily non-similar by Proposition 2.2. Therefore,

$$\mathcal{Z}_f(n) = \mathcal{Z}_L(n) = \mathcal{Z}_{L^1}(n) \cup \dots \cup \mathcal{Z}_{L^\ell}(n)$$

is a union of ℓ distinct hypersurfaces for large n by Lemma 2.1. Since f is an atom, Proposition 2.7(3) implies $\ell = 1$. Hence, L is irreducible.

Let $f^{-1} = 1 + \tilde{c}^* \widetilde{L}^{-1} \widetilde{\mathbf{b}}$ be a minimal FM realization. Since f is an atom, $\widetilde{L}$ is irreducible by Proposition 2.7(3), and $\mathcal{Z}_{\widetilde{L}} = \mathcal{Z}_f = \mathcal{Z}_L$ by Proposition 2.7(1). By [46, Theorem 3.11], the pencils L and $\widetilde{L}$ are of the same size d and there exists $P \in \mathrm{GL}_d(\mathbb{C})$ such that $\widetilde{L} = P^{-1} L P$. Therefore, f^{-1} admits the minimal FM realization

$$f^{-1} = 1 + c^* L^{-1} \mathbf{b},$$

where $\mathbf{b} = P\widetilde{\mathbf{b}}$ and $c = P^{-*}\tilde{c}$. $\square$

Combining Propositions 3.3 and 3.2, and using the fact that if $\mathcal{D}_f$ is convex, then there is a minimal hermitian monic pencil L such that $\mathcal{D}_f = \mathcal{D}_L$ [37], proves a bit more than claimed in Theorem 1.5.

Corollary 3.4 *Suppose $f \in \mathbb{C}\langle x, x^* \rangle$ is a hermitian atom and $f(0) \succ 0$. If $\mathcal{D}_f$ is proper and convex, then f has degree two and is concave.*

Further, normalizing $f(0) = 1$, if L is a minimal hermitian monic pencil such that $\mathcal{D}_f = \mathcal{D}_L$, then L is irreducible, f is a Schur complement of L and there exist vectors $c, b_1, \dots, b_{2g}$ such that

$$f^{-1} = 1 + c^* L^{-1} \mathbf{b}$$

is a minimal FM realization.

Remark 3.5 The properness in Corollary 3.4 ensures that a minimal hermitian monic pencil for $\mathcal{D}_f$ has size at least 1, so Proposition 3.3 applies. For the description of $f \in \mathbb{C}\langle x, x^* \rangle$ satisfying $f \succ 0$ globally, see [45, Remark 5.1].

Remark 3.6 From the proof of Theorem 1.5, we also obtain a bound on d, the size of L. Since $\widetilde{A} = 0$, the lower right $(d-1) \times (d-1)$ entries in the $\mathbb{C}$-algebra generated by A and A^* are spanned by $S = \{st^* : s, t \in \{u_1, \dots, u_g, v_1, \dots, v_g\}\}$. Since L is irreducible, this span is all of $\mathrm{M}_{d-1}(\mathbb{C})$ and hence $(d-1)^2$ is at most the maximal cardinality of S, namely $(2g)^2$. Hence, $d \le 2g + 1$. $\square$

4 Proof of Theorem 1.1 and Algorithms: Corollary 1.3

In this section, we prove Theorem 1.1 and explore algorithmic consequences. In particular, we present, stated as Corollary 1.3, a constructive version of the main result of [37].

4.1 Proof of Theorem 1.1

It suffices to prove item (ii) implies item (iii). Let L be the pencil appearing in a minimal FM realization for f^{-1}, and let $L^1, \ldots, L^\ell$ be its diagonal blocks as in (4). By Remark 2.6(4) , $\mathcal{K}_f = \mathcal{K}_L$. By assumption there exists a minimal hermitian monic pencil $\widetilde{L}$ such that $\mathcal{K}_L = \mathcal{D}_{\widetilde{L}}$. By $\partial\mathcal{K}_L(n)$, we denote the topological boundary of $\mathcal{K}_L(n)$. Thus,

$$\mathcal{Z}_L(n) \supseteq \partial\mathcal{K}_L(n) = \partial\mathcal{D}_{\widetilde{L}}(n)$$

for every n.

For $S \subseteq \mathrm{M}_n(\mathbb{C})^g$, let $\overline{S}^{\mathrm{Zar}}$ denote its Zariski closure. For n sufficiently large,

$$\mathcal{Z}_L(n) \supseteq \overline{\partial\mathcal{K}_L(n)}^{\mathrm{Zar}} = \overline{\partial\mathcal{D}_{\widetilde{L}}(n)}^{\mathrm{Zar}} = \mathcal{Z}_{\widetilde{L}}(n)$$

by Proposition 2.3. Note that $\mathcal{Z}_L(n)$ and $\mathcal{Z}_{\widetilde{L}}(n)$ are hypersurfaces. Therefore, the set of irreducible components of $\mathcal{Z}_L(n)$ contains the set of irreducible components of $\mathcal{Z}_{\widetilde{L}}(n)$. Since

$$\mathcal{Z}_L = \mathcal{Z}_{L^1} \cup \cdots \cup \mathcal{Z}_{L^\ell}$$

and the $\mathcal{Z}_{L^i}(n)$ are irreducible hypersurfaces for all n large enough by Lemma 2.1, there exist indices $1 \leq i_1 < \cdots < i_s \leq \ell$ such that the L^{i_k} are pairwise non-similar and

$$\overline{\partial\mathcal{K}_L(n)}^{\mathrm{Zar}} = \mathcal{Z}_{\widetilde{L}}(n) = \mathcal{Z}_{L^{i_1}}(n) \cup \cdots \cup \mathcal{Z}_{L^{i_s}}(n) \tag{13}$$

for all n large enough. Since $\widetilde{L}$ is minimal, it is (up to a unitary change of basis) equal to a direct sum of irredundant irreducible hermitian monic pencils $\widetilde{L}^k$ by Proposition 2.2. Each of them corresponds to an irreducible component in (13) by Proposition 2.3. Therefore, $\widetilde{L} = \widetilde{L}^1 \oplus \cdots \oplus \widetilde{L}^s$ and, after reindexing if needed, $\mathcal{Z}_{\widetilde{L}^k} = \mathcal{Z}_{L^{i_k}}$ for $k = 1, \ldots, s$. Then, $\mathcal{K}_{L^{i_k}} = \mathcal{D}_{\widetilde{L}^k}$ is convex for every k and therefore

$$\mathcal{K}_L = \bigcap_k \mathcal{K}_{L^{i_k}} = \bigcap_k \mathcal{D}_{\widetilde{L}^k} = \mathcal{D}_{\widetilde{L}^1 \oplus \cdots \oplus \widetilde{L}^s}. \tag{14}$$

Moreover, L^{i_k} is similar to $\widetilde{L}^k$ by [46, Theorem 3.11].

Recall that $\widehat{L}$ is the direct sum of irreducible blocks L^k that are similar to a hermitian monic pencil, and $\check{L}$ is the direct sum of the rest. Then, every L^{i_k} appears as a direct summand in $\widehat{L}$. Now let L^m be an arbitrary pencil appearing in $\widehat{L}$. If it is not similar to L^{i_k} for any k, then (13) implies

$$\bigcap_k \mathcal{K}_{L^{i_k}} \subseteq \mathcal{K}_{L^m}.$$

Hence, $\mathcal{K}_f = \mathcal{D}_{\widehat{L}}$ holds by (14). □

Remark 4.1 Given a factorization of f into atomic factors $f = f_1 \cdots f_t$ with $f_j(0) = I$, one can use the proof of Theorem 1.1 to identify those factors f_j that determine $\mathcal{K}_f$.

By (13),

$$\mathcal{Z}_{L^{i_1}}(n) \cup \cdots \cup \mathcal{Z}_{L^{i_s}}(n) \subseteq \mathcal{Z}_{f_1}(n) \cup \cdots \cup \mathcal{Z}_{f_t}(n).$$

for all n. Since $\mathcal{Z}_{f_j}(n)$ is an irreducible surface for large n by Proposition 2.4, there exist indices $1 \le j_1 < \cdots < j_s \le t$ such that

$$\mathcal{Z}_{L^{i_k}} = \mathcal{Z}_{f_{j_k}}$$

for all $k = 1, \ldots, s$. Therefore

$$\mathcal{K}_f = \bigcap_k \mathcal{K}_{f_{j_k}}$$

by (14) and Proposition 2.5(1) .

To find the indices j_k, we first compute minimal realizations for $f_j^{-1} = I + c_j L_j^{-1} \mathbf{b}_j$, and put each L_j into a block upper triangular form as in (4). For every j, precisely one of the blocks on the diagonal of L_j is irreducible by Proposition 2.4. Then, we compare these blocks to the pencils L^{i_k} to determine j_k.

Proof of Corollary 1.2 ($\Leftarrow$) is trivial. For the converse, let $f = \prod_i f_i$ and consider a minimal FM realization $f^{-1} = I + c^* L^{-1} \mathbf{b}$. After a basis change, we may assume that L is of the form (4). As in Remark 4.1, for every i there exists j_i such that $\mathcal{Z}_{L^i} = \mathcal{Z}_{f_{j_i}}$, whence $\mathcal{K}_{L^i} = \mathcal{K}_{f_{j_i}}$. If some L^i is not similar to a hermitian monic pencil, then $\widetilde{L}$ is nontrivial and is invertible on int $\mathcal{K}_{\widehat{L}}$ by convexity of $\mathcal{K}$ and Theorem 1.1. Hence, f_{j_i} is redundant, contradicting the assumption. □

4.2 Finding an LMI Representation for a Convex $\mathcal{K}_f$

The main result of [37] states that for a hermitian matrix polynomial $f \in \mathrm{M}_\delta(\mathbb{C}\langle x, x^* \rangle)$ with $f(0) \succ 0$, the set $\mathcal{K}_f(n)$ is convex for all n if and only if $\mathcal{K}_f$ is a free spectrahedron. Actually, the version in [37] does this for hermitian f with bounded $\mathcal{K}_f$. However, these two assumptions are redundant. Indeed, the former can be enforced by replacing f by $f^* f$. The alternative proof of [37, Theorem 1.4] due to Kriel [47] is based on Nash functions in real algebraic geometry and the Fritz–Netzer–Thom characterization [25] of free spectrahedra via operator systems theory. It also works for unbounded $\mathcal{K}_f = \mathcal{D}_{f^* f}$.

4.2.1 Algorithm

We next explain how the machinery developed in this paper produces an explicit minimal LMI representation for a convex $\mathcal{K}_f$. This efficient algorithm only involves linear algebra and semidefinite programming (SDP) [8,69].

(a) Compute the minimal realization

$$I + c^* L^{-1}\mathbf{b}$$

for f^{-1}. To construct this realization, one uses the explicit state-space formulae, c.f. [3, Section 4], for addition and multiplication to construct a realization for f, applying the Kalman decomposition [3, Section 7] at each step to ensure minimality. Lastly, the formula for inversion (6) yields a minimal realization for f^{-1}. This process only uses linear algebra, and minimization after every step keeps the sizes of intermediate realizations from blowing up. *Mathematica* notebooks with rudimentary programs for computing minimal realizations are found in [30].

(b) Next we find the Burnside decomposition [11, Corollary 5.23] of L into

$$L = \begin{pmatrix} L^1 & \star & \star \\ & \ddots & \star \\ & & L^\ell \end{pmatrix},$$

where each L^i is either irreducible or the identity. This decomposition can be found using deterministic algorithms with polynomial time complexity. Let $\mathcal{A}$ be the unital matrix subalgebra generated by the coefficients of L. One first computes and mods out the radical of $\mathcal{A}$ (corresponding to the $\star$ entries) using the algorithm in [16, Section 3]; then, the algorithm of [18, Theorem 3.5] is applied to find the irreducible blocks L^j. Alternatively, [13, Theorem 6] gives an algorithm for decomposing $\mathcal{A}$ as a direct sum of minimal left ideals; after omitting the ideals contained in the radical using a linear test [24, Corollary 4.3], the remaining ideals are necessarily one-dimensional, and the union of bases of ranges of their generators is a basis in which L has the desired block structure.

(c) Considering only the irreducible blocks, choose one from each similarity class. Note that checking similarity of linear pencils amounts to checking whether the system of linear equations $PL^i = L^j P$ has an invertible solution P.

(d) Find all those L^i that are similar to a hermitian monic pencil. This uses SDP. Each solution to the feasibility semidefinite problem

$$Q \succeq I, \qquad Q(L^i)^* = L^i Q \tag{15}$$

leads to a hermitian monic pencil $\widetilde{L}^i = Q^{-\frac{1}{2}} L^i Q^{\frac{1}{2}}$. If (15) is infeasible, then L^i is not similar to a hermitian monic pencil.

(e) The direct sum $\widetilde{L}$ of the hermitian monic pencils $\widetilde{L}^i$ obtained in (d) satisfies

$$\mathcal{D}_{\widetilde{L}} = \mathcal{K}_f$$

by Theorem 1.1.

(f) Using the minimization algorithm described in [32, Subsection 4.6], which uses SDP to eliminate redundant blocks in $\widetilde{L}$, we can produce a minimal hermitian monic pencil $\widehat{L}$ with $\mathcal{D}_{\widehat{L}} = \mathcal{K}_f$.

4.3 Checking Whether $\mathcal{K}_f$ is Convex

As a side product of Theorem 1.1 and the algorithm in Sect. 4.2, we obtain a procedure for checking whether $\mathcal{K}_f$ is convex.

Given $f \in \mathrm{M}_\delta(\mathbb{C}\langle x, x^*\rangle)$ with $f(0) = I$, we construct the realization of f^{-1} and identify its irreducible blocks L^i, choosing one from each similarity class. Let $\widehat{L}$ be the direct sum of all the L^i that are similar to a hermitian monic pencil, and let $\widecheck{L}$ be the direct sum of the others. By Theorem 1.1, it suffices to present an algorithm for checking whether property (iv) of Theorem 1.1 holds, that is, whether $\widecheck{L}$ is invertible on the interior of $\mathcal{D}_{\widehat{L}}$. To this end, we first prove general statements about (rectangular) affine linear pencils being of full rank on the interior of a free spectrahedron (see also [26,45,55,67] for related results).

For the rest of this section, let L be a $d \times d$ hermitian monic pencil, and let $\widetilde{L}$ be a $\delta \times \varepsilon$ affine linear pencil (in x and x^*). Assume $\delta \geq \varepsilon$ and consider the following system:

$$\Re(D\widetilde{L}) = P_0 + \sum_k C_k^* L C_k, \quad P_0 \succeq 0 \tag{16}$$

for some $D \in \mathrm{M}_{\varepsilon\times\delta}(\mathbb{C})$, $C_k \in \mathrm{M}_{d\times\varepsilon}(\mathbb{C})$ and $P_0 \in \mathrm{M}_\varepsilon(\mathbb{C})$, where $\Re(M) = \frac{1}{2}(M + M^*)$ denotes the real part of a square matrix M. (If $\delta < \varepsilon$ we simply replace $\widetilde{L}$ by $\widetilde{L}^*$.) Note that $D = 0$, $P_0 = 0$, $C_k = 0$ is a trivial solution. We mention that (16) is related to the notion of a $\widetilde{L}$-real left module of [35].

Lemma 4.2 *Let $\delta \geq \varepsilon$. If there exists a solution of (16) satisfying*

$$\ker P_0 \cap \bigcap_k \ker C_k = \{0\}, \tag{17}$$

then $\widetilde{L}(X, X^*)$ is full rank for every X satisfying $L(X, X^*) \succ 0$.

Proof Suppose (16) holds and $X \in \mathrm{M}_n(\mathbb{C})^g$ satisfies $L(X, X^*) \succ 0$. If $\Re(D\widetilde{L})(X, X^*)v = 0$ for $v \in \mathbb{C}^{\varepsilon n}$, then (16) together with $P_0 \succeq 0$ and $L(X, X^*) \succ 0$ imply

$$(P_0 \otimes I)v = 0, \quad (C_k \otimes I)v = 0 \quad \text{for all } k.$$

Therefore, $v = 0$ by Eq. (17). Hence, $\Re(D\widetilde{L})(X, X^*)$ is positive definite, so $(D\widetilde{L})(X, X^*)$ is invertible. Consequently, $\widetilde{L}(X, X^*)$ has full rank. □

Proposition 4.3 *Let $\delta \geq \varepsilon$. If every solution of (16) satisfies*

$$P_0 = 0, \quad C_k = 0 \quad \textit{for all } k,$$

then there exists $X \in \mathrm{M}_{\max\{d,\varepsilon\}}(\mathbb{C})^g$ such that $L(X, X^*) \succ 0$ and $\ker \widetilde{L}(X, X^*) \neq \{0\}$.

Before proving Proposition 4.3, we introduce some notation. Let $\eta = \max\{d, \varepsilon\}$. For $\ell = 0, 1, 2,$, let $\mathcal{V}_\ell \subseteq \mathrm{M}_\eta(\mathbb{C}<x, x^*>)$ denote the subspace of polynomials of degree at most ℓ, and let

$$\mathcal{S} = \left\{ \sum_i L_i^* L_i : L_i \in \mathcal{V}_1 \right\},$$
$$\mathcal{C} = \left\{ \sum_k C_k^* L C_k : C_k \in \mathrm{M}_{d\times\eta}(\mathbb{C}) \right\},$$
$$\mathcal{U} = \left\{ \begin{pmatrix} D_1\widetilde{L} + \widetilde{L}^* E_1^* & \widetilde{L}^* E_2^* \\ D_2\widetilde{L} & 0 \end{pmatrix} : D_1, E_1 \in \mathrm{M}_{\varepsilon\times\delta}(\mathbb{C}),\ D_2, E_2 \in \mathrm{M}_{(\eta-\varepsilon)\times\delta}(\mathbb{C}) \right\}.$$

Also let $\mathcal{V}_2^{\mathrm{h}} \subseteq \mathcal{V}_2$ be the $\mathbb{R}$-subspace of hermitian matrix polynomials. Both $\mathcal{C}$ and $\mathcal{S}$ are convex cones in $\mathcal{V}_2^{\mathrm{h}}$, and $\mathcal{U}$ is a subspace in $\mathcal{V}_2$. Observe that

$$\mathcal{U} \cap \mathcal{V}_2^{\mathrm{h}} = \left\{ \begin{pmatrix} (D_1\widetilde{L}) & \widetilde{L}^* D_2^* \\ D_2\widetilde{L} & 0 \end{pmatrix} : D_1 \in \mathrm{M}_{\varepsilon\times\delta}(\mathbb{C}),\ D_2 \in \mathrm{M}_{(\eta-\varepsilon)\times\delta}(\mathbb{C}) \right\}$$

and $\mathcal{U} = (\mathcal{U} \cap \mathcal{V}_2^{\mathrm{h}}) + i(\mathcal{U} \cap \mathcal{V}_2^{\mathrm{h}})$. Using the standard argument involving Caratheodory's theorem on convex hulls [5, Theorem 2.3], it is easy to show that $\mathcal{C} + \mathcal{S}$ is closed in $\mathcal{V}_2^{\mathrm{h}}$; see e.g. [31, Proposition 3.1].

Lemma 4.4 *Keep the notation from above. If every solution of* (16) *satisfies*

$$P_0 = 0, \quad C_k = 0 \ \text{ for all } k,$$

then $\mathcal{U} \cap (\mathcal{C} + \mathcal{S}) = \{0\}$.

Proof Suppose

$$\begin{pmatrix} (D_1\widetilde{L}) & \widetilde{L}^* D_2^* \\ D_2\widetilde{L} & 0 \end{pmatrix} = \sum_i L_i^* L_i + \sum_k \begin{pmatrix} C_k^* \\ C_k'^* \end{pmatrix} L \begin{pmatrix} C_k & C_k' \end{pmatrix} \tag{18}$$

for $D_1 \in \mathrm{M}_{\varepsilon\times\delta}(\mathbb{C})$, $D_2 \in \mathrm{M}_{(\eta-\varepsilon)\times\delta}(\mathbb{C})$, $L_i \in \mathcal{V}_1$, $C_k \in \mathrm{M}_{d\times\varepsilon}(\mathbb{C})$ and $C_k' \in \mathrm{M}_{d\times(\eta-\varepsilon)}(\mathbb{C})$. By looking at the degrees on both sides, we obtain $L_i \in \mathcal{V}_0$; let us write

$$\sum_i L_i^* L_i = \begin{pmatrix} p_1 & p_2 \\ p_2^* & p_3 \end{pmatrix}.$$

Therefore, $(D_1\widetilde{L})$ satisfies (16), so $p_1 = 0$ and $C_k = 0$ by the hypothesis. Moreover, $p_2 = 0$ by positive semidefiniteness. Finally, since L is monic, (18) implies $p_3 = 0$ and $C_k' = 0$. □

To prove Proposition 4.3, we require a version of the Gelfand–Naimark–Segal (GNS) construction. Given a Hilbert space H, let $\mathrm{B}(H)$ denote the (bounded linear) operators on H.

Lemma 4.5 *Suppose $\lambda : \mathcal{V}_2 \to \mathbb{C}$ is a positive linear functional in the sense that $\lambda(f^* f) > 0$ for all $f \in \mathcal{V}_1 \setminus \{0\}$. Thus, the resulting scalar product $\langle f_1, f_2\rangle_\lambda := \lambda(f_2^* f_1)$ on $\mathcal{V}_1$ makes $\mathcal{V}_1$ a Hilbert space and $\mathcal{V}_0 \subseteq \mathcal{V}_1$ is a subspace. Let $\pi : \mathcal{V}_1 \to \mathcal{V}_0 = \mathrm{M}_\eta(\mathbb{C})$ denote the orthogonal projection. For $a \in \mathrm{M}_\eta(\mathbb{C})$, let $\ell_a \in \mathrm{B}(\mathcal{V}_0)$ denote the map $f \mapsto af$, and let $Y_j \in \mathrm{B}(\mathcal{V}_0)$ denote the map $f \mapsto \pi(x_j f)$. Then,*

1. *$\ell_a^* = \ell_{a^*}$;*
2. *$Y_j^* f = \pi(x_j^* f)$;*
3. *$\ell_a Y_j = Y_j \ell_a$ (and hence $\ell_a Y_j^* = Y_j^* \ell_a$);*
4. *there is a unitary mapping $U : \mathbb{C}^\eta \otimes \mathbb{C}^\eta \to \mathcal{V}_0$ such that $U^* \ell_a U = a \otimes I$;*
5. *there exists $X_j \in \mathrm{M}_\eta(\mathbb{C})$ such that $U^* Y_j U = I \otimes X_j$, and if $L = C + \sum_j A_j x_j + \sum_j B_j x_j^*$ is an affine linear pencil of size η, then*

$$UL(X, X^*)U^* = \ell_C + \sum_j \ell_{A_j} Y_j + \sum_j \ell_{B_j} Y_j^*.$$

Proof The proofs of the first three items are straightforward. To prove (4), since $\lambda|_{\mathcal{V}_0}$ is a linear functional on $\mathrm{M}_\eta(\mathbb{C}) = \mathcal{V}_0$, there is a matrix $P \in \mathrm{M}_\eta(\mathbb{C})$ such that $\lambda(f) = \mathrm{tr}(Pf)$. Further, since λ is positive, P is positive definite. Define U by $U(u \otimes v) = uv^{\mathrm{t}} P^{-\frac{1}{2}}$ and extend by linearity. By the definition of $\langle\cdot,\cdot\rangle_\lambda$,

$$\begin{aligned}
\langle U(u_1 \otimes v_1), U(u_2 \otimes v_2)\rangle_\lambda &= \lambda\left((u_2 v_2^{\mathrm{t}} P^{-\frac{1}{2}})^* u_1 v_1^{\mathrm{t}} P^{-\frac{1}{2}}\right) \\
&= \mathrm{tr}\left((u_1 v_1^{\mathrm{t}} P^{-\frac{1}{2}}) P (P^{-\frac{1}{2}} (u_2 v_2^{\mathrm{t}})^*)\right) \\
&= \mathrm{tr}(u_1 v_1^{\mathrm{t}} (v_2^*)^{\mathrm{t}} u_2^*) = \langle u_1, u_2\rangle \langle v_1, v_2\rangle,
\end{aligned}$$

so U is unitary. Similarly, for $a \in \mathrm{M}_\eta(\mathbb{C})$,

$$\begin{aligned}
\langle U^* \ell_a U(u_1 \otimes v_1), (u_2 \otimes v_2)\rangle_\lambda &= \mathrm{tr}\left(((au_1) v_1^{\mathrm{t}} P^{-\frac{1}{2}}) P (P^{-\frac{1}{2}} (u_2 v_2^{\mathrm{t}})^*)\right) \\
&= \langle au_1, u_2\rangle \langle v_1, v_2\rangle \\
&= \langle (a \otimes I)(u_1 \otimes v_1), u_2 \otimes v_2\rangle.
\end{aligned}$$

Since Y_j commutes with each ℓ_a, it follows that $U^* Y_j U$ commutes with each $a \otimes I$. Hence, there is a $X_j \in \mathrm{M}_\eta(\mathbb{C})$ such that $U^* Y_j U = I \otimes X_j$, and hence, $U^* Y_j^* U = I \otimes X_j^*$. Finally, observe that

$$A_j \otimes X_j = (A_j \otimes I)(I \otimes X_j) = U^* \ell_{A_j} Y_j U$$

and analogously $B_j \otimes X_j^* = U^* \ell_{B_j} Y_j^* U$. □

Proof of Proposition 4.3 By Lemma 4.4, $\mathcal{U}\cap(\mathcal{C}+\mathcal{S})=\{0\}$. Since $\mathcal{C}+\mathcal{S}$ is also closed and convex and since $\mathcal{U}$ is a subspace, by [43, Theorem 2.5] there exists an $\mathbb{R}$-linear functional $\lambda_0:\mathcal{V}_2^{\mathrm{h}}\to\mathbb{R}$ satisfying

$$\lambda_0\left((\mathcal{C}+\mathcal{S})\setminus\{0\}\right)=\mathbb{R}_{>0},\qquad \lambda_0(\mathcal{U}\cap\mathcal{V}_2^{\mathrm{h}})=\{0\}.$$

We extend λ_0 to $\lambda:\mathcal{V}_2\to\mathbb{C}$ by

$$\lambda(f)=\lambda_0\left(\frac{1}{2}(f+f^*)\right)+i\lambda_0\left(\frac{1}{2i}(f-f^*)\right).$$

Note that λ vanishes on $\mathcal{U}$. Since $\lambda(\mathcal{S}\setminus\{0\})=\mathbb{R}_{>0}$, λ is a positive functional, so Lemma 4.5 applies; we assume the notation therein.

Write $\widetilde{L}=\widetilde{C}+\sum\widetilde{A}_jx_j+\sum\widetilde{B}_jx_j^*$ for $\widetilde{C},\widetilde{A}_j,\widetilde{B}_j\in\mathrm{M}_{\delta\times\varepsilon}(\mathbb{C})$. For $D\in\mathrm{M}_{\eta\times(\delta+\eta-\varepsilon)}(\mathbb{C})$, let

$$F_D:=U(D\,(\widetilde{L}\oplus I_{\eta-\varepsilon})(X,X^*))U^*=\ell_{D(\widetilde{C}\oplus I)}+\sum_j\ell_{D(\widetilde{A}_j\oplus 0)}Y_j+\sum_j\ell_{D(\widetilde{B}_j\oplus 0)}Y_j^*; \tag{19}$$

the second equality in (19) holds by Lemma 4.5(5). Let u denote $I_\varepsilon\oplus 0\in\mathrm{M}_\eta(\mathbb{C})$ considered as a vector in $\mathcal{V}_0$. Then

$$\begin{aligned}F_Du&=\left(\ell_{D(\widetilde{C}\oplus I)}+\sum_j\ell_{D(\widetilde{A}_j\oplus 0)}Y_j+\sum_j\ell_{D(\widetilde{B}_j\oplus 0)}Y_j^*\right)u\\&=\pi\left(D(\widetilde{C}\oplus I)(I\oplus 0)+\sum_jD(\widetilde{A}_j\oplus 0)(I\oplus 0)x_j+\sum_jD(\widetilde{B}_j\oplus 0)(I\oplus 0)x_j^*\right)\\&=\pi(D(\widetilde{L}\oplus 0)).\end{aligned}$$

Hence for every $f\in\mathcal{V}_0$,

$$\langle F_Du,f\rangle_\lambda=\langle D(\widetilde{L}\oplus 0),f\rangle_\lambda=\lambda(f^*D(\widetilde{L}\oplus 0))=0,$$

since $f^*D(\widetilde{L}\oplus 0)\in\mathcal{U}$. Thus, $F_Du=0$ for all $D\in\mathrm{M}_{\eta\times(\delta+\eta-\varepsilon)}(\mathbb{C})$. Consequently,

$$(\widetilde{L}\oplus I)(X,X^*)U^*u=0$$

and hence $\ker\widetilde{L}(X,X^*)\neq\{0\}$.

Now fix $0\neq v\in\mathcal{V}_0=\mathrm{M}_\eta(\mathbb{C})$ and choose an isometry $V:\mathbb{C}^d\to\mathbb{C}^\eta$ such that $V^*v\neq 0$. If $L=I+\sum_jA_jx_j+\sum_jA_j^*x_j^*$, then

$$U((V\otimes I)L(X,X^*)(V^*\otimes I))U^*=\ell_{VV^*}+\sum_j\ell_{VA_jV^*}Y_j+\sum_j\ell_{VA_j^*V^*}Y_j^*$$

by Lemma 4.5(5), and thus,

$$\langle U((V\otimes I)L(X,X^*)(V^*\otimes I))U^*v, v\rangle_\lambda = \langle \pi(VLV^*v), v\rangle_\lambda = \lambda(v^*VLV^*v) > 0$$

since $v^*VLV^*v \in \mathcal{C}$ is nonzero. It follows that $L(X,X^*)$ is positive definite. □

Corollary 4.6 *Let L be a $d\times d$ hermitian monic pencil. If $\widetilde{L}$ is a $\delta\times\varepsilon$ affine linear pencil such that $\widetilde{L}(X,X^*)$ is full rank for every X in the interior of $\mathcal{D}_L(\max\{d,\delta,\varepsilon\})$, then $\widetilde{L}$ is full rank on the interior of $\mathcal{D}_L$.*

The proof of Corollary 4.6 given below, while not the most efficient, yields an algorithm presented in Sect. 4.3.1.

Proof Without loss of generality, suppose $\delta \geq \varepsilon$ and let $\sigma = \max\{d,\delta\}$.

Given $\eta \leq \delta$ and $\widetilde{L}$, an affine linear pencil of size $\delta\times\eta$ such that $\widetilde{L}(X,X^*)$ is full rank for each X in the interior of $\mathcal{D}_L(\sigma)$, consider solutions to the system (16), i.e.,

$$(D\widetilde{L}) = P_0 + \sum_k C_k^* L C_k, \quad P_0 \succeq 0, \tag{20}$$

and denote $V = \ker P_0 \cap \bigcap_k \ker C_k \subseteq \mathbb{C}^\eta$. If, for each solution, $V = \mathbb{C}^\eta$ (equivalently $P_0 = 0$, $C_k = 0$), then there exists $X \in \mathrm{M}_\sigma(\mathbb{C})^g$ such that $L(X,X^*) \succ 0$ and $\ker \widetilde{L}(X,X^*) \neq \{0\}$ by Proposition 4.3, contradicting the assumption on $\widetilde{L}$. Hence, there is a solution with $\dim(V) < \eta$.

We now argue by induction that, with δ fixed, for each $\eta \leq \delta$ and each $\delta\times\eta$ affine linear pencil L' such that $L'(X,X^*)$ is full rank for every X in the interior of $\mathcal{D}_L(\sigma)$, we have L' is full rank on the interior of $\mathcal{D}_L$.

In the case $\eta = 1$, there is a solution to the system (16) with $0 = \dim(V) < \eta = 1$. By Lemma 4.2, we conclude that $\widetilde{L}$ is full rank on the interior of $\mathcal{D}_L(\sigma)$. Hence, the result holds for $\eta = 1$.

Recall that $\varepsilon \leq \delta$ and suppose the result holds for each $\eta < \varepsilon$. Let $\widetilde{L}$ be a $\delta\times\varepsilon$ affine linear pencil that is full rank on the interior of $\mathcal{D}_L(\sigma)$. As seen above, there is a solution D of (16) with $\eta = \dim(V) < \varepsilon$. In the case $\eta = 0$, just as before, an application of Lemma 4.2 completes the proof. Accordingly, we assume $0 < \eta < \varepsilon$. Let $\widetilde{L}'$ denote the $\delta\times\eta$ pencil whose coefficients are the restrictions of the coefficients of $\widetilde{L}$ to V. Let X satisfy $L(X,X^*) \succ 0$ and suppose $\widetilde{L}(X,X^*)(u+u') = 0$ for $u \in V^\perp$ and $u' \in V$. Thus,

$$(u+u')^*(D\widetilde{L})(X,X^*)(u+u') = 0$$

and hence, by Eq. (20),

$$u^*\left(P_0 + \sum_k C_k^* L C_k\right)u = 0.$$

Thus, $u \in V$ and therefore $u = 0$. Consequently $\widetilde{L}'(X, X^*)u' = \widetilde{L}(X, X^*)u' = 0$. Therefore, for each X in the interior of $\mathcal{D}_L$,

$$\ker \widetilde{L}(X, X^*) \neq \{0\} \iff \ker \widetilde{L}'(X, X^*) \neq \{0\}. \tag{21}$$

In particular, by assumption if X is in the interior of $\mathcal{D}_L(\sigma)$, then $\ker \widetilde{L}(X, X^*) = \{0\}$. Hence, the same is true of $\widetilde{L}'$. By the induction hypothesis, $\widetilde{L}'$ is of full rank on the interior of $\mathcal{D}_L$. Therefore, $\widetilde{L}$ is of full rank on the interior of $\mathcal{D}_L$ by (21). □

4.3.1 Algorithm

Let L be a $d \times d$ hermitian monic pencil and let $\widetilde{L}$ be a $\delta \times \varepsilon$ affine linear pencil. Following the proof of Corollary 4.6, we describe an algorithm for checking whether $\widetilde{L}$ is of full rank on the interior of L.

Step 1. Solve the following feasibility SDP:

$$\begin{aligned} &\operatorname{tr}((D\widetilde{L})(0)) = 1 \\ &(D\widetilde{L}) = P_0 + \sum_k C_k^* L C_k \quad \text{for some } C_k,\, P_0,\ \text{with } P_0 \succeq 0. \end{aligned} \tag{22}$$

We note that (22) is a SDP. Indeed, the first equation is simply a linear constraint, and the second equation can be rewritten as a semidefinite constraint using (localized) moment matrices; see e.g. [12,57] for details.

Step 2. If (22) is infeasible, then $\widetilde{L}(X, X^*)$ is not of full rank for some X in the interior of $\mathcal{D}_L$ by Proposition 4.3.

Step 3. Otherwise, we have a solution with $V := \ker P_0 \cap \bigcap_k \ker C_k \subsetneq \mathbb{C}^\varepsilon$.

Step 3.1 If $V = (0)$, then $\widetilde{L}$ is of full rank on the interior of $\mathcal{D}_L$ by Lemma 4.2.

Step 3.2. If $\varepsilon' = \dim V > 0$, then let $\widetilde{L}'$ be the $\delta \times \varepsilon'$ affine linear pencil whose coefficients are the restrictions of coefficients of $\widetilde{L}$ to V. Then, $\widetilde{L}$ is of full rank on the interior of $\mathcal{D}_L$ if and only if $\widetilde{L}'$ is of full rank on the interior of $\mathcal{D}_L$. Now we apply Step 1 to $\widetilde{L}'$; since $\widetilde{L}'$ is of smaller size than $\widetilde{L}$, the procedure will eventually stop.

5 Examples

We say that a hermitian $f \in \mathbb{C}\langle x, x^*\rangle$ with $f(0) = 1$ is a minimal degree defining polynomial for $\mathcal{D}_f$ if $\deg h \geq \deg f$ for every hermitian $h \in \mathbb{C}\langle x, x^*\rangle$ such that $\mathcal{D}_f = \mathcal{D}_h$. In this section, we present examples of hermitian polynomials f such that $\mathcal{D}_f$ is a free spectrahedron, f is a minimal degree defining polynomial for $\mathcal{D}_f$, and f is of degree more than two. By Theorem 1.5 such an f necessarily factors, even if $\mathcal{D}_f$ corresponds to an irreducible pencil. The construction of such f relies on the following lemma.

Lemma 5.1 *Suppose $f_1, s \in \mathbb{C}\langle x, x^*\rangle$ are atoms and L is a hermitian monic pencil. If*

(1) $s(0) = 1 = f_1(0)$ *and* $\deg f_1 > 2$*;*

(2) *$\mathcal{Z}_{f_1} = \mathcal{Z}_L$, and thus, $\mathcal{K}_{f_1} = \mathcal{D}_L$;*
(3) *s is hermitian;*
(4) *$f_1 s = s f_1^*$;*
(5) *$s(X, X^*) \succ 0$ for all $(X, X^*) \in \mathcal{D}_L$,*

then $f := f_1 s$ is hermitian and $\mathcal{D}_f = \mathcal{D}_L$. Furthermore, a minimal degree defining polynomial for $\mathcal{D}_f$ has degree at least $1 + \deg f_1$.

Proof The polynomial f is hermitian by items (3) and (4), and $\mathcal{D}_f = \mathcal{D}_L$ holds by item (2) and (5). Now let h be an arbitrary hermitian polynomial satisfying $\mathcal{D}_h = \mathcal{D}_f$. Let $\widetilde{L}$ denote a minimal hermitian monic pencil such that $\mathcal{D}_{\widetilde{L}} = \mathcal{D}_L$. By Lemma 2.5(2) $\mathcal{Z}_h \supseteq \mathcal{Z}_{\widetilde{L}}$. Since $\mathcal{K}_{f_1} = \mathcal{D}_{\widetilde{L}}$, f_1 is an atom and $\widetilde{L}$ is minimal, $\mathcal{Z}_{f_1} = \mathcal{Z}_{\widetilde{L}}$. Thus, $\mathcal{Z}_h \supseteq \mathcal{Z}_{f_1}$. Since f_1 is an atom, h has an atomic factor of degree $\deg f_1$ by [36, Theorem 4.3(3)]. Thus, the degree of h exceeds two by item (1). Hence, h is not an atom by Theorem 1.5. It follows that $\deg h \geq 1 + \deg f_1$. $\square$

Remark 5.2 In general, Corollary 1.2 implies that $f \in \mathbb{C}\langle x, x^*\rangle$ with $f(0) \neq 0$ has convex $\mathcal{K}_f$ if and only if it admits a complete factorization $f = s_0 f_1 s_1 \cdots f_\ell s_\ell$, where $\mathcal{K}_{f_k}$ are convex (such f_k are characterized in Sect. 6) and $s_0 \cdots s_\ell$ is invertible on $\bigcap_k \mathcal{K}_{f_k} = \mathcal{K}_f$.

For the rest of this section, let $g = 1$ and $x = x_1$.

5.1 Example of Degree 4

Let

$$f_1 = 1 + x + x^* - 2xx^* - (x + x^*)xx^*, \qquad s = 1 + \frac{1}{2}(x + x^*)$$

and

$$L = \begin{pmatrix} 1 + x + x^* & 0 & x \\ 0 & 1 & x \\ x^* & x^* & 1 \end{pmatrix}.$$

Let us sketch how to verify the assumptions of Lemma 5.1. Clearly, s is an atom and items (1) and (3) of Lemma 5.1 hold. Using standard realization algorithms (e.g. as in [3]), one checks that L appears in a minimal realization of f_1^{-1}. Moreover, a direct computation shows that L is irreducible. Hence, f_1 is an atom by Proposition 2.7(3), and item (2) holds by Proposition 2.7(1). Next, item (4) is straightforward to verify. Finally, for every $(X, X^*) \in \mathcal{D}_L$ we have $I + X + X^* \succeq 0$ and consequently $I + \frac{1}{2}(X + X^*) \succ 0$, so item (5) holds.

By Lemma 5.1, $f = f_1 s$ is hermitian with $\mathcal{D}_f = \mathcal{D}_L$, and f is a minimal degree defining polynomial for $\mathcal{D}_f$ since $\deg f = 4 = \deg f_1 + 1$. Note that

$$\{(X, X^*) \colon f(X, X^*) \succeq 0\} \neq \mathcal{D}_L$$

in this case.

5.2 Example of Degree 5 or 6

Let

$$\begin{aligned} f_1 &= 1-(x+x^*)-2(x+x^*)^2-2x^*x+(x+x^*)^3+2(x+x^*)^2x^*x,\\ s &= 1-(x+x^*)^2 \end{aligned}$$

and

$$L=\begin{pmatrix} 1-\frac{1}{2}(x+x^*) & -\sqrt{2}(x+x^*) & \frac{1}{2}(x+x^*) & x^* \\ -\sqrt{2}(x+x^*) & 1 & 0 & 0 \\ \frac{1}{2}(x+x^*) & 0 & 1-\frac{1}{2}(x+x^*) & -x^* \\ x & 0 & -x & 1 \end{pmatrix}.$$

As in the previous example the only item of Lemma 5.1 that is not simple to verify is (5) . Observe that the upper 2×2 block of L depends only on the hermitian variable $h=x+x^*$. The same holds for $s=1-h^2$. Hence, it suffices to see that $s>0$ on $\mathcal{D}_L(1)$, which is true since

$$\det\begin{pmatrix} 1-\frac{\rho}{2} & -\sqrt{2}\rho \\ -\sqrt{2}\rho & 1 \end{pmatrix}\geq 0 \quad\Longrightarrow\quad 1-\rho^2>0$$

for $\rho\in\mathbb{R}$. If $f=f_1 s$, then $\mathcal{D}_f$ is a free spectrahedron domain whose minimal degree defining polynomial has degree at least 5. Note that $\deg f=6$, but we do not know whether f is a minimal degree defining polynomial.

Of course, by taking a Schur complement of L we obtain a quadratic 2×2 non-commutative polynomial q with $\mathcal{D}_q=\mathcal{D}_L$:

$$q=\begin{pmatrix} 1-\frac{x}{2}-\frac{x^*}{2}-2x^2-2xx^*-3x^*x-2(x^*)^2 & \frac{x}{2}+\frac{x^*}{2}+x^*x \\ \frac{x}{2}+\frac{x^*}{2}+x^*x & 1-\frac{x}{2}-\frac{x^*}{2}-x^*x \end{pmatrix}.$$

5.3 High Degree Atoms with Convex $\mathcal{K}_f$

In the previous two subsections, we obtained atoms f_1 of degree $3,4$ with convex $\mathcal{K}_{f_1}$ in agreement with the degree at most four conclusion of the main result of [17]. Nevertheless, it is easy to construct examples of such polynomials f of high degree.

For example, let

$$\begin{aligned} f &= 1+4(x+x^*)+2(x^2+(x^*)^2)-xx^*-7xx^*(x+x^*)-4x^*x(x+x^*)\\ &\quad -xx^*(x^2+(x^*)^2)+2xx^*(xx^*+x^*x)(x+x^*). \end{aligned}$$

That $\mathcal{K}_f = \mathcal{D}_L$, where

$$L = \begin{pmatrix} 1-x-x^* & x & -x-x^* & x & -x & x+x^* \\ x^* & 1 & 0 & 0 & 0 & 0 \\ -x-x^* & 0 & 1+x+x^* & -x & x & -x-x^* \\ x^* & 0 & -x^* & 1 & 0 & 0 \\ -x^* & 0 & x^* & 0 & 1 & 0 \\ x+x^* & 0 & -x-x^* & 0 & 0 & 1+2x+2x^* \end{pmatrix},$$

can be checked using realization theory.

5.4 Counterexample to a One-Term Positivstellensatz

One might hope that for polynomials whose semialgebraic sets are spectrahedra, there exists a one-term Positivstellensatz (cf. [31, Theorem 1.1]), meaning: if $\mathcal{D}_f = \mathcal{D}_L$ for a hermitian polynomial f with $f(0) > 0$ and a $d \times d$ hermitian monic pencil L, then there exists $W \in \mathrm{M}_{d\times d}(\mathbb{C}{<}x, x^*{>})$ such that

$$I_d \otimes f = f \oplus \cdots \oplus f = W^* L W. \tag{23}$$

We note that such a conclusion holds for f that are real parts of a noncommutative analytic function under natural irreducibility and minimality assumptions on L. For a proof, we refer the gentle reader to [2], where this fact is exploited to characterize bianalytic maps between free spectrahedra. However, with Example 5.1 we shall demonstrate that (23) does not hold in general.

Let us assume the notation of Example 5.1 and suppose there exists $W \in \mathbb{C}{<}x, x^*{>}^{3\times 3}$ such that

$$\begin{matrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{matrix} = W^* L W. \tag{24}$$

Let $\Omega^{(n)}$ and $\Upsilon^{(n)}$ be g-tuples of $n \times n$ generic matrices and consider evaluations of f, W, L at $(\Omega^{(n)}, \Upsilon^{(n)})$. Taking determinants of both sides of (24) gives

$$\left(\det f(\Omega^{(n)}, \Upsilon^{(n)})\right)^3 = \det W^*(\Omega^{(n)}, \Upsilon^{(n)}) \det L(\Omega^{(n)}, \Upsilon^{(n)}) \det W(\Omega^{(n)}, \Upsilon^{(n)}).$$

Since $\det L(\Omega^{(n)}, \Upsilon^{(n)}) = \det f_1(\Omega^{(n)}, \Upsilon^{(n)})$,

$$\left(\det f_1(\Omega^{(n)}, \Upsilon^{(n)})\right)^2 \left(\det s(\Omega^{(n)}, \Upsilon^{(n)})\right)^3 = \det W^*(\Omega^{(n)}, \Upsilon^{(n)}) \det W(\Omega^{(n)}, \Upsilon^{(n)}). \tag{25}$$

Recall that $s = 1+\frac{1}{2}(x+x^*)$, so $p = \det s(\Omega^{(n)}, \Upsilon^{(n)})$ is an irreducible polynomial for all $n \in \mathbb{N}$. Therefore, it divides $\det W^*(\Omega^{(n)}, \Upsilon^{(n)})$ or $\det W(\Omega^{(n)}, \Upsilon^{(n)})$ by (25). But s is a hermitian polynomial, so p divides $\det W^*(\Omega^{(n)}, \Upsilon^{(n)})$ and $\det W(\Omega^{(n)}, \Upsilon^{(n)})$. Therefore, the left-hand side of (25) is divisible by p^3 but not by p^4, while the highest power of p dividing the right-hand side of (25) is even, a contradiction.

5.5 High Degree Matrix atoms Defining Free Spectrahedra

It is fairly easy to produce examples of irreducible hermitian matrix polynomials F of arbitrary high degree such that $\mathcal{D}_F$ is a free spectrahedron. For example, let $p \in \mathrm{M}_\delta(\mathbb{C}<x, x^*>) \setminus \mathrm{M}_\delta(\mathbb{C})$ be arbitrary and let

$$F = \begin{pmatrix} I & 0 & x \\ 0 & I & p \\ x^* & p^* & I + p^* p \end{pmatrix}.$$

Then, $\deg F = 2 \deg p$ and $\det F(\Omega^{(n)}, \Upsilon^{(n)}) = \det(I - \Upsilon^{(n)} \Omega^{(n)})$ is irreducible for all $n \in \mathbb{N}$, so F is an atom. Further, $\mathcal{D}_F = \mathcal{D}_{1-x^*x}$ is a free spectrahedron.

6 Classifying Hermitian Flip-Poly Pencils

A by-product of investigations in earlier sections is a description of hermitian monic flip-poly pencils, which helped us construct Examples 5.1, 5.2 and 5.3. Since it is of independent interest, we present it here in more detail.

A $d \times d$ monic pencil $L = I - A \odot x$ is called flip-poly [36, Section 5.3] if

$$A_j = N_j + v_j u^*$$

where the N_j are jointly nilpotent $d \times d$ matrices and $u, v_j \in \mathbb{C}^d$. Such pencils are important for distinguishing free loci of polynomials among all free loci.

Proposition 6.1 [36, Corollary 5.5] *The set of free loci of polynomials coincides with the set of free loci of flip-poly pencils.*

In this section we further examine the structure of *hermitian* flip-poly pencils. If $L = I - A \odot x - A^* \odot x^*$ is a $d \times d$ flip-poly pencil, then by the definition above there exist jointly nilpotent matrices $N_1, \dots, N_g, \tilde{N}_1, \dots, \tilde{N}_g$ and vectors $u, v_1, \dots, v_g, \tilde{v}_1, \dots, \tilde{v}_g$ such that

$$A_j = N_j + v_j u^*, \qquad A_j^* = \tilde{N}_j + \tilde{v}_j u^*.$$

The following folklore statement is a consequence of Engel's theorem [39, Corollary 3.3] and the Gram–Schmidt process.

Lemma 6.2 *Given jointly nilpotent matrices, there is an orthonormal basis in which they are simultaneously strictly upper triangular.*

After a unitary change of basis (which preserves the hermitian property of L), we can therefore assume that $N_j, \tilde{N}_j$ are strictly upper triangular matrices. For every j,

$$0 = A_j - (A_j^*)^* = N_j + v_j u^* - \tilde{N}_j^* - u \tilde{v}_j^*,$$

or equivalently,

$$N_j - \tilde{N}_j^* = u\tilde{v}_j^* - v_j u^*. \tag{26}$$

On the left-hand side of (26), there is a matrix with diagonal identically 0. Looking at the right-hand side of (26), we then obtain

$$u^k \overline{\tilde{v}_j^k} = \overline{u^k} v_j^k, \tag{27}$$

for every $1 \le j \le g$ and $1 \le k \le d$, where v^k denotes the k^{th} component of v.

Conversely, let $u, v_1, \dots, v_g \in \mathbb{C}^d$ be arbitrary. Next we choose $\tilde{v}_1, \dots, \tilde{v}_g$ that satisfy Eq. (27). Observe that this can always be done: if $u^k \neq 0$, then $\tilde{v}_j^k$ is determined by u^k and v_j^k; and if $u^k = 0$, then we can choose an arbitrary value for $\tilde{v}_j^k$. Then, the matrices $u\tilde{v}_j^* - v_j u^*$ have diagonals identically 0. Hence by declaring N_j to be the strictly upper triangular part of $u\tilde{v}_j^* - v_j u^*$, we obtain matrices $A_j = N_j + v_j u^*$ such that $L = I - A \odot x - A^* \odot x^*$ is flip-poly.

Thus, we derived the following result.

Proposition 6.3 *Let $L = I - A \odot x - A^* \odot x^*$. Then, L is flip-poly if and only if there exist vectors $u, v_1, \dots, v_g$ such that, after a unitary change of coordinates, $A_j = N_j + v_j u^*$, with N_j being the strictly upper triangular part of the matrix $u\tilde{v}_j^* - v_j u^*$, where $\tilde{v}_j$ is a vector satisfying*

$$\tilde{v}_j^k = \frac{u^k \overline{v_j^k}}{\overline{u^k}} \qquad \textit{for } u^k \neq 0.$$

Remark 6.4 Note that vectors $\tilde{v}_j$ in Proposition 6.3 are uniquely determined if all the entries of u are nonzero. Furthermore, if one is only interested in symmetric pencils, i.e., hermitian pencils with real entries, then the form of L can be further simplified when $u \in (\mathbb{R} \setminus \{0\})^d$. Namely, in this case one has $\tilde{v}_j = v_j$ for all j. Moreover, since matrices $uv_j^* - v_j u^*$ are skew-symmetric, it follows by (26) that $A_j = A_j^*$ for all j. Thus in this situation, one has $L = I - A \odot (x + x^*)$ for symmetric A_j; in particular, $\mathcal{D}_L$ is unbounded. Of course, in general not every symmetric flip-poly pencil is of this form, see Examples 5.1 and 5.2. □

7 Hereditary Polynomials

We say that a noncommutative polynomial f is hereditary if it is a linear combination of words uv with $u \in <x^*>$ and $v \in <x>$. Furthermore, f is truly hereditary if it is not analytic or anti-analytic, i.e., $f \notin \mathbb{C}<x> \cup \mathbb{C}<x^*>$. Hereditary polynomials arise naturally in free function theory [28]; they are a tame analog of free real analytic functions. For example, the composite of an analytic polynomial (with no x^*) with an hermitian pencil, a heavily studied class of objects in the geometry of free convex sets (cf. [2]), is hereditary. Similarly, the hereditary functional calculus [1] is a powerful tool in operator theory and complex analysis.

In this section, we prove the following.

Theorem 7.1 *Let f be a hereditary polynomial and $f(0) = 1$. Then, f admits a unique factorization*

$$f = phq, \qquad p(0) = h(0) = q(0) = 1, \tag{28}$$

with p anti-analytic, q analytic, and h a truly hereditary atom or constant. If f is moreover hermitian, then $q = p^*$ and $h = h^*$.

The normalization $p(0) = h(0) = q(0) = 1$ is only required to avoid "uniqueness up to scaling." Before giving a proof of Theorem 7.1, we record the following corollary.

Corollary 7.2 *Any hereditary minimal degree defining polynomial for a free spectrahedron is an atom and hence has degree at most 2.*

Proof Let f be hereditary and minimal degree defining polynomial for $\mathcal{D}_f$, and let $\mathcal{D}_f = \mathcal{D}_L$ for a minimal hermitian monic pencil L. Therefore, $\partial\mathcal{D}_L \subseteq \mathcal{Z}_f$ and hence $\mathcal{Z}_L \subseteq \mathcal{Z}_f$ by Proposition 2.3. Furthermore, after a unitary change of basis we can assume that $L = L^1 \oplus \cdots \oplus L^\ell$, where the L^i are pairwise non-similar irreducible hermitian pencils. Observe that for each i and large enough n, the polynomial $\det L^i(\Omega^{(n)}, \Upsilon^{(n)})$ is irreducible by Proposition 2.4 and cannot be independent of $\Omega^{(n)}$ or $\Upsilon^{(n)}$ by [46, Proposition 3.3], where $\Omega^{(n)}$ and $\Upsilon^{(n)}$ are g-tuples of $n \times n$ generic matrices corresponding to evaluating variables x and x^*, respectively, in an involution-free way.

By Theorem 7.1, $f = a^*ha$, where a is analytic (contains only variables x), and h is a hermitian hereditary atom. Since

$$\mathcal{Z}_{L^1} \cup \cdots \cup \mathcal{Z}_{L^\ell} = \mathcal{Z}_L \subseteq \mathcal{Z}_f = \mathcal{Z}_a \cup \mathcal{Z}_h \cup \mathcal{Z}_{a^*},$$

for each i and large enough n, the irreducible polynomial $\det L^i(\Omega^{(n)}, \Upsilon^{(n)})$ divides one of the polynomials $\det a^*(\Upsilon^{(n)})$, $\det h(\Omega^{(n)}, \Upsilon^{(n)})$ or $\det a(\Omega^{(n)})$. By the previous paragraph, it cannot divide the first one or the last one, so $\mathcal{Z}_{L^i} \subseteq \mathcal{Z}_h$. Since h is an atom it follows that $\mathcal{Z}_{L^i} = \mathcal{Z}_h$. Because the L^i are pairwise non-similar irreducible pencils, we necessarily have $\ell = 1$, so L is irreducible. Therefore, $\mathcal{D}_h = \mathcal{D}_L$ by Proposition 2.5(3) . Thus, h is concave of degree at most two by Theorem 1.5. Finally, since f is of minimal degree, $a = 1$ and $f = h$. □

Corollary 7.3 *If $q \in \mathbb{C}\langle x\rangle$ and $\mathcal{D}_{q+q^*}$ is a free spectrahedron, then* $\deg(q) \leq 1$.

Proof Observe that $q+q^*$ is an atom in $\mathbb{C}\langle x, x^*\rangle$ for every non-constant $q \in \mathbb{C}\langle x\rangle$. Therefore, $q + q^*$ is of degree at most 2 and concave by Theorem 1.5, so

$$q + q^* = \alpha + \ell - \sum_k \ell_k^* \ell_k$$

for some $\alpha > 0$ and linear polynomials $\ell, \ell_k \in \mathbb{C}\langle x, x^*\rangle$. If some ℓ_k is nonzero, then $\ell - \sum_k \ell_k^* \ell_k$ has a term of the form $\alpha x_j x_j^*$ or $\alpha x_j^* x_j$ with $\alpha < 0$. On the other hand, there are no mixed terms in $q + q^*$, so we conclude that $\ell_k = 0$ for all k. Therefore, q is affine linear. □

7.1 Proof of Existence of the Factorization (28)

Lemma 7.4 *Suppose f is hereditary and $f = pq$. If $p \notin \mathbb{C}<x^*>$, then $q \in \mathbb{C}<x>$. If $f = a^*hb$ and $a, b \in \mathbb{C}<x>$, then h is hereditary.*

Proof To prove the first statement, suppose $p \notin \mathbb{C}<x^*>$ and $q \notin \mathbb{C}<x>$. Write, $p = \sum p_\alpha \alpha$ and $q = \sum q_\beta \beta$. There exists a word α' and a j such that α' contains x_j and $p_{\alpha'} \neq 0$; and there is a word β' and a k such that β' contains x_k^* and $q_{\beta'} \neq 0$. Without loss of generality, we may assume that the (total) degrees of α' and β' are maximal with these properties. Now,

$$f = \sum_\gamma \left(\sum_{\alpha\beta=\gamma} p_\alpha q_\beta \right) \gamma.$$

Let $\Gamma = \alpha'\beta'$ and note that this word is not hereditary. Thus,

$$\sum_{\alpha\beta=\Gamma} p_\alpha q_\beta = 0.$$

It follows that there exists words σ and τ such that $(\sigma, \tau) \neq (\alpha', \beta')$, $p_\sigma \neq 0$, $q_\tau \neq 0$ and $\Gamma = \sigma\tau = \alpha'\beta'$. It follows that either α' properly divides σ on the left, in which case σ contains x_j and $|\sigma| > |\alpha'|$, contradicting the choice of α'; or β_m properly divides τ on the right, in which case τ contains x_k^* and $|\tau| > |\beta'|$, contradicting the choice of β'.

The second statement can be proved in a similar fashion. Sketching the argument, write

$$h = \sum h_\beta \beta$$

and, arguing by contradiction, suppose there is a β' such that $h_{\beta'} \neq 0$ has an x to the left of an x^*. Let α' and γ' denote maximum degree terms in a^* and b. It follows that $\alpha'\beta'\gamma'$ must appear in a^*hb (and has largest degree amongst words in a^*hb containing an x to the left of an x^*), and thus, f is not hereditary. □

Proof of existence in Theorem 7.1 The hereditary polynomial p factors as

$$f = q_0 q_1 q_2 \dots q_s q_{s+1}, \qquad q_k(0) = 1,$$

where $q_0 = 1 = q_{s+1}$ and, for each $1 \leq j \leq s$, the factor q_j is an atom. Suppose, without loss of generality, that $f \notin \mathbb{C}<x>$. There is an $1 \leq r \leq s$ such that $q_{r+1} \cdots q_{s+1} \in \mathbb{C}<x>$, but $q_r q_{r+1} \cdots q_{s+1} \notin \mathbb{C}<x>$. By Lemma 7.4, $q_0 q_1 \cdots q_{r-1} \in \mathbb{C}<x^*>$ as $f = (q_0 q_1 \cdots q_{r-1})(q_r \cdots q_{s+1})$ is hereditary. Thus, $f = a^*hb$, where $a = (q_0 q_1 \cdots q_{r-1})^*$, $q_{r+1} \cdots q_{s+1} \in \mathbb{C}<x>$ and $h = q_r$. By the other half of Lemma 7.4, q_r is hereditary and the proof is complete. □

7.2 Proof of Uniqueness of the Factorization (28)

Proving uniqueness requires background from Cohn [14] which we now introduce.

Let $q_1, q_2, \widehat{q}_1, \widehat{q}_2 \in \mathbb{C}<x>$ and suppose

$$q_1 q_2 = \widehat{q}_1 \widehat{q}_2. \tag{29}$$

If

$$q_1 \mathbb{C}<x> + \widehat{q}_1 \mathbb{C}<x> = \mathbb{C}<x>, \qquad \mathbb{C}<x> q_2 + \mathbb{C}<x> \widehat{q}_2 = \mathbb{C}<x>,$$

then (29) is called a comaximal relation [14, Section 0.5]. If, moreover, $q_1, q_2, \widehat{q}_1, \widehat{q}_2$ are atoms and

$$q_1 \mathbb{C}<x> \cap \widehat{q}_1 \mathbb{C}<x> \text{ is a principal right ideal in } \mathbb{C}<x>,$$

then (29) is called a comaximal transposition [14, Section 3.2].

Next, $q_1, \widehat{q}_2$ are stably associated [14, Section 0.5] if

$$I_d \otimes \widehat{q}_2 = P(I_d \otimes q_1)Q,$$

for some $d \in \mathbb{N}$ and $P, Q \in \mathrm{GL}_{d+1}(\mathbb{C}<x>)$.

Proposition 7.5 [14, Proposition 0.5.6] *q_1 and $\widehat{q}_2$ are stably associated if and only if they appear in a comaximal relation (29) for some $q_2, \widehat{q}_1$.*

Finally, a factorization $f = f_1 \cdots f_\ell$ in $\mathbb{C}<x>$ is **complete** [14, Section 3.2] if the f_k are atoms. Two complete factorizations of f are identified if their factors only differ up to scalars. Note that a noncommutative polynomial can admit distinct complete factorizations, e.g.

$$(1 + x_1 x_2)x_1 = x_1(1 + x_2 x_1).$$

However, this relation is a comaximal transposition. In fact, the following holds.

Proposition 7.6 ([14, Proposition 3.2.9]) *Given two complete factorizations of a polynomial, one can pass between them by a finite sequence of comaximal transpositions on adjacent pairs of atomic factors (in particular, they have the same length).*

Let us illustrate what is meant by a *finite sequence of comaximal transpositions*. To say that $q_1q_2q_3q_4$ is a complete factorization that can be transformed to a different factorization by applying comaximal transpositions on positions $(2, 3)$, $(3, 4)$ and $(1, 2)$ (in this order) means there exists $\widehat{q}_2, \widehat{q}_3, \widehat{\widehat{q}}_3, \widehat{\widehat{q}}_2$ such that

$$q_1q_2q_3q_4 = q_1\widehat{q}_2\widehat{q}_3q_4 = q_1\widehat{q}_2\widehat{\widehat{q}}_3\widehat{q}_4 = \widehat{q}_1\widehat{\widehat{q}}_2\widehat{\widehat{q}}_3\widehat{q}_4,$$

where

$$q_2q_3 = \widehat{q}_2\widehat{q}_3, \qquad \widehat{q}_3q_4 = \widehat{\widehat{q}}_3\widehat{q}_4, \qquad q_1\widehat{q}_2 = \widehat{q}_1\widehat{\widehat{q}}_2$$

are comaximal transpositions.

Lemma 7.7 *Suppose $\ell h = f_1 f_2$ is a comaximal relation where $\ell \in \mathbb{C}<x^*>$, h is hereditary, $f_1, f_2 \in \mathbb{C}<x, x^*>$ and all are normalized to equal 1 at the origin. Then, $f_1, f_2, h \in \mathbb{C}<x^*>$.*

Analogously, if $hr = f_1 f_2$ is a comaximal relation with $r \in \mathbb{C}<x>$ and h hereditary, then $f_1, f_2, h \in \mathbb{C}<x>$.

Proof By Proposition 7.5, ℓ and f_2 are stably associated. Then by the definition of stable associativity, there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that

$$\det \ell(\varUpsilon^{(n)}) = \alpha^n \det f_2(\varOmega^{(n)}, \varUpsilon^{(n)})$$

for all $n \in \mathbb{N}$, where $\varOmega^{(n)}$ and $\varUpsilon^{(n)}$ are tuples of $n \times n$ generic matrices. By [36, Proposition 5.11], $f_2 \in \mathbb{C}<x^*>$. But $f_1 f_2 = \ell h$ is hereditary, so $f_1 \in \mathbb{C}<x^*>$ and consequently $h \in \mathbb{C}<x^*>$. □

Proof of uniqueness in Theorem 7.1 Suppose $f = phq = \widehat{p}\widehat{h}\widehat{q}$ are two factorizations as in Theorem 7.1. Let

$$p = p_1 \cdots p_k, \qquad \widehat{p} = \widehat{p}_1 \cdots \widehat{p}_{\widehat{k}}, \qquad q = q_1 \cdots q_\ell, \qquad \widehat{q} = \widehat{q}_1 \cdots \widehat{q}_{\widehat{\ell}}$$

be complete factorizations (with factors equal to 1 at the origin). Then

$$p_1 \cdots p_k h q_1 \cdots q_\ell \;=\; \widehat{p}_1 \cdots \widehat{p}_{\widehat{k}} \widehat{h} \widehat{q}_1 \cdots \widehat{q}_{\widehat{\ell}}. \tag{30}$$

and by Proposition 7.6 we can pass from the left-hand side to the right-hand side of (30) by a series of comaximal transpositions. The heart of the proof is that there cannot be any transposing around the "middle" factor h unless it is trivial. Since f and all the factors p, q, h are normalized to equal 1 at 0, we can apply Lemma 7.7 to conclude the proof: for if we can transpose $p_k h$, then $h \in \mathbb{C}<x^*>$ and so $h = 1$ since h is truly hereditary, likewise for hq_1. When h is not trivial, comaximal transpositions can therefore only occur among the first $k-1$ factors and last $\ell - 1$ factors of the left-hand side in (30). However, these comaximal transpositions preserve $p_1 \cdots p_k$ and $q_1 \cdots q_\ell$. Thus, we conclude that $p_1 \cdots p_k = \widehat{p}_1 \cdots \widehat{p}_{\widehat{k}}$ and $q_1 \cdots q_\ell = \widehat{q}_1 \cdots \widehat{q}_{\widehat{\ell}}$. Therefore, $p = \widehat{p}$ and $q = \widehat{q}$, and consequently $h = \widehat{h}$.

The last part of Theorem 7.1 is a direct consequence of the uniqueness. □

A Modification of the Theory: Rational Functions

For the reader familiar with nc rational functions as found in [14,40], we point out that Theorem 1.1 extends to matrix noncommutative rational functions in a straightforward way. Assume $\mathbb{r} \in \mathbb{C}\langle\!\!\langle x, x^*\rangle\!\!\rangle^{\delta\times\delta}$ is regular at the origin (that is, 0 is in the domain of $\mathbb{r}$) and $\mathbb{r}(0) = I$. Then, we define $\mathcal{K}_{\mathbb{r}} = \bigcup_n \mathcal{K}_{\mathbb{r}}(n)$, where $\mathcal{K}_{\mathbb{r}}(n)$ is the closure of the connected component of

$$\left\{(X, X^*) \in \mathrm{M}_n(\mathbb{C})^{2g} : \mathbb{r} \text{ is regular at } (X, X^*) \text{ and } \det \mathbb{r}(X, X^*) \neq 0\right\}$$

containing the origin.

Now let $I + c^*L^{-1}\mathbf{b}$ be a minimal FM realization for $\mathbb{r} \oplus \mathbb{r}^{-1} \in \mathbb{C}\langle\!\!\langle x\rangle\!\!\rangle^{2\delta\times 2\delta}$. Using Remark 2.6(4) we observe that $\mathcal{Z}_L$ is precisely the set of all (X, X^*) for which either $\mathbb{r}$ is not defined at (X, X^*) or $\mathbb{r}$ is regular at (X, X^*) and $\det \mathbb{r}(X, X^*) = 0$. By comparing this observation with the definition of $\mathcal{K}_{\mathbb{r}}$, we see that

$$\mathcal{K}_{\mathbb{r}} = \mathcal{K}_L. \tag{31}$$

Now we apply the proof of Theorem 1.1 to L.

Likewise, from (31) we deduce that Corollary 1.3 holds for rational functions $\mathbb{r}$. This leads to improvements and strengthening of recent positivity results for noncommutative rational functions [45,55]. For instance, a rational function $\mathbb{r}$ is positive definite on the interior of $\mathcal{D}_L$ if and only if $\mathbb{r}(0) \succ 0$ and $\widetilde{L}$ is invertible on $\operatorname{int}\mathcal{D}_L$, where $\widetilde{L}$ is the minimal pencil in an FM realization of $\mathbb{r} \oplus \mathbb{r}^{-1}$. The latter condition can be efficiently checked by the algorithm of Sect. 4.3.

In [55], Pascoe gives a Positivstellensatz certifying when a noncommutative rational function $\mathbb{r}$ that is defined on $\mathcal{D}_L$, is positive semidefinite on $\mathcal{D}_L$. For bounded $\mathcal{D}_L$ our algorithms provide means of verifying whether $\mathbb{r}$ is defined on $\mathcal{D}_L$. Let $\widetilde{L}$ be the minimal pencil in an FM realization of $\mathbb{r}$. Then, $\widetilde{L}$ is invertible on $\mathcal{D}_L$ if and only if there is $\varepsilon > 0$ such that $\widetilde{L}\widetilde{L}^* - \varepsilon$ is invertible on $\operatorname{int}\mathcal{D}_L$, and this is something that can be checked with a sequence of SDPs (cf. Sect. 4.3).

We conclude with a variant of Theorem 1.5 for rational functions. McMillan degree [40] of a rational function is the size of the linear pencil in its minimal FM realization. Lemma A.2 asserts that, given L a minimal hermitian monic pencil L, there exists a hermitian $\mathbb{s} \in \mathbb{C}\langle\!\!\langle x, x^*\rangle\!\!\rangle$ such that $\mathcal{K}_{\mathbb{s}} = \mathcal{D}_L$. We say that a hermitian $\mathbb{r} \in \mathbb{C}\langle\!\!\langle x, x^*\rangle\!\!\rangle$ is minimal (McMillan) degree defining for $\mathcal{D}_L$ if $\mathcal{K}_{\mathbb{r}} = \mathcal{D}_L$ and the McMillan degree of $\mathbb{r}$ is smallest amongst all hermitian $\mathbb{s}$ such that $\mathcal{K}_{\mathbb{s}} = \mathcal{D}_L$.

Proposition A.1 *Let $\mathbb{r} = \mathbb{r}^* \in \mathbb{C}\langle\!\!\langle x, x^*\rangle\!\!\rangle$ be regular at the origin and $\mathbb{r}(0) = 1$. Suppose that $\mathcal{K}_{\mathbb{r}}$ is a free spectrahedron $\mathcal{D}_L$ for an irreducible hermitian monic pencil L. If $\mathbb{r}$ is minimal McMillan degree defining for $\mathcal{D}_L$, then either $\mathbb{r}$ or $\mathbb{r}^{-1}$ is concave or convex with the pencil in its minimal FM realization being equal to L.*

Lemma A.2 *Suppose L is an irreducible hermitian monic pencil of size d and $0 \neq \widehat{c} \in \mathbb{C}^d$ is of norm < 1. Setting $\widehat{\mathbf{b}} = Ac \odot x + A^*c \odot x^*$ and $\widehat{\mathbb{r}} = 1 + \widehat{c}^* L^{-1}\widehat{\mathbf{b}}$,*

$$\mathcal{K}_{\widehat{\mathbb{r}}} = \mathcal{D}_L,$$

$\widehat{\mathbb{r}}^{-1}$ is defined on $\operatorname{int}\mathcal{D}_L$ *and $\widehat{\mathbb{r}}^* = \widehat{\mathbb{r}}$.*

Proof Since the converse of Lemma 3.1 evidently holds, $\mathbb{r}^* = \mathbb{r}$. By Remark 2.6(6) we have $\widehat{\mathbb{r}}^{-1} = 1 - \widehat{c}^* L_\times^{-1} \widehat{\mathbf{b}}$, where $L_\times = L + \widehat{\mathbf{b}}\widehat{c}^*$. Since L is irreducible and $\widehat{c} \neq 0$ and $\widehat{\mathbf{b}} \neq 0$, the realization $\widehat{\mathbb{r}} = 1 + \widehat{c}^* L^{-1} \widehat{\mathbf{b}}$ is observable and controllable, and thus minimal by Remark 2.6(1) . Consequently, $\widehat{\mathbb{r}}^{-1} = 1 - \widehat{c}^* L_\times^{-1} \widehat{\mathbf{b}}$ is also minimal. The pencil $L_\times$ is invertible on int $\mathcal{D}_L$ because

$$(I - \widehat{c}\widehat{c}^*)(L + \widehat{\mathbf{b}}\widehat{c}^*) = (I - \widehat{c}\widehat{c}^*)\widehat{c}\widehat{c}^* + (I - \widehat{c}\widehat{c}^*)L(I - \widehat{c}\widehat{c}^*).$$

By the definition of $\mathcal{K}_{\widehat{\mathbb{r}}}$, we have

$$\mathcal{K}_{\widehat{\mathbb{r}}} = \mathcal{K}_{L \oplus L_\times},$$

so invertibility of $L_\times$ on int $\mathcal{D}_L$ implies

$$\mathcal{K}_{\widehat{\mathbb{r}}} = \mathcal{D}_L.$$

Furthermore, the domain of $\widehat{\mathbb{r}}^{-1}$ is the complement of $\mathcal{Z}_{L_\times}$ by Remark 2.6(4) , so $\widehat{\mathbb{r}}^{-1}$ is defined on int $\mathcal{D}_L$. □

Proof of Proposition A.1 Let $L = I - A \odot x - A^* \odot x^*$ be of size d. Let $\mathbb{r} = 1 + c^* \widetilde{L}^{-1} \mathbf{b}$ be a minimal realization. Hence, $\mathbb{r}^{-1} = 1 - c^* \widetilde{L}_\times^{-1} \mathbf{b}$, where $\widetilde{L}_\times$ is the pencil appearing in Remark 2.6(6) , is a minimal realization for $\mathbb{r}^{-1}$. Since $\mathcal{K}_{\mathbb{r}} = \mathcal{D}_L$, the topological boundary of $\mathcal{D}_L$ is contained in

$$\{(X,Y) \colon \mathbb{r} \text{ is undefined at } (X,Y)\} \cup \{(X,Y) \colon \mathbb{r}^{-1} \text{ is undefined at } (X,Y)\} = \mathcal{Z}_{\widetilde{L}} \cup \mathcal{Z}_{\widetilde{L}_\times}.$$

Since L is an irreducible hermitian monic pencil, it is minimal. Thus, by Proposition 2.3, $\mathcal{Z}_L \subseteq \mathcal{Z}_{\widetilde{L}} \cup \mathcal{Z}_{\widetilde{L}_\times}$. Since L is irreducible, either $\mathcal{Z}_L \subseteq \mathcal{Z}_{\widetilde{L}}$ or $\mathcal{Z}_L \subseteq \mathcal{Z}_{\widetilde{L}_\times}$. Without loss of generality suppose $\mathcal{Z}_L \subseteq \mathcal{Z}_{\widetilde{L}}$ (otherwise replace $\mathbb{r}$ by $\mathbb{r}^{-1}$). Since L is irreducible, up to similarity (change of basis), $\widetilde{L}$ has the form (4), where one of the blocks equals L. On the other hand, by Lemma A.2, the size of $\widetilde{L}$ is no larger than the size of L. Hence, $\widetilde{L}$ is similar to L and we may assume, by modifying c, $\mathbf{b}$ and A appropriately, that $\widetilde{L} = L$. Therefore, as L is an irreducible hermitian monic pencil,

$$\mathbb{r} = 1 + \lambda c^* L^{-1}(Ac \odot x + A^* c \odot x^*) = 1 + \lambda(c^* L^{-1} c - c^* c)$$

for some $\lambda \in \mathbb{R} \setminus \{0\}$ by Lemma 3.1. Since L is monic and hermitian, $\mathbb{r}$ is concave or convex (depending on the sign of λ). □

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