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An Onsager singularity theorem for Leray solutions of incompressible Navier–Stokes

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Abstract

We study in the inviscid limit the global energy dissipation of Leray solutions of incompressible Navier–Stokes on the torus $\mathbb{T}^d$, assuming that the solutions have norms for Besov space $B_3^{\sigma,\infty}(\mathbb{T}^d)$, $\sigma \in (0,1]$, that are bounded in the L^3-sense in time, uniformly in viscosity. We establish an upper bound on energy dissipation of the form $O(\nu^{(3\sigma-1)/(\sigma+1)})$, vanishing as $\nu \to 0$ if $\sigma > 1/3$. A consequence is that Onsager-type 'quasi-singularities' are required in the Leray solutions, even if the total energy dissipation vanishes in the limit $\nu \to 0$, as long as it does so sufficiently slowly. We also give two sufficient conditions which guarantee the existence of limiting weak Euler solutions u which satisfy a local energy balance with possible anomalous dissipation due to inertial-range energy cascade in the Leray solutions. For $\sigma \in (1/3, 1)$ the anomalous dissipation vanishes and the weak Euler solutions may be spatially 'rough' but conserve energy.

Keywords: Onsager's conjecture, fluid turbulence, anomalous dissipation
Mathematics Subject Classification numbers: 35Q30, 35Q31, 76F02, 35Q35

1. Introduction

In a 1949 paper on turbulence in incompressible fluids [1], Onsager announced a result that spatial Hölder exponents $\leqslant 1/3$ are required of the velocity field for anomalous turbulent dissipation (that is, energy dissipation non-vanishing in the limit of zero viscosity). Onsager's original statement and most subsequent work [2–10] have involved the conjecture that the velocity field in the limit of infinite Reynolds number is a weak (distributional) solution of the incompressible Euler equations. In this short paper we show that the arguments employed to prove Onsager's claim about weak Euler solutions apply as well to Leray's solutions of the incompressible Navier–Stokes equation and can be used to prove a theorem that 'quasi-singularities' are required in those solutions in order to account for anomalous energy dissipation. In fact, such consequences follow even if the energy dissipation is vanishing in the limit of zero viscosity, as long as it goes to zero as slowly as $\sim\nu^{\alpha}$ for some $\alpha \in (0,1)$. In that case, we show that the Navier–Stokes solutions cannot have Besov norms, above a critical smoothness $\frac{1+\alpha}{3-\alpha}$, which are bounded uniformly in viscosity. This observation is important because empirical studies (e.g. see remark 4 below) cannot distinguish in principle between a dissipation rate which is independent of viscosity and one which is vanishing sufficiently slowly. Our results thus considerably strengthen the conclusion that quasi-singularities are necessary to account for the enhanced energy dissipation rates observed in turbulent flow. No assumption need be made in our proof about existence of limiting Euler solutions, but weak Euler solutions do arise as $\nu \to 0$ limits of the Leray solutions if some further natural conditions are satisfied.

Let $u^\nu \in L^\infty([0,T];L^2(\mathbb{T}^d)) \cap L^2([0,T];H^1(\mathbb{T}^d))$ for $\nu > 0$ be Leray solutions of the incompressible Navier–Stokes equations satisfying

$$\partial_t u^\nu + \nabla\cdot(u^\nu \otimes u^\nu) = -\nabla p^\nu + \nu\Delta u^\nu + f^\nu, \tag{1}$$

$$\nabla \cdot u^\nu = 0, \tag{2}$$

in the sense of distributions on $\mathbb{T}^d \times [0,T]$, with solenoidal initial conditions $u^\nu|_{t=0} = u_0^\nu \in L^2(\mathbb{T}^d)$ and solenoidal body forcing $f^\nu \in L^2([0,T];L^2(\mathbb{T}^d))$. A fundamental property of these solutions, first obtained by Leray [11], is the global energy inequality, which states that viscous energy dissipation cannot exceed the loss of energy by the flow plus the energy input by external force. This property may be reformulated as a global balance of kinetic energy:

$$\int_0^T \int_{\mathbb{T}^d} \varepsilon[u^\nu]\, \mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_{\mathbb{T}^d} |u_0^\nu|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{T}^d} |u^\nu(\cdot,T)|^2\mathrm{d}x + \int_0^T \int_{\mathbb{T}^d} u^\nu \cdot f^\nu\, \mathrm{d}x\mathrm{d}t, \tag{3}$$

for almost every $T \geqslant 0$, where the total energy dissipation rate is

$$\varepsilon[u^\nu] := \nu|\nabla u^\nu|^2 + D[u^\nu] \tag{4}$$

with $D[u^\nu]$ a non-negative distribution (Radon measure) that represents dissipation due to possible Leray singularities. See Duchon–Robert [4] and the proof of our lemma 1. Our main result is then:

Theorem 1. *Let $u^\nu \in L^\infty([0,T];L^2(\mathbb{T}^d)) \cap L^2([0,T];H^1(\mathbb{T}^d))$ for $\nu > 0$ be any Leray solutions of incompressible Navier–Stokes equations on $\mathbb{T}^d \times [0,T]$ with initial data $u_0^\nu \in B_2^{\sigma,\infty}(\mathbb{T}^d)$, and forcing $f^\nu \in L^2([0,T];B_2^{\sigma,\infty}(\mathbb{T}^d))$ for some $\sigma \in (0,1]$. Suppose that:*

$$\int_0^T \int_{\mathbb{T}^d} \varepsilon[u^\nu]\, \mathrm{d}x\mathrm{d}t \geqslant \nu^\alpha L(\nu), \qquad \alpha \in [0,1) \tag{5}$$

where $L : \mathbb{R}^+ \to \mathbb{R}^+$ is a function slowly-varying at $\nu = 0$ in the sense of Kuramata [12], i.e. so that $\lim_{\nu\to 0} L(\lambda\nu)/L(\nu) = 1$ for any $\lambda > 0$. Then, for any $\epsilon > 0$, the family $\{u^\nu\}_{\nu>0}$ of Leray solutions cannot have norms $\|u^\nu\|_{L^3([0,T];B_3^{\sigma_\alpha+\epsilon,\infty}(\mathbb{T}^d))}$ with $\sigma_\alpha := \frac{1+\alpha}{3-\alpha} \in [1/3,1)$ that are bounded uniformly in $\nu > 0$.

Theorem 1 follows easily from the following lemma:

Lemma 1. *Let $\{u^\nu\}_{\nu>0}$ be a family of Leray solutions with σ, u_0^ν, and f^ν as in theorem 1. Assume that $u^\nu \in L^3([0,T]; B_3^{\sigma,\infty}(\mathbb{T}^d))$ with all the above Besov norms bounded, uniformly in viscosity. Then, for a.e. $T \geqslant 0$, the energy dissipation is bounded for some ν-independent constant C by:*

$$\int_0^T \int_{\mathbb{T}^d} \varepsilon[u^\nu]\, \mathrm{d}x\mathrm{d}t \leqslant C\nu^{\frac{3\sigma-1}{\sigma+1}}. \tag{6}$$

To see that theorem 1 follows from lemma 1, note that if for any $\epsilon > 0$, $u^\nu \in L^3([0,T]; B_3^{\sigma_\alpha+\epsilon,\infty}(\mathbb{T}^d))$ with norms bounded uniformly in viscosity, then the inequality equation (6) together with equation (5) implies:

$$L(\nu) \leqslant C\nu^{\epsilon\frac{(3-\alpha)^2}{4+\epsilon(3-a)}}. \tag{7}$$

Since $\alpha \in [0,1)$, the exponent in the power-law on the righthand side of equation (7) is positive. This obviously leads to a contradiction since $\lim_{\nu\to 0} \nu^{-p} L(\nu) = +\infty$ for L slowly varying at $\nu = 0$ and for any $p > 0$.

In the context of lemma 1, we note that that if $\sigma \in [1/3, 1]$ then theorem 6.1 of [5] implies that $D[u^\nu] = 0$ and energy dissipation arises entirely from viscosity. The proof of this fact for $\sigma > 1/3$ and fixed $\nu > 0$ follows easily by the Constantin-E-Titi commutator argument [3] for weak solutions, after taking into account the Leray–Hopf regularity $L^2(0,T; H^1(\mathbb{T}^d))$. We conjecture that our theorem 1 is optimal for space dimensions $d > 2$ in the sense that, for some $\alpha \in [0,1)$, there should exist sequences of Leray solutions of Navier–Stokes u^ν for $\nu > 0$ that are uniformly bounded in $L^3([0,T]; B_3^{\sigma_\alpha-\epsilon,\infty}(\mathbb{T}^d))$ with any $\epsilon > 0$ and for which the lower bound equation (5) on dissipation holds as an asymptotic equality for $\nu \to 0$. The case $d = 2$ is different, because of the absence of vortex-stretching. This implies strong bounds on enstrophy for Leray solutions in $d = 2$, even with initial vorticity $\omega_0 \in L^p$ only for $p < 2$, and an essential improvement of the energy dissipation bounds in our lemma 1 for $d = 2$ [13].

Remark 1. The main condition on uniform Besov regularity in lemma 1 is physically natural. The Besov space $B_p^{\sigma,\infty}(\mathbb{T}^d)$ is made up of measurable functions $f: \mathbb{T}^d \to \mathbb{R}^d$ which are finite in the norm

$$\|f\|_{B_p^{\sigma,\infty}(\mathbb{T}^d)} := \|f\|_{L^p(\mathbb{T}^d)} + \sup_{r\in(0,1]^d} \frac{\|f(\cdot+r) - f(\cdot)\|_{\mathbb{T}^d}}{|r|^\sigma} \tag{8}$$

for $p \geqslant 1$ and $\sigma \in (0,1)$. See [37], section 3.5. These spaces can be equivalently explained in a way more familiar to fluid dynamicists by using structure functions. The pth-order structure functions $S_p^\nu(r)$ of spatial velocity-increments $\delta u^\nu(r; x, t) := u^\nu(x+r,t) - u^\nu(x,t)$ may be defined as usual by $S_p^\nu(r,t) := \langle |\delta u^\nu(r,t)|^p \rangle$, where $\langle\cdot\rangle$ denotes space average over $x \in \mathbb{T}^d$. The velocity field belongs to the Besov space $B_p^{\sigma,\infty}(\mathbb{T}^d)$ for $p \geqslant 1$, $\sigma \in (0,1)$ at time t if and only if

$$\langle |u^\nu(\cdot,t)|^p \rangle < C_0(t), \qquad S_p^\nu(r,t) \leqslant C_1(t) \left|\frac{r}{\ell_0}\right|^{\zeta_p}, \ \forall |r| \leqslant \ell_0 \tag{9}$$

with $\zeta_p = \sigma p$ and then the optimal constants $C_0(t), C_1(t) > 0$ in these upper bounds define a norm for the Besov space $B_p^{\sigma,\infty}(\mathbb{T}^d)$ by the identification $\|u^\nu(\cdot,t)\|_{B_p^{\sigma,\infty}(\mathbb{T}^d)} := [C_0(t) + C_1(t)]^{1/p}$. e.g. see [14]. Here any choice of length-scale $\ell_0 > 0$ defines the same function space $B_p^{\sigma,\infty}(\mathbb{T}^d)$ but for a physical identification of the constant $C_1(t)$ as the 'amplitude' of an inertial-range scaling law, one must take ℓ_0 to be the integral-length of the turbulent flow and independent of $\nu > 0$. The uniform boundedness of the family $\{u^\nu\}_{\nu>0}$ in $L^p([0,T];B_p^{\sigma,\infty}(\mathbb{T}^d))$ is equivalent to the condition that coefficients $C_0(t)$, $C_1(t)$ independent of $\nu > 0$ should exist so that the bounds equation (9) are satisfied for a.e. $t \in [0,T]$ and $\int_0^T \mathrm{d}t\ [C_0(t) + C_1(t)] < \infty$. The theorem 1 and lemma 1 apply *a fortiori* to solution spaces $L^p([0,T], B_p^{\sigma,\infty}(\mathbb{T}^d))$ with any $p \geqslant 3$ and not only to $p = 3$. As a consequence, energy dissipation vanishing with $\nu \to 0$ as slowly as equation (5) (or possibly not vanishing at all for $\alpha = 0$), implies $\zeta_p \leqslant \left(\frac{1+\alpha}{3-\alpha}\right) p$ for $p \geqslant 3$ as a constraint on possible structure-function scaling exponents in the inertial-range of any turbulent flow with enhanced dissipation of the form equation (5). This inequality is a precise statement on 'quasi-singularities' in the sequence of Leray solutions, in order to be consistent with the observed slow decrease of energy dissipation as $\nu \to 0$. The Navier–Stokes solutions (barring possible true, Leray-type singularities) are spatially C^∞ for any $\nu > 0$, but they cannot possess smoothness of the form equation (9) that is uniform in viscosity. The primary physical motivation of our result is turbulence in space dimensions $d > 2$, where a forward energy cascade is expected. However our theorem has some implications even for $d = 2$. For example, [13] considers Navier–Stokes solutions with initial vorticity $\omega_0 \in L^p(\mathbb{T}^2)$, $p \in (1,2]$ and obtains an upper bound on energy dissipation of the form (const.)ν^{α_p} for $\alpha_p := \frac{2(p-1)}{p} \in (0,1]$, vanishing as $\nu \to 0$. If this is the actual scaling of the dissipation for $p < 3/2$, the Onsager critical value of p for $d = 2$, then our theorem 1 implies that the family $\{u^\nu\}_{\nu>0}$ cannot be uniformly bounded in $L^3([0,T]; B_3^{\sigma_{\alpha_p}+\epsilon,\infty}(\mathbb{T}^2))$ with $\sigma_{\alpha_p} := \frac{3p-2}{p+1} \in (1/2, 1)$.

Remark 2. A small but useful technical improvement of theorem 1 can be easily provided by sharpening the spaces considered. First, recall that energy conservation for weak solutions of the Euler equations holds provided that $u \in B_3^{1/3,c_0}(\mathbb{T}^d)$, a subspace of $B_3^{1/3,\infty}(\mathbb{T}^d)$ that can be defined as follow

$$B_p^{\sigma,c_0}(\mathbb{T}^d) = \left\{ f \in L^p(\mathbb{T}^d) : \lim_{|r|\to 0} \frac{\|f(\cdot + r) - f(\cdot)\|_{L^p(\mathbb{T}^d)}}{|r|^\sigma} = 0 \right\}. \tag{10}$$

See [5]. Note that $B_p^{\sigma',\infty}(\mathbb{T}^d) \subset B_p^{\sigma,c_0}(\mathbb{T}^d) \subset B_p^{\sigma,\infty}(\mathbb{T}^d)$ for any $\sigma' > \sigma$. Define also

$$L^q(0,T; B_p^{\sigma,c_0}(\mathbb{T}^d)) = \left\{ f \in L^q(0,T; L^p(\mathbb{T}^d)) : \lim_{|r|\to 0} \frac{\|f(\cdot + r) - f(\cdot)\|_{L^q(0,T;L^p(\mathbb{T}^d)}}{|r|^\sigma} = 0 \right\}. \tag{11}$$

Theorem 1 then holds in a form in which one replaces all instances of $B_p^{\sigma,\infty}$ with B_p^{σ,c_0} and the conclusion reads that the family $\{u^\nu\}_{\nu>0}$ of Leray solutions cannot have norms $\|u^\nu\|_{L^3([0,T];B_3^{\sigma_\alpha,c_0}(\mathbb{T}^d))}$ with $\sigma_\alpha := \frac{1+\alpha}{3-\alpha} \in [1/3, 1)$. Note that the spaces B_p^{σ,c_0} allow us to remove the 'ϵ' appearing in the theorem statement. The proof is almost identical and therefore omitted. We are grateful to the anonymous referee for this remark.

We emphasize again that we do not need to assume that any 'singular' or 'rough' Euler solutions exist in order to draw these conclusions. However, under reasonable additional conditions, weak Euler solutions will exist as inviscid limits of the Leray solutions. For example:

Theorem 2. *Let $u^\nu \in L^\infty([0,T]; L^2(\mathbb{T}^d)) \cap L^2([0,T]; H^1(\mathbb{T}^d))$ be any Leray solutions of incompressible Navier–Stokes equations with $\nu > 0$ on $\mathbb{T}^d \times [0,T]$, for initial data $u_0^\nu \in L^2(\mathbb{T}^d)$ and forcing $f^\nu \in L^2([0,T]; L^2(\mathbb{T}^d))$, and assume either:*

(i) For some $\sigma \in (0,1]$ *the family* $\{u^\nu\}_{\nu>0}$ *is uniformly bounded in* $L^3([0,T];B_3^{\sigma,\infty}(\mathbb{T}^d))$*, and that* $f^\nu \to f$ *strongly in* $L^2([0,T];L^2(\mathbb{T}^d))$ *as* $\nu \to 0^+$*. Let u then be any strong limit of a subsequence* $u^{\nu_k} \in L^3([0,T];L^3(\mathbb{T}^d))$*.*

or

(ii) $u^\nu \in L^3([0,T];L^3(\mathbb{T}^d))$ *with norms bounded uniformly in viscosity and furthermore, that weak convergence as* $\nu \to 0$ *holds for a full-measure set of times:*

$$u^\nu(\cdot,t) \underset{L^3}{\rightharpoonup} u(\cdot,t), \quad (u^\nu \otimes u^\nu)(\cdot,t) \underset{L^{3/2}}{\rightharpoonup} (u\otimes u)(\cdot,t), \quad f^\nu(\cdot,t) \underset{L^2}{\rightharpoonup} f(\cdot,t) \quad \text{a.e. } t \in [0,T]. \tag{12}$$

Then u is a weak Euler solution which also satisfies, in the sense of distributions, the balance

$$\partial_t\left(\frac{1}{2}|u|^2\right) + \nabla\cdot\left[\left(\frac{1}{2}|u|^2 + p\right)u\right] = -D[u] + u\cdot f \tag{13}$$

on $\mathbb{T}^d \times [0,T]$*, with D[u] the distributional limit of nonlinear 'energy flux' for the Leray solutions:*

$$D[u] := \mathcal{D}'\text{-}\lim_{\ell\to 0}\ \mathcal{D}'\text{-}\lim_{\nu\to 0}\Pi_\ell[u^\nu]. \tag{14}$$

See definition equation (20) below. In particular, $D[u]=0$ *and energy conservation holds if* $\sigma > 1/3$*. Furthermore, under the condition (i)*

$$D[u] = \mathcal{D}'\text{-}\lim_{\nu\to 0}\varepsilon[u^\nu], \tag{15}$$

where total dissipation measure $\varepsilon[u^\nu]$ *for Leray solutions is defined in equation (4), and* $u \in L^3([0,T];B_3^{\sigma-\epsilon,c_0}(\mathbb{T}^d))$ *for any* $\epsilon > 0$*. Thus,* $D[u]=0$ *and local energy conservation holds when* $\sigma \in (1/3,1]$*.*

Remark 3. We owe the first condition of theorem 2 to Isett [15], reproduced here with permission. In particular, he pointed out that uniform boundedness of a family of weak Navier–Stokes solutions $\{u^\nu\}_{\nu>0}$ in $L^2([0,T];B_2^{\sigma,\infty}(\mathbb{T}^d))$ guarantees strong pre-compactness in $L^2(\mathbb{T}^d\times[0,T])$ by the Aubin–Lions–Simon lemma (see also [16]). Isett pointed out to us [17] that the uniform boundedness assumed in lemma 1 allows such an argument also for $p=3$. In the physical application this means that if energy dissipation is bounded below as in equation (5) but if also $\{u^\nu\}_{\nu>0}$ is uniformly bounded in $L^3([0,T];B_3^{\sigma_\alpha-\epsilon,\infty}(\mathbb{T}^d))$ for any $\epsilon>0$, then a limit Euler solution u will exist. Moreover, the limit will possess some spatial Besov regularity with exponent $\sigma_\alpha-\epsilon$ but not *a priori* with a higher exponent $\sigma_\alpha+\epsilon$ for any $\epsilon>0$. See remark 6 below.

The second part of the theorem slightly generalizes recent results of Constantin and Vicol [18] for wall-bounded domains Ω. There, it is proved that if $u^\nu \rightharpoonup u$ weakly in $L^2(\Omega)$ for a.e. t and if a second-order structure function $\mathcal{S}_2^\nu(r)$ defined as in our remark 1 (but also time-averaged) satisfies an inertial-range scaling bound like equation (9), then u is a weak solution to the Euler equations (see theorem 3.1 of [18]). Recently, the condition on weak-convergence at a.e. time t was removed in [31] in favor of assuming a structure function bound within a more precise 'inertial range'. Also, as pointed out in [18], remark 3.4, this condition may be removed by assuming a bound on the *space-time structure function* defined by $\mathcal{S}_p^\nu(r,s) := \langle\langle|\delta u^\nu(r,s)|^p\rangle\rangle$, where $\delta u^\nu(r,s;x,t) = u^\nu(x+r,t+s) - u^\nu(x,t)$ are space-time increments and where $\langle\langle\cdot\rangle\rangle$ denotes the space-time average over $(x,t)\in\Omega\times[0,T]$. Specifically, it is assumed in [18] for $p = 2$ that

$$\langle\langle |u^\nu|^p \rangle\rangle \leqslant \mathcal{C}_0 \qquad \mathcal{S}_p^\nu(r,s) \leqslant \mathcal{C}_1 \left[\left| \frac{r}{\ell_0} \right| + \left| \frac{s}{t_0} \right| \right]^{\zeta_p}, \quad \forall\, \eta(\nu) \leqslant |r| \leqslant \ell_0,\ \tau(\nu) \leqslant s \leqslant t_0 \tag{16}$$

with some $\zeta_p > 0$, ν-independent constants $\mathcal{C}_0$, $\mathcal{C}_1 > 0$, and any scales $\eta(\nu)$, $\tau(\nu)$ converging to 0 as $\nu \to 0$. If the bound equation (16) is assumed to hold for $\eta(\nu) = \tau(\nu) \equiv 0$, then equation (16) is the uniform regularity statement $\sup_{\nu>0} \|u^\nu\|_{B_2^{\sigma,\infty}(\Omega\times[0,T])} < \infty$ for some $\sigma \in (0,1)$ and compactness in $L^2(\Omega \times [0,T])$ with the strong topology is immediately implied by the Kolmogorov–Riesz theorem [19]. Thus, subsequences $\nu_k \to 0$ always exist for which $u^{\nu_k} \to u$ strongly in L^2 and the limit function u is automatically a weak Euler solution. We could likewise replace the condition (ii) at each time slice in theorem 2 by the assumption that equation (16) holds for $p = 3$, i.e. uniform third-order space-time structure function bounds in the inertial range, and take u to be any weak limit point of $u^\nu \in L^3(0,T;L^3(\mathbb{T}^d))$. Furthermore, the limiting Euler solution inherits the space-time regularity $u \in B_3^{\sigma,\infty}(\Omega \times [0,T])$ by an argument similar to that in remark 6.

An earlier theorem giving conditions for convergence of Navier–Stokes solutions to weak Euler solutions satisfying a global energy inequality is proved in the work of Chen and Glimm [20]. Their sufficient conditions involve the time-average energy spectrum, or $p = 2$, because all terms of the energy balance that are cubic in the velocity vanish when integrated over space.

Remark 4. It is worthwhile to review briefly here the empirical evidence regarding the global energy dissipation rate in boundary-free turbulent flow. Numerical simulations of fourier-truncated Navier–Stokes dynamics by pseudo-spectral method in a periodic box correspond mostly closely to the conditions of our theorem 1. Free-decay simulations with body-force $f^\nu = 0$ such as [21, 22] do show a non-vanishing energy flux in the inertial-range, consistent with $D[u] > 0$ as defined in equation (14), but there seems to have been no systematic study of the dependence of space-average $\langle \varepsilon^\nu(t) \rangle$ upon $\nu = 1/\mathrm{Re}$ in such simulations. Forced simulations with very smooth (large-scale) forces f^ν [23, 24] provide the best evidence for a space-time average $\langle \varepsilon^\nu \rangle$ which is nearly independent of $\nu = 1/\mathrm{Re}$ as $\mathrm{Re} \to \infty$. These simulations are nominally 'long-time steady-states' with $T \to \infty$, but in practice the time-averages are performed only over several large-eddy turnover times, so that our theorem 1 applies. Given the data plotted in figure 1 of [23] or figure 3 of [24] a reasonable inference is that the dissipation rate does not vanish as $\mathrm{Re} \to \infty$, or vanishes only weakly with viscosity. Accepting this as an empirical fact, our theorem 1 for $p = \infty$ implies that Onsager's prediction of Hölder exponents $h \leqslant 1/3$ [1] remains valid as a statement about 'quasi-singularities' of Leray solutions. If any of the reasonable conditions in the theorem 2 hold as well, then Onsager's conjecture on weak Euler solutions remains true, even if the dissipation rate is vanishing weakly as $\nu \to 0$. In the latter case the Euler solutions may be spatially 'singular' or 'rough', but conserve energy. It should be emphasized that the Euler singularities inferred by this argument need not develop in finite time from smooth initial data. A standard practice in such numerical simulations is the initialization $u^\nu(\cdot,0) = u^{\nu'}(\cdot,T')$ of the simulation at high Re by the final state at time T' of a smaller Reynolds-number $\mathrm{Re}' < \mathrm{Re}$ simulation performed at lower resolution, interpolated onto the finer grid of the Re-simulation (e.g. see p L21 of [24]). This practice of 'nested' initialization means that initial conditions $u^\nu(\cdot,0)$ have Kolmogorov-type spectra over increasing ranges of scales as ν decreases and do not correspond to uniformly smooth initial data.

Similar remarks apply to studies of dissipation rates in boundary-free flows by laboratory experiment. The most common experiments study turbulence produced downstream of wire-mesh grids in wind-tunnels or turbulent wakes generated by flows past other solid obstacles, such as plates, cylinders, etc [25–27]. These experiments measure the time-averaged kinetic energy $(1/2)\langle|u^\nu(x,\cdot)|^2\rangle$ at distances x down-stream of the obstacle. If the data are reinterpreted by 'Taylor's hypothesis' as space-averages $(1/2)\langle|u^\nu(\cdot,t)|^2\rangle$ at times $t = x/U$, with U the mean flow velocity, then these studies yield the space-average dissipation rate $\langle\varepsilon^\nu(t)\rangle$ by time-differentiation. The data plotted in [25–27] again provide corroboratory evidence that $\langle\varepsilon^\nu(t)\rangle$ is nearly independent of $\nu = 1/\mathrm{Re}$ as Re increases. These experiments are obviously not in the space-periodic framework of our theorem 1. Ignoring the effects of walls in the wind-tunnel, at some distance from the turbulent wake, these flows might be regarded as contained in some large box with zero velocities at the wall (and thus periodic). However, the creation of the turbulence by flow past solid obstacles implies that these experiments are closer to the setting of [18], with vorticity fed into the flow by viscous boundary layers that detach from the walls. Since the boundary layers become thinner as $\nu = 1/\mathrm{Re}$ decreases, the initial data of these experiments also cannot be considered to be smooth uniformly in $\nu > 0$.

Remark 5. In light of the discussion in remark 4, theoretically incorporating the effects of solid confining walls is of great practical importance. The experimental observations are rather different for wall-bounded turbulence, such as seen as in pipes, channels, closed containers, etc than those reviewed above for boundary-free flows. Energy dissipation in confined turbulent flows with rough walls tends to constant values for $\mathrm{Re} \gg 1$, whereas energy dissipation in flows with smooth walls is generally observed to vanish with increasing Re, yet much more slowly than the laminar rate $\sim 1/\mathrm{Re}$. For example, see the study [28] whose results are typical. Recently, there have been a number of papers proving Onsager-type theorems on necessary conditions for anomalous dissipation by weak solutions of the Euler equations on domains with solid boundaries [29, 30, 32]. The statements of energy dissipation are slightly more involved due to the fact that assumptions need to be made both in the interior and near the walls. The results of Drivas and Nguyen [32], which focus on vanishing viscosity limits of Leray solutions, may be modified to provide results in the same spirit of our theorem 1. In particular, section 2.4 of [32] provides a connection between the physical energy dissipation and coarse-grained fluxes as in lemma 2. If one supposes that the energy dissipation is lower bounded as in equation (5) and introduces quantitative versions of the near-wall assumptions (i.e. impose how rapidly the velocity itself of the near-wall dissipation vanishes within a viscous boundary layer as viscosity tends to zero), then theorems 2 and 3 of [32] can translated into constraints on uniform interior Besov regularity and boundary-layer behavior of Leray–Hopf solutions. Detailed implications are left for future investigation.

The proof our lemma 1 will be based on the same method employed by Constantin-E-Titi [3] to prove the original Onsager statement for weak Euler solutions, by means of a spatial mollification. Specifically, let G be a *standard mollifier*, with $G \in D(\mathbb{T}^d)$, $G \geqslant 0$, and also $\int_{\mathbb{T}^d} G(r)\mathrm{d}r = 1$. Without loss of generality, we can assume that supp(G) is contained in the Euclidean unit ball in d dimensions. Define the dilatation $G_\ell(r) = \ell^{-d}G(r/\ell)$ and space-reflection $\check{G}(r) = G(-r)$. For any $v \in D'(\mathbb{T}^d)$, we define its *coarse-graining at scale* ℓ by

$$\overline{v}_\ell = \check{G}_\ell * v \in C^\infty(\mathbb{T}^d). \tag{17}$$

Then, we have the following:

Lemma 2. *Let initial data* $u_0^\nu \in L^2(\mathbb{T}^d)$*, forcing* $f^\nu \in L^2([0,T];L^2(\mathbb{T}^d))$ *and* u^ν *be corresponding Leray solutions of the incompressible Navier–Stokes equations on* $\mathbb{T}^d \times [0,T]$ *for* $\nu > 0$*. Then, the following local resolved energy balance holds for any* $\ell > 0$*, for every* $x \in \mathbb{T}^d$ *and a.e.* $t \in [0,T]$

$$\partial_t \left(\frac{1}{2}|\overline{(u^\nu)}_\ell|^2\right) + \nabla \cdot J_\ell^\nu = -\Pi_\ell[u^\nu] - \nu|\nabla\overline{(u^\nu)}_\ell|^2 + \overline{(u^\nu)}_\ell \cdot \overline{(f^\nu)}_\ell, \tag{18}$$

with

$$J_\ell^\nu := \left(\frac{1}{2}|\overline{(u^\nu)}_\ell|^2 + \overline{(p^\nu)}_\ell\right)\overline{(u^\nu)}_\ell + \overline{(u^\nu)}_\ell \cdot \tau_\ell(u^\nu,u^\nu) - \nu\nabla\left(\frac{1}{2}|\overline{(u^\nu)}_\ell|^2\right) \tag{19}$$

where the coarse-graining cumulant is defined by $\tau_\ell(g,h) := \overline{(g\otimes h)}_\ell - \overline{g}_\ell \otimes \overline{h}_\ell$ *for* $g,h \in L^2(\mathbb{T}^d,\mathbb{R}^d)$*, the trace is denoted by* $\tau_\ell(g;h) := \operatorname{Tr}\tau_\ell(g,h)$ *and where*

$$\Pi_\ell[u^\nu] := -\nabla\overline{(u^\nu)}_\ell : \tau_\ell(u^\nu,u^\nu). \tag{20}$$

Furthermore, for a.e. $T \geqslant 0$ *and for any standard mollifier G and any* $\ell > 0$*, we have:*

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^d}\varepsilon[u^\nu]\,\mathrm{d}x\mathrm{d}t = &\int_0^T\int_{\mathbb{T}^d}\Pi_\ell[u^\nu]\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{T}^d}\nu|\nabla\overline{(u^\nu)}_\ell|^2\,\mathrm{d}x\mathrm{d}t\\ &+\frac{1}{2}\int_{\mathbb{T}^d}\tau_\ell(u_0^\nu;u_0^\nu)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{T}^d}\tau_\ell(u^\nu(\cdot,T);u^\nu(\cdot,T))\\ &+\int_0^T\int_{\mathbb{T}^d}\tau_\ell(u^\nu;f^\nu)\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{21}$$

The key ingredient of the proof of lemma 1 is a simple exact formula derived in [3] which expresses the 'energy flux' $\Pi_\ell[u^\nu]$ in terms of velocity increments. Our relation equation (14) can thus be interpreted as an extension of the celebrated Kolmogorov 4/5-law to infinite Reynolds-number limits of Leray solutions.

2. Proofs

Proof of lemma 2. Any Leray weak solution u^ν of Navier–Stokes satisfies point-wise in $x \in \mathbb{T}^d$ and distributionally in $t \in [0,T]$ the coarse-grained equations

$$\partial_t\overline{(u^\nu)}_\ell + \nabla\cdot[\overline{(u^\nu\otimes u^\nu)}_\ell] = -\nabla\overline{(p^\nu)}_\ell + \nu\Delta\overline{(u^\nu)}_\ell + \overline{(f^\nu)}_\ell. \tag{22}$$

We use here the velocity-pressure formulation of Leray solutions, with pressure $p^\nu \in W^{-1,\infty}(0,T;L^2(\mathbb{T}^d))$ (e.g. see theorem V.1.4 of [35]). The d equation (22) can then be obtained by mollifying the Navier–Stokes equations with (non-solenoidal) test functions φ_i, $i = 1,2,\ldots,d$, of the form $\varphi_i(r,t) := \psi(t)G_\ell(r-x)e_i$ where $\psi \in C_0^\infty((0,T))$, $G \in C^\infty(\mathbb{T}^d)$, and e_i is the unit vector in the ith coordinate direction.

We now show that the classical time derivative of $\overline{(u^\nu)}_\ell(x,t)$ exists for every $x \in \mathbb{T}^d$ and a.e. $t \in [0,T]$. See also Prop. 2 of [36]. Since Leray solutions satisfy $u^\nu \in L^\infty([0,T];L^2(\mathbb{T}^d))$, then for every $x \in \mathbb{T}^d$

$$
\begin{aligned}
\|\nabla\cdot[\overline{(u^\nu\otimes u^\nu)}_\ell](x,\cdot)\|_{L^\infty([0,T])} &\leqslant \frac{1}{\ell}\|(\nabla G)_\ell\|_\infty\|u\|^2_{L^\infty([0,T];L^2(\mathbb{T}^d))},\\
\|\nu\Delta\overline{(u^\nu)}_\ell(x,\cdot)\|_{L^\infty([0,T])} &\leqslant \frac{\nu}{\ell^2}\|(\Delta G)_\ell\|_2\|u\|_{L^\infty([0,T];L^2(\mathbb{T}^d))},
\end{aligned}
\tag{23}
$$

by Young's convolution inequality. The pressure-gradient term $\nabla\overline{(p^\nu)}_\ell(x,t)$ in equation (22) is determined using $\nabla\cdot f^\nu=0$ from the Poisson equation

$$
-\Delta\nabla\overline{(p^\nu)}_\ell(\cdot,t)=(\nabla\otimes\nabla\otimes\nabla):\overline{(u^\nu\otimes u^\nu)}_\ell(\cdot,t) \tag{24}
$$

and the righthand-side belongs to $C^\infty(\mathbb{T}^d)$ for a.e. time t and is bounded above by a constant of the form $(1/\ell^3)\|((\nabla\otimes\nabla\otimes\nabla)G)_\ell\|_\infty\|u(\cdot,t)\|^2_{L^2(\mathbb{T}^d)}$. The solution of the Poisson problem thus satisfies a similar estimate as equation (23), i.e. for some constant C and every $x\in\mathbb{T}^d$:

$$
\|\nabla\overline{(p^\nu)}_\ell(x,\cdot)\|_{L^\infty([0,T])}\leqslant\frac{C}{\ell^3}\|((\nabla\otimes\nabla\otimes\nabla)G)_\ell\|_\infty\|u\|^2_{L^\infty([0,T];L^2(\mathbb{T}^d))}. \tag{25}
$$

We thus see that, except for $\overline{(f^\nu)}_\ell(x,\cdot)$, every term in equation (22) for the distributional derivative $\partial_t\overline{(u^\nu)}_\ell(x,\cdot)$ belongs to $L^\infty([0,T])$. Since we assume that $f^\nu\in L^2([0,T];L^2(\mathbb{T}^d))$, we have for every $x\in\mathbb{T}^d$ at least:

$$
\|\overline{(f^\nu)}_\ell(x,\cdot)\|_{L^2([0,T])}\leqslant\|G_\ell\|_2\|f^\nu\|_{L^2([0,T];L^2(\mathbb{T}^d))}. \tag{26}
$$

It follows from equation (22) that $\partial_t\overline{(u^\nu)}_\ell(x,\cdot)\in L^2([0,T])$, so that $\overline{(u^\nu)}_\ell(x,\cdot)$ for every $x\in\mathbb{T}^d$ is absolutely continuous in time and the classical time-derivative exists and is given by equation (22) for a.e. $t\in[0,T]$.

Taking the Euclidean inner product of equation (22) with $\overline{(u^\nu)}_\ell(x,\cdot)$ for each $x\in\mathbb{T}^d$ and writing $\overline{(u^\nu\otimes u^\nu)}_\ell=\overline{(u^\nu)}_\ell\otimes\overline{(u^\nu)}_\ell+\tau_\ell(u^\nu,u^\nu)$ yields by the Leibniz product rule the 'resolved energy' balance:

$$
\partial_t\left(\frac{1}{2}|\overline{(u^\nu)}_\ell|^2\right)+\nabla\cdot J^\nu_\ell=-\Pi_\ell[u^\nu]-\nu|\nabla\overline{(u^\nu)}_\ell|^2+\overline{(u^\nu)}_\ell\cdot\overline{(f^\nu)}_\ell, \tag{27}
$$

with

$$
J^\nu_\ell:=\left(\frac{1}{2}|\overline{(u^\nu)}_\ell|^2+\overline{(p^\nu)}_\ell\right)\overline{(u^\nu)}_\ell+\overline{(u^\nu)}_\ell\cdot\tau_\ell(u^\nu,u^\nu)-\nu\nabla\left(\frac{1}{2}|\overline{(u^\nu)}_\ell|^2\right), \tag{28}
$$

which, again, holds for every $x\in\mathbb{T}^d$ and a.e. $t\in[0,T]$ (and thus distributionally in space-time as well). Since $|\overline{(u^\nu)}_\ell|^2(x,\cdot)/2$ is absolutely continuous in time, upon integrating we have:

$$
\begin{aligned}
&\frac{1}{2}|\overline{(u^\nu)}_\ell(x,T)|^2-\frac{1}{2}|\overline{(u^\nu_0)}_\ell(x)|^2\\
&\quad=\int_0^T\Big[-\nabla\cdot J^\nu_\ell-\Pi_\ell[u^\nu]-\nu|\nabla\overline{(u^\nu)}_\ell|^2+\overline{(u^\nu)}_\ell\cdot\overline{(f^\nu)}_\ell\Big](x,t)\,\mathrm{d}t
\end{aligned}
\tag{29}
$$

for every $T \geqslant 0$ and $x \in \mathbb{T}^d$. Since Leray solutions satisfy $u^\nu \in L^3([0,T];L^3(\mathbb{T}^d))$ and, consequently, $p^\nu \in L^{3/2}([0,T];L^{3/2}(\mathbb{T}^d))$ (see e.g. Proposition 1 of [4]), each term of the integrand inside the square brackets in equation (29) is easily checked by the definitions equations (19) and (20) to belong to $L^1([0,T];L^1(\mathbb{T}^d))$. The Fubini theorem then gives that $\int_{\mathbb{T}^d}\int_0^T \nabla\cdot J_\ell^\nu \,\mathrm{d}t\,\mathrm{d}x = \int_0^T\int_{\mathbb{T}^d} \nabla\cdot J_\ell^\nu \,\mathrm{d}x\,\mathrm{d}t = 0$ by space-periodicity, so that integrating equation (29) over $\mathbb{T}^d$, we obtain the global balance of resolved energy:

$$\frac{1}{2}\int_{\mathbb{T}^d} |\overline{(u^\nu)}_\ell(x,T)|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{T}^d} |\overline{(u_0)}_\ell(x)|^2 \mathrm{d}x + \int_0^T\int_{\mathbb{T}^d} \Pi_\ell[u^\nu]\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{T}^d} \nu|\nabla\overline{(u^\nu)}_\ell|^2\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_{\mathbb{T}^d} \overline{(u^\nu)}_\ell\cdot\overline{(f)}_\ell\,\mathrm{d}x\mathrm{d}t = 0. \tag{30}$$

We now show that any Leray solution satisfies the global energy balance equation (3) for almost every $T \geqslant 0$. Duchon and Robert [4] prove a local version of equation (3), i.e. they show that Leray solutions satisfy

$$\partial_t\left(\frac{1}{2}|u^\nu|^2\right) + \nabla\cdot\left[\left(\frac{1}{2}|u^\nu|^2 + p^\nu\right)u^\nu - \nu\nabla\left(\frac{1}{2}|u^\nu|^2\right)\right] = -\varepsilon[u^\nu] + u^\nu\cdot f \tag{31}$$

in the sense of distributions on space-time. We smear equation (31) with a test function of the form $\varphi^\epsilon(x,t) = \psi^\epsilon(t)\chi_{\mathbb{T}^d}(x)$, where $\psi^\epsilon(t)$ approximates the characteristic function of the time-interval $[0,T]$ and $\chi_{\mathbb{T}^d}(x)$ is the characteristic function of the whole torus (the constant function 1). This yields:

$$-\int_0^\infty \psi^{\epsilon\prime}\left(\int_{\mathbb{T}^d}\frac{1}{2}|u^\nu|^2\mathrm{d}x\right)\mathrm{d}t = -\int_0^\infty \psi^\epsilon\int_{\mathbb{T}^d}\varepsilon[u^\nu]\mathrm{d}x\mathrm{d}t + \int_0^\infty \psi^\epsilon\int_{\mathbb{T}^d} u^\nu\cdot f\,\mathrm{d}x\mathrm{d}t. \tag{32}$$

Recall that Leray solutions u^ν are right-continuous in time, strongly in $L^2(\mathbb{T}^d)$, for a.e. $t \geqslant 0$ and, in particular, at $t = 0$, as a consequence of the energy inequality (see remark 2 of [33]). To make use of this one-sided continuity, let $0 \leqslant \psi^\epsilon(t) \leqslant 1$ be supported on the interval $[0, T+\epsilon]$ and equal to 1 on $[\epsilon, T]$. The derivative $\psi^{\epsilon\prime}(t)$ gives the difference of two bump functions, one supported on $[T, T+\epsilon]$ and the other supported on $[0,\epsilon]$. Taking $\epsilon \to 0$ we obtain by the right-continuity that:

$$-\int_0^\infty \psi^{\epsilon\prime}\left(\int_{\mathbb{T}^d}\frac{1}{2}|u^\nu|^2\mathrm{d}x\right)\mathrm{d}t \to \int_{\mathbb{T}^d}\frac{1}{2}|u^\nu(x,T)|^2\mathrm{d}x - \int_{\mathbb{T}^d}\frac{1}{2}|u_0^\nu(x)|^2\mathrm{d}x, \qquad \text{a.e.}\quad T \geqslant 0. \tag{33}$$

The assumption $f^\nu \in L^2([0,T];L^2(\mathbb{T}^d))$, *a priori* estimate $u^\nu \in L^\infty([0,T];L^2(\mathbb{T}^d)) \cap L^2([0,T];H^1(\mathbb{T}^d))$ and the fact that $D[u^\nu]$ is a Radon measure permit the dominated convergence theorem to be applied to guarantee that as $\epsilon \to 0$

$$-\int_0^\infty \psi^\epsilon\int_{\mathbb{T}^d}\varepsilon[u^\nu]\mathrm{d}x\mathrm{d}t + \int_0^\infty \psi^\epsilon\int_{\mathbb{T}^d} u^\nu\cdot f\,\mathrm{d}x\mathrm{d}t \to -\int_0^T\int_{\mathbb{T}^d}\varepsilon[u^\nu]\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{T}^d} u^\nu\cdot f\,\mathrm{d}x\mathrm{d}t. \tag{34}$$

Thus, the global energy balance equation (3) is proved.

Adding to equation (3) the resolved energy balance equation (30) gives, for almost every $T \geqslant 0$,

$$\int_0^T \int_{\mathbb{T}^d} \varepsilon[u^\nu]\, \mathrm{d}x\mathrm{d}t = \int_0^T \int_{\mathbb{T}^d} \Pi_\ell[u^\nu]\, \mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}^d} \nu|\nabla\overline{(u^\nu)}_\ell|^2\, \mathrm{d}x\mathrm{d}t$$
$$- \frac{1}{2}\int_{\mathbb{T}^d}\left(|u^\nu(\cdot,T)|^2 - |\overline{(u^\nu(\cdot,T))}_\ell|^2\right)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^d}\left(|u_0|^2 - |\overline{(u_0)}_\ell|^2\right)\mathrm{d}x$$
$$+ \int_0^T \int_{\mathbb{T}^d}(u^\nu \cdot f - \overline{(u^\nu)}_\ell \cdot \overline{(f)}_\ell)\, \mathrm{d}x\mathrm{d}t.$$

Since, for integrable $g \in L^1(\mathbb{T}^d)$ one has $\int_{\mathbb{T}^d} \overline{g}_\ell(x)\mathrm{d}x = \int_{\mathbb{T}^d} g(x)\mathrm{d}x$, we arrive at identity equation (21). □

Proof of lemma 1. we first prove the upper bound on the total dissipation of Leray solutions. By lemma 2, the global energy dissipation is given by the formula equation (21). Note that $|\overline{(u^\nu)}_\ell|^2 \leqslant \overline{(|u^\nu|^2)}_\ell$ by convexity and thus the contribution from $\tau_\ell(u^\nu(\cdot,T);u^\nu(\cdot,T)) \geqslant 0$ in equation (21) is non-positive and we may drop it at the expense of an inequality:

$$\int_0^T \int_{\mathbb{T}^d} \varepsilon[u^\nu]\, \mathrm{d}x\mathrm{d}t \leqslant \int_0^T \int_{\mathbb{T}^d} \Pi_\ell[u^\nu]\, \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} \nu|\nabla\overline{(u^\nu)}_\ell|^2\, \mathrm{d}x\mathrm{d}t$$
$$+ \frac{1}{2}\int_{\mathbb{T}^d} \tau_\ell(u_0^\nu;u_0^\nu)\, \mathrm{d}x + \int_0^T \int_{\mathbb{T}^d} \tau_\ell(u^\nu;f^\nu)\, \mathrm{d}x\mathrm{d}t. \tag{35}$$

The inequality (35) then implies:

$$\int_0^T \int_{\mathbb{T}^d} \varepsilon[u^\nu]\, \mathrm{d}x\mathrm{d}t \leqslant \int_0^T \|\Pi_\ell[u^\nu]\|_1 \mathrm{d}t + \int_0^T \nu\|\nabla\overline{(u^\nu)}_\ell\|_2^2\, \mathrm{d}t$$
$$+ \frac{1}{2}\|\tau_\ell(u_0^\nu;u_0^\nu)\|_1 + \int_0^T \|\tau_\ell(u^\nu;f^\nu)\|_1 \mathrm{d}t. \tag{36}$$

The energy flux-through-scale is bounded using the Constantin-E-Titi commutator estimate [3]:

$$\int_0^T \|\Pi_\ell[u^\nu(t)]\|_1 \mathrm{d}t \leqslant C_G \ell^{3\sigma-1} \int_0^T \|u^\nu(t)\|^3_{B_3^{\sigma,\infty}(\mathbb{T}^d)} \mathrm{d}t = O(\ell^{3\sigma-1}). \tag{37}$$

Above, C_G is a constant depending on G but not on ℓ, ν and the 'big-O' notation denotes an upper bound with a constant prefactor depending only upon G and u. Next, using the nesting property $L^p(\mathbb{T}^d) \subseteq L^q(\mathbb{T}^d)$, $p \geqslant q$, we bound the resolved energy dissipation term

$$\int_0^T \nu\|\nabla\overline{(u^\nu)}_\ell\|_2^2\, \mathrm{d}t \leqslant \int_0^T \nu\|\nabla\overline{(u^\nu)}_\ell\|_3^2\, \mathrm{d}t \leqslant C'_G \nu \ell^{2(\sigma-1)} \int_0^T \|u^\nu(t)\|^2_{B_3^{\sigma,\infty}} \mathrm{d}t = O(\nu\ell^{2(\sigma-1)}). \tag{38}$$

The remaining terms in equation (36) are bounded using estimates for coarse-graining cumulants (see, e.g. [3, 34]):

$$\|\tau_\ell(u_0^\nu;u_0^\nu)\|_1 \leqslant C''_G \ell^{2\sigma} \sup_{\nu>0} \|u_0^\nu\|^2_{B_2^{\sigma,\infty}(\mathbb{T}^d)} = O(\ell^{2\sigma}), \tag{39}$$

$$\int_0^T \|\tau_\ell(u^\nu;f^\nu)\|_1 \mathrm{d}t \leqslant C''_G \ell^{2\sigma} \sup_{\nu>0} \|f^\nu\|_{L^2([0,T];B_2^{\sigma,\infty}(\mathbb{T}^d))} \sup_{\nu>0} \|u^\nu\|_{L^3([0,T];B_3^{\sigma,\infty}(\mathbb{T}^d))} = O(\ell^{2\sigma}). \tag{40}$$

Thus, combining the estimates equations (37)–(40) in the inequality equation (36), we find that:

$$\int_0^T \int_{\mathbb{T}^d} \varepsilon[u^\nu]\, \mathrm{d}x\mathrm{d}t = O(\ell^{3\sigma-1}) + O(\nu\ell^{2(\sigma-1)}). \tag{41}$$

Here a term $O(\ell^{2\sigma})$ has been absorbed into $O(\ell^{3\sigma-1})$, since for $\sigma \leqslant 1$ it is always smaller as $\ell \to 0$. Because $\ell > 0$ in equation (41) is arbitrary, we specify a relation between ℓ and ν which optimizes the upper bound by balancing the contribution of the nonlinear flux with the resolved dissipation. This fixes a relationship $\ell \sim \nu^{1/(\sigma+1)}$ and yields the final upper bound:

$$\int_0^T \int_{\mathbb{T}^d} \varepsilon[u^\nu]\, \mathrm{d}x\mathrm{d}t = O(\nu^{\frac{3\sigma-1}{\sigma+1}})$$

as claimed in equation (6). It is worth remarking that $\ell \sim \nu^{1/(\sigma+1)}$ is the expected scaling in phenomenological theory for the 'dissipation length' where nonlinear energy flux and viscous energy dissipation become comparable, when the velocity increments exhibit scaling $\delta u(\ell) \sim \ell^\sigma$. See [38, 39]. □

Proof of theorem 2. we now show under either condition (i) or (ii) that u is a weak solution of the Euler equations which satisfies distributionally the local energy balance:

$$\partial_t \left(\frac{1}{2}|u|^2\right) + \nabla \cdot \left[\left(\frac{1}{2}|u|^2 + p\right) u\right] = -D[u] + u \cdot f, \qquad D[u] := \mathcal{D}'\text{-}\lim_{\ell\to 0} \Pi_\ell[u]. \tag{42}$$

We prove these conclusions separately for condition (i) and for condition (ii):

Proof of theorem 2(i). We apply the Aubin–Lions–Simon lemma, stated as in theorem II.5.16 of [35], with $p = 3$, $r = 3/2$, $B_0 = B_3^{\sigma,\infty}(\mathbb{T}^d)$, $B_1 = L^3(\mathbb{T}^d)$, and $B_2 = B_{3/2}^{\sigma-2,\infty}(\mathbb{T}^d)$. The imbedding of $B_3^{\sigma,\infty}(\mathbb{T}^d)$ in $L^3(\mathbb{T}^d)$ is compact by the Kolmogorov–Riesz theorem and $L^3(\mathbb{T}^d) = F_3^{0,2}(\mathbb{T}^d)$, a Triebel–Lizorkin space (see [37], section 3.5), is continuously embedded in $B_{3/2}^{\sigma-2,\infty}(\mathbb{T}^d)$ (e.g. remark 3.5.1.4, [37]).

We now show that a distributional Navier–Stokes solution $u \in L^3([0,T]; B_3^{\sigma,\infty}(\mathbb{T}^d))$ has a weak time-derivative in the sense of definition II.5.7 of [35], which is given by

$$\frac{\mathrm{d}u^\nu}{\mathrm{d}t} = -\mathbb{P}\nabla \cdot (u^\nu \otimes u^\nu) + \nu\Delta u^\nu + f^\nu \in L^{3/2}([0,T]; B_{3/2}^{\sigma-2,\infty}(\mathbb{T}^d)), \tag{43}$$

with $\mathbb{P}$ the Leray projector. To see this, choose smooth test functions of the form $\varphi(t,x) = \psi(t)\phi(x)$ with $\psi \in C_0^\infty((0,T))$ and $\phi \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$, giving

$$\left\langle \int_0^T \partial_t\psi(t)u(t)\mathrm{d}t, \phi \right\rangle = -\left\langle \int_0^T \psi(t)\Big[-\mathbb{P}\nabla \cdot (u \otimes u)(t) + \nu\Delta u(t) + f^\nu(t)\Big]\mathrm{d}t, \phi \right\rangle, \tag{44}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual pairing between elements of $D'(\mathbb{T}^d)$ and $D(\mathbb{T}^d) = C^\infty(\mathbb{T}^d)$. We next observe that each term inside the square bracket on the righthand side of the previous equation belongs to $L^{3/2}([0,T]; B_{3/2}^{\sigma-2,\infty}(\mathbb{T}^d))$ with norms uniformly bounded in ν. First, by the Calderon–Zygmund inequality we have for some constant c_0 depending only on space dimension d the estimate

$$\|\mathbb{P}\nabla\cdot(u^\nu\otimes u^\nu)\|_{L^{3/2}([0,T];B^{\sigma-2,\infty}_{3/2}(\mathbb{T}^d))}\leqslant c_0\|u^\nu\otimes u^\nu\|_{L^{3/2}([0,T];B^{\sigma-1,\infty}_{3/2}(\mathbb{T}^d))}\leqslant c_0\|u^\nu\|^2_{L^3([0,T];B^{\sigma,\infty}_3(\mathbb{T}^d))}. \tag{45}$$

On the other hand,

$$\|\Delta u^\nu\|_{L^{3/2}([0,T];B^{\sigma-2,\infty}_{3/2}(\mathbb{T}^d))}\leqslant c_1\|u^\nu\|_{L^{3/2}([0,T];B^{\sigma,\infty}_{3/2}(\mathbb{T}^d))}\leqslant c_1\|u^\nu\|_{L^3([0,T];B^{\sigma,\infty}_3(\mathbb{T}^d))}. \tag{46}$$

Finally, because the sequence f^ν is strongly convergent, it is uniformly bounded in $L^2([0,T];L^2(\mathbb{T}^d))$ and

$$\|f^\nu\|_{L^{3/2}([0,T];B^{\sigma-2,\infty}_{3/2}(\mathbb{T}^d))}\leqslant\|f^\nu\|_{L^2([0,T];L^2(\mathbb{T}^d))}. \tag{47}$$

These bounds imply that the element of $D'(\mathbb{T}^d)$ which is paired with ϕ on the right side of equation (44) in fact belongs to $B^{\sigma-2,\infty}_{3/2}(\mathbb{T}^d)$. Moreover, $\int_0^T\partial_t\psi(t)u(t)\,\mathrm{d}t\in B^{\sigma,\infty}_3(\mathbb{T}^d)$ on the left side of equation (44). Since there is the Banach space duality $\left(B^{2-\sigma,1}_3(\mathbb{T}^d)\right)'=B^{\sigma-2,\infty}_{3/2}(\mathbb{T}^d)$ and $D(\mathbb{T}^d)$ is dense in $B^{2-\sigma,1}_3(\mathbb{T}^d)$ ([37], section 3.5.6), we can extend the relation equation (44) to $\phi\in B^{2-\sigma,1}_3(\mathbb{T}^d)$ by continuity and this implies the equality

$$\int_0^T\partial_t\psi(t)u(t)\mathrm{d}t=-\int_0^T\psi(t)\Big[-\mathbb{P}\nabla\cdot(u\otimes u)(t)+\nu\Delta u(t)+f^\nu(t)\Big]\mathrm{d}t, \tag{48}$$

as elements of $B^{\sigma-2,\infty}_{3/2}(\mathbb{T}^d)$. It follows that equation (43) holds in the sense of definition II.5.7 of [35].

By the estimates equations (45)–(47), one has furthermore

$$\left\|\frac{\mathrm{d}u^\nu}{\mathrm{d}t}\right\|_{L^{3/2}([0,T];B^{\sigma-2,\infty}_{3/2}(\mathbb{T}^d))}\leqslant c_0\|u^\nu\|^2_{L^3([0,T];B^{\sigma,\infty}_3(\mathbb{T}^d))}+\nu c_1\|u^\nu\|_{L^3([0,T];B^{\sigma,\infty}_3(\mathbb{T}^d))}+\|f^\nu\|_{L^2([0,T];L^2(\mathbb{T}^d))}. \tag{49}$$

In view of our assumptions (i) in theorem 2, the family of weak time-derivatives $\{\mathrm{d}u^\nu/\mathrm{d}t\}_{\nu>0}$ is uniformly bounded in $L^{3/2}([0,T];B^{\sigma-2,\infty}_{3/2}(\mathbb{T}^d))$. The conditions of the Aubin–Lions–Simon lemma are therefore satisfied, so that $\{u^\nu\}_{\nu>0}$ is relatively compact in $L^3([0,T],L^3(\mathbb{T}^d))$. Subsequences $\nu_k\to0^+$ thus always exist so that $u^{\nu_k}\to u$ strongly in $L^3(\mathbb{T}^d\times[0,T])$. For any such subsequence, we can apply the arguments of [4] to obtain the statements equations (13)–(15). □

Proof of theorem 2(ii). First we show any limit u is a weak Euler solution. Recall our assumptions equation (12): for $\nu\to0$

$$u^\nu(\cdot,t)\underset{L^3}{\rightharpoonup}u(\cdot,t),\quad(u^\nu\otimes u^\nu)(\cdot,t)\underset{L^{3/2}}{\rightharpoonup}(u\otimes u)(\cdot,t),\quad f^\nu(\cdot,t)\underset{L^2}{\rightharpoonup}f(\cdot,t)\quad\text{a.e. }t\in[0,T]. \tag{50}$$

These conditions imply that $\overline{(f^\nu)}_\ell\to\overline{(f)}_\ell$, $\overline{(u^\nu)}_\ell\to\overline{(u)}_\ell$ and $\overline{(u^\nu\otimes u^\nu)}_\ell\to\overline{(u\otimes u)}_\ell$ pointwise in space, a.e. t. Integrating the coarse-grained Navier–Stokes equations equation (22) against an arbitrary solenoidal test function $\varphi\in D([0,T]\times\mathbb{T}^d)$ yields:

$$-\langle\partial_t\varphi,\overline{(u^\nu)}_\ell\rangle=\langle\nabla\varphi,\overline{(u^\nu\otimes u^\nu)}_\ell\rangle+\nu\langle\Delta\varphi,\overline{(u^\nu)}_\ell\rangle+\langle\varphi,\overline{(f^\nu)}_\ell\rangle. \tag{51}$$

To show convergence as $\nu \to 0$, we obtain uniform bounds for all the integrands in equation (51) and apply Lebesgue dominated convergence. Such bounds are easily obtained by applying Young's inequality for convolutions:

$$|\overline{(u^\nu)}_\ell(x,t)| \leqslant \|G_\ell\|_{3/2}\|u^\nu(\cdot,t)\|_3 \lesssim \|u^\nu(\cdot,t)\|_3, \tag{52}$$

$$|\overline{(u^\nu \otimes u^\nu)}_\ell(x,t)| \leqslant \|G_\ell\|_3\|u^\nu \otimes u^\nu(\cdot,t)\|_{3/2} \lesssim \|u^\nu(\cdot,t)\|_3^2, \tag{53}$$

$$|\overline{(f^\nu)}_\ell(x,t)| \leqslant \|G_\ell\|_2\|f^\nu(\cdot,t)\|_2 \lesssim \|f^\nu(\cdot,t)\|_2, \tag{54}$$

where the notation $\lesssim$ indicates an upper bound with constant prefactor depending on G and ℓ, but not on ν. By our assumption $u^\nu \in L^3([0,T];L^3(\mathbb{T}^d))$ and $f^\nu \in L^2([0,T];L^2(\mathbb{T}^d))$ with norms uniformly bounded, all of the upper bounds equations (52)–(54) are in $L^1(\mathbb{T}^d \times [0,T])$ uniformly in $\nu > 0$. Note that the term in equation (51) with viscosity as a pre-factor vanishes as $\nu \to 0$

$$\nu\langle \Delta\varphi, \overline{(u^\nu)}_\ell\rangle \leqslant \nu\|\Delta\varphi\|_2\|u^\nu\|_{L^\infty([0,T];L^2(\mathbb{T}^d))} \underset{\nu\to 0}{\longrightarrow} 0. \tag{55}$$

We may therefore apply dominated convergence to obtain from equation (51) for fixed $\ell > 0$ that in the limit $\nu \to 0$

$$-\langle \partial_t\varphi, \bar{u}_\ell\rangle = \langle \nabla\varphi, \overline{(u\otimes u)}_\ell\rangle + \langle \varphi, \bar{f}_\ell\rangle.$$

The argument is completed by taking the limit $\ell \to 0$, using the fact that mollification can be removed strongly in L^p. Taking the limit of equation (56) thus shows that u is a weak Euler solution.

The energy balance equation (42) is proved by a very similar argument. Smearing the resolved energy balance equation (18) established in lemma 2 with an arbitrary test function $\varphi \in D([0,T] \times \mathbb{T}^d)$ yields:

$$\begin{aligned} -\langle \partial_t\varphi, \tfrac{1}{2}|\overline{(u^\nu)}_\ell|^2\rangle &= \langle \nabla\varphi, J_\ell^0[u^\nu]\rangle - \langle \Delta\varphi, \tfrac{\nu}{2}|\overline{(u^\nu)}_\ell|^2\rangle \\ &\quad + \langle \varphi, -\Pi_\ell[u^\nu] - \nu|\nabla\overline{(u^\nu)}_\ell|^2 + \overline{(u^\nu)}_\ell\cdot\overline{(f^\nu)}_\ell\rangle \end{aligned} \tag{56}$$

where $J_\ell^0[u^\nu]$ is the inviscid part of the energy current $J_\ell[u^\nu]$ defined in equation (19), or

$$J_\ell^0[u^\nu] := \left(\frac{1}{2}|\overline{(u^\nu)}_\ell|^2 + \overline{(p^\nu)}_\ell\right)\overline{(u^\nu)}_\ell + \overline{(u^\nu)}_\ell \cdot \tau_\ell(u^\nu, u^\nu).$$

First note that the terms involving viscosity as a pre-factor vanish pointwise in space-time:

$$\nu|\nabla\overline{(u^\nu)}_\ell(x,t)|^2 \leqslant \frac{\nu}{\ell^2}\|(\nabla G)_\ell\|_2^2\|u^\nu\|^2_{L^\infty([0,T];L^2(\mathbb{T}^d))} \underset{\nu\to 0}{\longrightarrow} 0, \tag{57}$$

$$\frac{\nu}{2}|\overline{(u^\nu)}_\ell(x,t)|^2 \leqslant \frac{\nu}{2}\|G_\ell\|_2^2\|u^\nu\|^2_{L^\infty([0,T];L^2(\mathbb{T}^d))} \underset{\nu\to 0}{\longrightarrow} 0. \tag{58}$$

The above bounds follow from Young's inequality for convolutions. Thus, the contribution from these terms will vanish in equation (56) for $\nu \to 0$ and we must now argue that the remaining terms converge.

In addition to the pointwise-in-x convergence of the mollified quantities discussed above, we have similarly that $\tau_\ell(u^\nu, u^\nu) \to \tau_\ell(u,u)$ pointwise in space for a.e. t. Moreover, by general theory of Calderón-Zygmund operators, the map $u^\nu \otimes u^\nu \to p^\nu$ is strongly continuous in $L^p(\mathbb{T}^d)$ for $p \in (1,\infty)$ (see e.g. [4]). In particular, for $p = 3/2$, the assumption on weak convergence of $u^\nu \otimes u^\nu$ in equation (50) implies that $p^\nu \rightharpoonup p$ weakly in $L^{3/2}(\mathbb{T}^d)$ a.e. t. Thus, all of the following terms converge pointwise in space, for a.e. t:

$$\frac{1}{2}|\overline{(u^\nu)}_\ell|^2 \to \frac{1}{2}|\bar{u}_\ell|^2, \quad J^0_\ell[u^\nu] \to J^0_\ell[u], \quad \Pi_\ell[u^\nu] \to \Pi_\ell[u], \quad \overline{(u^\nu)}_\ell \cdot \overline{(f^\nu)}_\ell \to \bar{u}_\ell \cdot \bar{f}_\ell \tag{59}$$

since they are made up of products of objects which converge pointwise.

Once again, convergence in the sense of distributions follows if integrable bounds can be obtained that allow us to infer limits of the smeared terms in equation (56) by dominated convergence. Recall by our assumptions that $u^\nu \in L^3([0,T]; L^3(\mathbb{T}^d))$ and $p^\nu \in L^{3/2}([0,T]; L^{3/2}(\mathbb{T}^d))$ not only for each $\nu > 0$ (as holds for every Leray solution) but also with norms bounded uniformly in $\nu > 0$. Using Young's inequality for convolutions and Hölder's inequality, we have pointwise in space-time:

$$|\nabla\overline{(u^\nu)}_\ell(x,t)| \leqslant \frac{1}{\ell}\|(\nabla G)_\ell\|_{3/2}\|u^\nu(\cdot,t)\|_3 \lesssim \|u^\nu(\cdot,t)\|_3 \tag{60}$$

$$|\tau_\ell(u^\nu,u^\nu)(x,t)| \leqslant \|G_\ell\|_3\|(u^\nu \otimes u^\nu)(\cdot,t)\|_{3/2} + \|G_\ell\|^2_{3/2}\|u^\nu(\cdot,t)\|^2_3 \lesssim \|u^\nu(\cdot,t)\|^2_3. \tag{61}$$

Likewise we have for the terms appearing in equation (56) that

$$\begin{aligned} &\frac{1}{2}|\overline{u^\nu(x,t)}_\ell|^2 \lesssim \|u^\nu(\cdot,t)\|^2_2, \qquad |J^0_\ell[u^\nu](x,t)]| \lesssim \|u^\nu(\cdot,t)\|^3_3 + \|p^\nu(\cdot,t)\|_{3/2}\|u^\nu(\cdot,t)\|_3, \\ &|\Pi_\ell[u^\nu](x,t)]| \lesssim \|u^\nu(\cdot,t)\|^3_3, \qquad |\overline{(u^\nu)}_\ell(x,t)\cdot\overline{(f^\nu)}_\ell(x,t)| \lesssim \|u^\nu(\cdot,t)\|_2\|f^\nu(\cdot,t)\|_2. \end{aligned} \tag{62}$$

Since all of the latter upper bounds are in $L^1(\mathbb{T}^d \times [0,T])$ uniformly in $\nu > 0$ under our assumptions, we can apply dominated convergence theorem to obtain from equation (56) for fixed $\ell > 0$ that in the limit $\nu \to 0$

$$\partial_t\left(\frac{1}{2}|\bar{u}_\ell|^2\right) + \nabla\cdot J^0_\ell[u] = -\Pi_\ell[u] + \bar{u}_\ell\cdot\bar{f}_\ell, \tag{63}$$

in the sense of space-time distributions. We note in particular that

$$\mathcal{D}'\text{-}\lim_{\nu\to 0}\Pi_\ell[u^\nu] = \Pi_\ell[u] := -\nabla\overline{(u)}_\ell : \tau_\ell(u,u). \tag{64}$$

The argument is completed by taking the limit $\ell \to 0$ of equation (63) and showing that equation (42) holds distributionally. This fact is proved in [4] using a somewhat different regularization. For all terms except $\Pi_\ell[u]$, distributional convergence follows directly from the strong continuity of shifts in L^p since $u \in L^3([0,T]; L^3(\mathbb{T}^d))$ and $p \in L^{3/2}([0,T]; L^{3/2}(\mathbb{T}^d))$. In particular, the term $\bar{u}_\ell \cdot \tau_\ell(u,u)$ in $J^0_\ell[u]$ vanishes by the commutator identity for $\tau_\ell(u,u)$ in [3]. Convergence of the flux $\Pi_\ell[u]$ is then inferred from the distributional equality:

$$-\mathcal{D}'\text{-}\lim_{\ell\to 0}\Pi_\ell[u] = \partial_t\left(\frac{1}{2}|u|^2\right) + \nabla\cdot\left[\left(\frac{1}{2}|u|^2 + p\right)u\right] - u\cdot f := D[u]. \tag{65}$$

Under condition (i), the limiting Euler solutions $u \in L^3(\mathbb{T}^d \times [0,T])$ have additional space-regularity. The uniform boundedness condition in (i) of theorem 2, $\sup_{\nu>0} \|u^\nu\|_{L^3([0,T];B_3^{\sigma,\infty}(\mathbb{T}^d))} < \infty$, implies that

$$\|u^\nu\|_{L^3(\mathbb{T}^d\times[0,T])} < C', \quad \|u^\nu(\cdot + r, \cdot) - u^\nu\|_{L^3(\mathbb{T}^d\times[0,T])} < C|r|^\sigma \tag{66}$$

with constants C, C' independent of viscosity. The inequalities equation (66) are preserved under strong limits in $L^3(\mathbb{T}^d \times [0,T])$ and thus the limiting Euler solutions u under condition (i) satisfy them as well. This yields immediately $u \in L^3([0,T], B_3^{\sigma',c_0}(\mathbb{T}^d))$ for any $\sigma' < \sigma$, with definitions as in remark 2. Finally, $D[u] = 0$ for $\sigma \in (1/3, 1]$ follows from the additional space-regularity by the results of [5]. □

Remark 6. Although not stated in the theorem, the inequalities equation (66) are again preserved in the limit if we add to condition (ii) the assumption that equation (66) holds with constants C, C' independent of viscosity. Weak lower-semicontinuity of the $L^3(\mathbb{T}^d)$-norm and of

$$\|u^\nu(\cdot + r, t) - u^\nu(\cdot, t)\|_3 = \sup_{\|w\|_{3/2}=1} |\langle w(\cdot - r) - w, u^\nu(\cdot, t)\rangle| \tag{67}$$

and Fatou's lemma in time, together with the assumption equation (66), guarantees that limiting Euler solutions u under this strengthened condition (ii) satisfy the same bound. This is analogous to remark 3.5 in [18].

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