Inviscid Limit of Vorticity Distributions in the Yudovich Class

PETER CONSTANTIN
Princeton University

THEODORE D. DRIVAS
Princeton University

AND

TAREK M. ELGINDI
UC San Diego

Abstract

We prove that given initial data $\omega_0 \in L^\infty(\mathbb{T}^2)$, forcing

$$g \in L^\infty(0,T;L^\infty(\mathbb{T}^2)),$$

and any $T > 0$, the solutions u^ν of Navier-Stokes converge strongly in

$$L^\infty(0,T;W^{1,p}(\mathbb{T}^2))$$

for any $p \in [1,\infty)$ to the unique Yudovich weak solution u of the Euler equations. A consequence is that vorticity distribution functions converge to their inviscid counterparts. As a by-product of the proof, we establish continuity of the Euler solution map for Yudovich solutions in the L^p vorticity topology. The main tool in these proofs is a uniformly controlled loss of regularity property of the linear transport by Yudovich solutions. Our results provide a partial foundation for the Miller-Robert statistical equilibrium theory of vortices as it applies to slightly viscous fluids. © 2020 Wiley Periodicals LLC

1 Introduction

In this paper we discuss the connection between Yudovich solutions of the Euler equations

$$\partial_t \omega + u \cdot \nabla \omega = g, \tag{1.1}$$

with bounded forcing $g \in L^\infty(0,T;L^\infty(\mathbb{T}^2))$, and initial data

$$\omega(0) = \omega_0 \in L^\infty(\mathbb{T}^2), \tag{1.2}$$

and the vanishing viscosity limit ($\lim_{\nu\to 0}$) of solutions of the Navier-Stokes equations,

$$\partial_t \omega^\nu + u^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu + g, \tag{1.3}$$

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with initial data

$$\omega^\nu(0) = \omega_0^\nu \in L^\infty(\mathbb{T}^2), \tag{1.4}$$

and the same forcing g. We consider uniformly bounded initial data

$$\sup_{\nu>0} \|\omega_0^\nu\|_{L^\infty(\mathbb{T}^2)} \le \Omega_{0,\infty} < \infty. \tag{1.5}$$

The solutions of (1.1), (1.2), (1.3), and (1.4) are uniformly bounded in $L^\infty(\mathbb{T}^2)$:

$$\sup_{\nu\ge 0}\sup_{0\le t\le T} \|\omega^\nu(t)\|_{L^\infty(\mathbb{T}^2)} \le \Omega_\infty = \Omega_{0,\infty} + \int_0^T \|g(t)\|_{L^\infty(\mathbb{T}^2)} dt. \tag{1.6}$$

This bound is valid in $\mathbb{T}^2$ or $\mathbb{R}^2$ but is not available if boundaries are present or in three dimensions. The bound will be used repeatedly below.

We are interested in the small viscosity behavior of vorticity distribution function $\pi_{\omega^\nu(t)}(\mathrm{d}y)$ defined by

$$\int f(y)\pi_{\omega^\nu(t)}(\mathrm{d}y) = \int f(\omega^\nu(t,x))\mathrm{d}x, \tag{1.7}$$

for all continuous functions (observables) f. If $\omega_0^\nu \to \omega_0$ we prove that the distributions convergence

$$\pi_{\omega^\nu(t)}(\mathrm{d}y) \xrightarrow{\nu\to 0} \pi_{\omega(t)}(\mathrm{d}y) = \pi_{\omega_0}(\mathrm{d}y), \tag{1.8}$$

where the time invariance of the vorticity distribution function for the Euler equations follows from Lagrangian transport $\omega(t) = \omega_0 \circ X_t^{-1}$ and volume preservation of the homeomorphism $A_t = X_t^{-1}$.

The statement (1.8) is a consequence of the strong convergence of the vorticity in $L^\infty(0,T;L^p(\mathbb{T}^2))$ for all $p\in[1,\infty)$ and for any $T>0$. We prove this fact here, extending previous work for vortex patch solutions with smooth boundary [6], and removing additional assumptions on the Euler path [7]. Implications of our result for equilibrium theories of decaying two-dimensional turbulence [17,19] are briefly discussed at the end of this paper. Our main result is the following.

THEOREM 1. Let ω be the unique Yudovich weak solution of the Euler equations with initial data $\omega_0 \in L^\infty(\mathbb{T}^2)$ and forcing $g\in L^\infty(0,T;L^\infty(\mathbb{T}^2))$. Let ω^ν be the solution of the Navier-Stokes equation with the same forcing and initial data $\omega_0^\nu \to \omega_0$ strongly in $L^2(\mathbb{T}^2)$. Then, for any $T>0$ and $p\in[1,\infty)$, the inviscid limit $\omega^\nu\to\omega$ holds strongly in $L^\infty(0,T;L^p(\mathbb{T}^2))$:

$$\lim_{\nu\to 0}\sup_{0\le t\le T} \|\omega^\nu(t)-\omega(t)\|_{L^p(\mathbb{T}^2)} = 0. \tag{1.9}$$

Consequently, the distributions converge,

$$\lim_{\nu\to 0} \pi_{\omega^\nu(t)}(\mathrm{d}y) = \pi_{\omega_0}(\mathrm{d}y) \tag{1.10}$$

for all $t\in[0,T]$.

REMARK 1. There are several senses in which this theorem is sharp. First, there can be no infinite time result as the Euler solution is conservative and the Navier-Stokes solution is dissipative. This is obvious if we consider the stationary solutions $\omega_0(x) = \sin(Nx)$ and $g = 0$. Secondly, there can be no rate without additional regularity assumptions on ω_0, as is the case for the heat equation. Thirdly, there can be no strong convergence in L^∞ because ω_0 may not be continuous while ω^ν is smooth for any $t > 0$. Finally, there can be no strong convergence for $p > 1$ in domains with boundaries if the boundary condition of the Navier-Stokes solutions is no slip, and the Euler solution has nonvanishing tangential velocity at the boundary, in other words, if there are boundary layers [14].

REMARK 2. One implication of Theorem 1 is that the dissipation of convex functions of vorticity must vanish,

$$\lim_{\nu\to 0} \nu \int_0^T \int_{\mathbb{T}^2} f''(\omega^\nu)|\nabla \omega^\nu|^2 \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{1.11}$$

In the special case when $f(x) = |x|^2/2$, the above is the enstrophy dissipation (palenstrophy). In fact, it was proved by Eyink that anomalous enstrophy dissipation requires that $\omega_0 \notin L^2(\mathbb{T}^2)$ [11, 13]. The idea is that, if $\omega_0 \in L^2(\mathbb{T}^2)$, the enstrophy remains uniformly-in-ν bounded since it is nonincreasing under the Navier-Stokes evolution. Applying the Aubin-Lions lemma yields weak convergence on subsequences to ω, a weak solution of the Euler equations (possibly nonunique). Thus $\omega^\nu \to \omega$ in $C(0, T; w - L^2(\mathbb{T}^2))$. Moreover, for such initial data, all weak Euler solutions can be shown to be renormalized in the sense of DiPerna-Lions and hence conservative [8]. Thus, by weak lower semicontinuity of the L^2 norm, the Navier-Stokes enstrophy balance implies also that norms converge and hence the convergence is strong in L^2, pointwise in time, i.e., $\omega^\nu(t) \to \omega$ in $L^2(\mathbb{T}^2)$ for each $t \in [0, T]$. In fact, whenever the vorticity converges weakly to a conservative weak Euler solution, one has strong convergence and there can be no anomaly. The convergence can be made uniform in time. This proof using compactness, however, inherently gives a qualitative statement, and one cannot extract information about rates of convergence. On the other hand, our proof is quantitative. Specifically, given information on, say, the spectrum of the initial vorticity at high wavenumber, one can obtain a rate of convergence. One class of examples that we discuss in Corollary 2 concerns vorticity in the space $\omega_0 \in L^\infty \cap B^s_{p,\infty}$ for $s > 0$. However, more generally, for any $\omega_0 \in L^\infty$ our proof provides a computable rate of convergence depending on ω_0.

A corollary of the proof of Theorem 1 and Lemma 4 is the continuity of the Yudovich solution map $\omega(t) = S_t(\omega_0)$ in the L^p topology when restricted to fixed balls in L^∞.

COROLLARY 1. For any $\omega_0, \omega_0^n \in L^\infty(\mathbb{T}^2)$ such that ω_0^n is uniformly bounded in $L^\infty(\mathbb{T}^2)$ and $\omega_0^n \to \omega_0$ as $n \to \infty$ strongly in $L^2(\mathbb{T}^2)$ we have

$$\lim_{n\to\infty} \big\| S_t(\omega_0^n) - S_t(\omega_0)\big\|_{L^p(\mathbb{T}^2)} = 0 \tag{1.12}$$

for each time $t > 0$.

The proof of Theorem 1 is based on the fact that linear transport by Yudovich solutions has a short-time uniformly controlled loss of regularity: it maps bounded sets in $W^{1,p}$, $p > 2$, to bounded sets in H^1, uniformly in viscosity. More precisely, we consider the Yudovich solutions $\omega(t)$ and $\omega^\nu(t)$ of the Euler and Navier-Stokes equations with initial data $\omega_0 \in L^\infty$ and denote their corresponding velocities by $u(t)$ and $u^\nu(t)$, respectively. We take a sequence of regularizations $\omega_{0,n} \in W^{1,\infty}$ of ω_0, which is uniformly bounded in $W^{1,p}$, $p > 2$, and is such that $\omega_{0,n} \to \omega_0$ strongly in L^2. We let $\omega_n(t)$ be the unique solutions of the linear transport problems

$$\partial_t \omega_n + u \cdot \nabla \omega_n = 0$$

and $\omega_n^\nu(t)$ of

$$\partial_t \omega_n^\nu + u^\nu \cdot \nabla \omega_n^\nu = \nu \Delta \omega_n^\nu.$$

On one hand, $\omega_n(t)$ remains close to $\omega(t)$ and $\omega_n^\nu(t)$ remains close to $\omega^\nu(t)$ in L^p spaces because linear transport by Yudovich velocities is clearly bounded in L^p. The essential additional ingredient we show is a controlled loss of regularity: $\omega_n(t)$ and $\omega_n^\nu(t)$ are bounded in H^1 on a short time interval by their initial norms in $W^{1,p}$, $p > 2$. This uses the fact that ∇u and ∇u^ν are exponentially integrable. The rest of the proof rests on these observations as well as energy estimates and a time splitting.

In the direction of propagating regularity, we also prove the fact that if additional smoothness is assumed on the data, then some degree of fractional smoothness in L^p can be propagated uniformly in viscosity. We consider the unforced case $g = 0$ and fix initial data $\omega_0^\nu = \omega_0$ for simplicity, the natural extension being straightforward.

PROPOSITION 1. Suppose $\omega_0 \in (L^\infty \cap B^s_{p,\infty})(\mathbb{T}^2)$ for some $s > 0$ and some $p \geq 1$. Then the solutions of the Navier-Stokes equations satisfy $\omega^\nu(t) \in (L^\infty \cap B^{s(t)}_{p,\infty})(\mathbb{T}^2)$ uniformly in ν, where

$$s(t) = s\exp(-Ct\|\omega_0\|_{L^\infty(\mathbb{T}^2)})$$

for some universal constant $C > 0$.

The proof of Proposition 1 relies on the fact that the velocity is log-Lipschitz uniformly in ν and shows that the exponential estimate with loss of [1] holds uniformly in viscosity. Our proof uses the stochastic Lagrangian representation formula of [5]:

$$\mathrm{d}X_t(x) = u^\nu(X_t(x), t)\mathrm{d}t + \sqrt{2\nu}\,\mathrm{d}W_t, \quad X_0(x) = x, \tag{1.13}$$

yielding the representation formula

$$\omega^\nu(t) = \mathbb{E}[\omega_0 \circ A_t] \tag{1.14}$$

where back-to-labels map is defined as $A_t = X_t^{-1}$. The noisy Lagrangian picture allows for a nearly direct application of the theorems and proofs of [1, 2] to the viscous case. We remark that the uniform Sobolev regularity can be established by similar arguments; if $\omega_0 \in (L^\infty \cap W^{s,p})(\mathbb{T}^2)$, then $\omega^\nu(t) \in (L^\infty \cap W^{s(t),p})(\mathbb{T}^2)$ with uniformly bounded norms.

The uniform regularity of Proposition 1 is used to deduce the following:

COROLLARY 2. Let $\omega_0 \in (L^\infty \cap B^s_{2,\infty})(\mathbb{T}^2)$ with $s > 0$, and let ω and ω^ν solve respectively (1.1) and (1.3), with the same initial data $\omega_0^\nu = \omega_0$. Then the L^p convergence of vorticity, for any $p \in [1, \infty)$ and any finite time $T > 0$, occurs at the rate

$$\sup_{t\in[0,T]} \|\omega^\nu(t) - \omega(t)\|_{L^p(\mathbb{T}^2)} \lesssim (\nu T)^{\frac{s\exp(-2CT\|\omega_0\|_\infty)}{p(1+s\exp(-CT\|\omega_0\|_\infty))}-}, \tag{1.15}$$

with the universal constant $C > 0$ in Proposition 1.

REMARK 3. Recently the estimate with loss of [1] was sharpened for fixed $p \in (1, \infty)$ in [3], where it is shown that the propagated regularity decays inversely with time rather than exponentially, i.e., $\widetilde{s}(t) = s/(1 + Ctps)$ for some universal constant $C > 0$. See corollary 1.4 of [3]. This improvement is accomplished by taking greater advantage of the uniform exponential integrability of the velocity gradient stated in Lemma 1 below. The stochastic representation can also be used to show uniform boundedness of the vorticity in $\omega^\nu(t) \in (L^\infty \cap B^{\widetilde{s}(t)}_{p,\infty})$ as was done in Proposition 1. We omit details here, which are straightforward extensions of the proofs of [3]. This extension can lead to an improved rate in Corollary 2.

Corollary 2 applies in particular to the inviscid limits of vortex patches with non-smooth boundary. Indeed, lemma 3.2 of [7] shows that if $\omega_0 = \chi_\Omega$ is the characteristic function of a bounded domain whose boundary has box-counting (fractal) dimension D not larger than the dimension of space $d = 2$, i.e., $d_F(\partial\Omega) := D < 2$, then $\omega_0 \in B^{(2-D)/p}_{p,\infty}(\mathbb{T}^2)$. Proposition 1 then shows that some degree of fractional Besov regularity of the solution $\omega^\nu(t)$ is retained uniformly in viscosity for any finite time $T < \infty$ and Corollary 2 provides a rate depending only D, T, and p at which the vanishing viscosity limit holds, removing therefore the need for the additional assumptions on the solution imposed in [7].

2 Proofs

PROOF OF THEOREM 1. It suffices to prove that

$$\lim_{\nu\to 0} \sup_{t\in[0,T]} \|\omega^\nu(t) - \omega(t)\|_{L^2(\mathbb{T}^2)} = 0. \tag{2.1}$$

Indeed, convergence in L^p for any $p \in [2, \infty)$ then follows from interpolation and boundedness in L^∞:

$$\|\omega^\nu(t) - \omega(t)\|_{L^p(\mathbb{T}^2)} \le 2\Omega_\infty^{\frac{p-2}{p}} \|\omega^\nu(t) - \omega(t)\|_{L^2(\mathbb{T}^2)}^{\frac{2}{p}}. \tag{2.2}$$

In order to establish strong $L^\infty_t L^2_x$ convergence for arbitrary finite times T, it is enough to the convergence for a short time which depends only on a uniform L^∞ bound on the initial vorticity:

PROPOSITION 2. Let ω and ω^ν solve (1.1) and (1.3) respectively, with initial data (1.2) and (1.4). Assume that the Navier-Stokes initial data converge uniformly in $L^2(\mathbb{T}^2)$

$$\lim_{\nu\to 0} \|\omega_0^\nu - \omega_0\|_{L^2(\mathbb{T}^2)} = 0. \tag{2.3}$$

Assume also that there exists a contant Ω_∞ such that the initial data are uniformly bounded in $L^\infty(\mathbb{T}^2)$:

$$\sup_{\nu>0} \|\omega_0^\nu\|_{L^\infty(\mathbb{T}^2)} \le \Omega_\infty. \tag{2.4}$$

Then there exists a constant C_* such that the vanishing viscosity limit holds,

$$\lim_{\nu\to 0} \sup_{t\in[0,T_*]} \|\omega^\nu(t) - \omega(t)\|_{L^2(\mathbb{T}^2)} = 0, \tag{2.5}$$

on the time interval $[0, T_*]$ where

$$T_* = (C_*\Omega_\infty)^{-1}. \tag{2.6}$$

Once this proposition is established, the proof of Theorem 1 follows by dividing the time interval $[0, T]$ in subintervals

$$[0, T] = [0, T_*] \cup [T_*, 2T_*] \cup \cdots$$

where T_* is determined from the uniform bound (1.6) and by applying Proposition 2 to each interval, with initial data $\omega(nT_*)$ and $\omega^\nu(nT_*)$ for Euler and Navier-Stokes, respectively. . As there is no required rate of convergence for the initial data in Proposition 2, Theorem 1 follows.

PROOF OF PROPOSITION 2. We introduce functions ω_ℓ and ω_ℓ^ν, which are the unique solutions of the following *linear* problems. We fix $\ell > 0$ and let

$$\partial_t \omega_\ell + u \cdot \nabla \omega_\ell = \varphi_\ell * g, \qquad \omega_\ell(0) = \varphi_\ell * \omega_0, \tag{2.7}$$

$$\partial_t \omega_\ell^\nu + u^\nu \cdot \nabla \omega_\ell^\nu = \nu\Delta\omega_\ell^\nu + \varphi_\ell * g, \quad \omega_\ell^\nu(0) = \varphi_\ell * \omega_0^\nu, \tag{2.8}$$

where φ_ℓ is a standard mollifier at scale ℓ and where u and u^ν are respectively the unique solutions of Euler and Navier-Stokes equations. Note that the solutions to the linear problems (2.7) and (2.8) exist globally and are unique because the Yudovich velocity field u is log-Lipshitz. We observe that we have

$$\|\omega^\nu(t) - \omega(t)\|_{L^2(\mathbb{T}^2)} \le \|\omega(t) - \omega_\ell(t)\|_{L^2(\mathbb{T}^2)} + \|\omega^\nu(t) - \omega_\ell^\nu(t)\|_{L^2(\mathbb{T}^2)} + \|\omega_\ell^\nu(t) - \omega_\ell(t)\|_{L^2(\mathbb{T}^2)}.$$

Because the equations for $\omega_\ell, \omega_\ell^\nu$ and, respectively, ω, ω^ν share the same incompressible velocities, we find

$$\|\omega(t) - \omega_\ell(t)\|_{L^2(\mathbb{T}^2)} \le \|\omega_0 - \varphi_\ell * \omega_0\|_{L^2(\mathbb{T}^2)} + \int_0^t \|g(s) - \varphi_\ell * g(s)\|_{L^2(\mathbb{T}^2)} \mathrm{d}s, \tag{2.9}$$

$$\|\omega^\nu(t) - \omega_\ell^\nu(t)\|_{L^2(\mathbb{T}^2)} \le \|\omega_0^\nu - \varphi_\ell * \omega_0^\nu\|_{L^2(\mathbb{T}^2)} + \int_0^t \|g(s) - \varphi_\ell * g(s)\|_{L^2(\mathbb{T}^2)} \mathrm{d}s. \tag{2.10}$$

As mollification can be removed strongly in L^p, the two terms in the right-hand sides converge to 0 as $\ell, \nu \to 0$ in any order. It remains to show that

$$\lim_{\nu\to 0} \sup_{t\in[0,T_*]} \left\|\omega_\ell^\nu(t) - \omega_\ell(t)\right\|_{L^2(\mathbb{T}^2)} \to 0 \tag{2.11}$$

for fixed ℓ. In order to establish this, we use two auxiliary results. The first one is a general statement about the Biot-Savart law in dimension two.

LEMMA 1. Let $\omega \in L^\infty(\mathbb{T}^2)$, and let u be obtained from ω by the Biot-Savart law

$$u = K[\omega] = \nabla^\perp(-\Delta)^{-1}\omega.$$

There exist constants $\gamma > 0$ (nondimensional and C_K (with units of area) such that

$$\int_{\mathbb{T}^2} \exp\{\beta|\nabla u(x)|\}\mathrm{d}x \le C_K \tag{2.12}$$

holds for any $\beta > 0$ such that

$$\beta\|\omega\|_{L^\infty(\mathbb{T}^2)} \le \gamma. \tag{2.13}$$

PROOF OF LEMMA 1. The bound (2.12) holds due to the fact that Calderon-Zygmund operators map L^∞ to BMO [21], $\omega \in L^\infty \mapsto \nabla u = \nabla K[u] \in \mathrm{BMO}$, and from the John-Nirenberg inequality [12] for BMO functions. We provide below a direct and elementary argument (modulo a fact about norms of singular intergal operators) for the sake of completeness.

We recall that there exists a constant C_* so that for all $p \ge 2$,

$$\|\nabla K[v]\|_{L^p(\mathbb{T}^2)} = \|\nabla \otimes \nabla(-\Delta)^{-1}v\|_{L^p(\mathbb{T}^2)} \le C_* p\|v\|_{L^p(\mathbb{T}^2)} \tag{2.14}$$

(see [21]). The dependence of (2.14) on p is the important point. Thus,

$$\int_{\mathbb{T}^2} e^{\beta|\nabla u|}\mathrm{d}x = \sum_{p=0}^\infty \beta^p \frac{\|\nabla u\|^p_{L^p(\mathbb{T}^2)}}{p!} \le \sum_{p=0}^\infty \frac{\left(C_*\beta\|\omega\|_{L^p(\mathbb{T}^2)}\right)^p p^p}{p!}$$
$$\le |\mathbb{T}^2| \sum_{p=0}^\infty \frac{\left(C_*\beta\|\omega\|_{L^\infty(\mathbb{T}^2)}\right)^p p^p}{p!}. \tag{2.15}$$

This is a convergent series provided $C_*\beta\|\omega\|_{L^\infty(\mathbb{T}^2)} < 1/e$. Indeed, this can be seen using Stirling's bound $n! \geq \sqrt{2\pi} n^{n+1/2} e^{-n}$, which yields

$$\sum_{p=0}^{\infty} \frac{c^p p^p}{p!} \leq 1 + \sum_{p=1}^{\infty} \frac{p^{-1/2}}{\sqrt{2\pi}} (ce)^p \leq \frac{1}{1-ce} \quad \text{provided } c \in [0, 1/e) \tag{2.16}$$

where $c := C_*\beta\|\omega\|_{L^\infty(\mathbb{T}^2)}$. We may take thus

$$\gamma = (2C_* e)^{-1}, \quad C_K = 2|\mathbb{T}^2|. \tag{2.17}$$

The constant γ depends on the Biot-Savart kernel and is nondimensional; the constant C_K then is proportional to the area of the domain. □

The second auxiliary result concerns scalars transported and amplified by a velocity with bounded curl in two dimensions.

LEMMA 2. Let $u := u(x,t)$ be divergence free and

$$\omega := \nabla^\perp \cdot u \in L^\infty(0,T; L^\infty(\mathbb{T}^2)) \quad \text{with} \sup_{0\leq t\leq T} \|\omega(t)\|_{L^\infty(\mathbb{T}^2)} \leq \Omega_\infty.$$

Consider a nonnegative scalar field $\theta := \theta(x,t)$ satisfying the differential inequality

$$\partial_t \theta + u \cdot \nabla\theta - \nu\Delta\theta \leq |\nabla u|\theta + f, \tag{2.18}$$

with initial data $\theta|_{t=0} = \theta_0 \in L^\infty(\mathbb{T}^2)$, and forcing $f \in L^\infty(0,T; L^\infty(\mathbb{T}^2))$. Let $\gamma > 0$ be the constant from Lemma 1. Then, for any $p > 1$ and the time $T(p) = \frac{\gamma(p-1)}{2p\Omega_\infty}$, it holds that

$$\sup_{t\in[0,T(p)]} \|\theta(t)\|_{L^2(\mathbb{T}^2)} \leq C_1 \|\theta_0\|^p_{L^{2p}(\mathbb{T}^2)} + C_2 \tag{2.19}$$

for some constants C_1, C_2 depending only on p, Ω_∞ and $\|f\|_{L^\infty(0,T;L^\infty(\mathbb{T}^2))}$.

PROOF OF LEMMA 2. Let $p := p(t)$ with $p(0) = p_0$ and time dependence of $p(t)$ to be specified below. Consider

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2} |\theta|^{2p(t)} \mathrm{d}x \\
&\quad = p'(t)\int_{\mathbb{T}^2} \ln|\theta||\theta|^{2p(t)}\mathrm{d}x + p(t)\int_{\mathbb{T}^2} |\theta|^{2p(t)-2}\theta\partial_t\theta\,\mathrm{d}x \\
&\quad \leq p'(t)\int_{\mathbb{T}^2} \ln|\theta||\theta|^{2p(t)}\mathrm{d}x - p(t)\int_{\mathbb{T}^2} |\theta|^{2p(t)-2}\theta u\cdot\nabla\theta\,\mathrm{d}x \\
&\qquad + \nu p(t)\int_{\mathbb{T}^2} |\theta|^{2p(t)-2}\theta\Delta\theta\,\mathrm{d}x + p(t)\int_{\mathbb{T}^2} |\theta|^{2p(t)-2}|\nabla u|\theta^2\,\mathrm{d}x \\
&\qquad + p(t)\int_{\mathbb{T}^2} |\theta|^{2p(t)-2}\theta f\,\mathrm{d}x.
\end{aligned} \tag{2.20}$$

We now use the following facts:

$$\int_{\mathbb{T}^2} |\theta|^{2p-2}\theta f \,\mathrm{d}x \le C\|f\|_{L^\infty(0,T;L^\infty(\mathbb{T}^2))}\|\theta\|_{2p}^{2p-1}, \tag{2.21}$$

$$p\int_{\mathbb{T}^2} |\theta|^{2p-2}\theta u\cdot\nabla\theta \,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{T}^2} u\cdot\nabla(|\theta|^{2p})\mathrm{d}x = 0, \tag{2.22}$$

$$\nu\int_{\mathbb{T}^2} |\theta|^{2p-2}\theta\Delta\theta \,\mathrm{d}x = -\nu(2p-1)\int_{\mathbb{T}^2} |\theta|^{2p-2}|\nabla\theta|^2 \,\mathrm{d}x \le 0. \tag{2.23}$$

In the second equality we used the fact that the velocity is divergence free. Altogether we thus find

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta(t)\|_{2p(t)}^{2p(t)}\mathrm{d}x \\
&\quad\le p'(t)\int_{\mathbb{T}^2} \ln|\theta||\theta|^{2p(t)}\mathrm{d}x \\
&\qquad + p(t)\int_{\mathbb{T}^2} |\theta|^{2p(t)}|\nabla u|\mathrm{d}x + p(t)\|f\|_{L^\infty}\|\theta\|_{2p}^{2p-1}.
\end{aligned} \tag{2.24}$$

We now use the following elementary inequality: for $a\in\mathbb{R}$ and $b>0$,

$$ab \le e^a + b\ln b - b. \tag{2.25}$$

In fact, we use only that $ab \le e^a + b\ln b$. The inequality (2.25) is proved via calculus and follows because the Legendre transform of the convex function $b\ln b - b + 1$ is $e^a - 1$. Setting $a = \beta|\nabla u|$ and $b = \frac{1}{\beta}|\theta|^{2p}$, and applying (2.25) and Lemma 1, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta(t)\|_{2p(t)}^{2p(t)} \\
&\le p'(t)\int_{\mathbb{T}^2} \ln|\theta||\theta|^{2p}\mathrm{d}x + \frac{p(t)}{\beta}\int_{\mathbb{T}^2} \ln(\beta^{-1}|\theta|^{2p})|\theta|^{2p}\,\mathrm{d}x \\
&\quad + p(t)\int_{\mathbb{T}^2} e^{\beta|\nabla u|}\mathrm{d}x + Cp(t)\|f\|_{L^\infty}\|\theta\|_{2p}^{2p-1} \\
&\le \left(p'(t) + \frac{2p(t)^2}{\beta}\right)\int_{\mathbb{T}^2} \ln|\theta||\theta|^{2p}\mathrm{d}x + \frac{p(t)}{\beta}\ln(\beta^{-1})\|\theta(t)\|_{2p}^{2p} \\
&\quad + p(t)C_K + Cp(t)\|f\|_{L^\infty}\|\theta\|_{2p}^{2p-1},
\end{aligned} \tag{2.26}$$

where C_K is the constant from Lemma 1 and $\beta = \frac{\gamma}{\Omega_\infty}$ depends on the bound for $\|\omega(t)\|_{L^\infty}$. We now choose p to evolve according to

$$p'(t) = -2\beta^{-1}p(t)^2,\ p(0) = p_0 \quad\Longrightarrow\quad p(t) = \frac{\beta p_0}{\beta + 2p_0 t}. \tag{2.27}$$

Note that $p(t)$ is a positive monotonically decreasing function of t. Let the time t_* defined by $t_* = T(p_0) := \beta(p_0-1)/2p_0$ be such that $p(t_*) = 1$. Then

$p(t) \in [1, p_0]$ for all $t \in [0, t_*]$. Note also from (2.27) that

$$\int_0^t p(s)ds = \log\left(\frac{p_0}{p(t)}\right)^{2\beta} = \log\left(1 + \frac{2p_0 t}{\beta}\right)^{\frac{2}{\beta}}.$$

Defining $m(t) = \frac{1}{2}\|\theta(t)\|_{2p(t)}^{2p(t)}$ and using (2.27) we have the differential inequality

$$m'(t) \le p(t)(C_1 m(t) + C_2) \implies C_1 m(t) + C_2 \le (C_1 m_0 + C_2)\left(1 + \frac{2p_0 t}{\beta}\right)^{\frac{2C_1}{\beta}}$$

with C_1 and C_2 depending on $\|f\|_{L^\infty(0,T;L^\infty(\mathbb{T}^2))}$, p_0, C_K, and β. Thus

$$m(t) \le m_0\left(1 + \frac{2p_0 t}{\beta}\right)^{2C_1/\beta} + \frac{C_2}{C_1}\left[\left(1 + \frac{2p_0 t}{\beta}\right)^{2C_1/\beta} - 1\right].$$

Note that $p_0/p(t) = 1 + 2p_0\beta^{-1}t$ is increasing on $[0, t_*]$ from 1 to $p_0/p(t_*) = p_0$. Consequently,

$$\|\theta(t)\|_{2p(t)} \le C_1\|\theta_0\|_{2p_0}^{p_0} + C_2 \tag{2.28}$$

where the constants C_1 and C_2 have been redefined but the dependence on parameters is the same. As $p(t) \in [1, p_0]$ for all $t \in [0, t_*]$ we have that $\|\theta(t)\|_2 \le \|\theta(t)\|_{2p(t)}$ and we obtain

$$\sup_{t\in[0,t_*]} \|\theta(t)\|_2 \le C_1\|\theta_0\|_{2p_0}^{p_0} + C_2,$$

which completes the proof. □

A similar idea to our Lemma 2 was used in [10, lemma 3]. We apply our two lemmas to the two-dimensional linearized Euler and Navier-Stokes equations to obtain uniform boundedness of vorticity gradients for short time.

LEMMA 3. Fix $\ell > 0$ and let ω_ℓ and ω_ℓ^ν solve (2.7) and (2.8), respectively. Then there exists a constant C_* and a constant $C_\ell < \infty$ depending only on ℓ, the forcing norm $\|g\|_{L^\infty(0,T;L^\infty(\mathbb{T}^2))}$, and the uniform bound on solutions given in (1.6) such that for $T_* \le (C_*\Omega_\infty)^{-1}$, we have that

$$\sup_{t\in[0,T_*]} \left(\|\omega_\ell(t)\|_{H^1} + \|\omega_\ell^\nu(t)\|_{H^1}\right) \le C_\ell.$$

PROOF. We focus on proving a viscosity independent bound for $\|\omega_\ell^\nu(t)\|_{H^1}$. The proof for $\|\omega_\ell(t)\|_{H^1}$ is the same, when setting $\nu = 0$. We show that $|\nabla\omega_\ell^\nu|$ obeys (2.18). Differentiating (2.8), we find

$$(\partial_t + u^\nu\cdot\nabla)\nabla\omega_\ell^\nu + \nabla u^\nu\cdot\nabla\omega_\ell^\nu = \nu\Delta(\nabla\omega_\ell^\nu) + \nabla(\varphi_\ell * g).$$

A standard computation shows that $|\nabla\omega_\ell^\nu|$ satisfies

$$(\partial_t + u^\nu\cdot\nabla - \nu\Delta)|\nabla\omega_\ell^\nu| \le |\nabla u||\nabla\omega_\ell^\nu| + |\nabla(\varphi_\ell * g)|,$$

which is a particular case of the scalar inequality (2.18) with $\theta = |\nabla \omega_\ell^\nu|$, initial data $\theta_0 = |\nabla(\varphi_\ell * \omega_0^\nu)| \in L^\infty(\mathbb{T}^2)$, and forcing

$$f = |\nabla(\varphi_\ell * g)| \in L^\infty(0, T; L^\infty(\mathbb{T}^2)),$$

as claimed. Applying Lemma 2, we find that for any $p > 1$ (e.g., $p = 2$) we have

$$\begin{aligned} \sup_{t\in[0,T_*]} \left\|\omega_\ell^\nu(t)\right\|_{H^1} &= C_1 \frac{1}{\ell^p}\left(\int_{\mathbb{T}^2} \left|\omega_0^\nu * (\nabla\varphi)_\ell\right|^{2p} \mathrm{d}x\right)^{1/2} + C_2 \\ &\lesssim C_\ell \left\|\omega_0^\nu\right\|^p_{L^\infty(\mathbb{T}^2)} \lesssim C_\ell \Omega_\infty^p. \end{aligned} \tag{2.29}$$

The constant C_ℓ depends on Ω_∞. It diverges with the mollification scale ℓ, through the prefactor ℓ^{-p} and through the dependence on $\|\nabla(\varphi_\ell * g)\|_{L^\infty} \lesssim \ell^{-1}\|g\|_{L^\infty}$. The important point, however, is that (2.29) holds uniformly in viscosity, completing the proof. □

We return now now to the proof of the main theorem. Using Lemma 3, the difference energy obeys

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|\omega_\ell^\nu - \omega_\ell\|^2_{L^2(\mathbb{T}^2)} &= -\int_{\mathbb{T}^2} (u^\nu - u)\cdot\nabla\omega_\ell^\nu(\omega_\ell^\nu - \omega_\ell)\mathrm{d}x \\ &\quad - \nu\int_{\mathbb{T}^2} \left|\nabla\omega_\ell^\nu\right|^2 \mathrm{d}x + \nu\int_{\mathbb{T}^2} \nabla\omega_\ell^\nu\cdot\nabla\omega_\ell \,\mathrm{d}x \\ &\le 4\Omega\|u^\nu - u\|_{L^2}\left\|\nabla\omega_\ell^\nu\right\|_{L^2} + \nu\left\|\nabla\omega_\ell^\nu\right\|_{L^2}\|\nabla\omega_\ell\|_{L^2} \\ &\lesssim C_\ell\|u^\nu - u\|_{L^\infty(0,T;L^2(\mathbb{T}^2))} + \nu C_\ell^2. \end{aligned}$$

Integrating we find

$$\begin{aligned} \left\|\omega_\ell^\nu - \omega_\ell\right\|^2_{L^2} &\lesssim \left\|\varphi_\ell * (\omega_0^\nu - \omega_0)\right\|^2_{L^2} \\ &\quad + C_\ell T\|u^\nu - u\|_{L^\infty(0,T;L^2(\mathbb{T}^2))} + \nu C_\ell^2 T. \end{aligned} \tag{2.30}$$

To conclude the proof we must show that, at fixed $\ell > 0$, we have

$$\lim_{\nu\to 0}\left\|\omega_\ell^\nu - \omega_\ell\right\|_{L^2(\mathbb{T}^2)} = 0.$$

Recall that by our assumption (2.3) we have that $\lim_{\nu\to 0}\|\omega_0^\nu - \omega_0\|_{L^2(\mathbb{T}^2)} \to 0$. So we need only establish strong convergence of the velocity in $L^2(0, T; L^2(\mathbb{T}^2))$. If $g = 0$ and $u_0^\nu = u_0$, this is a consequence of theorem 1.4 of [4]. Below is a generalization of [4] that applies in our setting and is proved by a different argument.

LEMMA 4. Let $\omega_0 \in L^\infty(\mathbb{T}^2)$. There exist constants U, Ω_2, and K (see below (2.34), (2.35), and (2.50)) depending on norms of the initial data and of the forcing such that the difference $v = u^\nu - u$ of velocities of solutions (1.1) and (1.3) obeys

$$\begin{aligned} &\|v(t)\|_{L^2}^2 \\ &\leq 3U^2 K^{\frac{5(t-t_0)\Omega_\infty}{\gamma}} \left(\frac{\|v(t_0)\|^2_{L^2(\mathbb{T}^2)}}{U^2} + \gamma \frac{\Omega_2^2}{U^2 \Omega_\infty} \nu \right)^{1-\frac{5(t-t_0)\Omega_\infty}{\gamma}} \end{aligned} \tag{2.31}$$

for all $0 \leq t_0 \leq t$. By iterating the above, we obtain

$$\begin{aligned} &\|v(t)\|_{L^2}^2 \\ &\leq 20U^2 K^{1-e^{-10t\Omega_\infty/\gamma}} \left(\frac{\|v(0)\|^2_{L^2(\mathbb{T}^2)}}{U^2} + \gamma \frac{\Omega_2^2}{U^2 \Omega_\infty} \nu \right)^{e^{-\frac{10t\Omega_\infty}{\gamma}}} \end{aligned} \tag{2.32}$$

provided that $\|v(0)\|^2_{L^2(\mathbb{T}^2)} + \gamma\nu\Omega_2^2/\Omega_\infty \leq 9KU^2$.

REMARK 4 (Continuity of Solution Map). At zero viscosity, Lemma 4 establishes Hölder continuity of the Yudovich (velocity) solution map. Specifically, denoting $u_t := S_t^\nu(u_0)$ and setting $\nu = 0$, a consequence of Lemma 4 is that

$$\left\| S_t^\nu(u_0) - S_t^\nu(u_0') \right\|_{L^2(\mathbb{T}^2)} \leq C \|u_0 - u_0'\|^{\alpha(t)}_{L^2(\mathbb{T}^2)}$$

where $\alpha(t) := e^{-ct}$ and $c, C > 0$ are appropriate constants. This fact is used to prove Corollary 1. It is worth noting that the condition on the data $\|v(0)\|^2_{L^2(\mathbb{T}^2)} \leq 9KU^2$ required for the above estimate to hold is $O(1)$ (data need not be taken very close).

PROOF OF LEMMA 4. The proof proceeds in two steps.

Step 1. Short time bound. The proof of the lemma starts from the equation obeyed by the difference v,

$$\partial_t v + u^\nu \cdot \nabla v + v \cdot \nabla u + \nabla p = \nu \Delta v + \nu \Delta u$$

leading to the inequality

$$\frac{d}{dt}\|v\|^2_{L^2} + \nu \|\nabla v\|^2_{L^2} \leq \nu \|\nabla u\|^2_{L^2} + 2 \int |\nabla u||v|^2 \, dx \tag{2.33}$$

which is a straightforward consequence of the equation, using just integration by parts. We use the bound Ω_∞ (1.6) for the vorticity of the Euler solution. We also use a bound for the L^2 norms

$$\sup_{0\leq t\leq T} \left(\|u^\nu(t)\|_{L^2(\mathbb{T}^2)} + \|u(t)\|_{L^2(\mathbb{T}^2)} \right) \leq U, \tag{2.34}$$

which is easily obtained from energy balance. We use also bounds for L^p norms of vorticity,

$$\Omega_p = \sup_{0\leq t\leq T} \|\omega(t)\|_{L^p(\mathbb{T}^2)} \leq \Omega_\infty. \tag{2.35}$$

We split the integral

$$\int |\nabla u||v|^2\,\mathrm{d}x = \int_B |\nabla u||v|^2\,\mathrm{d}x + \int_{\mathbb{T}^2\setminus B} |\nabla u||v|^2\,\mathrm{d}x$$

where

$$B = \{x \mid |v(x,t)| \ge MU\}$$

with M to be determined below. Although B depends in general on time, it has small measure if M is large,

$$|B| \le M^{-2}.$$

The constant M has dimensions of inverse length. We bound

$$2\int_B |\nabla u||v|^2\,\mathrm{d}x \le 2\|\nabla u\|_{L^2}\|v\|_{L^4}^2 \le 2|B|^{\frac14}\|\nabla u\|_{L^4}\|v(t)\|_{L^4}^2 \tag{2.36}$$

where we used $\int_B |\nabla u|^2\,dx \le |B|^{\frac12}\|\nabla u\|_{L^4}^2$. We now use the fact that we are in Yudovich class and Ladyzhenskaya inequality to deduce

$$\|v(t)\|_{L^4}^2 \le C\|v(t)\|_{L^2}[\|\omega_0\|_{L^2} + \|g\|_{L^1(0,T;L^2)}] \le CU\Omega_2,$$

and we also use

$$\|\nabla u\|_{L^4} \le [C\|\omega_0\|_{L^4} + \|g\|_{L^1(0,T;L^4)}] = \Omega_4$$

to bound (2.36) by

$$2\int_B |\nabla u||v|^2\,\mathrm{d}x \le CU\Omega_2\Omega_4 M^{-\frac12}, \tag{2.37}$$

We nondimensionalize by dividing by U^2 and we multiply by $\beta = \gamma/\Omega_\infty$. The quantity

$$y(t) = \frac{\|v(t)\|_{L^2(\mathbb{T}^2)}^2}{U^2} \tag{2.38}$$

obeys the inequality

$$\beta\frac{\mathrm{d}y}{\mathrm{d}t} \le \beta\nu\frac{\Omega_2^2}{U^2} + C\beta\Omega_4\frac{\Omega_2}{U}M^{-\frac12} + 2\int_{\mathbb{T}^2\setminus B}\beta|\nabla u|\frac{|v|^2}{U^2}\,\mathrm{d}x. \tag{2.39}$$

We write the term

$$2\int_{\mathbb{T}^2\setminus B}\beta|\nabla u||v|^2U^{-2}\,\mathrm{d}x = 2\int_{\mathbb{T}^2\setminus B}\left(\beta|\nabla u| + \log\epsilon + \log\frac{1}{\epsilon}\right)|v|^2U^{-2}\,\mathrm{d}x \tag{2.40}$$

with ϵ (with units of inverse area) to be determined below. We use the inequality (2.25) and Lemma 1 with

$$a = \beta|\nabla u| + \log\epsilon, \quad b = \frac{|v|^2}{U^2},$$

to deduce

$$2\int_{\mathbb{T}^2\setminus B} \beta |\nabla u||v|^2 U^{-2}\,\mathrm{d}x \le 2\epsilon C_K + 2\log\frac{M^2}{\epsilon} y(t). \tag{2.41}$$

Inserting (2.41) in (2.39) we obtain

$$\beta\frac{\mathrm{d}y}{\mathrm{d}t} \le F + \log\left(\frac{M^2}{\epsilon}\right) y(t) \tag{2.42}$$

with

$$F = \beta\nu\frac{\Omega_2^2}{U^2} + C\beta\Omega_4\frac{\Omega_2}{U}M^{-\frac{1}{2}} + 2\epsilon C_K. \tag{2.43}$$

Note that F and $\frac{M^2}{\epsilon}$ are nondimensional. From (2.42) we obtain immediately

$$y(t) \le \left(\frac{M^2}{\epsilon}\right)^{\frac{t-t_0}{\beta}} y(t_0) + \frac{F}{\log(\frac{M^2}{\epsilon})}\left(\left(\frac{M^2}{\epsilon}\right)^{\frac{t-t_0}{\beta}} - 1\right). \tag{2.44}$$

We choose M such that

$$C\beta\Omega_4\frac{\Omega_2}{U}M^{-\frac{1}{2}} = \beta\nu\frac{\Omega_2^2}{U^2} + y(t_0) \tag{2.45}$$

and we choose ϵ such that

$$2\epsilon C_K = \beta\nu\frac{\Omega_2^2}{U^2} + y(t_0). \tag{2.46}$$

These choices imply

$$F = 3\beta\nu\frac{\Omega_2^2}{U^2} + 2y(t_0). \tag{2.47}$$

Then we see that

$$\Gamma = \frac{M^2}{\epsilon} = 2C_K\left(C\beta\Omega_4\frac{\Omega_2}{U}\right)^4 \times \left(\beta\nu\frac{\Omega_2^2}{U^2} + y(t_0)\right)^{-5}. \tag{2.48}$$

Taking without loss of generality $\log\Gamma \ge 1$, we have from (2.44)

$$\begin{aligned} y(t) &\le 3\left(y(t_0) + \beta\nu\frac{\Omega_2^2}{U^2}\right)\Gamma^{\frac{t-t_0}{\beta}} \\ &\le 3\left(y(t_0) + \beta\nu\frac{\Omega_2^2}{U^2}\right)^{1-\frac{5(t-t_0)}{\beta}} \times \left(2C_K\left(C\beta\Omega_4\frac{\Omega_2}{U}\right)^4\right)^{\frac{5(t-t_0)}{\beta}}. \end{aligned} \tag{2.49}$$

Recalling that $\beta = \gamma/\Omega_\infty$ and denoting the nondimensional constant

$$K = 2C_K\left(C\beta\Omega_4\frac{\Omega_2}{U}\right)^4 \tag{2.50}$$

we established

$$\frac{\|v(t)\|^2}{U^2} \le 3K^{\frac{5(t-t_0)\Omega_\infty}{\gamma}} \left(\frac{\|v(t_0)\|^2_{L^2(\mathbb{T}^2)}}{U^2} + \beta\nu \frac{\Omega_2^2}{U^2} \right)^{1-\frac{5(t-t_0)\Omega_\infty}{\gamma}} . \tag{2.51}$$

Thus, we established (2.31).

Step 2. Long time bound. With (2.31) established, we now prove (2.32). Let $c = 5\Omega_\infty/\gamma$, $\Delta t = 1/2c$, and $t_i = t_{i-1} + \Delta t$ and $a_i = \|v(t_i)\|^2_{L^2}/U^2$ for $i \in \mathbb{N}$. Then (2.31) states

$$a_i \le C_1(a_{i-1} + C_2\nu)^{1/2}, \quad i = 1, 2, \dots \tag{2.52}$$

with $C_1 = 3K^{\frac{5\Omega_\infty}{2c\gamma}} = 3K^{\frac{1}{2}}$ and $C_2 = \beta\frac{\Omega_2^2}{U^2}$. We set

$$\delta_n = \frac{a_i + C_2\nu}{C_1^2} \tag{2.53}$$

and observe that (2.52) is

$$\delta_n \le \sqrt{\delta_{n-1}} + \widetilde{\nu} \tag{2.54}$$

where

$$\widetilde{\nu} = \frac{C_2\nu}{C_1^2} \tag{2.55}$$

is a nondimensional inverse Reynolds number. It follows then by induction that

$$\delta_n \le (\delta_0)^{2^{-n}} + \sum_{i=0}^{n-1} (\widetilde{\nu})^{2^{-i}}. \tag{2.56}$$

Indeed, the induction step follows from

$$\delta_{n+1} \le \sqrt{\delta_n} + \widetilde{\nu} \tag{2.57}$$

and the subadditivity of $\lambda \mapsto \sqrt{\lambda}$. If

$$\widetilde{\nu} \le \frac{1}{\sqrt{5}-1}, \tag{2.58}$$

then the iteration (2.54) starting from $0 < \delta_0 < r$, where r is the positive root of the equation $x^2 - x - \widetilde{\nu} = 0$, remains in the interval $(0, r)$, and for any n, δ_n obeys (2.56). We observe that

$$\sum_{i=0}^{n-1} (\widetilde{\nu})^{2^{-i}} = (\widetilde{\nu})^{2^{-n+1}} \left(1 + \dots + (\widetilde{\nu})^{2^{n-1}}\right) \le \frac{1}{1-\widetilde{\nu}} (\widetilde{\nu})^{2^{-n+1}}, \tag{2.59}$$

and therefore (2.32) follows from (2.56). We note that the iteration defined with equality in (2.54) converges as $n \to \infty$ to r. Fixing any $t > 0$ and letting $n = \lceil t/\Delta t \rceil = \lceil 2ct \rceil = \lceil 10t\Omega_\infty/\gamma \rceil$ establishes the bound. □

Due to assumption (2.3) we have that $\lim_{\nu\to 0}\|u_0^\nu - u_0\|_{L^2(\mathbb{T}^2)} \to 0$. Lemma 4 then allows us to conclude from (2.30) that $\lim_{\nu\to 0}\sup_{t\in[0,T_*]}\|\omega_\ell^\nu - \omega_\ell\|_{L^2(\mathbb{T}^2)} \to 0$ at fixed $\ell > 0$, and the proof of Proposition 2 is complete. □

With the proposition proved, the proof of the strong convergence of the vorticity in the L^p statement in the theorem is established. To obtain convergence of the distribution functions, see theorem 3.6 in [7]. □

PROOF OF PROPOSITION 1. This proof makes use of the the stochastic Lagrangian representation for Navier-Stokes solutions [5], together with the uniform-in-ν boundedness of vorticity. In light of the Lagrangian representation (1.13), (1.14), the key ingredient of propagating some degree of fractional regularity on the vorticity is the (uniform) Hölder regularity of the inverse flow A_t. Since the diffusion coefficients on the additive noise on (1.13) are spatially constant, it follows that the results of chapter 3 of [2] hold realization-by-realization for the stochastic flow X_t and its inverse A_t, uniformly in viscosity. This gives uniform bounds on the separation of two trajectories driven by the same realization of Brownian noise, independent of viscosity, thereby establishing spatial Hölder regularity of the flow. Although straightforward, we include a proof of this statement for completeness.

PROPOSITION 3. There exists a unique measure-preserving stochastic flow of homeomorphisms solving (1.13). This flow and the back-to-labels map are continuous flows X, A, which for all $t \in [0, T]$ are uniformly-in-ν of the class $C^{\alpha(t)}(\mathbb{T}^2)$ with $\alpha(t) = \exp(-Ct/\beta)$ with constants defined in (2.60).

PROOF OF PROPOSITION 3. We employ the log-Lipschitz property of u^ν; i.e., there exists an absolute constant $C > 2$ such that one has the uniform-in-viscosity estimate

$$|u^\nu(x,t) - u^\nu(y,t)| \le \frac{C}{\beta} d(x,y) \ln\left(\frac{CC_K}{d(x,y)^2}\right) \quad \forall x, y \in \mathbb{T}^2, \tag{2.60}$$

where β and C_K are the constants in Lemma 1, which depend only on $\|\omega_0\|_{L^\infty}$; see lemma A.1 of [3]. Here $d(x,y) := \min\{|x - y - k| : k \in \mathbb{Z}^d, |k| \le 2\}$ is the geodesic distance on the torus upon the identification $\mathbb{T}^d = [0,1)^d$. Now, due to the spatial uniformity of the noise on the trajectories,

$$\mathrm{d}X_t(x) = u(X_t(x), t)\mathrm{d}t + \sqrt{2\nu}\,\mathrm{d}W_t, \quad X_0(x) = x, \tag{2.61}$$

we find that the difference has no martingale part and satisfies

$$\mathrm{d}(X_t(x) - X_t(y)) = \big(u^\nu(X_t(x), t) - u^\nu(X_t(y), t)\big)\mathrm{d}t. \tag{2.62}$$

Upon integration, we obtain the inequality

$$\begin{aligned} &d(X_t(x), X_t(y)) \\ &\le d(x,y) + \frac{C}{\beta}\int_0^t d(X_s(x), X_s(y)) \ln\left(\frac{CC_K}{d(X_s(x), X_s(y))^2}\right)\mathrm{d}s \end{aligned} \tag{2.63}$$

for all $x, y \in \mathbb{T}^2$. The solution of this integro-inequality (with a possibly larger constant C) is

$$d(X_t(x), X_t(y)) \le (CC_K)^{1+e^{-Ct/\beta}} d(x,y)^{e^{-Ct/\beta}} \quad \text{a.s.} \tag{2.64}$$

Since the bound holds almost surely, this says that the map $X_t(\cdot)$ is Hölder continuous $C^{\alpha(t)}(\mathbb{T}^2)$ with $\alpha(t) = e^{-Ct/\beta}$ as claimed. We remark that deterministic trajectories in a log-Lipschitz field satisfying (2.60) satisfy precisely the same upper bound (2.64).

To obtain Hölder regularity of the back-to-labels map, it suffices to note that A_t can be identified with the backwards flow $X_{t,0}$, which solves the following backward stochastic differential equation

$$\widehat{\mathrm{d}} X_{t,s}(x) = u(X_{t,s}(x), s)\mathrm{d}s + \sqrt{2\nu}\, \widehat{\mathrm{d}} \widehat{W}_s, \quad X_{t,t}(x) = x, \tag{2.65}$$

where the $\widehat{\mathrm{d}}$ indicates that the backward differential and $\widehat{W}_s = W_{t-s} - W_t$ is a Brownian motion adapted to the backward filtration $\widehat{\mathscr{F}}_s^t := \sigma\{\widehat{W}_u, u \in [0,s]\}$. For a discussion of backward Itō equations; see, e.g., [16]. With this identification, one finds as above that for any $t > 0$ and all $s \in [0,t]$, one has

$$d(X_{t,s}(x), X_{t,s}(y)) \le (CC_K)^{1+e^{-C(t-s)/\beta}} d(x,y)^{e^{-C(t-s)/\beta}} \quad \text{a.s.} \tag{2.66}$$

By setting $s = 0$ we find that $A_t = X_{t,0}$ satisfies the same estimate (2.64) as X_t and therefore is Hölder continuous with the same exponentially decaying exponent. □

Proceeding forward to obtain uniform bounds we wish to make use of the representation formula (1.13)–(1.14). This requires some regularity on the initial condition, so we replace $\omega_0 \in L^\infty(\mathbb{T}^2)$ with a mollification of it, $\omega_0 * \varphi_\ell \in C^\infty(\mathbb{T}^2)$ for $\ell > 0$. All the bounds will be manifestly independent of ℓ, which can be taken to zero at the end, so we simplify the notation by writing "ω_0."

We continue by following closely the proof of theorem 3.32 of [2]. In particular, we introduce the space $F_p^s(\mathbb{T}^d)$ (which belongs to the family of Triebel-Lizorkin spaces $F_p^s = F_{p,\infty}^s$ provided $p > 1$) that is comprised of measurable functions $f \in L^p(\mathbb{T}^d)$ that are finite in the seminorm

$$[f]_{F_p^s} := \inf_{g \in L^p(\mathbb{T}^d)} \Big\{ \|g\|_{L^p(\mathbb{T}^d)} : |f(x) - f(y)| \le d(x,y)^\alpha (g(x) + g(y)) \quad \forall x, y \in \mathbb{T}^d \Big\} < \infty \tag{2.67}$$

where $d(x,y)$ is the distance function on the torus defined above; see definition 3.30 of [2]. The key to the argument is understanding how composition with a (uniformly) Hölder continuous stochastic diffeomorphism provided by Proposition 3 operates on F_p^s. Using the stochastic representation (1.14),

$$\omega^\nu(t) = \mathbb{E}[\omega_0 \circ A_t], \tag{2.68}$$

Jensen's inequality, Hölder continuity of the back-to-labels map, and the fact that $\omega_0 \in F^s_p$, we have

$$
\begin{aligned}
&\frac{|\omega^\nu(x,t) - \omega^\nu(y,t)|}{d(x,y)^{s\alpha}} \\
&\quad = \frac{|\mathbb{E}[\omega_0(A_t(x)) - \omega_0(A_t(y))]|}{d(x,y)^{s\alpha}} \\
&\quad \le \mathbb{E}\Big[\frac{|\omega_0(A_t(x)) - \omega_0(A_t(y))|}{d(x,y)^{s\alpha}}\Big] \\
&\quad = \mathbb{E}\Big[\frac{|\omega_0(A_t(x)) - \omega_0(A_t(y))|}{d(A_t(x),A_t(y))^s}\frac{d(A_t(x),A_t(y))^s}{d(x,y)^{s\alpha}}\Big] \\
&\quad \le \|A_t\|^s_{C^\alpha}\mathbb{E}[g(A_t(x)) + g(A_t(y))]
\end{aligned}
\tag{2.69}
$$

for any $g \in L^p(\mathbb{T}^2)$, where we used that $\omega_0 \in F^s_p$ together with the definition (2.67). Let $\widetilde{g}(x) := \mathbb{E}[g(A_t(x))]$. Note that, since A_t is measure preserving and we can apply Jensen's inequality, we have $\|\widetilde{g}\|_{L^p(\mathbb{T}^2)} \le \|g\|_{L^p(\mathbb{T}^2)} < \infty$. Thus $\widetilde{g} \in L^p(\mathbb{T}^2)$, and it follows by the linearity of the expectation that the right-hand-side of (2.69) is an L^p function. This shows that

$$
[\omega^\nu(t)]_{F^{s(t)}_p} \le (CK)^{s(1+e^{-Ct/\beta})}[\omega_0]_{F^s_p}, \quad s(t) = s\exp(-Ct/\beta),
\tag{2.70}
$$

where we used the explicit bound on the Hölder norm computed in (2.64). The bound (2.70) holds uniformly in viscosity. In order to connect to some $B^s_{p,\infty}$ (which is a larger space) we need to use an embedding for the initial data

$$
B^{s_3}_{p,\infty} \subset B^{s_2}_{p,1} \subset W^{s_2,p} \subset F^{s_1}_p
\tag{2.71}
$$

with $s_3 > s_2 > s_1$. The proposition follows from lemma 3.31 of [2], which shows that the Triebel–Lizorkin spaces are continuously embedded in the Besov spaces, i.e., $F^s_p(\mathbb{T}^d) \hookrightarrow B^s_{p,\infty}(\mathbb{T}^d)$. □

PROOF OF COROLLARY 2. We need the following elementary lemma:

LEMMA 5. For any $s > 0$ and $f \in B^s_{2,\infty}(\mathbb{T}^d)$, the following inequality holds for all $0 < s' < s$:

$$
\|f\|_{L^2(\mathbb{T}^d)} \le \|f\|^{s'/(1+s')}_{H^{-1}(\mathbb{T}^d)}\|f\|^{1/(1+s')}_{B^s_{2,\infty}(\mathbb{T}^d)}.
\tag{2.72}
$$

PROOF. First note that the interpolation inequality

$$
\|f\|_{L^2(\mathbb{T}^d)} \le \|f\|^{s'/(1+s')}_{H^{-1}(\mathbb{T}^d)}\|f\|^{1/(1+s')}_{H^{s'}(\mathbb{T}^d)},
\tag{2.73}
$$

which follows from the Hölder inequality and the Fourier definition of the Sobolev norm. The claim follows from the embedding $B^s_{p,q}(\mathbb{T}^d) \subset B^{s'}_{p,q'}(\mathbb{T}^d)$ for $s' < s$ and any q', q (see §2.3.2 of [22]) and the identification $H^s := B^s_{2,2}$. □

Proceeding with the proof, applying Lemma 5 for all $t \in [0, T]$ we have

$$\|\omega^\nu(t) - \omega(t)\|_{L^2(\mathbb{T}^2)} \leq \|\omega^\nu(t) - \omega(t)\|_{H^{-1}(\mathbb{T}^2)}^{\frac{s'}{1+s'}} \|\omega^\nu(t) - \omega(t)\|_{B^{s(t)}_{2,\infty}(\mathbb{T}^2)}^{\frac{1}{1+s'}}$$
$$\lesssim \sup_{t\in[0,T]} \|u^\nu(t) - u(t)\|_{L^2(\mathbb{T}^2)}^{\frac{s(t)}{1+s(t)}-}$$

for any $s' < s(t) := s\exp(-CT\|\omega_0\|_\infty)$. In the above, we appealed to Proposition 1 to establish uniform-in-ν boundedness of the solution ω^ν in the space $L^\infty(0, t; B^{s(t)}_{2,\infty}(\mathbb{T}^2))$. We now use Lemma 4 to conclude

$$\begin{aligned} &\|\omega^\nu(t) - \omega(t)\|_{L^p(\mathbb{T}^2)} \\ &\leq \|\omega^\nu - \omega\|_{L^\infty(0,T;L^\infty(\mathbb{T}^2))}^{\frac{p-2}{p}} \|\omega^\nu(t) - \omega(t)\|_{L^2(\mathbb{T}^2)}^{\frac{2}{p}} \\ &\lesssim \sup_{t\in[0,T]} \|u^\nu(t) - u(t)\|_{L^2(\mathbb{T}^2)}^{\frac{2s(t)}{p(1+s(t))}-} \lesssim (\nu T)^{\frac{s\exp(-2CT\|\omega_0\|_\infty)}{p(1+s\exp(-CT\|\omega_0\|_\infty))}-}. \end{aligned} \tag{2.74}$$

This completes our proof. □

REMARK 5. The stochastic Lagrangian representation of the vorticity also offers an expression for the enstrophy dissipation as the variance of the (randomly sampled) initial data

$$\nu \int_0^t \int_{\mathbb{T}^2} |\nabla\omega^\nu(t', x)|^2 \,\mathrm{d}x\, \mathrm{d}t' = \frac{1}{2} \int_{\mathbb{T}^2} \mathrm{Var}\big[\omega_0^\nu(A_t(x))\big] \mathrm{d}x. \tag{2.75}$$

The above is a special case of the Lagrangian fluctuation dissipation relation for active scalars derived in [9]. This relation is easily generalized to incorporate the effect of body forces. A consequence of Theorem 1 is that the enstrophy dissipation vanishes in the high Reynolds number limit, forcing also the variance to become 0. Thus, there is no "spontaneous stochasticity" of Lagrangian trajectories in the vanishing viscosity limit for 2D Navier-Stokes with initial data in the Yudovich class.

3 Discussion

Predicting the long-time vortex structures in two-dimensional turbulence is of longstanding interest, starting with the work on dynamics of point vortices by Onsager [18]. There have been a number of theories developed to this effect. We briefly review the celebrated mean-field theory of Miller [17] and Robert [19] to give context to our result. The idea is to describe an equilibrium configuration ω_{eq} satisfying

$$\omega(t) \xrightarrow{t\to\infty} \omega_{\mathrm{eq}} \tag{3.1}$$

in some sense. If $\omega(t)$ is an Euler path with bounded initial vorticity, then one has the following information:

(1) conservation of energy:

$$\|u(t)\|_{L^2(\mathbb{T}^2)} = \|u_0\|_{L^2(\mathbb{T}^2)}, \tag{3.2}$$

(2) conservation of vorticity "casmirs": for any continuous f,

$$I_f := \int_{\mathbb{T}^2} f(\omega(x,t))\mathrm{d}x = \int_{\mathbb{T}^2} f(\omega_0(x))\mathrm{d}x. \tag{3.3}$$

For long-time limits of Euler flows, there is a natural candidate object to describe $\omega_{\rm eq}$. In particular, provided only $\omega_0 \in L^\infty(\mathbb{T}^2)$, then $\omega(t) \in L^\infty(\mathbb{T}^2)$ is the unique solution of Euler [23], and in the weak-$*$ sense

$$\lim_{n\to\infty} \int_{\mathbb{T}^2} \varphi(x)\omega(x,t_n)\mathrm{d}x = \int_{\mathbb{T}^2} \varphi(x)\overline{\omega}(x)\mathrm{d}x \quad \forall \varphi \in L^1(\mathbb{T}^2) \tag{3.4}$$

for some $\overline{\omega} \in L^\infty(\mathbb{T}^2)$ and some subsequence $t_n \to \infty$ as $n \to \infty$. However, large oscillations can remain in this limit. In particular, the above convergence does not imply for all continuous functions f that $f(\omega(x,t_n))$ converges to $f(\overline{\omega}(x))$ in the same sense, so it is not clear how the information (3.2) and (3.3) can be retained and in what sense. On the other hand, the fundamental theorem of Young measures guarantees

$$\lim_{n\to\infty} \int_{\mathbb{T}^2} \varphi(x) f(\omega(x,t_n))\mathrm{d}x = \int_{\mathbb{T}^2} \varphi(x) \int_{-M}^{M} f(y)\nu_x(\mathrm{d}y)\mathrm{d}x \tag{3.5}$$

with $M = \|\omega_0\|_{L^\infty(\mathbb{T}^2)}$. Note that, having introduced the Young measure $\nu_x(\mathrm{d}y)$, the convergence (3.4) holds with

$$\overline{\omega}(x) = \int_{-M}^{M} y\nu_x(\mathrm{d}y) \quad \forall f \in C([-M,M]). \tag{3.6}$$

Kraichnan developed a theory for the equilibrium distribution $\overline{\omega}$ discarding most of the information on the casmirs, keeping only conservation of energy and enstrophy [15]. However, it was since recognized that invariants involving higher powers of vorticity should not be neglected on compact domains such as $\mathbb{T}^2$. In order to retain as much information about the Euler solution as possible, Miller [17] and Robert [19] independently suggested that the long-time vorticity distribution resulting from freely decaying two-dimensional turbulence is a Young measure of the form

$$\nu_x(\mathrm{d}y) = \rho(x,y)\mathrm{d}y. \tag{3.7}$$

These Young measures have the property that their marginal distribution is the (initial) vorticity distribution function (1.8), which is left invariant under the Euler flow. Thus, if a measure (3.7) with the above property can be constructed such that also the energy associated to $\overline{\omega}$ equals that of ω_0, then the information on all ideal invariants is retained at the level of the predicted equilibrium distribution. Miller and Robert provide such a construction.[1] Specifically, by a Boltzmann counting argument, they showed that the entropy associated with a given density $\rho(x, y)$ of the Young measure has a specific form. Assuming ergodicity at long times, i.e., that the 2D Euler flow is sufficiently chaotic in phase space, they suggested to maximize this entropy subject to the above constraints. The prediction of the theory is the long-time distribution is

$$\rho(x, y) = \frac{\exp\big(\beta\big[y\bar{\psi}(x) + \mu(y)\big]\big)}{\int_{-M}^{M} \exp\big(\beta\big[y\bar{\psi}(x) + \mu(y)\big]\big)\mathrm{d}y}, \tag{3.8}$$

where the "inverse temperature" β and "chemical potential" $\mu(y)$ are Lagrange multipliers to enforce energy conservation and the marginal density $\pi_{\omega_0}[\mathrm{d}y]$ respectively, and where the stream function $\bar{\psi}$ solves

$$\Delta\bar{\psi}(x) = \bar{\omega} = \frac{\int_{-M}^{M} y \exp\big(\beta\big[y\bar{\psi}(x) + \mu(y)\big]\big)\mathrm{d}y}{\int_{-M}^{M} \exp\big(\beta\big[y\bar{\psi}(x) + \mu(y)\big]\big)\mathrm{d}y}. \tag{3.9}$$

Thus, the prediction is that the expected (average or coarsened) vorticity solves a very particular steady Euler equation $\omega = F(\psi)$ where ψ is the stream function. The function F depends on the distribution $\pi_{\omega_0}[\mathrm{d}y]$ and the energy E_0. It is important to remark that conservation individual casmirs may not survive as $t \to \infty$, but that according to this theory, at a given energy E_0, they are forever remembered at the level of the equilibrium distribution. Some numerical simulations have provided corroboratory evidence supporting this theory over competitive ones such as the Onsager-Joyce-Montgomery theory, at least in situations where ω_0 is supported on a finite area [20]. Whether or not the theory rigorously applies is an open question.

There are two major questions remaining about the domain of applicability of the Miller-Robert theory. The first being whether or not 2D Euler possesses the requisite ergodicity properties to justify entropy maximization. The second, and the one that motivates the present study, is whether the theory should apply to 2D Navier-Stokes solutions at small viscosity. This is related to the issue of anomalies in ideally conserved quantities. For energy, there is no question since

$$E^{\nu}(t) := \frac{1}{2}\int_{\mathbb{T}^2} |u^{\nu}(t)|^2\,\mathrm{d}x \xrightarrow{\nu\to 0} E_0$$

[1] We remark that the Miller-Robert theory applies to any compact domain $\Omega \subset \mathbb{R}^2$ with smooth boundary, with the torus $\Omega = \mathbb{T}^2$ as a special case. It is worth noting that convergence of higher-order vorticity moments in the zero viscosity limit on domains with boundaries is, in general, false. In fact, if the Euler velocity u is not identically 0 along the boundary and no-slip Navier-Stokes solutions converge to these $u^{\nu} \rightharpoonup u$ weakly in $L^{\infty}(0, T; L^2(\Omega))$, then $\limsup_{\nu\to 0} \|\omega^{\nu}\|_{L^{\infty}(0,T;L^p(\Omega))} = \infty$ for all $p \in (1, \infty]$ (see theorem 3.1 of [14]). If weak convergence fails to hold, then by Kato's energy dissipation condition, we know $\limsup_{\nu\to 0} \|\omega^{\nu}\|_{L^2(0,T;L^2(\Omega))} = \infty$. Thus, unless the Euler solution is identically zero on the boundary, higher moments of vorticity must diverge in the inviscid limit, presenting a great difficulty for the Miller-Robert theory as it applies to inviscid limits.

for any finite time under the assumption that $\omega_0 \in L^\infty(\mathbb{T}^2)$. On the other hand, it has not been clear that high-order ideal moments such as $I_n^\nu = \int_{\mathbb{T}^2} |\omega^\nu(t)|^n \, dx$ for $n > 2$ will be conserved in the limit of zero viscosity or if there will be an associated anomaly due to fine-scale mixing of the vorticity field. If they are not, it seems unlikely that these casmirs should be remembered at the level of the equilibrium distribution of vorticity. Our theorem establishes that there can be no such anomalies of higher-order invariants on any finite time interval $[0, T]$ with T arbitrarily large. Thus, it shows that the dependence of F on viscosity is slow, which provides a partial foundation for the Miller-Robert theory as it applies to slightly viscous fluids.

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PETER CONSTANTIN
Department of Mathematics
Princeton University
Princeton, NJ 08544
USA
E-mail: const@math.princeton.edu

THEODORE D. DRIVAS
Department of Mathematics
Princeton University
Princeton, NJ 08544
USA
E-mail: tdrivas@math.princeton.edu

TAREK M. ELGINDI
Department of Mathematics
Duke University
Durham, NC
USA
E-mail: tarek.elgindi@duke.edu

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