SIAM J. MATH. ANAL.
Vol. 52, No. 3, pp. 3073–3092

ENTROPY HIERARCHIES FOR EQUATIONS OF COMPRESSIBLE FLUIDS AND SELF-ORGANIZED DYNAMICS*

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Abstract. We develop a method of obtaining a hierarchy of new higher-order entropies in the context of compressible models with local and nonlocal diffusion and isentropic pressure. The local viscosity is allowed to degenerate as the density approaches vacuum. The method provides a tool to propagate initial regularity of classical solutions provided no vacuum has formed and serves as an alternative to the classical energy method. We obtain a series of global well-posedness results for state laws in previously uncovered cases, including $p(\rho) = c_p\rho$. As an application we prove global well-posedness of collective behavior models with pressure arising from an agent-based Cucker–Smale system.

Key words. compressible Navier–Stokes, flocking, alignment, Cucker–Smale, fractional diffusion

AMS subject classifications. 92D25, 35Q35, 76N10

DOI. 10.1137/19M1278983

1. Introduction.

We consider a class of compressible fluid models in one space dimension with periodic boundary conditions

$$\partial_t \rho + \partial_x(u\rho) = 0, \tag{1}$$

$$\partial_t(\rho u) + \partial_x(\rho u^2) = -\partial_x p(\rho) + \mathcal{D}(u,\rho) + \rho f, \tag{2}$$

$$(\rho, u)|_{t=0} = (\rho_0, u_0). \tag{3}$$

Here, $f \in L^\infty(\mathbb{T}\times\mathbb{R}^+)$ is a bounded external force, $\mathcal{D}(u,\rho)$ is a diffusion operator, and the pressure $p := p(\rho)$ is a given function of density. The central feature of dissipation operators $\mathcal{D}$ considered here is the existence of another quantity, denoted by Q, which allows one to express the term $\rho^{-1}\mathcal{D}$ as a transport of Q:

$$\mathrm{D}_t Q := Q_t + uQ_x = \rho^{-1}\mathcal{D}(u,\rho). \tag{4}$$

One classical example is the local dissipation in divergence form

$$\mathcal{D}(u,\rho) = (\mu(\rho)u_x)_x. \tag{5}$$

This example embodies a broad family of models that appear in various physical phenomena, such as barotropic compressible fluids, slender jets, shallow water waves, etc. Here, $\mu(\rho)$ designates the dynamic viscosity of the fluid which is typically given by the constitutive power law

$$\mu(\rho) = c_\mu \rho^\alpha, \qquad c_\mu > 0, \qquad \alpha \geqslant 0. \tag{6}$$

*Received by the editors August 5, 2019; accepted for publication (in revised form) April 15, 2020; published electronically June 30, 2020.
https://doi.org/10.1137/19M1278983
Funding: The research of the first author was partially supported by NSF grant DMS-1713985. The research of the second author was partially supported by NSF grant DMS-1703997. The research of the third author was supported in part by NSF grants DMS 1515705 and DMS-1813351 and the Simons Foundation.
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In this case,

$$Q = \frac{\mu(\rho)\rho_x}{\rho^2}. \tag{7}$$

In compressible barotropic fluid models, the pressure equation of state is of the form

$$p(\rho) = c_p \rho^\gamma, \qquad c_p > 0, \qquad \gamma > 0. \tag{8}$$

Some special cases include a barotropic monatomic gas with $\alpha = 1/3$ and $\gamma = 5/3$, shallow water waves with $\alpha = 1$ and $\gamma = 2$, and viscous slender jet dynamics with $\alpha = 1$ and $\gamma = 1/2$ (but with negative pressure $c_p < 0$). See [5, 11, 16] for recent studies and further discussion.

Recently, another class of examples, referred to as singular Euler-alignment models, with *nonlocal* density-dependent dissipation appeared in the context of collective behavior [18]:

$$\mathcal{D}(u, \rho) = \rho[\mathcal{L}^s(\rho u) - u\mathcal{L}^s(\rho)].$$

Here, $\mathcal{L}^s := -(-\partial_{xx})^{s/2}$ for $s \in (0, 2)$ is the fractional Laplacian given by the integral representation

$$\mathcal{L}^s f(x) = \int_{\mathbb{T}} \phi_s(x-y)(f(y) - f(x))\,\mathrm{d}y, \qquad \phi_s(x) = c\sum_{k\in\mathbb{Z}} \frac{1}{|x + 2\pi k|^{1+s}}.$$

The transport representation of the dissipation comes as a consequence of the commutator structure in this case, where one finds

$$Q = \partial_x^{-1}\mathcal{L}^s \rho. \tag{9}$$

With regard to the pressure law, two cases arise naturally from the corresponding agent-based Cucker–Smale system introduced in [6, 7]. One is the pressureless case, $p = 0$, which presents itself as a limit to monokinetic concentration $f \to \rho\delta_{v=u}$ in kinetic formulation with a strong local alignment forcing; see [9, 12] for rigorous study. The second is an isentropic pressure given by

$$p(\rho) = c_p \rho, \qquad c_p \geqslant 0, \tag{10}$$

which arises from stochastically forced systems. In the strong alignment-dissipation limit, the probability density f converges to a Maxwellian distribution; see [13, 14, 15]. While the pressureless case is well understood by now and is covered in extensive studies [18, 19, 20, 17, 8], the pressured case has virtually been omitted in the literature, except for smooth communication [4]. Both cases fall under the general class covered in our present study.

Since the relation (4) is linear, one can access a family of hybrid local/nonlocal dissipation models as well:

$$\begin{aligned} \partial_t(\rho u) + \partial_x(\rho u^2) &= -\partial_x p(\rho) + c_{\mathrm{nl}}\mathcal{D}_{\mathrm{nl}}(u,\rho) + c_{\mathrm{loc}}\mathcal{D}_{\mathrm{loc}}(u,\rho) + \rho f, \\ \mathcal{D}_{\mathrm{loc}}(u,\rho) &= (\mu(\rho)u_x)_x, \quad \mathcal{D}_{\mathrm{nl}}(u,\rho) = \rho[\mathcal{L}^s(\rho u) - u\mathcal{L}^s(\rho)]. \end{aligned} \tag{11}$$

In the context of collective behavior, these encompass multiscale alignment models—classical power law at large scales and strong singular alignment at local small scales. Although multiscaling has already appeared on the kinetic level in the analysis of hydrodynamic limit performed in [14], the net effect of the local alignment considered there averages down to zero in the macroscopic formulation. We argue, however, that local dissipation, along with the plethora of constitutive laws (6), appears naturally as a singular limit $s \to 2$ of the so-called topological model introduced in [17]. Let us recall the construction. It is observed in many biological behavioral studies [2, 10] that "agents,"such as birds or fish, probe local environment, sensing only a fixed number of other agents around them. Consequently, the actual communication neighborhood is determined by topologies determined by the density of the flock rather than the classical Euclidean one. The topological density-dependent distance can be defined by the mass of intermediate segment between agents (see [17] for multidimensional construction):

$$d(x, y) = \left| \int_x^y \rho(z, t) \, \mathrm{d}z \right|.$$

Since communication in dense areas progresses slower, the mass-distance should decrease effective viscosity of the alignment, leading one to consider a kernel inversely dependent on d:

$$\phi(x, y) = \frac{h(x - y)}{|x - y|^{1+s-\tau} d^\tau(x, y)},$$

where $\tau > 0$ is a parameter that gauges the contribution of the topological part and h is a local cut-off function. The corresponding operator is given by

$$\mathcal{L}^{s,\tau} f(x) = \int_{\mathbb{T}} \phi(x, y)(f(y) - f(x)) \, \mathrm{d}y.$$

As $s \to 2$, the normalized operator formally converges to the local elliptic operator in divergence form:

$$(2 - s)\mathcal{L}^{s,\tau} f \to (\rho^{-\tau} f_x)_x.$$

Consequently, the alignment term converges to

$$(2 - s)\rho[\mathcal{L}^{s,\tau}(u\rho) - u\mathcal{L}^{s,\tau}\rho] \to (\rho^{2-\tau} u_x)_x. \tag{12}$$

Thus, we obtain an example of the local operator (5) with topological viscosity given by $\mu(\rho) = \rho^{2-\tau}$. To summarize, in the context of flocking, the hybrid model (11) describes a flock driven by a strong local topological alignment and global power law communication.

Returning to the general discussion, we make a key observation—once the transport quantity Q is identified, one can rewrite the entire system as a system of conservation laws with the momentum equation given by the transport equation

$$\mathrm{D}_t X = -h_x(\rho) + f, \qquad h'(r) := \frac{p'(r)}{r} \tag{13}$$

for the new quantity

$$X = u + Q.$$

We will exploit this structure to prescribe an algorithm of constructing a hierarchy of entropy-like quantities which are extremely useful in studying regularity of such systems. The first member in the hierarchy is given by the well-known Bresch–Desjardins entropy [1]

$$\mathcal{H}_0 = \frac{1}{2} \int_{\mathbb{T}} \rho X^2 \, \mathrm{d}x + \int_{\mathbb{T}} \pi_0(\rho) \, \mathrm{d}x, \tag{14}$$

where π_0 is the pressure potential given by

$$\pi_0(\rho) = \rho \int_{\bar{\rho}}^{\rho} \frac{p(s)}{s^2}\, \mathrm{d}s \quad \text{for some } \bar{\rho} > 0. \tag{15}$$

The algorithm, described in detail in section 2, gives rise to higher-order entropies, $\mathcal{H}_1, \mathcal{H}_2, \dots,$ each controlling corresponding higher-order derivatives $X_x, X_{xx}, \rho_x, \rho_{xx}$, etc. Note that in the pressureless case, in particular, X_x is precisely the "e-quantity" discovered in [3], which determines a threshold for regularity of solutions in the bounded kernel case.

With the use of this method we present, in a unified way, a range of global existence and continuation results for local, nonlocal, or hybrid models. Let us state our main results now. We denote by σ the order of the operator $\mathcal{D}(u,\rho)$, i.e..

$$\sigma = \begin{cases} 2, & c_{\mathrm{nl}} \geqslant 0, c_{\mathrm{loc}} > 0, \\ s, & c_{\mathrm{nl}} > 0, c_{\mathrm{loc}} = 0. \end{cases} \tag{16}$$

Theorem 1.1 (continuation criterion). *Consider the system* (1)–(3) *with*

$$\mathcal{D}(u,\rho) := c_{\mathrm{nl}}\mathcal{D}_{\mathrm{nl}}(u,\rho) + c_{\mathrm{loc}}\mathcal{D}_{\mathrm{loc}}(u,\rho), \tag{17}$$

with equations of state given by (6), (8), *and* $f \in L^\infty(R^+; C^n)$. *Consider the following cases:*

(1) *purely nonlocal:* $c_{\mathrm{loc}} = 0$, $c_{\mathrm{nl}} > 0$, *in which case we require* $s \in (\frac{5}{3}, 2)$, $\gamma > 0$*;*
(2) *purely local:* $c_{\mathrm{loc}} > 0$, $c_{\mathrm{nl}} = 0$, *in which case we require* $(\alpha \geqslant 0, \gamma > 1)$ *or* $(\alpha > \frac{1}{2}, \gamma > 0)$*;*
(3) *mixed:* $c_{\mathrm{loc}} > 0$, $c_{\mathrm{nl}} > 0$, *in which case we require*

$$\alpha \geqslant 0, \gamma > 1, s \in (0,2) \quad \text{or} \quad \alpha > \frac{1}{2}, \gamma > 0, s \in (0,2) \quad \text{or} \quad a \geqslant 0, \gamma > 0, s \in \left(\frac{3}{2}, 2\right).$$

Suppose $(u,\rho) \in H^{m+1-\sigma} \times H^m$, $m \geqslant 2$, *is a local solution on time interval* $(0,T)$. *Suppose also that*

$$\underline{\rho} := \inf_{t\in[0,T)} \min_{x\in\mathbb{T}} \rho(x,t) > 0. \tag{18}$$

Then the solution belongs to the class

$$u \in L^\infty(0,T; H^{m+1-\sigma}) \cap L^2(0,T;H^m), \tag{19}$$

$$\rho \in L^\infty(0,T;H^m) \cap L^2(0,T;H^{\sigma/2+m}) \tag{20}$$

and hence can be extended locally beyond T.

Remark 1.2. We note that the correspondence

$$(\rho \in H^m) \sim (u \in H^{m+1-\sigma}) \tag{21}$$

is natural for reasons to be clarified later.

The statement about purely local models in our Theorem 1.1 provides an alternate proof (and extension) of Theorem 1.1 in [5]. In that paper, an "active potential" w which satisfied a less-degenerate parabolic equation was used to propagate higher regularity provided the density nowhere vanished. In our work, we establish the same result by analyzing the entropy hierarchy.

Our next result establishes global well-posedness for a class of hybrid models, which is proved by propagating a lower bound on the density and appealing to Theorem 1.1.

THEOREM 1.3 (global existence). *Assume $f \in L^\infty(\mathbb{R}^+; C^n)$, $p \in C^{n+1}_{\mathrm{loc}}(\mathbb{R}^+)$ with $p'(r) > 0$ for any $r > 0$ and*

$$c_{\mathrm{nl}} \geqslant 0, c_{\mathrm{loc}} > 0, \qquad \alpha \in (0, 1/2). \tag{22}$$

Then any given local classical solution with nonvacuous initial data enjoys a priori lower bound (49) *on its interval of existence. Consequently, any nonvacuous initial condition $(u_0, \rho_0) \in H^{m+1-\sigma} \times H^m$, $m \geqslant 2$, gives rise to a unique global solution in the range of parameters stated in Theorem* 1.1.

In the purely local case, Theorem 1.3 provides an alternate proof to that of Mellet and Vasseur [16], who proved global well-posedness in the parameter range $\alpha < 1/2$ and $\gamma > 1$. As another application, we obtain global existence for collective behavior models.

COROLLARY 1.4. *Any collective behavior model, $\gamma = 1$, with multiscale diffusion $c_{\mathrm{nl}}, c_{\mathrm{loc}} > 0$ in the range of parameters $\alpha \in (0, 1/2)$, $s \in (3/2, 2)$ is globally well-posed for initial data in the class $(u_0, \rho_0) \in H^{m-1} \times H^m$, $m \geqslant 2$.*

Our final result concerns the long-time behavior of the velocity and density fields in models which possess a nonlocal dissipation component. In particular, we show that the energy inequality together with the nonlocal analogue of Bresch–Desjardins entropy implies flocking in an L^2-sense.

THEOREM 1.5 (nonlocal "second law" implies flocking). *Consider the forceless system* (1)–(3) *with $c_{\mathrm{nl}} > 0$, $c_{\mathrm{loc}} \geqslant 0$ and pressure law given by* (10). *Then any classical solution undergoes flocking behavior in the weighted L^2-sense:*

$$\int_{\mathbb{T}\times\mathbb{T}} |u(x) - u(y)|^2 \rho(x)\rho(y)\, dx\, dy + \left\| \rho(t) - \fint \rho_0\, dx \right\|^2_{L^1(\mathbb{T})} \lesssim \frac{\ln t}{t}. \tag{23}$$

In the pressurized case, as opposed to pressureless, the density always converges to a uniform state selected by its average. It is an indirect consequence of stochastic diffusion that leads to persistent mixing and eventual homogenization of the flock density. Analogous behavior was also observed in the study of Choi [4], under the assumption of globally bounded velocity field u. Note, however, that such an assumption is not guaranteed a priori due to the lack of the maximum principle in the pressured system.

Last, we note that all our results are cast in the settings of the periodic domain $\mathbb{T}$. This choice is motivated by purely technical reasons—a lower bound on the density ρ would ensure ellipticity of the dissipative operators at hand. However, periodic settings are also common in collective behavior models which focus on dynamics in the bulk of the flock and bypass issues with the boundary (as seen in the literature quoted above).

2. Hierarchy of entropies method.

As observed in the introduction, due to the transport formulation of the dissipation (4) one can rewrite the entire momentum equation (2) as a transport equation for the new quantity

$$X = u + Q,$$
$$\mathrm{D}_t X = -(h(\rho))_x + f, \quad \text{where } \; h'(r) := \frac{p'(r)}{r}. \tag{24}$$

Now (1)–(2) becomes a system of conservation law for the new pair of unknowns $(\rho, \rho X)$. We will exploit this structure to prescribe an algorithm of constructing a hierarchy of entropy-like quantities. First, let us make a general observation. If we have two quantities, X and ρ, one being transported and the other being conserved

$$\mathrm{D}_t X = 0, \quad \rho_t + (u\rho)_x = 0,$$

then the "energy" given by $\frac{1}{2}\int_{\mathbb{T}} \rho X^2 \,\mathrm{d}x$ is preserved for all time. In the presence of the pressure, such conservation is destroyed,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{T}} \rho X^2 \,\mathrm{d}x = -\int_{\mathbb{T}} X p_x \,\mathrm{d}x + \int_{\mathbb{T}} \rho f X \,\mathrm{d}x,$$

and the pressure term splits into two elements coming from X:

$$X p_x = u p_x + Q p_x.$$

It turns out that the Q-term is in fact dissipative in all the examples we considered so far. So the only term that needs to be eliminated is $up(\rho)_x$. This is done with the use of the pressure potential given by

$$\pi_0(\rho) = \rho \int_{\bar{\rho}}^{\rho} \frac{p(s)}{s^2} \,\mathrm{d}s \quad \text{for some } \bar{\rho} > 0. \tag{25}$$

Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \pi_0(\rho) \,\mathrm{d}x = \int_{\mathbb{T}} u p_x \,\mathrm{d}x. \tag{26}$$

We thus recover what is known as Bresch–Desjardins's entropy:

$$\mathcal{H}_0 = \frac{1}{2}\int_{\mathbb{T}} \rho X^2 \,\mathrm{d}x + \int_{\mathbb{T}} \pi_0(\rho) \,\mathrm{d}x. \tag{27}$$

According to the computations above, we obtain the following balance relation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_0 = \int_{\mathbb{T}} Q_x p \,\mathrm{d}x + \int_{\mathbb{T}} \rho f X \,\mathrm{d}x. \tag{28}$$

In parallel, due to (26), we obtain an energy balance relation for the energy of the system given by

$$\mathcal{E} = \frac{1}{2}\int_{\mathbb{T}} \rho |u|^2 \,\mathrm{d}x + \int_{\mathbb{T}} \pi_0(\rho) \,\mathrm{d}x, \tag{29}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = \int_{\mathbb{T}} u\mathcal{D}(u,\rho) \,\mathrm{d}x + \int \rho u f \,\mathrm{d}x. \tag{30}$$

In all cases of interest, the pressure term in (28) is sign-definite provided $p'(r) \geqslant 0$. Indeed, in the nonlocal case (9) we obtain

$$\int_{\mathbb{T}} Q_x p \,\mathrm{d}x = \int_{\mathbb{T}} p\mathcal{L}^s \rho \,\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{T}^2} \phi_{s,\rho}(x,y)(\rho(y)-\rho(x))^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant 0, \tag{31}$$

where

$$\phi_{s,\rho}(x,y) = \phi_s(x,y)\int_0^1 p'(\theta\rho(x) + (1-\theta)\rho(y)) \,\mathrm{d}\theta.$$

In the local case (7) we find

$$\int_{\mathbb{T}} Q_x p \, \mathrm{d}x = -\int_{\mathbb{T}} \frac{\mu(\rho)\rho_x^2 p'(\rho)}{\rho^2} \, \mathrm{d}x \leqslant 0. \tag{32}$$

Consequently, the initial entropy $\mathcal{H}_0$ gives control over $\rho X^2 \in L_t^\infty L_x^1$. Together with the energy conservation, this in turn controls the solo-density term:

$$\int_{\mathbb{T}} \rho |Q|^2 \, \mathrm{d}x \leqslant \int_{\mathbb{T}} \rho |u|^2 \, \mathrm{d}x + \int_{\mathbb{T}} \rho X^2 \, \mathrm{d}x,$$

which will be used to extract initial regularity information on the density in each of the local and nonlocal cases separately.

As to the hybrid case, we have $Q = Q_{\mathrm{nl}} + Q_{\mathrm{loc}}$. The key observation is that we can extract control on each of the terms separately. Indeed, writing

$$\int_{\mathbb{T}} \rho |Q_{\mathrm{nl}} + Q_{\mathrm{loc}}|^2 \, \mathrm{d}x = \int_{\mathbb{T}} \rho |Q_{\mathrm{nl}}|^2 \, \mathrm{d}x + \int_{\mathbb{T}} \rho |Q_{\mathrm{loc}}|^2 \, \mathrm{d}x + 2 \int_{\mathbb{T}} \rho Q_{\mathrm{nl}} Q_{\mathrm{loc}} \, \mathrm{d}x,$$

we observe that the integral of the cross-dissipation is nonnegative:

$$\int_{\mathbb{T}} \rho Q_{\mathrm{nl}} Q_{\mathrm{loc}} \, \mathrm{d}x = \int_{\mathbb{T}} \partial_x^{-1} \mathcal{L}^s \rho \frac{\mu(\rho)\rho_x}{\rho} \, \mathrm{d}x = \int_{\mathbb{T}} \partial_x^{-1} \mathcal{L}^s \rho \psi(\rho)_x \, \mathrm{d}x = -\int_{\mathbb{T}} \psi(\rho) \mathcal{L}^s \rho \, \mathrm{d}x \geqslant 0, \tag{33}$$

where $\psi'(r) = \mu(r)/r$. The nonnegativity holds, in fact, provided $\mu(r) \geqslant 0$, which is manifestly true for positive viscosities.

Coming back to the entropy construction, we now present the next step in the hierarchy. Note that in the pressureless case the quantity X_x would have satisfied the continuity equation and hence in combination with the density the new variable $Y = \frac{1}{\rho} X_x$ would have been transported. By analogy with the previous, we would then start construction with the pressureless term $\rho Y^2 = \rho^{-1} X_x^2$. Note that X_x satisfies

$$\partial_t X_x + (u X_x)_x = -h_{xx} + f_x. \tag{34}$$

Hence, in conjunction with mass conservation,

$$\mathrm{D}_t Y = -\rho^{-1} h_{xx} + \rho^{-1} f_x.$$

We thus obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_{\mathbb{T}} \rho Y^2 \, \mathrm{d}x = -\int_{\mathbb{T}} h_{xx} Y \, \mathrm{d}x + \int_{\mathbb{T}} f_x Y \, \mathrm{d}x. \tag{35}$$

The appropriate next-order pressure potential that eliminates the first term on the right-hand side is given by

$$\pi_1(\rho, \rho_x) = \frac{1}{2} h'(\rho) \frac{\rho_x^2}{\rho^2}. \tag{36}$$

The details of this computation will be provided in the sections below. We thus arrive at the next entropy:

$$\mathcal{H}_1 = \frac{1}{2} \int_{\mathbb{T}} \rho Y^2 \, \mathrm{d}x + \int_{\mathbb{T}} \pi_1(\rho, \rho_x) \, \mathrm{d}x. \tag{37}$$

The algorithm is now clear. For the second-order entropy, we denote $Z = \frac{1}{\rho} Y_x$ and define

$$\begin{aligned} \mathcal{H}_2 &= \frac{1}{2}\int_{\mathbb{T}} \rho Z^2 \,\mathrm{d}x + \int_{\mathbb{T}} \pi_2 \,\mathrm{d}x, \\ \pi_2 &= \frac{1}{2} h'(\rho)\frac{\rho_{xx}^2}{\rho^4}. \end{aligned} \tag{38}$$

Continuing in the same fashion, we can design an entropy-like quantity of any order, where the pressure potential is given by

$$\pi_n = \frac{1}{2} h'(\rho)\frac{(\partial_x^n \rho)^2}{\rho^{2n}},$$

while the kinetic term is constructed inductively:

$$X_n = \frac{1}{\rho}\partial_x X_{n-1}.$$

We form the nth entropy accordingly:

$$\mathcal{H}_n = \frac{1}{2}\int_{\mathbb{T}} \rho X_n^2 \,\mathrm{d}x + \int_{\mathbb{T}} \pi_n \,\mathrm{d}x.$$

With the help of this hierarchy we establish a direct control over any higher-order regularity of solution, consistent with that of the initial datum, where the relative smoothness of u and ρ mentioned in (21) naturally equilibrates the order of the velocity and density terms as they enter into an expression for X_n. It should be noted, however, that these entropies do not decay precisely as the first element $\mathcal{H}_0$. Instead, they satisfy ODEs with residual terms. Controlling those residual terms presents the main technical component of the method.

3. Global existence and flocking. We start by presenting less technical proofs of Theorems 1.3 and 1.5. We begin by remarking that the proof of Theorem 1.3 along with the continuation criterion of Theorem 1.1 require local well-posedness of the model equations.

PROPOSITION 3.1 (local well-posedness). *Let $c_{\mathrm{loc}} \geqslant 0$ and $c_{\mathrm{nl}} \geqslant 0$. Assume that $p : \mathbb{R}^+ \to \mathbb{R}$ and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ are C^∞ functions away from zero. Assume $s \in (0,2)$. Let $(u_0, \rho_0) \in H^{m+1-\sigma} \times H^m$, $m \geqslant 2$, such that $\underline{\rho} := \min_{x\in\mathbb{T}} \rho_0 > 0$. Suppose that for all $T > 0$*

$$f \in L^2(0,T; H^{m-\sigma}(\mathbb{T})).$$

Then there exists a time $T_0 > 0$ depending only on $\|(\rho_0, u_0)\|_{H^m(\mathbb{T})\times H^{m+1-\sigma}(\mathbb{T})}$, $\underline{\rho}$, and f, and a unique strong solution (ρ, u) to (1)–(3) on $[0, T_0]$ with data (ρ_0, u_0), such that

$$u \in C(0,T_0; H^{m+1-\sigma}(\mathbb{T})) \cap L^2(0,T_0; H^m(\mathbb{T})), \tag{39}$$

$$\rho \in C(0,T_0; H^m(\mathbb{T})) \cap L^2(0,T_0; H^{\sigma/2+m}(\mathbb{T})) \tag{40}$$

and $\rho(x,t) > \frac{r_0}{2}$ for all $(x,t) \in \mathbb{T} \times [0, T_0]$.

It should be remarked that this local well-posedness result covers all cases discussed in Theorems 1.1 and 1.3 but it holds in far greater generality. In particular, we do not require power-law forms for the pressure and viscosity constitutive laws. We do not produce a proof of Proposition 3.1 here, which is standard. In fact, the purely local case was established in Appendix II of [5]. On the other hand, the purely nonlocal case follows from a general proof which works in higher dimensions and is provided in Appendix A of [15]; see also [18] for the singular kernel pressureless case. The mixed case is a routine exercise, so it is omitted. With local well-posedness in hand, we proceed with the proof of global well-posedness.

Proof of Theorem 1.3. By Proposition 3.1, we have a local strong solution on some interval $[0, T_0]$. We aim to show that T_0 may be taken infinite by establishing a lower bound on the density and appealing to the no-vacuum continuation criteria established in Theorem 1.1.

Let us recall from the previous section that either in the local-only or in hybrid cases we establish control over the local term $\int_{\mathbb{T}} \rho |Q_{\mathrm{loc}}|^2 \,\mathrm{d}x$, the entropy $\mathcal{H}_0$, and energy $\mathcal{E}$. Both are bounded uniformly in time due to (28) and (30), where we can estimate the force term by

$$\begin{aligned} \left| \int_{\mathbb{T}} f \rho u \,\mathrm{d}x \right| &\leqslant |f|_\infty \left(\mathcal{M} + \int_{\mathbb{T}} \rho |u|^2 \,\mathrm{d}x \right) \leqslant C_1 + C_2 \mathcal{E}, \\ \left| \int_{\mathbb{T}} f \rho X \,\mathrm{d}x \right| &\leqslant |f|_\infty \left(\mathcal{M} + \int_{\mathbb{T}} \rho |X|^2 \,\mathrm{d}x \right) \leqslant C_1 + C_2 \mathcal{H}_0, \end{aligned} \tag{41}$$

where $\mathcal{M}$ is the total mass of the fluid which is conserved from the continuity equation (1):

$$\mathcal{M} = \int_{\mathbb{T}} \rho(x,t) \,\mathrm{d}x = \int_{\mathbb{T}} \rho_0(x) \,\mathrm{d}x. \tag{42}$$

For $\mu(r) = r^\alpha$ with $\alpha < 1/2$, this implies

$$\int_{\mathbb{T}} \rho |Q_{\mathrm{loc}}|^2 \,\mathrm{d}x = \int_{\mathbb{T}} \rho \left| \rho^{\alpha-2} \rho_x \right|^2 \,\mathrm{d}x = \int_{\mathbb{T}} |(\rho^{\alpha-\frac{1}{2}})_x|^2 \,\mathrm{d}x = \|\rho^{\alpha-\frac{1}{2}}\|_{\dot{H}^1} < \infty.$$

To establish a pointwise bound, we simply recall that the density has a conserved finite mass $\|\rho(t)\|_1 = \mathcal{M} > 0$. So, for each time t there exists a point $x_0(t)$ such that $\rho(x_0(t), t) > \mathcal{M}/2$. We find

$$\int_{x_0(t)}^{x} (\rho^{\alpha-1/2})_x \,\mathrm{d}x = \rho^{\alpha-1/2}(x,t) - \rho^{\alpha-1/2}(x_0(t)) \geqslant \rho^{\alpha-1/2}(x,t) - (\mathcal{M}/2)^{\alpha-1/2}. \tag{43}$$

It follows that $\rho^{\alpha-1/2} \in L_t^\infty L_x^\infty$, which implies $1/\rho \in L_t^\infty L_x^\infty$ since $\alpha < 1/2$. This finishes the proof. ∎

Proof of Theorem 1.5. Due to the energy-entropy law elucidated in the previous section, (28), (30), and the specific form of enstrophy coming from the nonlocal dissipation, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{H}_0 &\leqslant -c_1 \int_{\mathbb{T}\times\mathbb{T}} \phi_s(x-y)(\rho(y) - \rho(x))^2 \,\mathrm{d}x \,\mathrm{d}y, \\ \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E} &\leqslant -c_2 \int_{\mathbb{T}\times\mathbb{T}} \phi_s(x-y) |u(x) - u(y)|^2 \rho(x)\rho(y) \,\mathrm{d}y \,\mathrm{d}x. \end{aligned} \tag{44}$$

Note that under the linear pressure law (10), $\phi_s = \phi_{s,\rho}$. Moreover, by the Galilean invariance of the system and conservation of momentum, we may assume that the total momentum remains 0:

$$\int_{\mathbb{T}} \rho u \,\mathrm{d}x = 0.$$

Let us also assume $\mathcal{M} = 1$. From the entropy dissipation we have

$$\int_{\mathbb{T}\times\mathbb{T}} \phi_s(x-y)(\rho(y)-\rho(x))^2 \,\mathrm{d}x\,\mathrm{d}y \geqslant c_0|\rho-1|_2^2 \geqslant c_0 \int_{\mathbb{T}} \rho\log\rho\,\mathrm{d}x \geqslant c_0 \int_{\mathbb{T}} \pi_0(\rho)\,\mathrm{d}x.$$

This shows that $\int_0^\infty \int_{\mathbb{T}} \pi_0 \,\mathrm{d}x\,\mathrm{d}t < \infty$. Then

$$\frac{1}{2}\int_{\mathbb{T}\times\mathbb{T}} |u(x)-u(y)|^2 \rho(x)\rho(y)\,\mathrm{d}y\,\mathrm{d}x = \mathcal{M}\int_{\mathbb{T}} \rho|u|^2\,\mathrm{d}x = c_1\mathcal{E} - c_2\int_{\mathbb{T}} \pi_0\,\mathrm{d}x. \tag{45}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \leqslant -c_1\mathcal{E} + F(t),$$

where $F \in L^1(\mathbb{R}_+)$. By Duhamel, we obtain

$$\mathcal{E}(t) \leqslant e^{-c_1 t}\mathcal{E}_0 + \int_0^t e^{-c_1(t-s)}F(s)\,\mathrm{d}s.$$

The asymptotics of convergence $\mathcal{E} \to 0$ is based on the convolution integral. We can estimate it as follows. Let us define a sequence of times by

$$t_{m+1} = t_m + \frac{\lambda}{c_1}\ln m \quad \text{for some } \lambda > 1.$$

Then $t_m \sim \frac{\lambda}{c_1}\ln(m!)$, and by Stirling approximation, $t_m \sim m\ln m$. Let $K = \int_0^\infty F(s)\,\mathrm{d}s$. Then for every natural $n \in \mathbb{N}$ there exists an $m \in [n,(K+1)n]$ such that

$$\int_{t_{m-1}}^{t_m} F(s)\,\mathrm{d}s \leqslant \frac{1}{n}.$$

Indeed, otherwise, $\int F\,\mathrm{d}s > K$. At time t_m we then have an estimate

$$\mathcal{E}(t_m) \leqslant \frac{1}{m^{cm}} + \int_0^{t_{m-1}} e^{-c_1(t_m-s)}F(s)\,\mathrm{d}s + \int_{t_{m-1}}^{t_m} e^{-c_1(t_m-s)}F(s)\,\mathrm{d}s \leqslant \frac{1}{m^{cm}} + \frac{K}{m^\lambda} + \frac{1}{n}.$$

But $1/n \sim 1/m$, which appears to be the leading-order term. Recalling that $t_m \sim m\ln m$, we conclude that $1/m \lesssim \ln t_m/t_m$. So, we have

$$\mathcal{E}(t_m) \lesssim \frac{\ln t_m}{t_m}.$$

For any other $t_m < t < t_{m+1}$, we have by monotonicity of the energy

$$\mathcal{E}(t) \leqslant \mathcal{E}(t_m) \lesssim \frac{\ln t_m}{t_m} \sim \frac{\ln t_{m+1}}{t_{m+1}} \leqslant \frac{\ln t}{t}.$$

This establishes the desired asymptotic.

Finally, by the Csiszar–Kullback inequality, $\int \pi_0\,\mathrm{d}x \geqslant |\rho-1|_1^2$, and (45), we obtain a flocking statement in the weighted L^2-sense:

$$\int_{\mathbb{T}\times\mathbb{T}} |u(x)-u(y)|^2\rho(x)\rho(y)\,\mathrm{d}y\,\mathrm{d}x + |\rho-1|_1^2 \lesssim \frac{\ln t}{t}. \tag{46}$$ □

4. Continuation of nonvacuous solutions. This section contains the main technical ingredients of the hierarchy method and provides the proof of Theorem 1.1.

Consider the evolution equations

$$\begin{aligned} &\partial_t \rho + \partial_x(u\rho) = 0, \\ &\partial_t(\rho u) + \partial_x(\rho u^2) = -\partial_x p(\rho) + c_{\mathrm{nl}}\mathcal{D}_{\mathrm{nl}}(u,\rho) + c_{\mathrm{loc}}\mathcal{D}_{\mathrm{loc}}(u,\rho) + \rho f, \end{aligned} \tag{47}$$

where, recall, the nonlocal and local dissipative operators are defined as

$$\mathcal{D}_{\mathrm{nl}}(u,\rho) := \rho[\mathcal{L}^s(\rho u) - u\mathcal{L}^s(\rho)], \qquad \mathcal{D}_{\mathrm{loc}}(u,\rho) = (\mu(\rho)u_x)_x.$$

We will be considering $c_{\mathrm{nl}} \geqslant 0$ and $c_{\mathrm{loc}} \geqslant 0$. The discussion of various cases requires us to break up our argument. In all cases, we consider the power law for the pressure, i.e., $p(\rho) = c_p\rho^\gamma$ for some $\gamma > 0$ and $c_p > 0$. In this case, we have the explicit formula

$$\pi_0(\rho) = c_p\rho\int_{\bar\rho}^{\rho} s^{\gamma-2}\mathrm{d}s = \begin{cases} \frac{c_p}{\gamma-1}\rho^\gamma, & \gamma > 1,\ \bar\rho = 0, \\ c_p\rho\log(\rho/\bar\rho), & \gamma = 1,\ \bar\rho = \frac{1}{2\pi}\mathcal{M}, \\ \frac{c_p}{1-\gamma}(1-\rho^\gamma) & \gamma < 1,\ \bar\rho = 1. \end{cases} \tag{48}$$

Thus, if $\gamma > 1$, then $\pi_0(\rho) \geqslant 0$ is nonnegative pointwise. If $\gamma = 1$, which is of particular relevance in the context of flocking (10), then upon spatial integration it is nonnegative by the Csiszar–Kullback inequality, i.e., $\int \pi_0 \,\mathrm{d}x \geqslant |\rho - \bar\rho|_1^2$. This nonnegativity will be repeatedly used for extracting information from the a priori estimates arising from the Bresch–Desjardins entropy balance. When $\gamma < 1$, we simply note that $\int \pi_0 \,\mathrm{d}s$ is bounded by the mass.

In what follows, we fix a regular local solution on time interval $[0,T)$ as in Proposition 3.1 with no-vacuum condition

$$\underline{\rho} := \inf_{t\in[0,T)} \min_{x\in\mathbb{T}} \rho(x,t) > 0. \tag{49}$$

4.1. Energy and $\mathcal{H}_0$-entropy. Recall next that the basic energy balance (30), which in this case reads as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = -c_{\mathrm{nl}}\int_{\mathbb{T}\times\mathbb{T}} \frac12\phi_s(x-y)|u(x)-u(y)|^2\rho(x)\rho(y)\mathrm{d}y\mathrm{d}x - c_{\mathrm{loc}}\int_{\mathbb{T}}\mu(\rho)|u_x|^2\mathrm{d}x + \int_{\mathbb{T}} f\rho u\mathrm{d}x, \tag{50}$$

holds for any $t \in [0,T)$. In view of the estimate (41), this establishes uniform control in the energy space $u \in L^\infty L^2 \cap L^2 H^{\sigma/2}$ on the given time interval.

The Bresch–Desjardins entropy (27) that satisfies the balance (28), (31),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_0 = -c_{\mathrm{nl}}\int_{\mathbb{T}^2}\frac12\phi_{s,\rho}(x,y)(\rho(y)-\rho(x))^2\,\mathrm{d}x\,\mathrm{d}y - c_{\mathrm{loc}}\int_{\mathbb{T}^2}\frac{\mu(\rho)p'(\rho)}{\rho}|\rho_x|^2\,\mathrm{d}x + \int_{\mathbb{T}}\rho f X\,\mathrm{d}x \tag{51}$$

holds for any $t \in [0,T)$. Again, using (41), we find that $\mathcal{H}_0$ remains bounded. We now derive conclusions from these balance in two cases.

Purely nonlocal dissipation. Given the finite energy and controllability of $\int \pi_0 \,\mathrm{d}x$ discussed above, we obtain an L^2 bound on $Q = \partial_x^{-1}\mathcal{L}^s\rho$, and hence $\rho \in L^\infty H^{s-1}$. A further condition coming from the dissipation term $\rho \in L^2 H^{s/2}$ follows from the lower bound on the density and p' under the assumptions of Theorem 1.1. Provided $s > 3/2$, we have the embedding $L^1 \cap \dot H^{s-1} = H^{s-1} \subset L^\infty$, and so the density is uniformly bounded: $\rho \in L^\infty(0,T;L^\infty)$.

Purely local or mixed dissipation. When the local or both components are active, the uniform bound on the density comes from several sources. Indeed, in light of the nonnegativity of the mixed term (33) discussed in the introduction, we have control over the two terms $\int \rho Q_{\rm nl}^2 {\rm d}x$ and $\int \rho Q_{\rm loc}^2 {\rm d}x$ separately from the Bresch–Desjardins entropy. Boundedness of the density is then established in the following parameter regimes:

(1) $c_{\rm nl} > 0$ and $c_{\rm loc} \geqslant 0$ with $\gamma > 0$, $s > 3/2$, and $\alpha \geqslant 0$. Then $\rho \in L^\infty(0,T;L^\infty)$ by the nonlocal argument above.

(2) $c_{\rm nl} \geqslant 0$ and $c_{\rm loc} > 0$ with $\gamma > 0$, $s \in (0,2)$, and $\alpha > 1/2$. Following the proof of Theorem 1.3, we find

$$\int_{\mathbb{T}} \rho |Q_{\rm loc}|^2 \,{\rm d}x = \|\rho^{\alpha - \frac{1}{2}}\|_{\dot H^1} < \infty.$$

Boundedness of the density follows, as in the theorem, from the conserved finite mass.

(3) $c_{\rm nl} \geqslant 0$ and $c_{\rm loc} > 0$ with $\gamma > 1$, $s \in (0,2)$, and $\alpha > 0$. This follows from the bound

$$\int \left|\partial_x \left(\rho^{\frac{\gamma-1}{2}}\right)\right| {\rm d}x \leqslant \|\rho\|_{L^\gamma}^{\gamma/2} \left(\int \rho^{-3}|\rho_x|^2 {\rm d}x\right)^{1/2} \leqslant c\|\rho\|_{L^\gamma}^{\gamma/2} \left(\int \rho Q_{\rm loc}^2 {\rm d}x\right)^{1/2} < \infty \tag{52}$$

with a constant $c := c(\underline{\rho})$. Thus, since $\rho \in L^\gamma$, it follows that $\rho^{\frac{\gamma-1}{2}} \in L^1$ so that combined with the above we have $\rho^{\frac{\gamma-1}{2}} \in L^\infty(0,T;W^{1,1}(\mathbb{T}))$. Boundedness follows from Sobolev embedding.

Once an upper bound on the density is established in any of the above cases, then the local part of the Bresch–Desjardins entropy gives the control $\rho \in L^\infty(0,T;H^1)$. This control trumps what can be obtained from the dissipation of the Bresch–Desjardins entropy.

All the a priori bounds we have obtained so for in either of the cases can be summarized as follows:

$$u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^{\sigma/2}), \tag{53}$$

$$\rho \in L^\infty(0,T;L^\infty) \cap L^\infty(0,T;H^{\sigma-1}) \cap L^2(0,T;H^{\sigma/2}) \tag{54}$$

with norms depending on $\underline{\rho}$.

4.1.1. $\mathcal{H}_1$-entropy and its consequences. We now work out a detailed balance relation for the $\mathcal{H}_1$-entropy. Let us recall the definitions

$$\mathcal{H}_1 = \frac{1}{2}\int_{\mathbb{T}} \rho Y^2 \,{\rm d}x + \int_{\mathbb{T}} \pi_1(\rho,\rho_x)\,{\rm d}x,$$
$$Y = \frac{1}{\rho}(u_x + Q_x), \quad \pi_1(\rho,\rho_x) = \frac{1}{2}h'(\rho)\frac{\rho_x^2}{\rho^2}.$$

According to (35), (37) we have

$$\frac{\rm d}{{\rm d}t}\mathcal{H}_1 = -\int_{\mathbb{T}} h_{xx}(\rho) Y \,{\rm d}x + \frac{\rm d}{{\rm d}t}\int_{\mathbb{T}} \pi_1(\rho,\rho_x)\,{\rm d}x + \int_{\mathbb{T}} f_x Y \,{\rm d}x. \tag{55}$$

Looking ahead at the argument below, we remark that expansion of the pressure potential term π_1 on the right-hand side of this enstrophy budget will produce a dissipation term:

$$-c_{\rm nl}\|\rho\|_{H^{\frac{s}{2}+1}}^2 - c_{\rm loc}\|\rho\|_{H^2}^2.$$

We will be using it repeatedly to absorb various residual terms by interpolation using uniform bounds on the density from below and above and in $H^{\sigma-1}$ by (54).

So, first, let us estimate the forcing

$$\left|\int_{\mathbb{T}} f_x Y \,\mathrm{d}x\right| \leqslant c(|f|_{C^1}) + \mathcal{H}_1.$$

Next, we expand the Y-term:

$$-\int_{\mathbb{T}} h_{xx}(\rho) Y \,\mathrm{d}x = -\int_{\mathbb{T}} h'(\rho)\rho_{xx} Y \,\mathrm{d}x - \int_{\mathbb{T}} h''(\rho)\rho_x^2 Y \,\mathrm{d}x. \tag{56}$$

The second term can be estimated by

$$\left|\int_{\mathbb{T}} h''(\rho)\rho_x^2 Y \,\mathrm{d}x\right| \leqslant \mathcal{H}_1^{\frac{1}{2}} |\rho_x|_4^2, \tag{57}$$

and using that $|\rho_x|_4 \leqslant \|\rho\|_{\dot H^{\sigma/2+1}}^{\frac{9-4\sigma}{2(4-\sigma)}}$, and $\frac{9-4\sigma}{4-\sigma} \leqslant 1$ as long as $\sigma \geqslant \frac{5}{3}$, we obtain

$$\left|\int_{\mathbb{T}} h''(\rho)\rho_x^2 Y \,\mathrm{d}x\right| \leqslant c(\varepsilon)\mathcal{H}_1 + \varepsilon \|\rho\|^2_{H^{\frac{\sigma}{2}+1}}. \tag{58}$$

In view of the remark above, we can absorb the term $\varepsilon\|\rho\|^2_{H^{\frac{\sigma}{2}+1}}$ into the upcoming dissipative contribution from the pressure potential. The remaining residual term $-\int_{\mathbb{T}} h'(\rho)\rho_{xx} Y \,\mathrm{d}x$ will in fact be completely canceled out by another contribution of the pressure potential on which we focus for the remainder of the proof. First, we introduce a couple of shortcuts that greatly simplify the exposition:

- Throughout this proof, we will routinely drop integral signs for brevity. All equalities are intended to hold only upon spatially integrating over $\mathbb{T}$.
- We denote $g(r) = \frac{1}{2}h'(r)/r^2$ so that $\pi_1 = g(\rho)\rho_x^2$.
- All inequalities hold up to adimensional constants and constants depending on $\underline{\rho} > 0$ and $|\rho|_\infty$ which at this point we have under uniform control on the interval $[0, T)$.

Let us compute the potential

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\pi_1 &= -g'\rho_x^2(u\rho)_x - 2g\rho_x(u\rho)_{xx} = g'\rho_x^2(u\rho)_x + 2g\rho_{xx}(u\rho)_x \\ &= 2g\rho\rho_{xx}u_x + \rho g'\rho_x^2 u_x + g'\rho_x^3 u + 2g\rho_{xx}\rho_x u,\end{aligned}$$

integrating by parts in the last term:

$$\begin{aligned}&= 2g\rho\rho_{xx}u_x + \rho g'\rho_x^2 u_x + g'\rho_x^3 u - g'\rho_x^3 u - g\rho_x^2 u_x \\ &= 2g\rho\rho_{xx}u_x + \rho g'\rho_x^2 u_x - g\rho_x^2 u_x.\end{aligned} \tag{59}$$

The last two terms are of the form $q(\rho)\rho_x^2 u_x$, where q is a smooth function on $\mathbb{R}^+$. We can estimate any such term by replacing

$$u_x = \rho Y - c_{\mathrm{nl}}\mathcal{L}^s\rho - c_{\mathrm{loc}}(\mu(\rho)\rho_x/\rho^2)_x. \tag{60}$$

The residual term $q(\rho)\rho\rho_x^2 Y$ enjoys the same estimate as in (58). The local term breaks up into two: $q(\rho)\rho_x^2\rho_{xx}$ and $q(\rho)\rho_x^4$ for smooth q (which we redefine line by line). Using interpolation inequality

$$|\rho_x|_4 \leqslant |\rho_{xx}|_2^{\frac{1}{4}} |\rho_x|_2^{\frac{3}{4}} \tag{61}$$

and the fact that $|\rho_x|_2$ is under uniform control (see (54)) with $\sigma = 2$, we estimate

$$\left|\int_{\mathbb{T}} q(\rho)\rho_x^2\rho_{xx}\,\mathrm{d}x\right| \leqslant |\rho_x|_4^2\|\rho\|_{\dot H^2} \lesssim \|\rho\|_{\dot H^2}^{\frac{3}{2}} \leqslant \varepsilon\|\rho\|_{\dot H^2}^2 + c(\varepsilon), \tag{62}$$

and similarly, we have

$$\left|\int_{\mathbb{T}} q(\rho)\rho_x^4\,\mathrm{d}x\right| \leqslant |\rho_x|_4^4 \lesssim \|\rho\|_{\dot H^2} \leqslant \varepsilon\|\rho\|_{\dot H^2}^2 + c(\varepsilon). \tag{63}$$

To estimate the nonlocal part $q(\rho)\rho_x^2\mathcal{L}^s\rho$, we symmetrize in the integral representation of $\mathcal{L}^s$ and estimate according to the following:

$$\left|\int_{\mathbb{T}} q(\rho)\rho_x^2\mathcal{L}^s\rho\,\mathrm{d}x\right| \leqslant |\rho_x|_\infty^2\|\rho\|_{H^{s/2}}^2 + |\rho_x|_\infty\|\rho\|_{H^{s/2+1}}\|\rho\|_{H^{s/2}}. \tag{64}$$

By the Gagliardo–Nirenberg inequality,

$$|\rho_x|_\infty \leqslant \|\rho\|_{\dot H^{s/2+1}}^{\frac{5-2s}{4-s}}\|\rho\|_{\dot H^{s-1}}^{\frac{s-1}{4-s}}, \qquad \|\rho\|_{\dot H^{s/2}} \leqslant \|\rho\|_{\dot H^{s/2+1}}^{\frac{2-s}{4-s}}\|\rho\|_{\dot H^{s-1}}^{\frac{2}{4-s}}.$$

Thus,

$$\left|\int_{\mathbb{T}} q(\rho)\rho_x^2\mathcal{L}^s\rho\,\mathrm{d}x\right| \leqslant \|\rho\|_{\dot H^{s/2+1}}^{\frac{14-6s}{4-s}} + \|\rho\|_{\dot H^{s/2+1}}^{\frac{11-4s}{4-s}} \leqslant \varepsilon\|\rho\|_{\dot H^{s/2+1}}^2 + c(\varepsilon),$$

due to both powers being less than or equal to 2 as long as $s > \frac{3}{2}$.

Finally, for the first term on the right-hand side of (59) we have

$$2g\rho\rho_{xx}u_x = 2g\rho^2\rho_{xx}Y - 2c_{\mathrm{nl}}g\rho\rho_{xx}\mathcal{L}^s\rho - 2c_{\mathrm{loc}}g\rho\rho_{xx}(\mu(\rho)\rho_x/\rho^2)_x.$$

Notice that $2g\rho^2\rho_{xx}Y = h'(\rho)\rho_{xx}Y$, which cancels with the first term on the right-hand side of (56). The main contribution of the last two terms is dissipation. Indeed, omitting constants and integrating by parts,

$$-g\rho\rho_{xx}\mathcal{L}^s\rho = g'\rho_x^2\mathcal{L}^s\rho + g\rho_x\mathcal{L}^s\rho_x.$$

The first one we already estimated. The second, after symmetrization, is bounded by

$$g\rho_x\mathcal{L}^s\rho_x \leqslant -c_0\|\rho\|_{\dot H^{s/2+1}}^2 + |\rho_x|_\infty\|\rho\|_{H^{s/2+1}}\|\rho\|_{H^{s/2}},$$

with the latter already being treated in (64). The last local term splits into

$$-g\rho\rho_{xx}(\mu(\rho)\rho_x/\rho^2)_x = -q_1|\rho_{xx}|^2 + q_2|\rho_x|^2\rho_{xx}, \qquad q_1 > 0. \tag{65}$$

With the use of (62), we estimate it by

$$-g\rho\rho_{xx}(\mu(\rho)\rho_x/\rho^2)_x \leqslant -c\|\rho\|_{H^2}^2 + c(\varepsilon).$$

Collecting the estimates, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_1 \leqslant c_1\mathcal{H}_1 + c_2 - c_3\|\rho\|_{\dot H^{\sigma/2+1}}^2. \tag{66}$$

This implies a uniform bound on $\mathcal{H}_1$ on the time interval at question along with integrability of $\|\rho\|_{\dot H^{\sigma/2+1}}^2$. As a consequence of the positivity of π_1, we obtain $\rho \in L^\infty H^1$, and

$$\rho Y = u_x + c_{\mathrm{nl}}\mathcal{L}^s\rho + c_{\mathrm{loc}}(\mu(\rho)\rho_x/\rho^2)_x \in L^\infty L^2. \tag{67}$$

In the purely nonlocal case ($c_{\mathrm{loc}} = 0$), if we apply $1 - s$ derivatives on this expression, we still obtain a function in L^2, yet $\rho_x \in L^2$ by the previous. This places u into H^{2-s} uniformly. However, the L^2-in-time class improves only to H^1. In the mixed or local cases, no further information is extracted from this computation. We obtain another series of a priori bounds:

$$u \in L^\infty(0,T;H^{2-\sigma}) \cap L^2(0,T;H^1), \tag{68}$$

$$\rho \in L^\infty(0,T;H^1) \cap L^2(0,T;H^{\sigma/2+1}), \tag{69}$$

where we identify $L^2 = H^0$. In addition, we record

$$X \in L^\infty(0,T;H^1). \tag{70}$$

Note that in the local/mixed case, the only improvement comes from the $\mathcal{H}_1$-entropy at the level of the L^2-in-time for ρ, as well as the boundedness of the X quantity (70). The latter point is important in continuing our procedure.

4.1.2. $\mathcal{H}_2$-entropy and its consequences. Let us denote $Z = \rho^{-1} Y_x$. Then Z satisfies

$$\mathrm{D}_t Z = -\rho^{-1}(\rho^{-1} h_{xx})_x + \rho^{-1}(\rho^{-1} f_x)_x. \tag{71}$$

We thus define our next entropy by

$$\begin{aligned} \mathcal{H}_2 &= \frac{1}{2}\int_{\mathbb{T}} \rho Z^2 \,\mathrm{d}x + \int_{\mathbb{T}} \pi_2 \,\mathrm{d}x, \\ \pi_2 &= \frac{1}{2} h'(\rho) \frac{\rho_{xx}^2}{\rho^4}. \end{aligned} \tag{72}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{T}} \rho Z^2 \,\mathrm{d}x = -\int_{\mathbb{T}} (\rho^{-1} h_{xx})_x Z \,\mathrm{d}x + \int_{\mathbb{T}} (\rho^{-1} f_x)_x Z \,\mathrm{d}x.$$

The main term we need to eliminate is in fact $\rho^{-1} h_{xxx} Z$. The remaining ones coming from h_{xx} and ρ^{-1} end up being bounded by $|\rho_x|^2 + |\rho_x||\rho_{xx}|$. This can be estimated by

$$\int_{\mathbb{T}} |Z|(|\rho_x|^2 + |\rho_x||\rho_{xx}|) \,\mathrm{d}x \leqslant \mathcal{H}_2^{1/2}(|\rho_x|_\infty |\rho_{xx}|_2 + |\rho_x|_4^2). \tag{73}$$

Keep in mind that at this stage the dissipation term will be in $H^{2+\sigma/2}$ (see the discussion at the start of the section on $\mathcal{H}_1$-entropy). Moreover, the density being uniform in H^1, we obtain, by interpolation between H^1 and $H^{2+\sigma/2}$,

$$|\rho_x|_\infty |\rho_{xx}|_2 + |\rho_x|_4^2 \lesssim \|\rho\|_{H^{2+\sigma/2}}^{\frac{3}{\sigma+2}} + \|\rho\|_{H^{2+\sigma/2}}^{\frac{1}{\sigma+2}}. \tag{74}$$

Here, obviously, $\frac{3}{\sigma+2} \leqslant 1$ if $\sigma = 2$ or $\sigma = s$ in our range of s. Hence the term can be hidden in the dissipation:

$$\int_{\mathbb{T}} |Z|(|\rho_x|^2 + |\rho_x||\rho_{xx}|) \,\mathrm{d}x \leqslant c(\varepsilon)\mathcal{H}_2 + \varepsilon \|\rho\|_{H^{2+\sigma/2}}^2. \tag{75}$$

In the main term $\rho^{-1}h_{xxx}Z$, the worst part comes when all derivatives fall on the density:

$$h_{xxx} = h'\rho_{xxx} + 3h''\rho_{xx}\rho_x + h'''\rho_x^3.$$

Indeed, the term in the middle repeats (75), while the last one admits

$$(76)\qquad \int_{\mathbb{T}} |Z||\rho_x|^3 \,\mathrm{d}x \leqslant \mathcal{H}_2^{1/2}|\rho_x|_6^3 \leqslant \mathcal{H}_2^{1/2}\|\rho\|_{H^{2+\sigma/2}}^{\frac{2}{\sigma+2}} \leqslant c(\varepsilon)\mathcal{H}_2 + \varepsilon\|\rho\|^2_{H^{2+\sigma/2}}.$$

Here and throughout, we repeatedly use interpolation between H^1 and $H^{2+\sigma/2}$ for the density terms. Thus, the worst part of the original term $-\int_{\mathbb{T}}(\rho^{-1}h_{xx})_x Z\,\mathrm{d}x$ gets reduced to just

$$(77)\qquad -\int_{\mathbb{T}} \rho^{-1}h'(\rho)\rho_{xxx}Z\,\mathrm{d}x.$$

As on the $\mathcal{H}_1$-step, we expect this term to be canceled by a contribution coming from the pressure potential π_2. Let us examine it next.

It will be convenient to denote $g(r) = \frac{1}{2}h'(r)/r^4$ and simply write $\pi_2 = g(\rho)\rho_{xx}^2$. Integrating by parts, we obtain

$$(78)\quad \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \pi_2\,\mathrm{d}x = g\rho\rho_{xxx}u_{xx} + g'\rho\rho_{xx}\rho_x u_{xx} - 2g\rho_{xx}\rho_x u_{xx} - \frac{1}{2}g'\rho\rho_{xx}^2u_x - \frac{5}{2}g\rho_{xx}^2u_x.$$

In the course of subsequent computations, we will encounter a number of similar terms. They can be sorted into two groups—local ones involving u and X and nonlocal ones involving operator $\mathcal{L}^s$. All terms come with a prefactor of the form $q(\rho)$ for smooth q which can be ignored. The local ones are

$$(79)\qquad \rho_{xx}^2u_x, \quad \rho_{xx}\rho_x^2u_x, \quad \rho_{xx}\rho_x^2X_x, \quad \rho_{xx}^2X_x,$$

$$(80)\qquad \rho_x^2\rho_{xx}^2, \quad \rho_{xx}\rho_x^4, \quad \rho_{xx}^3,$$

the nonlocals are $q(\rho)$-multiples of

$$(81)\qquad \rho_{xx}^2\mathcal{L}^s\rho, \quad \rho_{xx}\rho_x^2\mathcal{L}^s\rho, \quad \rho_{xx}\rho_x\mathcal{L}^s\rho_x.$$

Substituting $u_x = \rho Y - c_{\mathrm{nl}}\mathcal{L}^s\rho - c_{\mathrm{loc}}(\mu(\rho)\rho_x/\rho^2)_x$ in the first two local terms, we reduce it to the next local and nonlocal ones. We now use interpolation and boundedness of X_x in L^2 uniformly to estimate the local terms as follows:

$$(82)\qquad \int_{\mathbb{T}} |\rho_{xx}^2X_x|\,\mathrm{d}x \leqslant |X_x|_2|\rho_{xx}|_4^2 \leqslant c_1\|\rho\|_{H^{2+\sigma/2}}^{\frac{10}{2\sigma+4}} \leqslant c_2 + \varepsilon\|\rho\|^2_{H^{2+\sigma/2}},$$

$$\int_{\mathbb{T}} |\rho_{xx}\rho_x^2X_x|\,\mathrm{d}x \leqslant |X_x|_2|\rho_{xx}|_2|\rho_x|_\infty^2 \leqslant c_3\|\rho\|_{H^{2+\sigma/2}}^{\frac{2}{\sigma+2}}\|\rho\|_{H^{2+\sigma/2}}^{\frac{2}{\sigma+2}}$$

$$(83)\qquad = c_3\|\rho\|_{H^{2+\sigma/2}}^{\frac{4}{\sigma+2}} \leqslant c_4 + \varepsilon\|\rho\|^2_{H^{2+\sigma/2}}.$$

The other local terms are

$$\left|\int_{\mathbb{T}} \rho_{xx}^3\,\mathrm{d}x\right| \leqslant |\rho_{xx}|_\infty\|\rho\|_{\dot{H}^3}\|\rho\|_{\dot{H}^1} + |\rho_{xx}|_\infty^2\|\rho\|^2_{\dot{H}^1} \leqslant c_5\|\rho\|_{\dot{H}^3}^{\frac{7}{4}} + c_6\|\rho\|_{\dot{H}^3}^{\frac{3}{2}} \leqslant c_7 + \varepsilon\|\rho\|^2_{H^3},$$

which follows after noting that $\|\rho\|_{\dot H^1}$ is uniformly bounded and $|\rho_{xx}|_\infty \leqslant c\|\rho\|_{\dot H^3}^{\frac34}$. In the next two terms, we simply use the Hölder inequality:

$$\left|\int_{\mathbb{T}} \rho_{xx}^2\rho_x^2 \,\mathrm{d}x\right| \leqslant |\rho_x|_\infty^2 \|\rho\|_{H^2}^2 \leqslant \|\rho\|_{\dot H^3}^{\frac12}\|\rho\|_{\dot H^3} = \|\rho\|_{\dot H^3}^{\frac32} \leqslant c_7 + \varepsilon\|\rho\|_{H^3}^2,$$

$$\left|\int_{\mathbb{T}} \rho_{xx}\rho_x^4 \,\mathrm{d}x\right| \leqslant |\rho_{xx}|_2|\rho_x|_\infty\|\rho\|_{H^3} \leqslant \|\rho\|_{\dot H^3}^{\frac12}\|\rho\|_{\dot H^3}^{\frac14}\|\rho\|_{\dot H^3} = \|\rho\|_{\dot H^3}^{\frac74} \leqslant c_7 + \varepsilon\|\rho\|_{H^3}^2.$$

Let us turn to nonlocal ones. In the first one, we symmetrize in the operator $\mathcal{L}^s$, which produces increments of the other factors that come with it. Thus, we have

$$\left|\int_{\mathbb{T}} q(\rho)\rho_{xx}^2\mathcal{L}^s\rho \,\mathrm{d}x\right| \leqslant |\rho_{xx}|_\infty\|\rho\|_{\dot H^{s/2+2}}\|\rho\|_{\dot H^{s/2}} + |\rho_{xx}|_\infty^2\|\rho\|_{\dot H^{s/2}}^2$$

and noting that $\|\rho\|_{\dot H^{s/2}}$ is uniformly bounded and $|\rho_{xx}|_\infty \leqslant c\|\rho\|_{\dot H^{s/2+2}}^{\frac{3}{s+2}}$,

$$\leqslant c_5\|\rho\|_{\dot H^{s/2+2}}^{1+\frac{3}{s+2}} + c_6\|\rho\|_{\dot H^{s/2+2}}^{\frac{6}{s+2}} \leqslant c_7 + \varepsilon\|\rho\|_{H^{2+\sigma/2}}^2.$$

In the next one, we simply use the Hölder inequality:

$$\left|\int_{\mathbb{T}} q(\rho)\rho_{xx}\rho_x^2\mathcal{L}^s\rho \,\mathrm{d}x\right| \leqslant |\rho_{xx}|_2|\rho_x|_\infty^2\|\rho\|_{H^s} \leqslant \|\rho\|_{\dot H^{s/2+2}}^{\frac{2}{s+2}}\|\rho\|_{\dot H^{s/2+2}}^{\frac{2}{s+2}}\|\rho\|_{\dot H^{s/2+2}}^{\frac{2s-2}{s+2}}$$
$$= \|\rho\|_{\dot H^{s/2+2}}^{\frac{2s+2}{s+2}} \leqslant c_7 + \varepsilon\|\rho\|_{H^{2+\sigma/2}}^2.$$

The same strategy applies for the last one:

$$\left|\int_{\mathbb{T}} q(\rho)\rho_{xx}\rho_x\mathcal{L}^s\rho_x \,\mathrm{d}x\right| \leqslant |\rho_{xx}|_2|\rho_x|_\infty\|\rho\|_{H^{s+1}} \leqslant \|\rho\|_{\dot H^{s/2+2}}^{\frac{2}{s+2}}\|\rho\|_{\dot H^{s/2+2}}^{\frac{1}{s+2}}\|\rho\|_{\dot H^{s/2+2}}^{\frac{2s}{s+2}}$$
$$= \|\rho\|_{\dot H^{s/2+2}}^{\frac{2s+3}{s+2}} \leqslant c_7 + \varepsilon\|\rho\|_{H^{2+\sigma/2}}^2.$$

Let us now get back to (78). The last two terms are obviously of same local type. The two terms in the middle are of the same type, too. There we replace u_{xx} with

$$u_{xx} = \rho^2 Z + \rho^{-1}\rho_x u_x + c_{\mathrm{loc}}(\rho^{-1}\rho_x\mathcal{L}^s\rho - \mathcal{L}^s\rho_x) + c_{\mathrm{nl}}(q_1\rho_x^3 + q_2\rho_x\rho_{xx} - q_3\rho_{xxx}), \tag{84}$$

where q's are some functions of ρ and, most importantly, $q_3 = \mu(\rho)/\rho^2 > 0$. This results into $\rho_{xx}\rho_x Z$, already estimated in (75), and the series of terms

$$\rho_{xx}\rho_x^2 u_x,\ \rho_{xx}\rho_x^2\mathcal{L}^s\rho,\ \rho_{xx}\rho_x\mathcal{L}^s\rho_x,\ \rho_x^2\rho_{xx}^2,\ \rho_{xx}\rho_x^4,\ \rho_{xx}^3,$$

all of which have been protocoled above. Finally, in the first and main term in (78) we use (84) to obtain

$$g\rho\rho_{xxx}u_{xx} = \rho^{-1}h'(\rho)\rho_{xxx}Z + g\rho_{xxx}(\rho_x u_x + \rho_x\mathcal{L}^s\rho - \rho\mathcal{L}^s\rho_x) + g\rho_{xxx}(q_1\rho_x^3 + q_2\rho_x\rho_{xx} - q_3\rho_{xxx}).$$

The first term is precisely the one that cancels with (77). In what follows, we integrate by parts to relieve one derivative from ρ_{xxx} in all but the final term above, which is strictly dissipative. In the local terms, we integrate by parts to relieve one derivative from ρ_{xxx}, and letting $g_i(r) := g(r)q_i(r)$, we obtain

$$-g\rho_{xxx}\rho_x u_x = g'\rho_{xx}\rho_x^2 u_x + g\rho_{xx}^2 u_x + g\rho_{xx}\rho_x u_{xx},$$
$$-g_1\rho_{xxx}\rho_x^3 = g_1'\rho_{xx}\rho_x^4 + 3g_1\rho_{xx}^2\rho_x^2,$$
$$-g_2\rho_{xxx}\rho_x\rho_{xx} = \frac12 g_2\rho_{xx}^3 + \frac12 g_2'\rho_x^2\rho_{xx}^2$$

upon integration. We obtain (up to the opposite sign)

$$
\begin{aligned}
(85)\quad & g'\rho_{xx}\rho_x^2 u_x + g\rho_{xx}^2 u_x + g\rho_{xx}\rho_x u_{xx} + g'\rho_{xx}\rho_x^2\mathcal{L}^s\rho + g\rho_{xx}^2\mathcal{L}^s\rho + g\rho_{xx}\rho_x\mathcal{L}^s\rho_x \\
& - g'\rho_{xx}\rho_x\rho\mathcal{L}^s\rho_x - g\rho_{xx}\rho_x\mathcal{L}^s\rho_x - g\rho_{xx}\rho\mathcal{L}^s\rho_{xx} \\
& + g\rho_{xx}\rho_x u_{xx} + g_1'\rho_{xx}\rho_x^4 + \left(3g_1 + \frac{1}{2}g_2'\right)\rho_{xx}^2\rho_x^2 + \frac{1}{2}g_2\rho_{xx}^3 - g_3\rho_{xxx}^2.
\end{aligned}
$$

All of the terms have been included in the lists, except $g\rho_{xx}\rho\mathcal{L}^s\rho_{xx}$ and $gq_3\rho_{xxx}^2$, which are dissipative. This is obvious for the local term:

$$
(86)\qquad \int_{\mathbb{T}} g_3(\rho)\rho_{xxx}^2\,\mathrm{d}x \geqslant c_1\|\rho\|_{\dot{H}^3}^2.
$$

As to the nonlocal term, we perform the same argument. By symmetrization, and noting that variation of $\rho g(\rho)$ results in a variation of ρ, we find

$$
\int_{\mathbb{T}} \rho g(\rho)\rho_{xx}\mathcal{L}^s\rho_{xx}\,\mathrm{d}x \leqslant -c_1\|\rho\|_{\dot{H}^{2+s/2}}^2 + |\rho_{xx}|_\infty\|\rho\|_{\dot{H}^{s/2}}\|\rho\|_{\dot{H}^{2+s/2}}
$$

since $H^{s/2}$ norm is uniformly bounded:

$$
(87)\qquad \leqslant -c_1\|\rho\|_{\dot{H}^{2+s/2}}^2 + \|\rho\|_{\dot{H}^{2+s/2}}^{\frac{3}{2+\alpha}+1} \leqslant -c_8\|\rho\|_{H^{2+s/2}}^2 + c_9.
$$

Collecting the obtained estimates, we arrive at

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_2 \leqslant -c'\|\rho\|_{H^{2+\sigma/2}}^2 + c''\mathcal{H}_2 + c'''.
$$

This proves uniform boundedness of $\mathcal{H}_2$ and $\rho \in L^2H^{2+\sigma/2}$. Also, from the corrector we obtain $\rho \in L^\infty H^2$. Since $Z \in L^2$, this translates into $u_{xx} + \partial_x\mathcal{L}^s\rho + \rho_{xxx} \in L^2$ uniformly. This puts $u \in L^\infty H^{3-\sigma} \cap L^2H^2$. Collectively, we obtain

$$
(88)\qquad u \in L^\infty(0,T;H^{3-\sigma}) \cap L^2(0,T;H^2),
$$

$$
(89)\qquad \rho \in L^\infty(0,T;H^2) \cap L^2(0,T;H^{\sigma/2+2}).
$$

4.1.3. $\mathcal{H}_n$-entropy: Closing the argument. Let us note that in the previous calculations the requirements on s relax to just $s > 1$. It is now clear that we can construct a hierarchy of higher-order entropies in the form

$$
(90)\qquad
\begin{aligned}
\mathcal{H}_n &= \frac{1}{2}\int_{\mathbb{T}} \rho Z_n^2\,\mathrm{d}x + \int_{\mathbb{T}} \pi_n\,\mathrm{d}x, \\
\pi_n &= \frac{1}{2}h'(\rho)\frac{(\partial_x^n\rho)^2}{\rho^{2n}},
\end{aligned}
$$

where at the core of Z_n is the term $\partial_x^n u + c_{\mathrm{nl}}\partial_x^{n-1}\mathcal{L}^s\rho + c_{\mathrm{loc}}q(\rho)\partial_x^{n+1}\rho$. The argument above extends easily with identical steps to this general case. As a result, we obtain uniform boundedness of the nth entropy and corresponding L^2-integrability of the enstrophy. This puts our solution into the classes

$$
(91)\qquad u \in L^\infty(0,T;H^{n+1-\sigma}) \cap L^2(0,T;H^n),
$$

$$
(92)\qquad \rho \in L^\infty(0,T;H^n) \cap L^2(0,T;H^{\sigma/2+n}).
$$

We thus have shown that

$$\sup_{T\in[0,T)} \|\rho\|_{L^\infty(0,T;H^n)} + \sup_{T\in[0,T)} \|\rho\|_{L^2(0,T;H^{\sigma/2+n})} + \sup_{T\in[0,T)} \|u\|_{L^\infty(0,T;H^{n+1-\sigma})} + \sup_{T\in[0,T)} \|u\|_{L^2(0,T;H^n)} \leqslant F\left(\|(\rho_0,u_0)\|_{H^n(\mathbb{T})\times H^{n+1-\sigma}(\mathbb{T})}, \|f\|_{L^\infty(0,T;C^n)}, \frac{1}{\underline{\rho}}, T\right) < \infty \tag{93}$$

for $n \geqslant 2$. Appealing to local existence, established by Proposition 3.1, the solution can be extended past T.

Acknowledgment. The third author thanks Princeton University for its hospitality during the work on the paper.

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