Nonlinearity

PAPER

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To cite this article: Theodore D Drivas and Alexei A Mailybaev 2021 *Nonlinearity* **34** 2296

IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **34** (2021) 2296–2326 https://doi.org/10.1088/1361-6544/abbe60

'Life after death' in ordinary differential equations with a non-Lipschitz singularity

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Received 25 May 2020, revised 20 August 2020
Accepted for publication 5 October 2020
Published 17 February 2021

Abstract

We consider a class of ordinary differential equations featuring a non-Lipschitz singularity at the origin. Solutions exist globally and are unique up until the first time they hit the origin. After 'blowup', infinitely many solutions may exist. To study continuation, we introduce physically motivated regularizations: they consist of smoothing the vector field in a ν-ball. We show that the limit $\nu \to 0$ can be understood using a certain autonomous dynamical system obtained by a solution-dependent renormalization. This procedure maps the pre-blowup dynamics to the solution ending at infinitely large renormalized time. The asymptotic behavior near blowup is described by an attractor. The post-blowup dynamics is mapped to a different renormalized solution starting infinitely far in the past and, consequently, it is associated with another attractor. The regularization establishes a relation between these two different 'lives' of the renormalized system and generically selects a restricted family of solutions, not depending on the regularization.

Keywords: selection principle, non-Lipschitz singularity, non-uniqueness in ordinary differential equations, blowup

Mathematics Subject Classification numbers: 34A12 37C83.

(Some figures may appear in colour only in the online journal)

1. Introduction

Non-Lipschitz singularities in differential equations arise naturally in numerous physical applications. For example, such singularities are the collision points in the N-body problem [16]. Non-Lipschitz singularities, which are reached in finite time, are wide spread in partial differential equations, where they are often termed as blowup [27]. A classical example is shock

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formation in conservation laws and ideal compressible fluid systems, where the velocity fields dynamically form jump discontinuities [13]. Another important example arises in the setting of incompressible fluid dynamics: the celebrated Kolmogorov (K41) theory stipulates that an ‘ideal’ turbulent flow field is only 1/3-Hölder continuous [33, 38].

A basic understanding of such non-smooth systems can be obtained from the textbook example

$$\dot{x} = x^{1/3}, \quad x(t_0) = x_0. \tag{1}$$

This is a particular case of the equation $\dot{x} = \mathrm{sgn}(x)|x|^{\alpha}$ for $\alpha = 1/3$. The exact solution of (1) for $t \geqslant t_0$ is easily obtained by separation of variables as

$$x(t) = x_0\left[1 + \frac{2}{3}|x_0|^{-2/3}(t - t_0)\right]^{3/2}, \quad x_0 \neq 0, \tag{2}$$

which is the unique solution for $x_0 \neq 0$. The right-hand side of (1) is not Lipschitz continuous at $x_0 = 0$ and, as a result, the solution is non-unique:

$$x(t) = \pm\left[\frac{2}{3}(t - t_0)\right]^{3/2}, \quad x_0 = 0. \tag{3}$$

We remark that by Kneser’s theorem, whenever a solution of the initial value problem is non-unique, then there are a continuous infinity of them (see section 2.4 of [34]). Such solutions are obtained in our case combining the trivial evolution $x(t) \equiv 0$ in an arbitrary interval $t_0 \leqslant t \leqslant t_1$ with any of nontrivial solutions (3), where t_0 is replaced by t_1. This simple example has deep physical significance within the Richardson picture of particle separation by a turbulent flow [49]. In fact, equation (1) with x denoting the separation between two particles can serve as a toy model for Richardson dispersion in an ‘ideal’ K41 velocity field. The solution (3) implies that particles starting at exactly the same point $x_0 = 0$ can split and reach finite distances apart in finite time [31]; related phenomena have been addressed in numerous numerical studies, e.g., [5, 8, 28, 51].

Due to their ubiquitous nature, it is important to develop an understanding of such non-smooth systems, when the classical theory of differential equations fails to provide uniqueness. A particularly important question is whether or not there is a natural way to select a unique solution or, at least, to provide a non-trivial restriction? For example, this question is answered positively for binary collisions in the n-body problem, where solutions continuously dependent on initial conditions are obtained using the so-called collision manifold [3, 16, 46]. An example in partial differential equations is the theory of one-dimensional conservation laws, where the entropy condition can be used to select a unique weak solution after blowup [13]; this is a shock solution arising in the vanishing viscosity limit [39]. However, a natural analogue of the entropy condition—that a weak solution dissipates energy—is known to fail as a selection principle for three-dimensional incompressible Euler equations [10, 14, 36]. Non-smooth velocity fields provide interesting interplay between transport equations and ordinary differential equations [1, 17], where non-unique solutions can also be considered backward in time [30]. The question remains as to what extent any physically relevant criteria may serve to select unique solutions.

There are several notions of regularization used in the theory of non-smooth ordinary differential equations, e.g., analytic regularization and block regularization (or regularization by surgery) which require that extended solutions are continuous with respect to initial conditions [21]. The regularization used in our work is closely related to the concept of ϕ-regularization which can be used to give robust notion of a flow in discontinuous dynamical systems, see e.g. [40, 47].

The goal of the present paper is to investigate fundamental constraints on possible 'selected' solutions, when the non-uniqueness is caused by an isolated non-Lipschitz singularity. Our study exploits a class of ordinary differential equations described in section 2, which involve vector fields which are smooth away from the origin. At the origin, the equations are merely Hölder continuous (providing a cartoon of fields arising in fluid dynamics problems) or even unbounded (mimicking collisional problems for particles with Newtonian potentials). In generic situations, solutions enter the origin in finite time, which we call blowup. Since infinite number of solutions also start at the origin, continuation past the blowup is strongly non-unique.

In these models we uncover some universal characteristics of non-uniqueness in the context of a 'physically relevant' choice. To accomplish this, we introduce a set of renormalized phase-time variables, such that leaving the singularity takes infinite time in the new representation. Thereby, the solution splits into two different infinitely long evolutions before and after blowup. Section 3 shows how the first evolution determines the universal asymptotic behavior before the blowup. Section 4 studies continuation after blowup, when the uniqueness is lost. Here we introduce in a regularization scheme, which smooths the vector field inside a small ν-neighborhood of a singularity and consider the limit $\nu \to 0$. For a wide class of regularizations, we associate such limiting solutions with an attractor of the renormalized system. Section 5 describes the case of a fixed-point attractor, when a unique post-blowup solution is selected. Remarkably, this solution is not sensitive to a particular choice of regularization, i.e. the relevant choice is foreseen by a singular differential equation. Section 6 investigates a different situation, when the attractor is a periodic limit cycle. In this case limiting solutions are non-unique, but the regularization imposes a highly non-trivial constraint on possible continuations, again independent of the choice of regularization.

As a consequence, we establish a connection between the theory of attractors in autonomous dynamical systems with the problem of non-uniqueness and continuation at the non-Lipschitz singularity. It shows that in a generic case the equation may impose a 'destiny' on a solution even after the uniqueness is lost. Our theorems are coupled with concrete examples illustrating the non-uniqueness and specific choices of solutions after blowup. We end with the discussion section, where we also speculate on similar phenomena in models of fluid dynamics.

2. Equations with an isolated non-Lipschitz singularity

We study the Cauchy problem for an autonomous system of ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{4}$$

where $\mathbf{x} \in \mathbb{R}^d$ and the derivative is taken with respect to time $t \in \mathbb{R}$. We assume that $\mathbf{f}$ has an isolated singular (non-Lipschitz) point, which we assign to the origin, $\mathbf{x} = \mathbf{0}$. We will consider a self-similar form of the singularity, when

$$\mathbf{f}(\mathbf{x}) = r^{\alpha}\mathbf{F}(\mathbf{y}), \quad r = |\mathbf{x}|, \quad \mathbf{y} = \mathbf{x}/r \tag{5}$$

with $\alpha < 1$ and a smooth (continuously differentiable) function $\mathbf{F}(\mathbf{y})$ on a unit sphere $\mathbf{y} \in S^{d-1}$. The function $\mathbf{f}(\mathbf{x})$ is smooth everywhere except at the origin, where it is discontinuous for negative α and Hölder continuous, $\mathbf{f} \in C^{\alpha}(\mathbb{R}^d)$ for $\alpha \in (0, 1)$. We will separate the radial and spherical components of this function as

$$\mathbf{F}(\mathbf{y}) = F_{\mathrm{r}}(\mathbf{y})\mathbf{y} + \mathbf{F}_{\mathrm{s}}(\mathbf{y}) \tag{6}$$

with $F_r(\mathbf{y}) = \mathbf{F}(\mathbf{y}) \cdot \mathbf{y}$ and $\mathbf{F}_s$ defined implicitly by (6). The models (5) can be thought of as multi-dimensional generalizations of the toy model (1) for $\alpha = 1/3$.

Homogeneity of the field in (5) reflects, in a simplified form, the fundamental property of scale invariance ubiquitous in real-world applications. We wish to emphasize that our system should be understood as a zeroth-order (isolated singular point) model for non-smooth dynamics. This model gives useful insight into more general systems that feature higher-dimensional singular sets and, thus, cannot be cast as (5). A canonic example would be the n-body problem, in which the Newtonian force has a singularity with $\alpha = -2$ at points with equal coordinates and arbitrary momenta of any two bodies.

For $\alpha \in (0, 1)$, the function $\mathbf{f}(\mathbf{x})$ is continuous and bounded by $|\mathbf{f}(\mathbf{x})| < r^\alpha \max \mathbf{F}(\mathbf{y})$, where the maximum is taken on the unit sphere $\mathbf{y} \in S^{d-1}$. Hence, the Cauchy problem (4) possesses a solution globally in time for any initial condition [34]. However, the solutions that pass through the singularity at the origin, are not necessarily unique.

Let us denote by t_b the maximal time such that $r(t) \neq 0$ for $t \in [t_0, t_b)$. This provides the largest time interval, where the uniqueness of the solution is guaranteed due to the Lipschitz continuity. In particular, we have the global uniqueness in the case of $t_b = \infty$. When $t_b < \infty$, the solution reaches the singularity at the origin in finite time, $\lim_{t\to t_b} r(t) = 0$, and we call this scenario the *finite-time blowup*. This definition can be related to collisions or, alternatively, can be motivated by the blowup concept for partial differential equations, where it is linked to the breakdown of Lipschitz continuity. Recall, for example, that solutions of the inviscid Burgers equation have the local form $u(x) \propto -x^{1/3}$ at the blowup time [48].

So long as $r(t) > 0$, we can define the logarithmic radial coordinate as $z(t) = \ln r(t)$. The original solution is then expressed as

$$\mathbf{x}(t) = e^{z(t)}\mathbf{y}(t). \tag{7}$$

Equations for $z(t)$ and $\mathbf{y}(t)$ are obtained directly from (4) and read:

$$\dot{\mathbf{y}} = e^{-(1-\alpha)z}\mathbf{F}_s(\mathbf{y}), \quad \mathbf{y}(t_0) = \mathbf{y}_0, \tag{8}$$

$$\dot{z} = e^{-(1-\alpha)z}F_r(\mathbf{y}), \quad z(t_0) = z_0. \tag{9}$$

where the initial conditions are given by $\mathbf{y}_0 = \mathbf{x}_0/|\mathbf{x}_0|$ and $z_0 = \ln|\mathbf{x}_0|$.

3. Pre-blowup dynamics

For $t < t_b$, we introduce the a new temporal variable $s = s(t)$ defined by

$$s(t) = \int_{t_0}^{t} e^{-(1-\alpha)z(t')}\mathrm{d}t'. \tag{10}$$

The map $s(t)$ is monotonically increasing with $s(t_0) = 0$, and let us denote $s_b = \lim_{t\to t_b} s(t)$. Hence, the inverse monotonically increasing function $t = \tilde{t}(s)$ exists with $\tilde{t}(0) = t_0$ and $t_b = \lim_{s\to s_b} \tilde{t}(s)$. We denote $\tilde{\mathbf{y}}(s) := \mathbf{y}(\tilde{t}(s))$ and $\tilde{z}(s) := z(\tilde{t}(s))$; here and below we use tildes to denote functions of the new time variable s. By the inverse function theorem, the derivative of $\tilde{t}(s)$, obtained from (10), can be written in the form $\mathrm{d}\tilde{t}/\mathrm{d}s = e^{(1-\alpha)\tilde{z}(s)}$ and integrated as

$$\tilde{t}(s) = t_0 + \int_0^s e^{(1-\alpha)\tilde{z}(s')}\mathrm{d}s'. \tag{11}$$

Using $s(t)$ as the new temporal variable in system (8) and (9), a simple computation yields the renormalized system in the form

$$\mathrm{d}\tilde{\mathbf{y}}/\mathrm{d}s = \mathbf{F}_\mathrm{s}(\tilde{\mathbf{y}}), \quad \tilde{\mathbf{y}}(0) = \mathbf{y}_0, \tag{12}$$

$$\mathrm{d}\tilde{z}/\mathrm{d}s = F_\mathrm{r}(\tilde{\mathbf{y}}), \quad \tilde{z}(0) = z_0. \tag{13}$$

In this system, the first equation (12) is uncoupled, and the second equation (13) is integrated as

$$\tilde{z}(s) = z_0 + \int_0^s F_\mathrm{r}(\tilde{\mathbf{y}}(s'))\mathrm{d}s'. \tag{14}$$

Since the functions $\mathbf{F}_\mathrm{s}$ and F_r are continuous on the unit sphere S^{d-1} and therefore bounded, solutions $\tilde{\mathbf{y}}(s)$ and $\tilde{z}(s)$ exist globally in renormalized time and are unique. The solution $\mathbf{x}(t)$ of the original system (4) for $t \in [t_0, t_\mathrm{b})$ can be recovered from the global histories $\tilde{\mathbf{y}}(s)$ and $\tilde{z}(s)$ with $s = s(t)$ given by (10) through the transformations $\mathbf{x}(t) = \tilde{\mathbf{x}}(s(t))$ and $\tilde{\mathbf{x}}(s) = e^{\tilde{z}(s)}\tilde{\mathbf{y}}(s)$.

Proposition 1. *Let us define the following upper and lower renormalized-time averages:*

$$\underline{F_\mathrm{r}} = \liminf_{s\to\infty} \frac{1}{s}\int_0^s F_\mathrm{r}(\tilde{\mathbf{y}}(s'))\mathrm{d}s', \qquad \overline{F_\mathrm{r}} = \limsup_{s\to\infty} \frac{1}{s}\int_0^s F_\mathrm{r}(\tilde{\mathbf{y}}(s'))\mathrm{d}s'. \tag{15}$$

These averages are finite, because F_r is bounded, and characterize blowup as

(*a*) *If $\underline{F_\mathrm{r}} > 0$, then $t_\mathrm{b} = \infty$ and $\lim_{t\to\infty} r(t) = \infty$.*
(*b*) *If $\overline{F_\mathrm{r}} < 0$, then the solution $\mathrm{x}(t)$ blows up at finite time $t_\mathrm{b} < \infty$, i.e., $\lim_{t\to t_\mathrm{b}} r(t) = 0$.*

Proof. First, let us show that $s_\mathrm{b} = \infty$. In the case of blowup, $t_\mathrm{b} < \infty$, we have $z(t) = \ln r(t) \to -\infty$ as $t \to t_\mathrm{b}$. Since the function F_r is bounded, the corresponding behavior $\tilde{z}(s) \to -\infty$ as $s \to s_\mathrm{b}$ in (14) yields $s_\mathrm{b} = \infty$. Therefore, using $\tilde{t}(s)$ from (11), one has

$$t_\mathrm{b} = \lim_{s\to\infty} \tilde{t}(s) = t_0 + \int_0^\infty e^{(1-\alpha)\tilde{z}(s')}\mathrm{d}s'. \tag{16}$$

The same conclusion can be drawn in the case of no blowup, $t_\mathrm{b} = \infty$, because $\tilde{t}(s)$ is finite for any finite value of s.

Next, by definition, for any $\varepsilon > 0$ there exists an $s_\varepsilon > 0$ such that

$$\underline{F_\mathrm{r}} - \varepsilon/2 < \frac{1}{s}\int_0^s F_\mathrm{r}(\tilde{\mathbf{y}}(s'))\mathrm{d}s' < \overline{F_\mathrm{r}} + \varepsilon/2 \quad \text{for } s > s_\varepsilon. \tag{17}$$

The value of s_ε can be chosen sufficiently large such that

$$|z_0| < \varepsilon s_\varepsilon/2. \tag{18}$$

Using (17) and (18) in expression (14), we find

$$\left(\underline{F_\mathrm{r}} - \varepsilon\right)s < \tilde{z}(s) < \left(\overline{F_\mathrm{r}} + \varepsilon\right)s \quad \text{for } s > s_\varepsilon. \tag{19}$$

(a) If $\underline{F_\mathrm{r}} > 0$, then the first inequality in (19) with $\varepsilon = \underline{F_\mathrm{r}}/2$ yields

$$\tilde{z}(s) > \underline{F_\mathrm{r}}s/2 \quad \text{for } s > s_\varepsilon. \tag{20}$$

Using this estimate in (11), we obtain the lower bound:

$$\tilde{t}(s) > \tilde{t}(s_\varepsilon) + \int_{s_\varepsilon}^{s} e^{(1-\alpha)\underline{F_r} s'/2} \mathrm{d}s' \quad \text{for } s > s_\varepsilon. \tag{21}$$

Since $\alpha < 1$ and $\underline{F_r} > 0$, the integral in (21) diverges as $s \to \infty$ and we have $t_b = \lim_{s\to\infty} \tilde{t}(s) = \infty$. This proves that the solution $\mathbf{x}(t)$ does not blow up in finite time. From equation (20), it follows that $\lim_{t\to\infty} r(t) = \lim_{s\to\infty} e^{\tilde{z}(s)} = \infty$.

(b) If $\overline{F_r} < 0$, a similar argument with $\varepsilon = -\overline{F_r}/2 > 0$ and the second inequality in (19) yields

$$\tilde{z}(s) < \overline{F_r} s/2 \quad \text{for } s > s_\varepsilon. \tag{22}$$

This guarantees that $\lim_{s\to\infty} \tilde{z}(s) = -\infty$ and the integral in (11) converges to a finite value $t_b = \lim_{s\to\infty} \tilde{t}(s) < \infty$. Thus, the solution reaches the origin $\lim_{t\to t_b} r(t) = 0$ in a finite time t_b. □

Proposition 1 demonstrates that the key property of the renormalization is to extend the blowup time (if it exists) to infinity, while maintaining the equations in autonomous form (12) and (13). Hence, it is natural to associate the blowup with a dynamical attractor $\mathcal{A}$ of system (12). Recall that the attractor $\mathcal{A}$ is a nonempty, compact, invariant set that has a neighborhood U_0 with the property $\cap_{s\geqslant 0} U_s = \mathcal{A}$, where U_s is the neighborhood U_0 transported by the system flux at time $s > 0$, see, e.g., [26, 35]. The basin of attraction, $\mathcal{B}(\mathcal{A})$, is the set of all points that approach $\mathcal{A}$ asymptotically under $s \to \infty$. According to proposition 1, if $\mathbf{y}_0 \in \mathcal{B}(\mathcal{A})$, then the blowup dynamics is controlled by the average of the function F_r on the attractor.

Definition 1. We say that the attractor of the system (12) is *focusing* if $\overline{F_r} < 0$ for any $\mathbf{y}_0 \in \mathcal{B}(\mathcal{A})$. Similarly, we call the attractor *defocusing* if $\underline{F_r} > 0$ for any $\mathbf{y}_0 \in \mathcal{B}(\mathcal{A})$.

For simple attractors like an asymptotically stable fixed point or a limit cycle, there exists an average value, $\langle F_r \rangle = \underline{F_r} = \overline{F_r}$, which is independent of the initial condition $\mathbf{y}_0 \in \mathcal{B}(\mathcal{A})$. For example, if $\mathcal{A}$ is a fixed point $\{\mathbf{y}_*\}$, then $\langle F_r \rangle = F_r(\mathbf{y}_*)$. If $\mathcal{A}$ is a limit cycle, then $\langle F_r \rangle$ is the average of $F_r(\tilde{\mathbf{y}}(s))$ over one period of the attractor. The average $\langle F_r \rangle$ can also exist in more complex situations, in the case of quasi-periodic and chaotic attractors under proper ergodicity assumptions. For example for sufficiently mixing flows, $\langle F_r \rangle$ exists and is computed as an average of $F_r(\tilde{\mathbf{y}}(s))$ over the attractor, with respect to the SRB measure; see [9, 26] for precise definitions.

With the well-defined average value, the property of the attractor to be focusing ($\langle F_r \rangle < 0$) or defocusing ($\langle F_r \rangle > 0$) is robust. The only exception is given by the degeneracy condition $\langle F_r \rangle = 0$. The immediate consequence of proposition 1 is:

Theorem 1. *If $\mathbf{y}_0 \in \mathcal{B}(\mathcal{A})$ for a defocusing attractor $\mathcal{A}$ of system (12), then $t_b = \infty$ and $\lim_{t\to\infty} r(t) = \infty$. If $\mathbf{y}_0 \in \mathcal{B}(\mathcal{A})$ for a focusing attractor $\mathcal{A}$, then the solution $\mathbf{x}(t)$ blows up in finite time and the attractor describes the asymptotics of the spherical variables: $\lim_{t\to t_b} \mathrm{dist}(\mathbf{y}(t), \mathcal{A}) = 0$.*

This result provides a natural tool for characterizing and classifying possible types of blowup. In general, when $\langle F_r \rangle < 0$, the integrals (11) and (14) yield the estimates

$$\tilde{z}(s) \sim \langle F_r \rangle s, \quad t_b - \tilde{t}(s) \sim \exp\left[(1-\alpha)\langle F_r \rangle s\right] \quad \text{for } s \to \infty, \tag{23}$$

which lead to the power-law asymptotic for $r = e^z$ as

$$r(t) \sim (t_b - t)^{\frac{1}{1-\alpha}} \quad \text{for } t \to t_b. \tag{24}$$

Thus, near the singularity the solution behaves as a power law conforming to the scaling symmetry of the equation and is, in this sense, *asymptotically self-similar*.

We remark that the discussed scenarios of blowup in the finite-dimensional singular system (4) and (5) closely resemble the blowup dynamics in partial differential equations [27, 48] and infinite-dimensional turbulence models [18, 41, 42], where the asymptotic blowup dynamics is associated to the attractor of a renormalized system. Similar renormalized systems are also known as the blowing-up construction for classification of vector field singularities and local bifurcation theory [22, 23, 52]; notice that the notion of 'blow up' in this approach refers to the unfolding transformation of original equations rather than to a finite-time singularity addressed here.

To conclude this section, we give a concrete example.

Example 1. In the simplest case an attractor $\mathcal{A}$ is a fixed point, say $\mathcal{A} = \{\mathbf{y}_*\}$ with $\mathbf{F}_\mathrm{s}(\mathbf{y}_*) = \mathbf{0}$. Assuming that $\mathbf{y}_0 \in \mathcal{B}(\{\mathbf{y}_*\})$, we find the average value $\langle F_\mathrm{r}\rangle = F_\mathrm{r}(\mathbf{y}_*)$. A negative value of $F_\mathrm{r}(\mathbf{y}_*)$ guarantees the finite-time blowup. For $\mathbf{x}_0 = r_0\mathbf{y}_*$ and $r_0 = e^{z_0}$, the fixed-point solution $\tilde{\mathbf{y}}(s) \equiv \mathbf{y}_*$ with expressions (14) and (11) yield

$$\tilde{z}(s) = z_0 + F_\mathrm{r}(\mathbf{y}_*)s, \qquad \tilde{t}(s) = t_\mathrm{b} - \frac{\exp\left[(1-\alpha)(z_0 + F_\mathrm{r}(\mathbf{y}_*)s)\right]}{(\alpha-1)F_\mathrm{r}(\mathbf{y}_*)}, \tag{25}$$

where the blowup time (16) is given by

$$t_\mathrm{b} = t_0 + \frac{\exp[(1-\alpha)z_0]}{(\alpha-1)F_\mathrm{r}(\mathbf{y}_*)} = t_0 + \frac{r_0^{1-\alpha}}{(\alpha-1)F_\mathrm{r}(\mathbf{y}_*)}. \tag{26}$$

Recall that $(\alpha-1)F_\mathrm{r}(\mathbf{y}_*) > 0$ for negative $F_\mathrm{r}(\mathbf{y}_*)$ and $\alpha < 1$. The second expression in (25) can be used to solve the equation $t = \tilde{t}(s)$ with respect to $s = s(t)$. Then the solution $\mathbf{x}(t) = r(t)\mathbf{y}(t)$ with $\mathbf{y}(t) \equiv \mathbf{y}_*$ and $r(t) = e^{\tilde{z}(s(t))}$ is obtained in the form

$$\mathbf{x}(t) = r(t)\mathbf{y}_*, \quad r(t) = \left[(\alpha-1)F_\mathrm{r}(\mathbf{y}_*)(t_b - t)\right]^{\frac{1}{1-\alpha}}. \tag{27}$$

In the more general case, when $\mathbf{y}_0$ belongs to the basin of attraction of $\mathbf{y}_*$, expression (27) provides the asymptotic form of the solution before the blowup, as $t \to t_\mathrm{b}$.

Specifically, let us consider system (4) and (5) for any $\alpha < 1$ and $\mathbf{F}(\mathbf{y}) = (F_1, F_2)$ with

$$F_1(\mathbf{y}) = y_1^2 + y_1y_2 + y_1y_2^2, \qquad F_2(\mathbf{y}) = y_1y_2 + y_2^2 - y_1^2y_2. \tag{28}$$

Phase diagram of this system is shown in figure 1(a). For all initial conditions on the left half-plane $x_1 < 0$ and semi-axis $x_1 = 0$, $x_2 < 0$, solutions enter the singularity at the origin in finite time (blowup). On the other hand, all solutions in the upper half-place $x_2 > 0$ and semi-axis $x_1 > 0$, $x_2 = 0$ originate at the origin. The curves in the fourth quadrant $x_1 > 0$, $x_2 < 0$ never hit the singularity.

We can express $\mathbf{y} = (\cos\varphi, \sin\varphi)$, where the real variable $\varphi \in S^1$ describes the angular dynamics. Then the radial and circular components of the vector field are introduced as

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} F_\mathrm{r} + \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} F_\mathrm{s}, \qquad F_\mathrm{r}(\varphi) = \sqrt{2}\ \sin\left(\varphi + \frac{\pi}{4}\right), \\ F_\mathrm{s}(\varphi) = -\frac{\sin\ 2\varphi}{2}. \tag{29}$$

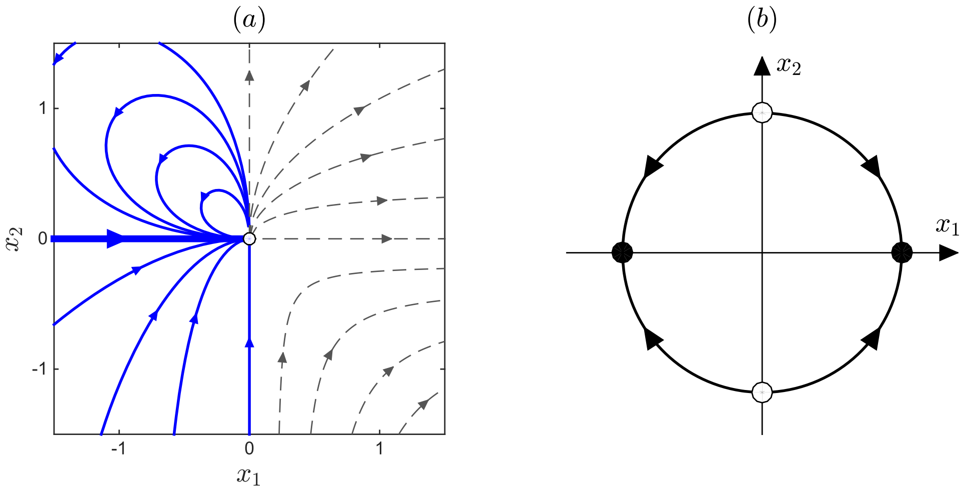

Figure 1. (a) Vector field of equations (4), (5) and (28) with representative solutions. Curves entering the origin in finite time (blowup) are colored blue. (b) Phase portrait of the renormalized system on a circle. Attractors are shown by black circles and repellers by white circles.

For the renormalized system (12) and (13) we obtain

$$\mathrm{d}\tilde{\varphi}/\mathrm{d}s = F_{\mathrm{s}}(\tilde{\varphi}), \quad \mathrm{d}\tilde{z}/\mathrm{d}s = F_{\mathrm{r}}(\tilde{\varphi}), \tag{30}$$

for $\tilde{\varphi}(s) := \varphi(\tilde{t}(s))$. Dynamics of the equation for $\tilde{\varphi}$ in (30) is very simple as it possesses only two fixed-point attractors at $\tilde{\varphi} = 0$ and π, see figure 1(b). The corresponding basins of attraction are separated by the two unstable fixed points at $\tilde{\varphi} = \pm\pi/2$. Since $F_{\mathrm{r}}(0) = 1 > 0$ for the first attractor, it does not correspond to the blowup. The second attractor with $\tilde{\varphi} = \pi$ has $F_{\mathrm{r}}(\pi) = -1 < 0$ with the basin of attraction $\pi/2 < \tilde{\varphi} < 3\pi/2$. Therefore, it describes the blowup from the initial condition at any point of the left half-plane $x_1 < 0$ (basin of attraction). This is confirmed in figure 1(a) demonstrating the phase portrait of original system (4) and (5). All blue curves enter the origin in finite time.

The stable fixed-point defines the blowup solution (27), which in our example takes the form

$$\mathbf{x}(t) = -\begin{pmatrix} [(1-\alpha)(t_{\mathrm{b}} - t)]^{\frac{1}{1-\alpha}} \\ 0 \end{pmatrix}, \quad 0 \leqslant t \leqslant t_b. \tag{31}$$

By theorem 1, the solution (31) is asymptotic for all solutions that end at the singularity from the left half-plane $x_1 < 0$; see the bold blue line in figure 1(a). There also exists a single solution corresponding to the unstable fixed point $\tilde{\varphi} = 3\pi/2$, see figure 1(b). This solution is similarly found as

$$\mathbf{x}(t) = -\begin{pmatrix} 0 \\ [(1-\alpha)(t_{\mathrm{b}} - t)]^{\frac{1}{1-\alpha}} \end{pmatrix}, \quad 0 \leqslant t \leqslant t_b. \tag{32}$$

However, this solution corresponds to a zero measure set of initial conditions. Thus, expression (31) describes asymptotically the only generic blowup scenario in the system.

4. Post-blowup dynamics

Our system is not Lipschitz continuous at the origin and, hence, there can be multiple solutions starting at the singularity; see, for example figure 1(a), where all solutions with $x_2 > 0$ originate at the singularity at finite time. This prevents one from uniquely defining the solution globally in time and motivates the study of regularized problems for the search of selection principles.

4.1. ν-regularization and 'inviscid' limit

In this section, we consider a class of regularized problems $\dot{\mathbf{x}} = \mathbf{f}^\nu(\mathbf{x})$ which have unique global solutions and provide a selection rule by taking the limit $\nu \to 0$ in which the regularization is removed. *A priori*, there are infinitely many different ways to regularize the vector field. For example, one can take the convolution $\mathbf{f}^\nu = G^\nu * \mathbf{f}$ with any smooth scaled mollifier G^ν which approximates the identity $G^\nu \to \delta$ as $\nu \to 0$. Often, a physical application determines a relevant regularization.

For analytical convenience, we consider the family $\mathbf{f}^\nu(\mathbf{x})$ obtained by smoothing the function $\mathbf{f}(\mathbf{x})$ in equation (5) inside a sphere of radius ν centered at the origin. Then the vector field is constructed by patching together the regularized and the original fields inside and outside the sphere. Due to the self-similar form of $\mathbf{f}(\mathbf{x})$, it is convenient to define the regularization for all $\nu > 0$ with the same function $\mathbf{G}(\mathbf{x})$ scaled properly inside the ν-sphere. More specifically, we say $\mathbf{x} = \mathbf{x}^\nu(t)$ is a solution of a *ν-regularized problem* if it solves

$$\dot{\mathbf{x}} = \mathbf{f}^\nu(\mathbf{x}), \quad \mathbf{f}^\nu(\mathbf{x}) := \begin{cases} r^\alpha \mathbf{F}(\mathbf{x}/r) & r > \nu; \\ \nu^\alpha \mathbf{G}(\mathbf{x}/\nu) & r \leqslant \nu, \end{cases} \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{33}$$

where the function $\mathbf{G}(\mathbf{x})$ is designed so that $\mathbf{f}^\nu(\mathbf{x})$ is continuously differentiable everywhere. Such regularization is demonstrated schematically in figure 2(a). We remark that there are non-trivial topological constraints on possible vector fields $\mathbf{G}$ that can be chosen inside this ν-ball. These issues are discussed in [12, 25]. It is easy to see that the regularized solution $\mathbf{x}^\nu(t)$ exists and is unique globally in time.

There is some similarity between our regularization and the role of viscosity in fluid dynamics: both change the system at small scales, which are responsible to the blowup. Motivated by this analogy, the limit $\nu \to 0$ can be termed the *inviscid limit*. We now show that $\mathbf{x}^\nu(t)$ converges (along subsequences) to a solution of the original singular system (4).

Theorem 2. *Given any initial condition $\mathbf{x}_0$ and any finite time interval $[t_0, t_1]$, there exists a vanishing subsequence $\lim_{n\to\infty} \nu_n \to 0$ such that the ν-regularized solutions $\{\mathbf{x}^{\nu_n}(t)\}_{n\geqslant 0}$ with $\mathbf{x}^{\nu_n}(t_0) = \mathbf{x}_0$ converge uniformly to a limit*

$$\mathbf{x}(t) = \lim_{n\to\infty} \mathbf{x}^{\nu_n}(t) \quad \text{for } t \in [t_0, t_1]. \tag{34}$$

The limiting function $\boldsymbol{x}(t)$ is a solution of the original system (4) with $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$.

Proof. Let us first consider the case $\alpha \in [0, 1)$. Equation (33) provides the estimate for the time derivative of the norm $r^\nu(t) = |\mathbf{x}^\nu(t)|$ as

$$\dot{r}^\nu \leqslant A \ \max(r^\nu, \nu)^\alpha \leqslant A(r^\nu + \nu)^\alpha, \tag{35}$$

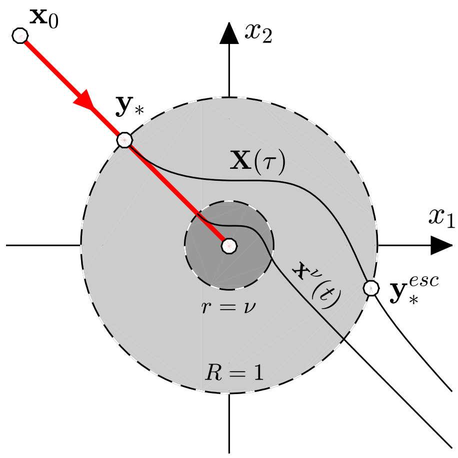

Figure 2. (a) Outline of the regularized system (33): the red line corresponds to blowup solution for $\nu = 0$. The curve $\mathbf{x}^\nu(t)$ shows the regularized solution with the equation modified in the disk of small radius $\nu > 0$. The other curve $\mathbf{X}(\tau)$ is a solution of the rescaled system (48), which corresponds to $\nu = 1$.

where $A \geqslant 0$ is the maximum value of the norms $|\mathbf{F}(\mathbf{y})|$ for $\mathbf{y} \in S^{d-1}$ and $|\mathbf{G}(\mathbf{x})|$ for $|\mathbf{x}| \leqslant 1$. The extremal solution $r^\nu_{\text{ext}}(t)$ of differential inequality (35) can be found as

$$r^\nu_{\text{ext}}(t) = \left[A(1-\alpha)(t-t_0) + (r_0+\nu)^{1-\alpha}\right]^{1/(1-\alpha)} - \nu, \tag{36}$$

where $r_0 = r^\nu(t_0) = |\mathbf{x}_0|$. The function (36) in the interval $t \in [t_0, t_1]$ attains its maximum at the final time t_1. In particular, given the initial value r_0, a finite interval $[t_0, t_1]$ and fixing an arbitrary $\nu_0 > 0$, there is a ball of finite radius $r_b = \left[A(1-\alpha)(t_1-t_0) + (r_0+\nu_0)^{1-\alpha}\right]^{1/(1-\alpha)}$ such that

$$r^\nu(t) \leqslant r_b \text{ for any } t \in [t_0, t_1], \quad \nu \in (0, \nu_0]. \tag{37}$$

In particular, solutions of (33) are *bounded uniformly in* ν.

We will now show that the family $\{\mathbf{x}^\nu(t)\}_{\nu>0}$ is equicontinuous. To see this, note that solutions of (33) satisfy the integral form of the equation:

$$\mathbf{x}^\nu(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}^\nu(\mathbf{x}^\nu(t'))\mathrm{d}t', \quad \nu > 0. \tag{38}$$

Let B be the maximum value of the norm $|\mathbf{f}^\nu(\mathbf{x})|$ in the compact set $\{\mathbf{x} : |\mathbf{x}| \leqslant r_b\}$ and $\nu \in [0, \nu_0]$. Then the relations (37) and (38) guarantee that

$$|\mathbf{x}^\nu(t) - \mathbf{x}^\nu(t')| \leqslant B|t-t'| \text{ for any } t, t' \in [t_0, t_1], \quad \nu \in (0, \nu_0]. \tag{39}$$

This inequality proves that the family of solutions $\{\mathbf{x}^\nu(t)\}_{\nu>0}$ is *equicontinuous*. Thus, by the Arzelà–Ascoli theorem (see, e.g. [50]), there exists a subsequence $\nu_0, \nu_1, \nu_2, \ldots$, such that $\lim_{n\to\infty} \nu_n = 0$ and the corresponding solutions converge uniformly to the continuous function (34). Finally, since the function $\mathbf{f}^\nu(\mathbf{x})$ is uniformly continuous in the compact set $|\mathbf{x}| \leqslant r_b$, $\nu \in [0, \nu_0]$, one can pass to the limit in equation (38), which yields

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}(\mathbf{x}(t'))\mathrm{d}t'. \tag{40}$$

This implies that the limiting solution $\mathbf{x}(t)$ solves the original equation (4).

In the case of negative α we use the transformation

$$\mathbf{x}_{\text{new}} = r^{-\alpha}\mathbf{x}, \qquad \nu_{\text{new}} = \nu^{1-\alpha}, \tag{41}$$

reducing system (33) to the form

$$\dot{\mathbf{x}}_{\text{new}} = \begin{cases} \mathbf{F}_{\text{new}}(\mathbf{y}) & r_{\text{new}} > \nu_{\text{new}}; \\ \mathbf{G}_{\text{new}}(\mathbf{x}_{\text{new}}/\nu_{\text{new}}) & r_{\text{new}} \leqslant \nu_{\text{new}}, \end{cases} \qquad \mathbf{x}_{\text{new}}(t_0) = |\mathbf{x}_0|^{1-\alpha}\mathbf{y}_0, \tag{42}$$

where

$$\begin{aligned} \mathbf{F}_{\text{new}}(\mathbf{y}) &= (1-\alpha)F_{\text{r}}(\mathbf{y})\mathbf{y} + \mathbf{F}_{\text{s}}(\mathbf{y}), \\ \mathbf{G}_{\text{new}}(\mathbf{x}) &= r^{-\beta}\left[(1-\alpha)G_{\text{r}}\left(r^{\beta}\mathbf{y}\right)\mathbf{y} + \mathbf{G}_{\text{s}}\left(r^{\beta}\mathbf{y}\right)\right], \end{aligned} \tag{43}$$

with $\beta = \alpha/(1-\alpha)$. Here $G_{\text{r}}(\mathbf{x}) = \mathbf{y}\cdot\mathbf{G}(\mathbf{x})$ and $\mathbf{G}_{\text{s}}(\mathbf{x}) = \mathbf{G}(\mathbf{x}) - G_{\text{r}}(\mathbf{x})\mathbf{y}$ are the radial and spherical parts of $\mathbf{G}(\mathbf{x})$. The function $\mathbf{G}_{\text{new}}$ in (43) is continuous in the case of $\alpha < 0$. It remains to notice that the system (42) corresponds to the case $\alpha = 0$, for which the statement of the theorem was already proved; note that this proof required only the continuity of $\mathbf{G}$. □

4.2. Selection by renormalization

Although convergent subsequences are guaranteed by theorem 2, the limits are generally not unique after the blowup time t_{b}. Different sequences $\nu_n \to 0$ may lead to different limits, as we show with explicit examples below. Despite this non-uniqueness, we will see in this section that the regularization procedure drastically decreases the number of 'choices' that the system can make after the blowup. Most importantly, we will show a counterintuitive fact that this set of possible choices is controlled primarily by the properties of the original singular system, rather than by a specific form of the regularization.

In this paper we only address the post-blowup dynamics in the simplest case, when the blowup is asymptotically self-similar (see theorem 1 and example 1), i.e., it corresponds to the fixed-point attractor $\mathbf{y}_*$ of the renormalized system (12) with $F_{\text{r}}(\mathbf{y}_*) < 0$. Thus, the angular part of our initial condition is assumed to be in the basin of attraction $\mathcal{B}(\{\mathbf{y}_*\})$.

We start by focusing on a specific initial condition $\mathbf{x}_0 = r_0\mathbf{y}_*$ for some $r_0 > 0$. As we showed in the previous section, this initial condition leads to solution (27), which blows up at finite time t_{b} given in (26). The same solution is valid for the regularized system (33) in the interval $t_0 \leqslant t \leqslant t^{\nu}_{\text{ent}}$, where

$$t^{\nu}_{\text{ent}} := t_{\text{b}} - \frac{\nu^{1-\alpha}}{F_{\text{r}}(\mathbf{y}_*)(\alpha-1)} \tag{44}$$

is the time when solution (27) hits the sphere $\{\mathbf{x} : r = \nu\}$ and starts to be affected by the regularization. First, we show that the limit $\nu \to 0$ reduces to the scaling limit for a single function $\mathbf{X}(\tau)$ defined by

$$\mathbf{x}^{\nu}(t) = \nu\mathbf{X}(\tau^{\nu}(t)), \quad \tau^{\nu}(t) := \nu^{-(1-\alpha)}(t - t_{\text{b}}). \tag{45}$$

By matching the representation (45) with (27), we obtain

$$\mathbf{X}(\tau) = \left[F_{\text{r}}(\mathbf{y}_*)(1-\alpha)\tau\right]^{\frac{1}{1-\alpha}}\mathbf{y}_*, \quad \tau_0 \leqslant \tau \leqslant \tau_{\text{ent}}, \tag{46}$$

where τ_0 and τ_{ent} correspond to t_0 and t^ν_{ent} via

$$\tau_0 := -\nu^{-(1-\alpha)}(t_\text{b} - t_0), \quad \tau_{\text{ent}} := -\nu^{-(1-\alpha)}(t_\text{b} - t^\nu_{\text{ent}}) = -\frac{1}{F_\text{r}(\mathbf{y}_*)(\alpha - 1)}. \tag{47}$$

Past the time $\tau > \tau_{\text{ent}}$, we find the behavior of $\mathbf{X}(\tau)$ by substituting (45) into (33) to obtain:

$$\frac{\text{d}\mathbf{X}}{\text{d}\tau} = \begin{cases} R^\alpha \mathbf{F}(\mathbf{Y}), & R > 1 \\ \mathbf{G}(\mathbf{X}), & R \leqslant 1 \end{cases}, \quad \mathbf{X}(\tau_{\text{ent}}) = \mathbf{y}_*, \tag{48}$$

where $R = |\mathbf{X}|$ and $\mathbf{Y} = \mathbf{X}/R$. Thus both the equation (48) and initial conditions determining $\mathbf{X}(\tau)$ are completely independent of ν, and $\mathbf{x}^\nu$ may be genuinely obtained by the scaling (45). Equation (48) has a unique global solution $\mathbf{X}(\tau)$. Due to the scaling property (45), the inviscid limit $\nu \to 0$ depends on the behavior of the solution $\mathbf{X}(\tau)$ at large τ, see figure 2.

Clearly, for times $t < t_\text{b}$ before the singularity, we can use continuous dependence of the solution $\mathbf{x}^\nu(t)$ on the parameter ν. Hence, the limit $\nu \to 0$ exists, it is unique and equal to the solution $\mathbf{x}(t)$ of the original system (4). By continuity, we extend this statement to the singular point, i.e., for the full time interval $t \in [t_0, t_\text{b}]$. The following definition and theorem distinguish two opposite scenarios in the inviscid limit for $t > t_\text{b}$, namely, when the solution is trapped at or immediately leaves the singular point.

Definition 2. Given the solution $\mathbf{X}(\tau)$ of (48), we say that the regularization is *expelling*, if there exists a finite time $\tau_{\text{esc}} > \tau_{\text{ent}}$ such that $R(\tau_{\text{esc}}) = 1$ and the solution is outside the regularization region, $R(\tau) > 1$, for all $\tau > \tau_{\text{esc}}$. We say that the regularization is *trapping*, if the solution stays (or returns to) the regularization region, $R(\tau) \leqslant 1$, for arbitrarily large τ and, additionally, remains bounded, $R(\tau) \leqslant R_\text{b} < \infty$ for all $\tau > \tau_{\text{ent}}$.

Theorem 3. *Let $\mathbf{x}^\nu(t)$ be a solution to the regularized problem (33) for the initial condition $\mathbf{x}_0 = r_0\mathbf{y}_*$ satisfying $\mathbf{F}_\text{s}(\mathbf{y}_*) = 0$ and $F_\text{r}(\mathbf{y}_*) < 0$.*

(*a*) *For the trapping regularization, the inviscid limit is trivial: $\boldsymbol{x}(t) = \lim_{\nu\to 0} \boldsymbol{x}^\nu(t) = 0$ for all $t > t_\text{b}$.*

(*b*) *For the expelling regularization, let $\nu_n \to 0$ be a subsequence providing (by theorem 2) the uniformly convergent solutions $\mathbf{x}^{\nu_n}(t) \to \mathbf{x}(t)$ in a given time interval $[t_\text{b}, t_1]$. Let $\mathbf{y}^{esc}_* \in \mathcal{B}(\mathcal{A}')$ for a defocusing attractor $\mathcal{A}'$ (see definition 1) and assume that the solution $\tilde{\mathbf{Y}}(S)$ of system (52) introduced below satisfies the condition*

$$\lim \sup_{S\to\infty} \int_0^S \exp\left[-(1-\alpha)\int_{S'}^S F_\text{r}(\tilde{\mathbf{Y}}(S''))\text{d}S''\right]\text{d}S' < \infty. \tag{49}$$

Then $r(t) > 0$ and $\mathbf{y}(t) \in \mathcal{A}'$ for all $t > t_\text{b}$.

Proof.

From equation (45), we express $r^\nu = \nu R$ for $t = t_\text{b} + \nu^{1-\alpha}\tau^\nu > t_\text{b} + \nu^{1-\alpha}\tau_{\text{ent}}$. In case of trapping regularization, the renormalized solution is bounded, $R(\tau) \leqslant R_\text{b}$. With these properties, in the limit $\nu \to 0$ we obtain $r = 0$ for $t > t_\text{b}$.

In the case of expelling regularization, equation (48) reduces to $\dot{\mathbf{X}} = R^\alpha \mathbf{F}(\mathbf{Y})$ for $\tau \geqslant \tau_{\text{esc}}$. Thus, we can follow the same steps as in (7), (10) and define

$$Z(\tau) = \ln\ R(\tau), \quad S(\tau) = \int_{\tau_{\text{esc}}}^{\tau} e^{-(1-\alpha)Z(\tau')}\text{d}\tau'. \tag{50}$$

We denote by $\tau = \tilde{\tau}(S)$ the inverse of the monotonously increasing function $S = S(\tau)$ and define $\tilde{\mathbf{Y}}(S) := \mathbf{Y}(\tilde{\tau}(S))$ and $\tilde{Z}(S) := Z(\tilde{\tau}(S))$. Similarly to (11), we have

$$\tilde{\tau}(S) = \tau_{\text{esc}} + \int_0^S e^{(1-\alpha)\tilde{Z}(S')}\text{d}S'. \tag{51}$$

In analogy to (12) and (13), the renormalized equations for $\tilde{\mathbf{Y}}(S)$ and $\tilde{Z}(S)$ are written as

$$\text{d}\tilde{\mathbf{Y}}/\text{d}S = \mathbf{F}_\text{s}(\tilde{\mathbf{Y}}), \quad \tilde{\mathbf{Y}}(0) = \mathbf{y}_*^{\text{esc}}, \tag{52}$$

$$\text{d}\tilde{Z}/\text{d}S = F_\text{r}(\tilde{\mathbf{Y}}), \quad \tilde{Z}(0) = 0, \tag{53}$$

where the initial conditions follow from the assumption $\mathbf{X}(\tau_{\text{esc}}) = \mathbf{y}_*^{\text{esc}}$ with $Z(\tau_{\text{esc}}) = \ln|\mathbf{X}(\tau_{\text{esc}})| = 0$ and $S(\tau_{\text{esc}}) = 0$. The solution of (52) and (53) exists globally for $S \geqslant 0$. The solution $\mathbf{X}(\tau)$ is recovered from (50) and (51) via $\mathbf{X}(\tau) = \tilde{\mathbf{X}}(S(\tau))$ with $\tilde{\mathbf{X}}(S) = e^{\tilde{Z}(S)}\tilde{\mathbf{Y}}(S)$.

Since system (52) has the same form (apart from initial conditions) as (12) in the previous section, we can use the same terminology of focusing and defocusing attractors given by definition 1. Recall that the attractors describe the dynamics of spherical components $\tilde{\mathbf{Y}}(S)$, while the property of being focusing or defocusing characterizes the radial variable $\tilde{R}(S) = e^{\tilde{Z}(S)}$. In the focusing case, orbits of the ideal system starting in the basin of attraction converge exponentially to the origin $\tilde{R}(S) \to 0$ as $S \to \infty$, and in the defocusing case they diverge exponentially to infinity.

Consider a subsequence $\lim_{n\to\infty} \nu_n \to 0$ given by theorem 2. It provides the uniformly convergent limit $\mathbf{x}(t) = \lim_{n\to\infty} \mathbf{x}^{\nu_n}(t)$, which solves equations (4) and (5). Let us fix some time $t > t_\text{b}$. Since $\alpha < 1$, the sequence of corresponding values of $\tau_n = \tau^{\nu_n}(t)$ given by (45) diverges: $\lim_{n\to\infty} \tau_n = \infty$. The corresponding values of the renormalized time $S_n = S(\tau_n)$ can be obtained from (50), or implicitly by inverting (51). Here the value of $\tilde{Z}(S) = \ln\ \tilde{R}(S) \geqslant 0$ is bounded from below because $\tilde{R}(S) \geqslant 1$ for all $S > 0$ by the assumptions of the expelling regularization. The upper bound is obtained from equation (53) as $\tilde{Z}(S) \leqslant SM$, where $M := \max_{\mathbf{Y}\in S^{d-1}} F_\text{r}(\mathbf{Y})$. We must have $M > 0$ for the expelling regularization. These estimates applied to the relation (51) yield

$$\tau_{\text{esc}} + S \leqslant \tilde{\tau}(S) \leqslant \tau_{\text{esc}} + \frac{\exp\left[(1-\alpha)SM\right] - 1}{(1-\alpha)M} \quad \text{for } S \geqslant 0. \tag{54}$$

Substituting $S = S(\tau)$ into the second inequality of (54), which yields $\tilde{\tau}(S(\tau)) = \tau$, after simple manipulation we have

$$S(\tau) \geqslant \frac{\ln\left[(1-\alpha)M(\tau - \tau_{\text{esc}}) + 1\right]}{(1-\alpha)M} \quad \text{for } \tau \geqslant \tau_{\text{esc}}. \tag{55}$$

As we already mentioned, $\lim_{n\to\infty} \tau_n = \infty$. The inequality (55) proves that $\lim_{n\to\infty} S_n = \infty$.

From (45) and (51), using the fact that $\tilde{\tau}(S) = \tau^\nu(t)$ when $S = S(\tau^\nu(t))$, we express

$$\nu^{1-\alpha} = \left(t - t_\text{b} - \nu^{1-\alpha}\tau_{\text{esc}}\right)\left(\int_0^{S(\tau^\nu(t))} e^{(1-\alpha)\tilde{Z}(S')}\text{d}S'\right)^{-1}. \tag{56}$$

Using the relations $r = \nu R = \nu e^Z$ with $Z = \tilde{Z}(S(\tau^\nu(t)))$ and substituting ν from (56), we obtain

$$r(t) = \left(t - t_\text{b} - \nu^{1-\alpha}\tau_{\text{esc}}\right)^{\frac{1}{1-\alpha}}\left(\int_0^{S(\tau^\nu(t))} e^{-(1-\alpha)(\tilde{Z}(S(\tau^\nu(t))) - \tilde{Z}(S'))}\text{d}S'\right)^{-\frac{1}{1-\alpha}}. \tag{57}$$

In the inviscid limit $\nu_n \to 0$, the first factor in (57) tends to $(t - t_b)^{\frac{1}{1-\alpha}} > 0$. Integrating equation (53) as

$$\tilde{Z}(S(\tau^{\nu}(t))) - \tilde{Z}(S') = \int_{S'}^{S(\tau^{\nu}(t))} F_r(\tilde{\mathbf{Y}}(S''))\mathrm{d}S'' \tag{58}$$

and using (49), we conclude that expression (57) provides $r(t) > 0$ for $t > t_b$ in the limit $\nu_n \to 0$.

For the angular variables, we write

$$\mathbf{y}(t) = \lim_{n\to\infty} \mathbf{y}^{\nu_n}(t) = \lim_{n\to\infty} \mathbf{Y}(\tau_n) = \lim_{n\to\infty} \tilde{\mathbf{Y}}(S_n). \tag{59}$$

Since $\lim_{n\to\infty} S_n = \infty$ and the initial condition $\tilde{\mathbf{Y}}(0) = \mathbf{y}_*^{\mathrm{esc}} \in \mathcal{B}(\mathcal{A}')$ is assumed to be in the basin of attraction, then $\lim_{n\to\infty} \tilde{\mathbf{Y}}(S_n) \in \mathcal{A}'$ and we conclude that $\mathbf{y}(t)$ belongs to the attractor $\mathcal{A}'$. □

Theorem 3 constitutes severe restriction on the limiting solutions: the spherical component of the solution must belong to the attractor $\mathcal{A}'$ given by the ideal system, independently on a particular form of regularization function $\mathbf{G}$. Below we provide a proposition with an example of concrete condition on the function F_r that guarantees that property (49) holds.

Proposition 2. *Inequality (49) in theorem 3 holds if there are constants $c_1 > 0$ and $c_0 \in \mathbb{R}$ such that the inequality*

$$\int_{S'}^{S} F_r(\tilde{\mathbf{Y}}(S''))\mathrm{d}S'' > c_1(S - S') + c_0 \tag{60}$$

is satisfied for any $S > S' \geqslant 0$. In particular, this is the case when $F_r(\mathbf{Y}) > 0$ for all $\mathbf{Y} \in \mathcal{A}'$.

Proof. Since $\alpha < 1$, we can use (60) in the estimate

$$\int_0^S \exp\left[-(1-\alpha)\int_{S'}^{S} F_r(\tilde{\mathbf{Y}}(S''))\mathrm{d}S''\right] dS' < e^{-(1-\alpha)c_0}\int_0^S e^{-c_1(1-\alpha)(S-S')}\mathrm{d}S' < \frac{e^{-(1-\alpha)c_0}}{(1-\alpha)c_1}. \tag{61}$$

This implies (49). In the case, when $F_r(\mathbf{Y}) > 0$ for all $\mathbf{Y} \in \mathcal{A}'$ on the attractor, we can choose $c_1 = \frac{1}{2}\min_{\mathbf{Y}\in\mathcal{A}'} F_r(\mathbf{Y}) > 0$. Then, for any solution $\tilde{\mathbf{Y}}(S)$ attracted to $\mathcal{A}'$, there exists an $S_1 < \infty$ such that $F_r(\tilde{\mathbf{Y}}(S)) > c_1$ for all $S \geqslant S_1$. Thus, inequality (60) is satisfied for $S > S' \geqslant S_1$ with $c_0 = 0$. One can verify that this inequality can be extended to the intervals with $S > S' \geqslant 0$ and

$$c_0 = -c_1 S_1 - \int_0^{S_1} |F_r(\tilde{\mathbf{Y}}(S''))|\mathrm{d}S''. \tag{62}$$

□

4.3. Extension to generic initial conditions

Now let us return to the generic case, when the initial condition in (4) has the angular part $\mathbf{y}_0$ belonging to the basin of attraction of the fixed point, $\mathcal{B}(\{\mathbf{y}_*\})$, instead of being exactly $\mathbf{y}_*$. By theorem 1, the corresponding solution blows up in finite time t_b with the angular part tending to the fixed-point attractor, $\mathbf{y}(t) \to \mathbf{y}_*$, as the time increases to the moment of blowup, $t \to t_b$. For the ν-regularized problem this implies that, for a sufficiently small ν, there is time $t_{\mathrm{ent}}^{\nu} \in [t_0, t_b)$, when the solution enters the regularization region. Furthermore, the value $\mathbf{y}_{\mathrm{ent}}^{\nu} = \mathbf{y}(t_{\mathrm{ent}}^{\nu})$ can be made arbitrarily close to $\mathbf{y}_*$. Therefore, the inviscid limit $\nu \to 0$ corresponds to the problem (48), where $\mathbf{y}_*$ in the initial condition is substituted by an arbitrarily close state $\mathbf{y}^\nu_{\text{ent}}$ depending on ν.

For the rigorous study of the inviscid limit, let us enhance the notions of expelling and trapping regularizations. We call the regularization *locally expelling* if all solutions $\mathbf{X}(t)$ of (48) with the initial conditions $\mathbf{X}(\tau_{\text{ent}}) = \mathbf{y}_{\text{ent}}$ in some neighborhood of $\mathbf{y}_*$ are expelled in the sense of definition 2; additionally, we require that the point $\mathbf{y}_{\text{esc}} = \mathbf{X}(\tau_{\text{esc}})$ depends smoothly on $\mathbf{y}_{\text{ent}}$. The latter condition is provided naturally by the smooth dependence on initial conditions, if the vector $\mathbf{F}(\mathbf{y}^{\text{esc}}_*)$ is transversal to the unit sphere, where $\mathbf{y}^{\text{esc}}_*$ corresponds to the solution starting exactly at $\mathbf{y}_*$. Similarly, we call the regularization *locally trapping*, if all solutions $\mathbf{X}(t)$ of (48) with the initial conditions in some neighborhood of $\mathbf{y}_*$ are trapped in the sense of definition 2 with the same bound $R(\tau) \leqslant R_{\text{b}} < \infty$ for $\tau > \tau_{\text{ent}}$. Now we can formulate the straightforward extension of theorem 3 as

Corollary 1. *Let* $\mathbf{x}^\nu(t)$ *be a solution to the regularized problem* (33) *for the initial condition* $\mathbf{x}_0 = r_0\mathbf{y}_0$ *with* $\mathbf{y}_0 \in \mathcal{B}(\{\mathbf{y}_*\})$*, where* $\mathbf{F}_{\text{s}}(\mathbf{y}_*) = 0$ *and* $F_{\text{r}}(\mathbf{y}_*) < 0$.

(*a*) *For the locally trapping regularization, the inviscid limit is trivial:* $\text{x}(t) = \lim_{\nu\to 0} \text{x}^\nu(t) = 0$ *for all* $t > t_{\text{b}}$.

(*b*) *For the locally expelling regularization, let* $\nu_n \to 0$ *be a subsequence providing* (*by theorem* 2) *the uniformly convergent solutions* $\mathbf{x}^{\nu_n}(t) \to \mathbf{x}(t)$ *in a given time interval* $[t_{\text{b}}, t_1]$. *Let* $\mathbf{y}^{esc}_*$ *belong to the basin of attraction* $\mathcal{B}(\mathcal{A}')$ *for a defocusing attractor* $\mathcal{A}'$ *of system* (52). *Additionally, we assume that any solution* $\tilde{\mathbf{Y}}(S)$ *starting in some neighborhood of* $\mathbf{y}^{esc}_*$ *satisfies condition* (49). *Then* $r(t) > 0$ *and* $\mathbf{y}(t) \in \mathcal{A}'$ *for all* $t > t_{\text{b}}$.

Corollary 1 extends the conclusion of theorem 3 to an open subset of initial conditions $\mathbf{x}_0$: the spherical component of the limiting solutions after blowup must belong to the attractor $\mathcal{A}'$ given by the ideal singular system, independently on a particular form of the regularization function $\mathbf{G}$. This restriction can be very strong, providing a constructive selection rule, as we describe in the next sections for the examples of fixed-point and periodic attractors $\mathcal{A}'$.

5. Fixed-point attractor and the unique inviscid limit

Let us consider the simplest case when $\mathcal{A}' = \{\mathbf{y}'_*\}$ is a fixed-point attractor, which is the case when $\mathbf{F}_{\text{s}}(\mathbf{y}'_*) = 0$. With the additional condition $F_{\text{r}}(\mathbf{y}'_*) > 0$, we guarantee that the attractor is defocusing and, by proposition 2, the inequality (49) holds. The part (b) of theorem 3 and corollary 1 state that the limiting solution for $t > t_{\text{b}}$ must satisfy the original equation (4) with $\mathbf{y}(t) = \mathbf{y}'_*$ and $r(t) > 0$. Such a solution is unique and can be derived similarly to (27) in the form

$$\mathbf{x}(t) = \left[(1-\alpha)F_{\text{r}}(\mathbf{y}'_*)(t-t_{\text{b}})\right]^{\frac{1}{1-\alpha}}\mathbf{y}'_*, \quad t > t_{\text{b}}. \tag{63}$$

Combining these arguments, we have

Theorem 4. *Let* $\mathbf{x}^\nu(t)$ *be a solution to the regularized problem* (33) *for the initial condition* $\mathbf{x}_0 = r_0\mathbf{y}_0$ *with* $\mathbf{y}_0 \in \mathcal{B}(\{\mathbf{y}_*\})$*, where* $\mathbf{F}_{\text{s}}(\mathbf{y}_*) = 0$ *and* $F_{\text{r}}(\mathbf{y}_*) < 0$. *Assume that the regularization is locally expelling and* $\mathbf{y}^{esc}_*$ *is in the basin of attraction for* $\mathcal{A}' = \{\mathbf{y}'_*\}$ *with* $\mathbf{F}_{\text{s}}(\mathbf{y}'_*) = 0$ *and* $F_{\text{r}}(\mathbf{y}'_*) > 0$. *Then the inviscid limit* $\mathbf{x}(t) = \lim_{\nu\to 0}\mathbf{x}^\nu(t)$ *exists and is given by* (63).

We see that the inviscid limit in the case of a fixed-point attractor is unique for all times and it is fully determined by the properties of the ideal system: a specific form of the regularization function $\mathbf{G}$ has no effect on the limiting solution as far as the generic conditions of theorem

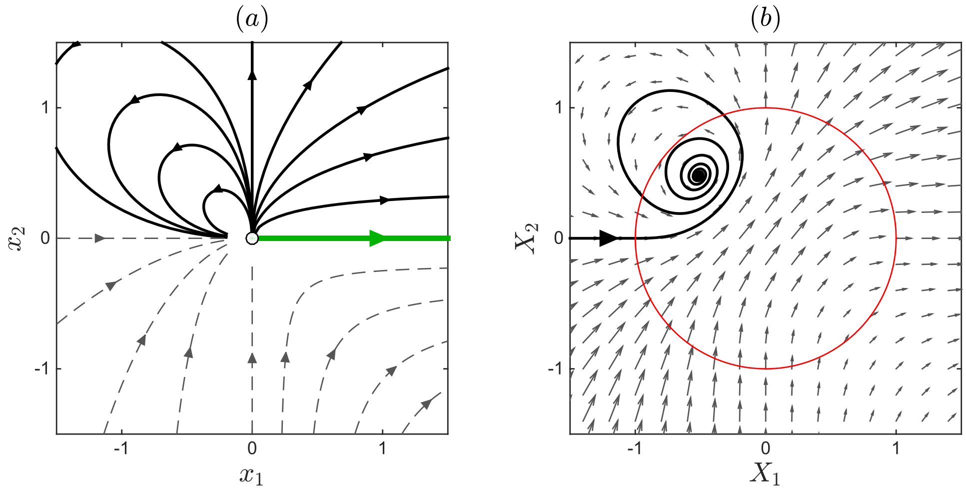

Figure 3. (a) Solid black curves show an infinite number of trajectories starting at the origin for the singular system. The bold green line indicates the unique solution chosen by a generic expelling regularization, see also figure 4(c). (b) Solution $\mathbf{X}(\tau)$ of the system with trapping regularization confined to the red circle $r \leqslant 1$.

4 are satisfied. We will illustrate this rather counterintuitive property with the following two examples.

Example 2. Let us investigate continuation past blowup in the system of example 1, assuming that $\alpha = 1/3$. As shown in figure 3(a), there are an infinite number of solutions which start at the singularity. Also, for any time $t_2 > t_b$ there exist solutions which remain at the origin, $\mathbf{x}(t) = \mathbf{0}$ for $t \in [t_b, t_2)$ and for $t > t_2$ escape the origin along any nontrivial path. We will now see how the regularization provides a specific choice of the solution for $t > t_b$.

For the regularized system (33), we define

$$\mathbf{G}(\mathbf{x}) = \xi(r)\mathbf{f}(\mathbf{x}) + (1 - \xi(r))\mathbf{G}_0, \quad r = |\mathbf{x}| \leqslant 1, \tag{64}$$

where $\mathbf{G}_0$ is a constant vector and $\xi(r) = 3r^2 - 2r^3$ smoothly interpolates between $\xi(0) = 0$ and $\xi(1) = 1$. Let us first choose $\mathbf{G} = (1, 1.3)$. The corresponding solution $\mathbf{X}(\tau)$ of the regularized system (48) with $\mathbf{y}_* = (-1, 0)$ is obtained numerically and presented in figure 3(b). Clearly, the proposed regularization is trapping. By corollary 1, for any initial condition in the left half-plane $x_1 < 0$, the solution blows up in finite time t_b and the inviscid limit provides the trivial solution, $\mathbf{x}(t) = \mathbf{0}$, for $t \geqslant t_b$.

As the second choice, we take $\mathbf{G} = (1, -2)$. The corresponding solution $\mathbf{X}(\tau)$ of the regularized system (48) is shown in figure 4(a). This regularization is expelling: the solution enters the regularization region $r \leqslant 1$ through the point $\mathbf{y}_*$ and exits forever at $\mathbf{y}_*^{\mathrm{esc}}$. The state $\mathbf{y}_*^{\mathrm{esc}}$ belongs to the basin of attraction of the fixed point $\mathbf{y}'_* = (1, 0)$ (i.e., $\varphi = 0$) of the renormalized system (52) with $F_{\mathrm{r}}(\mathbf{y}'_*) = 1 > 0$, see figure 1(b). By theorem 4, the inviscid limit defines the unique solution

$$\mathbf{x}(t) = \begin{pmatrix} [(1-\alpha)(t-t_b)]^{\frac{1}{1-\alpha}} \\ 0 \end{pmatrix}, \quad t \geqslant t_b. \tag{65}$$

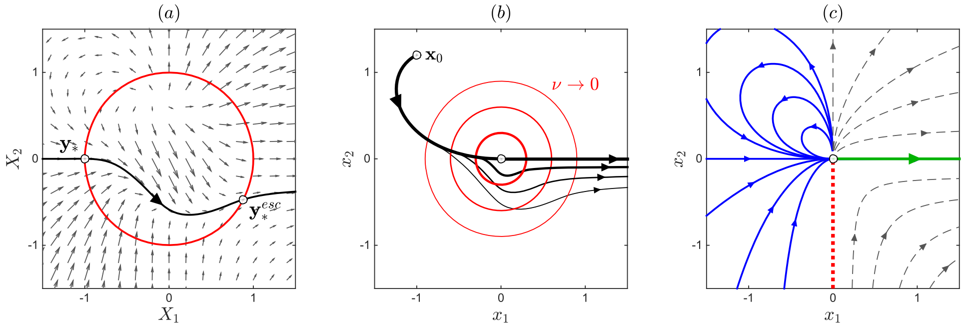

Figure 4. Expelling regularization: (a) solution $\mathbf{X}(\tau)$ of the rescaled regularized problem; the red circle indicates the regularization region $R \leqslant 1$. (b) Inviscid limit $\nu \to 0$ of the regularized solutions $\mathbf{x}^\nu(t)$ from the same initial condition. The black curves depict solutions for different values of ν. Together with the corresponding circular regularization regions $r \leqslant \nu$, they are distinguished by the line width. (c) Solutions of the singular system obtained in the inviscid limit. Solid blue curves correspond to solutions that blow up in finite time, continued identically past the singularity (green line). Dashed lines correspond to solutions that do not blow up, when initial conditions are taken on the right half-plane. The red dotted line corresponds to initial conditions that may lead to a non-generic behavior after blowup.

Exactly the same solution after blowup is obtained for any initial condition in the half-plane $x_1 < 0$, because these solutions blow up in finite time t_b with the same asymptotic behavior. This result is confirmed in figure 4(b) showing the sequence of regularized solutions $\mathbf{x}^\nu(t)$ with $\nu \to 0$.

We conclude that the regularization provides a unique global-in-time solution to the problem (4) and (5) for generic initial condition, see figure 4(c). In particular, all solutions with $x_1(t_0) < 0$ blow up in finite time and continue past the singularity in exactly the same way (65). The solutions with $x_1(t_0) > 0$, as well as $x_1(t_0) = 0$ and $x_2(t_0) > 0$ do not blowup. Finally, there is a set of initial conditions, $x_1(t_0) = 0$ and $x_2(t_0) \leqslant 0$, which requires a separate analysis. However, this set has zero measure, i.e., the corresponding initial conditions are not generic.

It is crucial that, for generic initial conditions, the inviscid limit $\mathbf{x}(t) = \lim_{\nu\to 0}\mathbf{x}^\nu(t)$ is defined primarily by the properties of the original singular system, namely, by attractors of the renormalized equations. Thus, the solution does not depend on fine details of the regularization. This means that the solution $\mathbf{x}(t)$ remains exactly the same under any (sufficiently small) deformation of the regularization function $\mathbf{G}$. However, very different regularizations (e.g., trapping vs expelling) may lead to different results. Further choices may appear in case of multiple attractors, as we demonstrate in the next simple example.

Example 3. Consider the one-dimension system (1) from the introduction, which is often used as a prototypical example of non-uniqueness. This system has no blowup. However, the equation possesses non-unique solutions starting exactly at the origin. One such solution is $x(t) \equiv 0$ for all $t \geqslant t_0$, and there are two extremal solutions (3), which leave the origin immediately. Furthermore, there are a continuum infinity of solutions, which stay at the origin until an arbitrary time $t_1 \geqslant t_0$ and peal off as $x(t) = \pm\left(\frac{2}{3}(t - t_1)\right)^{3/2}$ for $t > t_1$.

The renormalized system for (1) is trivial, since $y = x/|x| = \pm 1$ can take only two discrete values for any $x \neq 0$. Both these values can be seen as fixed-point attractors in the terminology of theorem 4, with the corresponding limiting solutions given by (3). Let us modify the problem

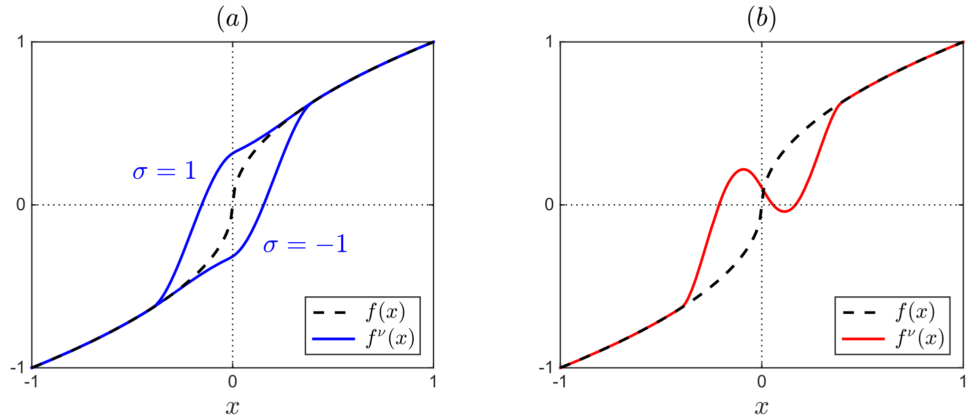

Figure 5. The function $f(x) = x^{1/3}$ (dashed black) and its ν-regularizations: (a) that expel solutions to the right ($\sigma = 1$) or to the left ($\sigma = -1$) in blue and (b) that traps the solutions in red. The regularization region is $|x| \leqslant \nu$ with $\nu = 0.4$.

by replacing (1) with a ν-regularized dynamics (33) as discussed in section 4.1. Similarly to (64), we consider the ν-regularization with

$$G(x) = \xi(r)f(x) + [1 - \xi(r)]\frac{\sigma + x}{2}, \tag{66}$$

where $r = |x|$ and the sign $\sigma = \pm 1$ defines two different regularizations. The resulting regularized functions $f^\nu(x)$ are shown in figure 5(a) by the blue lines. This regularization is expelling for the solution with $x(t_0) = 0$, and the inviscid limit $\nu \to 0$ yields the extreme solution (3) with the same sign σ. Another example is given by

$$G(x) = \xi(r)f(x) + [1 - \xi(r)]\frac{1 - 8x}{6}, \tag{67}$$

with the function $f^\nu(x)$ shown in figure 5(b) by the red line. In this case, the solution starting at the origin is attracted to a fixed-point located slightly to the right from the origin (trapping regularization). Thus, the inviscid-limit solution is $x(t) \equiv 0$.

We see that only three (out of the infinite number of) solutions can be selected by a generic ν-regularization. We remark that the two solutions appear simultaneously for the system with additive-noise regularization:

$$\mathrm{d}x = \mathrm{sgn}(x)|x|^\alpha \mathrm{d}t + \sqrt{2\kappa}\mathrm{d}W_t, \quad x(0) = 0, \tag{68}$$

where W_t is the standard Wiener process and $\alpha \in (0, 1)$. In this case, the zero-noise limit ($\kappa \to 0$) selects a non-trivial probability measure (spontaneously stochastic solution), which is distributed symmetrically between the two extremal solutions (3); see [2, 24, 32].

6. Limit cycle attractor and the non-unique inviscid limit

In this section we consider the case when the post-blowup dynamics in theorem 3(b) is governed by a limit cycle attractor $\mathcal{A}'$. The limit cycle is represented (up to a phase shift in S) by a periodic solution $\tilde{\mathbf{Y}}_{\mathrm{p}}(S)$ of system $\mathrm{d}\tilde{\mathbf{Y}}/\mathrm{d}S = \mathbf{F}_{\mathrm{s}}(\tilde{\mathbf{Y}})$:

$$\tilde{\mathbf{Y}}_{\mathrm{p}}(S) = \tilde{\mathbf{Y}}_{\mathrm{p}}(S+T), \tag{69}$$

with period T. We define the mean value of the radial function $F_{\mathrm{r}}(\mathbf{y})$ on the limit cycle as

$$\langle F_{\mathrm{r}}\rangle := \frac{I(0,T)}{T}, \qquad I(S_1,S_2) = \int_{S_1}^{S_2} F_{\mathrm{r}}(\tilde{\mathbf{Y}}_{\mathrm{p}}(S))\mathrm{d}S. \tag{70}$$

For further use, the integral function is extended for $S_1 > S_2$ as $I(S_1,S_2) = -I(S_2,S_1)$. Note that the function $I(S_1,S_2) - \langle F_{\mathrm{r}}\rangle(S_2 - S_1)$ is T-periodic with respect to both S_1 and S_2. Hence,

$$\langle F_{\mathrm{r}}\rangle(S_2 - S_1) + C_m < I(S_1,S_2) < \langle F_{\mathrm{r}}\rangle(S_2 - S_1) + C_M \tag{71}$$

for some constants C_m and C_M.

First, let us describe a family of solutions of the original problem (4) and (5), which start at the singularity and are induced by the limit cycle (69).

Proposition 3. *Consider a periodic solution (69) with a positive mean value, $\langle F_{\mathrm{r}}\rangle > 0$. Then there is a family of solutions of system (4) and (5) starting at the singularity $\mathbf{x} = 0$ at $t = t_{\mathrm{b}}$ and having the form*

$$\mathbf{x}(t) = (t - t_{\mathrm{b}})^{\frac{1}{1-\alpha}}\mathbf{X}_{\mathrm{p}}\left(\ln\left[(t - t_{\mathrm{b}})^{\frac{1}{1-\alpha}}\right] + \zeta\right), \tag{72}$$

where $\zeta \in \mathbb{R}$ is a constant parameter and the function $\mathbf{X}_{\mathrm{p}}$ has period $T\langle F_{\mathrm{r}}\rangle$ with respect to its argument; this function is defined as $\mathbf{X}_p := \mathbf{x}_p \circ \psi^{-1}$, where

$$\mathbf{x}_{\mathrm{p}}(s) = e^{-\varphi(s,s)}\tilde{\mathbf{Y}}_{\mathrm{p}}(s), \tag{73}$$

$$\varphi(s_1,s_2) = \frac{1}{1-\alpha}\ \ln\left[\int_{-\infty}^{0} e^{-(1-\alpha)I(s_1+s',s_2)}\mathrm{d}s'\right] \tag{74}$$

and $\psi^{-1} : \mathbb{R} \mapsto \mathbb{R}$ is the well defined inverse map of the function $\psi(s) := \varphi(s,0)$.

Proof. Following relations (11)–(13), solution $\mathbf{x}(t)$ of (4) at time $t = \tilde{t}(s)$ for $s \geqslant 0$ can be written as

$$\mathbf{x}(\tilde{t}(s)) = e^{\tilde{z}(s)}\tilde{\mathbf{y}}(s) \tag{75}$$

where the functions $\tilde{z}(s)$, $\tilde{\mathbf{y}}(s)$ and $\tilde{t}(s)$ satisfy the equations

$$\frac{\mathrm{d}\tilde{\mathbf{y}}}{\mathrm{d}s} = \mathbf{F}_{\mathrm{s}}(\tilde{\mathbf{y}}), \qquad \frac{\mathrm{d}\tilde{z}}{\mathrm{d}s} = F_{\mathrm{r}}(\tilde{\mathbf{y}}), \qquad \frac{\mathrm{d}\tilde{t}}{\mathrm{d}s} = e^{(1-\alpha)\tilde{z}(s)}, \tag{76}$$

and s is the auxiliary variable. By the assumption, the first equation possesses the periodic solution

$$\tilde{\mathbf{y}}(s) = \tilde{\mathbf{Y}}_{\mathrm{p}}(s). \tag{77}$$

The second equation in (76) is integrated as

$$\tilde{z}(s) = -I(s,0) - \zeta, \tag{78}$$

where I is given in (70) and ζ is an arbitrary integration constant.

Notice that

$$\lim_{s\to-\infty} \tilde{z}(s) = -\infty, \quad \lim_{s\to+\infty} \tilde{z}(s) = +\infty, \tag{79}$$

because of (71) with $\langle F_{\mathrm{r}}\rangle > 0$. Hence, $|\tilde{\mathbf{x}}(s)| = e^{\tilde{z}(s)} \to 0$ in the limit $s \to -\infty$, i.e., the solution tends to the singularity. Let us choose the solution of the last equation in (76) as

$$\tilde{t}(s) = t_{\mathrm{b}} + \int_{-\infty}^{s} e^{(1-\alpha)\tilde{z}(s')}\mathrm{d}s', \tag{80}$$

which has the property $\tilde{t}(s) \to t_{\mathrm{b}}$ as $s \to -\infty$. Using (78), this yields

$$\tilde{t}(s) = t_{\mathrm{b}} + \int_{-\infty}^{s} e^{-(1-\alpha)\left(I(s',0)+\zeta\right)}\mathrm{d}s', \tag{81}$$

where the integral converges because of (71) for $\langle F_{\mathrm{r}}\rangle > 0$. Expression (81) can be rewritten after changing the integration variable $s' \mapsto s' + s$ and using (74) as

$$\tilde{t}(s) = t_{\mathrm{b}} + e^{(1-\alpha)(\varphi(s,0)-\zeta)}. \tag{82}$$

Recalling the notation $\psi(s) := \varphi(s,0)$, we write (82) in the form

$$\psi(s) = \ln\left[(\tilde{t}(s) - t_{\mathrm{b}})^{\frac{1}{(1-\alpha)}}\right] + \zeta. \tag{83}$$

Substituting the expression $I(s+s',0) = I(s+s',s) + I(s,0)$ into the formula (74) for $\varphi(s,0)$ and then expressing $I(s,0)$ from (78), one can deduce that

$$\psi(s) = \varphi(s,0) = \varphi(s,s) - I(s,0) = \varphi(s,s) + \tilde{z}(s) + \zeta. \tag{84}$$

In view of (83) and (84), we have

$$\tilde{z}(s) = \ln\left[(\tilde{t}(s) - t_{\mathrm{b}})^{\frac{1}{1-\alpha}}\right] - \varphi(s,s). \tag{85}$$

Using (77), (85) and the definitions (73) and (75), we express

$$\mathbf{x}(\tilde{t}(s)) = (\tilde{t}(s) - t_{\mathrm{b}})^{\frac{1}{1-\alpha}}\mathbf{x}_{\mathrm{p}}(s), \quad s \in \mathbb{R}. \tag{86}$$

The function $t = \tilde{t}(s)$ is monotonically increasing with a positive derivative; see the last equation in (76). Due to the properties (79), this function maps $\tilde{t} : \mathbb{R} \mapsto (t_{\mathrm{b}}, \infty)$. Thus, the inverse function $s = s(t)$, which maps $(t_{\mathrm{b}}, \infty) \mapsto \mathbb{R}$, is well defined. This allows one to rewrite (86) in the form

$$\mathbf{x}(t) = (t - t_{\mathrm{b}})^{\frac{1}{1-\alpha}}\mathbf{x}_{\mathrm{p}}(s(t)), \quad t \in (t_{\mathrm{b}}, \infty). \tag{87}$$

The same properties of $\tilde{t}(s)$ imply that the function $\psi(s)$ in (83) is monotonically increasing with a positive derivative, and it maps $\mathbb{R} \mapsto \mathbb{R}$. Hence, the inverse function exists: $\psi^{-1} : \mathbb{R} \mapsto \mathbb{R}$. Using ψ^{-1} in (83) evaluated at $s = s(t)$, we have

$$s(t) = \psi^{-1}\left(\ln\left[(t - t_{\mathrm{b}})^{\frac{1}{1-\alpha}}\right] + \zeta\right). \tag{88}$$

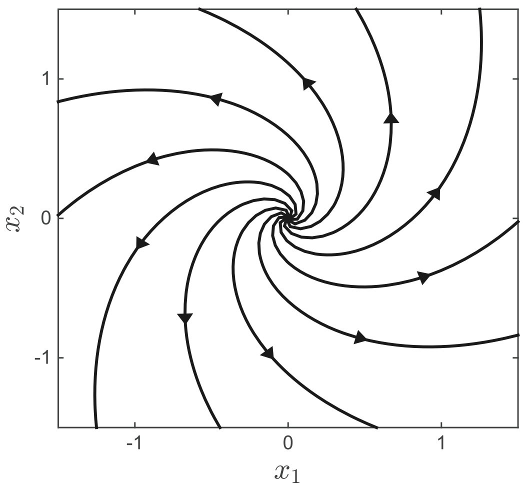

Figure 6. Vector field and non-unique trajectories emanating from the origin for equation (92).

Composing (87) and (88), we have

$$\mathbf{x}(t) = (t - t_b)^{\frac{1}{1-\alpha}} (\mathbf{x}_p \circ \psi^{-1}) \left(\ln \left[(t - t_b)^{\frac{1}{1-\alpha}} \right] + \zeta \right) \tag{89}$$

which yields (72) as claimed.

Because of the T-periodicity of $\tilde{\mathbf{Y}}_p(s)$ in the expressions (70), one has

$$I(s + T + s', s + T) = I(s + s', s), \qquad I(s + T + s', 0) = I(s + s', 0) - \langle F_r \rangle T. \tag{90}$$

Using these formulas in (74), one obtains for $\varphi(s, s)$ and $\psi(s) := \varphi(s, 0)$:

$$\varphi(s, s) = \varphi(s + T, s + T), \quad \psi(s) = \psi(s + T) - \langle F_r \rangle T. \tag{91}$$

The former equality implies that $\mathbf{x}_p(s)$ in (73) is T-periodic, while the latter yields that the composition $\mathbf{X}_p = \mathbf{x}_p \circ \psi^{-1}$ has period $T\langle F_r \rangle$. □

Example 4. To illustrate proposition 3 with a simple example, let us consider the system

$$\dot{\mathbf{x}} = r^{\alpha-1}(x_1 - x_2,\ x_1 + x_2), \tag{92}$$

with $\mathbf{x} = (x_1, x_2)$ and $\alpha < 1$. The corresponding vector field is shown in figure 6, demonstrating that all solutions emanate from the singularity at the origin. We will find these solutions explicitly assuming that they start from the singularity $\mathbf{x}(t_b) = \mathbf{0}$ at time $t = t_b$.

The right-hand side of system (92) can be written as $r^\alpha \mathbf{F}(\mathbf{y})$ with $\mathbf{y} = (y_1, y_2) = \mathbf{x}/r$ and $\mathbf{F}(\mathbf{y}) = (y_1 - y_2, y_1 + y_2)$. As in example 1, it is convenient to work with the angle variable $\varphi \in S^1$ on the circle $\mathbf{y} = (\cos\varphi,\ \sin\varphi)$. Then the radial and circular components of $\mathbf{F}$ are given by

$$F_r(\varphi) = F_s(\varphi) = 1. \tag{93}$$

For the renormalized system (30), the first equation $\mathrm{d}\tilde{\varphi}/\mathrm{d}s = F_s(\tilde{\varphi})$ has the particular solution $\tilde{\varphi}(s) = s$. In the original variables, this yields the 2π-periodic solution

$$\tilde{\mathbf{Y}}_p(s) = (\cos\, s, \sin\, s). \tag{94}$$

Since $F_r \equiv 1$, we have $I(s_1, s_2) = s_2 - s_1$ from equation (70). A simple calculation using the definition (74) yields

$$\begin{aligned} &\varphi(s,s) = -\frac{\ln(1-\alpha)}{1-\alpha}, \quad \psi(s) := \varphi(s,0) = s - \frac{\ln(1-\alpha)}{1-\alpha}. \\ &\psi^{-1}(\xi) = \xi + \frac{\ln(1-\alpha)}{1-\alpha}. \end{aligned} \tag{95}$$

Then, formulas (72) and (73) of proposition 3 yield the explicit solutions

$$\mathbf{x}(t) = [(1-\alpha)(t-t_b)]^{\frac{1}{1-\alpha}}(\cos\,\xi, \sin\,\xi), \quad \xi = \ln\left[(t-t_b)^{\frac{1}{1-\alpha}}\right] + \zeta_1, \tag{96}$$

where $\zeta_1 = \zeta + (1-\alpha)^{-1}\ln(1-\alpha)$ is an arbitrary constant parameter.

We obtained a family of solutions with the same initial condition at the singularity, which depend 2π-periodically on the constant phase parameter ζ_1. When $\alpha \in (0,1)$, one can also construct solutions that sit at the origin for an arbitrary period of time prior to being shed off following any of the paths (96). This describes all (non-unique) solutions that start at the singularity at finite time. For system (92), all such solutions are related to the limit cycle in the renormalized equation. This is not the case in general, as we will see in the next example: for systems of higher dimension, $d > 2$, solutions of proposition 3 form a zero-measure subset of all solutions originating from the singular point.

Now let us consider the ν-regularized problem. The results of theorem 3 establish that any solution after blowup obtained by a sequence of expelling regularizations must have angular part that lives on an attractor of the renormalized system. We now prove that, for limit cycle attractors, solutions of proposition 3 are the only possibility arising from 'inviscid limit' of expelling regularizations.

We assume that all characteristic exponents of the linearized problem near the limit cycle $\tilde{\mathbf{Y}}_p(S)$ have negative real parts, except for the single vanishing exponent responsible to time-translations, $\tilde{\mathbf{Y}}_p(S + \delta S)$. In this case the corresponding attractor $\mathcal{A}' = \{\mathbf{Y} = \tilde{\mathbf{Y}}_p(S) : S \in [0,\, T)\}$ is exponentially stable. More specifically, every solution $\tilde{\mathbf{Y}}(S)$ with the initial conditions from the basin of attraction, $\tilde{\mathbf{Y}}(0) = \mathbf{y}_*^{\mathrm{esc}} \in \mathcal{B}(\mathcal{A}')$, approaches the limit cycle exponentially fast

$$\tilde{\mathbf{Y}}(S) = \tilde{\mathbf{Y}}_p(S + S_1) + \boldsymbol{\varepsilon}_Y(S) \tag{97}$$

with

$$|\boldsymbol{\varepsilon}_Y(S)| < C_Y e^{-\lambda S} \quad \text{for } S \geqslant 0 \tag{98}$$

for some constant phase $0 \leqslant S_1 < T$ and $\lambda > 0$, $C_Y > 0$; see, e.g. [34, p 254]. Since $\tilde{\mathbf{Y}}(S)$ tends to the limit cycle, we can define the average value

$$\langle F_r \rangle = \lim_{S\to\infty} \frac{1}{S}\int_0^S F_r(\tilde{\mathbf{Y}}(S'))\mathrm{d}S' = \frac{1}{T}\int_0^T F_r(\tilde{\mathbf{Y}}_p(S))\mathrm{d}S, \tag{99}$$

which is the same as the average (70) on the periodic attractor $\mathcal{A}'$. Recall that $\langle F_r \rangle \geqslant 0$ is a necessary condition for the expelling regularization.

The next theorem characterizes all solutions that can be obtained in the inviscid limit after the blowup for a specific type of initial conditions. The result is a one-parameter family of solutions, depending on the viscous subsequence. First, we consider initial conditions of the self-similar blowup, which is governed by a fixed-point attractor $\mathbf{F}_s(\mathbf{y}_*) = 0$ with the property $F_r(\mathbf{y}_*) < 0$, see section 3. Later, we will extend the results to more general initial conditions.

Theorem 5. *Let $\mathbf{x}^\nu(t)$ be a solution to the regularized problem (33) for the initial condition $\mathbf{x}_0 = r_0\mathbf{y}_*$, where $\mathbf{F}_s(\mathbf{y}_*) = 0$ and $F_r(\mathbf{y}_*) < 0$. Assume that the regularization is expelling and $\mathbf{y}_*^{esc}$ is in the basin of attraction $\mathcal{B}(\mathcal{A}')$ for an exponentially stable limit cycle $\mathcal{A}'$ with the average $\langle F_r\rangle > 0$. Then the inviscid limit $\mathbf{x}(t) = \lim_{n\to\infty} \mathbf{x}^{\nu_n}(t)$ exists for the sequence (geometric progression)*

$$\nu_n = e^{-T\langle F_r\rangle n+\chi} \tag{100}$$

with an arbitrary fixed χ. After the blowup, for $t > t_b$, the limiting solution coincides with the one given by proposition 3 with $\zeta = c - \chi$ for some regularization-dependent constant c.

Proof. As described in section 4.2, for the times $t \in [t_0, t_b]$, the inviscid limit is given by the blowup solution (26) and (27) of the original system (4) and (5). For the times after the blowup, $t > t_b$, the limit can be studied using the relation (45), where $\mathbf{X}(\tau)$ is the solution of the ν-independent regularized system (48). After leaving the regularization region, $\tau \geqslant \tau_{esc}$, this system is equivalent to equations (51)–(53) providing separately $\tilde{\tau}(S)$ and $\tilde{\mathbf{X}}(S) = e^{\tilde{Z}(S)}\tilde{\mathbf{Y}}(S)$ as functions of the auxiliary variable S. Thus, we start by studying the solutions $\tilde{\mathbf{Y}}(S)$, $\tilde{Z}(S)$ and $\tilde{\tau}(S)$, where behavior of $\tilde{\mathbf{Y}}(S)$ is already described by relation (97).

Since the function $F_r(\mathbf{Y}) : S^{d-1} \mapsto \mathbb{R}$ is smooth and defined on the sphere, there is a positive constant c_{var} bounding the variation of this function as

$$|F_r(\mathbf{Y}) - F_r(\mathbf{Y}')| < c_{var}|\mathbf{Y} - \mathbf{Y}'| \tag{101}$$

for all $\mathbf{Y}, \mathbf{Y}' \in S^{d-1}$. Using this property with the relations (97) and (98), we have

$$F_r(\tilde{\mathbf{Y}}(S)) = F_r(\tilde{\mathbf{Y}}_p(S + S_1)) + \varepsilon_r(S), \tag{102}$$

where the exponentially decaying correction term is bounded as

$$|\varepsilon_r(S)| < C_r e^{-\lambda S} \quad \text{for } S \geqslant 0 \tag{103}$$

and the positive coefficient $C_r = c_{var}C_Y$.

The solutions $\tilde{Z}(S)$ of equation (53) can be written using (102) as

$$\tilde{Z}(S) = \int_0^S F_r(\tilde{\mathbf{Y}}(S'))\mathrm{d}S' = \int_0^S F_r(\tilde{\mathbf{Y}}_p(S' + S_1))\mathrm{d}S' + \int_0^S \varepsilon_r(S')\mathrm{d}S'. \tag{104}$$

Using the integral notation from (70) and introducing the quantities

$$c = I(0, S_1) - \int_0^\infty \varepsilon_r(S')\mathrm{d}S', \qquad \varepsilon_Z(S) = -\int_S^\infty \varepsilon_r(S')\mathrm{d}S', \tag{105}$$

we write (104) after changing the integration variable $S' \mapsto S' + S_1$ as

$$\tilde{Z}(S) = I(0, S + S_1) - c + \varepsilon_Z(S) = -I(S + S_1, 0) - c + \varepsilon_Z(S). \tag{106}$$

Notice that the integrals in (105) converge because of the bound (103). This bound guarantees also that

$$|\varepsilon_Z(S)| < C_Z e^{-\lambda S} \quad \text{for } S \geqslant 0 \tag{107}$$

and the positive constant $C_Z = C_r/\lambda$.

The function $\tilde{\tau}(S)$ is given by equation (51), which we write using (106) in the form

$$\begin{aligned}\tilde{\tau}(S) &= \tau_{\text{esc}} + \int_0^S e^{(1-\alpha)\left[-I(S'+S_1,0)-c+\varepsilon_Z(S')\right]}\mathrm{d}S' \\ &= \tau_{\text{esc}} + \int_{-S}^0 e^{(1-\alpha)\left[-I(S'+S+S_1,0)-c+\varepsilon_Z(S+S')\right]}\mathrm{d}S',\end{aligned} \tag{108}$$

where we changed the integration variable $S' \mapsto S + S'$ in the last expression. This can be recast as

$$\tilde{\tau}(S) = \tau_{\text{esc}} + e^{-(1-\alpha)c}\left[1 + \varepsilon_1(S) + \varepsilon_2(S) + \varepsilon_3(S)\right]\int_{-\infty}^0 e^{-(1-\alpha)I(S'+S+S_1,0)}\mathrm{d}S', \tag{109}$$

where we introduced

$$\varepsilon_1(S) = -\left(\int_{-\infty}^0 e^{-(1-\alpha)I(S'+S+S_1,0)}\mathrm{d}S'\right)^{-1}\int_{-\infty}^{-S} e^{-(1-\alpha)I(S'+S+S_1,0)}\mathrm{d}S', \tag{110}$$

$$\varepsilon_2(S) = \left(\int_{-\infty}^0 e^{-(1-\alpha)I(S'+S+S_1,0)}\mathrm{d}S'\right)^{-1}\int_{-S}^{-S/2} e^{-(1-\alpha)I(S'+S+S_1,0)}\left(e^{\varepsilon_Z(S+S')} - 1\right)\mathrm{d}S', \tag{111}$$

$$\varepsilon_3(S) = \left(\int_{-\infty}^0 e^{-(1-\alpha)I(S'+S+S_1,0)}\mathrm{d}S'\right)^{-1}\int_{-S/2}^{0} e^{-(1-\alpha)I(S'+S+S_1,0)}\left(e^{\varepsilon_Z(S+S')} - 1\right)\mathrm{d}S'. \tag{112}$$

Using the function φ defined in (74), we reduce (109) to the form

$$\tilde{\tau}(S) = \tau_{\text{esc}} + e^{(1-\alpha)\left[\varphi(S+S_1,0)-c\right]}\left[1 + \varepsilon_\tau(S)\right], \tag{113}$$

where $\varepsilon_\tau(S) = \varepsilon_1(S) + \varepsilon_2(S) + \varepsilon_3(S)$.

We now show that

$$\lim_{S\to\infty} \varepsilon_1(S) = 0, \quad \lim_{S\to\infty} \varepsilon_2(S) = 0, \quad \lim_{S\to\infty} \varepsilon_3(S) = 0, \tag{114}$$

which implies

$$\lim_{S\to\infty} \varepsilon_\tau(S) = 0. \tag{115}$$

From the bounds (71), for arbitrary $S_a < S_b$, it follows that

$$A_m e^{\beta(S+S_1)}\left(e^{\beta S_b} - e^{\beta S_a}\right) < \int_{S_a}^{S_b} e^{-(1-\alpha)I(S'+S+S_1,0)}\mathrm{d}S' < A_M e^{\beta(S+S_1)}\left(e^{\beta S_b} - e^{\beta S_a}\right). \tag{116}$$

with $\beta = (1-\alpha)\langle F_{\rm r}\rangle > 0$ and some positive constants A_m and A_M. Using these inequalities in the definition (110) yields

$$|\varepsilon_1(S)| < A_M e^{-\beta S}/A_m. \tag{117}$$

From this estimate, the first limit in (114) easily follows. Absolute value of the last factor in (111) has the form $\left|e^{\varepsilon_Z(S+S')}-1\right|$ with $-S \leqslant S' \leqslant -S/2$, and it can be bounded by unity for large S using (107). Then the second limit in (114) is obtained similarly using (116). For $-S/2 \leqslant S' \leqslant 0$ and sufficiently large S, one can use the same relation (107) and elementary analysis to show

$$\left|e^{\varepsilon_Z(S+S')}-1\right| < \left|e^{C_Z e^{-\lambda S/2}}-1\right| < 2C_Z e^{-\lambda S/2}. \tag{118}$$

Using (116) and (118) in the expression (112) proves the last limit in (114).

Now let us return to the regularized solution $\mathbf{x}^\nu(t)$. The functions $\tilde{\mathbf{Y}}(S)$, $\tilde{Z}(S)$ and $\tilde{\tau}(S)$ given by (97), (106) and (113) define implicitly the function $\mathbf{X}(\tau)$ with $\tau = \tilde{\tau}(S)$ and $\mathbf{X}(\tilde{\tau}(S)) = \tilde{\mathbf{X}}(S) = e^{\tilde{Z}(S)}\tilde{\mathbf{Y}}(S)$. Then, relations (45) provide the implicit representation for the function $\mathbf{x}^\nu(t)$ with $t = \tilde{t}^\nu(S)$ and $\mathbf{x}^\nu(\tilde{t}^\nu(S)) = \tilde{\mathbf{x}}^\nu(S)$ defined by

$$\tilde{\mathbf{x}}^\nu(S) = \nu\tilde{\mathbf{X}}(S) = \nu e^{\tilde{Z}(S)}\tilde{\mathbf{Y}}(S), \quad \tilde{t}^\nu(S) = t_b + \nu^{(1-\alpha)}\tilde{\tau}(S). \tag{119}$$

Recall that $\tilde{\tau}(S)$ is the unbounded strictly increasing function with the fixed initial value $\tau(0) = \tau_{\rm esc}$; see (51). Hence, given a fixed time $t > t_{\rm b}$ and sufficiently small $\nu > 0$, there exists a unique S satisfying

$$t = \tilde{t}^\nu(S) = t_{\rm b} + \nu^{(1-\alpha)}\tilde{\tau}(S). \tag{120}$$

We denote by S_n the solution corresponding to $\nu_n = e^{-T\langle F_r\rangle n+\chi}$ from (100). Since $\nu_n \to 0$ as $n\to\infty$, the solution S_n exists and is unique for large n. For this solution, we rewrite equation (120) using (113) as

$$t = t_{\rm b} + \nu_n^{(1-\alpha)}\tau_{\rm esc} + e^{(1-\alpha)\left[\varphi(S_n+S_1,0)-T\langle F_{\rm r}\rangle n - c+\chi\right]}\left(1+\varepsilon_\tau(S_n)\right). \tag{121}$$

One can see from (70) and (74) that

$$\begin{aligned} I(S_n+S_1,0)+T\langle F_{\rm r}\rangle n &= I(S_n+S_1-nT,0),\\ \varphi(S_n+S_1,0)-T\langle F_{\rm r}\rangle n &= \varphi(S_n+S_1-nT,0). \end{aligned} \tag{122}$$

Therefore, equation (121) takes the form

$$t = t_{\rm b} + \nu_n^{(1-\alpha)}\tau_{\rm esc} + e^{(1-\alpha)[\varphi(\xi_n,0)-\zeta]}\left(1+\varepsilon_\tau(S_n)\right), \tag{123}$$

where we defined

$$\xi_n = S_n + S_1 - nT, \quad \zeta = c - \chi. \tag{124}$$

Expressing $\psi(\xi_n) := \varphi(\xi_n, 0)$, we have

$$\psi(\xi_n) = \frac{1}{1-\alpha}\ \ln\left[\frac{t-t_{\rm b}-\nu_n^{(1-\alpha)}\tau_{\rm esc}}{1+\varepsilon_\tau(S_n)}\right] + \zeta. \tag{125}$$

Using the inverse map ψ^{-1} defined in proposition 3 and solving (125) for ξ_n yields

$$\xi_n = \psi^{-1}\left(\frac{1}{1-\alpha}\ \ln\left[\frac{t - t_b - \nu_n^{(1-\alpha)}\tau_{\text{esc}}}{1+\varepsilon_\tau(S_n)}\right] + \zeta\right). \tag{126}$$

Recall that, since ψ is continuously differentiable and monotonically increasing, ψ^{-1} is also continuous. Thus, it is possible to take the limit $n \to \infty$ in the right-hand side with both $\nu_n \to 0$ and $\varepsilon_\tau(S_n) \to 0$, which we denote as

$$s = \lim_{n\to\infty} \xi_n = \psi^{-1}\left(\ln\left[(t-t_b)^{\frac{1}{1-\alpha}}\right] + \zeta\right). \tag{127}$$

Finally, from the first relation of (119) with expressions (97) and (106), we have

$$\tilde{\mathbf{x}}^\nu(S) = \nu e^{-I(S+S_1,0)-c+\varepsilon_Z(S)}\left[\tilde{\mathbf{Y}}_p(S+S_1) + \boldsymbol{\varepsilon}_Y(S)\right]. \tag{128}$$

Taking this relation for $\nu = \nu_n = e^{-T\langle F_r\rangle n+\chi}$ and $S = S_n$, yields

$$\tilde{\mathbf{x}}^{\nu_n}(S_n) = e^{-I(S_n+S_1,0)-T\langle F_r\rangle n+\chi-c+\varepsilon_Z(S_n)}\left[\tilde{\mathbf{Y}}_p(S_n+S_1) + \boldsymbol{\varepsilon}_Y(S_n)\right]. \tag{129}$$

Adopting notations (124) and using relations (69), (122), we write

$$\tilde{\mathbf{x}}^{\nu_n}(S_n) = e^{-I(\xi_n,0)-\zeta+\varepsilon_Z(\xi_n-S_1+nT)}\left[\tilde{\mathbf{Y}}_p(\xi_n) + \boldsymbol{\varepsilon}_Y(\xi_n - S_1 + nT)\right]. \tag{130}$$

We can define the limit

$$\tilde{\mathbf{x}}(s) := \lim_{n\to\infty} \tilde{\mathbf{x}}^{\nu_n}(S_n) = e^{-I(s,0)-\zeta}\tilde{\mathbf{Y}}_p(s), \tag{131}$$

which was computed using (127) and (98), (107). Recalling the relation (78), we see that the relations (127) and (131) provide exactly the solution $\mathbf{x}(t)$ of proposition 3 in the implicit form. Therefore, we proved that the values of the regularized solution $\mathbf{x}^{\nu_n}(t)$ for given $t > t_b$ (these values are determined by the auxiliary variable $S = S_n$) converge to $\mathbf{x}(t)$. □

It is now straightforward to extend the result of theorem 5 from the specific initial data $\mathbf{x}_0 = r_0\mathbf{y}_*$ to an arbitrary point from the basin of attraction of $\mathbf{y}_*$. As we know from section 3, all such initial points lead to the blowup with the same asymptotic form. The next statement describes their continuation past the blowup time.

Corollary 2. *The statement of theorem 5 remains valid for arbitrary initial condition* $\mathbf{x}_0 = r_0\mathbf{y}_0$, *where* $\mathbf{y}_0$ *belongs to the basin of attraction* $\mathcal{B}(\{\mathbf{y}_*\})$ *of the fixed-point attractor.*

Proof. In theorem 5, we proved our statement for the specific initial condition with $\mathbf{y}(t_0) = \mathbf{y}_*$. Consider now the case of initial conditions with arbitrary $\mathbf{y}_0 = \mathcal{B}(\{\mathbf{y}_*\})$, i.e., for any initial condition leading to the same asymptotic form of self-similar blowup. In this case, both the time t_{ent}^ν and the point $\mathbf{y}_{\text{ent}}^\nu$, at which the solution enters the regularization region, depend on ν. Let us write the regularized solution $\mathbf{x}^\nu(t)$ in the form (45), where the function $\mathbf{X}(\tau)$ satisfies the same ν-independent equation (48) but for the ν-dependent initial condition $\mathbf{X}(\tau_{\text{ent}}^\nu) = \mathbf{y}_{\text{ent}}^\nu$. From the results of section 3 it follows that $\mathbf{y}_{\text{ent}}^\nu \to \mathbf{y}_*$ in the inviscid limit $\nu \to 0$. Also, the corresponding rescaled time τ_{ent}^ν converges to τ_{ent} given in (47).

Continuous dependence on initial conditions guarantees that, for any fixed τ, the ν-dependent solution $\mathbf{X}(\tau)$ converges to the analogous ν-independent solution considered in the proof of theorem 5 as $\nu \to 0$. Recall that the solution can be represented as $\mathbf{X}(\tau) = e^{Z(\tau)}\mathbf{Y}(\tau)$ given implicitly by the renormalized equations (51)–(53) outside the regularization region. The

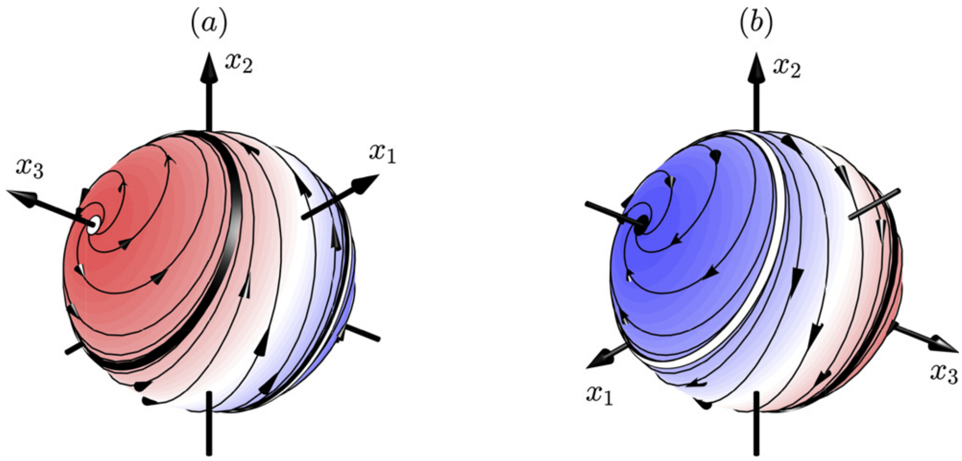

Figure 7. Vector field $\mathbf{F}_s(\mathbf{y})$ on the sphere $\mathbf{y} \in S^2$ from two different points of view. The color indicates the sign and magnitude of the axial component $F_r(\mathbf{y})$ with blue corresponding to $F_r < 0$ and red to $F_r > 0$. (a) Fixed-point repeller (white dot) and periodic attractor (black strip). (b) Fixed-point attractor (black dot) and periodic repeller (white strip).

convergence for Z and $\mathbf{Y}$ in the inviscid limit is uniform with respect to τ, which follows from the exponential stability of periodic solutions; see, e.g. [34, p 254]. One can verify that such uniform convergence is sufficient for extending the proof of theorem 5 to the more general case under consideration. □

The non-uniqueness of post-blowup dynamics described in theorem 5 and corollary 2 was observed after the blowup in the infinite dimensional shell model of turbulence in [43], where the periodic attractor has the form of a periodic wave in the renormalized system. Below we provide a much simpler illustrative example of finite dimension.

Example 5. Consider the system of three equation (4) with $\mathbf{x} = (x_1, x_2, x_3)$ and the right-hand side

$$\mathbf{f}(\mathbf{x}) = r^{\alpha}\left[\mathbf{a} + \frac{y_3}{2}\mathbf{b} + \left(y_3^2 - \frac{1}{4}\right)\mathbf{c}\right], \quad \alpha = \frac{1}{3}, \tag{132}$$

where $r = |\mathbf{x}|$, $\mathbf{y} = \mathbf{x}/r$ and

$$\mathbf{a} = (-y_2, y_1, 0), \quad \mathbf{b} = (y_1, y_2, y_3), \quad \mathbf{c} = \mathbf{a} \times \mathbf{b}. \tag{133}$$

In this case the angular and radial parts of the vector field take the form

$$\mathbf{F}_s(\mathbf{y}) = \mathbf{a} + \left(y_3^2 - \frac{1}{4}\right)\mathbf{c}, \quad F_r(\mathbf{y}) = \frac{y_3}{2}. \tag{134}$$

Phase portrait of the system $\mathrm{d}\mathbf{y}/\mathrm{d}s = F_s(\mathbf{y})$ on the sphere $\mathbf{y} \in S^2$ is shown in figure 7. There are two fixed-point solutions (attractor and repeller) and two periodic solutions (attractor and repeller), which confine the qualitative behavior of all other solutions. With the color in figure 7 we indicate the regions with $F_r < 0$ (blue) and $F_r > 0$ (red).

Let us consider the fixed-point attractor $\mathbf{y}_* = (0, 0, -1)$, show as a black dot in figure 7(b). All solutions with $\mathbf{y}_0 \in \mathcal{B}(\{\mathbf{y}_*\})$ blow up in finite time. The basin of attraction $\mathcal{B}(\{\mathbf{y}_*\})$ is

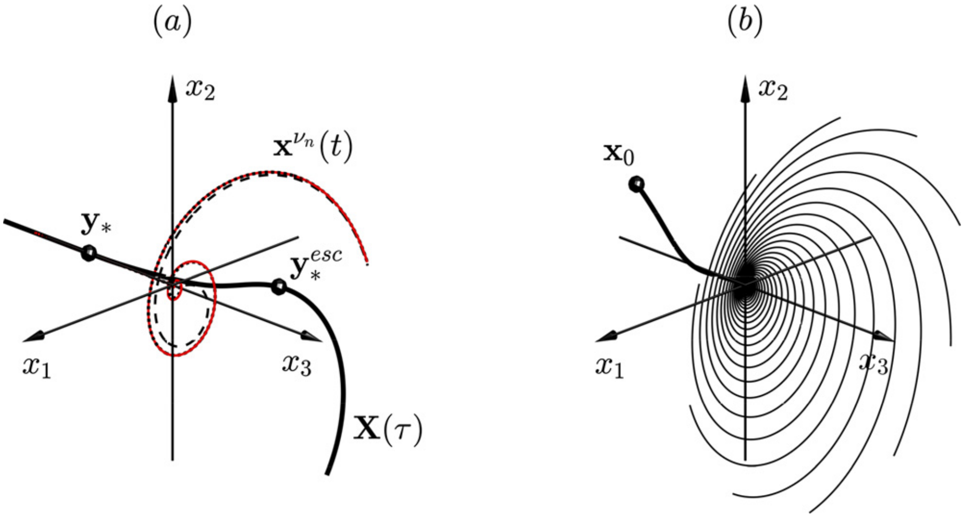

Figure 8. (a) Solid bold line shows the solution $\mathbf{X}(\tau)$ of the rescaled regularized problem (48). This solution enters the unit ball at $\mathbf{y}_*$ and exists at $\mathbf{y}_*^{\text{esc}}$. Regularized solutions $\mathbf{x} = \mathbf{x}^\nu(t)$ with the same initial condition are shown for $\nu = \nu_n$ from (100) with $\chi = 0$ and $n = 1$ (dashed line), $n = 2$ (dotted line) and $n = 3$ (red solid line). These solutions converge to the solution of the singular problem as $n \to \infty$. (b) Different inviscid-limit solutions with the same initial condition $\mathbf{x}_0$ obtained for the subsequences (100) with $\chi/T = 0, 0.1, \ldots, 0.9$.

bounded by the unstable limit cycle (white strip) in figure 7(b). To define solutions after the blowup, let us consider the regularization (64) with $\mathbf{G}_0 = (0, 0.1, 1)$. The corresponding solution $\mathbf{X}(\tau)$ of the regularized system (48) is shown by the bold line in figure 8(a). The regularization is expelling: the solution $\mathbf{X}(\tau)$ leaves the unit ball at the point $\mathbf{y}_*^{\text{esc}}$. This point belongs to the basin of attraction $\mathcal{B}(\mathcal{A}')$, where $\mathcal{A}'$ is the stable limit cycle shown with the black strip in figure 7(a).

By theorem 5 and corollary 2, solutions in the inviscid limit are given implicitly by expressions (72) obtained by solving equation (76) on the limit cycle attractor. In our case, integration of the first equation in (76) with the first expression of (134) yields

$$\tilde{\mathbf{Y}}_{\text{p}}(s) = \left(\frac{\sqrt{3}}{2}\ \cos s, \frac{\sqrt{3}}{2}\ \sin s, \frac{1}{2}\right), \tag{135}$$

where the period $T = 2\pi$. The respective average value, $\langle F_{\text{r}} \rangle = T^{-1} \int_0^T F_{\text{r}}(\tilde{\mathbf{Y}}_{\text{p}}(s))\text{d}s = 1/4$. As in example 4, from (72) we obtain solutions of the form

$$\mathbf{x}(t) = \left[\frac{(1-\alpha)(t-t_{\text{b}})}{4}\right]^{\frac{1}{1-\alpha}} \left(\frac{\sqrt{3}}{2}\ \cos s, \frac{\sqrt{3}}{2}\ \sin s, \frac{1}{2}\right), s = 4\ \ln\left[(t-t_{\text{b}})^{\frac{1}{1-\alpha}}\right] + \zeta, \tag{136}$$

where ζ is an arbitrary constant parameter.

The relation (100) of theorem 5 is verified in figure 8(a), which presents the solutions computed numerically for $n = 1, 2, 3$ and $\chi = 0$. These solutions converge to the solution (136) for a specific value of ζ. A family of solutions (136) for different $\zeta \in [0, T)$ span the conical surface as shown in figure 8(b). Only these solutions are selected in the inviscid limit of the regularization $\mathbf{G}(\mathbf{x})$ under consideration, as well as of regularizations obtained by any (not too large) deformation of $\mathbf{G}(\mathbf{x})$. We stress that solutions (136) represent only a small (zero-measure) subset of all solutions starting at the singularity $\mathbf{x} = 0$. In fact, one can show that all solutions inside the cone in figure 8(b) also originate at the singularity.

We conclude that the inviscid limit in this example is non-unique (depends on the particular subsequence $\nu_n \to 0$) for times after the blowup. However, all possible inviscid limits are restricted to a very small (zero measure) subset of all possible solutions. Once again, remarkably, this subset is selected by the ideal system, i.e., it is not sensitive to the details of the regularization procedure.

7. Discussion

In the present work, we studied a class of singular ordinary differential equations with an isolated non-Lipschitz point, when a continuation of solutions past singularity (termed as blowup) is infinitely non-unique. We showed that solutions chosen by a generic 'viscous' regularization procedure, which first smooths the vector field in ν-vicinity of a singular point and then sends $\nu \to 0$, remain highly constrained by the underlying singular equation and these constraints are (nearly) independent of regularization procedure. Such constraints are obtained from the solution-dependent renormalization, which maps the pre-blowup and post-blowup dynamics into two different infinite evolutions in the new phase-time variables. This describes the asymptotic form of blowup by the attractor of the first evolution, and the post-blowup continuation by the attractor in the second evolution. The viscous regularization acts as a bridge between these two infinitely long 'lives' of the renormalized solution.

The restrictions imposed in this way on a selected non-unique solution depend crucially on the type of attractors. For the pre-blowup dynamics, the fixed-point attractors describe an asymptotically self-similar power-law dynamics closely resembling self-similar finite-time singularities in partial differential equations [27]. As for more sophisticated attractors, we can mention analogies with the chaotic Belinsky–Khalatnikov–Lifshitz singularity in general relativity [4, 37] or chaotic blowup in turbulence models [11, 15, 42]. Attractors play even more decisive role for post-blowup dynamics, because they describe the exact dynamics rather than its asymptotic form. It is appealing to compare the case of a fixed-point attractor, when the unique solution is selected, with shock wave formation in conservation laws. In both cases the viscous regularization chooses a specific unique solution, which is not sensitive (within certain limits) to the details of this regularization. For example, the same shock in the Burgers equation results from the limit of vanishing viscosity or hyper-viscosity. A similar renormalization procedure on both sides of blowup can be formally introduced for the Burgers equation, where the attractors take the form of traveling waves propagating in log–log space-time coordinates [27, 44].

The case of a periodic attractor offers non-unique choices within a one-parameter family, resulting from different geometric subsequences of vanishing viscous parameters ν. Though we are not aware of real-world physical phenomena that have this property, such periodic non-uniqueness was readily observed in a popular infinite-dimensional model of turbulence (shell model) [43]. We remark that the same shell model demonstrates a different regime, when the attractor is chaotic [44]. As we prove in a separate companion paper [20], the proper selection of post-blowup solutions in the chaotic case requires a source of randomness, which must be introduced and removed through the regularization procedure. Then, the limit exists in a stochastic sense, selecting different post-blowup solutions with a well-defined probability. Such behavior resembles the spontaneous stochasticity observed in models of passive advection [6, 19], particle trajectories in presence of shocks [30], simple quantum systems [29], Rayleigh–Taylor turbulence [7, 45] and singular vortex sheets [53].

One may think of a large variety of interesting situations, when attractors governing both pre- and post-blowup dynamics are not fixed points. We expect that a degree of sensitivity to a regularization varies from case to case, which require separate studies. Nevertheless, according to theorem 1, the asymptotic behavior still must 'shadow' some trajectories on the attractors and, therefore, remain highly constrained independently of the regularization details.

Acknowledgments

The authors are grateful to Gregory Eyink and Enrique Ramiro Pujals for fruitful discussions. Research of TD is partially supported by NSF-DMS Grant 1703997, and AM by CNPq Grants 303047/2018-6, 406431/2018-3.

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