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On quasisymmetric plasma equilibria sustained by small force

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We construct smooth, non-symmetric plasma equilibria which possess closed, nested flux surfaces and solve the magnetohydrostatic (steady three-dimensional incompressible Euler) equations with a small force. The solutions are also 'nearly' quasisymmetric. The primary idea is, given a desired quasisymmetry direction ξ, to change the smooth structure on space so that the vector field ξ is Killing for the new metric and construct ξ–symmetric solutions of the magnetohydrostatic equations on that background by solving a generalized Grad–Shafranov equation. If ξ is close to a symmetry of Euclidean space, then these are solutions on flat space up to a small forcing.

Key words: fusion plasma

1. Introduction

Let $T \subset \mathbb{R}^3$ be a domain with smooth boundary. The three-dimensional magneto hydrostatic (MHS) equations on T read

$$J \times B = \nabla P + f, \quad \text{in } T, \tag{1.1}$$

$$\nabla \cdot B = 0, \quad \text{in } T, \tag{1.2}$$

$$B \cdot \hat{n} = 0, \quad \text{on } \partial T, \tag{1.3}$$

where $J = \nabla \times B$ is the current, f is an external force and P is the pressure. The solution B to (1.1)–(1.3) can be interpreted as either a stationary fluid velocity field which solves the time-independent Euler equation, or as a steady self-supporting magnetic field in a continuous medium with trivial flow velocity. The latter interpretation is robust across a variety of magnetohydrodynamic models (e.g. compressible, incompressible, non-ideal) and makes the system (1.1)–(1.3) central to the study of plasma confinement fusion.

In view of this, there is a long-standing scientific program to identify and construct MHS equilibria which are effective at confining ions during a nuclear fusion reaction. The most basic requirement for confinement is the existence of a 'flux function' ψ, whose level sets foliate the domain T and which satisfies $B \cdot \nabla\psi = 0$. To first approximation, ions move along the integral curves of B and so this condition ensures that particle trajectories are approximately constrained to the level sets of ψ. For this reason, it is desirable to seek

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equilibria with nested flux surfaces (isosurfaces of ψ) which foliate the plasma domain. When T is the axisymmetric torus, it is natural to look for such solutions in the form of axisymmetric magnetic fields. If (R, Φ, Z) denote the usual cylindrical coordinates on $\mathbb{R}^3$ and the centreline of the torus lies in the $Z = 0$ plane, axisymmetric solutions take the form

$$B_0 = \frac{1}{R^2}\left(C_0(\psi_0) R e_\Phi + R e_\Phi \times \nabla \psi_0\right), \tag{1.4}$$

with flux function ψ_0. In order for B_0 to satisfy (1.1) with $P_0 = P_0(\psi_0)$, taking $f = 0$ momentarily for simplicity and taking T to be the torus with inner radius $R_0 - 1$ and outer radius $R_0 + 1$, say, the flux function needs to satisfy the axisymmetric Grad–Shafranov equation (Shafranov 1966; Grad 1967)

$$\partial_R^2 \psi_0 + \partial_Z^2 \psi_0 - \frac{1}{R}\partial_R \psi_0 + R^2 P_0'(\psi_0) + C_0 C_0'(\psi_0) = 0, \quad \text{in } D_0, \tag{1.5}$$

$$\psi_0 = \text{const.} \quad \text{on } \partial D_0, \tag{1.6}$$

where D_0 denotes the cross-section of the torus (unit disk) in the $\Phi = 0$ half-plane centred at $R = R_0$. Conversely, if ψ_0 is any solution[1] to (1.5) with $e_\Phi \cdot \nabla \psi_0 = 0$ then the vector field B_0 defined in (1.4) is divergence-free and satisfies (1.1). If ψ_0 is constant on ∂T, B_0 satisfies (1.3).

Unfortunately, these tokamak equilibria come with a slew of problems from the point of view of plasma confinement fusion (Landreman 2019). For example, to achieve improved confinement it is desirable for the magnetic field to 'twist' as it wraps around the torus and this can only be accomplished in axisymmetry with a large plasma current, J. Such plasma configurations are hard to control in practice. One approach to finding equilibria with better confinement properties is to consider equilibria in geometries which have the desired twist built in. This is the basic design principle behind the stellarator (Garren & Boozer 1991). It is still desirable for these configurations to possess a form of symmetry, which is known as quasisymmetry.

DEFINITION 1.1 (Weak quasisymmetry, Rodriguez, Helander & Bhattacharjee (2020)). *Let ξ be a non-vanishing vector field tangent to ∂T. We say that ξ is a quasisymmetry and the field B is quasisymmetric with respect to ξ if*

$$\operatorname{div} \xi = 0, \quad \text{in } T, \tag{1.7}$$

$$B \times \xi = -\nabla \psi, \quad \text{in } T, \tag{1.8}$$

$$\xi \cdot \nabla |B| = 0, \quad \text{in } T, \tag{1.9}$$

for some function $\psi : T \to \mathbb{R}$.

The significance of the condition (1.8) is that it implies $B \cdot \nabla \psi = 0$ and $\xi \cdot \nabla \psi = 0$ and so quasisymmetric solutions possess flux functions which are symmetric with respect to ξ. In Rodriguez *et al.* (2020), the authors argue that (1.7)–(1.9) form sufficient conditions that ensure first-order (in gyroradius) particle confinement, hence the terminology of weak quasisymmetry. In the confinement fusion literature (Landreman 2019; Burby, Kallinikos

[1]We remark that it could be that this equation admits 'large' solutions with non-trivial dependence on Φ, see the work of Garabedian Garabedian (2006).

& MacKay 2020), one encounters the following alternative definition which is actually stronger than the above. It replaces (1.9) with

$$\xi \times J = \nabla (B \cdot \xi) \quad \text{in } T. \tag{1.10}$$

We term this set of conditions strong quasisymmetry. When $f = 0$ it is this stronger form of quasisymmetry which is equivalent to other definitions in the plasma fusion literature involving Boozer angles, see § 8 of Landreman (2019). If $\mathrm{div} B = 0$ then (1.9) requires only that a single component of (1.10) vanish, $B \cdot (\xi \times J - \nabla (B \cdot \xi)) = 0$.[2] In light of this, the additional content of strong quasisymmetry (1.7), (1.8) and (1.10) is the assumption that the other two components of $\xi \times J - \nabla (B \cdot \xi)$ vanish. It turns out that when there is no force and the equilibria are toroidal, strong quasisymmetry is equivalent to Definition 1.1.[3]

From (1.8), if $\xi \cdot B$ is constant on surfaces of constant ψ, $\xi \cdot B = C(\psi)$ (by a result in Burby *et al.* (2020), any solution of (1.1) with $f = 0$ which satisfies (1.7)–(1.9) satisfies this condition), it follows that B is of the form

$$B = \frac{1}{|\xi|^2} \left(C(\psi) \xi + \xi \times \nabla \psi \right), \tag{1.12}$$

and when $f = 0$, the requirement that (1.1) holds implies that ψ must satisfy the quasisymmetric Grad–Shafranov equation (introduced in Burby *et al.* (2020)) which reads

$$\Delta \psi - \frac{\xi \times \mathrm{curl} \xi}{|\xi|^2} \cdot \nabla \psi + \frac{\xi \cdot \mathrm{curl} \xi}{|\xi|^2} C(\psi) + CC'(\psi) + |\xi|^2 P'(\psi) = 0, \quad \text{in } T, \tag{1.13}$$

$$\psi = \text{const.} \quad \text{on } \partial T. \tag{1.14}$$

Equations (1.2), (1.7) and (1.9) can be thought of as constraints relating ψ to the deformation tensor of ξ, the symmetric two-tensor $\mathcal{L}_\xi \delta$ defined by

$$(\mathcal{L}_\xi \delta)(X, Y) = \nabla_X \xi \cdot Y + \nabla_Y \xi \cdot X, \tag{1.15}$$

where ∇ denotes covariant differentiation with respect to the Euclidean metric. Recall that ξ generates an isometry of Euclidean space if and only if $\mathcal{L}_\xi \delta = 0$, in which case ξ is called a Killing field for the metric δ. Assuming that $\xi \cdot \nabla \psi = 0$ and $\mathrm{div} \xi = 0$, from (1.12) we find

$$\mathrm{div} B = C(\psi)(\mathcal{L}_\xi \delta)(\xi, \xi) + (\mathcal{L}_\xi \delta)(\xi, \xi \times \nabla \psi), \tag{1.16}$$

and expanding the condition (1.9) we find

$$\frac{1}{2} \mathcal{L}_\xi |B|^2 = (\mathcal{L}_\xi \delta)(\xi, \xi) + \frac{2}{C(\psi)} (\mathcal{L}_\xi \delta)(\xi, \xi \times \nabla \psi) + \frac{1}{C(\psi)^2} (\mathcal{L}_\xi \delta)(\xi \times \nabla \psi, \xi \times \nabla \psi), \tag{1.17}$$

see lemma C.6 of appendix C. Equation (1.17) is a complicated relationship between ψ, $C(\psi)$ and ξ but notice that it holds trivially (assuming only that $\xi \cdot \nabla \psi = 0$) whenever when ξ is a Killing field. It is well known that in Euclidean space the only Killing fields

[2] To see this, using standard vector calculus identities, we write

$$\xi \times \mathrm{curl} B - \nabla (B \cdot \xi) = B \cdot \nabla \xi + \xi \cdot \nabla B + B \times \mathrm{curl} \xi. \tag{1.11}$$

Taking the inner product with B results in $B \cdot (\xi \times \mathrm{curl} B - \nabla (B \cdot \xi)) = \frac{1}{2} \xi \cdot \nabla |B|^2 + B \cdot \nabla \xi \cdot B$. The argument is completed by using the elementary identity $\mathcal{L}_\xi B = \mathrm{curl}(B \times \xi) + \mathrm{div} B \xi - \mathrm{div} \xi B = \mathrm{curl}(B \times \xi)$. This yields $B \cdot (\xi \times \mathrm{curl} B - \nabla (B \cdot \xi)) = \xi \cdot \nabla |B|^2$.

[3] M. Landreman, private communication.

are linear combinations of translations and rotations. Therefore, up to a multiplicative constant, the only such field compatible with the geometry of the axisymmetric torus is $\xi = Re_\Phi$ and as mentioned above, such solutions have problematic confinement properties. We have arrived at the following problem.

Problem. Given a toroidal domain T, construct a function $\psi : T \to \mathbb{R}$ with nested flux surfaces and a divergence-free vector field ξ which does not generate an isometry of $\mathbb{R}^3$ and is tangent to ∂T, so that (1.13), (1.9), the nonlinear constraints (1.16), (1.17) and $\xi \cdot \nabla \psi = 0$ all hold.

It is not clear that there are any smooth solutions ψ, ξ to the above problem. In fact, in 1967 (long before the above notion of quasisymmetry was introduced), Grad & Rubin (1958) and Grad (1967, 1985) conjectured that the only smooth solutions to (1.1)–(1.3) possessing a good flux function have a Euclidean symmetry,[4] and this would in particular rule out any solutions of the above type. Since Grad's work, there have been some constructions of non-symmetric equilibria an infinite cylindrical domains (Salat & Kaiser 1995; Kaiser & Salat 1997). As these are unbounded in extent, they have limited practical appeal for the perspective of confinement. No such examples of smooth solutions have been rigorously demonstrated on toroidal domains, although there has been some work on suggestive formal near-axis expansions (Bernardin, Moses & Tataronis 1986; Weitzner 2014; Jorge, Sengupta & Landreman 2019) and non-symmetric weak solution equilibria with pressure jumps have been rigorously constructed (Bruno & Laurence 1996) which may have practical implications for the confinement fusion program (Hudson *et al.* 2011, 2012).[5]

We do not address Grad's conjecture here and our goal is instead to present a robust method for constructing solutions to (1.1) with small force and which are approximately quasisymmetric with respect to a given vector field ξ (sufficiently close to the axisymmetric vector field $\xi_0 = Re_\Phi$), in the sense that (1.8) holds but that (1.9) holds up to a small error.

In addition to the non-trivial constraint (1.9), there are two serious difficulties in constructing solutions to (1.1)–(1.3) of the form (1.12) with given symmetry direction ξ. The first is that by (1.16), unlike in the axisymmetric setting, vector fields of the form (1.12) need not be divergence-free. The second difficulty is that for arbitrary ξ, it is not at all clear that the equations (1.13) and (1.14) admit any solutions with $\xi \cdot \nabla \psi = 0$, since the coefficients appearing in (1.13)–(1.14) need not be invariant under ξ. Both of these difficulties can be traced to the fact that ξ need not be a Killing field with respect to the Euclidean metric. To circumvent these issues, inspired by Lichtenfelz, Misiolek & Preston (2019) and Burby, Kallinikos & MacKay (2020), we replace the metric structure of $(\mathbb{R}^3, \delta)$ with $(\mathbb{R}^3, g)$ for a metric g for which ξ is a Killing field. The resulting magnetic field will not satisfy the usual MHS equations (1.1), but provided ξ is sufficiently close to Killing for the Euclidean metric, the error will be small. We now explain the idea.

Let us suppose that given ξ, we can find a metric g on $\mathbb{R}^3$ for which $\mathcal{L}_\xi g = 0$, that is, for which ξ generates an isometry (we give an explicit construction of such metrics for a large class of vector fields ξ after the upcoming statement of theorem 1.3). We then consider the

[4]Specifically, in Grad (1967) Grad conjectures that there no families of smooth solutions to (1.1)–(1.3), each possessing a flux function with closed level sets that foliate the domain T, other than the axisymmetric solutions. This leaves open the possibility of isolated non-axisymmetric steady states, far from symmetry.

[5]See Lortz (1970) for a construction of a non-axisymmetric toroidal equilibrium which nevertheless enjoys plane reflection symmetry (forcing all magnetic field lines to be closed).

following generalization of the ansatz (1.12), introduced in Burby *et al.* (2020):

$$B_g = \frac{1}{|\xi|_g^2}\left(C(\psi)\xi + \sqrt{|g|}\xi \times_g \nabla_g \psi\right). \tag{1.18}$$

Here, $|\xi|_g$, $\times_g$, ∇_g denote the analogues of the usual Euclidean quantities $|\xi|$, $\times\nabla$ with respect to the metric g (see appendix B). In lemma C.3 we use the fact that $\mathcal{L}_\xi g = 0$ to show that vector fields of this form are divergence free assuming only that $\xi \cdot \nabla\psi = 0$,

$$\mathrm{div} B_g = 0 \tag{1.19}$$

and also that ψ is a flux function for B_g,

$$\xi \times \nabla\psi = B_g. \tag{1.20}$$

We emphasize the somewhat surprising fact that even though the definition (1.18) involves the metric g in a non-trivial way, it is designed that way so that the identities (1.19) and (1.20) involve only Euclidean quantities. We remark that B_g will not be divergence-free with respect to the g metric.

We then seek B_g of the form (1.18) which satisfy the MHS with respect to the metric g,

$$\mathrm{curl}_g B_g \times_g B_g = \nabla_g P. \tag{1.21}$$

This ansatz leads to the generalized Grad–Shafranov equation for ψ,

$$\mathrm{div}_g\left(\sqrt{|g|}\frac{\nabla_g\psi}{|\xi|_g^2}\right) - C(\psi)\frac{\xi}{|\xi|_g^2}\cdot_g \mathrm{curl}_g\left(\frac{\xi}{|\xi|_g^2}\right) + \frac{C(\psi)C'(\psi)}{\sqrt{|g|}|\xi|_g^2} + \frac{P'(\psi)}{\sqrt{|g|}} = 0, \quad \text{in } T, \tag{1.22}$$

$$\psi = (\text{const.}), \quad \text{on } \partial T, \tag{1.23}$$

where $|\xi|_g$, $\cdot_g$, curl_g denote the magnitude, dot product and curl with respect to the metric g (see appendix B for the definitions and appendix C for the derivation of (1.22) from (1.18) and (1.21)). Note that (1.22) and (1.23) reduces to (1.13) and (1.14) when $g = \delta$, and when g is the circle-averaged metric, it agrees with the equation derived in Burby *et al.* (2020). As shown in Burby *et al.* (2020), all solutions of MHS (1.1)–(1.3) without force and non-vanishing pressure gradient must have a flux function satisfying (1.22) where g is the circle-averaged metric discussed below. In light of this, the study of the generalized Grad–Shafranov equation (1.22) is of fundamental importance in the study of solutions to MHS with a generalized symmetry.

As another consequence of the fact that $\mathcal{L}_\xi g = 0$, the coefficients in (1.22) are invariant under ξ and so (1.22), unlike (1.14), is consistent with the requirement $\xi \cdot \nabla\psi = 0$. The downside is that the equation (1.21) does not agree with (1.1) unless $g = \delta$ and so B_g will not satisfy the original MHS equations. However, if we can arrange for the metric g to be sufficiently close to the Euclidean metric δ, then B_g will satisfy the usual MHS equations $\mathrm{curl} B_g \times B_g - \nabla P = 0$ up to a small error. Our approach will be to solve the generalized Grad–Shafranov equation (1.22) by deforming an appropriate solution ψ_0 of the axisymmetric Grad–Shafranov equation (1.5), using the methods from Constantin, Drivas & Ginsberg (2020). In particular, we seek a diffeomorphism $\gamma : D_0 \to D$ and requiring that $\psi = \psi_0 \circ \gamma^{-1}$. It turns out (see § 2) that this reduces to a system of nonlinear elliptic equations for the components of γ which can be solved by a iteration.

In what follows, T_0 denotes the axisymmetric torus

$$T_0 = \{(R, \Phi, Z) \mid (R - R_0)^2 + Z^2 \le a, 0 \le \Phi \le 2\pi\}, \tag{1.24}$$

with thickness $0 < a \ll R_0$. Let $\xi_0 = Re_\Phi$ be the generator of rotations in the $Z = 0$ plane. Let D be any domain in the half-plane $\{\Phi = 0\}$ sufficiently close to D_0. Suppose that ξ is a vector field which is sufficiently close to the rotation field ξ_0 with the property that all the orbits of ξ starting from D are periodic (with possibly different period $\tau(p)$). In this case we define the toroidal domain

$$T = \{\varphi_s(p) \mid p \in D,\ s \in [0, \tau(p))\}, \tag{1.25}$$

where $\varphi_s(p)$ denotes the time-s flow of ξ starting from $p \in D$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\varphi_s(p) = \xi(\varphi_s(p)), \quad \varphi_0(p) = p \in D. \tag{1.26a,b}$$

In this setting we say that the toroidal domain T is swept out by ξ from D.

Our first result is that, given a toroidal domain T swept out by a vector field ξ as above, sufficiently close to the axisymmetric torus T_0, we can find a flux function satisfying the generalized Grad–Shafranov equation (1.22). The proof is constructive and relies on deforming a known axisymmetric steady state satisfying mild conditions (H1)–(H2) stated in § 2.

THEOREM 1.2. *Fix $k \ge 0, \alpha > 0$ and let $\xi \in C^{k+2,\alpha}(\mathbb{R}^3)$ be a divergence-free vector field, sufficiently close in $C^{k+2,\alpha}$ to the rotation vector field $\xi_0 = Re_\Phi$. Let D be a domain sufficiently close to D_0 in $C^{k+2,\alpha}$ in the sense that $D = \{(r, \theta) \mid 0 \le r \le \mathfrak{b}(\theta), \theta \in \mathbb{S}^1\}$ for a function $\mathfrak{b} : \mathbb{S}^1 \to \mathbb{R}$ sufficiently close to 1 in $C^{k+2,\alpha}(\mathbb{S}^1)$. Let $\psi_0 \in C^{k+2,\alpha}(D_0)$ be a solution of (1.5)–(1.6) with pressure $P_0 \in C^{k+1,\alpha}(\mathbb{R})$ and with $C_0 \in C^{k+1,\alpha}(\mathbb{R})$ satisfying* (H1)–(H2).

Suppose, moreover, that ξ has closed integral curves that sweep out a toroidal domain T from D. Suppose that there is a metric $g \in C^{k+2,\alpha}(\mathbb{R}^3)$ with the property $\mathcal{L}_\xi g = 0$ which is sufficiently close to the Euclidean metric. Then, for any given $C \in C^{k+1,\alpha}(\mathbb{R})$ sufficiently close to C_0, there is a flux function $\psi \in C^{k+2,\alpha}(T)$, and a pressure $P = P(\psi) \in C^{k+1,\alpha}(T)$ so that ψ satisfies the generalized Grad–Shafranov equation (1.22) and the boundary condition (1.23). Moreover, the level sets of ψ are diffeomorphic to the level sets of ψ_0.

As a consequence of the above theorem, we are able to produce magnetic fields with nested flux surfaces and a global symmetry that solve MHS up to a small force whose magnitude is controlled by the deviation of the symmetry from being Euclidean. These fields satisfy two of the three quasisymmetry conditions, the third holding approximately. The resulting magnetic field possesses flux surfaces which have the same topology as the axisymmetric base state.

THEOREM 1.3. *Suppose the hypotheses of the previous theorem hold. For any given $C \in C^{k+1,\alpha}(\mathbb{R})$ sufficiently close to C_0, there is a flux function $\psi \in C^{k+2,\alpha}(T)$, and a pressure $P = P(\psi) \in C^{k+1,\alpha}(T)$ so that the magnetic field B defined by (1.18) satisfies $B \cdot \nabla\psi = 0$, $B \times \xi = \nabla\psi$ as well as MHS (1.1)–(1.3) with a force f obeying*

$$\|f\|_{C^{k,\alpha}(T)} \le c\|\delta - g\|_{C^{k+2,\alpha}(T)}, \tag{1.27}$$

where $c := c(\|\xi\|_{C^{k+2,\alpha}(T)}, \|\psi\|_{C^{k+2,\alpha}(T)}, \|P_0\|_{C^{k+1,\alpha}(\mathbb{R})}, \|C_0\|_{C^{k+1,\alpha}(\mathbb{R})})$. Moreover, the flux surfaces of B (isosurfaces of ψ) are diffeomorphic to the isosurfaces of ψ_0, and ψ is a solution of the generalized Grad–Shafranov equation (1.22).

The point of the bound (1.27) is that if ξ is a Killing field for the Euclidean metric δ then we can take $g = \delta$ in the above and by (1.27), the B is then an exact solution of the MHS equations (1.1) with $f = 0$. In this sense, (1.27) shows that one can construct approximate solutions to MHS with symmetry direction ξ with error proportional to how far ξ is from being a symmetry of $\mathbb{R}^3$. We remark that the proof is quantitative in that all the small parameters can be explicitly defined in terms of the inputs ψ_0, C_0, P_0 and ξ. Let us also remark that one is not free to choose P from the outset and it is instead determined in the course of the proof to enforce a certain compatibility condition, see § 2.

The above theorem is perturbative, in the sense that the resulting magnetic field will be approximately axisymmetric and have a flux function close to a given ψ_0 satisfying the axisymmetric Grad–Shafranov equation (1.5). As will be discussed in the upcoming section, the result follows from a theorem in Constantin *et al.* (2020) by deforming the given solution ψ_0 of the axisymmetric Grad–Shafranov equation (1.5) into a solution of the generalized Grad–Shafranov equation (1.22). The same theorem from Constantin *et al.* (2020) in fact allows one to deform a given solution ψ_1 to (1.22) for given $\xi = \xi_1$ into a solution ψ_2 to (1.22) with nearby, but different, $\xi = \xi_2$. Given a desired ξ, if one can produce a sequence of vector fields $\xi_0, \xi_1, \ldots, \xi_{N-1}, \xi_N = \xi$ in such a way that the resulting solutions $\psi_0, \psi_1, \ldots, \psi_{N-1}$ all satisfy the conditions (H1)–(H2) this would produce a flux function satisfying (1.22) far from axisymmetry. Note, however, that the resulting force in (1.1) could be quite large.

If one is only interested in constructing approximate equilibria, this can be achieved simply by pushing forward a given axisymmetric state by a volume-preserving diffeomorphishm. The resulting flux function need not satisfy the generalized Grad–Shafranov equation (1.22). On the other hand, the construction in theorem 1.2 does ensure that the generalized Grad–Shafranov equation is exactly satisfied. In light of the fact that Burby *et al.* (2020) show that all unforced solutions to (1.1) and (1.3) must satisfy the equation (1.22), our theorem may provide a path towards obtaining non-axisymmetric solutions without force.

We now describe how to produce a base state ψ_0 and metric g which are suitable inputs for theorem 1.2. In appendix A, we provide an example of a base state ψ_0 satisfying (H1)–(H2) living on a large aspect ratio torus. This is obtained as a perturbation of an explicit profile on an 'infinite aspect ratio' torus. It should be stressed that the conditions (H1)–(H2) are not very stringent and should hold for a wide class of axisymmetric solutions that possess simple nested flux surfaces (which could, for example, be numerically obtained). Next, we describe two large classes of vector fields ξ and metrics g satisfying the hypotheses of our theorem.

Remark (Designer metrics). We provide two possible ways of constructing a 'near' Euclidean metric given a 'near' isometry ξ.

(i) (Deformed metric) Suppose that the torus T is given by $T = f(T_0)$ where f is a diffeomorphism defined in a neighbourhood of T_0 and which is sufficiently close to the identity. Then we can take $\xi = \mathrm{d}f(\xi_0)$ where df denotes the differential and let $g = f^*\delta$ denote the pullback of the Euclidean metric δ by f. Because the Lie derivative is invariant under diffeomorphisms, we have ξ is a Killing field for g since $\mathcal{L}_\xi g = f^*(L_{\xi_0}\delta) = 0$.

(ii) (Circle-averaged metric) Suppose the orbits of ξ starting from D are all 2π-periodic. In this case we say that ξ generates a circle action. Defining the circle-averaged metric g,

$$g = \frac{1}{2\pi}\int_0^{2\pi} \varphi_s^*\delta \,\mathrm{d}s, \tag{1.28}$$

it follows by a simple computation that $\mathcal{L}_\xi g = 0$. Moreover, when ξ is a Killing field for Euclidean space, $g = \delta$. This metric was introduced by Burby *et al.* (2020). As motivation for the appearance of this particular metric, consider the MHS in terms of one-forms $\mathcal{L}_B(B^\flat) = \mathrm{d}P$ (see Arnold & Khesin 1999). In this representation, it is clear that the metric appears linearly (in the definition of $\flat$). Therefore, if B and P are invariant under the flow of ξ, then one finds $\mathcal{L}_B(B^{\bar\flat}) = \mathrm{d}P$ where $\bar\flat$ denotes the operation of lowering the index with respect to the circle-averaged metric. Raising indices with g, we find that any such MHS solution on Euclidean space is also a solution of the circle-averaged equation (MHS with respect to the metric g).

We conclude with some remarks about achieving exact quasisymmetry. By construction, the magnetic field B from the previous theorem will satisfy (1.8) but will only approximately satisfy the property (1.9) of quasisymmetry. Thus, our fields confine particles to zeroth but not first order in the guiding centre approximation (Rodriguez *et al.* 2020). The error from being an exact weak quasisymmetry can be easily quantified; for a vector field B_g of the form (1.18), assuming that g is such that $\mathcal{L}_\xi g = 0$ the condition (1.9) reads

$$\xi \cdot \nabla |B_g| = \frac{C^2(\psi)}{2|\xi|_g^4 |B_g|} \left[(\mathcal{L}_\xi \delta)(\xi, \xi) + 2C^{-1}(\psi)(\mathcal{L}_\xi \delta)(\xi, \nabla_g^\perp \psi) \right. \\ \left. + C^{-2}(\psi)(\mathcal{L}_\xi \delta)(\nabla_g^\perp \psi, \nabla_g^\perp \psi) \right], \tag{1.29}$$

see lemma C.6. Since (1.29) involves the Euclidean deformation tensor alone, it is controlled by the deviation of ξ from being a Euclidean isometry and our solution will have $\xi \cdot \nabla |B_g|$ small. The error from being a strong quasisymmetry is also quantifiably small.

It is worth remarking that there are additional freedoms in our construction that could, in principle, be used to further constrain the constructed solution. Specifically, in our theorem, we treat ξ as a fixed vector field sufficiently close to ξ_0 and we made the somewhat arbitrary choice that the map γ should be volume preserving. The results in Constantin *et al.* (2020) actually allow one to construct the map γ so that $\det \nabla\gamma := \rho$ is any given function, sufficiently close to one; in fact by iterating that result, one can additionally achieve that $\det \nabla\gamma = X(\phi, \eta, \partial\phi, \partial\eta, \partial\partial_s\phi, \partial\partial_s\eta)$ for a suitable nonlinearity X sufficiently close to one when $\phi, \eta = 0$. Using this freedom, it is possible to show that, under some (possibly restrictive and undesirable) assumptions on the field ξ, the Jacobian ρ can be used to achieve exact quasisymmetry on a slice of the torus (namely on the cross-section D). Ensuring this property holds seems of little practical interest for ion confinement in a stellarator, since particles starting on the slice will immediately leave. In contrast, Garren and Boozer Garren & Boozer (1991) and Plunk & Helander (2018) show that exact quasisymmetry is possible to achieve on one flux surface while maintaining the MHS force balance, which is of greater relevance to confinement in a stellarator. It is unknown whether or not quasisymmetry can be achieved in a volume. We leave open the question of whether or not, using our approach, a carefully designed field ξ (perhaps constructed dynamically alongside the solution) can be used to ensure quasisymmetry on a flux surface or a volume.

2. Proof of theorem 1.3

We start by giving an outline of the arguments used to establish the main theorem. All details can be found in Constantin *et al.* (2020).

Let D_0, D, ψ_0, ξ, g be as in the statement of theorem 1.3. We will start by constructing a solution ψ to the generalized Grad–Shafranov equation (1.22) of the form $\psi = \psi_0 \circ$

γ^{-1} for a diffeormorphism $\gamma : D_0 \to D$ which is to be determined. With the toroidal coordinates (r, θ, φ) defined as in (A 1*a–c*), for functions η, ϕ independent of φ, write $\nabla\eta = \partial_r \eta e_r + (1/r)\partial_\theta \eta e_\theta$ and $\nabla^\perp \phi = -\partial_r \phi e_\theta + (1/r)\partial_\theta f e_r$. We will look for $\gamma(r, \theta) = (r, \theta) + \nabla\eta(r, \theta) + \nabla^\perp\phi(r, \theta)$ and the functions η, ϕ are the unknowns. For simplicity, using the assumption that Vol $D =$ Vol D_0, we will require that $\det \nabla\gamma^{-1} = 1$. After a short calculation, this condition reads

$$\Delta\eta = \mathcal{N}_\eta[\phi, \eta], \tag{2.1}$$

where $\mathcal{N}_\eta[\phi, \eta] = \mathcal{N}_\eta(\partial\phi, \partial\eta, \partial^2\phi, \partial^2\eta)$ is a quadratic nonlinearity. We will pose boundary conditions momentarily. We think of this equation as determining η at the linear level from ϕ and it remains to determine ϕ in a such a way that $\psi = \psi_0 \circ \gamma^{-1}$ is a solution to (1.22). We now describe how this is done.

The Grad–Shafranov equation (1.5) is of the form

$$L_0\psi_0 = F_0(\psi_0) + G_0(r, \theta, \psi_0), \quad \text{in } D_0, \tag{2.2}$$

with nonlinearities $F_0 = P_0'$, $G_0 = (1/R^2)C_0C_0'(\psi_0)$ and where the operator L_0 is elliptic. Similarly, we write the generalized Grad–Shafranov (1.22) in the form

$$L\psi = F(\psi) + G(r, \theta, \psi), \quad \text{in } D. \tag{2.3}$$

At this stage, the function F (which is related to the pressure of the solution in our application) is actually undetermined and will be chosen momentarily, while G can be chosen to be any function sufficiently close to G_0.

A calculation (see appendix B of Constantin *et al.* (2020)) shows that provided $\det \nabla\gamma^{-1} = 1$, we have

$$(\nabla\psi) \circ \gamma^{-1} = \nabla\psi_0 + \nabla\partial_s\phi - \nabla^\perp\partial_s\eta + \nabla^\perp\phi \cdot \nabla^2\psi_0 + \nabla\eta \cdot \nabla^2\psi_0, \tag{2.4}$$

where we have introduced the notation $\partial_s = \nabla\psi_0 \cdot \nabla^\perp$ for the 'streamline derivative'. Then ∂_s is tangent to level sets of ψ_0. After a computation, composing both sides of (2.3) with γ^{-1} and using (2.4), (2.3) takes the form

$$\mathscr{L}_{\psi_0}\partial_s\phi = \mathcal{N}_\phi[\phi, \eta] + F(\psi_0) - F_0(\psi_0) + G(r, \theta, \psi_0) - G_0(r, \theta, \psi_0), \tag{2.5}$$

where $\mathcal{L}_{\psi_0}$ is the linearization of L_0 around ψ_0, for a function $\mathcal{N}_\phi[\phi, \eta] = \mathcal{N}_\phi(\partial\phi, \partial\eta, \partial\partial_s\phi, \partial\partial_s\eta)$ (whose explicit form can be found in appendix B of Constantin *et al.* (2020)), which consists of terms which are linear in η and its derivatives, and either nonlinear or weakly linear in derivatives of ϕ, meaning it involves terms which can bounded by $\epsilon|\partial\phi|$, for example. This latter point is a consequence of the assumption that $\xi - \xi_0$ is sufficiently small. Notice that at the linear level this is an equation for $\partial_s\phi$ and not ϕ itself. In order for this equation to be solvable for ϕ at the linear level (given appropriate boundary conditions), there are two requirements. The first is that $\mathscr{L}_{\psi_0}$ should be invertible. The second is a somewhat subtle condition which is easiest to understand in the simple model case. In order to solve the problem $\Delta\partial_\theta u = f$, in the unit disk, say (with arbitrary boundary conditions), it is clearly necessary that $\int_0^{2\pi} f \, \mathrm{d}\theta = 0$. We will now impose a condition on (2.5) which is analogous to this one and which will determine the function F at the linear level. Assume that the Dirichlet problem for $\mathscr{L}_{\psi_0}$,

$$\left.\begin{aligned} \mathscr{L}_{\psi_0} u = f, &\quad \text{in } D_0, \\ u = 0, &\quad \text{on } \partial D_0, \end{aligned}\right\} \tag{2.6}$$

has a unique solution for $f \in L^2$, say. Writing $\mathscr{L}_{\psi_0}^{-1} f = u$, if we apply $\mathscr{L}_{\psi_0}^{-1}$ to both sides of (2.5) and integrating with respect to ds over the streamline $\{\psi_0 = c\}$ (considered as a subset of the two-dimensional set D_0) we find

$$0 = \oint_{\{\psi_0 = c\}} \mathscr{L}_{\psi_0}^{-1} (F(\psi_0) - F_0(\psi_0))\, \mathrm{d}s + \oint_{\{\psi_0 = c\}} \mathscr{L}_{\psi_0}^{-1} (G - G_0 + \mathcal{N})\, \mathrm{d}s. \tag{2.7}$$

This is an equation which must be solved for F. Writing $T(c)q = \oint_{\{\psi_0=c\}} \mathscr{L}_{\psi_0}^{-1} q\, \mathrm{d}s$, given R depending only on the streamline $R = R(c)$, we would like to be able to find $q = q(c)$ with $T(c)q = R$. This is a complicated problem which would be hard to address directly, however, in Constantin *et al.* (2020) we show that such q can be found, assuming that the following hypotheses hold.

Hypothesis 1 (H1) The operator $\mathscr{L}_{\psi_0}$ is positive definite.

Hypothesis 2 (H2) There exists a constant $C > 0$ such that we have

$$\mu(c) = \oint_{\{\psi_0 = c\}} \frac{\mathrm{d}\ell}{|\nabla \psi_0|} \leq C \quad \text{for } c \in \operatorname{rang}(\psi_0), \tag{2.8}$$

where ℓ is the arclength parameter.

Notice that the hypothesis (H1), in particular, ensures that the operator $\mathscr{L}_{\psi_0}^{-1}$ is well-defined. This is a condition on P_0, C_0. Hypothesis (H2) concerns the travel time $\mu(c)$ for a particle governed by the Hamiltonian system $\dot{x} = \nabla^\perp \psi_0(x)$ and moving along the streamline of $\{\psi_0 = c\}$. It is easy to see that it holds provided ψ_0 has at most one critical point in $D_0 = T_0 \cap \{\varphi = 0\}$ and that it vanishes no faster than to first order there. We remark that (H2) is trivially satisfied if $|\nabla \psi_0|$ is bounded below in the domain D_0. This could be accomplished if, for example, one worked on a 'hollowed out' toroidal domain.

We now discuss the boundary conditions. Assume that D is the interior of a Jordan curve B,

$$\partial D = \{p \in \mathbb{R}^2 \mid \mathfrak{b}(p) = 0\}. \tag{2.9}$$

We also write $\partial D_0 = \{p \in \mathbb{R}^2 \mid \mathfrak{b}_0(r, \theta) = 0\}$ where $\mathfrak{b}_0$ is chosen with $|\nabla \mathfrak{b}_0| = 1$, $\nabla \psi_0 \cdot \nabla \mathfrak{b}_0 > 0$. We write $\gamma - \mathrm{id} = \nabla \eta + \nabla^\perp \phi = \alpha e_x + \beta e_y$ where (x, y) are rectangular coordinates. Using that $\mathfrak{B}_0|_{\partial D_0} = 0$, the requirement that $\gamma : \partial D_0 \to \partial D$ can be written as

$$\begin{aligned} 0 = \mathfrak{b} \circ \gamma|_{\partial D_0} &= \mathfrak{b}_0 \circ \gamma|_{\partial D_0} + (\delta \mathfrak{b}) \circ \gamma|_{\partial D_0} \\ &= \alpha \partial_x \mathfrak{b}_0|_{\partial D_0} + \beta \partial_y \mathfrak{b}_0|_{\partial D_0} + \mathfrak{b}_1(\alpha, \beta)|_{\partial D_0}, \end{aligned} \tag{2.10}$$

where $\delta \mathfrak{b} = \mathfrak{b} - \mathfrak{b}_0$, and where the remainder $\mathfrak{b}_1$ is

$$\mathfrak{b}_1(\alpha, \beta, x, y) = \mathfrak{b}_0 \circ \gamma - \mathfrak{b}_0 - \alpha \partial_1 \mathfrak{b}_0 - \beta \partial_2 \mathfrak{b}_0 + (\delta \mathfrak{b}) \circ \gamma. \tag{2.11}$$

Returning to ϕ, η, we have

$$\alpha \partial_1 \mathfrak{b}_0 + \beta \partial_2 \mathfrak{b}_0 = \nabla \phi \cdot \nabla^\perp \mathfrak{b}_0 + \nabla \eta \cdot \nabla \mathfrak{b}_0. \tag{2.12}$$

By the choice of $\mathfrak{b}_0$, we have $\nabla \eta \cdot \nabla \mathfrak{b}_0 = \partial_n \eta$ where n is the outward-facing normal to ∂D_0. Additionally, using that ψ_0 is constant on the boundary we have $\nabla^\perp \mathfrak{b}_0 = \nabla^\perp \psi_0 / |\nabla \psi_0|$, and using (2.12) and these observations, the formula (2.10) becomes

$$\frac{1}{|\nabla\psi_0|}\partial_s\phi + \partial_n\eta = -\mathfrak{b}_1(\phi,\eta), \quad \text{on } \partial D_0. \tag{2.13}$$

This is one boundary condition for the two functions ϕ, η. Again we need to ensure that this equation is compatible with the requirement $\oint_{\{\psi_0=\psi_0|_{\partial D_0}\}} \partial_s\phi \, \mathrm{d}s = 0$. We therefore take $\partial_n\eta$ constant on the boundary and impose the following nonlinear boundary conditions:

$$\partial_n\eta = -\frac{\oint_{\partial D_0} \mathfrak{b}_1(\phi,\eta)\,\mathrm{d}\ell}{\mathrm{length}(\partial D_0)} \quad \text{on } \partial D_0, \tag{2.14}$$

$$\partial_s\phi = |\nabla\psi_0| \left(-\mathfrak{b}_1(\phi,\eta) + \frac{\oint_{\partial D_0} \mathfrak{b}_1(\phi,\eta)\,\mathrm{d}\ell}{\mathrm{length}(\partial D_0)}\right) \quad \text{on } \partial D_0. \tag{2.15}$$

We now summarize the result of the above calculation. The function $\psi = \psi_0 \circ \gamma^{-1}$ is a solution of the equation (2.3) in D with constant boundary value provided the diffeomorphism γ is of the form $\gamma = \mathrm{id} + \nabla\eta + \nabla^{\perp}\phi$ and the functions $\eta, \phi : D_0 \to \mathbb{R}$ satisfy the elliptic equations

$$\left.\begin{aligned} &\Delta\eta = \mathcal{N}_\eta[\phi,\eta] && \text{in } D_0, \\ &\mathscr{L}_{\psi_0}\partial_s\phi = \mathcal{N}_\phi[\phi,\eta] + F(\psi_0) - F_0(\psi_0) + G(\psi_0,r,\theta) - G_0(\psi_0,r,\theta), && \text{in } D_0, \end{aligned}\right\} \tag{2.16}$$

where F is determined by solving

$$\oint_{\{\psi_0=c\}} \mathscr{L}_{\psi_0}^{-1} F \,\mathrm{d}s = \oint_{\{\psi_0=c\}} \mathscr{L}_{\psi_0}^{-1}(G_0 - G - \mathcal{N}_\phi + F_0)\,\mathrm{d}s, \tag{2.17}$$

and where η, ϕ satisfy the boundary conditions (2.14) and (2.15).

This nonlinear system can be solved by the following iteration scheme. Given η^{N-1}, ϕ^{N-1}, define $F^N = F^N(c)$ by solving

$$\oint_{\{\psi_0=c\}} \mathscr{L}_{\psi_0}^{-1} F^N \,\mathrm{d}s = \oint_{\{\psi_0=c\}} \mathscr{L}_{\psi_0}^{-1}(G_0 - G - \mathcal{N}_\phi[\phi^{N-1},\eta^{N-1}] + F_0)\,\mathrm{d}s. \tag{2.18}$$

Then solve for η^N, Φ^N satisfying

$$\left.\begin{aligned} &\Delta\eta^N = \mathcal{N}_\eta[\phi^{N-1},\eta^{N-1}] && \text{in } D_0, \\ &\mathscr{L}_{\psi_0}\Phi^N = \mathcal{N}_\phi[\phi^{N-1},\eta^{N-1}] + F^N - F_0 + G - G_0 && \text{in } D_0, \end{aligned}\right\} \tag{2.19}$$

with boundary conditions

$$\left.\begin{aligned} \partial_n\eta^N &= \int_{D_0} \mathcal{N}_\eta[\phi^{N-1},\eta^{N-1}]\,\mathrm{d}x, \\ \Phi^N &= |\nabla\psi_0| \left(-\mathfrak{b}_1(\phi^{N-1},\eta^{N-1}) + \frac{\oint_{\partial D_0} \mathfrak{b}_1(\phi^{N-1},\eta^{N-1})\,\mathrm{d}\ell}{\mathrm{length}(\partial D_0)}\right) \quad \text{on } \partial D_0. \end{aligned}\right\} \tag{2.20}$$

The boundary condition for η^N has been chosen so that the Neumann problem (2.19) and (2.20) is solvable. Once Φ^N has been found, as a consequence of the choice of F^N it can be shown that $\oint_{\{\psi_0=c\}} \Phi^N \,\mathrm{d}s = 0$ for all c, and so $\Phi^N = \partial_s \phi^N$ for a function ϕ^N which is determined up to a constant, which can be fixed throughout the iteration by requiring that $\int_{D_0} \phi^N = 0$. In Constantin *et al.* (2020) we prove that this iteration converges in a suitable topology. We remark that the boundary condition (2.20) is not the same as the boundary condition in (2.14) but as a consequence of Vol $D_0 =$ Vol D, they agree after taking $N \to \infty$.

Proof of theorems 1.2 and 1.3. We first reduce the problem to solving a certain elliptic problem on the domain D. Given a (local) coordinate system (x_1, x_2) defined on a neighbourhood of D, we can extend it to a (local) coordinate system on a neighbourhood of the torus T by pulling back along the flow of ξ. Explicitly, given $p \in T$, there is a unique $p_0 \in D$ and a unique smallest $x_3 > 0$ so that $\Phi_{x_3}(p_0) = p$ where $\Phi_s(p_0)$ denotes the time-s flow of ξ starting from a point $p_0 \in D$, because the integral curves of ξ are closed. Then the map $\Psi : T \to D \times \mathbb{R}$ defined by $\Psi(p) = (p_0, x_3)$ is a (local) diffeomorphism onto its image. In these coordinates, $\xi \cdot \nabla f = (\partial/\partial x^3) f$ for any function f. We now express the given metric g in this coordinates, $g = \sum_{i,j=1}^3 g_{ij}(x_1, x_2, x_3)\,\mathrm{d}x^i\,\mathrm{d}x^j$, where $g_{ij} = g(\partial_{x^i}, \partial_{x^j})$. By the definition of the Lie derivative of the metric we have

$$(\mathcal{L}_\xi g)_{ij} = (\xi \cdot \nabla) g_{ij} - g([\xi, \partial_{x^i}], \partial_{x^j}) - g([\xi, \partial_{x^j}], \partial_{x^i}) = \frac{\partial}{\partial x^3} g_{ij} = 0, \tag{2.21}$$

since by assumption $\mathcal{L}_\xi g = 0$ and since $[\xi, \partial_{x^\ell}] = [\partial_{x^3}, \partial_{x^\ell}] = 0$ by construction. In this coordinate system, the Grad–Shafranov equation is the following three-dimensional elliptic equation:

$$L\psi := \sum_{i,j=1}^{3} a^{ij}_{\xi,g} \partial_{x^i} \partial_{x^j} \psi + \sum_{i=1}^{3} b^i_{\xi,g} \partial_{x^i} \psi + G_{\xi,g}(x_1, x_2, x_3, C, \psi) + \frac{1}{\sqrt{|g|}} P'(\psi) = 0, \quad \text{in } T, \tag{2.22}$$

for coefficients $a^{ij}_{\xi,g}, b^j_{\xi,g}$ and a function $G_{\xi,g}$, depending on x_1, x_2, x_3 which are all computed explicitly in appendix D. The crucial point is that all of these quantities are independent of x_3, because they involve algebraic functions of components of the metric.

We can therefore look for a two-dimensional solution, $\bar{\psi} = \bar{\psi}(x_1, x_2)$, of the equation

$$\sum_{i,j=1}^{2} a^{ij}_{\xi,g}(x_1, x_2) \partial_{x^i} \partial_{x^j} \bar{\psi} + \sum_{i=1}^{2} b^i_{\xi,g}(x_1, x_2) \partial_{x^i} \bar{\psi} + G_{\xi,g}(x_1, x_2, C, \bar{\psi}) + \frac{1}{\sqrt{|g|}} P'(\bar{\psi}) = 0, \tag{2.23}$$

in D with $\bar{\psi}$ constant on ∂D. Given such $\bar{\psi}$, we can recover ψ satisfying (2.22) by setting $\psi(x_1, x_2, x_3) = \bar{\psi}(x_1, x_2)$, i.e. by extending $\bar{\psi}$ to be constant along integral curves of ξ. Since the integral curves of ξ are closed it follows that ψ is as smooth as $\bar{\psi}$, and by construction we have $\mathcal{L}_\xi \psi = 0$. Also since ξ is tangent to ∂T it follows that the resulting ψ is constant there.

Supposing that we have a solution $\bar{\psi}$ as above, by lemma C.3, defining B as in (1.18) provides a magnetic field satisfying $\mathrm{div} B = 0$ and which satisfies the MHS equations with respect to g, $\mathrm{curl}_g B \times_g B_g = \nabla_g P$ exactly, by lemma C.5. As a consequence, B satisfies the usual MHS equations with forcing,

$$f = (\mathrm{curl}_g - \mathrm{curl}) B \times B + \mathrm{curl}_g B (\times_g - \times) B + (\nabla_g - \nabla) P. \tag{2.24}$$

From the formulae for curl_g, $\times_g$ in appendix B, it is clear this satisfies a bound of the form (1.27), completing the proof of theorem 1.3.

We have therefore reduced the problem to solving the generalized Grad–Shafranov equation for a function $\bar{\psi}: D \to \mathbb{R}$. In what follows we will abuse notation and just write $\psi = \bar{\psi}$. Using, for example, variational methods (proposition 11.4 of Taylor (1996)), in principle one can find a weak solution to this equation in H^1_0. Unfortunately, these solutions need not be smooth and, more importantly, the structure of the level sets cannot be specified. In particular, the flux function may possess 'magnetic islands'. We provide here an explicit construction of classical solutions which allows for control of flux surfaces, based on the approach of Constantin *et al.* (2020). As explained above, the method there is to deform a solution ψ_0 to the axisymmetric Grad–Shafranov equations into a solution to (2.23) and this has the benefit of ensuring that the level sets of the resulting ψ are tori, as well as providing a simple algorithm to compute the solution.

Let $\psi_0 : D_0 \to \mathbb{R}$ be a solution to the axisymmetric Grad–Shafranov equation (1.5) with $\psi_0|_{\partial D_0}$ constant, satisfying the mild hypotheses (H1) and (H2) (see appendix A for an example of such a flux function). We begin by writing the axisymmetric Grad–Shafranov equation on the unit disk D_0 in the same coordinate system (x_1, x_2) as above. Letting h_{ij} denote the components of the Euclidean metric restricted to D_0 in this coordinate system, on D_0 we have

$$L_0\psi_0 := \sum_{i,j=1}^{2} \tilde{a}^{ij}_{\xi_0,\delta}(x_1, x_2)\partial_{x^i}\partial_{x^j}\psi_0 + \sum_{i=1}^{2} \tilde{b}^{i}_{\xi_0,\delta}(x_1, x_2)\partial_{x^i}\psi_0 + \tilde{G}_{\xi_0,\delta}(x_1, x_2, C_0, \psi_0) + \frac{1}{\sqrt{|h|}}P_0'(\psi_0) = 0, \tag{2.25}$$

where $|h|$ denotes the determinant of the matrix h_{ij}, $\xi_0 = Re_\Phi$ is the generator of rotations in the $Z = 0$ plane, and $\tilde{a}^{ij}_{\xi_0,\delta}$, $\tilde{b}^{i}_{\xi_0,\delta}$, $\tilde{G}_{\xi_0,\delta}$ can be computed explicitly by changing variables in (1.5),

$$\left.\begin{aligned} \tilde{a}^{ij}_{\xi_0,\delta} &= \frac{\sqrt{|h|}}{|\xi_0|^2}h^{ij}, \quad \tilde{b}^{i}_{\xi_0,\delta} = \sum_{j=1,2}\frac{1}{\sqrt{|h|}}\partial_{x^i}\left(\frac{\sqrt{|h|}}{|\xi_0|^2}h^{ij}\right), \\ \tilde{G}_{\xi_0,\delta} &= \frac{C(\psi)}{|\xi_0|^2}\left(\frac{C'(\psi)}{\sqrt{|h|}} - \xi_0\cdot \mathrm{curl}\xi_0\right). \end{aligned}\right\} \tag{2.26}$$

In order to appeal to the results of Constantin *et al.* (2020) we need that the coefficients of L are close to those of L_0, that $G_{\xi_0,\delta}$ is close to $G_{\xi,g}$ and that the domain D is close to D_0. For simplicity, we take the function C in (2.23) to just be C_0, though this is not essential. From the formulae in appendix D we have

$$\sum_{i,j=1}^{2} \|a^{ij}_{\xi,g} - \tilde{a}^{ij}_{\xi_0,\delta}\|_{C^{k,\alpha}} + \sum_{i=1}^{2}\|b^{i}_{\xi,g} - \tilde{b}^{i}_{\xi_0,\delta}\|_{C^{k,\alpha}} + \|G_{\xi,g} - \tilde{G}_{\xi_0,\delta}\|_{C^{k,\alpha}} \le c\|g-\delta\|_{C^{k+1,\alpha}} + c\|\xi - \xi_0\|_{C^{k+1,\alpha}}, \tag{2.27}$$

where c is a constant depending on $k, \alpha, \sum_{i,j=1}^{3}\|g\|_{C^{k+2,\alpha}}$ and $\|C_0\|_{C^{k+3,\alpha}}$. Here, and in what follows, we are writing $C^{k+2,\alpha} = C^{k+2,\alpha}(U)$ where U is a domain containing both D and D_0. By theorem 3.1 from Constantin *et al.* (2020), there is $\epsilon > 0$ depending on k, α, ψ_0, D_0

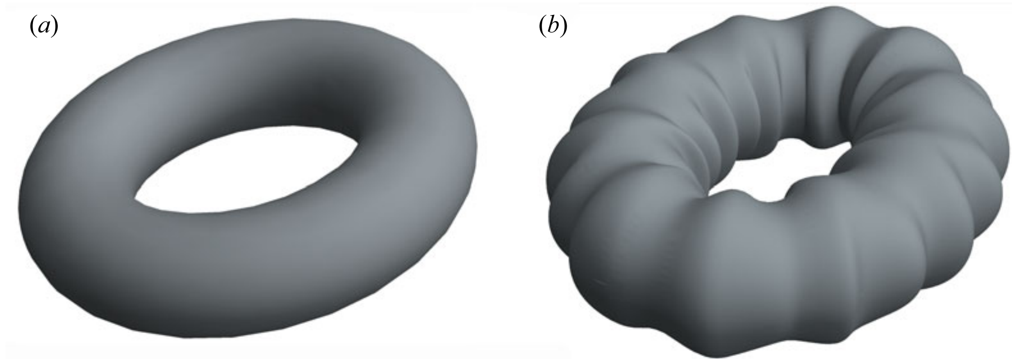

FIGURE 1. Tokamak and stellarator geometries.

so that if the following holds,

$$\sum_{i,j=1}^{2} \|a^{ij}_{\xi,g} - \tilde{a}^{ij}_{\xi_0,\delta}\|_{C^{k,\alpha}} + \sum_{i=1}^{2} \|b^{i}_{\xi,g} - \tilde{b}^{i}_{\xi_0,\delta}\|_{C^{k,\alpha}} + \|G_{\xi,g} - \tilde{G}_{\xi_0,\delta}\|_{C^{k,\alpha}} + \|\mathfrak{b} - \mathfrak{b}_\circ\|_{C^{k+2,\alpha}} \leq \epsilon \tag{2.28}$$

and the hypotheses (H1) and (H2) hold, there is a function $\psi \in C^{k,\alpha}$ of the form $\psi = \psi_0 \circ \gamma^{-1}$ where $\gamma : D_0 \to D$ is a diffeomorphism and ψ satisfies the generalized Grad–Shafranov equation (1.22) for some pressure profile P which is close to P_0. We now take $\|g - \delta\|_{C^{k+1,\alpha}}$, $\|\xi - \xi_0\|_{C^{k+2,\alpha}}$ and $\|\mathfrak{b} - \mathfrak{b}_0\|_{C^{k+2,\alpha}}$ small enough that (2.28) holds and let ψ be the flux function guaranteed by theorem 3.1 from Constantin *et al.* (2020). This completes the proof of theorem 1.2. □

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Declaration of interests

The authors report no conflict of interest.

Appendix A. Flux function satisfying our hypotheses

The purpose of this section is to give a simple example of a flux function satisfying the hypotheses (H1)–(H2). We will work in toroidal coordinates (r, θ, φ) defined by

$$R = R_0 + r\cos\theta, \quad Z = r\sin\theta, \quad \Phi = \varphi, \tag{A 1a–c}$$

where (R, Z, Φ) are the usual cylindrical coordinates on $\mathbb{R}^3$. In these coordinates, (1.5) becomes

$$\partial_r^2\psi_0 + \frac{1}{r}\partial_r\psi_0 + \frac{1}{r^2}\partial_\theta^2\psi_0 - \frac{1}{R}\left(\cos\theta\,\partial_r\psi_0 - \frac{\sin\theta}{r}\partial_\theta\psi_0\right) + R^2p_0'(\psi_0) + C_0C_0'(\psi_0) = 0. \tag{A 2}$$

The flux function we exhibit is not an exact solution of (A2) but satisfies it when the aspect ratio of the torus is taken to infinity. Using theorem 3.1 from Constantin *et al.* (2020), one can show that there exist solutions on the true axisymmetric torus with large aspect ratio nearby this example. Although they do not have a simple analytical form, they will continue to satisfy (H1)–(H2) as these are open conditions.

We consider the torus where r ranges in $[0, r_0]$ with $0 < r_0 < R_0$ for $R_0 > 1$ and solve the equation (A2) with the choices

$$\psi_0(r) = \bar{\psi}\left(1 - (r/r_0)^2\right), \qquad C_0(\psi) = \bar{c}\sqrt{\bar{\psi} - \psi + \epsilon}, \quad P_0(\psi) = \bar{p}(r_0R_0)^{-2}\psi, \tag{A 3a–c}$$

for $\epsilon \ll 1$ (this is to regularize the square root) and for some constants $\bar{\psi}, \bar{c}$ and $\bar{p}$. The functions C_0 and P_0 are both infinitely differentiable functions of ψ. Note that the pressure vanishes at the outer boundary where ψ_0 is zero, and so this boundary may be interpreted as vacuum. For special choices of constants, ψ_0 solves the 'infinite aspect ratio' Grad–Shafranov equation ((A2) as $R_0 \gg 1$)

$$\partial_r^2\psi_0 + \frac{1}{r^2}\partial_\theta^2\psi_0 = -R_0^2P_0'(\psi_0) - C_0C_0'(\psi_0), \tag{A 4}$$

since $\partial_r^2\psi_0 = -2\bar{\psi}/r_0^2$, $R_0^2P_0'(\psi_0) = \bar{p}/r_0^2$ and $C_0C_0'(\psi_0) = \bar{c}^2$. Thus ψ_0 is a solution if $\bar{p} = 2\bar{\psi} - (\bar{c}r_0)^2$.

Appendix B. Geometric identities

In this section we recall some basic definitions and facts from Riemannian geometry which will be used in the upcoming sections. These are standard and we include the details for the convenience of the reader. Throughout we fix a Riemannian metric g. In our applications, we will take either $g = \delta$, the Euclidean metric, or g will be a metric with $\mathcal{L}_\xi g = 0$ for a given vector field ξ. We let $\flat, \sharp$ denote the usual operations of lowering and raising indices with respect to g. If $X = X^i\partial_{x^i}$ is a vector field and $\beta = \beta_i\,\mathrm{d}x^i$ is a one-form, where $\{x^i\}_{i=1}^3$ are arbitrary local coordinates, then

$$X^\flat = g_{ij}X^j\,\mathrm{d}x^i, \quad \beta^\sharp = g^{ij}\beta_j\partial_i. \tag{B 1a,b}$$

We write $\nabla_g f$ for the gradient of f with respect to the metric g,

$$\nabla_g f = (\mathrm{d}f)^\sharp, \quad (\nabla_g f)^i = g^{ij}\partial_j f. \tag{B 2a,b}$$

In an arbitrary coordinate system $\{x^i\}_{i=1}^3$, if $X = X^i\partial_i$ is a vector field and $\beta = \beta_i\,\mathrm{d}x^i$ is a one-form then $\nabla X, \nabla\beta$ have components

$$\nabla_i X^j = \frac{\partial}{\partial x^i}X^j + \Gamma_{ik}^j X^k, \quad \nabla_i\beta_j = \frac{\partial}{\partial x^i}\beta_j - \Gamma_{ij}^k\beta_k, \tag{B 3a,b}$$

where Γ_{jk}^i are the Christoffel symbols in this coordinate system, defined by

$$\Gamma_{ij}^k = \frac{1}{2}g^{k\ell}\left(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}\right). \tag{B 4}$$

Here we are writing g_{ij} for the components of the metric in this corodinate system and g^{ij} for the components of the inverse metric. The Γ are symmetric in the lower indices,

$$\Gamma^k_{ij} = \Gamma^k_{ji}. \tag{B 5}$$

We also note that covariant differentiation commutes with lowering and raising indices since

$$\nabla_i g_{jk} = \nabla_i g^{jk} = 0. \tag{B 6}$$

Let us also recall that the divergence of a vector field can be written as

$$\mathrm{div}_g X = \nabla_i X^i = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} X^i), \tag{B 7}$$

where $|g| = \det g$ denotes the determinant of the matrix with components g_{ij}.

We let $\mathcal{L}_X$ denote the Lie derivative in the direction X. If f is a function then $\mathcal{L}_X f$ is defined by

$$\mathcal{L}_X f = X^i \partial_i f = Xf. \tag{B 8}$$

For a vector field Y, $\mathcal{L}_X Y$ is the commutator $\mathcal{L}_X Y = [X, Y]$. In an arbitrary coordinate system, $\mathcal{L}_X Y = (\mathcal{L}_X Y)^i \partial_i$ with

$$(\mathcal{L}_X Y)^i = X^j \partial_j Y^i - Y^j \partial_j X^i. \tag{B 9}$$

Many of our results will be stated in terms of the deformation tensor of X, denoted $\mathcal{L}_X g$, which is the $(0, 2)$ tensor defined by the formula

$$X\,(g(Y, Z)) = (\mathcal{L}_X g)(Y, Z) + g(\mathcal{L}_X Y, Z) + g(Y, \mathcal{L}_X Z). \tag{B 10}$$

In an arbitrary coordinate system, $\mathcal{L}_X g = \mathcal{L}_X g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ and a standard calculation shows that

$$\mathcal{L}_X g_{ij} = \nabla_i X_j + \nabla_j X_i, \quad X_k = g_{k\ell} X^\ell, \tag{B 11a,b}$$

where ∇ denotes covariant differentiation (B 3a,b). We will often abuse notation and write $\mathcal{L}_X g(Y, \cdot)$ for the vector field with components

$$(\mathcal{L}_X g(Y, \cdot))^i = g^{ij} (\nabla_j X_k + \nabla_k X_j) Y^k. \tag{B 12}$$

Let $*_g$ denote the Hodge star with respect to the Riemannian volume form $\mathrm{d}\mu = \sqrt{|g|}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. For the general definition, see Lee (2013). For our purposes we will only need to compute $*_g \omega$ when ω is a two-form. With ϵ_{ijk} denoting the Levi–Civita symbol, so that ϵ_{ijk} denotes the sign of the permutation taking $(1, 2, 3)$ to (i, j, k), we have

$$*_g\,(\mathrm{d}x^i \wedge \mathrm{d}x^j) = \sqrt{|g|}g^{ik}g^{j\ell}\epsilon_{k\ell m}\,\mathrm{d}x^m. \tag{B 13}$$

If $\beta = \beta_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j$ is a two-form then from the above formula,

$$*_g\,\beta = \sqrt{|g|}\beta^{k\ell}\epsilon_{k\ell m}\,\mathrm{d}x^m, \quad \beta^{k\ell} = g^{ik}g^{j\ell}\beta_{ij}. \tag{B 14a,b}$$

Let d denote exterior differentiation. If β is a one-form then $d\beta$ is defined by

$$\mathrm{d}\beta = \partial_i\beta_j\,\mathrm{d}x^i \wedge \mathrm{d}x^j. \tag{B 15}$$

We will use the following identity relating $*_g$, d and covariant differentiation ∇. If $\omega = \omega_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ is a $(0, 2)$-tensor then

$$*_g\,d*_g\,\omega = \delta_g\omega, \quad (\delta_g\omega)_i := g^{kj}\nabla_j\omega_{ik}. \tag{B 16a,b}$$

Given vector fields X, Y, let $X\times_g Y$ be the vector field

$$X\times_g Y = (*_g X^\flat \wedge Y^\flat)^\sharp. \tag{B 17}$$

Explicitly, $X\times_g Y = (X\times_g Y)^\ell\partial_\ell$ with $(X\times_g Y)^k = \sqrt{|g|}g^{k\ell}\epsilon_{ij\ell}X^iY^j$. The curl of a vector field, curl_gX, is then defined by

$$\mathrm{curl}_gX = (*_g\,\mathrm{d}X^\flat)^\sharp, \tag{B 18}$$

or, in components,

$$(\mathrm{curl}_gX)^m = \sqrt{|g|}g^{mn}g^{ik}g^{j\ell}\epsilon_{k\ell n}\partial_iX_j = \sqrt{|g|}g^{mn}g^{ik}\epsilon_{k\ell n}\nabla_iX^\ell, \tag{B 19}$$

where the second equality follows from a direct calculation involving the formula for the Christoffel symbols (B4).

We now collect some basic vector calculus identities.

LEMMA B.1. *Define $\times_g$ by (B17), curl$_g$ by (B18) and ∇_g by (B 2a,b). Suppose that M is a subset of R^3. Let $\times, \cdot$ denote the usual cross and dot products in Euclidean space. Then we have*

$$\left.\begin{aligned} (X\times_g Y)\cdot_g Z &= \sqrt{|g|}X\times Y\cdot Z,\\ curl_g(fX) &= \nabla_g f\times_g X + f\,curl_gX,\\ curl_g\nabla_g f &= 0, \end{aligned}\right\} \tag{B 20}$$

$$(X\times_g Y)\times_g Z = (X\cdot_g Y)Z - (X\cdot_g Z)Y. \tag{B 21}$$

Proof. The first three identities are immediate. The last identity is proved by changing coordinates as in the proof of the upcoming identity (B.3), and we omit the proof. □

We will also need the following slightly more complicated identities in the next section.

LEMMA B.2. *Let ∇_g denote covariant differentiation. For any vector fields X, Y,*

$$curl_g(X \times_g Y) = X div_g Y - Y div_g X + \mathcal{L}_Y X, \tag{B 22}$$

$$\nabla_g |X|_g^2 = 2\nabla_X X - 2X \times_g curl_g X, \tag{B 23}$$

$$\mathcal{L}_X(X^\flat) = (\nabla_X X)^\flat + \frac{1}{2}\nabla_g |X|_g^2, \tag{B 24}$$

$$div_g(X \times_g Y) = Y \cdot_g curl_g X - X \cdot_g curl_g Y, \tag{B 25}$$

$$X \times_g curl_g X = \nabla_g |X|^2 + (\mathcal{L}_\xi g(X, \cdot))^\sharp, \tag{B 26}$$

where $\nabla_X := X^i \nabla_i$, and where $\mathcal{L}_X g$, defined in (B 11a,b) denotes the deformation tensor of X, and we are using the notation (B12).

Proof. We begin by writing

$$(\text{curl}_g(X \times_g Y))^\flat = *_g d *_g (X^\flat \wedge Y^\flat) = \delta_g(X^\flat \wedge Y^\flat). \tag{B 27}$$

Using (B 16*a*,*b*) and writing $X_i = g_{ij}X^j$, $Y_k = g_{k\ell}Y^\ell$,

$$\delta_g(X^\flat \wedge Y^\flat)_i = g^{kj}\nabla_j(X_i Y_k - X_k Y_i) = X_i g^{kj}\nabla_j Y_k - Y_i g^{kj}\nabla_j X_k + Y_k g^{kj}\nabla_j X_i - X_k g^{kj}\nabla_j Y_i. \tag{B 28}$$

The first two terms are $X_i \text{div}_g Y - Y_i \text{div}_g X$. If we raise the index on the last two terms (using (B6)) and use (B5) then we see

$$Y_k g^{kj}\nabla_j X^i - X_k g^{kj}\nabla_j Y^i = Y^j \nabla_j X^i - X^j \nabla_j Y^i = Y^j \partial_j X^i - X^j \partial_j Y^i, \tag{B 29}$$

which gives (B22). To prove (B23) we start by computing $X \times_g \text{curl}_g X$. Writing $\beta = X^\flat$, a direct calculation using (B19) shows that

$$\beta \wedge (*_g \,\text{d}\beta) = \sqrt{|g|} g^{ik} g^{j\ell} \epsilon_{k\ell m} \nabla_i X_j X_n \,\text{d}x^n \wedge \text{d}x^m \tag{B 30}$$

and that

$$*_g \left(\beta \wedge (*_g d\beta)\right) = |g| g^{ik} g^{j\ell} g^{nr} g^{mq} \epsilon_{k\ell m} \epsilon_{rqp} \nabla_i X_j X_n \,\text{d}x^p = |g| g^{mq} \epsilon_{k\ell m} \epsilon_{rqp} \nabla^k X^\ell X^r \,\text{d}x^p. \tag{B 31}$$

The identity (B23) follows at any given point P after changing coordinates near P so that expressed in these coordinates, the metric is given by diag(1, 1, 1).

To prove (B24), we write

$$(\mathcal{L}_X X^\flat)_j = X^i \partial_i X_j + X_i \partial_j X^i = X^i \partial_i X_j + \frac{1}{2}\partial_j \left(g_{i\ell} X^\ell X^i\right) - \frac{1}{2}\left(\partial_j g_{i\ell}\right) X^\ell X^i, \tag{B 32}$$

and so it follows from the definition of the covariant derivative that

$$\mathcal{L}_X X_j - \nabla_X X_j - \frac{1}{2}\partial_j |X|_g^2 = \Gamma_{ij}^k X^i X_k - \frac{1}{2}\left(\partial_j g_{i\ell}\right) X^\ell X^i \tag{B 33}$$

and expanding the definition of the Christoffel symbols, the right-hand side is

$$\Gamma_{ij}^k X^i X_k - \frac{1}{2}\left(\partial_j g_{i\ell}\right) X^\ell X^i = \frac{1}{2} X^i X^\ell \left(\partial_i g_{\ell j} + \partial_j g_{\ell i} - \partial_\ell g_{ij}\right) - \frac{1}{2}\left(\partial_j g_{i\ell}\right) X^\ell X^i = 0. \tag{B 34}$$

To prove (B25), we use (B6) and write

$$\begin{aligned}\operatorname{div}_g(X \times_g Y) = \nabla_i(\sqrt{|g|}g^{ij}\epsilon_{jk\ell}X^kY^\ell) &= Y^\ell(\sqrt{|g|}g^{ij}\epsilon_{jk\ell}\nabla_iX^k) + X^k(\sqrt{|g|}g^{ij}\epsilon_{jk\ell}\nabla_iY^\ell) \\ &= Y^k(\operatorname{curl}_gX)_k - X^k(\operatorname{curl}_gY)_k. \end{aligned} \tag{B 35}$$

The final identity (B26) follows from (B23). □

The following identity involving $\times$ and $\times_g$ is crucial for proving that the vector field B_g defined in (1.18) possesses a flux function. This result follows directly from standard vector calculus identities when $g = \delta$.

LEMMA B.3. *If φ is a function with $\mathcal{L}_X\varphi = 0$, then with ∇ denoting the Euclidean gradient,*

$$X \times (X \times_g \nabla_g\varphi) = -\frac{1}{\sqrt{|g|}}|X|_g^2\nabla\varphi. \tag{B 36}$$

Proof. This follows from a straightforward but tedious argument; we include the details for the convenience of the reader. With ω the quantity on the left-hand side of (B36), from the definitions we have

$$\omega^i = \sqrt{|g|}\delta^{ij}g^{\ell m}g^{pq}\epsilon_{jk\ell}\epsilon_{mnp}X^kX^n\partial_q\varphi. \tag{B 37}$$

Fix any $P \in M$ and choose coordinates (Z^1, Z^2, Z^3) near P, so that at P we have

$$g_{ij}\frac{\partial x^i}{\partial Z^\alpha}\frac{\partial x^j}{\partial Z^\beta} = \delta_{\alpha\beta}. \tag{B 38}$$

We note the following relation which will be useful in what follows: at P, we have

$$g^{\ell m} = \delta^{\alpha\beta}\frac{\partial x^\ell}{\partial Z^\alpha}\frac{\partial x^m}{\partial Z^\beta}. \tag{B 39}$$

Expressing ω in these coordinates, $\omega = \omega^a\partial_{Z^a}$ with $\omega^a = (\partial Z^a/\partial x^i)\omega^i$, evaluating at P and using (B39) to rewrite $g^{\ell m}$ and g^{pq}, from (B37) we have

$$\begin{aligned}\omega^a &= \sqrt{|g|}\frac{\partial Z^a}{\partial x^i}\delta^{ij}g^{\ell m}g^{pq}\epsilon_{jk\ell}\epsilon_{mnp}X^kX^n\partial_q\varphi \\ &= \sqrt{|g|}\frac{\partial Z^a}{\partial x^i}\frac{\partial x^\ell}{\partial Z^b}\frac{\partial x^m}{\partial Z^c}\frac{\partial x^p}{\partial Z^d}\frac{\partial x^k}{\partial Z^{b'}}\frac{\partial x^n}{\partial Z^{c'}}\delta^{bc}\delta^{dd'}\epsilon_{jk\ell}\epsilon_{mnp}X^{b'}X^{c'}\partial_{Z^{d'}}\varphi, \end{aligned} \tag{B 40}$$

writing, e.g. $X^k = (\partial x^k/\partial Z^{b'})X^{b'}$. Now we note that

$$\epsilon_{mnp}\frac{\partial x^m}{\partial Z^c}\frac{\partial x^n}{\partial Z^{c'}}\frac{\partial x^p}{\partial Z^d} = \epsilon_{cc'd}\det(\partial_Zx), \qquad \epsilon_{jk\ell}\frac{\partial x^j}{\partial Z^e}\frac{\partial x^k}{\partial Z^{b'}}\frac{\partial x^\ell}{\partial Z^b} = \epsilon_{eb'b}\det(\partial_Zx). \tag{B 41a,b}$$

Indeed, the quantity on the left-hand side of, for example, the first equality is antisymmetric in all three indices, and so is a multiple of $\epsilon_{cc'd}$ and evaluating at $c = 1$, $c' =$

2, $d=3$ gives the result. Therefore (B40) reads

$$\omega^a = \sqrt{|g|}\,\det(\partial_Z x)^2\delta^{ij}\frac{\partial Z^a}{\partial x^i}\frac{\partial Z^e}{\partial x^j}\delta^{dd'}\delta^{bc}\epsilon_{cc'd}\epsilon_{beb'}X^{b'}X^{c'}\partial_{Z^{d'}}\varphi$$
$$= \sqrt{|g|}\,\det(\partial_Z x)^2\delta^{ij}\frac{\partial Z^a}{\partial x^i}\frac{\partial Z^e}{\partial x^j}\delta^{dd'}\left(\delta_{c'e}\delta_{db'}-\delta_{c'b'}\delta_{de}\right)X^{b'}X^{c'}\partial_{Z^{d'}}\varphi, \tag{B 42}$$

using a well known identity for the Levi–Civita symbol. Now we note that

$$\delta^{dd'}\delta_{db'}\partial_{Z^{d'}}\varphi X^{b'} = X\cdot\nabla\varphi = 0, \tag{B 43}$$

by assumption, and so

$$\omega^a = -\sqrt{|g|}(\det\partial_Z x)^2\delta^{ij}\frac{\partial Z^a}{\partial x^i}\frac{\partial Z^e}{\partial x^j}\partial_{Z^e}\varphi\delta_{c'b'}X^{c'}X^{b'} = -\sqrt{|g|}(\det\partial_Z x)^2\delta^{ij}\frac{\partial Z^a}{\partial x^i}\partial_{x^j}\varphi|X|_g^2. \tag{B 44}$$

From (B38), we have $\sqrt{|g|}(\det\partial_Z x)^2 = 1/\sqrt{|g|}$ and so at P we find

$$\omega^i = -\frac{1}{\sqrt{|g|}}\delta^{ij}\partial_{x^j}\varphi|X|_g^2, \tag{B 45}$$

and since P was arbitrary we get the result. □

We finally record some useful formulae involving Lie derivatives along g Killing fields.

LEMMA B.4. *Let X, Y be vector fields and let ξ be a Killing vector field for the metric g. Then*

$$\mathcal{L}_\xi(X\times_g Y) = \mathcal{L}_\xi X\times_g Y + X\times_g\mathcal{L}_\xi Y, \tag{B 46}$$

$$\mathcal{L}_\xi curl_g X = curl_g\mathcal{L}_\xi X. \tag{B 47}$$

Proof. To prove (B46), we start from the following fact, which can be found on p. 177 of Fecko (2006). If ξ_0 is a Killing field for a metric g, $\mathcal{L}_\xi g = 0$, then $\mathcal{L}_\xi *_g\alpha = *_g\mathcal{L}_\xi\alpha$. Similarly, $\mathcal{L}_\xi X^\flat = (\mathcal{L}_\xi X)_\flat$, where $\flat$ denotes lowering indices with g. For any vector field ξ, if Φ_s denotes its flow, we have the following identity $\Phi_s^* *_g\alpha = *_{\Phi_s^* g}\Phi_s^*\alpha$. If ξ is a Killing field for g then this becomes $\Phi_s^* *_g = *_g\Phi_s^*\alpha$. Differentiating this at $s=0$ and using the definition of the Lie derivative gives the result. Now, to get the formula for $\mathcal{L}_\xi(X\times_g Y)$ we then recall that $(X\times_g Y)_\flat = *_g(X^\flat\wedge Y^\flat)$. Using that $\mathcal{L}_\xi$ commutes with $\sharp$, $\flat$, and $*_g$,

$$\mathcal{L}_\xi(X\times_g Y) = \mathcal{L}_\xi *_g\left(X^\flat\wedge Y_\flat\right)^\sharp = *_g\left(\mathcal{L}_\xi(X_\flat\wedge Y_\flat)\right)^\sharp = *_g\left((\mathcal{L}_\xi X)^\flat\wedge Y_\flat + X_\flat\wedge(\mathcal{L}_\xi Y)_\flat\right)^\sharp$$
$$= \mathcal{L}_\xi X\times_g Y + X\times_g\mathcal{L}_\xi Y. \tag{B 48}$$

To prove (B47), recall that $\mathrm{curl}_g X = (*_g\,\mathrm{d}X^\flat)^\sharp$, where $\flat$ and $\sharp$ denote lowering and raising the index with g. Recall that if $\mathcal{L}_\xi g = 0$ then

$$\mathcal{L}_\xi *_g = *_g\mathcal{L}_\xi,\quad \mathcal{L}_\xi X^\flat = (\mathcal{L}_\xi X^\flat),\quad \mathcal{L}_\xi\alpha^\sharp = (\mathcal{L}_\xi\alpha)^\sharp. \tag{B 49a–c}$$

After lowering the index on $\mathrm{curl}_g X$ and using the fact that Lie derivatives commute with exterior differentiation, $\mathcal{L}_\xi d = \mathrm{d}\mathcal{L}_\xi$, we obtain $\mathcal{L}_\xi *_g(\mathrm{d}X_\flat) = *_g d(\mathcal{L}_\xi X)_\flat$. Raising the index with g and using (B 49*a–c*) again we get the result. □

Appendix C. Generalized quasisymmetric Grad–Shafranov equation

In this section we summarize the relationship between quasisymmetry and the MHS equation (1.1). Recall that the deformation tensor $\mathcal{L}_\xi \delta$ is defined by

$$(\mathcal{L}_\xi \delta)(X, Y) = X \cdot (\nabla \xi + (\nabla \xi)^{\mathrm{T}}) \cdot Y. \tag{C 1}$$

We begin by showing that the ansatz (1.18) is automatically Euclidean divergence-free, has flux surfaces and we write the condition for weak quasisymmetry explicitly for such fields.

PROPOSITION C.1 (Characterization of quasisymmetric B_g MHS solutions). *Let ξ be a non-vanishing and divergence-free vector field tangent to ∂T and let g be any metric with $\mathcal{L}_\xi g = 0$. Let $\psi : T \to \mathbb{R}$ satisfy $\mathcal{L}_\xi \psi = 0$ and $|\nabla \psi| > 0$. Then B_g given by (1.18) is (Euclidean) divergence-free, satisfies (1.8) and is tangent to ∂T. Moreover, B_g is weakly quasisymmetric if and only if*

$$(\mathcal{L}_\xi \delta)(\xi, \xi) + 2C^{-1}(\psi)(\mathcal{L}_\xi \delta)(\xi, \nabla_g^{\perp} \psi) + C^{-2}(\psi)(\mathcal{L}_\xi \delta)(\nabla_g^{\perp} \psi, \nabla_g^{\perp} \psi) = 0. \tag{C 2}$$

The field B_g additionally solves MHS with forcing f if and only if $f \cdot_g \nabla_g^{\perp} \psi = f \cdot_g \xi = 0$, and ψ satisfies the generalized Grad–Shafranov equation

$$div_g \left(\sqrt{|g|} \frac{\nabla_g \psi}{|\xi|_g^2} \right) - C(\psi) \frac{\xi}{|\xi|_g^2} \cdot_g curl_g \left(\frac{\xi}{|\xi|_g^2} \right) + \frac{C(\psi) C'(\psi)}{\sqrt{|g|} |\xi|_g^2} + \frac{P'(\psi)}{\sqrt{|g|}} = \frac{f \cdot_g \nabla_g \psi}{\sqrt{|g|} |\nabla_g \psi|_g^2}. \tag{C 3}$$

This section will build up to the proof of proposition C.1 by developing the following lemmas C.4 and C.5. The proof is a straightforward combination of these results. First we record some elementary vector identities.

LEMMA C.2. *Fix a metric g. Let ξ be a vector field with $|\xi| \neq 0$ and let $\psi : T \to \mathbb{R}$ be a function satisfying $\mathcal{L}_\xi \psi = 0$. Then we have*

$$\xi \times_g \nabla_g \psi = \nabla_g^{\perp} \psi, \quad \nabla_g \psi \times_g \nabla_g^{\perp} \psi = |\nabla_g \psi|_g^2 \xi, \quad \nabla_g^{\perp} \psi \times_g \xi = |\xi|_g^2 \nabla_g \psi, \tag{C 4a–c}$$

where we have introduced $\nabla_g^{\perp} \psi = \xi \times_g \nabla_g \psi$. Thus, the triple $(\nabla_g \psi, \nabla_g^{\perp} \psi, \xi)$ forms an orthogonal basis of $\mathbb{R}^3$ at each $x \in T$ where $|\nabla_g \psi|_g > 0$.

Proof. Follows from the identity (B21). □

The following are the main results in this section and are proved at the end of the section.

LEMMA C.3 (Structural properties of B_g). *Fix a metric g. Let $\xi : T \to \mathbb{R}^3$ be a (Euclidean) divergence-free vector field with $|\xi|_g \neq 0$ which is tangent to ∂T. Let $\psi : T \to \mathbb{R}$ be a function satisfying $\mathcal{L}_\xi \psi = 0$ which is constant on ∂T. Fix $C : \mathbb{R} \to \mathbb{R}$. Then B_g defined in (1.18) satisfies*

$$\xi \times B_g = -\nabla \psi, \tag{C 5}$$

$$div B_g = -\frac{1}{|\xi|_g^2} (\mathcal{L}_\xi g)(\xi, B_g), \tag{C 6}$$

$$\mathcal{L}_\xi B_g = -\frac{1}{|\xi|_g^2} (\mathcal{L}_\xi g)(\xi, B_g) \xi, \tag{C 7}$$

$$B_g \cdot \hat{n}|_{\partial T} = 0. \tag{C 8}$$

For the proof, see § 2. The crucial point here is, despite the fact that B_g is defined in terms of an arbitrary metric g, the identities (C5) and (C6) involve the Euclidean metric.

We now begin the derivation of the Grad–Shafranov equation (1.22) which involves a somewhat lengthy calculation using the above identities. The most important and complicated ingredient is the following formula, which is a direct consequence of lemma C.7, below.

LEMMA C.4 (Curl of B_g). *Fix a metric g. Let ξ be a vector field with $|\xi| \neq 0$ and let $\psi : T \to \mathbb{R}$ be a function satisfying $\mathcal{L}_\xi \psi = 0$. Fix a function $C : \mathbb{R} \to \mathbb{R}$. Then B_g defined in (1.18) satisfies*

$$curl_g B_g = F\nabla_g \psi + G\nabla_g^\perp \psi + H\xi, \tag{C 9}$$

where $curl_g$ is with respect to the metric g, defined in (B18), and with F, G, H defined by

$$F := -\frac{\sqrt{|g|}}{|\xi|_g^4 |\nabla_g \psi|_g^2}\left[\frac{C(\psi)}{\sqrt{|g|}}(\mathcal{L}_\xi g)(\xi, \nabla_g^\perp \psi) + 2|\xi|_g^2 |\nabla_g \psi|_g^2 \frac{\mathcal{L}_\xi \sqrt{|g|}}{\sqrt{|g|}} - |\xi|_g^2 (\mathcal{L}_\xi g)(\nabla_g \psi, \nabla_g \psi) - |\nabla_g \psi|_g^2 (\mathcal{L}_\xi g)(\xi, \xi)\right], \tag{C 10}$$

$$G := \frac{\sqrt{|g|}}{|\xi|_g^2}\frac{1}{|\nabla_g \psi|_g^2}\left[(\mathcal{L}_\xi g)(B_g, \nabla_g \psi) - |\nabla_g \psi|_g^2 \frac{C'(\psi)}{\sqrt{|g|}}\right], \tag{C 11}$$

$$H := div_g\left(\sqrt{|g|}\frac{\nabla_g \psi}{|\xi|_g^2}\right) + \frac{1}{|\xi|_g^4} C(\psi)\xi \cdot_g curl_g \xi + \frac{\sqrt{|g|}}{|\xi|_g^2}(\mathcal{L}_\xi g)(\nabla_g \psi, \xi). \tag{C 12}$$

LEMMA C.5 (MHS for B_g). *Fix a metric g. Let ξ be a vector field with $|\xi| \neq 0$ and let $\psi : T \to \mathbb{R}$ be a function satisfying $\mathcal{L}_\xi \psi = 0$. Fix a function $C : \mathbb{R} \to \mathbb{R}$. Then B_g defined in (1.18) satisfies*

$$curl_g B_g \times_g B_g - \nabla_g P = \left(C(\psi)G - H - P'\right)\nabla_g \psi - \frac{C(\psi)}{|\xi|_g^2} F\nabla_g^\perp \psi + \frac{|\nabla_g \psi|_g^2}{|\xi|_g^2} F\xi, \tag{C 13}$$

with F, G and H defined by (C10), (C11) and (C12). In particular, if $\mathcal{L}_\xi g = 0$ then B satisfies the MHS equation with force f

$$curl_g B_g \times_g B_g = \nabla_g P + f, \tag{C 14}$$

if and only if $f \cdot_g \nabla_g^\perp \psi = f \cdot_g \xi = 0$, and ψ satisfies the generalized Grad–Shafranov equation

$$div_g\left(\sqrt{|g|}\frac{\nabla_g \psi}{|\xi|_g^2}\right) - C(\psi)\frac{\xi}{|\xi|_g^2} \cdot_g curl_g\left(\frac{\xi}{|\xi|_g^2}\right) + \frac{C(\psi)C'(\psi)}{\sqrt{|g|}|\xi|_g^2} + \frac{P'(\psi)}{\sqrt{|g|}} = \frac{f \cdot_g \nabla_g \psi}{\sqrt{|g|}|\nabla_g \psi|_g^2}. \tag{C 15}$$

Proof. Follows from lemma C.4, (C.2), standard vector identities and $\mathcal{L}_\xi \psi = 0$. □

The generalized Grad–Shafranov equation (C15) for vector fields of the form (1.18) was first derived in Burby *et al.* (2020) when g was taken to be the circle-averaged metric.

LEMMA C.6 (Quasisymmetry of B_g). *Fix a metric g with $\mathcal{L}_\xi g = 0$. Let ξ be a vector field with $|\xi| \neq 0$ and let $\psi : T \to \mathbb{R}$ be a function satisfying $\mathcal{L}_\xi \psi = 0$. Fix $C : \mathbb{R} \to \mathbb{R}$. Then B_g satisfies*

$$\mathcal{L}_\xi |B_g|^2 = \frac{C^2(\psi)}{|\xi|_g^4}\left[(\mathcal{L}_\xi \delta)(\xi,\xi) + 2C^{-1}(\psi)(\mathcal{L}_\xi\delta)(\xi, \nabla_g^\perp\psi) + C^{-2}(\psi)(\mathcal{L}_\xi\delta)(\nabla_g^\perp\psi, \nabla_g^\perp\psi)\right]. \tag{C 16}$$

C.1. *Auxiliary lemmas*

We collect some calculations which are useful for the proofs of the other lemmas in the following statement.

LEMMA C.7. *Fix a metric g. Let ξ be a vector field with $|\xi| \neq 0$ and let $\psi : T \to \mathbb{R}$ be a function satisfying $\mathcal{L}_\xi \psi = 0$. Fix a function $C : \mathbb{R} \to \mathbb{R}$. Then*

$$div_g\left(C(\psi)\frac{\xi}{|\xi|_g^2}\right) = \frac{C(\psi)}{|\xi|_g^2}\left(div_g\xi - \frac{1}{|\xi|_g^2}(\mathcal{L}_\xi g)(\xi,\xi)\right), \tag{C 17}$$

$$div_g\left(\sqrt{|g|}\frac{\nabla_g^\perp\psi}{|\xi|_g^2}\right) = -\frac{\sqrt{|g|}}{|\xi|_g^4}(\mathcal{L}_\xi g)(\xi, \nabla_g^\perp\psi) + \frac{1}{|\xi|_g^2}\mathcal{L}_{\nabla_g^\perp\psi}\sqrt{|g|}, \tag{C 18}$$

$$\begin{aligned} curl_g\left(C(\psi)\frac{\xi}{|\xi|_g^2}\right) &= \frac{1}{|\xi|_g^4}C(\psi)(\xi \cdot_g curl_g\xi)\xi \\ &+ \frac{1}{|\xi|_g^2}\left(\frac{C(\psi)}{|\xi|_g^2|\nabla_g\psi|_g^2}(\mathcal{L}_\xi g)(\xi,\nabla_g\psi) - C'(\psi)\right)\nabla_g^\perp\psi \\ &- \frac{C(\psi)}{|\xi|_g^4|\nabla_g\psi|_g^2}(\mathcal{L}_\xi g)(\xi,\nabla_g^\perp\psi)\nabla_g\psi, \end{aligned} \tag{C 19}$$

$$\begin{aligned} &curl_g\left(\sqrt{|g|}\frac{\nabla_g^\perp\psi}{|\xi|_g^2}\right) \\ &= \left(div_g\left(\sqrt{|g|}\frac{\nabla_g\psi}{|\xi|_g^2}\right) + \frac{\sqrt{|g|}}{|\xi|_g^4}(\mathcal{L}_\xi g)(\nabla_g\psi,\xi)\right)\xi \\ &+ \frac{\sqrt{|g|}}{|\xi|_g^4}\left(\mathcal{L}_\xi g(\xi,\xi) - 2|\xi|_g^2\frac{\mathcal{L}_\xi\sqrt{|g|}}{\sqrt{|g|}} + \frac{|\xi|_g^2}{|\nabla_g\psi|^2}(\mathcal{L}_\xi g)(\nabla_g\psi,\nabla_g\psi)\right)\nabla_g\psi \\ &+ \frac{\sqrt{|g|}}{|\xi|_g^4|\nabla_g\psi|^2}(\mathcal{L}_\xi g)(\nabla_g\psi,\nabla_g^\perp\psi)\nabla_g^\perp\psi. \end{aligned} \tag{C 20}$$

Proof. We will repeatedly use the product rule (B10) as well as the commutator identity

$$\mathcal{L}_\xi\nabla_g f = \nabla_g\mathcal{L}_\xi f - (\mathcal{L}_\xi g)(\nabla_g f, \cdot). \tag{C 21}$$

Step 1: identity (C17). To prove (C17) we note

$$\mathrm{div}_g\left(C(\psi)\frac{\xi}{|\xi|_g^2}\right) = C(\psi)\frac{\mathrm{div}_g\xi}{|\xi|_g^2} - |\xi|_g^{-4}C(\psi)\mathcal{L}_\xi|\xi|_g^2 = \frac{C(\psi)}{|\xi|_g^2}\mathrm{div}_g\xi - \frac{C(\psi)}{|\xi|_g^4}(\mathcal{L}_\xi g)(\xi,\xi), \tag{C 22}$$

using the product rule (B20).

Step 2: identity (C18). First note that

$$\mathrm{div}_g\left(\sqrt{|g|}\frac{\nabla_g^\perp\psi}{|\xi|_g^2}\right) = \sqrt{|g|}\mathrm{div}_g\left(\frac{\nabla_g^\perp\psi}{|\xi|_g^2}\right) + \frac{1}{|\xi|_g^2}\mathcal{L}_{\nabla_g^\perp\psi}\sqrt{|g|}. \tag{C 23}$$

Next we compute

$$\begin{aligned}\mathrm{div}_g\left(\frac{\nabla_g^\perp\psi}{|\xi|_g^2}\right) &= \frac{1}{|\xi|_g^2}\left(\mathrm{div}_g\nabla_g^\perp\psi + |\xi|_g^2\mathcal{L}_{\nabla_g^\perp\psi}|\xi|_g^{-2}\right)\\ &= \frac{1}{|\xi|_g^2}\left(\mathrm{div}_g(\xi\times_g\nabla_g\psi) - |\xi|_g^{-2}(\xi\times_g\nabla_g\psi)\cdot_g\nabla_g|\xi|_g^2\right)\\ &= \frac{1}{|\xi|_g^2}\left(\mathcal{L}_{\mathrm{curl}_g\xi}\psi - \xi\cdot_g\mathrm{curl}_g\nabla_g\psi - |\xi|_g^{-2}(\nabla_g|\xi|_g^2\times_g\xi)\cdot_g\nabla_g\psi\right)\\ &= \frac{1}{|\xi|_g^2}\left(\mathcal{L}_{\mathrm{curl}_g\xi}\psi - |\xi|_g^{-2}(\nabla_g|\xi|_g^2\times_g\xi)\cdot_g\nabla_g\psi\right).\end{aligned} \tag{C 24}$$

We now simplify the second term in the above. First note the identity (which follows from (B26))

$$\begin{aligned}\xi\times_g\mathrm{curl}_g\xi &= \frac{1}{2}\nabla_g|\xi|_g^2 - (\xi\cdot_g\nabla_g)\xi\\ &= \nabla_g|\xi|_g^2 - ((\xi\cdot_g\nabla_g)\xi + \nabla_g\xi\cdot_g\xi)\\ &= \nabla_g|\xi|_g^2 - (\mathcal{L}_\xi g)\cdot_g\xi,\end{aligned} \tag{C 25}$$

so that

$$\begin{aligned}\nabla_g|\xi|_g^2\times_g\xi &= (\xi\times_g\mathrm{curl}_g\xi)\times_g\xi + ((\mathcal{L}_\xi g)\cdot_g\xi)\times_g\xi\\ &= |\xi|_g^2\mathrm{curl}_g\xi - (\xi\cdot_g\mathrm{curl}_g\xi)\xi + ((\mathcal{L}_\xi g)\cdot_g\xi)\times_g\xi,\end{aligned} \tag{C 26}$$

where we have used the elementary identity

$$(\xi\times_g\mathrm{curl}_g\xi)\times_g\xi = |\xi|_g^2\mathrm{curl}_g\xi - (\xi\cdot_g\mathrm{curl}_g\xi)\xi. \tag{C 27}$$

Noting finally that

$$(((\mathcal{L}_\xi g)\cdot_g\xi)\times_g\xi)\cdot_g\nabla_g\psi = ((\mathcal{L}_\xi g)\cdot_g\xi)\cdot_g(\xi\times_g\nabla_g\psi) = (\mathcal{L}_\xi g)(\xi,\nabla_g^\perp\psi), \tag{C 28}$$

using that $\mathcal{L}_\xi\psi = 0$ we have

$$|\xi|_g^{-2}(\nabla_g|\xi|_g^2\times_g\xi)\cdot_g\nabla_g\psi = \mathcal{L}_{\mathrm{curl}_g\xi}\psi + |\xi|_g^{-2}(\mathcal{L}_\xi g)(\xi,\nabla_g^\perp\psi). \tag{C 29}$$

Putting this together with (C24), we obtain the identity (C18).

Step 3: identity (C19). To prove (C19), we note

$$\mathrm{curl}_g(C(\psi)\xi) = C'(\psi)\nabla_g\psi \times_g \xi + C(\psi)\mathrm{curl}_g\xi = -C'(\psi)\nabla_g^{\perp}\psi + C(\psi)\mathrm{curl}_g\xi. \quad (C\,30)$$

Using this formula and (C25), we find that

$$\begin{aligned}\mathrm{curl}_g\left(C(\psi)\frac{\xi}{|\xi|_g^2}\right) &= \frac{1}{|\xi|_g^2}\left(-C'(\psi)\nabla_g^{\perp}\psi + C(\psi)\mathrm{curl}_g\xi\right) - |\xi|_g^{-4}C(\psi)\nabla|\xi|_g^2 \times_g \xi \\ &= \frac{1}{|\xi|_g^2}\left(-C'(\psi)\nabla_g^{\perp}\psi + C(\psi)\mathrm{curl}_g\xi\right) \\ &\quad - |\xi|_g^{-4}C(\psi)\left(|\xi|_g^2\mathrm{curl}_g\xi - (\xi \cdot_g \mathrm{curl}_g\xi)\xi + (\mathcal{L}_\xi g)\cdot\xi \times_g \xi\right) \\ &= -\frac{C'(\psi)}{|\xi|_g^2}\nabla_g^{\perp}\psi + |\xi|_g^{-4}C(\psi)\left((\xi \cdot_g \mathrm{curl}_g\xi)\xi - (\mathcal{L}_\xi g)\cdot\xi \times_g \xi\right).\end{aligned} \quad (C\,31)$$

Note finally using lemma C.2 that

$$\begin{aligned}(\mathcal{L}_\xi g)\cdot_g \xi \times_g \xi &= ((\mathcal{L}_\xi g)\cdot_g \xi \times_g \xi)\cdot_g \widehat{\nabla_g\psi}\,\widehat{\nabla_g\psi} + ((\mathcal{L}_\xi g)\cdot_g \xi \times_g \xi)\cdot_g \widehat{\nabla_g^{\perp}\psi}\,\widehat{\nabla_g^{\perp}\psi} \\ &= \frac{1}{|\nabla_g\psi|_g^2}(\mathcal{L}_\xi g)(\xi,\nabla_g^{\perp}\psi)\nabla_g\psi - \frac{1}{|\nabla_g\psi|_g^2}(\mathcal{L}_\xi g)(\xi,\nabla_g\psi)\nabla_g^{\perp}\psi,\end{aligned} \quad (C\,32)$$

where we used the identity (C28) in passing to the second line together with

$$(((\mathcal{L}_\xi g)\cdot_g \xi)\times_g \xi)\cdot_g \nabla_g^{\perp}\psi = ((\mathcal{L}_\xi g)\cdot_g \xi)\cdot_g(\xi \times_g \nabla_g^{\perp}\psi) = -|\xi|_g^2(\mathcal{L}_\xi g)(\xi,\nabla\psi). \quad (C\,33)$$

Combining this with (C31) gives

$$\begin{aligned}\mathrm{curl}_g\left(C(\psi)\frac{\xi}{|\xi|_g^2}\right) &= -\frac{C'(\psi)}{|\xi|_g^2}\nabla_g^{\perp}\psi + |\xi|_g^{-4}C(\psi)(\xi \cdot_g \mathrm{curl}_g\xi)\xi \\ &\quad + \frac{1}{|\xi|_g^2}\left(\frac{C(\psi)}{|\xi|_g^2|\nabla_g\psi|_g^2}(\mathcal{L}_\xi g)(\xi,\nabla_g\psi)\nabla_g^{\perp}\psi\right. \\ &\quad \left. - \frac{C(\psi)}{|\xi|_g^2|\nabla_g\psi|_g^2}(\mathcal{L}_\xi g)(\xi,\nabla_g^{\perp}\psi)\nabla_g\psi\right).\end{aligned} \quad (C\,34)$$

Rearrangement establishes (C19).

Step 4: identity (C20). First note that

$$\begin{aligned}\mathrm{curl}_g\left(\sqrt{|g|}\frac{\nabla_g^{\perp}\psi}{|\xi|_g^2}\right) &= \sqrt{|g|}\mathrm{curl}_g\left(\frac{\nabla_g^{\perp}\psi}{|\xi|_g^2}\right) + \frac{1}{|\xi|_g^2}\nabla\sqrt{|g|} \times_g \nabla_g^{\perp}\psi \\ &= \sqrt{|g|}\mathrm{curl}_g\left(\frac{\nabla_g^{\perp}\psi}{|\xi|_g^2}\right) + \frac{1}{|\xi|_g^2}(\mathcal{L}_{\nabla_g\psi}\sqrt{|g|})\xi - \frac{1}{|\xi|_g^2}(\mathcal{L}_\xi\sqrt{|g|})\nabla\psi.\end{aligned} \quad (C\,35)$$

Now, by the identity (B22),

$$\begin{aligned}\mathrm{curl}_g \nabla_g^{\perp}\psi = \mathrm{curl}_g(\xi \times_g \nabla_g \psi) &= \xi \Delta_g \psi - \nabla_g \psi \, \mathrm{div}_g \xi + \mathcal{L}_{\nabla_g \psi}\xi \\ &= \xi \Delta_g \psi - \nabla_g \psi \, \mathrm{div}_g \xi - \mathcal{L}_{\xi}\nabla_g \psi \\ &= \xi \Delta_g \psi - \nabla_g \psi \, \mathrm{div}_g \xi + (\mathcal{L}_{\xi} g)(\nabla_g \psi, \cdot),\end{aligned} \tag{C 36}$$

where we used (C21) and $(\mathcal{L}_{\xi} g)(\nabla_g \psi, \cdot)$ is defined as in (B12). Therefore

$$\begin{aligned}\sqrt{|g|}\mathrm{curl}_g\left(\frac{\nabla_g^{\perp}\psi}{|\xi|_g^2}\right) &= \frac{\sqrt{|g|}}{|\xi|_g^2}\big(\xi \Delta_g \psi - \mathrm{div}_g \xi \, \nabla_g \psi + (\mathcal{L}_{\xi} g)(\nabla_g \psi, \cdot)\big) - \frac{\sqrt{|g|}}{|\xi|_g^4}\nabla_g |\xi|_g^2 \times \nabla_g^{\perp}\psi \\ &= \frac{\sqrt{|g|}}{|\xi|_g^2}\big(\xi \Delta_g \psi - \mathrm{div}_g \xi \, \nabla_g \psi + (\mathcal{L}_{\xi} g)(\nabla_g \psi, \cdot)\big) \\ &\quad + \frac{\sqrt{|g|}}{|\xi|_g^4}\big((\mathcal{L}_{\xi}|\xi|_g^2)\nabla_g \psi - (\nabla_g \psi \cdot_g \nabla_g |\xi|_g^2)\xi\big) \\ &= \sqrt{|g|}\mathrm{div}_g\left(\frac{\nabla_g \psi}{|\xi|_g^2}\right)\xi + \frac{\sqrt{|g|}}{|\xi|_g^2}(\mathcal{L}_{\xi} g)(\nabla_g \psi, \cdot) \\ &\quad + \frac{\sqrt{|g|}}{|\xi|_g^4}\left(\mathcal{L}_{\xi} g(\xi, \xi) - |\xi|_g^2 \mathrm{div}_g \xi\right)\nabla_g \psi \\ &= \mathrm{div}_g\left(\sqrt{|g|}\frac{\nabla_g \psi}{|\xi|_g^2}\right)\xi - \frac{1}{|\xi|_g^2}(\mathcal{L}_{\nabla_g \psi}\sqrt{|g|})\xi \\ &\quad + \frac{\sqrt{|g|}}{|\xi|_g^2}(\mathcal{L}_{\xi} g)(\nabla_g \psi, \cdot) + \frac{1}{|\xi|_g^4}\left(\sqrt{|g|}\mathcal{L}_{\xi} g(\xi, \xi) - |\xi|_g^2 \mathcal{L}_{\xi}\sqrt{|g|}\right)\nabla_g \psi,\end{aligned} \tag{C 37}$$

where we used (C41) to say $\sqrt{|g|}\mathrm{div}_g \xi = \mathcal{L}_{\xi}\sqrt{|g|}$ as well as the identity

$$\nabla_g |\xi|_g^2 \times \nabla_g^{\perp}\psi := \nabla_g |\xi|_g^2 \times (\xi \times_g \nabla_g \psi) = (\nabla_g |\xi|_g^2 \cdot_g \nabla_g \psi)\xi - (\nabla_g |\xi|_g^2 \cdot_g \xi)\nabla_g \psi. \tag{C 38}$$

Finally, note that we can express

$$\begin{aligned}(\mathcal{L}_{\xi} g)(\nabla_g \psi, \cdot) &= \frac{1}{|\xi|_g^2}(\mathcal{L}_{\xi} g)(\nabla_g \psi, \xi)\xi + \frac{1}{|\xi|_g^2 |\nabla_g \psi|_g^2}(\mathcal{L}_{\xi} g)(\nabla_g \psi, \nabla_g^{\perp}\psi)\nabla_g^{\perp}\psi \\ &\quad + \frac{1}{|\nabla_g \psi|_g^2}(\mathcal{L}_{\xi} g)(\nabla_g \psi, \nabla_g \psi)\nabla_g \psi.\end{aligned} \tag{C 39}$$

This completes the derivation. □

C.2. *Proof of lemma C.3*

The result follows from direct computation as follows.

Step 1: identity (C5). The property of having a flux function (C5) follows from lemma B.3.

Step 2: identity (C6). For the divergence (C6), lemma C.6 gives

$$\mathrm{div}_g B_g = \frac{1}{|\xi|_g^2}\big(C(\psi)\mathrm{div}_g\xi - (\mathcal{L}_\xi g)(\xi, B_g)\big) + \frac{1}{|\xi|_g^2}\mathcal{L}_{\nabla_g^\perp\psi}\sqrt{|g|}. \tag{C 40}$$

Next recall the relation between the divergence on flat and curved backgrounds

$$\mathrm{div}X = \mathrm{div}_g X - \frac{1}{\sqrt{|g|}}\mathcal{L}_X\sqrt{|g|}. \tag{C 41}$$

Applying this identity to convert (C40) to the divergence using the Euclidean metric, we have

$$\mathrm{div}B_g = \mathrm{div}_g B_g - \frac{1}{\sqrt{|g|}}\mathcal{L}_{B_g}\sqrt{|g|} = \frac{1}{|\xi|_g^2}\big(C(\psi)\mathrm{div}_g\xi - (\mathcal{L}_\xi g)(\xi, B)\big) - \frac{1}{\sqrt{|g|}}\frac{C(\psi)}{|\xi|_g^2}\mathcal{L}_\xi\sqrt{|g|}. \tag{C 42}$$

Using $\mathrm{div}\xi = 0$ and (C41) again we find $\sqrt{|g|}\mathrm{div}_g\xi = \mathcal{L}_\xi\sqrt{|g|}$, and get the claimed result.

Step 3: identity (C7). We have the identity

$$\begin{aligned}\mathcal{L}_\xi B_g &:= \xi\cdot\nabla B_g - B_g\cdot\nabla\xi \\ &= \mathrm{curl}(B_g\times\xi) + (\mathrm{div}B_g)\xi - (\mathrm{div}\xi)B_g = -\frac{1}{|\xi|_g^2}(\mathcal{L}_\xi g)(\xi, B_g)\xi,\end{aligned} \tag{C 43}$$

and the result follows from (C5), (C6) and the assumption $\mathrm{div}\xi = 0$.

Step 4: identity (C8). Let $\hat{n}$ be the unit outward normal vector to ∂T. Then we have

$$B_g\cdot\hat{n} = \frac{1}{|\xi|_g^2}\sqrt{|g|}\,(\xi\times_g\nabla_g\psi)\cdot\hat{n}, \tag{C 44}$$

since $\xi\cdot\hat{n} = 0$ by assumption. Now, for any vector field X and scalar function f we have

$$X\cdot\nabla f = \delta_{ij}X^i\delta^{jk}\partial_k f = \delta_i^k X^i\partial_k f = g_{im}g^{km}X^i\partial_k f = g_{im}X^i(\nabla_g f)^m = X\cdot_g\nabla_g f. \tag{C 45}$$

As a result, since ψ is assumed constant on the boundary, we can choose $\hat{n} = \nabla\psi/|\nabla\psi|$ on the boundary and a standard vector identity shows that $(\xi\times_g\nabla_g\psi)\cdot\hat{n} = 0$.

C.3. *Proof of lemma C.6*

Proof. Direct computation shows

$$|B_g|^2 = \frac{1}{|\xi|_g^4}\left[C(\psi)|\xi|^2 + 2C(\psi)\xi\cdot\nabla_g^\perp\psi + |\nabla_g^\perp\psi|^2\right]. \tag{C 46}$$

Since $\mathcal{L}_\xi g = 0$, from (C7) it follows that $\mathcal{L}_\xi B_g = 0$. Thus we have

$$\mathcal{L}_\xi\nabla_g^\perp\psi = \mathcal{L}_\xi\big(|\xi|_g^2 B_g - C(\psi)\xi\big) = 0. \tag{C 47}$$

Using $\mathcal{L}_\xi|\xi|_g^2 = 0$, $\mathcal{L}_\xi\xi = 0$, $\mathcal{L}_\xi\psi = 0$, $\mathcal{L}_\xi\nabla_g^\perp\psi = 0$ and $\mathcal{L}_\xi|\xi|^2 = (\mathcal{L}_\xi\delta)(\xi,\xi)$ completes the proof. □

Appendix D. Explicit expression for the generalized Grad–Shafranov equation

Fix a domain D in the $\{\Phi = 0\}$ half-plane and let ξ be a vector field whose orbits starting from D are all periodic (with possibly different period). Fix an arbitrary local coordinate system on D and extend it to a coordinate system (x_1, x_2, x_3) on the torus T defined in (1.25) by pulling back along the flow of ξ. In these coordinates we have $\xi \cdot \nabla f = (\partial/\partial x^3) f$. In this section we express the coefficients appearing in the generalized Grad–Shafranov equation (1.22) in these coordinates. The most complicated part of the calculation is contained in the following lemma.

LEMMA D.1. *Let g be an arbitrary metric on T and let (x_1, x_2, x_3) be a coordinate system on T as above. Then*

$$\begin{aligned} curl_g\xi \cdot_g \xi = |g|^{1/2} \big(& (\partial_1 g_{23} - \partial_2 g_{13})(g^{11}g^{22} - (g^{12})^2) \\ & +(\partial_3 g_{13} - \partial_1 g_{33})(g^{21}g^{23} - g^{22}g^{13}) + (\partial_3 g_{23} - \partial_2 g_{33})(g^{11}g^{23} - g^{12}g^{13})\big). \end{aligned} \tag{D 1}$$

Proof. We use the formula $\mathrm{curl}_g\xi \cdot_g \xi = i_\xi(\mathrm{curl}_g\xi)_\flat = i_\xi(*_g\, \mathrm{d}\alpha)$, where $*_g$ is the Hodge star in terms of g and $\alpha = \xi_\flat$ denotes the one-form which is dual to ξ with respect to g. Explicitly $\alpha = \alpha_i\, \mathrm{d}x^i = g_{ij}\xi^j\, \mathrm{d}x^i = g_{i3}\, \mathrm{d}x^i$. We now compute the terms on the right-hand side of (D1) explicitly and the main step is computing $*_g d\alpha$. Acting on two-forms, $*_g$ is defined by linearity and the rule

$$*_g\, (\mathrm{d}x^i \wedge \mathrm{d}x^j) = |g|^{1/2} g^{ik} g^{j\ell} \epsilon_{k\ell m}\, \mathrm{d}x^m, \tag{D 2}$$

where $|g| = \det g$ and $\epsilon_{k\ell m}$ is the Levi–Civita symbol.

Since in our coordinate system $\xi = \partial_3$ we have $i_\xi\, \mathrm{d}x^m = \mathrm{d}x^m(\partial_3) = \delta^{m3}$ and so

$$i_\xi *_g (\mathrm{d}x^i \wedge \mathrm{d}x^j) = |g|^{1/2} g^{ik} g^{j\ell} \epsilon_{k\ell 3}. \tag{D 3}$$

A straightforward calculation shows

$$\left.\begin{aligned} i_\xi *_g (\mathrm{d}x^1 \wedge \mathrm{d}x^2) &= |g|^{1/2}\left(g^{11}g^{22} - (g^{12})^2\right), \\ i_\xi *_g (\mathrm{d}x^2 \wedge \mathrm{d}x^3) &= |g|^{1/2}\left(g^{21}g^{32} - g^{22}g^{31}\right), \\ i_\xi *_g (\mathrm{d}x^1 \wedge \mathrm{d}x^3) &= |g|^{1/2}\left(g^{11}g^{32} - g^{12}g^{31}\right). \end{aligned}\right\} \tag{D 4}$$

Since $\mathrm{d}\alpha = (\partial_1\alpha_2 - \partial_2\alpha_1)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 + (\partial_1\alpha_3 - \partial_3\alpha_1)\, \mathrm{d}x^1 \wedge \mathrm{d}x^3 + (\partial_2\alpha_3 - \partial_3\alpha_2)\, \mathrm{d}x^2 \wedge \mathrm{d}x^3$, we have

$$\begin{aligned} \mathrm{curl}_g\xi \cdot_g \xi &= (\partial_1\alpha_2 - \partial_2\alpha_1) i_\xi *_g (\mathrm{d}x^1 \wedge \mathrm{d}x^2) - \partial_1\alpha_3 i_\xi *_g (\mathrm{d}x^1 \wedge \mathrm{d}x^3) + \partial_2\alpha_3 i_\xi *_g (\mathrm{d}x^2 \wedge \mathrm{d}x^3) \\ &= |g|^{1/2}\big((\partial_1\alpha_2 - \partial_2\alpha_1)(g^{11}g^{22} - (g^{12})^2) + (\partial_3\alpha_1 - \partial_1\alpha_3)(g^{21}g^{23} - g^{22}g^{13}) \\ &\quad + (\partial_3\alpha_2 - \partial_2\alpha_3)(g^{11}g^{23} - g^{12}g^{13})\big) \end{aligned} \tag{D 5}$$

which gives (D1) since $\alpha_i = g_{i3}$. □

The next lemma follows from the previous one and (C15) after noting that $|\xi|^2 = g(\xi, \xi) = g_{33}$.

LEMMA D.2. *Fix a vector field* ξ *and a metric* g *with* $\mathcal{L}_\xi g = 0$. *Let* (x_1, x_2, x_3) *be any coordinate system as in the statement of the previous lemma. Then* (C15) *with* $f = 0$ *takes the form*

$$\sum_{i,j=1}^{3} a^{ij}_{\xi,g}\partial_i\partial_j\psi + \sum_{i=1}^{3} b^{i}_{\xi,g}\partial_i\psi + G_{\xi,g}(x_1, x_2, x_3, C, \psi) + \frac{P'(\psi)}{\sqrt{|g|}} = 0, \tag{D 6}$$

where

$$\left.\begin{aligned} a^{ij}_{\xi,g} &= \frac{\sqrt{|g|}}{g_{33}}g^{ij}, \quad b^{i}_{\xi,g} = \sum_{j=1,2}\frac{\sqrt{|g|}}{g_{33}}\partial_j\left(\sqrt{|g|}g^{ij}\right) + g^{ij}\partial_j\left(\frac{\sqrt{|g|}}{g_{33}}\right), \\ G_{\xi,g} &= \frac{C(\psi)}{g_{33}}\left(\frac{C'(\psi)}{\sqrt{|g|}} - \xi\cdot_g curl_g\xi\right), \end{aligned}\right\} \tag{D 7}$$

where $\xi\cdot_g curl_g\xi$ *is given by* (D1).

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