Commun. Math. Phys. 385, 521–563 (2021)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-021-04048-4

Communications in Mathematical Physics

Flexibility and Rigidity in Steady Fluid Motion

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Received: 18 July 2020 / Accepted: 20 February 2021
Published online: 11 March 2021 –

Abstract: Flexibility and rigidity properties of steady (time-independent) solutions of the Euler, Boussinesq and Magnetohydrostatic equations are investigated. Specifically, certain Liouville-type theorems are established which show that suitable steady solutions with no stagnation points occupying a two-dimensional periodic channel, or axisymmetric solutions in (hollowed out) cylinder, must have certain structural symmetries. It is additionally shown that such solutions can be deformed to occupy domains which are themselves small perturbations of the base domain. As application of the general scheme, Arnol'd stable solutions are shown to be structurally stable.

1. Introduction

In this paper, we address two fundamental questions pertaining to steady configurations of fluid motion (modeled here as solutions to the two-dimensional Euler and Boussinesq equations or the three-dimensional Euler equations or Magnetohydrostatic (MHS) equations). Specifically,

- *Rigidity:* Given a domain D_0 with symmetry, to what extent must steady fluid states u_0 conform to the symmetries of the domain?
- *Flexibility:* Given domains D_0, D which are "close" in some sense, and a solution u_0 of a steady fluid equation in D_0, can one find a steady solution u in D nearby u_0?

To study the rigidity, we show that steady fluid configurations with no stagnation points (non-vanishing velocity) confined to (topologically) annular regions have the following special property: any quantity which is steadily transported (such as vorticity in two-dimensions or temperature in the Boussinesq fluid) can be constructed as a 'nice' function of the streamfunction. This fact is then exploited by recognizing that, as a consequence, the streamfunction of such a flow must solve a certain nonlinear elliptic equation. As such, Liouville theorems are used to constrain the possible behavior of all sufficiently regular solutions. For the two-dimensional Euler equations on the periodic channel, this is the result of Hamel and Nadirashvili [18,19] that all steady flows without stagnation points are shears. A similar statement can be made for a Boussinesq fluid with a certain types of stratification profiles. For stationary three-dimensional axisymmetric Euler (e.g. pipe flow), we show that non-degenerate solutions must be purely radial for a large class of pressure profiles. That there should be a strong form of rigidity for stationary solutions of 3d Euler was already envisioned by Harold Grad [15] who conjectured that all smooth solutions (with a certain topological structure) must conform to a symmetry.

To study the flexibility, we modify an idea introduced by Wirosoetisno and Vanneste [28]. To illustrate the general scheme we note that steady states u_0 which are tangent to the boundary can be constructed from a scalar stream function $\psi_0 : D_0 \to \mathbb{R}$ which solves

$$\begin{aligned} \mathcal{L}\psi_0 &= \mathcal{N}_0(\psi_0), \quad &&\text{in } D_0, \\ \psi_0 &= (\text{const.}) \quad &&\text{on } \partial D_0, \end{aligned}$$

where $\mathcal{L}$ is some linear elliptic operator and $\mathcal{N}_0$ is some nonlinear function. We seek a solution in a nearby domain D by imposing that the stream function ψ have the form

$$\psi = \psi_0 \circ \gamma^{-1}, \tag{1.1}$$

where $\gamma : D_0 \to D$ is a diffeomorphism to be determined. We furthermore require that ψ satisfies

$$\mathcal{L}\psi = \mathcal{N}(\psi), \quad \text{in } D, \tag{1.2}$$

for a function $\mathcal{N} = \mathcal{N}_0 + \chi$ with χ conforming to the structure of the steady equations to be determined. Note that by construction, we automatically have $\psi = (\text{const.})$ on ∂D since $\psi_0 = (\text{const.})$ on ∂D_0 and $\gamma : \partial D_0 \to \partial D$. Thus, if such a diffeomorphism can be found, ψ defines a stream function for a steady solution in D which is tangent to the boundary.

Having fixed the form of ψ by (1.1), we regard (1.2) as an equation for the diffeomorphism γ. To solve it, we transform it into an equation in D_0 by composing with γ,

$$\mathcal{L}_\gamma \psi_0 = \mathcal{N}(\psi_0), \quad \text{where} \quad \mathcal{L}_\gamma f := \mathcal{L}\big(f \circ \gamma^{-1}\big) \circ \gamma. \tag{1.3}$$

Under certain conditions on the original domain D_0 and the base steady state ψ_0, the equation (1.3) becomes a non-degenerate, nonlinear elliptic equation for the components of the map γ which can be solved provided the deformations are sufficiently small. This scheme to produce a γ has an infinite-dimensional degree of freedom which can be removed by fixing the Jacobian of the diffeomorphism $\rho = \det \nabla\gamma$. This steady state will solve (1.2) potentially with a modified nonlinearity $\mathcal{N}$ which is completely determined in the construction of the map γ with a given ρ.

Theorem 3.1 in § 3 is our main result in this direction. We illustrate its consequences in the following three cases

- *2d Euler* for domains close to the periodic channel (see Fig. 1) and Arnol'd stable steady states on compact Riemannian manifolds.
- *2d Boussinesq* for domains close to the periodic channel (see Fig. 1).
- *3d axisymmetric Euler* for domains close to the cylinder (see Fig. 2).

We now describe these settings in greater detail and state our Theorems for each case before proceeding to the proofs.

Two-dimensional Euler Equations

Given a bounded domain $D_0 \subset \mathbb{R}^2$ with smooth boundary, a steady solution of $2d$ Euler satisfies

$$
\begin{aligned}
u_0 \cdot \nabla u_0 &= -\nabla p_0, && \text{in } D_0, && (1.4)\\
\nabla \cdot u_0 &= 0, && \text{in } D_0, && (1.5)\\
u_0 \cdot \hat{n} &= 0, && \text{on } \partial D_0. && (1.6)
\end{aligned}
$$

As a consequence of incompressibility, the solutions u_0 to the above can be constructed from a stream function ψ_0 via the formula $u_0 = \nabla^\perp \psi_0$ with $\nabla^\perp = (-\partial_2, \partial_1)$. Since $u_0 = \nabla^\perp \psi_0$, if the velocity is tangent to the boundary then ψ_0 must be constant along ∂D_0. We consider here steady solutions with additional structure, namely that the vorticity ω_0 is a Lipschitz function of the stream function ψ_0 through $\omega_0 = F_0(\psi_0)$. As it turns out (see Lemma 2.1 below), all sufficiently regular flows in annular domains and without stagnation points have vorticity satisfying this property for some F_0. Together with $\omega_0 = \nabla^\perp \cdot u_0$ this means ψ_0 satisfies

$$
\begin{aligned}
\Delta \psi_0 &= F_0(\psi_0), && \text{in } D_0, && (1.7)\\
\psi_0 &= (\text{const.}), && \text{on } \partial D_0. && (1.8)
\end{aligned}
$$

On the other hand, clearly any solution of the above for a Lipschitz F_0 is the stream function of a steady solution to 2d Euler which is tangent to the boundary.

When the domain D_0 is a channel

$$
\begin{aligned}
D_0 &= \{(y_1, y_2) \mid y_1 \in \mathbb{T},\ y_2 \in [0, 1]\},\\
\partial D_0^{\text{top}} &= \{y_2 = 1\}, \quad \partial D_0^{\text{bot}} = \{y_2 = 0\},
\end{aligned} \tag{1.9}
$$

solutions of the Euler equations exhibit a certain remarkable rigidity, Theorem 1.1 of [18]:[1]

Theorem 1.1 (Rigidity of non-stagnant Euler flows). *Let D_0 be a periodic channel given by* (1.9) *and suppose that $u_0 : D_0 \to \mathbb{R}^2$ be a $C^2(D_0)$ solution of* (1.4)–(1.6) *with the property that* $\inf_{D_0} u_0 > 0$. *Then u_0 is a shear flow, namely $u_0(y_1, y_2) = (v(y_2), 0)$ for some scalar function $v(y_2)$.*

Theorem 1.1 is an example of a Liouville theorem for solutions of the incompressible Euler equations. It shows that *any* smooth steady solution of the Euler equations in the channel which never vanishes must be a shear flow, isolating such configurations from non-shear steady states. It should be emphasized that there are many examples of non-trivial flows with stagnation points which are non-shear (e.g. cellular flows). In fact, Lin and Zeng [22] shows that there exist Cat's–eye vortices arbitrarily close to Couette flow $u_0(y) = (y, 0)$ in the H^s, $s < 3/2$ topology. Similar results to Theorem 1.1 hold when the domain is the annulus or the disk (under some additional conditions on the solution) [19]. We remark that the very interesting recent work of Coti Zelati, Elgindi, and Widmayer [7] shows that the assumption of non-degeneracy is not always necessary

[1] We remark that Theorem 1.1, in effect, combines the results of [18,19]. In the former, they establish the Theorem on an infinite strip for velocities with non-trivial inflow/outflow and in the latter they show solutions in annular domains must be radial (i.e. solutions in periodic channels must be shears). See also the interesting complementary work [13] which establishes that solutions with *single-signed* vorticity (possibly possessing stagnation points) on $\mathbb{R}^2$ must be radial.

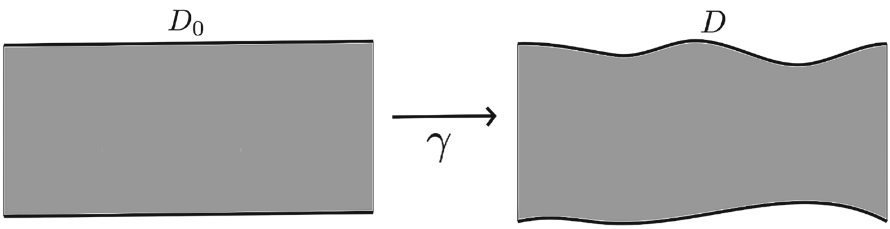

Fig. 1. Deformation of periodic channel D_0 by γ

for such a Liouville theorem by establishing a similar rigidity of 2d Euler solutions near Poiseuille flow $u_0(y) = (y^2 - c, 0)$ for $c \geq 0$ which stagnates at $y = \pm\sqrt{c}$.

In light of the rigidity result of [18], we show in Theorem 1.2 that we can perturb away from *any non-vanishing solution* of the two-dimensional Euler equations in the channel u_0 to domains

$$\begin{aligned} D &= \{(x_1, x_2) \mid x_1 \in \mathbb{T},\ b_0(x_1) \leq x_2 \leq 1 + b_1(x_1)\}, \\ \partial D^{\text{top}} &= \{x_2 = 1 + b_1(x_1)\}, \quad \partial D^{\text{bot}} = \{x_2 = b_0(x_1)\}, \end{aligned} \tag{1.10}$$

for suitably small b_0, b_1 (see Fig. 1) and obtain a steady solution u in D:

$$\begin{aligned} u \cdot \nabla u &= -\nabla p, && \text{in } D, \\ \nabla \cdot u &= 0, && \text{in } D, \\ u \cdot \hat{n} &= 0, && \text{on } \partial D. \end{aligned}$$

Theorem 1.2 (Flexibility of non-stagnant Euler flows). *Let D_0 be defined by* (1.9) *and D be defined by* (1.10)*. Suppose $\psi_0 : D_0 \to \mathbb{R}$ with $U_0 := \inf_{D_0} |\nabla\psi_0| > 0$ and $\psi_0 \in C^{k,\alpha}(D_0)$ for some $\alpha > 0$, $k \geq 3$ such that $u_0 = \nabla^\perp\psi_0$ satisfies* (1.5)–(1.6)*. Then there is an $F_0 \in C^{k-2,\alpha}(\mathbb{R})$ such that ψ_0 satisfies* (1.7)–(1.8) *and constants $\varepsilon_1, \varepsilon_2$ depending only on U_0, D_0, F_0 and $\|\psi_0\|_{C^{k,\alpha}}$ such that if $b_0, b_1 : \mathbb{R} \to \mathbb{R}$ and $\rho : D_0 \to \mathbb{R}$ with $\int_{D_0} \rho = \mathrm{Vol}(D)$ satisfy*

$$\begin{aligned} \|b_0\|_{C^{k,\alpha}(\mathbb{R})} + \|b_1\|_{C^{k,\alpha}(\mathbb{R})} &\leq \varepsilon_1, \\ \|1 - \rho\|_{C^{k,\alpha}(D_0)} &\leq \varepsilon_2, \end{aligned}$$

there is a diffeomorphism $\gamma : D_0 \to D$ with Jacobian $\det(\nabla\gamma) = \rho$, and a function $F : \mathbb{R} \to \mathbb{R}$ close in $C^{k-2,\alpha}$ to F_0 in so that $\psi = \psi_0 \circ \gamma^{-1} \in C^{k,\alpha}(D)$ and ψ satisfies $\Delta\psi = F(\psi)$. Thus, $u = \nabla^\perp\psi$ is an Euler solution in D nearby u_0.

This theorem is a generalization of Theorem 1 of Wirosoetisno and Vanneste [28] to include non-volume preserving maps γ and follows from a much more general theorem which we prove, Theorem 3.1. The freedom of choosing the Jacobian of the map gives an additional mechanism to reach nearby other steady states. When b_0 and b_1 are zero, the perturbed domain is again a channel and the solution *must* be a shear flow, which is a consequence of the Theorem of [18] discussed above. Nevertheless, due to the fact that the Jacobian is an arbitrary function near unity, our procedure picks out different solutions, allowing us to “slide" along the space of shear flows in the channel. Note also that radial solutions on the annulus or the disk can also be deformed. In the case of a simply connected domain such as the disk, the base state must have a stagnation point and Theorem 1.2 applies provided that ψ_0 satisfies the non-degeneracy Hypothesis **(H2)** below. We finally we make some remarks about the case where Hypothesis **(H2)** is violated. In this case, if one has that the linearized operator $\Delta - F_0'(\psi_0)$ has a trivial kernel, then a standard implicit function theorem argument produces 'nearby' stationary states for 'nearby' vorticity profiles F. This condition is implied by Arnol'd stability (see discussion below) and it holds also, for example, for Couette flow whose vorticity is constant so that $F_0' = 0$. On the other hand, this argument does not give much information on the structure of the solution, whereas Theorem 1.2 and the other below allow one to understand and control the geometry of the streamlines to a certain extent.

A different class of flows which display a remarkably general form of rigidity and flexibility on any domains with a symmetry are so-called Arnol'd stable steady states. Recall that a stationary state on a domain $\Omega \subset \mathbb{R}^2$ is called Arnol'd stable if the vorticity of an Euler solution $\omega = F(\psi)$ satisfies either of the following two conditions

$$-\lambda_1 < F'(\psi) < 0, \quad \text{or} \quad 0 < F'(\psi) < \infty \tag{1.11}$$

where $\lambda_1 := \lambda_1(\Omega) > 0$ is the smallest eigenvalue of $-\Delta$ in Ω. See [2] or Theorem 4.3 of [3]. The above two ranges are referred to type I and type II Arnol'd stability conditions. These conditions ensure that the steady state is either a minimum or a maximum of the energy (the action) for a fixed vorticity distribution and guarantee that such states are orbitally stable in the topology of $L^2(\Omega)$ of the vorticity.

To emphasize the generality of what follows, let (M, g) be a two-dimensional Riemannian manifold with smooth boundary ∂M and let ξ be a Killing field for g. Suppose that ξ is tangent to ∂M. Consider a solution ψ of

$$\Delta_g \psi = F(\psi), \qquad \text{in } M, \tag{1.12}$$
$$\psi = (\text{const.}), \quad \text{on } \partial M, \tag{1.13}$$

where Δ_g is the Laplace–Beltrami operator on M. The velocity constructed from ψ by

$$u^i = -\sqrt{\det g}\, g^{ij} \epsilon_{jk} g^{kl} \frac{\partial \psi}{\partial x^l} =: \nabla_g^\perp \psi$$

is automatically a solution of the Euler equation on M with vorticity $\omega = \frac{\epsilon_{ij}}{\sqrt{\det g}} \frac{\partial}{\partial x^i}(g_{jk} u^k) = F(\psi)$. In the language of differential forms $u = (*_g d\psi)^\sharp$ where $*_g$ and $\sharp$ denote the Hodge star and musical isomorphism associated to the metric g and d denotes the differential. Since ξ is Killing for g, the commutator of the Lie derivative $\mathcal{L}_\xi$ with the Laplace–Beltrami operator vanishes $[\mathcal{L}_\xi, \Delta_g] = 0$ (applied to tensors of any rank). Moreover, since ξ is tangent to $\partial\Omega$, on which ψ takes constant values, we have that $\mathcal{L}_\xi \psi = 0$ on $\partial\Omega$. Thus, differentiating (1.12)–(1.13) we obtain the equation

$$\Big(\Delta_g - F'(\psi)\Big)\mathcal{L}_\xi \psi = 0, \qquad \text{in } M,$$
$$\mathcal{L}_\xi \psi = 0, \qquad \text{on } \partial M.$$

Clearly if the operator $\Delta_g - F'(\psi)$ has a trivial kernel in H_0^1, then $\mathcal{L}_\xi \psi = 0$. A sufficient condition to ensure this is that $F'(\psi) > -\lambda_1$ where λ_1 is the first eigenvalue of $-\Delta_g$ on M. Both type I and type II Arnol'd stability conditions ensure this. Thus, we obtain

Proposition 1.1 (Rigidity of Arnol'd stable states). *Let (M, g) be a compact two-dimensional Riemannian manifold with smooth boundary ∂M and let ξ be a Killing field for g. Suppose that ξ is tangent to ∂M. Let $u : M \to \mathbb{R}^2$ be a $u = \nabla_g^\perp \psi \in C^2(M)$ Arnol'd stable state. Then $\mathcal{L}_\xi \psi = 0$.*

In some simple cases, Proposition 1.1 implies

- on the periodic channel with $\xi = e_{y_1}$ and $\zeta = e_{y_2}$, all Arnol'd stable stationary solutions are shears $u = v(y_2)e_{y_1}$.
- on the disk (or annulus), with $\xi = e_\theta$ and $\zeta = e_r$, all Arnol'd stable stationary solutions are radial $u = v(r)e_\theta$.
- on a spherical cap[2] with $\xi = e_\lambda$ and $\zeta = e_\phi$, all Arnol'd stable stationary solutions are zonal (functions of latitude) $u = v(\phi)e_\lambda$.
- on manifolds without boundary possessing two transverse Killing fields (e.g. the two-torus or the sphere), there can be no Arnol'd stable steady states (see [29]).

We remark that in fact the statement for the periodic channel or annulus hold whether or not the state is Arnol'd stable, see [18,19], provided that the flow has no stagnation points.

Thus, Proposition 1.1 reveals a strong form of rigidity of Arnol'd stable steady states. However, we also show that are also flexible in the sense that nearby stable steady states exist on wrinkled domains (slight changes of the background metric) with wiggled boundaries.

Specifically, consider a steady solution on $M_0 \subset \mathbb{R}^2$ satisfying the following hypotheses:

(H1) The vorticity $\omega_0 = F_0(\psi_0)$ satisfies $F_0'(\psi_0) > -\lambda_1(M_0)$.
(H2) There is a constant $c_{\psi_0} > 0$ such that for all $c \in \mathrm{im}(\psi_0)$ we have

$$\oint_{\{\psi_0 = c\}} \frac{\mathrm{d}\ell}{|\nabla \psi_0|} \leq \frac{1}{c_{\psi_0}}. \tag{1.14}$$

Hypothesis **(H1)** is ensured for type I and II Arnol'd states by (1.11). Hypothesis **(H2)** ensures that the period of rotation of fluid parcels along streamlines (left-hand-side of (1.14)) is bounded and is automatically satisfied for any base flow without stagnation points on annular domains and it holds on simply connected domains if there is non-vanishing vorticity $F_0 \neq 0$ at the critical point. We call flows satisfying **(H2)** *non-degenerate*. With these, our result is:

Theorem 1.3 (Structural stability of Arnol'd stable states). *Let $\alpha \in (0, 1)$ and $k \geq 3$. Let (M_0, g_0) be a compact two-dimensional Riemannian submanifold of $\mathbb{R}^2$ with $C^{k,\alpha}$ boundary. Suppose $\psi_0 \in C^{k,\alpha}(M_0)$ is a non-degenerate, Arnol'd stable steady state on (M_0, g_0) with vorticity profile $F_0 \in C^{k-2,\alpha}(\mathbb{R})$. Then there are constants $\varepsilon_1, \varepsilon_2, \varepsilon_3$ depending only on M_0, F_0, g_0 and $\|\psi_0\|_{C^{k,\alpha}}$ such that if (M, g) is a compact Riemannian manifold and $\rho : M_0 \to \mathbb{R}$ with $\int_{D_0} \rho \, \mathrm{dvol}_{g_0} = \mathrm{Vol}_g(D)$ and $g : M_0 \to \mathbb{R}^2$ satisfy*

$$\begin{aligned} \|\partial M - \partial M_0\|_{C^{k,\alpha}} &\leq \varepsilon_1, \\ \|\rho - 1\|_{C^{k,\alpha}(M_0)} &\leq \varepsilon_2, \\ \|g - g_0\|_{C^{k,\alpha}(M_0)} &\leq \varepsilon_3, \end{aligned}$$

there is a diffeomorphism $\gamma : M_0 \to M$ with Jacobian $\det(\nabla\gamma) = \rho$, and a function $F : \mathbb{R} \to \mathbb{R}$ close in $C^{k-2,\alpha}$ to F_0 so that $\psi = \psi_0 \circ \gamma^{-1} \in C^{k,\alpha}(M)$ and ψ satisfies (1.12)–(1.13) on (M, g). Thus, $u = \nabla_g^\perp \psi$ is a non-degenerate, Arnol'd stable steady Euler solution on (M, g) nearby u_0 whose vorticity $\omega = F(\psi)$.

[2] On the cap of a sphere of radius R, we use spherical coordinates $x = (R, \lambda, \phi)$, where $\lambda \in [-\pi, \pi]$ is longitude and $\phi \in [-\pi/2, \pi/2]$ is latitude, with the poles at $\phi = \pm\pi/2$.

We remark that hypothesis necessary to run Theorem 1.3, Hypothesis **(H1)**, is weaker than Arnol'd stability since it allows the deformation of states of constant vorticity $F'(\psi_0) = 0$. Theorem 1.3 shows that such steady states are non-isolated from other stable stationary states, even fixing the domain and metric, since the Jacobian ρ can be freely chosen. A very interesting open issue is whether any time-independent solution of the two-dimensional Euler equation can be isolated from other steady solutions. Note that that the deformation scheme can be repeated to deformation between two "far apart" domains provided along the path of steady states Hypothesis **(H1)** and **(H2)** remain true. Finally, as discussed above, if one is not interested in controlling aspects of the streamline geometry of the new steady states, then an implicit function argument can be used to dispense with the non-degeneracy hypothesis **(H2)** and allow the construction of solutions with nearby vorticity profiles F. See, for example, the work of Choffrut and Sverák [8].

Two-dimensional Boussinesq equations

Given a domain $D_0 \subset \mathbb{R}^2$ with smooth boundary, steady states of the Boussinesq system satisfy

$$\begin{aligned} u_0 \cdot \nabla u_0 &= -\nabla p_0 + \theta_0 e_2, && \text{in } D_0, \\ u_0 \cdot \nabla \theta_0 &= 0, && \text{in } D_0, \\ \nabla \cdot u_0 &= 0, && \text{in } D_0, \qquad (1.15) \\ u_0 \cdot \hat{n} &= 0, && \text{on } \partial D_0. \qquad (1.16) \end{aligned}$$

Introducing the vorticity $\omega_0 = \nabla^\perp \cdot u_0$, Eq. (1.15) can be written as

$$\begin{aligned} \omega_0 u_0^\perp &= \nabla P_0 + \theta_0 e_2, \\ -P_0 &:= p_0 + \frac{1}{2}|u_0|^2. \qquad (1.17) \end{aligned}$$

Letting $u_0 = \nabla^\perp \psi_0$, $\omega_0 = \Delta \psi_0$ and $u_0^\perp = -\nabla \psi_0$ Eq. (1.17) reads

$$-\Delta \psi_0 \nabla \psi_0 = \nabla P_0 + \theta_0 e_2.$$

The Grad–Shafranov-like equation (analogous to Eqs. (1.7)–(1.8) of 2d Euler) is obtained by assuming that θ_0, P_0 can be constructed from the stream function, in the sense that

$$\theta_0(y_1, y_2) = \Theta_0(\psi_0(y_1, y_2)), \qquad (1.18)$$
$$P_0(y_1, y_2) = -y_2 \Theta_0(\psi_0(y_1, y_2)) - G_0(\psi_0(y_1, y_2)), \qquad (1.19)$$

for smooth functions G_0, Θ_0. This again turns out to be completely general provided that u_0 never vanishes, see Lemma 2.1. In this case, provided that G_0, Θ_0 are sufficiently smooth the stream function must satisfy

$$\Delta \psi_0 - y_2 \Theta_0'(\psi_0) - G_0'(\psi_0) = 0, \qquad \text{in } D_0, \qquad (1.20)$$
$$\psi_0 = (\text{const.}), \qquad \text{on } \partial D_0. \qquad (1.21)$$

Given a solution ψ_0 to (1.20)–(1.21), the function $u_0 = \nabla^\perp \psi_0$ solves (1.15)–(1.16) with temperature θ_0 determined by (1.18) and pressure P_0 determined by (1.19).

As for 2d Euler, if D_0 is a periodic channel the Boussinesq equations have a certain rigidity. Specifically, *all* smooth steady states with nowhere vanishing velocity must be shear flows and the temperature and pressures must satisfy the equations of state (1.18), (1.19) for some Lipschitz functions Θ_0 and G_0:

Theorem 1.4 (Rigidity of non-stagnant Boussinesq flows). *Let D_0 be a periodic channel given by (1.9) and suppose that $u_0 : D_0 \to \mathbb{R}^2$ and $\theta_0 : D_0 \to \mathbb{R}$ be a $C^2(D_0)$ solution of (1.15)–(1.16) with $\inf_{D_0} u_0 > 0$. Then there exists Lipschitz functions $\Theta_0, G_0 : \mathbb{R} \to \mathbb{R}$ such that (1.18), (1.19) hold and if furthermore*

$$\Theta_0'(\psi_0) \leq 0,$$

then u_0 is a shear flow, namely $u_0(x, y) = (v(y), 0)$ for some scalar function $v(y)$.

We next establish the flexibility of Boussinesq solutions by proving the existence of steady solutions on the channel to solutions on nearby domains. Fix D_0 to be a channel defined by (1.9) and fix functions ψ_0, Θ_0, G_0 satisfying (1.20)–(1.21) on D_0. Given a function Θ, we then deform ψ_0 to obtain a steady solution ψ to (1.20) defined on D given by (1.10) (for suitably small b_0, b_1) with temperature profile Θ and some vorticity profile G. As a result, $u = \nabla^\perp \psi$ and $\theta = \Theta(\psi)$ satisfy the steady Boussinesq equations on D:

$$\begin{aligned} u \cdot \nabla u &= -\nabla p + \theta e_2, && \text{in } D, \\ u \cdot \nabla \theta &= 0, && \text{in } D, \\ \nabla \cdot u &= 0, && \text{in } D, \\ u \cdot \hat{n} &= 0, && \text{on } \partial D. \end{aligned}$$

Specifically, we prove

Theorem 1.5 (Flexibility of non-stagnant Boussinesq flows). *Let D_0 be defined by (1.9) and D be defined by (1.10). Suppose $\psi_0 : [0, 1] \to \mathbb{R}$ is a shear with $\psi_0 \in C^{k,\alpha}(D_0)$ for some $\alpha > 0$, $k \geq 3$ satisfying (1.20)–(1.21) for a given $G_0, \Theta_0 \in C^{k-1,\alpha}(\mathbb{R})$ having no stagnation points $U_0 := \inf_{D_0} |\nabla \psi_0| > 0$. Then there are constants $\varepsilon_1, \varepsilon_2, \varepsilon_3$ depending only on U_0, D_0, Θ_0, G_0 and $\|\psi_0\|_{C^{k,\alpha}}$ such that if $b_0, b_1 : \mathbb{R} \to \mathbb{R}$, $\Theta : \mathbb{R} \to \mathbb{R}$ and $\rho : D_0 \to \mathbb{R}$ with $\int_{D_0} \rho = \mathrm{Vol}(D)$ satisfy*

$$\begin{aligned} \|b_0\|_{C^{k,\alpha}(\mathbb{R})} + \|b_1\|_{C^{k,\alpha}(\mathbb{R})} &\leq \varepsilon_1, \\ \|1 - \rho\|_{C^{k,\alpha}(D_0)} &\leq \varepsilon_2, \\ \|\Theta_0' - \Theta'\|_{C^{k-2,\alpha}(\mathbb{R})} &\leq \varepsilon_3, \end{aligned}$$

there is a diffeomorphism $\gamma : D_0 \to D$ with Jacobian $\det(\nabla\gamma) = \rho$, and a function $G : \mathbb{R} \to \mathbb{R}$ close to G_0 so that $\psi = \psi_0 \circ \gamma^{-1} \in C^{k,\alpha}(D)$ and ψ satisfies

$$\Delta\psi - x_2 \Theta'(\psi) - G'(\psi) = 0 \quad in \quad D.$$

Thus, $u = \nabla^\perp \psi$ and $\theta = \Theta(\psi)$ defines a Boussinesq solution in D nearby $u_0 = \nabla^\perp \psi_0$, $\theta_0 = \Theta_0(\psi_0)$.

Note that, in light of Theorem 1.4, if the base state ψ_0 has no stagnation points and $\Theta_0' \leq 0$, all smooth steady states are shears and so the assumption that ψ_0 is a shear is automatic.

Three-dimensional axisymmetric Euler

Let $T_0 \subset \mathbb{R}^3$ be a domain with smooth boundary. The three-dimensional steady Euler equations (or, equivalently, the three-dimensional steady Magnetohydrostatic equations) read

$$\begin{aligned} \omega_0 \times u_0 &= \nabla P_0, \quad \text{in } T_0, \\ \nabla \cdot u_0 &= 0, \quad \text{in } T_0, \qquad (1.22) \\ u_0 \cdot \hat{n} &= 0, \quad \text{on } \partial T_0, \qquad (1.23) \end{aligned}$$

where $\omega_0 = \nabla \times u_0$ denotes the vorticity and P_0 denotes the pressure.

The issue of existence of solutions to (1.22)–(1.23) is of fundamental importance to the problem of magnetic confinement fusion. In particular, one strategy to achieve fusion is to drive a plasma contained in an axisymmetric toroidal domain (tokamak) towards an equilibrium configuration which is (ideally) stable and enjoys certain properties that make it suitable for confining particles which, to first approximation, travel along its magnetic field lines well inside the domain. Once such a suitable steady state is identified, the control of the plasma to remain near this state is a very important and challenging engineering problem. However, as Grad remarked in [15], "Almost all stability analyses are predicated on the existence of an equilibrium state that is then subject to perturbation. But a more primitive reason than instability for lack of confinement is the absence of an appropriate equilibrium state." Grad goes on to write that there are exactly four known symmetries for which smooth toroidal plasma equilibria with nested magnetic surfaces can exist. These are: two-dimensional, axial, helical and reflection symmetries. He asserts in [16] that "no additional exceptions have arisen since 1967, when it was conjectured that toroidal existence... of smooth solutions with simple nested surfaces admits only these . . . exceptions. . . . The proper formulation of the nonexistence statement is that, other than stated symmetric exceptions, there are no families of solutions depending smoothly on a parameter." We formalize this statement as a rigidity property of solutions of three-dimensional Euler (Magnetohydrostatics):

Conjecture 1 (H. Grad [15,16]). *Any non-isolated and non-vanishing (away from the "magnetic axis") smooth unforced MHS equilibrium on a (topologically toroidal or cylindrical) domain $T \subset \mathbb{R}^3$ that has a pressure possessing nested level sets which foliate T has either plane-reflection, axial or helical symmetry.*

By an *isolated stationary state*, we mean that, in some suitably topology, there are no nearby steady states aside from those which correspond to a trivial rescaling or translation of the original. It is possible that no such object exists. The qualifier is included to make precise Grad's assertion that the conjecture apply to solutions which appear in continuous"families".

Complementary to Grad's conjecture, here we prove that solutions with symmetry can also be severely restricted to conform to a stronger form of symmetry. Specifically, we consider periodic-in-z solutions in the (hollowed out) axisymmetric cylinder (see right half of Fig. 2)

$$T_0 = D_0 \times \mathbb{T}, \qquad D_0 = \{(r, z) \in [1/2, 1] \times \mathbb{T}\}, \tag{1.24}$$

which are axisymmetric in the sense that $u_0 = u_0(r, z)$. We remark that solutions with this symmetry on this domain are not suitable for the confinement of a plasma in a tokamak and instead describe steady flow in a pipe. To find solutions with this symmetry, we make the ansatz

$$u_0 = \frac{1}{r} e_\theta \times \nabla \psi_0 + \frac{1}{r} C_0(\psi_0) e_\theta \tag{1.25}$$

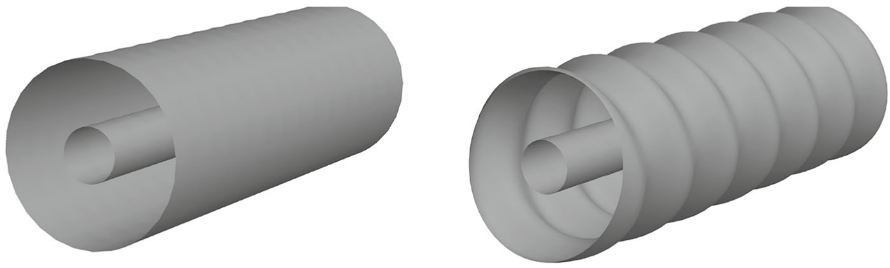

Fig. 2. Axisymmetric deformation of the cylinder (corrugated pipe)

for a function $C_0 : \mathbb{R} \to \mathbb{R}$ and $\psi_0 = \psi_0(r, z)$ is to be determined. In fact, by the results in [5], any sufficiently smooth solution to (1.22)–(1.23) possessing symmetry in the θ direction and $\mathrm{curl}(u_0 \times e_\theta) = 0$ and possessing a nowhere vanishing pressure gradient is necessarily of the form (1.25). If we seek a solution with pressure of the form $P_0 = \Pi_0(\psi_0)$ for some profile function $\Pi_0 : \mathbb{R} \to \mathbb{R}$, then (1.25) satisfies (1.22)–(1.23) provided ψ_0 satisfies

$$\frac{\partial^2}{\partial r^2}\psi_0 + \frac{\partial^2}{\partial z^2}\psi_0 - \frac{1}{r}\frac{\partial}{\partial r}\psi_0 = -r^2\Pi_0'(\psi_0) + C_0C_0'(\psi_0), \quad \text{in } D_0, \tag{1.26}$$

$$\psi_0 = (\text{const.}), \quad \text{on } \partial D_0. \tag{1.27}$$

The Eq. (1.26) is known in plasma physics as the Grad–Shafranov equation [14,24].[3]

Here we prove that all solutions whose pressure has a certain property must be radial.

Theorem 1.6 (Rigidity of axisymmetric pipe flows). *Let D_0 be given by* (1.24). *Suppose $\Pi_0, C_0 : \mathbb{R} \to \mathbb{R}$ are Lipchitz functions and that $\psi_0 : D_0 \to \mathbb{R}$ is $C^2(D_0)$ solution of the Grad–Shafranov equation* (1.26)–(1.27) *with* $\inf_{D_0} |\partial_r \psi_0| > 0$. *If furthermore Π_0 satisfies*

$$\Pi_0'(\psi_0) \geq 0, \tag{1.28}$$

then ψ_0 is radial, i.e. $\psi_0(r, z) = \psi_0(r)$.

Physically, Theorem 1.6 says that in order to support some non-trivial structure in pipe flow, the pressure cannot satisfy (1.28). It is conceivable that this has some bearing for identifying good flow configurations from the point of view of drag reduction.

Liouville theorems constraining axisymmetric solutions of three-dimensional fluid equations have appeared previously in the work of Shvydkoy for Euler on $\mathbb{R}^3$ (§5 of [25]) and by Koch–Nadirashvili–Seregin–Sverák [21] for ancient solutions of Navier-Stokes on $\mathbb{R}^3$. Establishing similar rigidity results for the full three-dimensional problem outside of symmetry—which is necessary to address Grad's conjecture—seems to be out of reach of existing techniques.

[3] In fact, (1.26) has been derived long before by Hicks in 1898 [20]. Consequently, in the fluid dynamics community, the same equation is known as the Hicks equation and also as the Bragg–Hawthorne equation [4] and the Squire–Long equation [23,26] due to independent re-derivations. One can derive versions of (1.26) for other symmetries as well; see [5] for a generalization of (1.26) in this direction.

We prove also the complementary flexibility result for periodic-in-z solutions. Specifically, we construct solutions of

$$\begin{aligned} \frac{\partial^2}{\partial r^2}\psi + \frac{\partial^2}{\partial z^2}\psi - \frac{1}{r}\frac{\partial}{\partial r}\psi &= -r^2\Pi'(\psi) + CC'(\psi), \quad &&\text{in } D \\ \psi &= (\text{const.}), &&\text{on } \partial D. \end{aligned} \tag{1.29}$$

where the domain occupied by the fluid is given by

$$T = D \times \mathbb{T}, \qquad D = \{(r, z) \in \ [b_0(z), b_1(z)] \times \mathbb{T}\}. \tag{1.30}$$

See the left half of Fig. 2 for a depiction of a possible domain.

Theorem 1.7 (Flexibility of axisymmetric pipe flows). *Let D_0 be given by (1.24) and D defined by (1.30). Fix $k \geq 3$, $\alpha \in (0, 1)$. Suppose $\psi_0 \in C^{k,\alpha}(D_0)$ is a solution to the axisymmetric Grad–Shafranov equation (1.26)–(1.27) for some $\Pi_0, C_0 \in C^{k-1,\alpha}(\mathbb{R})$, having no stagnation points in the sense that $U_0 := \inf |\nabla\psi_0| > 0$ in D_0. Suppose that additionally Π_0, C_0 and ψ_0 satisfy*

$$\Pi_0'(\psi_0) > \tfrac{1}{r^2}(C_0C_0')'(\psi_0).$$

Then there are constants $\varepsilon_1, \varepsilon_2, \varepsilon_3$ depending only on U_0 and $\|\psi_0\|_{C^{k,\alpha}(D_0)}$, $\|\Pi_0\|_{C^{k-1,\alpha}}$, and $\|C_0\|_{C^{k-1,\alpha}}$ such that if $b_0, b_1 : \mathbb{R} \to \mathbb{R}$, $\Pi : \mathbb{R} \to \mathbb{R}$ and $\rho : D_0 \to \mathbb{R}$ with $\int_{D_0} \rho = \mathrm{Vol}(D)$ satisfy

$$\begin{aligned} \|b_0\|_{C^{k,\alpha}(\mathbb{R})} + \|b_1\|_{C^{k,\alpha}(\mathbb{R})} &\leq \varepsilon_1, \\ \|1 - \rho\|_{C^{k,\alpha}(D_0)} &\leq \varepsilon_2, \\ \|\Pi_0' - \Pi'\|_{C^{k-2,\alpha}(\mathbb{R})} &\leq \varepsilon_3, \end{aligned}$$

there is a diffeomorphism $\gamma : D_0 \to D$ and a function $C \in C^{k-1,\alpha}(\mathbb{R})$ so that $\psi = \psi_0 \circ \gamma^{-1}$ satisfies (1.29) in D with pressure Π and swirl C. In particular,

$$u = \frac{1}{r}e_\theta \times \nabla\psi + \frac{1}{r}C(\psi)e_\theta$$

satisfies the Euler equation (1.22)–(1.23) with pressure $P = \Pi(\psi)$ in the domain $T = D \times \mathbb{T}$.

We remark, for the deforming scheme, that it is not necessary to cut out the inner part of the cylinder. Theorem 1.7 applies provided that **(H2)** on non-degeneracy is satisfied by ψ_0.

2. Rigidity: Liouville Theorems

To establish the claimed Liouville theorems, we first show that functions which satisfy steady transport by a velocity $u_0 = \nabla^\perp\psi_0$ *with no stagnation points* can be constructed from the streamfunction ψ_0 via a 'nice' equation of state. This is the content of Lemma 2.1 below.

Lemma 2.1. *Fix $k \geq 3$ and let D_0 be diffeomorphic to the annulus and $\psi_0 : D_0 \to \mathbb{R}$ satisfy*

- $\psi_0 \in C^k(D_0)$,
- $\psi_0|_{\partial D_0^{\rm bot}} = c_0$ *and* $\psi_0|_{\partial D_0^{\rm top}} = c_1$ *for constants* $c_0 \neq c_1$.
- $|\nabla \psi_0| \neq 0$ *in* D_0.

Suppose that $\theta \in C^{k-2}(D_0)$ *satisfies*

$$\nabla^\perp \psi_0 \cdot \nabla \theta = 0, \quad in \quad D_0.$$

Then, there exists a $(k-2)$*–times continuously differentiable function* $\Theta : \mathbb{R} \to \mathbb{R}$ *such that*

$$\theta(x, y) = \Theta(\psi_0(x, y)), \quad in \quad D_0.$$

Proof of Lemma 2.1. Our Lemma 2.1 essentially appears as Lemma 2.4 of [19] in the case when D_0 is the channel. We summarize the argument here for the sake of completeness. Given $p \in D_0$ and let $\xi_p = \xi_p(t)$ denote the integral curve of $\nabla^\perp \psi$ starting at p at "time" $t = 0$, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi_p(t) = \nabla^\perp \psi_0(\xi_p(t)), \qquad \xi_p(0) = p, \qquad t \in \mathbb{R}.$$

By Lemma 2.2 of [19], $\xi_p(t)$ is uniquely defined for all $t \in \mathbb{R}$ and is periodic in t and moreover the curve $\xi_p(\mathbb{R})$ passes through each $x \in [0, 2\pi)$. Identifying the periodic channel with the annulus, this means that the curve $\xi_p(\mathbb{R})$ surrounds the inner disc. Given $q \in D_0$ we also let σ_q denote the integral curve of $\nabla \psi$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_q(t) = \nabla \psi_0(\sigma_q(t)), \qquad \sigma_q(0) = q, \qquad t \in \mathbb{R}.$$

We now fix any point $q = (q_1, 0)$ at the bottom of the channel $\{y = 0\}$. As a consequence of the fact that the vector field $\nabla \psi_0$ points normal to the boundary, it is shown in [19] that there is a $t_q < \infty$ so that $\sigma_q(t)$ lies at the top of the channel, $\sigma_q(t_q) = (q_2, 1)$. Writing $g(t) = \psi(\sigma_q(t))$, we have $g'(t) = |\nabla \psi_0(\sigma_q(t))|^2 > 0$ so it follows that g is invertible with C^{k-2} inverse. We define Θ by

$$\Theta(\tau) = \theta(\sigma_q(g^{-1}(\tau))).$$

Then Θ is C^{k-2} and $\Theta(\psi_0(\sigma_q(s))) = \theta(\sigma_q(s))$ for any s. Finally, fix now any point $p \in D_0$. For large enough t, there is an s so that $\xi_p(t) = \sigma_q(s)$. Since $\nabla^\perp \psi_0 \cdot \nabla \theta = 0$, we have $\theta(p) = \theta(\xi_p(t)) = \theta(\sigma_q(s))$. This completes the proof since then we have

$$\theta(p) = \theta(\sigma_q(s)) = \Theta(\psi_0(\sigma_q(s))) = \Theta(\psi_0(p)).$$

□

We will need the following result which ensures that the stream function takes different values at the top and bottom, and ranges between these values in the interior:

Lemma 2.2 (Lemma 2.6 of [18], Lemma 2.1 of [19]). *Let* D_0 *be diffeomorphic to the annulus and let* $\psi_0 \in C^3(D_0)$ *with* $|\nabla \psi_0| \neq 0$ *on* D_0 *satisfy*

$$\psi_0|_{\partial D_0^{\rm bot}} = c_0, \qquad \psi_0|_{\partial D_0^{\rm top}} = c_1,$$

for some $c_0, c_1 \in \mathbb{R}$. *Then* $c_0 \neq c_1$ *and*

$$\min\{c_0, c_1\} < \psi_0 < \max\{c_0, c_1\}, \quad on \quad D_0 \setminus \partial D_0.$$

The proof of this result can be found in the cited references. There, (2.2) is established when D_0 is periodic channel (1.9), but an inspection of the proof shows that the result holds more generally.

Finally, we prove the corresponding Liouville theorem which modestly generalizes Theorem 1.6 of [18] to accommodate the additional terms arising in the settings of the Boussinesq and axisymmetric Euler equations.

Theorem 2.1 (Liouville Theorem). *Let $D_0 = \mathbb{T} \times [1/2, 1]$ and let $f = f(y), g = g(y, \psi), h = h(\psi)$ be Lipschitz functions. Let $\psi \in C^2(D_0)$ be a solution to*

$$\Delta\psi + f(y)\partial_y\psi + g(y, \psi) + h(\psi) = 0, \quad \text{in } D_0,$$

where ψ is periodic in $x \in \mathbb{T}$ with boundary conditions

$$\psi(x, 1/2) = 0, \qquad \psi(x, 1) = c > 0.$$

Suppose that one of the following conditions holds

- $g_y, f_y \geq 0$, *and* $0 < \psi < c$ *in* D_0,
- $g_y, f_y \geq 0$, *and* $\psi_y \geq 0$ *on* D_0.

Then ψ is independent of x, namely $\psi := \psi(y)$.

Proof of Theorem 2.1. The proof is nearly identical to the one in [18] with minor extension to accommodate f and g. For the sake of completeness we include a proof here. Fix $\xi \in \mathbb{R}^2$ with $\xi = (\xi_1, \xi_2)$ with $\xi_2 > 0$. For $\tau \in (0, 1/\xi_2)$, set

$$D_0{}^\tau = \mathbb{T} \times (1/2, 1 - \tau\xi_2),$$

and

$$w^\tau(x) = \psi(x + \tau\xi) - \psi(x), \quad x \in \overline{D_0}^\tau. \tag{2.1}$$

Then the main ingredient for the proof of Theorem 2.1 is the following lemma

Lemma 2.3. *For w^τ defined by (2.1) we have*

$$w^\tau > 0 \quad \text{in } \overline{D_0}^\tau, \quad \text{for all } \tau \in (1/2, 1/\xi_2).$$

We first prove Theorem 2.1 assuming the result of this lemma.

Note that $\psi \geq 0$ on D_0, since $\psi_y \geq 0$ on D_0 by assumption and the boundary values are $\psi|_{y=1/2} = 0$ and $\psi|_{y=1} = c > 0$. Taking $\xi_2 \to 0$ in the inequality $w^\tau > 0$ shows that

$$\psi(x + \tau\xi_1, y) \geq \psi(x, y). \tag{2.2}$$

This holds for any $\xi_1 \in \mathbb{R}$ and we claim that this implies that we actually have equality in (2.2). Indeed, suppose that there are x, τ, ξ_1, y so that $\psi(x + \tau\xi_1, y) > \psi(x, y)$. Applying (2.2) we have

$$\psi(x, y) = \psi(x - \tau\xi_1 + \tau\xi_1, y) \geq \psi(x + \tau\xi_1, y) > \psi(x, y),$$

a contradiction. This completes the proof. □

Proof of Lemma 2.3. Set

$$\tau_* = \inf\{\tau \in (1/2, 1/\xi_2) \text{ such that } w^{\tau'} > 0 \text{ in } \overline{D_0}_{\tau'} \text{ whenever } \tau' \in (\tau, 1/\xi_2)\}.$$

By the maximum principle for narrow domains [11] we have $\tau_* < 1/\xi_2$. We are going to prove that $\tau_* = 1/2$. Suppose that instead $\tau_* > 1/2$. Then $w^{\tau^*} \geq 0$ in $\overline{D_0}^{\tau_*}$ and there are sequences $\tau^k \in (1/2, \tau_*]$ and $(x^k, y^k) \in D_0$ so that

$$(x^k, y^k) \in \overline{D_0}^{\tau_k}, \quad \text{and} \quad w^{\tau_k}(x^k, y^k) \leq 0.$$

Define

$$\psi_k(x, y) = \psi(x + x^k, y), \qquad \text{for} \qquad (x, y) \in D_0.$$

The functions ψ_k are uniformly bounded in $C^{2,\alpha}(D_0{}^{\tau_*})$, and so we can extract a convergent subsequence with $\psi_k \to \Psi \in C^2(D_0{}^{\tau^*})$. Taking $k \to \infty$ we see that $0 \leq \Psi \leq c$. We now show that these inequalities are strict. Taking $k \to \infty$ in the equation for ψ and differentiating in y we see that

$$\Delta \Psi_y + f(y)\partial_y \Psi_y + \big(f_y(y) + g_\Psi(y, \Psi) + h_\Psi(\Psi)\big)\Psi_y = -g_y(y, \Psi) \leq 0,$$

by assumption. Since we also have $\Psi_y \geq 0$ on the boundary, it follows from the maximum principle for non-negative functions that $\Psi_y > 0$ in the interior as well, and so

$$0 < \Psi < c. \tag{2.3}$$

Next, the points y^k are bounded and so we can extract a convergent subsequence $y^k \to \tilde{y}$. We have

$$\Psi(\tau_*\xi_2, \tilde{y} + \tau_*\xi_2) = \Psi(0, \tilde{y}), \tag{2.4}$$

because $w^{\tau_*} \geq 0$ in $D_0{}^{\tau_*}$ and $w^{\tau_k}(x^k, y^k) \leq 0$. If $(0, \tilde{y}) \in \partial D_0{}^{\tau_*}$ then either $\tilde{y} = 0$ or $\tilde{y} = 1 - \tau_*\xi_2$. But by (2.4) and (2.3) neither of these are possible. The only possibility left is $(0, \tilde{y}) \in D_0{}^{\tau_*}$. Set

$$W(x) = \Psi(x + \tau_*\xi) - \Psi(x),$$

then writing $\Psi_{\tau_*}(x) = \Psi(x + \tau_*\xi)$ we see that in $D_0^{\tau_*}$, since f, g are Lipschitz in ψ there is an L^∞ function $c = c(x, y)$ so that

$$\begin{aligned}&\Delta W + f(y)\partial_y W + c(x, y)W \\ &\quad = \big(f(y) - f(y + \tau_*\xi_2)\big)\partial_y \Psi_{\tau^*} + g(y, \Psi_{\tau^*}) - g(y + \xi_2\tau_*, \Psi_{\tau^*}) \leq 0\end{aligned}$$

because $\xi_2 \geq 0$ and that $\partial_y \Psi \geq 0$ (which is only needed f is nonzero) and that f, g are increasing in y. Also we have $W \geq 0$ in D_0, $W \geq 0$ on ∂D_0. By the maximum principle for non-negative functions this implies that $W \equiv 0$ and in particular $W = 0$ on ∂D_0. As we have shown that this is impossible, we conclude $\tau_* = \frac{1}{2}$. □

2.1. Proof of Theorem 1.1. We assume $\psi_0 \in C^3(D_0)$. Note that, since the vorticity satisfies

$$u_0 \cdot \nabla\omega_0 = 0$$

and $|u_0| \neq 0$ and $\omega_0 \in C^1(D_0)$, Lemma 2.1 implies that there exists a $C^1(\mathbb{R})$ function F_0 such that $\omega_0 = F_0(\psi_0)$. Consequently, the stream function ψ_0 satisfies the elliptic equation

$$\Delta\psi_0 = F_0(\psi_0) \quad \text{in } D_0,$$
$$\psi|_{\partial D_0^{\text{top}}} = c_1, \qquad \psi|_{\partial D_0^{\text{top}}} = c_2,$$

for some constants c_1 and c_2 with $c_1 \neq c_2$ by Lemma 2.2. Without loss of generality, we may take $c_1 = 0$ and $c_2 > 0$ by shifting $\psi_0 \mapsto \psi_0 - c_1$, sending $\psi_0 \mapsto -\psi_0$ if $c_2 < 0$, and replacing $F_0(\psi_0)$ with $\pm F_0(\pm\psi_0 + c_1)$. Moreover, by Lemma 2.2 we have

$$0 < \psi_0 < c_2 \quad \text{in } D_0.$$

Applying Theorem 2.1 with $b = 0$, $f = 0$ and $g = -F_0$ gives the result.

2.2. Proof of Theorem 1.4. We argue as in the proof of Theorem 1.1, but we apply Theorem 2.1 with $f = 0$, $g(y, \psi) = y\Theta_0'(\psi)$ and $h(\psi) = -G_0(\psi)$.

2.3. Proof of Theorem 1.6. Assuming (1.28), the proof follows as in Theorem 1.1, but now $f = -\frac{1}{r}$, $g(y, \psi) = r^2\Pi_0'(\psi)$ and $h(\psi) = C_0C_0'(\psi)$.

3. Flexibility: Deforming Domains

We prove here a more general theorem, which covers the specific settings of Theorems 1.2, 1.5 and 1.7. We now outline the general setup. Consider two bounded domains $D_0, D \subset \mathbb{R}^2$ given by the zero level sets of functions $B_0, B : \mathbb{R}^2 \to \mathbb{R}$:

$$\partial D_0 = \{B_0 = 0\}, \qquad \partial D = \{B = 0\}. \tag{3.1}$$

It is convenient to denote points in D_0 by $y = (y_1, y_2)$ and points in D by $x = (x_1, x_2)$. We will consider the problem of solving a certain elliptic equation on D by deforming a solution of a 'nearby' elliptic equation on D_0.

Elliptic equation on D_0: Consider a second-order elliptic operator on D_0 of the form

$$L_0 = \sum_{i,j=1}^{2} a_0^{ij}(y)\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j} + \sum_{i=1}^{2} b_0^i(y)\frac{\partial}{\partial y^i}, \tag{3.2}$$

where a_0^{ij}, b_0^i are smooth functions defined on $\mathbb{R}^2$ and where the matrix a_0^{ij} satisfies

$$a_0^{ij} z_i z_j \geq M|z|^2, \qquad \forall z \in \mathbb{R}^2 \tag{3.3}$$

for some $M > 0$. We assume that we have a solution ψ_0 to the following nonlinear equation

$$\begin{aligned} L_0\psi_0 &= F_0(\psi_0) + G_0(y, \psi_0), && \text{in } D_0, \\ \psi_0 &= (\text{const.}), && \text{on } \partial D_0, \end{aligned} \tag{3.4}$$

with functions $F_0 : \mathbb{R} \to \mathbb{R}$ and $G_0 : D_0 \times \mathbb{R} \to \mathbb{R}$.

Elliptic equation on D: Given coefficients a^{ij}, b^i defined on $\mathbb{R}^2$, we set

$$L = \sum_{i,j=1}^{2} a^{ij}(x)\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} + \sum_{i=1}^{2} b^i(x)\frac{\partial}{\partial x^i},$$

which is assumed to be elliptic as in (3.3). Consider the following equation for ψ

$$\begin{aligned} L\psi &= F(\psi) + G(x, \psi), && \text{in } D, \\ \psi &= (\text{const.}), && \text{on } \partial D, \end{aligned} \tag{3.5}$$

with functions $F : \mathbb{R} \to \mathbb{R}$ and $G : D \times \mathbb{R} \to \mathbb{R}$.

Problem *Let D and D_0 by two nearby domains (in the sense that B and B_0 are close) Let F_0, G_0 and a solution ψ_0 to* (3.4) *on D_0 be given. Let G be a given function close to G_0. Find a diffeomorphism $\gamma : D_0 \to D$ and a function F close to F_0 so that the function*

$$\psi = \psi_0 \circ \gamma^{-1} \tag{3.6}$$

is a solution to (3.5).

The important observation of [28] is that if we write $\gamma = \text{id} + \nabla\eta + \nabla^{\perp}\phi$ for functions η, ϕ, then plugging (3.6) into (3.5) leads to an Dirichlet problem for $\partial_s\phi := \nabla^{\perp}\psi_0 \cdot \nabla\phi$. The function η is free in the problem but if one wants to fix the value of the Jacobian determinant $\rho := \det \nabla\gamma$, η can be determined by solving a Neumann problem. We formalize this in the following Proposition.

Proposition 3.1 (Elliptic system for diffeomorphism). *Fix two domains $D_0, D \subseteq \mathbb{R}^2$ as in* (3.1) *and a solution to* (3.4) *$\psi_0 : D_0 \to \mathbb{R}$. Let F_0, G_0 and G be given. Let $\rho : D_0 \to \mathbb{R}$ be a given continuous function such that $\int_{D_0} \rho = \text{Vol}(D)$. Suppose that $\eta, \phi : D_0 \to \mathbb{R}$ satisfy*

$$\begin{aligned} \Delta\eta &= \rho - 1 + \mathcal{N}_\eta, \\ (L_0 - \Lambda)\partial_s\phi &= (F - F_0)(\psi_0) + \mathcal{L}_\phi + \mathcal{N}_\phi, \end{aligned} \tag{3.7}$$

for some $F = F(\psi_0)$, where L_0 is as in (3.2), *$\Lambda := F_0'(\psi_0) + (\partial_\psi G_0)(y, \psi_0)$, where $\partial_s\phi = \nabla^{\perp}\psi_0 \cdot \nabla\phi$, and where*

$$\mathcal{L}_\phi = \mathcal{L}_\phi(\delta a, \delta b, \delta F, \partial\delta G, \partial a_0, \partial b_0, \partial^3\eta, \partial\rho, \partial^2\partial_s\phi, \partial_{\psi_0}\phi; \partial^3\psi_0)$$

is defined by (B.7) *consists of terms which are linear in ϕ and η (and their derivatives), multiplied by small factors, where*

$$\begin{aligned} \mathcal{N}_\eta &= \mathcal{N}_\eta(\partial^2\eta, \partial^2\phi) \\ \mathcal{N}_\phi &= \mathcal{N}_\phi(\partial^2 a_0, \partial^2 b_0, \partial^2\phi, \partial^2\eta, \partial\rho, \partial^2\partial_s\phi; \partial^3\psi_0) \end{aligned}$$

are nonlinearities with $\mathcal{N}_\eta$ is defined by (B.1) *and $\mathcal{N}_\phi$ by* (B.8)*, and where $\delta a = a - a_0$ and similarly for $\delta b, \delta F, \delta G$. If $\gamma = \text{id} + \nabla^{\perp}\phi + \nabla\eta$ is a diffeomorphism $\gamma : D_0 \to D$ with $\det\nabla\gamma = \rho$, then the function $\psi = \psi_0 \circ \gamma^{-1}$ is a solution of* (3.5) *in D.*

This Proposition is proved in § B (see Lemma B.2). With this in hand, we address the above problem by constructing solutions with one (infinite dimensional) degree of freedom fixed by choosing the Jacobian of the map. This requires three hypotheses on ψ_0 and the quantities in (3.5).

We need one hypothesis on the invertibility of the operator appearing in Proposition 3.1 so the $\partial_s\psi$ can be recovered from Eq. (3.7) at the linear level. We view $L_0 : H_0^1 \cap H^2 \to L^2$ and require:

Hypothesis 1 (**H1**): Let $\Lambda = F_0'(\psi_0) + (\partial_\psi G_0)(y, \psi_0)$. The problem

$$\begin{aligned} (L_0 - \Lambda)\, u &= 0 \qquad \text{in } \ D_0, \\ u &= 0 \qquad \text{on } \ \partial D_0, \end{aligned}$$

admits only the trivial solution in $H_0^1(D_0)$.

It is easy to see that Hypothesis (**H1**) is guaranteed if Λ avoids the discrete spectrum of $-L_0$, an open condition. In light of this, a stronger but easier to verify hypothesis that implies (**H1**) is

Hypothesis 1′ (**H1′**): The operator $(L_0 - \Lambda)$ is positive definite, i.e. for all $f \in H_0^1(D_0)$ there is a constant $C > 0$ such that $\langle (L_0 - \Lambda)\, f, f\rangle_{L^2(D_0)} \geq C\|f\|^2_{H^1(D_0)}$.

This holds in the case of the 2d Euler equation if the base state is Arnol'd stable or if it is a shear flow without stagnation points (see Lemma 4.1).

The next two hypotheses are needed in order to recover ϕ from $\partial_s\phi$ once the latter is obtained by solving Eq. (3.7) using (**H1**). Since $\partial_s = \nabla^\perp\psi_0 \cdot \nabla$, in order to recover ϕ, we must be able to integrate along streamlines of ψ_0 which requires a certain non-degeneracy of the base state. On a multiply connected domain diffeomorphic to the annulus, the base state must have no stagnation points (points at which $\nabla\psi_0 = 0$). On a simply connected domain diffeomorphic to a disc, there must be exactly one stagnation point. This is quantified by the following hypothesis on the "travel-time" μ of a parcel moving at speed $|\nabla\psi|$ to make a complete revolution on a streamline:

Hypothesis 2 (**H2**): Let $I = \mathrm{im}(\psi_0)$. There exists a constant $C > 0$ so that

$$\mu(c) = \oint_{\{\psi_0 = c\}} \frac{\mathrm{d}\ell}{|\nabla\psi_0|} \leq C \qquad \text{for all } \ c \in I$$

where ℓ is the arc-length parameter. Note if $\psi_0 \in C^{k,\alpha}(D_0)$ then $\mu \in C^{k-1,\alpha}(I)$.

Finally, we need an additional hypothesis that allows us to recover ϕ once we solve Eq. (3.7) for $\partial_s\phi$. Specifically, at the linear level, ϕ needs to be chosen to satisfy $(L_0 - \Lambda)\partial_s\phi = F + N$, for a given function N and for F to be determined. An obvious necessary condition for solvability is that $(L_0 - \Lambda)^{-1}_{\mathrm{hbc}}(F + N)$ should have integral zero along streamlines. We must therefore be able to choose F to satisfy this condition while maintaining that $F = F(\psi_0)$ is a function *only of the stream function* in order for the resulting ψ to solve the correct equation.

Hypothesis 3 (**H3**): Fix $k \geq 2$, $\alpha \in (0, 1)$ and let $I = \mathrm{im}(\psi_0)$. Let $K_{\psi_0} : C^{k-2,\alpha}(I) \to C^{k,\alpha}(I)$ be

$$(K_{\psi_0} u)(c) := \frac{1}{\mu(c)} \oint_{\{\psi_0 = c\}} (L_0 - \Lambda)^{-1}_{\mathrm{hbc}}[u \circ \psi_0]\, \frac{\mathrm{d}\ell}{|\nabla\psi_0|}, \qquad c \in I.$$

For any $g \in C^{k,\alpha}(I)$ such that $g(\psi_0(\partial D_0)) = 0$, there exists a $u \in C^{k-2,\alpha}(I)$ such that $K_{\psi_0} u = g$. Moreover, $\|u\|_{C^{k-2,\alpha}(I)} \lesssim \|g\|_{C^{k,\alpha}(I)}$.

It turns out that (**H3**) is a consequence of (**H1′**) and (**H2**). To prepare for the proof, we define the streamline projector $\mathbb{P}_{\psi_0}$ which maps functions on D_0 to functions which are constant on level sets of the streamfunction ψ_0,

$$(\mathbb{P}_{\psi_0} f)(c) := \frac{1}{\mu(c)} \oint_{\{\psi_0=c\}} f \, \mathrm{d}s, \qquad \text{for all } \ c \in I$$

where $\mathrm{d}s = \mathrm{d}\ell/|\nabla\psi_0|$. This operation is well defined on functions which can be integrated on curves (e.g. functions that are in $H^1(D_0)$) by Hypothesis (**H2**). With this notation we have $K_{\psi_0} u := \mathbb{P}_{\psi_0}(L_0 - \Lambda)^{-1}_{\text{hbc}}[u]$. Note that if f, g are such that $\mathbb{P}_{\psi_0} f = 0$ and $\mathbb{P}_{\psi_0} g = g$ then

$$\int_{D_0} fg = \int_I \left(\oint_{\{\psi_0=c\}} fg \mathrm{d}s \right) \mathrm{d}c = \int_I g \left(\oint_{\{\psi_0=c\}} f \mathrm{d}s \right) \mathrm{d}c = 0. \tag{3.8}$$

Here we use the fact that ψ_0 satisfies (**H2**) and therefore has streamlines which foliate D_0 so we can use action-angle coordinates to compute the integral (3.8). For further discussion see § E herein or the textbook [1]. It follows that $\mathbb{P}_{\psi_0}$ is orthogonal in $L^2(D_0)$, i.e. for any $h \in C(D_0)$ we have

$$\|h\|^2_{L^2} = \int_{D_0} \left(|\mathbb{P}_{\psi_0} h|^2 + 2(\mathbb{P}_{\psi_0} h)(\mathbb{Q}_{\psi_0} h) + |\mathbb{Q}_{\psi_0} h|^2 \right) = \|\mathbb{P}_{\psi_0} h\|^2_{L^2} + \|\mathbb{Q}_{\psi_0} h\|^2_{L^2}$$

where $\mathbb{Q}_{\psi_0} = \mathbf{1} - \mathbb{P}_{\psi_0}$. In light of these properties, $\mathbb{P}_{\psi_0}$ is a projection on L^2.

The motivation for Lemma 3.1 in a Hilbert space H is that if P is a projection ($P^2 = P$ and $P^* = P$) and A is bounded positive operator then the compression PAP is positive in PH since

$$\langle PAPx, x\rangle_H = \langle APx, Px\rangle_H \geq C\langle Px, Px\rangle_H.$$

The fact that A is bounded is used only to make sure that PH is included in the domain of A.

Lemma 3.1. *Fix $k \geq 2$ and suppose Hypotheses (**H1′**) and (**H2**) hold. Then (**H3**) holds.*

The proof is deferred to § A. We remark that, invertibility of $\mathcal{L}_0 - \Lambda$ alone (Hypothesis (**H1**)) cannot be expected to imply Hypothesis (**H3**) itself as is easily demonstrated in finite dimensions. Positive definiteness is a crucial point in our argument. We finally note that if we further know that $(L_0 - \Lambda)g(\psi_0)$ is itself a function of ψ_0, which is the case when the base solution and the operator L_0 enjoy some mutual symmetry, we can find the solution of Hypothesis (**H3**) explicitly

Lemma 3.2. *Suppose for any $f \in C^{k,\alpha}(I)$, the function $(L_0 - \Lambda) f(\psi_0)$ depends only on the value of the stream function, $(L_0 - \Lambda) f(\psi_0) = h(\psi_0)$ for some $h \in C^{k-2,\alpha}(I)$. Then (**H3**) holds with $u = (L_0 - \Lambda)g$.*

Our main theorem on deforming solutions of elliptic equations is a quantitative version of the implicit function theorem. As stated above, this generalizes the setup and results of Wirosoetisno and Vanneste [28]. It is also similar in spirit to the result of Choffrut and Sverák [8] which, on annular domains, establishes a one-to-one correspondence between vorticity distribution functions and steady states of two-dimensional Euler nearby a solution satisfying a version of (**H1**) (see also [9]). In our theorem, $F := F(\psi)$ (which plays the role of the vorticity distribution function for 2d Euler) is not chosen ahead of time but rather accommodates the deformation of the other parameters (boundary, coefficients, Jacobian) so as the resulting streamfunction remain a solution. We prove

Theorem 3.1 (Deforming solutions of elliptic equations). *Let $\alpha \in (0, 1)$ and $k \geq 2$. Fix two domains $D_0, D \subseteq \mathbb{R}^2$ with $C^{k,\alpha}$ boundaries given by (3.1) and a solution $\psi_0 : D_0 \to \mathbb{R}$ to (3.4) on D_0 with $\psi_0 \in C^{k,\alpha}(D_0)$ and $F_0 \in C^{k-1,\alpha}(\mathbb{R})$. Suppose in addition that* (**H1**)*,* (***H2***) *and* (***H3***) *are satisfied. Let $\rho : D_0 \to \mathbb{R}$ such that $\rho \in C^{k-1,\alpha}(D_0)$ and $\int_{D_0} \rho = \mathrm{Vol}(D)$. Suppose that for sufficiently small $\varepsilon > 0$, $|\mathrm{Vol}(D) - \mathrm{Vol}(D_0)| \lesssim \varepsilon$ as well as*

$$
\begin{aligned}
\|B - B_0\|_{C^{k,\alpha}} &\leq \varepsilon, &\qquad \|\rho - 1\|_{C^{k-1,\alpha}} &\leq \varepsilon \\
\|a - a_0\|_{C^{k,\alpha}} &\leq \varepsilon, &\qquad \|b - b_0\|_{C^{k,\alpha}} &\leq \varepsilon, \\
\|G - G_0\|_{C^{k-2,\alpha}} &\leq \varepsilon.
\end{aligned}
$$

Then, for ε sufficiently small, there exists a diffeomorphism $\gamma : D_0 \to D$ such that $\det \nabla\gamma = \rho$ and a function $F : \mathbb{R} \to \mathbb{R}$ satisfying $\|F - F_0\|_{C^{k-2,\alpha}} \lesssim \varepsilon$ such that the function $\psi = \psi_0 \circ \gamma^{-1}$ satisfies the Eq. (3.5) in D. The diffeomorphism γ is of the form $\gamma = \mathrm{id} + \nabla\eta + \nabla^{\perp}\phi$ and η, ϕ satisfy the estimates

$$
\begin{aligned}
&\|\partial_s\eta\|_{C^{k,\alpha}} + \|\partial_s\phi\|_{C^{k,\alpha}} + \|\eta\|_{C^{k,\alpha}} + \|\phi\|_{C^{k,\alpha}} \\
&\quad \leq C_{k,\alpha}\big(\|\rho - 1\|_{C^{k-1,\alpha}} + \|a - a_0\|_{C^{k,\alpha}} + \|b - b_0\|_{C^{k,\alpha}} \\
&\qquad + \|G - G_0\|_{C^{k-2,\alpha}} + \|B - B_0\|_{C^{k,\alpha}}\big)
\end{aligned}
\tag{3.9}
$$

for constants $C_{k,\alpha}$ depending on k, α, D_0, the ellipticity constant M and $\|a_0\|_{C^{k,\alpha}}$, $\|b_0\|_{C^{k,\alpha}}$.

Theorem 3.1 is used in [6] to construct approximate solutions to the Magnetohydrostatic equations on wobbled tori which are nearly quasisymmetric. As in [27], one may think about these deformations arising dynamically from a slow adiabatic deformation of the boundary, though we do not establish this point here. We remark also that (**H1**) may not be strictly needed for the Theorem 3.1 provided kernel of $L_0 - \Lambda$ is very well understood. This is demonstrated in a slightly different context by the recent work [7] for Kolmogorov flow $u_0 = (\sin(y), 0)$ which is a shear with stagnation points so that Lemma 4.1 does not apply and the corresponding operator $\Delta - F_0'(\psi_0)$ has a nontrivial kernel (consisting of linear combinations of $\{\sin(y), \cos(y), \sin(x), \cos(x)\}$). To deal with this degeneracy, extra degrees of freedom are introduced in the contraction scheme.

We note that we can iterate the above theorem to impose a nonlinear constraint on ρ. Specifically, given a function $X = X(y, \phi, \eta, \nabla\phi, \nabla\eta, \nabla\partial_s\phi, \nabla\partial_s\eta)$ with $X|_{\phi,\eta=0}$ sufficiently close to one, we can solve for the diffeomorphism γ so that $\rho = X$, at the expense of slightly modifying the domain. Fixing notation, we consider

$$
X : D_0 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2, \quad X = X(y, q_1, q_2, p_1, p_2, p_3, p_4) \tag{3.10}
$$

and write

$$
X_0(y) = X|_{(q,p)=(0,0)} \qquad DX_0(y) = (\nabla_q X, \nabla_p X)|_{(q,p)=(0,0)} \tag{3.11}
$$

with $q = (q_1, q_2)$, $p = (p_1, p_2, p_3, p_4)$.

Theorem 3.2 (Deforming with an imposed nonlinear constraint). *Let* $\alpha \in (0,1)$ *and* $k \geq 2$. *Fix two domains* $D_0, D \subseteq \mathbb{R}^2$ *with* $C^{k,\alpha}$ *boundaries given by* (3.1) *and a solution* $\psi_0 : D_0 \to \mathbb{R}$ *to* (3.4) *on* D_0 *with* $\psi_0 \in C^{k,\alpha}(D_0)$ *and* $F_0 \in C^{k-2,\alpha}(\mathbb{R})$. *Let* X *be as in* (3.10) *and satisfy* $X \in C^{k,\alpha}$.

Suppose that for sufficiently small $\varepsilon, \varepsilon_X > 0$, $|\mathrm{Vol}(D) - \mathrm{Vol}(D_0)| \leq \varepsilon$ *and that*

$$\|B - B_0\|_{C^{k,\alpha}} \leq \varepsilon, \qquad \|a - a_0\|_{C^{k,\alpha}} \leq \varepsilon,$$
$$\|b - b_0\|_{C^{k,\alpha}} \leq \varepsilon, \qquad \|G - G_0\|_{C^{k-2,\alpha}} \leq \varepsilon,$$

and with notation as in (3.11),

$$\|X_0 - 1\|_{C^{k,\alpha}} + \|DX_0\|_{C^{k-1,\alpha}} \leq \varepsilon_X. \tag{3.12}$$

Then there exists a $\sigma > 0$ *and a diffeomorphism* $\gamma : D_0 \to D_\sigma$ *where* $D_\sigma = \sigma D$ *is a dilation-by-*σ *of the domain* D *satisfying*

$$\det \nabla\gamma =: \rho = X(y, \eta, \phi, \partial_s\phi, \partial_s\eta, \nabla\partial_s\eta, \nabla\partial_s\phi), \tag{3.13}$$

with the dilation factor $\sigma > 0$ *given by* $\sigma^2 := \mathrm{Vol} D / \int_{D_0} \rho$. *Moreover, there is a function* $F : \mathbb{R} \to \mathbb{R}$ *satisfying* $\|F - F_0\|_{C^{k-2,\alpha}} \lesssim \varepsilon$ *such that* $\psi = \psi_0 \circ \gamma^{-1}$ *is a solution to the* (3.5) *in* D_σ.

We do not apply Theorem 3.2 in the present paper. We record it here since it exploits a freedom in the construction and may be useful to build solutions with additional desirable properties (such as quasisymmetry in the context of plasma confinement fusion, see [6]).

4. Applications to Fluid Systems

4.1. Proof of Theorem 1.2. In the case of two-dimensional Euler equations on the channel, we apply Theorem 3.1 with $a_0^{ij} = a^{ij} \equiv \delta^{ij}, b_0^i = b^i \equiv 0, c_0 = c = 0, G = G_0 = 0$ and $F_0 = F_0$, the vorticity of the base state. As a result of Theorem 1.1, our base state $u_0 = (v_0(y), 0)$ is a shear where $v_0(y) = -\psi_0'(y)$ never vanishes. As a consequence, it satisfies (1.7) with $F_0(\psi) = \psi_0''(\psi_0^{-1}(\psi))$. We now show that all the hypotheses are met.

We first claim that in this setting Hypothesis **(H1′)** is a consequence of the nondegeneracy of the base shear flow. This follows immediately from the following Lemma (see e.g. [17])

Lemma 4.1. *Let* Ω *the periodic channel and let* $u_0 = (v_0(y_2), 0)$ *be a shear flow steady Euler solution and suppose* $\inf_\Omega |v_0| > 0$. *For all* u *such that* $u|_{\partial\Omega} = 0$, *the following holds*

$$\int_\Omega u\Big(\Delta - F_0'(\psi_0)\Big)u \,\mathrm{d}y_1\mathrm{d}y_2 = -\int_\Omega |v_0(y_2)|^2 \left|\nabla\left(\frac{u}{|v_0(y_2)|}\right)\right|^2 \mathrm{d}y_1\mathrm{d}y_2.$$

Proof. Note that $F_0'(\psi_0(y_2)) = (v_0''/v_0)(y_2)$. The result follows from direct computation. □

Hypothesis **(H2)** follows by our assumption that there are no stagnation points.

4.2. Proof of Theorem 1.3. In the case of two-dimensional Euler equations on (M_0, g_0) we apply Theorem 3.1 $L_0 = \Delta_{g_0}$ and $L = \Delta_g$. The coefficients can be computed directly in terms of the metrics and it is clear that the hypotheses on the closeness of a_0 and a (resp b_0 and b) hold when g_0 is close to g. Note also that under our hypotheses, $\mathcal{L}_\xi \psi = 0$ according to Theorem 1.1.

Hypothesis **(H1′)** follows by the assumption of Arnol'd stability.

Hypothesis **(H2)** follows by our assumption on the base states that they are non-degenerate.

Persistence of stability follows because Arnol'd stability conditions are open and our perturbation is small.

Remark (Hypothesis (**H3**). on Domains with Symmetry) We remark that if the domain admits a symmetry direction tangent to the boundary (so that all Arnol'd stable solutions enjoy the same symmetry according to Proposition 1.1), we may apply Lemma 3.2 to write explicitly the solution in Hypothesis (**H3**). Specifically, we apply the Lemma with $L_0 = \Delta_g$ and $\Lambda = F'$, and appeal to the following result

Lemma 4.2. *Let Δ_g be the Laplace–Beltrami operator on (M, g). Suppose ξ is a non-vanishing Killing field for g which is tangent to ∂M. Assume for $\psi \in C^{k,\alpha}(M)$ satisfies* **(H2)** *and that $\mathcal{L}_\xi \psi = 0$. Then for any function $f \in C^{k,\alpha}(\mathbb{R})$, we have $\Delta_g f(\psi) = G(\psi)$ for some function $G \in C^{k-2,\alpha}(\mathbb{R})$.*

Proof. First, by assumption the integral curves of ξ foliate M. Since $\mathcal{L}_\xi \psi = 0$, we know that ψ is constant on integral curves of ξ. Moreover, since **(H2)** guarantees that $|\nabla_g \psi| > 0$ except at one point (if the domain is simply connected, and nowhere otherwise), ψ takes different values on different integral curves of ξ. Since ξ is a Killing field, $\mathcal{L}_\xi \Delta_g f(\psi) = \Delta_g(f'(\psi)\mathcal{L}_\xi \psi) = 0$ and therefore $\mathcal{L}_\xi \Delta_g f(\psi)$ is constant on integral curves of ξ and thus a function of ψ. □

Therefore, Lemmas 3.2 and 4.2 show that Hypothesis (**H3**) holds with an explicit u in the symmetric setting.

4.3. Proof of Theorem 1.5. In the case of two-dimensional Boussinesq equations on the channel, we have $a_0^{ij} = a^{ij} \equiv \delta^{ij}$, $b_0^i = b^i \equiv 0$, $c_0 = c = 0$, $G_0 = y\Theta_0'(\psi_0)$, $G = y\Theta'(\psi)$, and $F_0 = G_0'$. Then $L_0 = \Delta$ and $\Lambda = G_0'(\psi_0) + y_2\Theta_0'(\psi_0)$.

Hypothesis **(H1′)** is verified for the following reason. Since ψ_0 is a shear $\psi_0 = \psi_0(y_2)$. Given this, we know that $\Delta\psi_0 = \psi_0''(y_2)$ so that $G_0(c)+y_2\Theta_0(c) = \psi_0''(\psi_0^{-1}(c))$. Thus

$$\Lambda = G_0'(\psi_0) + y_2\Theta_0'(\psi_0) = \frac{v_0''(y_2)}{v_0(y_2)}$$

where $v_0 = \psi_0'$. Thus Lemma 4.1 is applicable and the hypothesis follows.

Hypothesis **(H2)** follows by our assumption that there are no stagnation points.

The result of the deformation defines a stream function ψ for the Boussinesq equations with velocity $u = \nabla^\perp \psi$ and temperature profile $\theta = \Theta(\psi)$ (note Θ is recovered from Θ' up to a constant, which can be absorbed into the pressure).

4.4. Proof of Theorem 1.7. Hypothesis **(H1)** is verified when

$$\operatorname{im}\Big((C_0C_0')'(\psi_0) - r^2 P_0'(\psi_0)\Big) \notin \operatorname{Spec}\left(-\Delta + \frac{1}{r}\partial_r\right).$$

Since $-\Delta + \frac{1}{r}\partial_r$ is a positive operator Hypothesis **(H1′)** is verified when

$$(C_0C_0')'(\psi_0) - r^2 P_0'(\psi_0) < 0.$$

Hypothesis **(H2)** follows by our assumption that there are no stagnation points.

Acknowledgements. We thank T. M. Elgindi and H. Q. Nguyen for insightful discussions. We thank T. M. Elgindi in particular for pointing out Lemma 3.1. The work of PC was partially supported by NSF Grant DMS-1713985 and by the Simons Center for Hidden Symmetries and Fusion Energy award # 601960. Research of TD was partially supported by NSF Grant DMS-1703997. Research of DG was partially supported by the Simons Center for Hidden Symmetries and Fusion Energy.

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Appendix A. Proof of Lemma 3.1

We aim to solve $K_{\psi_0}[u] = g$ for a $g := g(\psi_0)$ with $g(\partial D_0) = 0$. To avoid technical difficulties of defining the trace of a function which is just in L^2, we solve this equation assuming $g \in H^1$ for an $u \in H^{-1}$. Define the spaces

$$S^{(k)} = \{f \in H^k(D_0) \mid \mathbb{Q}_{\psi_0} f = 0\},$$
$$S_0^{(k)} = \{f \in H^k(D_0) \mid \mathbb{Q}_{\psi_0} f = 0, \quad f|_{\partial D_0} = 0\}.$$

Note that for $k \geq 1$ the operator $\mathbb{P}_{\psi_0} : (H^k \cap H_0^1)(D_0) \to S_0^{(k)}$ is a continuous operator (which follows from (E.2)) and that therefore $S_0^{(k)}$ is a closed subspace of $H^k \cap H_0^1$. We also remark that for all $f \in S^k$, we know that $\nabla^\perp \psi_0 \cdot \nabla f = 0$ and therefore by (a minor extension of) Lemma 2.1, there exists a function $F \in H^k(I)$ such that $f = F \circ \psi_0$.

Recall now that for $u \in S^{(k)}$ and with this notation we have $K_{\psi_0} u := \mathbb{P}_{\psi_0}(L_0 - \Lambda)^{-1}_{\text{hbc}}[u]$. For $k \geq 1$, this operator $K_{\psi_0} : S^{(k-2)} \to S_0^{(k)}$ is continuous since $(L_0 - \Lambda)^{-1}_{\text{hbc}} : H^{k-2}(D_0) \to (H^k \cap H_0^1)(D_0)$ is continuous by Hypothesis **(H1′)** together with the fact that $\mathbb{P}_{\psi_0}$ is continuous. We remark that for $f \in S^{(-1)}$, we have $(L_0 - \Lambda)^{-1}_{\text{hbc}}[f] \in H^1$. Define now

$$S_K := \{K_{\psi_0} f \mid f \in S^{(-1)}\} \subseteq S_0^{(1)}.$$

We aim to show that $S_K = S_0^{(1)}$, that is, $K_{\psi_0} : S^{(-1)} \to S_0^{(1)}$ is onto. Since $K_{\psi_0} : S^{(k-2)} \to S_0^{(k)}$ is continuous, S_K is a closed subspace of $S_0^{(1)}$ in the $H^1(D_0)$ topology.[4]

[4] By the continuity of K_{ψ_0}, it suffices to prove that if $K_{\psi_0}[f^n] \to g$ in H^1 with $f^n \in H^{-1}$, then f^n converges in H^{-1}. By (A.2) we have

$$c_0 \|f^n - f^m\|^2_{H^{-1}} \leq \langle f^n - f^m, K_{\psi_0}[f^n - f^m]\rangle_{L^2} \leq C\|f^n - f^m\|_{H^{-1}} \|K_{\psi_0}[f^n - f^m]\|_{H^1},$$

which can be justified by an approximation. From this, we conclude that the sequence $\{f^n\}_{n\geq 0}$ is Cauchy in H^{-1}.

Thus, $S_0^{(1)} = S_K \oplus S_K^\perp$ where

$$S_K^\perp := \{f \in S_0^{(1)} \mid \langle f, g\rangle_{S^{(1)}} = 0 \text{ for all } g \in S_K\}, \tag{A.1}$$

where the $S^{(1)}$ topology is equivalent to the H^1 topology and will be defined shortly. Note that for any $h \in S_K^\perp \subset S^{(1)}$, the function $K_{\psi_0} h \in S_K$, it follows from (A.1) that $\langle h, K_{\psi_0} h\rangle_{S^{(1)}} = 0$. Thus, to conclude that $S_K^\perp = \{0\}$, we show that K_{ψ_0} has a trivial kernel in $S_0^{(1)}$. This is accomplished by designing a topology on $S^{(1)}$ which is equivalent to the H^1 topology and showing that for all $h \in S^{(1)}$, there is a constant $c > 0$ depending only on ψ_0 such that

$$\langle h, K_{\psi_0} h\rangle_{S^{(1)}} \geq c\|h\|^2_{L^2(D_0)}, \qquad \forall h \in S_0^{(1)}.$$

We now design the topology. First, we equip $S^{(0)}$ with the $L^2(D_0)$ topology:

$$\langle f, g\rangle_{S^{(0)}} = \int_{D_0} fg.$$

Note that, using orthogonality of the projection (3.8), by Hypothesis (**H1′**) we have

$$\langle h, K_{\psi_0} h\rangle_{S^{(0)}} := \langle h, \mathbb{P}_{\psi_0}(L_0 - \Lambda)^{-1}_{\text{hbc}} h\rangle_{S^{(0)}} = \langle h, (L_0 - \Lambda)^{-1}_{\text{hbc}} h\rangle_{L^2} \geq c_0\|h\|^2_{\dot{H}^{-1}} \geq 0 \tag{A.2}$$

for some $c_0 > 0$ where $\|h\|_{\dot{H}^{-1}} = \|\nabla g\|_{L^2}$ where $\Delta g = h$ and $g = 0$ at the boundary.

Now let $\partial_{\psi_0} = \frac{\nabla\psi_0}{|\nabla\psi_0|^2} \cdot \nabla$. Recalling for any $h \in S^k$ there exists $H \in H^k(I)$ such that $h = H \circ \psi_0$, we note that $\partial_{\psi_0} h = H'(\psi_0)$. Now let

$$\langle f, g\rangle_{S^{(1)}} = \langle \partial_{\psi_0} f, \partial_{\psi_0} g\rangle_{S^{(0)}} + M\langle f, g\rangle_{S^{(0)}}.$$

for some large constant M to be specified shortly. This topology is obviously equivalent to that of H^1 on S^1. To see this, denoting $\hat{x} = x/|x|$, we can write $\nabla = \widehat{\nabla^\perp\psi_0}\widehat{\nabla^\perp\psi_0} \cdot \nabla + \widehat{\nabla\psi_0}\,\widehat{\nabla\psi_0} \cdot \nabla$ and notice that on any $f \in S^1$, $\nabla f = \nabla\psi_0 \partial_{\psi_0} f$. Now note that, since $h = 0$ and $\mathbb{P}_{\psi_0}(L_0 - \Lambda)^{-1}_{\text{hbc}} h = 0$ on the boundary, we have

$$\begin{aligned}
&\langle h, \mathbb{P}_{\psi_0}(L_0 - \Lambda)^{-1}_{\text{hbc}} h\rangle_{S^{(1)}} \\
&\quad = \langle \partial_{\psi_0} h, \partial_{\psi_0}\mathbb{P}_{\psi_0}(L_0 - \Lambda)^{-1}_{\text{hbc}} h\rangle_{S^{(0)}} + M\langle h, \mathbb{P}_{\psi_0}(L_0 - \Lambda)^{-1}_{\text{hbc}} h\rangle_{S^{(0)}} \\
&\quad = -\langle \partial^2_{\psi_0} h, \mathbb{P}_{\psi_0}(L_0 - \Lambda)^{-1}_{\text{hbc}} h\rangle_{S^{(0)}} + M\langle h, (L_0 - \Lambda)^{-1}_{\text{hbc}} h\rangle_{S^{(0)}} \\
&\quad = \langle \partial_{\psi_0} h, \partial_{\psi_0}(L_0 - \Lambda)^{-1}_{\text{hbc}} h\rangle_{S^{(0)}} + M\langle h, (L_0 - \Lambda)^{-1}_{\text{hbc}} h\rangle_{S^{(0)}}.
\end{aligned}$$

In the above we repeatedly used that $\mathbb{P}_{\psi_0}$ is an orthogonal projection on L^2 so that $\langle f, \mathbb{P}_{\psi_0} g\rangle_{S^{(0)}} = \langle \mathbb{P}_{\psi_0} f, g\rangle_{S^{(0)}}$. Thus, when paired with functions only of ψ_0 such as h or $\partial^2_{\psi_0} h$, the projector is the identity. Introducing $f = (L_0 - \Lambda)^{-1}_{\text{hbc}} h$, we have

$$\langle \partial_{\psi_0} h, \partial_{\psi_0}(L_0 - \Lambda)^{-1}_{\text{hbc}} h\rangle_{S^{(0)}} = \langle \partial_{\psi_0}(L_0 - \Lambda) f, \partial_{\psi_0} f\rangle_{S^{(0)}} = -\langle (L_0 - \Lambda) f, \partial^2_{\psi_0} f\rangle_{S^{(0)}}$$

since $(L_0 - \Lambda) f = h$ which is zero at the boundary. Now, by (c.f. §7.2, pg 390 of [10]) we have

$$\langle (L_0 - \Lambda) f, \partial^2_{\psi_0} f\rangle_{S^{(0)}} \geq \beta\|f\|^2_{H^2} - \gamma\|f\|^2_{L^2},$$

for some constants $\beta, \gamma > 0$. Moreover, by our hypotheses, we have for some constant $c > 0$ that

$$c\|f\|_{H^1}^2 \le \langle (L_0 - \Lambda)f, f\rangle_{S^{(0)}}.$$

Combining the above bounds we obtain

$$\langle h, \mathbb{P}_{\psi_0}(L_0-\Lambda)^{-1}_{\mathrm{hbc}}h\rangle_{S^{(1)}} \ge \beta\|f\|_{H^2}^2 - \gamma\|f\|_{L^2}^2 + cM\|f\|_{H^1}^2.$$

It follows that by choosing M sufficiently large that for some $c_1 > 0$ depending only on ψ_0 we have

$$\langle h, \mathbb{P}_{\psi_0}(L_0-\Lambda)^{-1}_{\mathrm{hbc}}h\rangle_{S^{(1)}} \ge c_1\|f\|_{H^2}^2$$

and we deduce $\langle h, \mathbb{P}_{\psi_0}(L_0-\Lambda)^{-1}_{\mathrm{hbc}}h\rangle_{S^{(1)}}$ is coercive.

Thus we have established that for all $g \in S_0^{(1)}$, there exists a unique $u \in S^{(-1)}$ such that

$$K_{\psi_0}[u] = g.$$

Now we want to show that for $k \ge 1$, if $g \in S_0^{(k)}$, then $u \in S^{(k-2)}$. Let $g \in S_0^{(2)}$. We know there is a solution $u \in S_0^{(-1)}$. We wish to show that actually $u \in S_0^{(0)}$. To see this, we formally differentiate:

$$\partial_{\psi_0}K_{\psi_0}[u] = \partial_{\psi_0}g.$$

The following formal apriori calculation can be made rigorous by an approximation argument. We compute the commutator of derivative with K_{ψ_0}:

$$\begin{aligned}
[\partial_{\psi_0}, K_{\psi_0}]f &= \partial_{\psi_0}K_{\psi_0}[f] - K_{\psi_0}[\partial_{\psi_0}f]\\
&= \partial_{\psi_0}\mathbb{P}_{\psi_0}(L_0-\Lambda)^{-1}_{\mathrm{hbc}}f - \mathbb{P}_{\psi_0}(L_0-\Lambda)^{-1}_{\mathrm{hbc}}\partial_{\psi_0}f\\
&= \mathbb{P}_{\psi_0}\partial_{\psi_0}(L_0-\Lambda)^{-1}_{\mathrm{hbc}}f - \mathbb{P}_{\psi_0}(L_0-\Lambda)^{-1}_{\mathrm{hbc}}\partial_{\psi_0}f + [\mathbb{P}_{\psi_0}, \partial_{\psi_0}](L_0-\Lambda)^{-1}_{\mathrm{hbc}}f\\
&= \mathbb{P}_{\psi_0}[\partial_{\psi_0}, (L_0-\Lambda)^{-1}_{\mathrm{hbc}}]f + [\mathbb{P}_{\psi_0}, \partial_{\psi_0}](L_0-\Lambda)^{-1}_{\mathrm{hbc}}f.
\end{aligned}$$

Now note that by Lemma E.2 we have

$$[\mathbb{P}_{\psi_0}, \partial_{\psi_0}]g = -\mathbb{P}_{\psi_0}\Big[\Big(\frac{\mu'}{\mu^2} + \frac{\Delta\psi_0 - 2\kappa|\nabla\psi_0|}{|\nabla\psi_0|^2}\Big)g\Big].$$

Thus, commutator of derivative with streamline projector is of zero order:

$$\|[\mathbb{P}_{\psi_0}, \partial_{\psi_0}]g\|_{L^2} \le C\|g\|_{L^2}.$$

Also the commutator of derivative and the inverse operator is zero smoothing of degree -2. Specifically, note that with $f = (L_0-\Lambda)^{-1}_{\mathrm{hbc}}g$ we have

$$\begin{aligned}
[\partial_{\psi_0}, (L_0-\Lambda)^{-1}_{\mathrm{hbc}}]g &= \partial_{\psi_0}f - (L_0-\Lambda)^{-1}_{\mathrm{hbc}}\partial_{\psi_0}(L_0-\Lambda)f\\
&= (L_0-\Lambda)^{-1}_{\mathrm{hbc}}[\partial_{\psi_0}, L_0-\Lambda]f.
\end{aligned}$$

The commutator $[\partial_{\psi_0}, L_0-\Lambda]$ is a differential operator of order 2. Thus we obtain

$$\|[\partial_{\psi_0}, (L_0-\Lambda)^{-1}_{\mathrm{hbc}}]g\|_{L^2} \le C\|(L_0-\Lambda)^{-1}_{\mathrm{hbc}}g\|_{L^2},$$

and we obtain the estimate

$$\|[\partial_{\psi_0}, K_{\psi_0}]f\|_{L^2} \leq C\|(L_0 - \Lambda)^{-1}_{\text{hbc}} f\|_{L^2}.$$

Moreover, $[\partial_{\psi_0}, K_{\psi_0}]f$ is zero on the boundary. Then if $u \in S^{(-1)}$ and $g \in S_0^{(2)}$

$$K_{\psi_0}[\partial_{\psi_0} u] = [\partial_{\psi_0}, K_{\psi_0}]u + \partial_{\psi_0} g \in S_0^{(1)}.$$

It follows that $\partial_{\psi_0} u \in S^{(-1)}$ and thus $u \in S^{(0)}$. Higher regularity follows by similar arguments.

Appendix B. Proof of Proposition 3.1

Let D_0, D be two nearby domains and let $\gamma : D_0 \to D$ be a diffeomorphism. Denote points in D by (x_1, x_2) and points in D_0 by (y_1, y_2). Decompose the diffeomorphism

$$\gamma = \text{id} + (\alpha, \beta) = \text{id} + \nabla^\perp \phi + \nabla \eta.$$

where $\nabla^\perp = (-\partial_2, \partial_1)$. Let $\rho = \det \nabla\gamma$ and $0 < |\rho| < \infty$. Write $\psi = \psi_0 \circ \gamma^{-1}$. More explicitly

$$x_1 = y_1 + \alpha(y_1, y_2), \qquad x_2 = y_2 + \beta(y_1, y_2).$$

and

$$\alpha(y_1, y_2) = -\partial_{y_2}\phi + \partial_{y_1}\eta, \qquad \beta(y_1, y_2) = \partial_{y_1}\phi + \partial_{y_2}\eta.$$

We have so $\nabla_y \psi_0 = \nabla\gamma \cdot (\nabla_x \psi) \circ \gamma$, and

$$(\nabla_x \psi) \circ \gamma = (\nabla\gamma)^{-1} \cdot \nabla_y \psi_0, \qquad \nabla\gamma = I + \begin{pmatrix} \partial_{y_1}\alpha & \partial_{y_1}\beta \\ \partial_{y_2}\alpha & \partial_{y_2}\beta \end{pmatrix}.$$

Jacobian of Transformation: Note that, with $\rho = \det \nabla\gamma$ we find

$$\rho = 1 + \partial_{y_1}\alpha + \partial_{y_2}\beta + (\partial_{y_1}\alpha\partial_{y_2}\beta - \partial_{y_2}\alpha\partial_{y_1}\beta) = 1 + \Delta\eta - \mathcal{N}_\eta(\partial^2\eta, \partial^2\phi)$$

where

$$\mathcal{N}_\eta := \partial_{y_1}\alpha\partial_{y_2}\beta - \partial_{y_2}\alpha\partial_{y_1}\beta = -\nabla\alpha \cdot \nabla^\perp\beta. \tag{B.1}$$

Inverse Gradient: A useful expression for the inverse gradient of the transformation is

$$\begin{aligned}
(\nabla\gamma)^{-1} &= \frac{1}{\rho}\left(I + \begin{pmatrix} \partial_{y_2}\beta & -\partial_{y_1}\beta \\ -\partial_{y_2}\alpha & \partial_{y_1}\alpha \end{pmatrix}\right) = \frac{1}{\rho}\left(I + \begin{pmatrix} \partial_{y_2}\partial_{y_1}\phi + \partial^2_{y_2}\eta & -\partial^2_{y_1}\phi - \partial_{y_1}\partial_{y_2}\eta \\ \partial^2_{y_2}\phi - \partial_{y_1}\partial_{y_2}\eta & -\partial_{y_2}\partial_{y_1}\phi + \partial^2_{y_1}\eta \end{pmatrix}\right) \\
&= \frac{1}{\rho}\left(I + \begin{pmatrix} \partial_{y_2}\partial_{y_1}\phi & -\partial^2_{y_1}\phi \\ \partial^2_{y_2}\phi & -\partial_{y_2}\partial_{y_1}\phi \end{pmatrix} + \begin{pmatrix} \partial^2_{y_2}\eta & -\partial_{y_1}\partial_{y_2}\eta \\ -\partial_{y_1}\partial_{y_2}\eta & \partial^2_{y_1}\eta \end{pmatrix}\right) \\
&= \frac{1}{\rho}\left(I - \begin{pmatrix} \partial_{y_1}\nabla^\perp_y\phi \\ \partial_{y_2}\nabla^\perp_y\phi \end{pmatrix} + \begin{pmatrix} -\partial_{y_2}\nabla^\perp_y\eta \\ \partial_{y_1}\nabla^\perp_y\eta \end{pmatrix}\right).
\end{aligned}$$

Derivatives of $\psi := \psi_0 \circ \gamma^{-1}$: In the above, $\partial_{y_1} \nabla_y^\perp \phi$ and similar terms are understood as row vectors forming the matrices. Thus

$$(\nabla_x \psi) \circ \gamma = \frac{1}{\rho} \Big(\nabla_y \psi_0 + \nabla_y \partial_s \phi + (\nabla_y^\perp \phi \cdot \nabla_y) \nabla_y \psi_0 \Big) - \frac{1}{\rho} \Big(\nabla_y^\perp \partial_s \eta + (\nabla_y \eta \cdot \nabla_y) \nabla_y \psi_0 \Big),$$

where we introduced the notation for streamline derivatives $\partial_s = \nabla^\perp \psi_0 \cdot \nabla_y$. In the future, we will bin terms involving η in

$$\mathcal{L}_0(\partial^2 \eta, \rho; \partial^2 \psi_0) := -\frac{1}{\rho} \Big(\nabla_y^\perp \partial_s \eta + (\nabla_y \eta \cdot \nabla_y) \nabla_y \psi_0 \Big). \tag{B.2}$$

Note we track only the highest number derivatives in the notation on the left. We now obtain a formula for the Hessian in terms of ψ_0 and the diffeomorphism. First note that

$$(\nabla_x \otimes \nabla_x \psi) \circ \gamma = (\nabla_y \gamma)^{-1} \nabla_y \Big((\nabla_x \psi) \circ \gamma \Big).$$

The right-hand-side of the above is calculated as

$$\begin{aligned} \nabla_y \Big((\nabla_x \psi) \circ \gamma \Big) = \rho \nabla_y \rho^{-1} \otimes (\nabla_x \psi) \circ \gamma \\ &+ \frac{1}{\rho} \Big(\nabla_y \otimes \nabla_y \psi_0 + (\nabla_y \otimes \nabla_y) \partial_s \phi + (\nabla_y \otimes \nabla_y^\perp \phi)(\nabla_y \otimes \nabla_y \psi_0) \\ &+ (\nabla_y^\perp \phi \cdot \nabla) \nabla_y \otimes \nabla_y \psi_0 - (\nabla_y \otimes \nabla_y^\perp) \partial_s \eta \\ &- (\nabla_y \otimes \nabla_y \eta)(\nabla_y \otimes \nabla_y \psi_0) - (\nabla_y \eta \cdot \nabla) \nabla_y \otimes \nabla_y \psi_0 \Big). \end{aligned}$$

Thus we obtain

$$\begin{aligned} (\nabla_x \otimes \nabla_x \psi) \circ \gamma = \frac{1}{\rho^2} \nabla_y \otimes \nabla_y \psi_0 \ + \frac{1}{\rho^2} \Big((\nabla_y \otimes \nabla_y) \partial_s \phi - (\nabla_y \phi \cdot \nabla^\perp) \nabla_y \otimes \nabla_y \psi_0 \Big) \\ + \mathcal{L}_1(\partial^3 \eta, \partial \rho; \partial^3 \psi_0) + \mathcal{N}_1(\partial^2 \phi, \partial^2 \partial_s \phi, \partial^3 \eta, \partial \rho; \partial^3 \psi_0), \end{aligned}$$

where we have grouped the terms linear in η and $\nabla \rho$

$$\begin{aligned} &\mathcal{L}_1(\partial^3 \eta, \partial \rho; \partial^3 \psi_0) \\ &\quad := -\frac{1}{\rho^2} \Big(\nabla_y \rho \otimes (\nabla_x \psi_0) \circ \gamma + (\nabla_y \otimes \nabla_y^\perp) \partial_s \eta + (\nabla_y \eta \cdot \nabla) \nabla_y \otimes \nabla_y \psi_0 \Big), \end{aligned} \tag{B.3}$$

as well as all terms which are quadratic in combinations of $(\phi, \eta, \nabla\rho)$:

$$\begin{aligned}
&\mathcal{N}_1(\partial^2\phi, \partial^2\partial_s\phi, \partial^3\eta, \partial\rho; \partial^3\psi_0)\\
&\quad := -\frac{1}{\rho^2}\nabla_y\rho\otimes\Big(\nabla_y\partial_s\phi + (\nabla_y^\perp\phi\cdot\nabla_y)\nabla_y\psi_0 - \nabla_y^\perp\partial_s\eta - (\nabla_y\eta\cdot\nabla_y)\nabla_y\psi_0\Big)\\
&\qquad + \frac{1}{\rho^2}\left(-\begin{pmatrix}\partial_{y_1}\nabla_y^\perp\phi\\ \partial_{y_2}\nabla_y^\perp\phi\end{pmatrix} + \begin{pmatrix}-\partial_{y_2}\nabla_y^\perp\eta\\ \partial_{y_1}\nabla_y^\perp\eta\end{pmatrix}\right)\Big[-\nabla_y\rho\otimes(\nabla_x\psi_0)\circ\gamma +\\
&\qquad - \nabla_y\rho\otimes\Big(\nabla_y\partial_s\phi + (\nabla_y^\perp\phi\cdot\nabla_y)\nabla_y\psi_0 - \nabla_y^\perp\partial_s\eta - (\nabla_y\eta\cdot\nabla_y)\nabla_y\psi_0\Big)\\
&\qquad + \Big((\nabla_y\otimes\nabla_y)\partial_s\phi + (\nabla_y\otimes\nabla_y^\perp\phi)(\nabla_y\otimes\nabla_y\psi_0) + (\nabla_y^\perp\phi\cdot\nabla)\nabla_y\otimes\nabla_y\psi_0\\
&\qquad - (\nabla_y\otimes\nabla_y^\perp)\partial_s\eta - (\nabla_y\otimes\nabla_y\eta)(\nabla_y\otimes\nabla_y\psi_0) - (\nabla_y\eta\cdot\nabla)\nabla_y\otimes\nabla_y\psi_0\Big)\Big].
\end{aligned} \tag{B.4}$$

We note the important point the nonlinearity involves third derivatives of ϕ only through $\partial^2\partial_s\phi$. We now introduce stream function coordinates. Note the following formula for ∇_y in terms of derivative along and transverse to streamlines, i.e. $\partial_s = \nabla^\perp\psi_0\cdot\nabla$ and $\partial_{\psi_0} = \nabla\psi_0\cdot\nabla$,

$$\nabla_y = \frac{1}{|\nabla\psi_0|^2}\Big[\nabla\psi_0\partial_{\psi_0} + \nabla^\perp\psi_0\partial_s\Big], \qquad \nabla_y^\perp = \frac{1}{|\nabla\psi_0|^2}\Big[\nabla^\perp\psi_0\partial_{\psi_0} - \nabla\psi_0\partial_s\Big].$$

With this, we have

$$\nabla_y\phi\cdot\nabla^\perp = \frac{1}{|\nabla\psi_0|^2}\Big((\partial_s\phi)\partial_{\psi_0} - (\partial_{\psi_0}\phi)\partial_s\Big).$$

We arrive at

Lemma B.1. *The following formulae hold*

$$\begin{aligned}
(\nabla_x\psi)\circ\gamma &= \frac{1}{\rho}\nabla_y\psi_0 + \frac{1}{\rho}\left(\nabla_y\partial_s\phi + \frac{1}{|\nabla\psi_0|^2}\Big[(\partial_s\phi)\partial_{\psi_0} - (\partial_{\psi_0}\phi)\partial_s\Big]\nabla_y\psi_0\right)\\
&\quad + \mathcal{L}_0(\partial^3\eta, \rho; \partial^3\psi_0),\\
(\nabla_x\otimes\nabla_x\psi)\circ\gamma &= \frac{1}{\rho^2}\nabla_y\otimes\nabla_y\psi_0 - \frac{1}{\rho^2|\nabla\psi_0|^2}\Big[(\partial_s\phi)\partial_{\psi_0} - (\partial_{\psi_0}\phi)\partial_s\Big]\nabla_y\otimes\nabla_y\psi_0\\
&\quad + \frac{1}{\rho^2}(\nabla_y\otimes\nabla_y)\partial_s\phi + \mathcal{L}_1(\partial^3\eta, \partial\rho; \partial^3\psi_0)\\
&\quad + \mathcal{N}_1(\partial^2\phi, \partial^2\partial_s\phi, \partial^3\eta, \partial\rho; \partial^3\psi_0),
\end{aligned}$$

where $\mathcal{L}_0$, $\mathcal{L}_1$ *and* $\mathcal{N}_1$ *are defined by* (B.2), (B.3) *and* (B.4). *Let* $a_\gamma = a\circ\gamma$, $b_\gamma = b\circ\gamma$ *and*

$$L := a : \nabla_x\otimes\nabla_x + b\cdot\nabla_x, \qquad L_\gamma = a_\gamma : \nabla_y\otimes\nabla_y + b_\gamma\cdot\nabla_y.$$

Then we have

$$(L\psi)\circ\gamma = \frac{1}{\rho^2}L_\gamma\psi_0 + \frac{1}{\rho^2}L_\gamma\partial_s\phi - \frac{1}{\rho^2|\nabla\psi_0|^2}\Big[(\partial_s\phi)\partial_{\psi_0} - (\partial_{\psi_0}\phi)\partial_s\Big]L_\gamma\psi_0$$
$$+ \frac{1}{\rho^2|\nabla\psi_0|^2}\bigg(\Big[(\partial_s\phi)\partial_{\psi_0}a_\gamma - (\partial_{\psi_0}\phi)\partial_s a_\gamma\Big] : \nabla_y\otimes\nabla_y\psi_0$$
$$+ \Big[(\partial_s\phi)\partial_{\psi_0}b_\gamma - (\partial_{\psi_0}\phi)\partial_s b_\gamma\Big]\cdot\nabla_y\psi_0\bigg)$$
$$+ a : \mathcal{L}_1(\partial^3\eta, \partial\rho; \partial^3\psi_0) + a : \mathcal{N}_1(\partial^2\phi, \partial^2\partial_s\phi, \partial^3\eta, \partial\rho; \partial^3\psi_0)$$
$$+ b\cdot\mathcal{L}_0(\partial^3\eta, \partial\rho; \partial^3\psi_0).$$

We now simplify these formulae in the setting where ψ_0 and ψ satisfy (3.4), (3.5). Recall $L_0 := a_0 : \nabla_y\otimes\nabla_y + b_0\cdot\nabla_y$, so that

$$\begin{aligned}L_\gamma - L_0 &= (a_\gamma - a_0) : \nabla_y\otimes\nabla_y + (b_\gamma - b_0)\cdot\nabla_y\\ &= (a - a_0)_\gamma : \nabla_y\otimes\nabla_y\\ &\quad + (b - b_0)_\gamma\cdot\nabla_y + ((a_0)_\gamma - a_0) : \nabla_y\otimes\nabla_y + ((b_0)_\gamma - b_0)\cdot\nabla_y.\end{aligned}$$

We now denote the nonlinearities arising in expanding by

$$\mathcal{R}_{a_0}(\partial\phi, \partial\eta, \partial^2 a_0) := (a_0)_\gamma - a_0 - (\gamma - y)\cdot\nabla a_0,$$
$$\mathcal{R}_{b_0}(\partial\phi, \partial\eta, \partial^2 b_0) := (b_0)_\gamma - b_0 - (\gamma - y)\cdot\nabla b_0.$$

Note that the dependences are a consequence of Taylor's formula. Then

$$\begin{aligned}L_\gamma - L_0 &= (a - a_0)_\gamma : \nabla_y\otimes\nabla_y + (b - b_0)_\gamma\cdot\nabla_y\\ &\quad + (\gamma - y)\cdot\nabla a_0 : \nabla_y\otimes\nabla_y + (\gamma - y)\cdot\nabla b_0\cdot\nabla_y\\ &\quad + \mathcal{R}_{a_0}(\partial\phi, \partial\eta, \partial^2 a_0) : \nabla_y\otimes\nabla_y + \mathcal{R}_{b_0}(\partial\phi, \partial\eta, \partial^2 b_0)\cdot\nabla_y\\ &= (a - a_0)_\gamma : \nabla_y\otimes\nabla_y + (b - b_0)_\gamma\cdot\nabla_y\\ &\quad + \nabla^\perp\phi\cdot\nabla a_0 : \nabla_y\otimes\nabla_y + \nabla^\perp\phi\cdot\nabla b_0\cdot\nabla_y\\ &\quad + \nabla\eta\cdot\nabla a_0 : \nabla_y\otimes\nabla_y + \nabla\eta\cdot\nabla b_0\cdot\nabla_y\\ &\quad + \mathcal{R}_{a_0}(\partial\phi, \partial\eta, \partial^2 a_0) : \nabla_y\otimes\nabla_y + \mathcal{R}_{b_0}(\partial\phi, \partial\eta, \partial^2 b_0)\cdot\nabla_y.\end{aligned}$$

Thus we have

$$\begin{aligned}(L_\gamma - L_0)\psi_0 &= -\frac{1}{|\nabla\psi_0|^2}\big((\partial_s\phi)\partial_{\psi_0}a_0 - (\partial_{\psi_0}\phi)\partial_s a_0\big) : \nabla_y\otimes\nabla_y\psi_0\\ &\quad - \frac{1}{|\nabla\psi_0|^2}\big((\partial_s\phi)\partial_{\psi_0}b_0 - (\partial_{\psi_0}\phi)\partial_s b_0\big)\cdot\nabla_y\psi_0\\ &\quad + \mathcal{L}_2(\delta a, \delta b, \partial a_0, \partial b_0, \partial\eta, \gamma; \partial^2\psi_0)\\ &\quad + \mathcal{N}_2(\partial^2 a_0, \partial^2 b_0, \partial\phi, \partial\eta, \gamma; \partial^2\psi_0),\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{L}_2(\delta a, \delta b, \partial a_0, \partial b_0, \partial\eta, \gamma; \partial^2\psi_0)\\
&\quad := (a - a_0)_\gamma : \nabla_y \otimes \nabla_y \psi_0 + (b - b_0)_\gamma \cdot \nabla_y \psi_0\\
&\qquad + \nabla\eta \cdot \nabla a_0 : \nabla_y \otimes \nabla_y \psi_0 + \nabla\eta \cdot \nabla b_0 \cdot \nabla_y \psi_0,\\
&\mathcal{N}_2(\partial^2 a_0, \partial^2 b_0, \partial\phi, \partial\eta, \gamma; \partial^2\psi_0)\\
&\quad := \mathcal{R}_{a_0}(\partial\phi, \partial\eta, \partial^2 a_0) : \nabla_y \otimes \nabla_y \psi_0 + \mathcal{R}_{b_0}(\partial\phi, \partial\eta, \partial^2 b_0) \cdot \nabla_y \psi_0.
\end{aligned}$$

Additionally we find

$$\begin{aligned}
&(L_\gamma - L_0)\partial_s\phi\\
&\quad = \mathcal{L}_3(\delta a, \delta b, \partial a_0, \partial b_0, \partial\eta, \gamma, \partial^2\partial_s\phi) + \mathcal{N}_3(\partial^2 a_0, \partial^2 b_0, \partial\phi, \partial\eta, \gamma, \partial^2\partial_s\phi).
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{L}_3(\delta a, \delta b, \partial a_0, \partial b_0, \partial\eta, \gamma, \partial^2\partial_s\phi)\\
&\quad := (a - a_0)_\gamma : \nabla_y \otimes \nabla_y \partial_s\phi + (b - b_0)_\gamma \cdot \nabla_y \partial_s\phi\\
&\mathcal{N}_3(\partial^2 a_0, \partial^2 b_0, \partial\phi, \partial\eta, \gamma, \partial^2\partial_s\phi)\\
&\quad := \nabla^\perp\phi \cdot \nabla a_0 : \nabla_y \otimes \nabla_y \partial_s\phi + \nabla^\perp\phi \cdot \nabla b_0 \cdot \nabla_y \partial_s\phi\\
&\qquad + \nabla\eta \cdot \nabla a_0 : \nabla_y \otimes \nabla_y \partial_s\phi + \nabla\eta \cdot \nabla b_0 \cdot \nabla_y \partial_s\phi\\
&\qquad + \mathcal{R}_{a_0}(\partial\phi, \partial\eta, \partial^2 a_0) : \nabla_y \otimes \nabla_y \partial_s\phi + \mathcal{R}_{b_0}(\partial\phi, \partial\eta, \partial^2 b_0) \cdot \nabla_y \partial_s\phi.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\rho^2(L\psi)\circ\gamma = L_0\psi_0 + L_0\partial_s\phi &- \frac{1}{|\nabla\psi_0|^2}\left((\partial_s\phi)\partial_{\psi_0}a_0 - (\partial_{\psi_0}\phi)\partial_s a_0\right) : \nabla_y \otimes \nabla_y \psi_0\\
&- \frac{1}{|\nabla\psi_0|^2}\left((\partial_s\phi)\partial_{\psi_0}b_0 - (\partial_{\psi_0}\phi)\partial_s b_0\right) \cdot \nabla_y \psi_0\\
&- \frac{1}{|\nabla\psi_0|^2}\Big[(\partial_s\phi)\partial_{\psi_0} - (\partial_{\psi_0}\phi)\partial_s\Big]L_0\psi_0\\
&+ \frac{1}{|\nabla\psi_0|^2}\Big[(\partial_s\phi)\partial_{\psi_0}a - (\partial_{\psi_0}\phi)\partial_s a\Big] : \nabla_y \otimes \nabla_y \psi_0\\
&+ \frac{1}{|\nabla\psi_0|^2}\Big[(\partial_s\phi)\partial_{\psi_0}b - (\partial_{\psi_0}\phi)\partial_s b\Big] \cdot \nabla_y \psi_0\\
&+ \mathcal{L}_5(\delta a, \delta b, \partial a_0, \partial b_0, \partial^3\eta, \gamma, \partial^2\partial_s\phi; \partial^3\psi_0)\\
&+ \mathcal{N}_5(\partial^2 a_0, \partial^2 b_0, \partial^2\phi, \partial^2\eta, \partial\rho, \partial^2\partial_s\phi; \partial^3\psi_0)
\end{aligned}$$

where the linear and nonlinear terms are

$$\mathcal{L}_5 = \sum_{i=0}^{4}\mathcal{L}_i, \qquad \mathcal{N}_5 = \sum_{i=1}^{4}\mathcal{N}_i. \tag{B.5}$$

Rearranging this, we have

$$\begin{aligned}
\rho^2(L\psi)\circ\gamma = L_0\psi_0 &+ (L_0 - \lambda_1)\partial_s\phi + \lambda_2\partial_{\psi_0}\phi\\
&+ \mathcal{L}_5(\delta a, \delta b, \partial a_0, \partial b_0, \partial^3\eta, \partial\rho, \partial^2\partial_s\phi; \partial^3\psi_0)\\
&+ \mathcal{N}_5(\partial^2 a_0, \partial^2 b_0, \partial^2\phi, \partial^2\eta, \partial\rho, \partial^2\partial_s\phi; \partial^3\psi_0)
\end{aligned}$$

where $\lambda_1 := \frac{\partial_{\psi_0} L_0\psi_0}{|\nabla\psi_0|^2}$ and $\lambda_2 := \frac{\partial_s L_0\psi_0}{|\nabla\psi_0|^2}$. To get the desired equation for $\partial_s\phi$ we must use the equations that ψ_0 and ψ satisfy. Recall

$$L\psi = F(\psi) + G(x, \psi), \qquad L_0\psi = F_0(\psi_0) + G_0(y, \psi_0),$$

Then we have

$$(L\psi)\circ\gamma - F(\psi_0) + G(\gamma, \psi_0) = 0,$$

Thus upon substitution we obtain

$$\begin{aligned}(L_0 + \Lambda)\partial_s\phi &= L_0\psi_0 - \rho^2(L\psi)\circ\gamma + \Lambda_2\partial_{\psi_0}\phi + \mathcal{L}_5 + \mathcal{N}_5\\ &= F_0(\psi_0) - F(\psi_0) + G_0(y, \psi_0) - G(\gamma, \psi_0)\\ &\quad + \Lambda_2\partial_{\psi_0}\phi + \mathcal{L}_5 + \mathcal{N}_5.\end{aligned}$$

Introducing the notation

$$\mathcal{R}_G(\partial\phi, \partial\eta, \partial_Y^2 G) := G_0(y, \psi_0) - G(\gamma, \psi_0) - (\gamma - y)\cdot(\nabla_y G)\circ\gamma,$$

we further express the nonlinear G terms as follows

$$\begin{aligned}G_0(y, \psi_0) - G(\gamma, \psi_0) &= \frac{1}{|\nabla\psi_0|^2}\Big((\partial_s\phi)(\partial_{\psi_0}G)(\gamma, \psi_0) - (\partial_{\psi_0}\phi)(\partial_s G)(\gamma, \psi_0)\Big)\\ &\quad - \nabla\eta\cdot(\nabla_y G)(\gamma, \psi_0) + (G - G_0)(\gamma, \psi_0) + \mathcal{R}_G. \qquad \text{(B.6)}\end{aligned}$$

Together, we obtain

$$\begin{aligned}(L_0 - \Lambda)\partial_s\phi &= \Lambda_2\partial_{\psi_0}\phi + (F - F_0)(\psi_0) + \mathcal{L}_\phi(\delta a, \delta b, \partial a_0, \partial b_0, \partial^3\eta, \partial\rho, \partial^2\partial_s\phi; \partial^3\psi_0)\\ &\quad + \mathcal{N}(\partial^2 a_0, \partial^2 b_0, \partial^2\phi, \partial^2\eta, \partial\rho, \partial^2\partial_s\phi; \partial^3\psi_0)\end{aligned}$$

where we have defined

$$\mathcal{L}_\phi = \mathcal{L}_5, \qquad \text{(B.7)}$$
$$\mathcal{N}_\phi = \mathcal{N}_5 + \text{(B.6)} \qquad \text{(B.8)}$$

where $\mathcal{L}_5$ and $\mathcal{N}_5$ are defined in (B.5) and where

$$\Lambda_1 := \frac{1}{|\nabla\psi_0|^2}\Big(\partial_{\psi_0}L_0\psi_0 - \partial_{\psi_0}G(y, \psi_0)\Big), \qquad \Lambda_2 := \frac{1}{|\nabla\psi_0|^2}\Big(\partial_s L_0\psi_0 - \partial_s G(y, \psi_0)\Big).$$

Note that we separate out $(F - F_0)(\psi_0)$ since we will use F to fix $\partial_s\phi$ as mean-zero on streamlines during the construction. We note now that

$$\partial_{\psi_0}L_0\psi_0 = |\nabla\psi_0|^2 F_0'(\psi_0) + \partial_{\psi_0}G + |\nabla\psi_0|^2 G_0'(y, \psi_0), \quad \partial_s L_0\psi_0 = \partial_s G_0,$$

where G_0' denotes differentiation with respect to its ψ_0 argument. Thus, introducing

$$\Lambda := F_0'(\psi_0) + G_0'(y, \psi_0), \qquad \text{(B.9)}$$

we obtain

Lemma B.2. *If ψ_0 solves (3.4) and $\psi = \psi_0 \circ \gamma$ solves (3.5) then $\partial_s \phi$ satisfies*

$$\Delta\eta = \rho - 1 + \mathcal{N}_\eta(\partial^2\eta, \partial^2\phi),$$
$$(L_0 - \Lambda)\partial_s\phi = (F - F_0)(\psi_0) + \mathcal{L}_\phi(\delta a, \delta b, \delta F, \partial\delta G, \partial a_0, \partial b_0, \partial^3\eta, \partial\rho, \partial^2\partial_s\phi, \partial_{\psi_0}\phi; \partial^3\psi_0) + \mathcal{N}_\phi(\partial^2 a_0, \partial^2 b_0, \partial^2\phi, \partial^2\eta, \partial\rho, \partial^2\partial_s\phi; \partial^3\psi_0),$$

where Λ is given by (B.9), $\mathcal{L}_\phi$ (defined by (B.7)) are all the collected terms which are linear in ϕ and η (and their derivatives), but all multiplied by small factors, $\mathcal{N}_\eta$ is defined by (B.1) and $\mathcal{N}_\phi$ collects the nonlinear terms above (defined by (B.8)) .

Appendix C. Proof of Theorem 3.1

C.1. Perturbative assumptions. We will make the following assumptions that ensure that various quantities we will encounter can be treated perturbatively.

- The density ρ satisfies
$$\|\rho - 1\|_{C^{k,\alpha}(D_0)} \le \epsilon_1. \tag{C.1}$$
- The boundary ∂D is given by $\{B = 0\}$ and ∂D_0 is given by $\{B_0 = 0\}$ where B, B_0 are smooth functions defined in a neighborhood of ∂D_0 and
$$\|B - B_0\|_{C^{k,\alpha}} \le \epsilon_2. \tag{C.2}$$
- The operators L_0, L are close in the sense that the coefficients satisfy
$$\|a - a_0\|_{C^{k,\alpha}(D_0)} + \|b - b_0\|_{C^{k,\alpha}(D_0)} \le \epsilon_3. \tag{C.3}$$
- The nonlinearities/forcings are close in the sense that
$$\|G - G_0\|_{C^{k-2,\alpha}(D_0)} \le \epsilon_4. \tag{C.4}$$

The size of the parameters $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ will be set in Lemma C.5 and depends on D_0, the base solution ψ_0, and the operator L_0.

C.2 Boundary conditions. Suppose D_0 is given as the interior of a Jordan curve B_0 in $\mathbb{R}^2$,

$$\partial D_0 = \{p \in \mathbb{R}^2 \mid B_0(p) = 0\}.$$

For convenience we will assume, without loss of generality, that B_0 is given so that $|\nabla B_0| = 1$ and $\nabla\psi_0 \cdot \nabla B_0 > 0$. Suppose that D is given as the interior of a Jordan curve B,

$$\partial D = \{p \in \mathbb{R}^2 \mid B(p) = 0\}.$$

If $\gamma : D_0 \to D$ is of the form $\gamma = \mathrm{id} + (\alpha, \beta)$, then using that $B_0|_{\partial D_0} = 0$, the requirement that $\gamma : \partial D_0 \to \partial D$ can be written as

$$\begin{aligned} 0 = B \circ \gamma|_{\partial D_0} &= B_0 \circ \gamma|_{\partial D_0} + (\delta B) \circ \gamma|_{\partial D_0} \\ &= \alpha\partial_1 B_0|_{\partial D_0} + \beta\partial_2 B_0|_{\partial D_0} + B_1(\alpha, \beta)|_{\partial D_0} \end{aligned} \tag{C.5}$$

where the remainder B_1 is

$$B_1(\alpha, \beta, x, y) = B_0 \circ \gamma - B_0 - \alpha \partial_1 B_0 - \beta \partial_2 B_0 + (\delta B) \circ \gamma \tag{C.6}$$

which will be small, $O(\alpha^2, \beta^2, \delta B)$, provided $\alpha, \beta, \delta B$ are small and that $B_0 \in C^2$, say. It is convent to write α, β in terms of a gradient and skew gradient of ϕ, η,

$$(\alpha, \beta) = \nabla^\perp \psi + \nabla \eta.$$

In this case,

$$\alpha \partial_1 B_0 + \beta \partial_2 B_0 = \nabla^\perp B_0 \cdot \nabla \phi + \nabla B_0 \cdot \nabla \eta.$$

Since $|\nabla B_0| = 1$, to follows that ∇B_0 is the outward-facing unit normal vector field, $\hat{n}$ to D_0 and $\nabla^\perp B_0$ is the unit tangential vector field forming a right-handed basis with ∇B_0. Since ψ_0 is constant on ∂B_0 we in fact have $\nabla^\perp B_0 = \frac{\nabla^\perp \psi_0}{|\nabla \psi_0|}$. Using this, we re-write (C.5) as the condition

$$\frac{1}{|\nabla \psi_0|} \partial_s \phi + \partial_n \eta = -B_1(\phi, \eta), \qquad \text{on} \qquad \partial D_0.$$

We will choose η so that $\partial_n \eta$ is constant on the boundary and so that $\partial_s \phi$ has zero average along streamlines, i.e. $\oint_{\psi_0} \partial_s \phi \, \mathrm{d}s = 0$ where $\mathrm{d}s = \mathrm{d}\ell / |\nabla \psi_0|$ and ℓ is the arc-length parameter. We will construct η, ϕ so that they satisfy

$$\partial_n \eta = -\frac{\oint_{\partial D_0} B_1(\phi, \eta) \, d\ell}{\text{length}(\partial D_0)} \qquad \text{on } \partial D_0, \tag{C.7}$$

$$\partial_s \phi = |\nabla \psi_0| \left(-B_1(\phi, \eta) + \frac{\oint_{\partial D_0} B_1(\phi, \eta) \, d\ell}{\text{length}(\partial D_0)} \right) \qquad \text{on } \partial D_0. \tag{C.8}$$

This choice is made so that the integral of the right-hand side of (C.8) along streamlines is zero.

C.3 Governing equations for η and $\partial_s \phi$. By Proposition 3.1, if $\det \nabla \gamma = \rho$, we have

$$\begin{aligned} \Delta \eta &= \rho - 1 + \mathcal{N}_\eta, && \text{in } D_0, \\ \partial_n \eta &= -\frac{\oint_{\partial D_0} B_1(\phi, \eta) \, d\ell}{\text{length}(\partial D_0)} && \text{on } \partial D_0, \end{aligned} \tag{C.9}$$

where B_1 is defined as in (C.6), and where $\mathcal{N}_\eta$ is a homogeneous quadratic polynomial depending on $\partial^2 \eta, \partial^2 \phi$ defined in (B.1). The equation for $\partial_s \phi$ takes the form

$$\big(L_0 - \Lambda\big) \partial_s \phi = (F - F_0) + \mathcal{L}_\phi + \mathcal{N}_\phi, \qquad \text{in } D_0, \tag{C.10}$$

$$\partial_s \phi = |\nabla \psi_0| \left(-B_1(\phi, \eta) + \frac{\oint_{\partial D_0} B_1(\phi, \eta) \, d\ell}{\text{length}(\partial D_0)} \right) \qquad \text{on } \partial D_0, \tag{C.11}$$

where L_0 is the elliptic operator defined in (3.4) and $\mathcal{L}_\phi$ are terms which are linear in derivatives of ϕ and η and ρ multiplied by small factors and $\mathcal{N}_\phi$ are quadratically nonlinear terms in derivatives of ϕ and η and ρ. In this formulation, the function F is unknown and will need to be chosen to be consistent with the fact that $\oint \partial_s \phi d\ell = 0$. This point will be explained in detail in the next section. We emphasize that $\mathcal{L}$ and $\mathcal{N}$ do not involve arbitrary third derivatives of η, ϕ and it is only $\partial^2 \partial_s \eta$, $\partial^2 \partial_s \phi$ that enter, which will be important in what follows. The following estimates are immediate consequences of the definitions of the terms on the right-hand sides of (C.9)–(C.10) in which can be found in (B.1), (B.7) and (B.8). Note that these quantities involve three derivatives of ψ_0 but we are assuming $F_0 \in C^{k-1,\alpha}$ so by standard elliptic estimates $\|\psi_0\|_{C^{k+1,\alpha}}$ is finite.

Lemma C.1. *If the bounds in § C.1 hold, then we have*

$$
\begin{aligned}
\|\mathcal{L}_\phi\|_{C^{k-2,\alpha}(D_0)} &\le C_{k,\alpha}\big(\|\rho - 1\|_{C^{k-1,\alpha}(D_0)} \\
&\qquad + \|\eta\|_{C^{k,\alpha}(D_0)} + \|\partial_s \eta\|_{C^{k,\alpha}(D_0)} + \epsilon \|\partial_s \phi\|_{C^{k-1,\alpha}(D_0)}\big),\\
\|\mathcal{N}_\phi\|_{C^{k-2,\alpha}(D_0)} &\le C_{k,\alpha}\big(\|\rho - 1\|_{C^{k-1,\alpha}(D_0)} \\
&\qquad + (\|\eta\|_{C^{k,\alpha}(D_0)} + \|\partial_s \eta\|_{C^{k,\alpha}(D_0)} + \|\partial_s \phi\|_{C^{k,\alpha}(D_0)})^2\big),\\
\|\mathcal{N}_\eta\|_{C^{k-2,\alpha}(D_0)} &\le C_{k,\alpha}\big(\|\eta\|_{C^{k,\alpha}(D_0)} + \|\phi\|_{C^{k,\alpha}(D_0)}\big)^2,\\
\|B_1\|_{C^{k-1,\alpha}(\partial D_0)} &\le C_{k,\alpha}\Big(\|\eta\|_{C^{k,\alpha}(D_0)} + \|\phi\|_{C^{k,\alpha}(D_0)}\Big),
\end{aligned}
$$

where $\epsilon = \max\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$.

We also need Lipschitz bounds for the operators $\mathcal{L}_\phi, \mathcal{N}_\phi, \mathcal{N}_\eta$. Given functions ϕ_1, η_1, ϕ_1, η_2 we write $u_1 = (\phi_1, \eta_1)$, $u_2 = (\phi_2, \eta_2)$ and let $\mathcal{L}^i_\phi, \mathcal{N}^i_\phi, \mathcal{N}^i_\eta$ for $i = 1, 2$ denote the operators $\mathcal{L}_\phi, \mathcal{N}_\phi, \mathcal{N}_\eta$ defined in Proposition 3.1 evaluated at (ϕ_i, η_i). The following estimates are then straightforward consequences of the definitions.

Lemma C.2. *If the bounds in § C.1 hold, then we have*

$$
\begin{aligned}
\|\mathcal{L}^1_\phi - \mathcal{L}^2_\phi\|_{C^{k-2,\alpha}(D_0)} &\le C_{k,\alpha}\big(\|\eta_1 - \eta_2\|_{C^{k,\alpha}(D_0)} + \|\partial_s \eta_1 - \partial_s \eta_2\|_{C^{k,\alpha}(D_0)} \\
&\qquad + \epsilon \|\partial_s \phi_1 - \partial_s \phi_2\|_{C^{k,\alpha}(D_0)}\big),\\
\|\mathcal{N}^1_\phi - \mathcal{N}^2_\phi\|_{C^{k-2,\alpha}(D_0)} &\le C_{k,\alpha}\big(\|\eta_1 - \eta_2\|_{C^{k,\alpha}(D_0)} + \|\partial_s \eta_1 - \partial_s \eta_2\|_{C^{k,\alpha}(D_0)} \\
&\qquad + \|\partial_s \phi_1 - \partial_s \phi_2\|_{C^{k,\alpha}(D_0)}\big)^2,\\
\|\mathcal{N}^1_\eta - \mathcal{N}^2_\eta\|_{C^{k-2,\alpha}(D_0)} &\le C_{k,\alpha}\big(\|\eta_1 - \eta_2\|_{C^{k,\alpha}(D_0)} + \|\phi_1 - \phi_2\|_{C^{k,\alpha}(D_0)}\big)^2.
\end{aligned}
$$

C.4 Recovering ϕ from $\partial_s \phi$. In the construction, a solution Φ is obtained by solving (C.10)–(C.11) for "$\partial_s \phi = \nabla^\perp \psi_0 \cdot \nabla \psi$". Consistent with $\Phi = \partial_s \phi$, we will construct a solution Φ with the property that its integral on each streamline is zero. This is done further in the proof and requires the use of (**H3**). To verify that it is indeed the "streamline derivative" and to recover the periodic function ϕ, we appeal to the following lemma.

Lemma C.3. *Suppose* $\Phi \in C^{k,\alpha}$ *satisfies*

$$
\oint_{\{\psi_0 = c\}} \Phi = 0 \qquad \text{for all} \qquad c \in \operatorname{im}(\psi_0).
$$

Then $\Phi = \partial_s \phi$ *for a unique function* $\phi = \phi(\psi_0, \theta)$ *which is a zero-mean, periodic function on streamlines of* ψ_0*, i.e.* $\phi(\psi_0, 0) = \phi(\psi_0, 2\pi)$ *and* $\oint_{\psi_0} \phi = 0$*. Moreover,* ϕ *enjoys the bound*

$$
\|\phi\|_{k,\alpha} \le C \|\partial_s \phi\|_{k,\alpha}, \tag{C.12}
$$

where the constant C *depends on* $C^{k,\alpha}$ *norms of* ψ_0.

Proof. First note that all that enters in the formulation of the problem is $\partial_s\phi$ and not ϕ itself and so we are free to modify ϕ by adding an arbitrary function of ψ_0. To be more precise, let ℓ be the arc-length along curves $\{\psi = c\}$ and introduce the notation

$$\mathrm{d}s = \frac{\mathrm{d}\ell}{|\nabla\psi_0|}.$$

We the fix the freedom in defining ϕ by enforcing that, on each streamline,

$$\oint_{\{\psi_0=c\}} \phi \,\mathrm{d}s = 0, \qquad \forall c \in \mathrm{im}(\psi_0).$$

Assuming that this holds, we have $\|\phi\|_{k,\alpha} \le C\|\partial_s\phi\|_{k,\alpha}$, a fact that we use repeatedly in what follows. To be more precise, we introduce an "angular coordinate" along streamlines as

$$\theta(x) = \frac{2\pi}{\mu(\psi_0(x))}\int_{\Gamma_{x_0(\psi_0),x}} \mathrm{d}s, \qquad \mu(c) = \oint_{\{\psi_0=c\}} \mathrm{d}s$$

where μ is the travel time of a particle along a streamline and where, for each $x \in D_0$ the line integral is taken counterclockwise from an arbitrary point $x_0(\psi_0)$ on the streamline to the point x. This point $x_0(\psi_0)$ can be obtain by flowing an arbitrary point $p \in D_0$ by the vector field $\nabla\psi_0$ which is orthogonal to streamlines. This segment is denoted by $\Gamma_{x_0(\psi_0),x}$. Then $\theta(x)$ is a 2π–periodic parametrization of the streamline with value $\psi_0(x)$.

Now, given a Φ which is mean zero on streamlines, note that for an arbitrary $\theta_0 \in [0, 2\pi]$

$$\phi(\psi_0, \theta) = \phi(\psi_0, \theta_0) + \mu(\psi_0)\int_{\theta_0}^{\theta} \Phi \,\mathrm{d}\theta'.$$

Integrating this expression along the streamline in s_0 we ψ_0 =const.

$$\phi(y) = \phi(\psi_0(y), \theta(y)), \qquad \phi(\psi_0, \theta) = \mu(\psi_0)\int_{\mathbb{T}}\left(\int_{\theta_0}^{\theta} \Phi \,\mathrm{d}\theta'\right)\mathrm{d}\theta_0.$$

One can check that $\nabla^\perp\psi_0 \cdot \nabla\theta = \mu^{-1}$. Thus, for the quantity defined above we have that

$$\Phi = \partial_s\phi.$$

The definitions of θ and μ as functions on D_0 we obtain the estimate (C.12). □

C.5. The iteration to solve the nonlinear elliptic system. We use the following iteration. Given $\eta^n, \phi^n \in C^{k-1,\alpha}(D_0)$ with $\partial_s\eta^n, \partial_s\phi^n \in C^{k-1,\alpha}(D_0)$ and $\oint_{\psi_0}\phi^n = 0$, set

$$\mathcal{N}_\eta^n := \mathcal{N}_\eta(\eta^n, \phi^n),$$

with $\mathcal{N}_\eta$ defined in (B.1), which satisfies the following bound

$$\|\mathcal{N}_\eta^n\|_{C^{k-2,\alpha}(D_0)} \le C_{k,\alpha}\big(\|\phi^n\|_{C^{k,\alpha}(D_0)} + \|\eta^n\|_{C^{k,\alpha}(D_0)}\big)^2.$$

By Lemma D.2 the following problem has a unique solution $\eta^{n+1} \in C^{k-1,\alpha}$,

$$\Delta\eta^{n+1} = \rho - 1 + \mathcal{N}_\eta^n, \qquad \text{in } D_0, \tag{C.13}$$

$$\partial_n\eta^{n+1} = \kappa^{n+1} \equiv \int_{D_0} \left(\rho - 1 + \mathcal{N}_\eta^n\right) \qquad \text{on } \partial D_0. \tag{C.14}$$

Moreover, the iterate η^{n+1} enjoys the following estimate

$$\|\eta^{n+1}\|_{C^{k,\alpha}(D_0)} \leq C_{k,\alpha}\big(\|\rho - 1\|_{C^{k-2,\alpha}(D_0)} + \|\mathcal{N}_\eta^n\|_{C^{k-2,\alpha}(D_0)} + \kappa^{n+1}\big).$$

We note that (C.14) does not agree with (C.7) but instead has been chosen to ensure that the Neumann problem is solvable. In the upcoming Lemma C.7 we show that provided η^n, ϕ^n converge, the limit η will satisfy (C.7) as a consequence of the assumption that $\mathrm{Vol}(D) = \int_{D_0} \rho$.

In order to get an estimate for $\|\partial_s\eta^{n+1}\|_{C^{k,\alpha}(D_0)}$ we commute the Eqs. (C.13)–(C.14) with ∂_s. Applying ∂_s to (C.13), using

$$[\partial_s, \Delta] = -2\nabla^\perp\Delta\psi_0 \cdot \nabla - \nabla \otimes \nabla^\perp\psi_0 : \nabla \otimes \nabla,$$

we note that the right-hand side involves highest-order derivatives falling on $\partial_s\eta$ and lower-order terms. By the estimates for the Neumann problem from Lemma D.2 and using that $\partial_s\eta = 0$ on the boundary since ∂_s is a tangential derivative, we have

$$\|\partial_s\eta^{n+1}\|_{C^{k,\alpha}(D_0)} \leq C_{k,\alpha}\|\eta^{n+1}\|_{C^{k,\alpha}(D_0)} + C_{k,\alpha}\big(\|\partial_s(\rho - 1)\|_{C^{k,\alpha}(D_0)} + \|\partial_s\mathcal{N}_\eta^n\|_{C^{k-2,\alpha}(D_0)}\big).$$

With η^{n+1} defined, we now set

$$\mathcal{N}_\phi^n = \mathcal{N}_\phi(\eta^{n+1}, \phi^n),$$
$$B_1^n = B_1(\eta^{n+1}, \phi^n),$$

with $\mathcal{N}_\phi$ defined in (C.10) and B_1 defined in (C.6). Using that $\|\phi^n\|_{C^{k-1,\alpha}(D_0)} \leq \|\partial_s\phi^n\|_{C^{k-1,\alpha}(D_0)}$ from Lemma C.3, we have that

$$\begin{aligned}\|\mathcal{N}_\phi^n\|_{C^{k-2,\alpha}(D_0)} \leq C_{k,\alpha}\big(&\|\rho - 1\|_{C^{k,\alpha}(D_0)} + (\|\eta^{n+1}\|_{C^{k,\alpha}(D_0)} + \|\partial_s\eta^{n+1}\|_{C^{k,\alpha}(D_0)}\\ &+\|\partial_s\phi^n\|_{C^{k,\alpha}(D_0)})^2\\ &+\|\eta^{n+1}\|_{C^{k,\alpha}(D_0)} + \|\partial_s\eta^{n+1}\|_{C^{k,\alpha}(D_0)} + \epsilon\|\partial_s\phi^n\|_{C^{k-1,\alpha}(D_0)}\big),\end{aligned}$$

with $\epsilon = \max\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$. We note that it is crucial that the estimate for the nonlinearity $\mathcal{N}_\phi^n$ only requires a bound for $\|\partial_s\eta^{n+1}\|_{C^{k,\alpha}(D_0)}$ and not the full norm $\|\eta^{n+1}\|_{C^{k+1,\alpha}(D_0)}$ since we could only get a bound for this term by differentiating the equation for η^{n+1} in all directions and this would require a bound for $\|\phi^{n+1}\|_{C^{k+1,\alpha}(D_0)}$ instead of just $\|\partial_s\phi^{n+1}\|_{C^{k,\alpha}(D_0)}$. The boundary operator satisfies the estimate

$$\|B_1^n\|_{C^{k,\alpha}(D_0)} \leq C_{k,\alpha}\Big(\|\eta^{n+1}\|_{C^{k,\alpha}(D_0)} + \|\delta B\|_{C^{k,\alpha}(D_0)}\Big).$$

We now envoke hypothesis (**H3**) and define F^n by the requirement that the right-hand side of (3.7) has zero average along streamlines with ϕ, η replaced by ϕ^n, η^n. Consider the problem

$$\begin{aligned}(L_0 - \Lambda)u &= g \qquad \text{in } D_0\\ u &= u_b \qquad \text{on } \partial D_0.\end{aligned}$$

Letting G be the Green's function for the Dirichlet problem for $L_0 - \Lambda$, we have

$$u(x) = \int_{D_0} G(x, x')g(x')\mathrm{d}x' + \oint_{\partial D_0} \partial_n G(x, x')u_b(x')\mathrm{d}\ell.$$

We define

$$(L_0 - \Lambda)^{-1}_{\mathrm{hbc}} g := \int_{D_0} G(x, x')g(x')\mathrm{d}x',$$

so that $f = (L_0 - \Lambda)^{-1}_{\mathrm{hbc}} g$ means $(L_0 - \Lambda)f = g$ and $f = 0$ on ∂D_0. If the Eqs. (C.10)–(C.11) are to hold then since $\oint_{\psi_0=c} \partial_s \phi = 0$ we must ensure that

$$K_{\psi_0}[F^n - F_0] = -\oint_{\psi_0} (L_0 - \Lambda)^{-1}_{\mathrm{hbc}}[\mathcal{L}^n_\phi + \mathcal{N}^n_\phi]\mathrm{d}s + \oint_{\psi_0} \oint_{\partial D_0} \partial_n G(x, x')\partial_s \phi(x')\mathrm{d}\ell \mathrm{d}s, \tag{C.15}$$

with K_{ψ_0} defined in the statement of (**H3**). Notice that the right-hand-side is a function only ψ_0, and the boundary conditions for $\partial_s \phi$ are chosen exactly so that the contribution on the boundary of the final term in (C.15) is zero. Thus, with $g^n(\psi_0)$ defined as the right-hand-side of equation (C.15), we see that $g^n(\psi_0(\partial D_0)) = 0$. Moreover, $g \in C^{k,\alpha}(I)$ with $I = \mathrm{im}(\psi_0)$, which follows from Lemma E.2 Appendix E. These verify that we are in the setting of (**H3**) and so by assumption there is an $F^n = F^n(\psi_0) \in C^{k-2,\alpha}(I)$ which ensures that (C.15) holds. Moreover,

$$\begin{aligned}\|F^n - F_0\|_{C^{k-2,\alpha}(I)} &\lesssim \|K_{\psi_0}[\mathcal{L}^n_\phi + \mathcal{N}^n_\phi]\|_{C^{k,\alpha}} \lesssim \|(L_0 - \Lambda)^{-1}_{\mathrm{hbc}}\Big(\mathcal{L}^n_\phi + \mathcal{N}^n_\phi\Big)\|_{C^{k,\alpha}}\\ &\lesssim \|\mathcal{L}^n_\phi + \mathcal{N}^n_\phi\|_{C^{k-2,\alpha}(D_0)}.\end{aligned} \tag{C.16}$$

The second inequality above follows from Lemma C.15.

By Lemma D.1 the following problem has a unique solution $\Phi^{n+1} \in C^{k,\alpha}$,

$$(L_0 - \Lambda)\Phi^{n+1} = (F^n - F_0)(\psi_0) + \mathcal{L}^n_\phi + \mathcal{N}^n_\phi, \qquad \text{in } D_0, \tag{C.17}$$

$$\Phi^{n+1} = |\nabla\psi_0| \left(-B_1(\phi^n, \eta^n) + \frac{\oint_{\partial D_0} B_1(\phi^n, \eta^n)\, d\ell}{\mathrm{length}(\partial D_0)}\right) \qquad \text{on } \partial D_0. \tag{C.18}$$

By our choice for F^n and the above discussion, the solution Φ^{n+1} has zero average along streamlines and so by Lemma C.3 it follows that $\Phi^{n+1} = \partial_s \phi^{n+1}$ for a unique function ϕ^{n+1} with zero average along streamlines. From (C.17)–(C.18) and (C.16) we have

$$\begin{aligned}&\|\partial_s \phi^{n+1}\|_{C^{k,\alpha}(D_0)} + \|\phi^{n+1}\|_{C^{k,\alpha}(D_0)}\\ &\quad\le C_{k,\alpha}\big(\|\mathcal{L}^n_\phi\|_{C^{k-2,\alpha}(D_0)} + \|\mathcal{N}^n_\phi\|_{C^{k-2,\alpha}(D_0)} + \|B^n_1\|_{C^{k,\alpha}(\partial D_0)}\big).\end{aligned}$$

In summary, using Lemma C.1, we have shown

Lemma C.4. *Suppose that (**H1**)–(**H3**) and the assumptions* (C.1)*,* (C.2)*,* (C.3) *and* (C.4) *hold. Let* $\phi^n, \eta^n \in C^{k,\alpha}(D_0)$ *with* $\partial_s\phi^n, \partial_s\eta^n \in C^{k,\alpha}(D_0)$ *be given functions. With* $\mathcal{N}^n_\eta, \mathcal{N}^n_\phi, B^n_1$ *defined as above, and with* F^n *defined implicitly by* (C.15)*, the problems* (C.13)–(C.14) *and* (C.17)–(C.18) *have a unique solution* η^{n+1}, ϕ^{n+1} *satisfying* $\oint_{\psi_0} \phi^{n+1}\, d\ell = 0$*, and we have the estimates*

$$\begin{aligned}\|\partial_s \eta^{n+1}\|_{C^{k,\alpha}} + \|\eta^{n+1}\|_{C^{k,\alpha}} &\le C_{k,\alpha}\big(\|\partial_s(\rho-1)\|_{C^{k,\alpha}} + \|\rho-1\|_{C^{k-2,\alpha}} \\ &\quad + (\|\partial_s\eta^n\|_{C^{k,\alpha}} + \|\eta^n\|_{C^{k,\alpha}})\|\partial_s\phi^n\|_{C^{k,\alpha}}, \qquad \text{(C.19)}\\ \|\partial_s \phi^{n+1}\|_{C^{k,\alpha}} + \|\phi^{n+1}\|_{C^{k,\alpha}} &\le C_{k,\alpha}\big(\|\rho-1\|_{C^{k,\alpha}} + (\|\partial_s\eta^{n+1}\|_{C^{k,\alpha}} + \|\eta^{n+1}\|_{C^{k,\alpha}})\|\partial_s\phi^n\|_{C^{k-1,\alpha}} \\ &\quad + (\|\partial_s\eta^{n+1}\|_{C^{k,\alpha}} + \|\eta^{n+1}\|_{C^{k,\alpha}}) \\ &\quad + \epsilon\|\partial_s\phi^n\|_{C^{k,\alpha}} + \|B^n_1\|_{C^{k,\alpha}(\partial D_0)}, \\ \|B^n_1\|_{C^{k,\alpha}(\partial D_0)} &\le C_{k,\alpha}\Big(\|\eta^n\|_{C^{k,\alpha}} + \|\phi^n\|_{C^{k,\alpha}}\Big). \qquad \text{(C.20)}\end{aligned}$$

C.6. Uniform estimates for the iterates. We now set $\eta^0 = \phi^0 = 0$. Given η^ℓ, ϕ^ℓ, using Lemma C.4 let $\eta^{\ell+1}$ satisfy (C.13)–(C.14) and let $\Phi^{\ell+1} = \partial_s\phi^{\ell+1}$ satisfy (C.17)–(C.18). In this section we prove that the sequences (η^ℓ, ϕ^ℓ), $(\partial_s\eta^\ell, \partial_s\phi^\ell)$ are uniformly bounded in $C^{k,\alpha}(D_0)$.

Lemma C.5. *There* $\epsilon_0 = \epsilon_0(D_0, k, \alpha, \theta) > 0$ *so that if the assumptions* (C.1)–(C.3) *hold with* $\epsilon_1 + \epsilon_2 + \epsilon_3 \le \epsilon_0/2$*, if the sequence* ϕ^ℓ, η^ℓ *is defined as above, then*

$$\|\partial_s\eta^\ell\|_{C^{k,\alpha}(D_0)} + \|\partial_s\phi^\ell\|_{C^{k,\alpha}(D_0)} + \|\eta^\ell\|_{C^{k,\alpha}(D_0)} + \|\phi^\ell\|_{C^{k,\alpha}(D_0)} \le 1.$$

Proof. Let $C_{k,\alpha}$ be as in (C.19)–(C.20) and set

$$\epsilon_0 = \min(1, 1/(4(C_{k,\alpha} + C^2_{k,\alpha}))).$$

Let $\epsilon = \epsilon_1 + \epsilon_2 + \epsilon_3$ and set $M = 4C_{k,\alpha}\epsilon$. We claim that if $\epsilon \le \epsilon_0/2$ then the iterates ϕ^ℓ, η^ℓ satisfy

$$\|\eta^\ell\|_{C^{k,\alpha}(D_0)} + \|\phi^\ell\|_{C^{k,\alpha}(D_0)} + \|\partial_s\eta^\ell\|_{C^{k,\alpha}(D_0)} + \|\partial_s\phi^\ell\|_{C^{k,\alpha}(D_0)} \le M \le 1.$$

This certainly holds for $\ell = 0$. If it holds for $\ell = 0, \dots, m-1$ then by (C.19)–(C.20) we have

$$\begin{aligned}&\|\eta^m\|_{C^{k,\alpha}(D_0)} + \|\phi^m\|_{C^{k,\alpha}(D_0)} + \|\partial_s\eta^m\|_{C^{k,\alpha}(D_0)} \\ &+\|\partial_s\phi^m\|_{C^{k,\alpha}(D_0)} \le C_{k,\alpha}(M^2 + \epsilon M + \epsilon) \le M,\end{aligned}$$

since $C_{k,\alpha}M^2 \le \frac{1}{2}M$, $C_{k,\alpha}\epsilon M \le \frac{1}{4}M^2 \le \frac{1}{4}M$ and $C_{k,\alpha}\epsilon \le \frac{1}{4}M$ if $\epsilon \le \epsilon_0/2$. The result follows. □

C.7. Cauchy estimates for the iterates.

Lemma C.6. *There is $\epsilon_0' = \epsilon_0'(D_0, k, \alpha, \theta) > 0$ with the following property. If the assumptions (C.1)–(C.3) hold with $\epsilon_1 + \epsilon_2 + \epsilon_3 \le \epsilon_0'/2$, then with the sequence $\{\phi^\ell, \eta^\ell\}$ defined as in the previous lemma, if we set*

$$D_{N,M} = \|\partial_s \eta^N - \partial_s \eta^M\|_{C^{k-1,\alpha}(D_0)} + \|\partial_s \phi^N - \partial_s \phi^M\|_{C^{k-1,\alpha}(D_0)} + \|\eta^N - \eta^M\|_{C^{k-1,\alpha}(D_0)} + \|\phi^N - \eta^M\|_{C^{k-1,\alpha}(D_0)},$$

then $D_{N,M} \le \frac{1}{2} D_{N-1,M-1}$. In particular, with $d_1 = \|\partial_s \eta^1\|_{C^{k-1,\alpha}(D_0)} + \|\partial_s \phi^1\|_{C^{k-1,\alpha}(D_0)} + \|\eta^1\|_{C^{k-1,\alpha}(D_0)} + \|\phi^1\|_{C^{k-1,\alpha}(D_0)}$, we have that

$$D_{N,M} \le 2^{1-\min(N,M)} d_1.$$

Proof. This is proved in nearly the same way as the previous lemma, but relies on Lemma C.2 in place of Lemma C.1. □

C.8. Convergence of the boundary term.

Lemma C.7. *Let D_0, D be domains in $\mathbb{R}^2$ and suppose that for some function ρ,*

$$\text{Area}(D) = \int_{D_0} \rho \, \mathrm{d}y.$$

Suppose that $\partial D = \{B(x) = 0\}$ for some function B defined in a tubular neighborhood of ∂D and has non-vanishing gradients there. Let γ be a diffeomorphism of the form $\gamma = \mathrm{id} + \nabla^\perp \phi + \nabla \eta$ where

$$\partial_s \phi = |\nabla \psi_0| \left(-B_1(\phi, \eta) + \frac{\oint_{\partial D_0} B_1(\phi, \eta)\, d\ell}{\text{length}(\partial D_0)} \right), \qquad \text{on } \partial D_0,$$

with B_1 defined as in (C.6). and where $\partial_n \eta$ is constant on ∂D_0. Then in fact

$$\partial_n \eta = -\frac{\oint_{\partial D_0} B_1(\phi, \eta)\, d\ell}{\text{length}(\partial D_0)}, \qquad \text{on } \partial D_0,$$

and as a consequence, $\gamma : \partial D_0 \to \partial D$.

Proof. Recall that, by the definition of the map γ, we have

$$B \circ \gamma|_{\partial D_0} = \frac{1}{|\nabla \psi_0|} \partial_s \phi + \partial_n \eta + B_1(\phi, \eta), \qquad \text{on} \qquad \partial D_0.$$

By the above assumptions, this implies that, for some constant c,

$$B \circ \gamma|_{\partial D_0} = c.$$

This says that γ maps ∂D_0 to the level set $\{B = c\}$. We wish to conclude that $c = 0$ based on the fact that the area of $\gamma(D_0)$ is the same as $D = \overline{\{B = 0\}}$. Note that the area enclosed by the level set, $\overline{\{B = c\}}$, has the property that

$$\frac{\mathrm{d}}{\mathrm{d}c} \text{Area}(\overline{\{B = c\}}) = \oint_{\{B=c\}} \frac{\mathrm{d}\ell}{|\nabla B|}.$$

By our assumptions that $|\nabla B|$ is non-vanishing in a neighborhood of the zero set, then the level sets are Jordan curves in a neighborhood of 0 and the area enclosed must change in accord with the above formula. Thus, the unique value of c such that

$$\mathrm{Area}(\overline{\{B=c\}}) = \mathrm{Area}(\overline{\{B=0\}})$$

is $c = 0$ and we are done. □

C.9. Proof of Theorem 3.1. By Lemmas C.5 and C.6 it follows that the iterates (η^ℓ, ϕ^ℓ) form a Cauchy sequence in $C^{k+2,\alpha}(D_0)$ and so they converge to functions $(\eta, \phi) \in C^{k+2,\alpha}(D_0)$ with a corresponding statement for $\partial_s \eta^\ell, \partial_s \phi^\ell$. We then set $\gamma = \mathrm{id} + \nabla\eta + \nabla^\perp\phi$. It remains to show that $\gamma(D_0) = D$, and by Lemma C.7 and (C.7)–(C.8) it follows that $\gamma|_{\partial D_0} = \partial D$ as required. The estimate (3.9) follows from the proof of Lemma C.5.

C.10. Proof of Theorem 3.2. We define a sequence of diffeomorphisms $\{\gamma^N\}$ as follows. Given a domain D_{N-1} and a diffeomorphism $\gamma^{N-1} : D_0 \to D_{N-1}$ of the form $\gamma^{N-1} = \mathrm{id} + \nabla\eta^{N-1} + \nabla^\perp\phi^{N-1}$, define ρ^N by

$$\rho^N(y) = X(y, \eta^{N-1}, \phi^{N-1}, \nabla\eta^{N-1}, \nabla\phi^{N-1}, \nabla\partial_s\eta^{N-1}, \nabla\partial_s\phi^{N-1})$$

and define $\sigma^N > 0$ by $\sigma_N^2 = \frac{\int_{D_0}\rho^N}{Vol D_0}$ so that with $D_N = \sigma_N D_0$, we have $Vol D_N = \sigma_N^2 Vol D_0 = \int_{D_0}\rho^N$. By Theorem 3.1 there is a diffeomorphism $\gamma^N : D_0 \to D_N$ with $\det\nabla\gamma^N = \rho^N$ and where γ^N is of the form $\gamma^N = \mathrm{id} + \nabla\eta^N + \nabla^\perp\phi^N$ so that $\psi^N = \psi_0 \circ \gamma^N$ satisfies (3.5), and we have the estimates

$$\begin{aligned} &\|\partial_s\eta^N\|_{C^{k-1,\alpha}} + \|\partial_s\phi^N\|_{C^{k-1,\alpha}} + \|\eta^N\|_{C^{k-1,\alpha}} + \|\phi^N\|_{C^{k-1,\alpha}} \\ &\quad \le C_{k,\alpha}\big(\|\rho^N - 1\|_{C^{k-1,\alpha}} + \varepsilon\big). \end{aligned} \tag{C.21}$$

Taylor expanding $\rho^N = X(y, \eta^{N-1}, \phi^{N-1}, \nabla\eta^{N-1}, \nabla\phi^{N-1}, \nabla\partial_s\eta^{N-1}, \nabla\partial_s\phi^{N-1})$ around $(\eta, \phi) = (0, 0)$ and using the bound (3.12) we have

$$\begin{aligned} \|\rho^N - 1\|_{C^{k-1,\alpha}} \le C\epsilon_X\big(&\|\partial_s\eta^{N-1}\|_{C^{k-1,\alpha}} + \|\partial_s\phi^{N-1}\|_{C^{k-1,\alpha}} \\ &+\|\eta^{N-1}\|_{C^{k-1,\alpha}} + \|\phi^{N-1}\|_{C^{k-1,\alpha}}\big) \\ &+C\big(\|\partial_s\eta^{N-1}\|_{C^{k-1,\alpha}} + \|\partial_s\phi^{N-1}\|_{C^{k-1,\alpha}} + \|\eta^{N-1}\|_{C^{k-1,\alpha}} \\ &+\|\phi^{N-1}\|_{C^{k-1,\alpha}}\big)^2, \end{aligned} \tag{C.22}$$

provided $\|\partial_s\eta^{N-1}\|_{C^{k-1,\alpha}} + \|\partial_s\phi^{N-1}\|_{C^{k-1,\alpha}} + \|\eta^{N-1}\|_{C^{k-1,\alpha}} + \|\phi^{N-1}\|_{C^{k-1,\alpha}} \le 1$, say. We now prove that the sequence γ^N is uniformly bounded provided ε_X is taken sufficiently small. Let $C_{k,\alpha}$ be the constant in (3.9) and take ε_X so small that $4C_{k,\alpha}C'_{k,\alpha}\varepsilon_X \le 1$, and suppose that

$$\|\partial_s\eta^N\|_{C^{k-1,\alpha}} + \|\partial_s\phi^N\|_{C^{k-1,\alpha}} + \|\eta^N\|_{C^{k-1,\alpha}} + \|\phi^N\|_{C^{k-1,\alpha}} \le 2C_{k,\alpha}\varepsilon \le 1.$$

By (C.22), we then have

$$\begin{aligned} &\|\partial_s\eta^{N+1}\|_{C^{k-1,\alpha}} + \|\partial_s\phi^{N+1}\|_{C^{k-1,\alpha}} + \|\eta^{N+1}\|_{C^{k-1,\alpha}} + \|\phi^{N+1}\|_{C^{k-1,\alpha}} \\ &\quad \le C_{k,\alpha}\big(C'_{k,\alpha}\varepsilon_X(2\varepsilon)^{k+1+\alpha} + \varepsilon\big) \le 2C_{k,\alpha}C'_{k,\alpha}\varepsilon\varepsilon_X + C_{k,\alpha}\varepsilon \le 2C_{k,\alpha}\varepsilon, \end{aligned}$$

and it follows that the sequence $\{\gamma^N\}_{N=0}^\infty$ is uniformly bounded in $C^{k+1,\alpha}$. Using a similar argument it is straightforward to see that this sequence is also a Cauchy sequence in $C^{k+1,\alpha}$ and so $\gamma^N \to \gamma \in C^{k+1,\alpha}$ which by construction satisfies (3.13).

Appendix D. Elliptic Estimates

In this appendix, we collect some well-posedness results from elliptic theory for the Dirichlet and Neumann problems. The first is concerning the Dirichlet problem can be found in e.g. Theorem 6.6 of [12] when $k = 0$ and Problem 6.2 of [12] when $k \geq 1$:

Lemma D.1. *Fix $k \geq 2$ and $\alpha \in (0, 1)$ and $f \in C^{k-2,\alpha}(D_0)$, $g \in C^{k,\alpha}(\partial D_0)$. Let a^{ij}, b^i, c be smooth coefficients and set*

$$L = \sum_{i,j=1}^{2} a^{ij} \partial_i \partial_j + \sum_{i=1}^{2} b^i \partial_i.$$

Suppose that the only solution to

$$(L + c)v = 0, \qquad v \in H_0^1(D_0)$$

is $v = 0$. Then the Dirichlet problem

$$\begin{aligned} (L + c)u &= f, \quad \text{in } D_0, \\ u &= g, \quad \text{on } \partial D_0, \end{aligned}$$

has a unique solution $u \in C^{k,\alpha}(D_0)$, and there is a constant $C_0 = C_0(D_0, \|b\|_{k-2,\alpha})$ with

$$\|u\|_{k,\alpha} \leq C_0 \big(\|f\|_{k-2,\alpha} + |g|_{k,\alpha} \big).$$

For the Neumann problem, compatibility is also required.

Lemma D.2. *Fix $k \geq 2$ and $\alpha \in (0, 1)$, and $f \in C^{k-2,\alpha}(D_0)$, $g \in C^{k-1,\alpha}(\partial D_0)$ satisfying*

$$\int_{D_0} f = \int_{\partial D_0} g.$$

Then the Neumann problem

$$\begin{aligned} \Delta u &= f, \quad \text{in } D_0, \\ \partial_n u &= g, \quad \text{on } \partial D_0, \end{aligned}$$

has a unique solution $u \in C^{k,\alpha}(D_0)$ and there is a constant $C_1 = C_1(D_0)$ with

$$\|u\|_{k,\alpha} \leq C_1 \big(\|f\|_{k-2,\alpha} + |g|_{k-1,\alpha} \big).$$

Appendix E. Streamline Geometry

In this appendix, we prove some formulae which are useful for our deformation scheme which uses streamline coordinates. First, we state some relations between the curvature and vorticity of along a given streamline.

Lemma E.1. *Let* $\hat{n} = \nabla\psi/|\nabla\psi|$ *and* $\hat{\tau} = \nabla^{\perp}\psi/|\nabla\psi|$*. The following formulae hold*

$$\hat{\tau}\cdot\nabla\otimes\nabla\psi\cdot\hat{\tau} = |\nabla\psi|\kappa, \tag{E.1}$$

$$\hat{n}\cdot\nabla\otimes\nabla\psi\cdot\hat{n} = \Delta\psi - |\nabla\psi|\kappa, \tag{E.2}$$

where $\kappa := \hat{\tau}\cdot\nabla\hat{n}\cdot\hat{\tau}$ *is the curvature of the streamline.*

Proof. We being by noticing that

$$\Delta\psi = \mathrm{tr}\nabla\otimes\nabla\psi = \hat{n}\cdot\nabla\otimes\nabla\psi\cdot\hat{n} + \hat{\tau}\cdot\nabla\otimes\nabla\psi\cdot\hat{\tau}.$$

Next, a direct calculation gives

$$|\nabla\psi|\nabla\hat{n} = \nabla\otimes\nabla\psi - (\hat{n}\cdot\nabla\otimes\nabla\psi\cdot\hat{n})\hat{n}\otimes\hat{n} - (\hat{n}\cdot\nabla\otimes\nabla\psi\cdot\hat{\tau})\hat{n}\otimes\hat{\tau},$$

so that $|\nabla\psi|\hat{\tau}\cdot\nabla\hat{n}\cdot\hat{\tau} = \hat{\tau}\cdot\nabla\otimes\nabla\psi\cdot\hat{\tau}$ yielding (E.1) as claimed. Combining with the above we obtain (E.2). □

Before stating the next required lemma, we briefly review action-angle coordinates. For an in depth discussion, see Arnol'd [1], pg 297. The streamfunction ψ plays the role of a Hamiltonian for tracer dynamics since $u = \nabla^{\perp}\psi$. We assume that the level sets $\{\psi = c\}$ are simply connected Jordan curves, so that all the integral curves of u (solutions of $\dot{X} = u \circ X$) are periodic orbits. This system allows for a canonical transformation to action-angle variables, $(x, y) \mapsto (J, \theta)$ which satisfy the following criteria

(1) $\psi(x, y) = \Psi(J(x, y))$ for all $(x, y) \in \Omega$ and some function Ψ
(2) $\int_{\{\psi=c\}} \mathrm{d}\theta = 1$,
(3) $\nabla^{\perp}J \cdot \nabla\theta = 1$.

Introduce the frequency $\mu^{-1} = \Psi'(J)$. The phase flow satisfies

$$\frac{\mathrm{d}J}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \mu^{-1}.$$

The first is simply because the system travels along paths of fixed J. The latter follows from

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}\theta}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} = \nabla^{\perp}\psi\cdot\nabla\theta = \Psi'(J)(J_x\theta_y - J_y\theta_x) = \mu^{-1}(\nabla^{\perp}J\cdot\nabla\theta) = \mu^{-1}.$$

The period for each orbit $\{\psi = c\}$ is the travel time $\mu := \mu(c)$ given by

$$\mu(c) = \oint_{\{\psi=c\}} \frac{\mathrm{d}\ell}{|\nabla\psi|},$$

and the line element for each orbit satisfies

$$\mathrm{d}\ell = \sqrt{\dot{x}^2 + \dot{y}^2} = |\nabla\psi|\mathrm{d}t = \mu^{-1}|\nabla J|\mathrm{d}t = |\nabla J|\mathrm{d}\theta.$$

We now give the rule for differentiating functions integrated over streamlines.

Lemma E.2. *For $f \in C^1(\Omega)$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}c}\oint_{\{\psi=c\}} f\frac{\mathrm{d}\ell}{|\nabla\psi|} = \oint_{\{\psi=c\}} \frac{\nabla\psi\cdot\nabla f - f(\omega - 2\kappa|\nabla\psi|)}{|\nabla\psi|^2}\frac{\mathrm{d}\ell}{|\nabla\psi|},$$

where $\kappa = \hat{\tau}\cdot\nabla\hat{n}\cdot\hat{\tau}$ is the curvature of the streamline and $\omega := \Delta\psi$.

Proof of Lemma E.2. First we show that for $g \in C^1(\Omega)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}c}\oint_{\{\psi=c\}} g|\nabla\psi|\mathrm{d}\ell = \oint_{\{\psi=c\}} \frac{1}{|\nabla\psi|}\Big(\nabla g\cdot\nabla\psi + g\Delta\psi\Big)\mathrm{d}\ell.$$

To establish this, set $F := g\nabla^\perp\psi$ and $\mathrm{d}l = (\dot{x}\mathrm{d}t, \dot{y}\mathrm{d}t)$. Then $F\cdot\mathrm{d}l = g|\nabla\psi|\mathrm{d}\ell$. By Green's theorem,

$$\oint_{\{\psi=c\}} F\cdot\mathrm{d}l = \iint_{\overline{\{\psi=c\}}} \nabla^\perp\cdot F\mathrm{d}x\mathrm{d}y = \iint_{\overline{\{\psi=c\}}} [\nabla g\cdot\nabla\psi + g\Delta\psi]\mathrm{d}x\mathrm{d}y.$$

Then, for two values $c_0 \le c_1$ in the range of ψ, we have

$$\begin{aligned}\oint_{\{\psi=c_1\}} F\cdot\mathrm{d}l - \oint_{\{\psi=c_0\}} F\cdot\mathrm{d}\ell &= \iint_{\{c_0\le\psi\le c_c\}} [\nabla g\cdot\nabla\psi + g\Delta\psi]\mathrm{d}x\mathrm{d}y \\ &= \iint_{\{c_0\le\psi\le c_c\}} [\nabla g\cdot\nabla\psi + g\Delta\psi]\mathrm{d}\theta\mathrm{d}J,\end{aligned}$$

where we made a change of variables to action angle coordinates (the Jacobian is unity). Finally,

$$\iint h\mathrm{d}\theta\mathrm{d}J = \iint h\mu(\psi)\mathrm{d}\theta\mathrm{d}\psi = \iint h\mathrm{d}t\mathrm{d}\psi = \iint \frac{h}{|\nabla\psi|}\mathrm{d}\ell\mathrm{d}\psi,$$

for any integrable h. The result follows from taking the coincidence limit $c_1 \to c_0$ of the difference quotients. The lemma then follows by applying the formula with $g = f/|\nabla\psi|^2$. This gives

$$\frac{\mathrm{d}}{\mathrm{d}c}\oint_{\{\psi=c\}} \frac{f}{|\nabla\psi|}\mathrm{d}\ell = \oint_{\{\psi=c\}} \frac{\nabla\psi\cdot\nabla f + f\left(\Delta\psi - 2\widehat{\nabla\psi}\cdot\nabla\otimes\nabla\psi\cdot\widehat{\nabla\psi}\right)}{|\nabla\psi|^3}\mathrm{d}\ell.$$

To work this into the stated form we appeal to Lemma E.1. □

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Communicated by A. Ionescu