A MORAWETZ INEQUALITY FOR GRAVITY-CAPILLARY WATER WAVES AT LOW BOND NUMBER

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ABSTRACT. This paper is devoted to the 2D gravity-capillary water waves equations in their Hamiltonian formulation, addressing the general question of proving Morawetz inequalities. We continue the analysis initiated in our previous work, where we have established local energy decay estimates for gravity waves. Here we add surface tension and prove a stronger estimate with a local regularity gain, akin to the smoothing effect for dispersive equations. Our main result holds globally in time and holds for genuinely nonlinear waves, since we are only assuming some very mild uniform Sobolev bounds for the solutions. Furthermore, it is uniform both in the infinite depth limit and the zero surface tension limit.

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1. INTRODUCTION

1.1. **The water-wave equations.** A classical topic in the mathematical theory of hydrodynamics concerns the propagation of water waves. The problem consists in studying the evolution of the free surface separating air from an incompressible perfect fluid, together with the evolution of the velocity field inside the fluid domain. We assume that the free surface $\Sigma(t)$ is a graph and that the fluid domain $\Omega(t)$ has a flat bottom, so that

$$\Omega(t) = \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid -h < y < \eta(t,x)\},$$
$$\Sigma(t) = \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid y = \eta(t,x)\},$$

where h is the depth and η is an unknown (called the free surface elevation). We assume that the velocity field $v\colon \Omega \to \mathbb{R}^2$ is irrotational, so that $v = \nabla_{x,y}\phi$ for some potential $\phi\colon \Omega \to \mathbb{R}$ satisfying

$$\begin{aligned} &\Delta_{x,y}\phi = 0 \quad \text{in } \Omega,\\ &\partial_t\phi + \frac{1}{2}\left|\nabla_{x,y}\phi\right|^2 + P + gy = 0 \quad \text{in } \Omega,\\ &\partial_y\phi = 0 \quad \text{on } y = -h, \end{aligned} \tag{1.1}$$

where $P\colon \Omega \to \mathbb{R}$ is the pressure, $g > 0$ is the acceleration of gravity, and $\Delta_{x,y} = \partial_x^2 + \partial_y^2$. Partial differentiations will often be denoted by suffixes so that $\phi_x = \partial_x\phi$ and $\phi_y = \partial_y\phi$.

The water-wave problem is described by two equations that hold on the free surface: firstly an equation describing the time evolution of Σ:

$$\partial_t \eta = \sqrt{1+\eta_x^2}\, \phi_n|_{y=\eta} = \phi_y(t,x,\eta(t,x)) - \eta_x(t,x)\phi_x(t,x,\eta(t,x)), \tag{1.2}$$

and secondly an equation for the balance of forces at the free surface:

$$P|_{y=\eta} = -\kappa H(\eta), \tag{1.3}$$

where κ is the coefficient of surface tension and $H(\eta)$ is the curvature given by

$$H(\eta) = \partial_x \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}} \right). \tag{1.4}$$

We begin by recalling the main features of the water wave problem.

- **Hamiltonian system.** Since ϕ is harmonic function with Neumann boundary condition on the bottom, it is fully determined by its trace on Σ. Set

$$\psi(t,x) = \phi(t,x,\eta(t,x)).$$

Zakharov discovered that η and ψ are canonical variables. Namely, he gave the following Hamiltonian formulation of the water-wave equations [62, 63], see also [19]:

$$\frac{\partial \eta}{\partial t} = \frac{\delta \mathcal{H}}{\delta \psi}, \qquad \frac{\partial \psi}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \eta},$$

where $\mathcal{H}$ is the energy, which reads

$$\mathcal{H} = \frac{g}{2}\int_{\mathbb{R}} \eta^2\, dx + \kappa \int_{\mathbb{R}} \left(\sqrt{1+\eta_x^2} - 1 \right) dx + \frac{1}{2}\iint_{\Omega(t)} \left| \nabla_{x,y}\phi \right|^2 dy dx. \tag{1.5}$$

The Hamiltonian is the sum of the gravitational potential energy, a surface energy due to stretching of the surface and the kinetic energy. One can give more explicit evolution equations by introducing the Dirichlet to Neumann operator associated to the fluid domain $\Omega(t)$, defined by

$$G(\eta)\psi = \sqrt{1+\eta_x^2}\, \phi_{n|_{y=\eta}} = (\phi_y - \eta_x \phi_x)_{|_{y=\eta}}.$$

Then (see [40] and also [20]), with the above notations, the water-wave system reads

$$\begin{cases} \partial_t \eta = G(\eta)\psi \\ \partial_t \psi + g\eta + \mathcal{N}(\eta)\psi - \kappa H(\eta) = 0, \end{cases} \tag{1.6}$$

where

$$\mathcal{N}(\eta)\psi := \frac{1}{2}\psi_x^2 - \frac{1}{2}\frac{(G(\eta)\psi + \eta_x \psi_x)^2}{1+\eta_x^2}. \tag{1.7}$$

Of course the Hamiltonian is conserved along the flow. Another conservation law that is essential in this paper is the conservation of the horizontal momentum,

$$\mathcal{M} = \int_{\mathbb{R}} \eta \psi_x \, dx. \tag{1.8}$$

From a Hamiltonian perspective, this can be seen as arising via Noether's theorem as the generator for the horizontal translations, which commute with the water wave flow.

- **Scaling invariance.** Another symmetry is given by the scaling invariance which holds in the infinite depth case (that is when $h = \infty$) when either $g = 0$ or $\kappa = 0$.

  If $\kappa = 0$ and ψ and η are solutions of the gravity water waves equations (1.6), then ψ_λ and η_λ defined by

$$\psi_\lambda(t,x) = \lambda^{-3/2}\psi(\sqrt{\lambda}t, \lambda x), \quad \eta_\lambda(t,x) = \lambda^{-1}\eta(\sqrt{\lambda}t, \lambda x),$$

  solve the same system of equations. The (homogeneous) Sobolev spaces invariant by this scaling correspond to η in $\dot{H}^{3/2}(\mathbb{R})$ and ψ in $\dot{H}^2(\mathbb{R})$.

  On the other hand if $g = 0$ and ψ and η are solutions of the capillary water waves equations (1.6), then ψ_λ and η_λ defined by

$$\psi_\lambda(t,x) = \lambda^{-\frac{1}{2}}\psi(\lambda^{\frac{3}{2}}t, \lambda x), \quad \eta_\lambda(t,x) = \lambda^{-1}\eta(\lambda^{\frac{3}{2}}t, \lambda x),$$

  solve the same system of equations. The (homogeneous) Sobolev spaces invariant by this scaling correspond to η in $\dot{H}^{3/2}(\mathbb{R})$ and ψ in $\dot{H}^1(\mathbb{R})$.
- This is a **quasi-linear** system of **nonlocal** equations. As a result, even the study of the Cauchy problem for smooth data is highly nontrivial. The literature on this topic is extensive by now, starting with the works of Nalimov [47] and Yosihara [61], who proved existence and uniqueness in Sobolev spaces under a smallness assumption. Without a smallness assumptions on the data, the well-posedness of the Cauchy problem was first proved by Wu [57, 58] without surface tension and by Beyer-Günther in [10] in the case with surface tension. Several extensions of these results were obtained by various methods and many authors. We begin by quoting recent results for gravity-capillary waves. For the local in time Cauchy problem we refer to [1, 4, 7, 13, 17, 35, 39, 40, 45, 51, 53, 52, 54], see also [33] and [28, 37, 56, 24] for global existence results for small enough initial data which are localized, and [11, 26] for results about splash singularities for large enough initial data. Let us recall that the Cauchy problem for the gravity-capillary water-wave equations is locally well-posed in suitable function spaces which are 3/2-derivative more regular than the scaling invariance, e.g. when initially

$$\eta \in H^{s+\frac{1}{2}}(\mathbb{R}), \qquad \psi \in H^s(\mathbb{R}), \qquad s > \frac{5}{2}.$$

  Actually, some better results hold using Strichartz estimates (see [23, 22, 48]). There are also many recent results for the equations without surface tension, and we refer the reader to the papers [3, 2, 5, 12, 18, 21, 27, 29, 30, 31, 32, 36, 59, 60].
- **Dispersive equation.** Consider the linearized water-wave equations:

$$\begin{cases} \partial_t \eta = G(0)\psi = |D| \tanh(h\,|D|)\psi, \qquad (|D| = \sqrt{-\partial_x^2}) \\ \partial_t \psi + g\eta - \kappa \partial_x^2 \eta = 0. \end{cases}$$

  Then the dispersion relationship reads $\omega^2 = k\tanh(hk)(g + \kappa k^2)$, which shows that water waves are dispersive waves. The dispersive properties have been studied for many different problems, including the global in time existence results alluded to before and also various problems about Strichartz estimates, local smoothing effect, control theory or the study of solitary waves (see e.g. [3, 2, 4, 14, 22, 23, 34, 64, 65]).

1.2. **Morawetz estimates.** Despite intensive researches on dispersive or Hamiltonian equations, it is fair to say that many natural questions concerning the dynamics of water waves are mostly open. Among these, we have initiated in our previous work [6] the study of Morawetz estimates for water waves. In this paragraph, we introduce this problem within the general framework of Hamiltonian equations.

Consider a Hamiltonian system of the form

$$\frac{\partial\eta}{\partial t}=\frac{\delta\mathcal{H}}{\delta\psi},\quad \frac{\partial\psi}{\partial t}=-\frac{\delta\mathcal{H}}{\delta\psi}\quad\text{with}\quad \mathcal{H}=\int e\,dx,$$

where the density of energy e depends only on η and ψ. This setting includes the water-wave equations, the Klein-Gordon equation, the Schrödinger equation, the Korteveg-de-Vries equation, etc. For the sake of simplicity, let us compare the following linear equations:

- (GWW) the gravity water-wave equation $\partial_t^2 u+|D_x|\,u=0$;
- (CWW) the capillary water-wave equation $\partial_t^2 u+|D_x|^3\,u=0$;
- (KG) the Klein-Gordon equation $\Box u+u=0$;
- (S) the Schrödinger equation $i\partial_t u+\Delta u=0$.

All of them can be written in the above Hamiltonian form with

$$\begin{aligned}
e_{\rm GWW}&=\eta^2+(|D_x|^{1/2}\,\psi)^2,\\
e_{\rm CWW}&=(|D_x|\,\eta)^2+(|D_x|^{1/2}\,\psi)^2,\\
e_{\rm KG}&=(\langle D_x\rangle^{1/2}\eta)^2+(\langle D_x\rangle^{1/2}\psi)^2,\\
e_{\rm S}&=(|D_x|\,\eta)^2+(|D_x|\,\psi)^2.
\end{aligned}$$

Here in the case of both gravity and capillary water waves one can think of u as simply $u=\eta$.

For such Hamiltonian systems, one can deduce from the Noether's theorem and symmetries of the equations some conserved quantities. For instance, the invariance by translation in time implies that the energy $\mathcal{H}$ is conserved, while the invariance by spatial translation implies that the momentum $\mathcal{M}=\int\eta\psi_x\,dx$ is conserved. Then, a Morawetz estimate is an estimate of the local energy in terms of a quantity which scales like the momentum. More precisely, given a time $T>0$ and a compactly supported function $\chi=\chi(x)\geq 0$, one seeks an estimate of the form

$$\int_0^T\int_{\mathbb{R}}\chi(x)e(t,x)\,dxdt\leq C(T)\sup_{t\in[0,T]}\left\|\eta(t)\right\|_{H^s(\mathbb{R})}\left\|\psi(t)\right\|_{H^{1-s}(\mathbb{R})},$$

for some real number s chosen in a balanced way depending on the given equation. If the constant $C(T)$ does not depend on time T, then we say that the estimate is global in time. The study of the latter estimates was introduced in Morawetz's paper [46] in the context of the nonlinear Klein-Gordon equation.

The study of Morawetz estimates is interesting for both linear and nonlinear equations. We begin by discussing the linear phenomena. For the Klein-Gordon equation, the local energy density $e_{\rm KG}$ measures the same regularity as the momentum density $\eta\psi_x$ so that only the global in time estimate is interesting. The latter result expresses the fact that the localized energy is globally integrable in time, and hence has been called a *local energy decay* result. Morawetz estimates have also been proved for the Schrödinger equation. Here the natural energy density $e_{\rm S}$ measures a higher regularity than the momentum density $\eta\psi_x$; for this reason the result is meaningful even locally in time and the resulting (local in time) Morawetz estimates have been originally called *local smoothing estimates*, see [16, 55, 50]. The same phenomena appears also for the KdV equation. In fact, the general study of Morawetz estimates has had a long history, which is too extensive to try to describe here. For further recent references we refer the reader to [43] for the wave equation, [42] for the Schrödinger equation, [49] for fractional dispersive equations. These estimates are very robust and also hold for nonlinear problems, which make them useful in the study of the Cauchy problem for nonlinear equations (see e.g. [15, 38, 50]).

We are now ready to discuss Morawetz estimates for water waves. One of our motivations to initiate their study in [6] is that this problem exhibits interesting new features. Firstly, since the problem is nonlocal, it is already difficult to obtain a global in time estimate for the linearized equations. The second and key observation is that, even if the equation is *quasilinear*, one can prove such global in time estimates for the nonlinear equations, assuming some very mild smallness assumption on the solution.

Given a compactly supported bump function $\chi = \chi(x)$, we want to estimate the local energy

$$\int_0^T\int_{\mathbb{R}} \chi(x-x_0)(g\eta^2+\kappa\eta_x^2)\,dxdt + \int_0^T\int_{\mathbb{R}}\int_{-h}^{\eta(t,x)} \chi(x-x_0)\,|\nabla_{x,y}\phi|^2\,dydxdt,$$

uniformly in time T and space location x_0, assuming only a uniform bound on the size of the solutions. In our previous paper [6], we have studied this problem for gravity water waves (that is for $\kappa = 0$). There the momentum is not controlled by the Hamiltonian energy, and as a result we have the opposite phenomena to local smoothing, namely a loss of 1/4 derivative in the local energy. This brings substantial difficulties in the low frequency analysis, in particular in order to prove a global in time estimate. In addition, we also took into account the effect of the bottom, which generates an extra difficulty in the analysis of the low frequency component.

In this article we assume that $\kappa \geq 0$ so that one can both study gravity water waves and gravity-capillary water waves (for $\kappa > 0$). Furthermore, we seek to prove Morawetz estimates uniformly with respect to surface tension κ as $\kappa \to 0$, and also uniformly with respect to the depth h as $h \to \infty$. In this context we will impose a smallness condition on the Bond number.

1.3. **Function spaces.** As explained above, the goal of the present paper is to be able to take into account surface tension effects i.e. the high frequency local smoothing, while still allowing for the presence of a bottom which yields substantial difficulties at low frequencies. In addition, one key point is that our main result holds provided that some mild Sobolev norms of the solutions remain small enough uniformly in time. Here we introduce the functional setting which will allow us to do so.

Precisely, in this subsection we introduce three spaces: a space E^0 associated to the energy, a space $E^{\frac{1}{4}}$ associated to the momentum, and a uniform in time control norm $\|\cdot\|_{X^\kappa}$ which respects the scaling invariance.

The above energy $\mathcal{H}$ (Hamiltonian) corresponds to the energy space for (η,ψ),

$$E^0 = \left(g^{-\frac{1}{2}}L^2(\mathbb{R}) \cap \kappa^{-\frac{1}{2}}\dot{H}^1(\mathbb{R})\right) \times \dot{H}_h^{\frac{1}{2}}(\mathbb{R}),$$

with the depth dependent $\dot{H}_h^{\frac{1}{2}}(\mathbb{R})$ space defined as

$$\dot{H}_h^{\frac{1}{2}}(\mathbb{R}) = \dot{H}^{\frac{1}{2}}(\mathbb{R}) + h^{-\frac{1}{2}}\dot{H}^1(\mathbb{R}).$$

We already observe in here the two interesting frequency thresholds in this problem, namely h^{-1}, determined by the depth, and $\sqrt{g/\kappa}$, determined by the balance between gravity and capillarity. The dimensionless connection between these two scales is described by the Bond number,

$$\mathbf{B} = \frac{\kappa}{gh^2}.$$

A key assumption in the present work is that

$$\mathbf{B} \ll 1.$$

This is further discussed in the comments following our main result.

Similarly, in order to measure the momentum, we use the space $E^{\frac{1}{4}}$, which is the h-adapted linear $H^{\frac{1}{4}}$-type norm for (η, ψ) (which corresponds to the momentum),

$$E^{\frac{1}{4}} := \left(g^{-\frac{1}{4}} H_h^{\frac{1}{4}}(\mathbb{R}) \cap \kappa^{-\frac{1}{4}} H_h^{\frac{3}{4}}(\mathbb{R})\right) \times \left(g^{\frac{1}{4}} \dot{H}_h^{\frac{3}{4}}(\mathbb{R}) + \kappa^{\frac{1}{4}} \dot{H}_h^{\frac{1}{4}}(\mathbb{R})\right), \tag{1.9}$$

with

$$H_h^s(\mathbb{R}) := \dot{H}^s(\mathbb{R}) \cap h^s L^2(\mathbb{R}), \qquad \dot{H}_h^s(\mathbb{R}) = \dot{H}^s(\mathbb{R}) + h^{s-1}\dot{H}^1(\mathbb{R}),$$

so that in particular we have

$$|\mathcal{M}| \lesssim \|(\eta, \psi)\|^2_{E^{\frac{1}{4}}}.$$

We remark that there is some freedom here in choosing the space $E^{\frac{1}{4}}$; the one above is not as in the previous paper [6], adapted to gravity waves, but is instead changed above the frequency threshold $\lambda_0 = \sqrt{g/\kappa}$ and adapted to capillary waves instead at high frequency.

For our uniform a-priori bounds for the solutions, we begin by recalling the set-up in [6] for the pure gravity waves. There we were able to use a scale invariant norm, which corresponds to the following Sobolev bounds:

$$\eta \in H_h^{\frac{3}{2}}(\mathbb{R}), \qquad \nabla\phi|_{y=\eta} \in g^{\frac{1}{2}} H_h^1(\mathbb{R}) \cap \kappa^{\frac{1}{2}} L^2(\mathbb{R}).$$

Based on this, we have introduced the homogeneous norm X_0 defined by

$$X_0 := L_t^\infty H_h^{\frac{3}{2}} \times g^{-\frac{1}{2}} L_t^\infty H_h^1,$$

and used it to define the uniform control norm X by

$$\|(\eta, \psi)\|_X := \|P_{\leq h^{-1}}(\eta, \nabla\psi)\|_{X_0} + \sum_{\lambda > h^{-1}} \|P_\lambda(\eta, \psi)\|_{X_0}.$$

Here we use a standard Littlewood-Paley decomposition[1] beginning at frequency $1/h$,

$$1 = P_{<1/h} + \sum_{1/h < \lambda \in 2^{\mathbb{Z}}} P_\lambda.$$

The uniform control norm we use in this paper, denoted by X^κ, matches the above scenario at low frequency, but requires some strengthening at high frequency. The threshold frequency in this context is

$$\lambda_0 = \sqrt{g/\kappa} \gg 1,$$

[1]See e.g. [41] and [8].

and describes the transition from gravity to capillary waves. Then we will complement the X bound with a stronger bound for η in the higher frequency range, setting

$$X_1 := \left(\frac{g}{\kappa}\right)^{\frac{1}{4}} L_t^\infty H^2.$$

One can verify that this matches the first component of X_0 exactly at frequency λ_0. Then our uniform control norm X^κ will be

$$\|(\eta,\psi)\|_{X^\kappa} := \|(\eta,\nabla\psi)\|_X + \|\eta\|_{X_1}.$$

Based on the expression (1.5) for the energy, we introduce the following notations for the local energy. Fix an arbitrary compactly supported nonnegative function χ. Then, the local energy centered around a point x_0 is

$$\|(\eta,\psi)\|^2_{LE^\kappa_{x_0}} := g\int_0^T\int_{\mathbb{R}} \chi(x-x_0)\eta^2\,dxdt + \kappa\int_0^T\int_{\mathbb{R}} \chi(x-x_0)\eta_x^2\,dxdt + \int_0^T\int_{\mathbb{R}}\int_{-h}^{\eta(t,x)} \chi(x-x_0)\left|\nabla_{x,y}\phi\right|^2\,dydxdt.$$

It is also of interest to take the supremum over x_0,

$$\|(\eta,\psi)\|^2_{LE^\kappa} := \sup_{x_0\in\mathbb{R}} \|(\eta,\psi)\|^2_{LE^\kappa_{x_0}}. \tag{1.10}$$

1.4. **The main result.** Our main Morawetz estimate for gravity-capillary water waves is as follows:

Theorem 1.1 (Local energy decay for gravity-capillary waves). *Let $s > 5/2$. There exist positive constants ϵ_0 and C_0 such that the following result holds. For all $T \in (0,+\infty)$, all $h \in [1,+\infty)$, all $g \in (0,+\infty)$, all $\kappa \in (0,+\infty)$ with[2] $\kappa \ll g$ and all solutions $(\eta,\psi) \in C^0([0,T]; H^s(\mathbb{R}) \times H^s(\mathbb{R}))$ of the water-wave system (1.6) satisfying*

$$\|(\eta,\psi)\|_{X^\kappa} \le \epsilon_0 \tag{1.11}$$

the following estimate holds

$$\|(\eta,\psi)\|^2_{LE^\kappa} \le C_0(\|(\eta,\psi)(0)\|^2_{E^{\frac{1}{4}}} + \|(\eta,\psi)(T)\|^2_{E^{\frac{1}{4}}}). \tag{1.12}$$

We continue with several remarks concerning the choices of parameters/norms in the theorem.

Remark 1.2 (Uniformity). One key feature of our result is that it is global in time (uniform in T) and uniform in $h \ge 1$ and $\kappa \ll g$. In particular our estimate is uniform in the infinite depth limit and the zero surface tension limit.

Remark 1.3 (Time scaling). Another feature of our result is that the statement of Theorem 1.1 is invariant with respect to the following scaling law (time associated scaling)

$$(\eta(t,x),\psi(t,x)) \to (\eta(\lambda t,x),\lambda\psi(\lambda t,x))$$
$$(g,\kappa,h) \to (\lambda^2 g,\lambda^2\kappa,h).$$

This implies that the value of g and κ separately are not important but their ratio is. By scaling one could simply set $g = 1$ in all the proofs. We do not do that in order to improve the readability of the article.

[2]This should be interpreted as $\kappa \le cg$ for a small universal constant c, which in particular does not depend on T and h

Remark 1.4 (Window size)**.** Here by window size we mean the size of the support of the bump function χ in the definition of the local energies; this is taken to be of size 1 in our main result. Once we have the local energy decay bounds for a unit window size, we also trivially have them for any higher window size $R > 1$, with a constant of R on the right in the estimate.

Remark 1.5 (Spatial scaling)**.** One can also rescale the spatial coordinates $x \to \lambda x$, and correspondingly

$$(\eta(t,x), \psi(t,x)) \to (\lambda^{-1}\eta(t,\lambda x), \lambda^{-\frac{3}{2}}\lambda\psi(t,\lambda x))$$
$$(g, \kappa, h, R) \to (g, \lambda{-}2\kappa, \lambda^{-1}h, \lambda^{-1}R).$$

Then combining this with respect to the previous remark, we can restate our hypothesis $\kappa/g \ll 1 \leq h^2$ as a constraint on the Bond number, $\mathbf{B} \ll 1$, together with a window size restriction as $\kappa/g \lesssim R^2 \lesssim h^2$.

Remark 1.6. As already explained, the uniform control norms in (1.11) are below the current local well-posedness threshold for this problem, and are instead closer to one might view as the critical, scale invariant norms for this problem. The dependence on h is natural as spatial scaling will also alter the depth h. In the infinite depth limit one recovers exactly the homogeneous Sobolev norms at low frequency. We also note that, by Sobolev embeddings, our smallness assumption guarantees that

$$|\eta| \lesssim \epsilon_0 h, \qquad |\eta_x| \lesssim \epsilon_0.$$

Our previous work [6] on gravity waves followed the same principles as in Morawetz's original paper ([46]), proving the results using the multiplier method, based on the momentum conservation law. One difficulty we encountered there was due to the nonlocality of the problem, which made it far from obvious what is the "correct" momentum density. Our solution in [6] was to work in parallel with two distinct momentum densities.

Some of the difficulties in [6] were connected to low frequencies, due both to the fact that the equations are nonlocal, and that they have quadratic nonlinearities. A key to defeating these difficulties was to shift some of the analysis from Eulerian coordinates and the holomorphic coordinates; this is because the latter provide a better setting to understand the fine bilinear and multilinear structure of the equations.

Here we are able to reuse the low frequency part of the analysis from [6]. On the other hand, we instead encounter additional high frequency issues arising from the surface tension contributions. As it turns out, these are also best dealt with in the set-up of holomorphic coordinates.

1.5. **Plan of the paper.** The key idea in our proof of the Morawetz estimate is to think of the energy as the flux for the momentum. Unfortunately this is not as simple as it sounds, as the momentum density is not a uniquely defined object, and the "obvious" choice does not yield the full energy as its flux, not even at leading order. To address this matter, in the next section, we review density flux pairs for the momentum. The standard density $\eta\psi_x$ implicit in (1.8) only allows one to control the local potential energy, while for the local kinetic energy we introduce an alternate density and the associated flux.

The two density-flux relations for the momentum are exploited in Section 3. There the proof of the Morawetz inequality is reduced to three main technical lemmas. However, proving these lemmas in the standard Eulerian setting seems nearly impossible; instead, our strategy is to first switch to holomorphic coordinates.

We proceed as follows. In Section 4 we review the holomorphic (conformal) coordinates and relate the two sets of variables between Eulerian and holomorphic formulation. In particular we show that the fixed energy type norms in the paper admit equivalent formulations in the two settings. In this we largely follow [31], [30] and [6], though several new results are also needed. In Section 5 we discuss the correspondence between the local energy norms in the two settings, as well as several other related matters.

The aim of the last section of the paper is to prove the three key main technical lemmas. This is done in two steps. First we obtain an equivalent formulation in the holomorphic setting, and then we use multilinear analysis methods to prove the desired estimates.

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2. Conservation of momentum and local conservation laws

A classical result is that the momentum, which is the following quantity,

$$\mathcal{M}(t) = \int_{\mathbb{R}} \int_{-h}^{\eta(t,x)} \phi_x(t,x,y)\, dydx,$$

is a conserved quantity:

$$\frac{d}{dt}\mathcal{M} = 0.$$

This comes from the invariance with respect to horizontal translation (see Benjamin and Olver [9] for a systematic study of the symmetries of the water-wave equations). We exploit later the conservation of the momentum through the use of density/flux pairs (I, S). By definition, these are pairs such that

$$\mathcal{M} = \int I\, dx, \tag{2.1}$$

and such that one has the conservation law

$$\partial_t I + \partial_x S = 0. \tag{2.2}$$

One can use these pairs by means of the multiplier method of Morawetz. Consider a function $m = m(x)$ which is positive and increasing. Multiplying the identity (2.2) by $m = m(x)$ and integrating over $[0, T] \times \mathbb{R}$ then yields (integrating by parts)

$$\iint_{[0,T]\times\mathbb{R}} S(t,x) m_x\, dxdt = \int_{\mathbb{R}} m(x) I(T,x)\, dx - \int_{\mathbb{R}} m(x) I(0,x)\, dx.$$

The key point is that, since the slope m_x is nonnegative, the later identity is favorable provided that S is non-negative.

There are three pairs (I, S) that play a role in our work. These pairs have already been discussed in [6, §2]. Here we keep the same densities; however, the associated fluxes will acquire an extra term due to the surface tension term in the equations.

Lemma 2.1. *The expression*

$$I_1(t,x) = \int_{-h}^{\eta(t,x)} \phi_x(t,x,y)\,dy,$$

is a density for the momentum, with associated density flux

$$S_1(t,x) := -\int_{-h}^{\eta(t,x)} \partial_t\phi\,dy - \frac{g}{2}\eta^2 + \kappa\left(1 - \frac{1}{\sqrt{1+\eta_x^2}}\right) + \frac{1}{2}\int_{-h}^{\eta(t,x)} (\phi_x^2 - \phi_y^2)\,dy.$$

Proof. With minor changes this repeats the computation in [6]. Given a function $f = f(t,x,y)$, we use the notation $\widetilde{f}$ to denote the function

$$\widetilde{f}(t,x) = f(t,x,\eta(t,x)).$$

Then we have

$$\partial_t I_1 = \partial_t \int_{-h}^{\eta} \phi_x\,dy = (\partial_t\eta)\widetilde{\phi_x} + \int_{-h}^{\eta} \partial_t\phi_x\,dy.$$

It follows from the kinematic equation for η and the Bernoulli equation for ϕ that

$$\partial_t I_1 = (\widetilde{\phi_y} - \eta_x\widetilde{\phi_x})\widetilde{\phi_x} - \int_{-h}^{\eta} \partial_x\left(\frac{1}{2}|\nabla_{x,y}\phi|^2 + P\right)dy,$$

so

$$\partial_t I_1 = \widetilde{\phi_y}\widetilde{\phi_x} + \frac{1}{2}\eta_x\widetilde{\phi_y^2} - \frac{1}{2}\eta_x\widetilde{\phi_x}^2 + \eta_x\widetilde{P} - \partial_x\int_{-h}^{\eta}\left(\frac{1}{2}|\nabla_{x,y}\phi|^2 + P\right)dy.$$

Using again the Bernoulli equation and using the equation for the pressure at the free surface, we find that

$$\partial_t I_1 = \widetilde{\phi_y}\widetilde{\phi_x} + \frac{1}{2}\eta_x\widetilde{\phi_y^2} - \frac{1}{2}\eta_x\widetilde{\phi_x}^2 - \kappa\eta_x\partial_x\left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right) + \partial_x\int_{-h}^{\eta}(\partial_t\phi + gy)\,dy.$$

Since

$$-\kappa\eta_x\partial_x\left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right) = \kappa\partial_x\frac{1}{\sqrt{1+\eta_x^2}} = \kappa\partial_x\left(\frac{1}{\sqrt{1+\eta_x^2}} - 1\right),$$

and since $\partial_x\int_{-h}^{\eta} gy\,dy = \partial_x(g\eta^2/2)$, to complete the proof, it is sufficient to verify that

$$\widetilde{\phi_y}\widetilde{\phi_x} + \frac{1}{2}\eta_x\widetilde{\phi_y^2} - \frac{1}{2}\eta_x\widetilde{\phi_x}^2 = \frac{1}{2}\partial_x\int_{-h}^{\eta}(\phi_y^2 - \phi_x^2)\,dy.$$

This in turn follows from the fact that

$$\int_{-h}^{\eta}(\phi_x\phi_{yx} - \phi_x\phi_{xx})\,dy = \int_{-h}^{\eta}(\phi_x\phi_{yx} + \phi_x\phi_{yy})\,dy = \int_{-h}^{\eta}\partial_y(\phi_x\phi_y)\,dy = \widetilde{\phi_x}\widetilde{\phi_y},$$

where we used $\phi_{xx} = -\phi_{yy}$ and the solid wall boundary condition $\phi_y(t,x,-h) = 0$. □

The above density-flux pair will not be directly useful because it has a linear component in it. However, we will use it as a springboard for the next two density-flux pairs.

Lemma 2.2. *The expression*

$$I_2(t,x) = \eta(t,x)\psi_x(t,x)$$

is a density for the momentum, with associated density flux

$$S_2(t,x) := -\eta\psi_t - \frac{g}{2}\eta^2 + \kappa\left(1 - \frac{1}{\sqrt{1+\eta_x^2}}\right) + \frac{1}{2}\int_{-h}^{\eta(t,x)}(\phi_x^2 - \phi_y^2)\,dy.$$

Proof. The proof is identical to the one of Lemma 2.3 in [6]. □

To define the third pair we recall from [6] two auxiliary functions defined inside the fluid domain. Firstly we introduce the stream function q, which is the harmonic conjugate of ϕ:

$$\begin{cases} q_x = -\phi_y, & \text{in } -h < y < \eta(t,x), \\ q_y = \phi_x, & \text{in } -h < y < \eta(t,x), \\ q(t,x,-h) = 0. \end{cases} \tag{2.3}$$

We also introduce the harmonic extension θ of η with Dirichlet boundary condition on the bottom:

$$\begin{cases} \Delta_{x,y}\theta = 0 & \text{in } -h < y < \eta(t,x), \\ \theta(t,x,\eta(t,x)) = \eta(t,x), \\ \theta(t,x,-h) = 0. \end{cases} \tag{2.4}$$

Now the following lemma contains the key density/flux pair for the momentum.

Lemma 2.3. *The expression*

$$I_3(t,x) = \int_{-h}^{\eta} \nabla\theta(t,x,y) \cdot \nabla q(t,x,y)\, dy$$

is a density for the momentum, with associated density flux

$$S_3(t,x) := -\frac{g}{2}\eta^2 - \int_{-h}^{\eta(t,x)} \theta_y \phi_t\, dy + \kappa\left(1 - \frac{1}{\sqrt{1+\eta_x^2}}\right) + \int_{-h}^{\eta(t,x)} \left(\frac{1}{2}(\phi_x^2 - \phi_y^2) + \theta_t\phi_y\right) dy.$$

Proof. The proof is identical to the one of Lemma 2.4 in [6]. □

In our analysis we will not only need the evolution equations restricted to the free boundary, but also the evolution of θ and ϕ within the fluid domain. To describe that, we introduce the operators H_D and H_N, which act on functions on the free surface $\{y = \eta(t,x)\}$ and produce their harmonic extension inside the fluid domain with zero Dirichlet, respectively Neumann boundary condition[3] on the bottom $\{y = -h\}$. With these notations, we have

$$\theta = H_D(\eta), \qquad \phi = H_N(\psi).$$

Recall that, given a function $f = f(t,x,y)$, we set $\widetilde{f}(t,x) := f(t,x,\eta(t,x))$.

Lemma 2.4. *The function ϕ_t is harmonic within the fluid domain, with Neumann boundary condition on the bottom, and satisfies*

$$\phi_t = -gH_N(\eta) - H_N\left(\widetilde{|\nabla\phi|^2}\right) + \kappa H_N(\partial_x(\eta_x/\sqrt{1+\eta_x^2})). \tag{2.5}$$

The function θ_t is harmonic within the fluid domain, with Dirichlet boundary condition on the bottom, and satisfies

$$\theta_t = \phi_y - H_D\left(\widetilde{\nabla\theta\cdot\nabla\phi}\right). \tag{2.6}$$

Proof. The equation for ϕ follows directly from the Bernoulli equation. The equation for θ is as in [6, Lemma 2.5]. □

[3]These two operators coicide in the infinite depth setting.

3. Local energy decay for gravity-capillary waves

In this section we prove our main result in Theorem 1.1, modulo three results (see Lemmas 3.1, 3.4, 3.5), whose proofs are in turn relegated to the last section of the paper.

We begin with a computation similar to [6]. We use the density-flux pairs (I_2, S_2) and (I_3, S_3) introduced in the previous section. Given $\sigma \in (0,1)$ to be chosen later on, we set

$$\mathcal{I}_m^\sigma(t) = \int_{\mathbb{R}} m(x)(\sigma I_2(x,t) + (1-\sigma) I_3(x,t))\, dx.$$

Here the function m is chosen so that m_x is a smooth non-negative bump function supported around $x_0 = 0$ and with integral one; it will be used to bound the local energy around $x_0 = 0$. This suffices due to the translation invariance.

Since $\partial_t I_j + \partial_x S_j = 0$, by integrating by parts, we have

$$\partial_t \mathcal{I}_m^\sigma(t) = \int_{\mathbb{R}} m_x(\sigma S_2(x,t) + (1-\sigma) S_3(x,t))\, dx.$$

Consequently, to prove Theorem 1.1, it is sufficient to establish the following estimates:

(i): Fixed time bounds,

$$\left| \int_{\mathbb{R}} m(x) I_2\, dx \right| \lesssim \|\eta\|_{g^{-\frac14} H_h^{\frac14} \cap \kappa^{-\frac14} H_h^{\frac34}} \|\psi_x\|_{g^{\frac14} H_h^{-\frac14} + \kappa^{\frac14} H_h^{-\frac34}}, \tag{3.1}$$

$$\left| \int_{\mathbb{R}} m(x) I_3\, dx \right| \lesssim \|\eta\|_{g^{-\frac14} H_h^{\frac14} \cap \kappa^{-\frac14} H_h^{\frac34}} \|\psi_x\|_{g^{\frac14} H_h^{-\frac14} + \kappa^{\frac14} H_h^{-\frac34}}. \tag{3.2}$$

(ii): Time integrated bound; the goal is to prove that there exist $\sigma \in (0,1)$ and $c < 1$ such that

$$\int_0^T \int_{\mathbb{R}} m_x(\sigma S_2(t) + (1-\sigma) S_3(t))\, dxdt \gtrsim \|(\eta,\psi)\|^2_{LE_0^\kappa} - c\|(\eta,\psi)\|^2_{LE^\kappa}, \tag{3.3}$$

where the local norms are as defined in (1.10).

We now successively discuss the two sets of bounds above.

(i) The fixed time bounds (3.1)-(3.2). These are similar to [6], except that in this case, one needs to take into account the threshold frequency $\lambda_0 = \sqrt{g/\kappa} \gg 1$. Since $I_2 = \eta\psi_x$, by a duality argument, to prove the bound in (3.1) it suffices to show that

$$\|m\eta\|_{g^{-\frac14} H_h^{\frac14} \cap \kappa^{-\frac14} H_h^{\frac34}} \lesssim \|\eta\|_{g^{-\frac14} H_h^{\frac14} \cap \kappa^{-\frac14} H_h^{\frac34}}. \tag{3.4}$$

The $H_h^{\frac14}$ bound was already proved in [6, (4.9)], so it remains to establish the $H_h^{\frac34}$ bound, which is stronger only at frequencies $\lambda > \lambda_0$. For this we use a paradifferential decomposition of the product $m\eta$ (see Métivier's book [44] for precise definitions):

$$m\eta = T_m \eta + T_\eta m + \Pi(m,\eta),$$

and estimate each term separately. Here $T_m\eta$ represents the multiplication between the low frequencies of m and the high frequencies of η, and $\Pi(m,\eta)$ is the *high-high* interaction operator.

We begin with the first term and note that since we are estimating the high frequencies in η, we do not have to deal with the λ_0 frequency threshold.

$$\|T_m\eta\|^2_{\dot{H}^{\frac{3}{4}}} \lesssim \sum_\lambda \lambda^{\frac{3}{2}}\|m_{<\lambda}\eta_\lambda\|^2_{L^2} \lesssim \sum_\lambda \lambda^{\frac{3}{2}}\|m_{<\lambda}\|^2_{L^\infty}\|\eta_\lambda\|^2_{L^2} \lesssim \|m\|^2_{L^\infty}\sum_\lambda \lambda^{\frac{3}{2}}\|\eta_\lambda\|^2_{L^2}$$
$$\lesssim \|m\|^2_{L^\infty}\|\eta\|_{\dot{H}^{\frac{3}{4}}}.$$

For the second term, as discussed above, we only need to bound (3.4) only at frequencies $\lambda > \lambda_0$. For this we rewrite $T_\eta m$

$$T_\eta m = \sum_\lambda \eta_{<\lambda} m_\lambda,$$

and then, due to the fact that we are not adding all frequencies (only the ones above λ_0), we get

$$\|T_\eta m\|_{\dot{H}^{\frac{3}{4}}} \lesssim \sum_{\lambda_0<\lambda} \|\eta_{<\lambda} m_\lambda\|^2_{\dot{H}^{\frac{3}{4}}},$$

and for each term we estimate using Plancherel and Bernstein's inequalities

$$\begin{aligned}
\sum_{\lambda_0<\lambda} \|\eta_{<\lambda} m_\lambda\|^2_{\dot{H}^{\frac{3}{4}}} &\lesssim \sum_{\lambda_0<\lambda} \lambda^{\frac{3}{2}}\|\eta_{<\lambda} m_\lambda\|^2_{L^2} \\
&\lesssim \sum_{\lambda_0<\lambda} \lambda^{\frac{3}{2}}\|\eta_{<\lambda}\|^2_{L^4}\|m_\lambda\|^2_{L^4} \\
&\lesssim \sum_{\lambda_0<\lambda} \lambda^{\frac{3}{2}}\|\eta\|^2_{\dot{H}^{\frac{1}{4}}}\lambda^{\frac{3}{2}}\|m_\lambda\|^2_{L^1} \\
&\lesssim \sum_{\lambda_0<\lambda} \lambda^{-1}\|\eta\|^2_{\dot{H}^{\frac{1}{4}}}\|m_{xx}\|^2_{L^1}.
\end{aligned}$$

The summation over λ is trivial. Finally, the bound for the final term is obtained in a similar fashion.

To obtain the second bound in (3.2), we begin by transforming I_3. Firstly, by definition of I_3 and q, we have

$$\int_{\mathbb{R}} mI_3\, dx = \iint_{\Omega(t)} m(\theta_y\phi_x - \theta_x\phi_y)\, dydx = \iint_{\Omega(t)} m\left(\partial_y(\theta\phi_x) - \partial_x(\theta\phi_y)\right)\, dydx.$$

Now we have

$$\iint_{\Omega(t)} m\partial_y(\theta\phi_x)\, dydx = \int_{\mathbb{R}} m(\theta\phi_x)|_{y=\eta}\, dx.$$

On the other hand, integrating by parts in x, we get

$$\iint_{\Omega(t)} m\partial_x(\theta\phi_y)\, dydx = -\iint_{\Omega(t)} m_x\theta\phi_y\, dydx - \int_{\mathbb{R}} \eta_x m(\theta\phi_y)|_{y=\eta}\, dx.$$

Consequently,

$$\int_{\mathbb{R}} mI_3\, dx = \int_{\mathbb{R}} m(\theta\phi_x + \eta_x\theta\phi_y)|_{y=\eta}\, dx + \iint_{\Omega(t)} m_x\theta\phi_y\, dydx.$$

Now, by definition of θ one has $\theta|_{y=\eta} = \eta$. Since $\phi_y = -q_x$ and since $(\phi_x + \eta_x\phi_y)|_{y=\eta} = \psi_x$, we end up with

$$\int_{\mathbb{R}} mI_3\, dx = \int_{\mathbb{R}} mI_2\, dx - \iint_{\Omega(t)} m_x\theta q_x\, dydx.$$

It remains to estimate the second part. This is a more delicate bound, which requires the use of holomorphic coordinates and is postponed for the last section of the paper. We state the desired bound as follows:

Lemma 3.1. *The following fixed estimate holds:*

$$\left| \iint_{\Omega} m_x \theta q_x \, dydx \right| \lesssim \|\eta\|_{g^{-\frac{1}{4}} H_h^{\frac{1}{4}} \cap \kappa^{-\frac{1}{4}} H_h^{\frac{3}{4}}} \|\psi_x\|_{g^{\frac{1}{4}} H_h^{-\frac{1}{4}} + \kappa^{\frac{1}{4}} H_h^{-\frac{3}{4}}}. \tag{3.5}$$

(ii) The time integrated bound (3.3)**.** We take $\sigma < 1/2$, but close to $1/2$. Using the expressions in Lemmas 2.2, 2.3 as well as the second equation in (1.6) and the relations (2.5) and (2.6) we write the integral in (3.3) as a combination of two leading order terms plus error terms

$$\int_0^T \int_{\mathbb{R}} m_x(\sigma S_2(t) + (1-\sigma) S_3(t)) \, dxdt = LE_\psi + gLE_\eta + \kappa LE_\eta^\kappa + Err_1 + gErr_2 - Err_3,$$

where[4], with $H(\eta)$ defined by (1.4),

$$LE_\psi := \frac{1}{2} \int_0^T \iint_{\Omega(t)} m_x [\sigma(\phi_x^2 - \phi_y^2) + (1-\sigma)|\nabla \phi|^2] \, dxdydt$$

$$LE_\eta := \int_0^T \left(\frac{\sigma}{2} \int_{\mathbb{R}} m_x \eta^2 \, dx - (1-\sigma) \iint_{\Omega(t)} m_x \theta_y (\theta - H_N(\eta)) \, dxdy \right) dt,$$

$$LE_\eta^\kappa = \int_0^T \left(\int_{\mathbb{R}} m_x \left(1 - \frac{1}{\sqrt{1+\eta_x^2}} - \sigma \eta H(\eta) \right) dx - (1-\sigma) \iint_{\Omega(t)} m_x \theta_y H_N(H(\eta)) dydx \right) dt,$$

and finally

$$Err_1 := \sigma \int_0^T \int_{\mathbb{R}} m_x \eta \mathcal{N}(\eta) \psi \, dxdt,$$

$$Err_2 := \frac{1-\sigma}{2} \int_0^T \iint_{\Omega(t)} m_x \theta_y H_N(\widetilde{|\nabla \phi|^2}) \, dxdydt,$$

$$Err_3 := \frac{1-\sigma}{2} \int_0^T \iint_{\Omega(t)} m_x \phi_y H_D(\widetilde{\nabla \theta \nabla \phi}) \, dxdydt.$$

The terms which do not involve the surface tension have already been estimated in [6]. We recall the outcome here:

Proposition 3.2 ([6])**.** *The following estimates hold:*

(i) Positivity estimates:

$$LE_\psi + LE_\eta \gtrsim \|(\eta, \psi)\|_{LE_0^\kappa}^2 - c\|(\eta, \psi)\|_{LE^\kappa}^2. \tag{3.6}$$

(ii) Error bounds:

$$|Err_1| + |Err_2| \lesssim \epsilon_0 (LE_\eta + LE_\psi). \tag{3.7}$$

(iii) Normal form correction in holomorphic coordinates:

$$|Err_3| \lesssim \epsilon_0 \left(LE_\psi + LE_\eta + \|\eta(0)\|_{H_h^{\frac{1}{4}}} \|\psi_x(0)\|_{H_h^{-\frac{1}{4}}} + \|\eta(T)\|_{H_h^{\frac{1}{4}}} \|\psi_x(T)\|_{H_h^{-\frac{1}{4}}} \right). \tag{3.8}$$

[4] Here for the last term in LE_η we also use $\int \theta_y \theta dy = \frac{1}{2}\eta^2$.

We recall that the bound for Err_3 is more complex because, rather than estimating it directly, in [6] we use a normal form correction to deal with the bulk of Err_3, and estimate directly only the ensuing remainder terms. Fortunately the normal form correction only uses the η equation, and thus does not involve at all the surface tension.

Thus, in what follows our remaining task is to estimate the contribution of LE^κ_η, which we describe in the following

Proposition 3.3. *The following estimate holds:*

$$LE^\kappa_\eta \gtrsim \int_{\mathbb{R}} m_x \eta_x^2 dx - \epsilon_0 LE^\kappa. \tag{3.9}$$

For the rest of the section we consider the main steps in the proof of the above proposition. We split the first integral in LE^κ_η in two, separating the $H(\eta)$ term, and consider the three terms in LE^κ_η separately. For the first one there is nothing to do. For the remaining two we recall that

$$H(\eta) = \partial_x \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}} \right).$$

The second term is easier. Integrating by parts we obtain

$$\sigma \int_0^T \int_{\mathbb{R}} m_x \left(\frac{\eta_x^2}{\sqrt{1+\eta_x^2}} + 1 - \frac{1}{\sqrt{1+\eta_x^2}} \right) + m_{xx} \frac{\eta \eta_x}{\sqrt{1+\eta_x^2}} \, dxdt.$$

The first term gives the positive contribution

$$c \int_0^T \int_{\mathbb{R}} m_x \eta_x^2 dxdt, \qquad c \approx \frac{3}{2}\sigma,$$

while the second is lower order and can be controlled by Cauchy-Schwarz using the gravity part of the local energy, provided that $\kappa \ll g$. This condition is invariant with respect to pure time scaling, but not with respect to space-time scaling. This implies that even if this condition is not satisfied, we still have local energy decay but with a window size larger than 1 (depending on the ratio κ/g), provided that $h^2 g \gg \kappa$.

Here we emphasize that while the local energy measured in (3.9) is centered around $x = 0$, the local energy on the right is taken as the supremum over all centers $x_0 \in \mathbb{R}$, see (1.10). Because of this, in the previous argument we do not need to directly control m_{xx} by m_x (which would be impossible), but rather by finitely many translates of m_x, as in

$$|m_{xx}(x)| \lesssim \sum_{|k| \leq 10} m_x(x+k).$$

The more difficult term is the last one, involving $H_N(H(\eta))$, namely

$$I^\kappa = - \int_0^T \iint_{\Omega(t)} m_x \theta_y H_N(H(\eta)) \, dydxdt.$$

The difficulty here is that, even though $H(\eta)$ is an exact derivative as a function of x, this property is lost when taking its harmonic extension since the domain itself is not flat. Thus in any natural expansion of $H_N(H(\eta))$ (e.g. in holomorphic coordinates where this is easier to see) there are quadratic (and also higher order) terms where no cancellation occurs in the high $\times$ high $\to$ low terms, making it impossible to factor out one derivative.

One can think of I^κ as consisting of a leading order quadratic part in η plus higher order terms. We expect the higher order terms to be perturbative because of our smallness condition, but not the quadratic term. Because of this, it will help to identify precisely the quadratic term. On the top we have, neglecting the quadratic and higher order terms,

$$H(\eta) \approx \eta_{xx} \approx \theta_{xx},$$

so one might think of replacing $H(\eta)$ with θ_{xx} modulo cubic and higher order terms. This is not entirely correct since θ_{xx} satisfies a Dirichlet boundary condition on the bottom, and not the Neumann boundary condition which we need. Nevertheless, we will still make this substitution, and pay the price of switching the boundary conditions. Precisely, we write

$$H_N(H(\eta)) = \theta_{xx} + (H_N(\theta_{xx}) - \theta_{xx}) + H_N(H(\eta) - \theta_{xx}) \tag{3.10}$$

and estimate separately the contribution of each term.

The contribution I_1^κ of the first term in (3.10) to I^κ is easily described using the relation $\theta_{xx} = -\theta_{yy}$,

$$\begin{aligned} I_1^\kappa &= \int_0^T \iint_{\Omega(t)} m_x \theta_y \theta_{yy}\, dydxdt \\ &= \frac{1}{2}\int_0^T \int_{\mathbb{R}} m_x \theta_y^2|_{y=\eta(t,x)}\, dxdt - \frac{1}{2}\int_0^T \int_{\mathbb{R}} m_x \theta_y^2|_{y=-h}\, dxdt \\ &:= \frac{1}{2} I^\kappa_{1,top} - \frac{1}{2} I^\kappa_{1,bottom} \end{aligned} \tag{3.11}$$

Here the main term $I^\kappa_{1,top}$ has the right sign, but the remaining expression $I^\kappa_{1,bottom}$ still has to be estimated.

In addition, it remains to estimate the integrals

$$I_2^\kappa = -\int_0^T \iint_{\Omega(t)} m_x \theta_y (H_N(\theta_{xx}) - \theta_{xx})\, dydxdt,$$

and

$$I_3^\kappa = -\int_0^T \iint_{\Omega(t)} m_x \theta_y H_N(H(\eta) - \theta_{xx})\, dydxdt.$$

For these three remaining integrals we will prove the following lemmas:

Lemma 3.4. *The integrals $I^\kappa_{1,bottom}$ and I_2^κ are estimated by*

$$|I^\kappa_{1,bottom}| + |I_2^\kappa| \lesssim h^{-2}\|\eta\|^2_{LE^0}. \tag{3.12}$$

respectively

Lemma 3.5. *The expression I_3^κ is estimated by*

$$|I_3^\kappa| \lesssim \epsilon\Big(\|\eta_x\|^2_{LE^0} + \frac{g}{\kappa}\|\eta\|^2_{LE^0}\Big). \tag{3.13}$$

Given the relation (3.11) and the last two lemmas, the desired result in Proposition 3.3 follows.

Unfortunately, proving these lemmas in the standard Eulerian setting seems to be a very difficult task, as it requires delicate estimates in depth, which are hard to establish when solving the Laplace equation within a moving domain. To address this difficulty, our strategy will be to first switch to holomorphic coordinates. In the next two sections we recall how the transition to holomorphic coordinates works, following [31], [33], [30] and [6]. Finally, in the last section of the paper we prove Lemmas 3.4, 3.5.

4. Holomorphic coordinates

4.1. **Harmonic functions in the canonical domain.** We begin by discussing two classes of harmonic functions in the horizontal strip $S = \mathbb{R} \times (-h, 0)$.

Given a function $f = f(\alpha)$ defined on the top, consider its harmonic extension with homogeneous Neumann boundary condition on the bottom,

$$\begin{cases} \Delta u = 0 \qquad \text{in } \ S \\ u(\alpha, 0) = f \\ \partial_\beta u(\alpha, -h) = 0. \end{cases} \tag{4.1}$$

It can be written in the form

$$u(\alpha, \beta) = P_N(\beta, D) f(\alpha) := \frac{1}{2\pi} \int p_N(\xi, \beta) \hat{f}(\xi) e^{i\alpha\xi} \, d\xi, \tag{4.2}$$

where p_N is a Fourier multiplier with symbol

$$p_N(\xi, \beta) = \frac{\cosh((\beta + h)\xi)}{\cosh(h\xi)}.$$

We will make use of the Dirichlet to Neumann map $\mathcal{D}_N$, defined by

$$\mathcal{D}_N f = \partial_\beta u(\cdot, 0),$$

as well as the Tilbert transform, defined by

$$\mathcal{T}_h f(\alpha) = -\frac{1}{2h} \lim_{\epsilon \downarrow 0} \int_{|\alpha - \alpha'| > \epsilon} \operatorname{cosech} \left(\frac{\pi}{2h} (\alpha - \alpha') \right) f(\alpha') \, d\alpha'. \tag{4.3}$$

Then the Tilbert transform is the Fourier multiplier

$$\mathcal{T}_h = -i \tanh(hD).$$

Notice that it takes real-valued functions to real-valued functions.

The inverse Tilbert transform is denoted by $\mathcal{T}_h^{-1}$; *a priori* this is defined modulo constants. It can be seen for instance as an operator from L^2 to $L^2 + \dot{H}^1$, or even from temperate distributions to temperate distributions modulo constants.

It follows that the Dirichlet to Neumann map can be written under the form

$$\mathcal{D}_N f = \mathcal{T}_h \partial_\alpha f.$$

We now consider a similar problem with the homogeneous Dirichlet boundary condition on the bottom

$$\begin{cases} \Delta v = 0 \qquad \text{in } \ S \\ v(\alpha, 0) = g \\ v(\alpha, -h) = 0. \end{cases} \tag{4.4}$$

Then

$$v(\alpha, \beta) = P_D(\beta, D) g(\alpha) := \frac{1}{2\pi} \int p_D(\xi, \beta) \hat{g}(\xi) e^{i\alpha\xi} \, d\xi,$$

where

$$p_D(\xi, \beta) = \frac{\sinh((\beta + h)\xi)}{\sinh(h\xi)}.$$

The Dirichlet to Neumann map $\mathcal{D}_D$ for this problem is given by

$$\partial_\beta v(\alpha, 0) = \mathcal{D}_D g = -\mathcal{T}_h^{-1} \partial_\alpha g.$$

The solution to (4.1) is related to the one of (4.4) by means of harmonic conjugates. Namely, given a real-valued solution u to (4.1), we consider its harmonic conjugate v, i.e., satisfying the Cauchy-Riemann equations

$$\begin{cases} u_\alpha = -v_\beta \\ u_\beta = v_\alpha \\ \partial_\beta u(\alpha, -h) = 0. \end{cases}$$

Then v is a solution to (4.4) provided that the Dirichlet data g for v on the top is determined by the Dirichlet data f for u on the top via the relation

$$g = -\mathcal{T}_h f.$$

Conversely, given v, there is a corresponding harmonic conjugate u (which is uniquely determined modulo real constants). Here u and v can be thought of as smooth functions within the strip S, whose traces on the top belong to the class of temperate distributions. Within the context of this paper, one can simply take $g \in L^2$ and $f \in L^2 + \dot{H}^1_h$.

4.2. **Holomorphic functions in the canonical domain.** Here we consider the real algebra of holomorphic functions w in the canonical domain $S := \{\alpha + i\beta \,:\, \alpha \in \mathbb{R},\ -h \leq \beta \leq 0\}$, which are real on the bottom $\{\mathbb{R} - ih\}$. Notice that such functions are uniquely determined by their values on the top $\{\beta = 0\}$, and can be expressed as

$$w = u + iv,$$

where u and v are harmonic conjugate functions satisfying the equations (4.1), respectively (4.4).

Hereafter, by definition, we will call functions on the real line holomorphic if they are the restriction on the real line of holomorphic functions in the canonical domain S which are real on the bottom $\{\mathbb{R} - ih\}$. Put another way, they are functions $w\colon \mathbb{R} \to \mathbb{C}$ so that there is an holomorphic function, still denoted by $w\colon S \to \mathbb{C}$, which satisfies

$$\operatorname{Im} w = -\mathcal{T}_h \operatorname{Re} w$$

on the top. The complex conjugates of holomorphic functions are called antiholomorphic.

4.3. **Holomorphic coordinates and water waves.** Recall that $\Omega(t)$ denotes the fluid domain at a given time $t \geq 0$, in Eulerian coordinates. In this section we recall following [31, 33] (see also [25, 30]) how to rewrite the water-wave problem in holomorphic coordinates.

We introduce holomorphic coordinates $z = \alpha + i\beta$, thanks to conformal maps

$$Z\colon S \to \Omega(t),$$

which associate the top to the top, and the bottom to the bottom. Such a conformal transformation exists by the Riemann mapping theorem. Notice that these maps are uniquely defined up to horizontal translations in S and that, restricted to the real axis, this provides a parametrization for the water surface Γ.

We set

$$W := Z - \alpha,$$

so that $W = 0$ if the fluid surface is flat i.e., $\eta = 0$. Because of the boundary condition on the bottom of the fluid domain the function W is holomorphic when $\alpha \in \mathbb{R}$.

Moving to the velocity potential ϕ, we consider its harmonic conjugate q and then the function $Q := \phi + iq$, taken in holomorphic coordinates, is the holomorphic counterpart of ϕ. Here q is exactly the stream function, see [6].

With this notations, the water-wave problem can be recast as an evolution system for (W, Q), within the space of holomorphic functions defined on the surface (again, we refer the reader to [31, 33, 25, 30] for the details of the computations). Here we recall the equations:

$$\begin{cases} W_t + F(1+W_\alpha) = 0 \\ Q_t + FQ_\alpha - g\mathcal{T}_h[W] + \mathbf{P}_h\left[\dfrac{|Q_\alpha|^2}{J}\right] + i\kappa\mathbf{P}_h\left[\dfrac{W_{\alpha\alpha}}{J^{1/2}(1+W_\alpha)} - \dfrac{\bar{W}_{\alpha\alpha}}{J^{1/2}(1+\bar{W}_\alpha)}\right] = 0, \end{cases} \tag{4.5}$$

where

$$J = |1+W_\alpha|^2, \qquad F = \mathbf{P}_h\left[\frac{Q_\alpha - \bar{Q}_\alpha}{J}\right].$$

Here $\mathbf{P}_h$ represents the orthogonal projection on the space of holomorphic functions with respect with the inner product in the Hilbert space $\mathfrak{H}_h$ introduced in [30]. This has the form

$$\langle u, v\rangle_{\mathfrak{H}_h} := \int \left(\mathcal{T}_h \operatorname{Re} u \cdot \mathcal{T}_h \operatorname{Re} v + \operatorname{Im} u \cdot \operatorname{Im} v\right)\, d\alpha,$$

and coincides with the L^2 inner product in the infinite depth case. Written in terms of the real and imaginary parts of u, the projection $\mathbf{P}_h$ takes the form

$$\mathbf{P}_h u = \frac{1}{2}\left[(1 - i\mathcal{T}_h)\operatorname{Re} u + i(1 + i\mathcal{T}_h^{-1})\operatorname{Im} u\right]. \tag{4.6}$$

Since all the functions in the system (4.5) are holomorphic, it follows that these relations also hold in the full strip S for the holomorphic extensions of each term.

We also remark that in the finite depth case there is an additional gauge freedom in the above form of the equations, in that $\operatorname{Re} F$ is a-priori only uniquely determined up to constants. This corresponds to the similar degree of freedom in the choice of the conformal coordinates, and will be discussed in the last subsection.

A very useful function in the holomorphic setting is

$$R = \frac{Q_\alpha}{1+W_\alpha},$$

which represents the "good variable" in this setting, and corresponds to the Eulerian function

$$R = \phi_x - i\phi_y.$$

We also remark that the function θ introduced in the previous section is described in holomorphic coordinates by

$$\theta = \operatorname{Im} W.$$

Also related to W, we will use the auxiliary holomorphic function

$$Y = \frac{W_\alpha}{1+W_\alpha}.$$

Another important auxiliary function here is the advection velocity

$$b = \operatorname{Re} F,$$

which represents the velocity of the particles on the fluid surface in the holomorphic setting.

It is also interesting to provide the form of the conservation laws in holomorphic coordinates. We begin with the energy (Hamiltonian), which has the form

$$\mathcal{H} = \int \frac{g}{2}|\operatorname{Im} W|^2(1+\operatorname{Re} W_\alpha) + \kappa(\sqrt{1+2\operatorname{Re} W_\alpha + |W_\alpha|^2} - 1)\, d\alpha - \frac{1}{4}\langle Q, \mathcal{T}_h^{-1}[Q_\alpha]\rangle_{\mathfrak{H}_h}.$$

One should compare this with (1.5); the two formulas are related via a change of variable. See also a similar computation in the Appendix of [31]. The momentum on the other has the form

$$\mathcal{M} = \frac{1}{2}\langle W, \mathcal{T}_h^{-1}Q_\alpha\rangle_{\mathfrak{H}_h} = \int_{\mathbb{R}} \mathcal{T}_h \operatorname{Re} W \cdot \operatorname{Re} Q_\alpha \, d\alpha = \int_{\mathbb{R}} \operatorname{Im} W \cdot \operatorname{Re} Q_\alpha \, d\alpha.$$

and does not depend on κ, see (1.8).

4.4. **Uniform bounds for the conformal map.** In order to freely switch computations between the Eulerian and holomorphic setting it is very useful to verify that our Eulerian uniform smallness assumption for the functions $(\eta, \nabla\phi|_{y=\eta})$ also has an identical interpretation in the holomorphic setting for the functions $(\operatorname{Im} W, R)$. Our main result is as follows:

Theorem 4.1. *Assume that the smallness condition* (1.11) *holds. Then we have*

$$\|(\operatorname{Im} W, R)\|_{X^\kappa} \lesssim \epsilon_0. \tag{4.7}$$

This result is in effect an equivalence between the two bounds. We state and prove only this half because that is all that is needed here.

Proof. The similar result for the X space corresponding to pure gravity waves was proved in [6], so we only need to add the X_1 component of the X^κ norm. We first recall some of the set-up in [6], and then return to X_1.

The X norm is described in [6] using the language of frequency envelopes. We define a *frequency envelope* for $(\eta, \nabla\phi_{|y=\eta})$ in X to be any positive sequence

$$\{c_\lambda : \quad h^{-1} < \lambda \in 2^{\mathbb{Z}}\}$$

with the following two properties:

(1) Dyadic bound from above,

$$\|P_\lambda(\eta, \nabla\phi_{|y=\eta})\|_{X_0} \leq c_\lambda.$$

(2) Slowly varying,

$$\frac{c_\lambda}{c_\mu} \leq \max\left\{\left(\frac{\lambda}{\mu}\right)^\delta, \left(\frac{\mu}{\lambda}\right)^\delta\right\}.$$

Here $\delta \ll 1$ is a small universal constant. Among all such frequency envelopes there exists a *minimal frequency envelope*. In particular, this envelope has the property that

$$\|(\eta, \nabla\phi_{|y=\eta})\|_X \approx \|c\|_{\ell^1}.$$

We set the notations as follows:

Definition 4.2. *By* $\{c_\lambda\}_{\lambda\geq 1/h}$ *we denote the minimal frequency envelope for* $(\eta, \nabla\phi|_{y=\eta})$ *in* X_0. *We call* $\{c_\lambda\}$ *the* control frequency envelope.

Since in solving the Laplace equation on the strip, solutions at depth β are localized at frequencies $\leq \lambda$ where $\lambda \approx |\beta|^{-1}$, we will also use the notation

$$c_\beta = c_\lambda, \qquad \lambda \approx |\beta|^{-1}.$$

This determines c_β up to an $1 + O(\epsilon)$ constant, which suffices for our purposes.

Using these notations, in [6] we were able to prove a stronger version of the above theorem for the X norm, and show that one can transfer the control envelope for $(\eta, \nabla\phi_{|y=\eta})$ to their counterpart $(\operatorname{Im} W, R)$ in the holomorphic coordinates.

Proposition 4.3. *Assume the smallness condition* (1.11)*, and let* $\{c_\lambda\}$ *be the control envelope as above. Then we have*

$$\|P_\lambda(\operatorname{Im} W, R)\|_{X_0} \lesssim c_\lambda. \tag{4.8}$$

As noted in [6], as a consequence of this proposition we can further extend the range of the frequency envelope estimates:

Remark 4.4. The X control envelope $\{c_\lambda\}$ is also a frequency envelope for

- $(\operatorname{Im} W, R)$ in X_0.
- W_α in $H_h^{\frac{1}{2}}$ and L^∞.
- Y in $H_h^{\frac{1}{2}}$.

We remark that this in particular implies, by Bernstein's inequality, the pointwise bound

$$\|W_\alpha\|_{L^\infty} \lesssim \epsilon_0. \tag{4.9}$$

This in turn implies that the Jacobian matrix for the change of coordinates stays close to the identity.

We now turn our attention to the X_1 component of the X^κ norm. From the X_1 component of the bound (1.11), we have the additional information that

$$\|\eta_{xx}\|_{L^2} \lesssim \epsilon_0 \left(\frac{g}{\kappa}\right)^{\frac{1}{4}}, \tag{4.10}$$

and we need to show that

$$\|W_{\alpha\alpha}\|_{L^2} \lesssim \epsilon_0 \left(\frac{g}{\kappa}\right)^{\frac{1}{4}}. \tag{4.11}$$

Using the relations $\eta = \operatorname{Im} W$ and $x = \alpha + \operatorname{Re} W$ on the top, we use the chain rule to compute for functions on the top

$$\partial_x = \frac{1}{1 + \operatorname{Re} W_\alpha}\partial_\alpha.$$

Therefore,

$$\eta_{xx} = \frac{1}{1 + \operatorname{Re} W_\alpha}\partial_\alpha\left(\frac{1}{1 + \operatorname{Re} W_\alpha}\operatorname{Im} W_\alpha\right),$$

which implies that

$$\|\eta_{xx}\|_{L^2} \approx \left\|\partial_\alpha\left(\frac{1}{1 + \operatorname{Re} W_\alpha}\operatorname{Im} W_\alpha\right)\right\|_{L^2}.$$

Thus we have

$$\|\operatorname{Im} W_{\alpha\alpha}\|_{L^2} \lesssim \|\eta_{xx}\|_{L^2} + \|W_\alpha\|_{L^\infty}\|W_{\alpha\alpha}\|_{L^2} \lesssim \epsilon_0 \left(\frac{g}{\kappa}\right)^{\frac{1}{4}} + \epsilon_0\|W_{\alpha\alpha}\|_{L^2}.$$

On the other hand, the real and imaginary parts of $W_{\alpha\alpha}$ have the same regularity at frequency $> h^{-1}$; more precisely, we have $\operatorname{Im} W = \mathcal{T}_h \operatorname{Re} W$, so we can estimate

$$\|\operatorname{Re} W_{\alpha\alpha}\|_{L^2} \lesssim \|\operatorname{Im} W_{\alpha\alpha}\|_{L^2} + h^{-1}\|\operatorname{Im} W_\alpha\|_{L^2} \lesssim \|\operatorname{Im} W_{\alpha\alpha}\|_{L^2} + h^{-\frac{1}{2}}\epsilon_0,$$

where the L^2 bound for $\operatorname{Im} W_\alpha$ comes from the X norm. Combining the last two bounds we get

$$\|W_{\alpha\alpha}\|_{L^2} \lesssim \epsilon_0 \left(\left(\frac{g}{\kappa}\right)^{\frac{1}{4}} + h^{-\frac{1}{2}} \right).$$

Then (4.11) follows from the relation $h^2 g \gtrsim \kappa$, which says that the Bond number stays bounded. □

4.5. **Fixed time bounds at the level of the momentum.** Our objective here is to relate the Eulerian norms of (η, ψ) at the momentum level in $E^{\frac{1}{4}}$ (see (1.9)) to their counterpart in the holomorphic setting for (W, R). Precisely, we have:

Lemma 4.5. *Assume that the condition* (1.11) *holds. Then we have the estimate*

$$\|(\operatorname{Im} W, \partial_\alpha^{-1} \operatorname{Im} R)\|_{E^{\frac{1}{4}}} \approx \|(\eta, \psi)\|_{E^{\frac{1}{4}}}. \tag{4.12}$$

Proof. Recalling that $\eta = \operatorname{Im} W$, for the first part of the equivalence we are bounding the same function but in different coordinates. As the change of coordinates is bi-Lipschitz, the L^2 and $\dot{H}^1$ norms are equivalent, and, by interpolation, all intermediate norms.

For the second part of the equivalence we use the relation $\psi = \operatorname{Re} Q$. By the same reasoning as above, we can switch coordinates to get

$$\|\psi\|_{g^{\frac{1}{4}} \dot{H}_h^{\frac{3}{4}}(\mathbb{R}) + \kappa^{\frac{1}{4}} \dot{H}_h^{\frac{1}{4}}(\mathbb{R})} \approx \|\operatorname{Re} Q\|_{g^{\frac{1}{4}} \dot{H}_h^{\frac{3}{4}}(\mathbb{R}) + \kappa^{\frac{1}{4}} \dot{H}_h^{\frac{1}{4}}(\mathbb{R})},$$

where the first norm is relative to the the Eulerian coordinate x and the second norm is relative to the the holomorphic coordinate α. It remains to relate the latter to the corresponding norm of $\partial^{-1} R$. Differentiating, we need to show that

$$\|Q_\alpha\|_{g^{\frac{1}{4}} H_h^{-\frac{1}{4}}(\mathbb{R}) + \kappa^{\frac{1}{4}} H_h^{-\frac{3}{4}}(\mathbb{R})} \approx \|R\|_{g^{\frac{1}{4}} H_h^{-\frac{1}{4}}(\mathbb{R}) + \kappa^{\frac{1}{4}} H_h^{-\frac{3}{4}}(\mathbb{R})}.$$

But here we can use the relation

$$Q_\alpha = R(1 + W_\alpha)$$

along with the multiplicative bound

$$\|fR\|_{g^{\frac{1}{4}} H_h^{-\frac{1}{4}}(\mathbb{R}) + \kappa^{\frac{1}{4}} H_h^{-\frac{3}{4}}(\mathbb{R})} \lesssim (\|f\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + (\kappa/g)^{\frac{1}{4}}\|f_\alpha\|_{L^2})\|R\|_{g^{\frac{1}{4}} H_h^{-\frac{1}{4}}(\mathbb{R}) + \kappa^{\frac{1}{4}} H_h^{-\frac{3}{4}}(\mathbb{R})}, \tag{4.13}$$

applied with $f = W_\alpha$ and then in the other direction with $f = Y$. Here the uniform Besov norm bound for W_α comes from the X component of X^κ; this also carries over to Y by Moser estimates.

It remains to prove (4.13). By duality we rephrase this as

$$\|fR\|_{g^{-\frac{1}{4}} H_h^{\frac{1}{4}}(\mathbb{R}) \cap \kappa^{-\frac{1}{4}} H_h^{\frac{3}{4}}(\mathbb{R})} \lesssim (\|f\|_{\dot{B}^{\frac{1}{2}}_{2,1}} + (\kappa/g)^{\frac{1}{4}}\|f_\alpha\|_{L^2})\|R\|_{g^{-\frac{1}{4}} H_h^{\frac{1}{4}}(\mathbb{R}) \cap \kappa^{-\frac{1}{4}} H_h^{\frac{3}{4}}(\mathbb{R})}, \tag{4.14}$$

which we approach in the same way as in the earlier proof of (3.4). In the paraproduct decomposition the terms $T_f R$ and $\Pi(f, R)$ are easy to estimate, using only the L^∞ bound for f. The term $T_R f$ is more interesting. At fixed frequency λ we estimate in L^2 the product

$$f_\lambda R_{<\lambda}.$$

We split into two cases:

a) $\lambda < \lambda_0 = (g/\kappa)^{\frac{1}{2}}$. Here we use Sobolev embeddings to write

$$\|f_\lambda R_{<\lambda}\|_{g^{-\frac{1}{4}} H_h^{\frac{1}{4}}(\mathbb{R})} \lesssim g^{\frac{1}{4}} \lambda^{\frac{1}{4}} \|f_\lambda\|_{L^4} \|R_{<\lambda}\|_{L^4} \lesssim \|f_\lambda\|_{\dot{B}^{\frac{1}{2}}_{2,1}} \|R\|_{g^{-\frac{1}{4}} H_h^{\frac{1}{4}}(\mathbb{R})},$$

which suffices.

b) $\lambda > \lambda_0 = (g/\kappa)^{\frac{1}{2}}$. Then we estimate

$$\|f_\lambda R_{<\lambda}\|_{L^2} \lesssim \|f_\lambda\|_{L^2} \|R_{<\lambda}\|_{L^\infty} \lesssim \lambda^{-1} (\kappa/g)^{-\frac{1}{8}} [(\kappa/g)^{\frac{1}{4}} \|f_\alpha\|_{L^2}] \|R\|_{g^{-\frac{1}{4}} H_h^{\frac{1}{4}}(\mathbb{R}) \cap \kappa^{-\frac{1}{4}} H_h^{\frac{3}{4}}(\mathbb{R})},$$

which again suffices.

□

4.6. **Vertical strips in Eulerian vs holomorphic coordinates.** In our main result, the local energy functionals are defined using vertical strips in Eulerian coordinates. On the other hand, for the multilinear analysis in our error estimates in the last section, it would be easier to use vertical strips in holomorphic coordinates. To switch from one to the other we need to estimate the horizontal drift between the two strips in depth. As the conformal map is biLipschitz, it suffices to compare the centers of the two strips. It is more convenient to do this in the reverse order, and compare the Eulerian image of the holomorphic vertical section with the Eulerian vertical section. This analysis was carried out in [6], and we recall the result here:

Proposition 4.6. *Let $(x_0, \eta(x_0)) = Z(\alpha_0, 0)$, respectively $(\alpha_0, 0)$ be the coordinates of a point on the free surface in Eulerian, respectively holomorphic coordinates. Assume that (1.11) holds, and let $\{c_\lambda\}$ be the control frequency envelope in Definition 4.2. Then we have the uniform bounds:*

$$|\operatorname{Re} Z(\alpha_0, \beta) - x_0 + \beta \operatorname{Im} W_\alpha(\alpha_0, \beta)| \lesssim c_\lambda, \qquad |\beta| \approx \lambda^{-1}. \tag{4.15}$$

As a corollary, we see that the distance between the two strip centers grows at most linearly:

Corollary 4.7. *Under the same assumptions as in the above proposition we have*

$$|\operatorname{Re} Z(\alpha_0, \beta) - x_0| \lesssim \epsilon_0 |\beta|. \tag{4.16}$$

4.7. **The horizontal gauge invariance.** Here we briefly discuss the gauge freedom due to the fact that $\operatorname{Re} F$ is a-priori only uniquely determined up to constants. In the infinite depth case this gauge freedom is removed by making the assumption $F \in L^2$. In the finite depth case (see [30]) instead this is more arbitrarily removed by setting $F(\alpha = -\infty) = 0$.

In the present paper no choice is necessary for our main result, as well as for most of the proof. However, such a choice was made for convenience in [6], whose results we also apply here. Thus we briefly recall it.

Assume first that we have a finite depth. We start with a point $x_0 \in \mathbb{R}$ where our local energy estimate is centered. Then we resolve the gauge invariance with respect to horizontal translations by setting $\alpha(x_0) = x_0$, which corresponds to setting $\operatorname{Re} W(x_0) = 0$. In dynamical terms, this implies that the real part of F is uniquely determined by

$$0 = \operatorname{Re} W_t(x_0) = \operatorname{Re}(F(1 + W_\alpha))(x_0),$$

which yields

$$\operatorname{Re} F(x_0) = \operatorname{Im} F(x_0) \frac{\operatorname{Im} W_\alpha(x_0)}{1 + \operatorname{Re} W_\alpha(x_0)}.$$

In the infinite depth case, the canonical choice for F is the one vanishing at infinity. This corresponds to a moving location in the α variable. We can still rectify this following the finite depth model, at the expense of introducing a constant component in both $\operatorname{Re} W$ and in F. We will follow this convention in the paper, in order to insure that our infinite depth computation is an exact limit of the finite depth case.

5. Local energy bounds in holomorphic coordinates

5.1. **Notations.** We begin by transferring the local energy bounds to the holomorphic setting. Recall that in the Eulerian setting, they are equivalently defined as

$$\|(\eta, \psi)\|_{LE^\kappa} := g^{\frac{1}{2}}\|\eta\|_{LE^0} + \kappa^{\frac{1}{2}}\|\eta_x\|_{LE^0} + \|\nabla\phi\|_{LE^{-\frac{1}{2}}},$$

where

$$\|\eta\|_{LE^0} := \sup_{x_0\in\mathbb{R}} \|\eta\|_{L^2(S(x_0))}, \qquad \|\nabla\phi\|_{LE^{-\frac{1}{2}}} := \sup_{x_0\in\mathbb{R}} \|\nabla\phi\|_{L^2(\mathbf{S}(x_0))}.$$

Here $S(x_0)$, respectively $\mathbf{S}(x_0)$ represent the Eulerian strips

$$S(x_0) := \{[0,T]\times[x_0-1, x_0+1]\}, \qquad \mathbf{S}(x_0) := S(x_0)\times[-h, 0].$$

In holomorphic coordinates the functions η and $\nabla\phi$ are given by $\operatorname{Im} W$ and R. Thus we seek to replace the above local energy norm with

$$\|(W, R)\|_{LE^\kappa} := g^{\frac{1}{2}}\|\operatorname{Im} W\|_{LE^0} + \kappa^{\frac{1}{2}}\|\operatorname{Im} W_\alpha\|_{LE^0} + \|R\|_{LE^{-\frac{1}{2}}},$$

with

$$\|\operatorname{Im} W\|_{LE^0} := \sup_{x_0\in\mathbb{R}} \|\operatorname{Im} W\|_{L^2(S_h(x_0))}, \qquad \|R\|_{LE^{-\frac{1}{2}}} := \sup_{x_0\in\mathbb{R}} \|R\|_{L^2(\mathbf{S}_h(x_0))}.$$

Here $S_h(x_0)$ and $\mathbf{S}_h(x_0)$ represent the holomorphic strips given by

$$S_h(x_0) := \{(t,\alpha) : t\in[0,T],\ \alpha\in[\alpha_0-1, \alpha_0+1]\}, \qquad \mathbf{S}_h(x_0) := S(x_0)\times[-h, 0],$$

where $\alpha_0 = \alpha_0(t, x_0)$ represents the holomorphic coordinate of x_0, which in general will depend on t.

We call the attention to the fact that, while the strips $S_h(x_0)$ on the top roughly correspond to the image of $S(x_0)$ in holomorphic coordinates, this is not the case for the strips $\mathbf{S}_h(x_0)$ relative to $\mathbf{S}(x_0)$. In depth, there may be a horizontal drift, which is estimated by means of Proposition 4.6.

We can now state the following equivalence.

Proposition 5.1. *Assuming the uniform bound (1.11), we have the equivalence:*

$$\|(\eta, \psi)\|_{LE^\kappa} \approx \|(W, R)\|_{LE^\kappa}. \tag{5.1}$$

Proof. Here the correspondence between the LE^0 norms of η and $\operatorname{Im} W$ is straightforward due to the bi-Lipschitz property of the conformal map. However, the correspondence between the $LE^{-\frac{1}{2}}$ norms of $\nabla\phi$ and R is less obvious, and was studied in detail in [6].

Moving on to the LE^0 norms of η_x and $\operatorname{Im} W$, we have

$$\eta_x = J^{-\frac{1}{2}} \operatorname{Im} W_\alpha.$$

Since $J = 1 + O(\epsilon)$ and the correspondence between the two sets of coordinates is bi-Lipschitz, it immediately follows that $\|\eta_x\|_{LE^0} \approx \|W_\alpha\|_{LE^0}$. □

One difference between the norms for $\operatorname{Im} W$ and for R is that they are expressed in terms of the size of the function on the top, respectively in depth. For the purpose of multilinear estimates later on we will need access to both types of norms. Since the local energy norms are defined using the unit spatial scale, in order to describe the behavior of functions in these spaces we will differentiate between high frequencies and low frequencies. We begin with functions on the top:

a) **High frequency characterization on top.** Here we will use local norms on the top, for which we will use the abbreviated notation

$$\|u\|_{L^2_t H^s_{loc}} := \sup_{x_0 \in \mathbb{R}} \|u\|_{L^2_t H^s_\alpha([\alpha_0-1,\alpha_0+1])},$$

where again $\alpha_0 = \alpha_0(x_0, t)$.

b) **Low frequency characterization on top.** Here we will use local norms on the top to describe the frequency λ or $\leq \lambda$ part of functions, where $\lambda < 1$ is a dyadic frequency. By the uncertainty principle such bounds should be uniform on the λ^{-1} spatial scale. Then it is natural to use the following norms:

$$\|u\|_{L^2_t L^\infty_{loc}(B_\lambda)} := \sup_{x_0 \in \mathbb{R}} \|u\|_{L^2_t L^\infty_\alpha(B_\lambda(x_0))},$$

where

$$B_\lambda(x_0) := \{\alpha \in \mathbb{R} : \ |\alpha - \alpha_0| \lesssim \lambda^{-1}\}.$$

We remark that the local norms in a) correspond exactly to the $B_\lambda(x_0)$ norms with $\lambda = 1$.

Next we consider functions in the strip which are harmonic extensions of functions on the top. To measure them we will use function spaces as follows:

a1) **High frequency characterization in strip.** Here we will use local norms on regions with depth at most 1, for which we will use the abbreviated notation

$$\|u\|_{L^2_t X_{loc}(A_1)} := \sup_{x_0 \in \mathbb{R}} \|u\|_{L^2_t X(A_1(x_0))},$$

where X will represent various Sobolev norms and

$$A_1(x_0) := \{(\alpha, \beta) : |\beta| \lesssim 1, \ |\alpha - \alpha_0| \lesssim 1\}.$$

b1) **Low frequency characterization in strip.** Here a frequency $\lambda < 1$ is associated with depths $|\beta| \approx \lambda^{-1}$. Thus, we define the regions

$$A_\lambda(x_0) = \{(\alpha, \beta) : |\beta| \approx \lambda^{-1}, \ |\alpha - \alpha_0| \lesssim \lambda^{-1}\}, \qquad \lambda < 1,$$

as well as

$$\begin{aligned} \mathbf{B}_1(x_0) &:= \{(\alpha, \beta); \ |\alpha - \alpha_0| \leq 1, \ \beta \in [-1, 0]\}, \\ \mathbf{B}_\lambda(x_0) &:= \{(\alpha, \beta); \ |\alpha - \alpha_0| \leq \lambda^{-1}, \ \beta \in [-\lambda^{-1}, 0]\}, \text{ for } \lambda < 1. \end{aligned}$$

In these regions we use the uniform norms,

$$\|u\|_{L^2_t L^\infty_{loc}(A_\lambda)} := \sup_{x_0 \in \mathbb{R}} \|u\|_{L^2_t L^\infty_{\alpha,\beta}(A_\lambda(x_0))},$$

and similarly for $\mathbf{B}^1$ and $\mathbf{B}_\lambda$.

5.2. **Multipliers and Bernstein's inequality in uniform norms.** Here we recall the results of [6] describing how multipliers act on the uniform spaces defined above. We will work with a multiplier $M_\lambda(D)$ associated to a dyadic frequency λ. In order to be able to use the bounds in several circumstances, we make a weak assumption on their (Lipschitz) symbols $m_\lambda(\xi)$:

$$|m_\lambda(\xi)| \lesssim \ (1+\lambda^{-1}|\xi|)^{-3}, \ \text{ and } \ |\partial_\xi^{k+1} m_\lambda(\xi)| \lesssim c_k \ |\xi|^{-k}(1+\lambda^{-1}|\xi|)^{-4}. \tag{5.2}$$

Examples of such symbols include

- Littlewood-Paley localization operators P_λ, $P_{\leq\lambda}$.
- The multipliers $p_D(\beta, D)$ and $p_N(\beta, D)$ in subsection 4 with $|\beta| \approx \lambda^{-1}$.

We will separately consider high frequencies, where we work with the spaces $L^2_t L^p_{loc}$, and low frequencies, where we work with the spaces $L^2_t L^p_{loc}(B_\lambda)$ associated with a dyadic frequency $1/h \leq \lambda \leq 1$.

A. High frequencies. Here we consider a dyadic high frequency $\lambda \geq 1$, and seek to understand how multipliers $M_\lambda(D)$ associated to frequency λ act on the spaces $L^2_t L^p_{loc}$.

Lemma 5.2 ([6], Lemma 6.2)**.** *Let $\lambda \geq 1$ and $1 \leq p \leq q \leq \infty$. Then*

$$\|M_\lambda(D)\|_{L^2_t L^p_{loc} \to L^2_t L^q_{loc}} \lesssim \lambda^{\frac{1}{p}-\frac{1}{q}}. \tag{5.3}$$

B. Low frequencies. Here we consider two dyadic low frequencies $1/h \leq \lambda_1, \lambda_2 \leq 1$, and seek to understand how multipliers $M_{\lambda_2}(D)$ associated to frequency λ_2 act on the spaces $L^2_t L^p_{loc}(B_{\lambda_1})$. For such multipliers we have:

Lemma 5.3 ([6], Lemma 6.3)**.** *Let $1/h \leq \lambda_1, \lambda_2 \leq 1$ and $1 \leq p \leq q \leq \infty$.*
a) Assume that $\lambda_1 \leq \lambda_2$. Then

$$\|M_{\lambda_2}(D)\|_{L^2_t L^p_{loc}(B_{\lambda_1}) \to L^2_t L^q_{loc}(B_{\lambda_1})} \lesssim \lambda_2^{\frac{1}{p}-\frac{1}{q}}. \tag{5.4}$$

b) Assume that $\lambda_2 \leq \lambda_1$. Then

$$\|M_{\lambda_2}(D)\|_{L^2_t L^p_{loc}(B_{\lambda_1}) \to L^2_t L^q_{loc}(B_{\lambda_2})} \lesssim \lambda_1^{\frac{1}{p}} \lambda_2^{-\frac{1}{q}}. \tag{5.5}$$

We remark that part (a) is nothing but the classical Bernstein's inequality in disguise, as the multiplier M_{λ_2} does not mix λ_1^{-1} intervals. Part (b) is the more interesting one, where the λ_1^{-1} intervals are mixed.

5.3. **Bounds for $\eta = \operatorname{Im} W$ and for $\eta_x = J^{-\frac{1}{2}} \operatorname{Im} W_\alpha$.** Here we have the straightforward equivalence

$$\|\eta\|_{LE^0} \approx \|\operatorname{Im} W\|_{LE^0}, \qquad \|\eta_x\|_{LE^0} \approx \|\operatorname{Im} W_\alpha\|_{LE^0}, \tag{5.6}$$

as η and $\operatorname{Im} W$ are one and the same function up to a bi-Lipschitz change of coordinates. We begin with a bound from [6] for the low frequencies of $\operatorname{Im} W$ on the top:

Lemma 5.4 ([6], Lemma 6.5)**.** *For each dyadic frequency $1/h \leq \lambda < 1$ we have*

$$\|\operatorname{Im} W_{\leq\lambda}\|_{L^2_t L^\infty_{loc}(B_\lambda)} \lesssim \|\operatorname{Im} W\|_{LE^0}. \tag{5.7}$$

Here one may also replace $\operatorname{Im} W$ by W_α,

$$\|W_{\alpha,\leq\lambda}\|_{L^2_t L^\infty_{loc}(B_\lambda)} \lesssim \|W_\alpha\|_{LE^0}. \tag{5.8}$$

Since

$$\|W_\alpha\|_{LE^0} \lesssim \|\operatorname{Im} W_\alpha\|_{LE^0} + h^{-1}\|\operatorname{Im} W\|_{LE^0},$$

we can also estimate the same expression in terms of $\operatorname{Im} W$,

$$\|W_{\alpha,\leq\lambda}\|_{L^2_t L^\infty_{loc}(B_\lambda)} \lesssim \lambda\|\operatorname{Im} W\|_{LE^0}. \tag{5.9}$$

On the other hand, for nonlinear estimates, we also need bounds in depth, precisely over the regions $A_\lambda(x_0)$. There, by [6], we have

Lemma 5.5 ([6], Lemma 6.6)**.** *For each dyadic frequency $\lambda < 1$ we have*

$$\|\operatorname{Im} W\|_{L^2_t L^\infty_{loc}(A_\lambda)} + \lambda^{-1}\|W_\alpha\|_{L^2_t L^\infty_{loc}(A_\lambda)} \lesssim \|\operatorname{Im} W\|_{LE^0}. \tag{5.10}$$

We will also need a mild high frequency bound:

Lemma 5.6. *The following estimate holds:*

$$\|\operatorname{Im} W\|_{L^2_t L^2_{loc}(A_1)} + \|\beta W_\alpha\|_{L^2_t L^\infty_{loc}(A_1)} \lesssim \|\operatorname{Im} W\|_{LE^0}. \tag{5.11}$$

Proof. For $\beta \in (-1, 0)$ we can define both $\operatorname{Im} W(\beta)$ and $\beta W_\alpha(\beta)$ in terms of $\operatorname{Im} W$ via zero order multipliers localized at frequency $\lesssim |\beta|^{-1}$, as follows:

$$\operatorname{Im} W(\beta) = P_D(D,\beta)\operatorname{Im} W, \qquad \beta \operatorname{Im} W(\beta) = i\beta D P_D(D,\beta)\operatorname{Im} W, \tag{5.12}$$

where we recall that

$$p_D(\xi,\beta) = \frac{\sinh((\beta+h)\xi)}{\sinh(h\xi)}$$

Both multipliers in (5.12) are bounded, with symbol type regularity, and rapidly decaying for $|\xi| \gg |\beta|^{-1}$. Hence a standard analysis shows that their respective kernels K_β are uniformly integrable and uniformly decaying outside the unit scale,

$$\int |K_\beta(\alpha)|\, d\alpha \lesssim 1, \qquad |K_\beta(\alpha)| \lesssim |\alpha|^{-2}$$

This easily implies the bounds (5.11).

□

5.4. **Estimates for $F(W_\alpha)$.** Here we consider a function F which is holomorphic in a neighbourhood of 0, with $F(0) = 0$ and prove local energy bounds for the auxiliary holomorphic function $F(W_\alpha)$.

Lemma 5.7 (Holomorphic Moser estimate, [6], Lemma 6.8)**.** *] Assume that $\|W_\alpha\|_{L^\infty} \ll 1$. Then*

a) For $\lambda > 1$ we have

$$\|F(W_\alpha)_\lambda\|_{L^2_t L^2_{loc}} \lesssim \lambda\|W\|_{L^2_t L^2_{loc}}. \tag{5.13}$$

b) For $\lambda \leq 1$ we have

$$\|F(W_\alpha)_\lambda\|_{L^2_t L^\infty_\alpha(B_\lambda(x_0))} \lesssim \lambda\|W\|_{L^2_t L^2_{loc}}. \tag{5.14}$$

In both cases, one should think of the implicit constant as depending on $\|W_\alpha\|_{L^\infty}$. We note that both estimates follow directly from Lemma 5.5 if $F(z) = z$. However to switch to an arbitrary F one would seem to need some Moser type inequalities, which unfortunately do not work in negative Sobolev spaces. The key observation is that in both of these estimates it is critical that W_α is holomorphic, and $F(W_\alpha)$ is an analytic function of W_α. This lemma was proved in [6] for the expression

$$Y = \frac{W_\alpha}{1+W_\alpha},$$

but the proof is identical in the more general case considered here.

6. The error estimates

The aim of this section is to use holomorphic coordinates in order to prove Lemmas 3.1, 3.4, 3.5, which for convenience we recall below.

Lemma 6.1. *The following fixed estimate holds:*

$$\left| \iint_{\Omega(t)} m_x(x-x_0)\theta q_x \, dydx \right| \lesssim \|\eta\|_{g^{-\frac{1}{4}} H_h^{\frac{1}{4}} \cap \kappa^{-\frac{1}{4}} H_h^{\frac{3}{4}}} \|\psi_x\|_{g^{\frac{1}{4}} H_h^{-\frac{1}{4}} + h^{\frac{1}{4}} H_h^{-\frac{3}{4}}}. \tag{6.1}$$

Lemma 6.2. *The integrals $I^\kappa_{1,bottom}$ and I^κ_2 are estimated by*

$$|I^\kappa_{1,bottom}| + |I^\kappa_2| \lesssim h^{-2}\|\eta\|^2_{LE^0}. \tag{6.2}$$

Lemma 6.3. *The expression I^κ_3 is estimated by*

$$|I^\kappa_3| \lesssim \epsilon\Big(\|\eta_x\|^2_{LE^0} + \frac{g}{\kappa}\|\eta\|^2_{LE^0}\Big). \tag{6.3}$$

We begin by expressing the quantities in the Lemma using holomorphic coordinates. We first recall that

$$\theta = \operatorname{Im} W, \qquad q_x = \operatorname{Im} R,$$

and, recalling the correspondence between Eulerian and holomorphic coordinates

$$x + iy = W + (\alpha + i\beta)$$

by the chain rule we have, within the strip S,

$$\theta_x = \operatorname{Im}\left(\frac{W_\alpha}{1+W_\alpha}\right), \qquad \theta_y = \operatorname{Re}\left(\frac{W_\alpha}{1+W_\alpha}\right). \tag{6.4}$$

A second use of chain rule yields

$$\theta_{xx} = \operatorname{Im}\left[\frac{1}{1+W_\alpha}\partial_\alpha\left(\frac{W_\alpha}{1+W_\alpha}\right)\right] = -\frac{1}{2}\partial_\alpha \operatorname{Im}\frac{1}{(1+W_\alpha)^2}. \tag{6.5}$$

Finally, $H(\eta)$ is expressed as

$$H(\eta) = -\frac{i}{1+W_\alpha}\partial_\alpha(J^{-\frac{1}{2}}(1+W_\alpha)).$$

We recall that the local energy norms are easily transferred, see Proposition 5.1

$$\|\eta\|_{LE^\kappa} \approx \|\operatorname{Im} W\|_{LE^\kappa}, \qquad \|\eta_x\|_{LE^\kappa} \approx \|\operatorname{Im} W_\alpha\|_{LE^\kappa}.$$

while, by Theorem 4.1, for the uniform bound we have

$$\|\operatorname{Im} W\|_{X^\kappa} \lesssim \|\eta\|_{X^\kappa}. \tag{6.6}$$

The last item we need to take into account in switching coordinates is that the image of the vertical strip in Euclidean coordinates is still a strip $S_{hol}(t)$ in the holomorphic setting with $O(1)$ horizontal size, but centered around $\alpha_0(t,\beta)$, where

$$|\alpha_0(t,\beta) - \alpha_0(t,0)| \lesssim \epsilon|\beta|.$$

This is from Proposition 4.6.

6.1. **Proof of Lemma 6.1.** We begin by rewriting our integral in holomorphic coordinates,

$$I_{t,x_0} = \iint_S Jm_x(x-x_0)\,\mathrm{Im}\,W\,\mathrm{Im}\,R\,d\alpha d\beta.$$

where we recall that in holomorphic coordinates $x = \alpha + \mathrm{Re}\,W$. In view of the norm equivalence in Lemma 4.5, for this integral we need to prove the bound

$$|I_{t,x_0}| \lesssim \|\mathrm{Im}\,W\|_{g^{-\frac14}H_h^{\frac14}\cap\kappa^{-\frac14}H_h^{\frac34}} \|\mathrm{Im}\,R\|_{g^{\frac14}H_h^{-\frac14}+\kappa^{\frac14}H_h^{-\frac34}} \tag{6.7}$$

where the norms on the right are taken on the top. Here we recall that m_x is a bounded, Lipschitz bump function with support in the strip $S_{hol}(t)$. This is all we will use concerning m_x. The strip $S_{hol}(t)$ is contained in the dyadic union

$$S_{hol}(t) \subset A_1(x_0) \bigcup \bigcup_{h^{-1}<\lambda<1} A_\lambda(x_0).$$

Correspondingly we split the integral as

$$I_{t,x_0} = I_1 + \sum_{h^{-1}<\lambda<1} I_\lambda.$$

For I_λ we directly estimate

$$|I_\lambda| \lesssim \lambda^{-1}\|\mathrm{Im}\,W\|_{L^\infty(A_\lambda(x_0))}\|\mathrm{Im}\,R\|_{L^\infty(A_\lambda(x_0))}.$$

For the pointwise bounds we recall that $\mathrm{Im}\,W(\beta) = P_N(\beta,D)\,\mathrm{Im}\,W$, and similarly for $\mathrm{Im}\,R$, where, for $\beta \approx \lambda^{-1}$, the multiplier $P_N(\beta,D)$ selects the frequencies $\leq \lambda$. Hence, harmlessly allowing rapidly decaying tails in our Littlewood-Paley truncations, we obtain using Bernstein's inequality

$$\begin{aligned}
\sum_{\lambda<1}|I_\lambda| &\lesssim \sum_{\lambda<1}\lambda^{-1}\|\mathrm{Im}\,W_{\leq\lambda}\|_{L^\infty}\|\mathrm{Im}\,R_{\leq\lambda}\|_{L^\infty} \\
&\lesssim \sum_{\lambda<1}\lambda^{-1}\sum_{\mu<\lambda}\mu^{\frac12}\|\mathrm{Im}\,W_\mu\|_{L^2}\sum_{\nu<\lambda}\nu^{\frac12}\|\mathrm{Im}\,R_\nu\|_{L^2} \\
&\lesssim \sum_{\mu,\nu<1}\min\left\{(\mu/\nu)^{\frac12},(\nu/\mu)^{\frac12}\right\}\|\mathrm{Im}\,W_\mu\|_{L^2}\|\mathrm{Im}\,R_\nu\|_{L^2} \\
&\lesssim \|\mathrm{Im}\,W_{<1}\|_{H_h^{\frac14}}\|\mathrm{Im}\,R_{\leq1}\|_{H_h^{-\frac14}} \\
&\lesssim \|\mathrm{Im}\,W\|_{g^{-\frac14}H_h^{\frac14}\cap\kappa^{-\frac14}H_h^{\frac34}}\|\mathrm{Im}\,R\|_{g^{\frac14}H_h^{-\frac14}+\kappa^{\frac14}H_h^{-\frac34}},
\end{aligned}$$

where the last step re-accounts for the rapidly decaying tails in the frequency localizations, which have been neglected through the above computation.

It remains to consider I_0, for which it suffices to estimate at fixed $\beta \in [0,1]$ (the norms for $\mathrm{Im}\,W$ and $\mathrm{Im}\,R$ at depth β are easily estimated by the similar norms on the top):

$$\left|\int_{\mathbb{R}} Jm_x\,\mathrm{Im}\,W\,\mathrm{Im}\,R\,d\alpha\right| \lesssim \|Jm_x\,\mathrm{Im}\,W\|_{g^{-\frac14}H_h^{\frac14}\cap\kappa^{-\frac14}H_h^{\frac34}}\|\mathrm{Im}\,R\|_{g^{\frac14}H_h^{-\frac14}+\kappa^{\frac14}H_h^{-\frac34}},$$

where for the first factor we further estimate as in the proof of (3.4),

$$\|Jm_x \operatorname{Im} W\|_{g^{-\frac{1}{4}}H_h^{\frac{1}{4}}\cap \kappa^{-\frac{1}{4}}H_h^{\frac{3}{4}}} \lesssim \|m_x\|_{W^{1,1}}\|J \operatorname{Im} W\|_{g^{-\frac{1}{4}}H_h^{\frac{1}{4}}\cap \kappa^{-\frac{1}{4}}H_h^{\frac{3}{4}}}$$
$$\lesssim (\|J\|_{L^\infty} + (\kappa/g)^{\frac{1}{4}}\|J_\alpha\|_{L^2})\|\operatorname{Im} W\|_{g^{-\frac{1}{4}}H_h^{\frac{1}{4}}\cap \kappa^{-\frac{1}{4}}H_h^{\frac{3}{4}}},$$

and for the L^2 norm of J_α we use our a-priori X^κ bound given by Theorem 4.1 to get (see also (4.11)).

$$(\kappa/g)^{\frac{1}{4}}\|J_\alpha\|_{L^2} \lesssim \epsilon.$$

6.2. **Proof of Lemma 6.2.** *a) The bound for $I^\kappa_{1,bottom}$.* By (6.4), on the bottom $\beta = -h$ we have $|\theta_y| \approx |\operatorname{Re} W_\alpha|$, so the integral $I^\kappa_{1,bottom}$ has size

$$I^\kappa_{1,bottom} \approx \int_{\beta=-h} m_x|\operatorname{Re} W_\alpha|^2 d\alpha \lesssim \|\operatorname{Re} W_\alpha\|^2_{L^2_t L^\infty_{\alpha,\beta}(A_{1/h}(x_0))}$$

Then the bound (3.12) for $I^\kappa_{1,bottom}$ follows directly from (5.5).

a) The bound for I^κ_2. Taking into account the above properties, we bound the integral I^κ_2 by

$$|I^\kappa_2| \lesssim \int_0^T \iint_{S_{hol}(t)} |\theta_y||H_N(\theta_{xx}) - \theta_{xx}|\, d\alpha d\beta dt.$$

As in [6], we split the integration region vertically into dyadic pieces, which are contained in the regions $A_1(x_0)$, respectively $A_\lambda(x_0)$ with $h^{-1} < \lambda < 1$ dyadic, and all of which are contained in $\mathbf{B}_{1/h}(x_0)$. We also take advantage of the fact that the second factor is smooth on the h scale and vanishes on the top in order to insert a β factor. Then we estimate

$$\begin{aligned}
|I^\kappa_2| \lesssim & \int_0^T \iint_{A_1(x_0)} |\beta\theta_y| d\alpha d\beta \sup_{A_1(x_0)} |\beta^{-1}(H_N(\theta_{xx}) - \theta_{xx})| + \\
& + \sum_{\lambda=1}^{h^{-1}} \lambda^{-1} \sup_{A_\lambda(x_0)} |\beta\theta_y| \sup_{A_\lambda(x_0)} |\beta^{-1}(H_N(\theta_{xx}) - \theta_{xx})|\, dt \\
\lesssim & \left(\|\beta\theta_y\|_{L^2_t L^\infty(A_1(x_0))} + \sum_{\lambda=1}^{h^{-1}} \lambda^{-1}\|\beta\theta_y\|_{L^2_t L^\infty(A_\lambda(x_0))}\right)\|\beta^{-1}(H_N(\theta_{xx}) - \theta_{xx})\|_{L^2_t L^\infty(\mathbf{B}^{1/h}(x_0))}.
\end{aligned}$$

Then it suffices to prove the following bounds:

$$\|\beta\theta_y\|_{L^2_t L^\infty(A_1(x_0))} \lesssim \|\operatorname{Im} W\|_{LE^0}, \tag{6.8}$$

$$\|\beta\theta_y\|_{L^2_t L^\infty(A_\lambda(x_0))} \lesssim \|\operatorname{Im} W\|_{LE^0}, \tag{6.9}$$

respectively

$$\|\beta^{-1}(H_N(\theta_{xx}) - \theta_{xx})\|_{L^2_t L^\infty(\mathbf{B}^{1/h}(x_0))} \lesssim h^{-3}\|\operatorname{Im} W\|_{LE^0}. \tag{6.10}$$

Given these three bounds, the conclusion of the Lemma easily follows. It remains to prove (6.8), (6.9), respectively (6.10).

Proof of (6.8), (6.9): In view of the representation (6.4) for θ_y, these bounds are direct consequences of Lemma 5.6, respectively Lemma 5.5.

Proof of (6.10)**:** Here we are subtracting the Dirichlet and Neumann extension of a given function. This we already had to do in [6], where the idea was that the only contributions come from very low frequencies $\leq h^{-1}$,

$$H_N(\theta_{xx}) - \theta_{xx} \approx P_{<1/h}\theta_{xx}.$$

If instead of θ_{xx} we had its principal part $\operatorname{Im} W_{\alpha\alpha}$ then the argument would be identical to [6], gaining two extra $1/h$ factors from the derivatives. The challenge here is to show that we can bound the very low frequencies of θ_{xx} in a similar fashion. But given the expression (6.5) for θ_{xx}, this is also a direct consequence of Lemma 5.7, applied to the function

$$F(W_\alpha) = \frac{1}{(1+W_\alpha)^2} - 1.$$

6.3. **Proof of Lemma 6.3.** As before we bound I_3^κ as

$$|I_3^\kappa| \lesssim \int_0^T \iint_{S_{hol}(t)} |\theta_y||H_N(H(\eta) - \theta_{xx})| \, d\alpha d\beta dt.$$

We write on the top

$$\begin{aligned} H(\eta) - \theta_{xx} = &\operatorname{Im}\left[\frac{1}{1+W_\alpha}\partial_\alpha(J^{-\frac{1}{2}}(1+W_\alpha)) - \frac{1}{1+W_\alpha}\partial_\alpha\left(\frac{W_\alpha}{1+W_\alpha}\right)\right] \\ = &\operatorname{Im}\left[W_{\alpha\alpha}\left(\frac{1}{(1+W_\alpha)^{\frac{3}{2}}(1+\bar{W}_\alpha)^{\frac{1}{2}}} - \frac{1}{(1+W_\alpha)^3}\right)\right], \end{aligned}$$

where the linear part cancels and we are left with a sum of expressions of the form

$$\operatorname{Im}[\partial_\alpha F_1(W_\alpha)G_1(\bar{W}_\alpha)], \qquad \operatorname{Im}[\partial_\alpha F_2(W_\alpha)],$$

where the subscript indicates the minimum degree of homogeneity. The first type of expression is the worst, as it allows all dyadic frequency interactions, whereas the second involves products of holomorphic functions which preclude high-high to low interactions. Since $F_1(W_\alpha)$ and $G_1(\bar{W}_\alpha)$ have the same regularity as W_α and $\bar{W}_\alpha$, to streamline the computation we simply replace them by that. Hence we end up having to bound trilinear expressions of the form

$$I = \int_0^T \iint_{S_{hol}(t)} |W_\alpha||H_N(W_{\alpha\alpha}\bar{W}_\alpha)| \, d\alpha d\beta dt. \tag{6.11}$$

Here we remark that the argument of H_N, discussed above, is only used on the top, whereas the $|W_\alpha|$ factor is used within the entire strip S in view of (6.4). We also note that the Neumann extension is identical in the Eulerian and the holomorphic setting.

To estimate this expression we use again a dyadic decomposition with respect to depth. We consider dyadic β regions $|\beta| \approx \lambda^{-1}$ associated to a frequency $\lambda \in (0, h^{-1}]$. But here we need to separate into three cases, depending on how λ compares to 1 and also to $\lambda_0 = \sqrt{g/\kappa} > 1$.

Case 1, $\lambda \geq \lambda_0 \gg 1$. The harmonic extension at depth λ is a multiplier which selects frequencies $\leq \lambda$, with exponentially decaying tails; this applies both to W_α and to the Neumann extension in (6.11). In addition the $S_{hol}(t)$ localization translates into a localization to A_1, therefore we need to estimate an integral of the form

$$I_\lambda = \int_0^T \iint_S 1_{A_1} 1_{|\beta|\approx\lambda^{-1}} |W_{\alpha,<\lambda}||P_{<\lambda}(W_{\alpha\alpha}\bar{W}_\alpha)|\, d\alpha d\beta dt, \tag{6.12}$$

where all the entries above are viewed as functions on the top. Then we use two local energies for the W_α factors and one apriori bound $W_{\alpha\alpha} \in \epsilon(g/\kappa)^{\frac{1}{4}} L^2$ arising from the X_1 norm (see (4.11)) plus Bernstein's inequality to bound this by

$$\begin{aligned} I_\lambda &\lesssim \lambda^{-1}\|W_\alpha\|_{LE^0}\|P_{<\lambda}(W_{\alpha\alpha}\bar{W}_\alpha)\|_{LE^0} \\ &\lesssim \lambda^{-1}\|W_\alpha\|_{LE^0}\lambda^{\frac{1}{2}}\|W_{\alpha\alpha}\|_{L^2}\|W_\alpha\|_{LE^0} \\ &\lesssim \epsilon\lambda^{-1}\lambda^{\frac{1}{2}}(g/\kappa)^{\frac{1}{4}}\|W_\alpha\|^2_{LE^0}, \end{aligned}$$

where the dyadic λ summation is trivial for $\lambda > \lambda_0$.

Case 2, $1 \leq \lambda < \lambda_0$**.** Here we still need to estimate an integral of the form (6.12) but we balance norms differently. Precisely, for the first W_α factor we use the local energy norm for $\operatorname{Im} W$ instead, as in (5.10), and otherwise follow the same steps as before. Then we bound the integral in (6.12) by

$$I_\lambda \lesssim \epsilon\lambda^{-1}\lambda\lambda^{\frac{1}{2}}(g/\kappa)^{\frac{1}{4}}\|\operatorname{Im} W\|_{LE^0}\|W_\alpha\|_{LE^0}.$$

Again the dyadic λ summation for $\lambda < \lambda_0$ is straightforward, and we conclude by the Cauchy-Schwarz inequality.

Case 3, $\lambda < 1$. Here the corresponding part of the integral (6.11) is localized in the intersection of the strip $S_h(x_0)$ with A_λ, and we estimate it using L^∞ norms in A_λ, by

$$\begin{aligned} I_\lambda &= \lambda^{-1}\int_0^T \sup_{A_\lambda(x_0)}|W_\alpha| \sup_{A_\lambda(x_0)}|H_N(W_{\alpha\alpha}\bar{W}_\alpha)|\, dt \\ &\lesssim \lambda^{-1}\|W_\alpha\|_{L^2_t L^\infty(A_\lambda)}\|H_N(W_{\alpha\alpha}\bar{W}_\alpha)\|_{L^2_t L^\infty_{loc}(A_\lambda)}. \end{aligned}$$

For the first factor we use the local energy of W, via (5.10),

$$\|W_\alpha\|_{L^2_t L^\infty_{loc}(A_\lambda)} \lesssim \lambda\|\operatorname{Im} W\|_{LE^0}.$$

For the bilinear factor we recall again that the harmonic extension at depth $\beta \approx \lambda^{-1}$ is a multiplier selecting frequencies $\leq \lambda$, see (4.2). Then we use the a-priori L^2 bound for $W_{\alpha\alpha}$ and the local energy of W_α, and apply the Bernstein inequality in Lemma 5.3:

$$\begin{aligned} \|P_{<\lambda}(W_{\alpha\alpha}\bar{W}_\alpha)\|_{L^2_t L^\infty(B_\lambda)} &\lesssim \lambda\|W_{\alpha\alpha}\bar{W}_\alpha\|_{L^2_t L^1_{loc}(B_\lambda)} \lesssim \lambda\|W_{\alpha\alpha}\|_{L^\infty_t L^2}\|\bar{W}_\alpha\|_{L^2_t L^2_{loc}(B_\lambda)} \\ &\lesssim \lambda\epsilon(g/\kappa)^{\frac{1}{4}}\lambda^{-\frac{1}{2}}\|W_\alpha\|_{LE^0}. \end{aligned}$$

Overall we obtain the same outcome as in Case 2.

References

[1] Siddhant Agrawal. Angled crested type water waves with surface tension: Wellposedness of the problem. arXiv:1909.09671, 2019.

[2] Albert Ai. Low regularity solutions for gravity water waves II: The 2D case. *arXiv e-prints*, page arXiv:1811.10504, Nov 2018.

[3] Albert Ai. Low Regularity Solutions for Gravity Water Waves. *Water Waves*, 1(1):145–215, May 2019.

[4] Thomas Alazard, Nicolas Burq, and Claude Zuily. On the water-wave equations with surface tension. *Duke Math. J.*, 158(3):413–499, 2011.

[5] Thomas Alazard and Jean-Marc Delort. Global solutions and asymptotic behavior for two dimensional gravity water waves. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(5):1149–1238, 2015.

[6] Thomas Alazard, Mihaela Ifrim, and Daniel Tataru. A Morawetz inequality for water waves. *arXiv e-prints*, page arXiv:1806.08443, Jun 2018.

[7] David M. Ambrose and Nader Masmoudi. The zero surface tension limit of two-dimensional water waves. *Comm. Pure Appl. Math.*, 58(10):1287–1315, 2005.

[8] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren der mathematischen Wissenschaften, 343, Springer (2011). Springer, Heidelberg, 2011.

[9] T. Brooke Benjamin and Peter J. Olver. Hamiltonian structure, symmetries and conservation laws for water waves. *J. Fluid Mech.*, 125:137–185, 1982.

[10] Klaus Beyer and Matthias Günther. On the Cauchy problem for a capillary drop. I. Irrotational motion. *Math. Methods Appl. Sci.*, 21(12):1149–1183, 1998.

[11] Angel Castro, Diego Córdoba, Charles Fefferman, Francisco Gancedo, and Javier Gómez-Serrano. Finite time singularities for water waves with surface tension. *J. Math. Phys.*, 53(11):115622, 26, 2012.

[12] Angel Castro, Diego Córdoba, Charles Fefferman, Francisco Gancedo, and Javier Gómez-Serrano. Finite time singularities for the free boundary incompressible Euler equations. *Ann. of Math. (2)*, 178(3):1061–1134, 2013.

[13] Robin Ming Chen, Jeremy L. Marzuola, Daniel Spirn, and J. Douglas Wright. On the regularity of the flow map for the gravity-capillary equations. *J. Funct. Anal.*, 264(3):752–782, 2013.

[14] Hans Christianson, Vera Mikyoung Hur, and Gigliola Staffilani. Strichartz estimates for the water-wave problem with surface tension. *Comm. Partial Differential Equations*, 35(12):2195–2252, 2010.

[15] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$. *Ann. of Math. (2)*, 167(3):767–865, 2008.

[16] Peter Constantin and Jean-Claude Saut. Local smoothing properties of dispersive equations. *J. Amer. Math. Soc.*, 1(2):413–439, 1988.

[17] Daniel Coutand and Steve Shkoller. Well-posedness of the free-surface incompressible Euler equations with or without surface tension. *J. Amer. Math. Soc.*, 20(3):829–930 (electronic), 2007.

[18] Daniel Coutand and Steve Shkoller. On the finite-time splash and splat singularities for the 3-D free-surface Euler equations. *Comm. Math. Phys.*, 325(1):143–183, 2014.

[19] Walter Craig, and David Nicholls. Travelling two and three dimensional capillary gravity water waves. *SIAM J. Math. Anal.* 32(2):323–359, 2000.

[20] W. Craig, C. Sulem and P-L. Sulem Nonlinear modulation of gravity waves: a rigorous approach, Nonlinearity 5, 1992 no.2, 497–522.

[21] Thibault de Poyferré. A priori estimates for water waves with emerging bottom. *Arch. Ration. Mech. Anal.*, 232(2):763–812, 2019.

[22] Thibault de Poyferré and Quang-Huy Nguyen. Strichartz estimates and local existence for the gravity-capillary waves with non-Lipschitz initial velocity. *J. Differential Equations*, 261(1):396–438, 2016.

[23] Thibault de Poyferré and Quang-Huy Nguyen. A paradifferential reduction for the gravity-capillary waves system at low regularity and applications. *Bull. Soc. Math. France*, 145(4):643–710, 2017.

[24] Yu Deng, Alexandru Ionescu, Benoît Pausader, and Fabio Pusateri. Global solutions of the gravity-capillary water-wave system in three dimensions. *Acta Math.* 219(2):213–402, 2017.

[25] Alexander I Dyachenko, Evgenii A Kuznetsov, MD Spector, and Vladimir E Zakharov. Analytical description of the free surface dynamics of an ideal fluid (canonical formalism and conformal mapping). *Physics Letters A*, 221(1-2):73–79, 1996.

[26] Charles Fefferman, Alexandru D. Ionescu, and Victor Lie. On the absence of splash singularities in the case of two-fluid interfaces. *Duke Math. J.*, 165(3):417–462, 2016.

[27] Pierre Germain, Nader Masmoudi, and Jalal Shatah. Global solutions for the gravity water waves equation in dimension 3. *Ann. of Math. (2)*, 175(2):691–754, 2012.

[28] Pierre Germain, Nader Masmoudi, and Jalal Shatah. Global existence for capillary water waves. *Comm. Pure Appl. Math.* 68(4):625–687, 2015.

[29] Yan Guo and Ian Tice. Decay of viscous surface waves without surface tension. *Anal. PDE*, 6(6):1429–1533, 2013.

[30] Benjamin Harrop-Griffiths, Mihaela Ifrim, and Daniel Tataru. The lifespan of small data solutions to the KP-I. *Int. Math. Res. Not. IMRN*, (1):1–28, 2017.

[31] Ifrim Mihaela Hunter, John and Daniel Tataru. Two dimensional water waves in holomorphic coordinates. *Comm. Math. Phys.*, 346(2):483–552, 2016. arXiv:1401.1252.

[32] Mihaela Ifrim and Daniel Tataru. Two dimensional water waves in holomorphic coordinates II: Global solutions. *Bull. Soc. Math. France*, 144(2):369–394, 2016.

[33] Mihaela Ifrim and Daniel Tataru. The lifespan of small data solutions in two dimensional capillary water waves. *Arch. Ration. Mech. Anal.*, 225(3):1279–1346, 2017.

[34] Mihaela Ifrim and Daniel Tataru. No solitary waves in 2-d gravity and capillary waves in deep water. *arXiv e-prints*, page arXiv:1808.07916, Aug 2018.

[35] Tatsuo Iguchi. A long wave approximation for capillary-gravity waves and an effect of the bottom. *Comm. Partial Differential Equations*, 32(1-3):37–85, 2007.

[36] Alexandru D. Ionescu and Fabio Pusateri. Global solutions for the gravity water waves system in 2d. *Invent. Math.*, 199(3):653–804, 2015.

[37] Alexandru D. Ionescu and Fabio Pusateri. Global regularity for 2D water waves with surface tension. *Mem. Amer. Math. Soc.*, 256(1227):v+124, 2018.

[38] Carlos E. Kenig, Gustavo Ponce, and Luis Vega. The Cauchy problem for quasi-linear Schrödinger equations. *Invent. Math.*, 158(2):343–388, 2004.

[39] David Lannes. A Stability Criterion for Two-Fluid Interfaces and Applications. *Arch. Ration. Mech. Anal.*, 208(2):481–567, 2013.

[40] David Lannes. *Water waves: mathematical analysis and asymptotics*, volume 188 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2013.

[41] John E. Littlewood, and Raymond E. A. C. Paley. Theorems on Fourier Series and Power Series. *J. London Math. Soc.* 6(3):230–233, 1931.

[42] Jeremy Marzuola, Jason Metcalfe, and Daniel Tataru. Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations. *J. Funct. Anal.*, 255(6):1497–1553, 2008.

[43] Jason Metcalfe and Daniel Tataru. Global parametrices and dispersive estimates for variable coefficient wave equations. *Math. Ann.*, 353(4):1183–1237, 2012.

[44] Guy Métivier. Para-differential calculus and applications to the Cauchy problem for nonlinear systems, volume 5 of Centro di Ricerca Matematica Ennio De Giorgi (CRM) Series. Edizioni della Normale, Pisa, 2008.

[45] Mei Ming and Chao Wang. Water waves problem with surface tension in a corner domain ii: the local well-posednes. arXiv:1812.09911, 2018.

[46] Cathleen S. Morawetz. Time decay for the nonlinear Klein-Gordon equations. *Proc. Roy. Soc. Ser. A*, 306:291–296, 1968.

[47] V. I. Nalimov. The Cauchy-Poisson problem. *Dinamika Splošn. Sredy*, (Vyp. 18 Dinamika Zidkost. so Svobod. Granicami):104–210, 254, 1974.

[48] Huy Quang Nguyen. A sharp Cauchy theory for the 2D gravity-capillary waves. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 34(7):1793–1836, 2017.

[49] Tohru Ozawa and Keith M. Rogers. A sharp bilinear estimate for the Klein-Gordon equation in $\mathbb{R}^{1+1}$. *Int. Math. Res. Not. IMRN*, (5):1367–1378, 2014.

[50] Fabrice Planchon and Luis Vega. Bilinear virial identities and applications. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(2):261–290, 2009.

[51] Frédéric Rousset and Nikolay Tzvetkov. Transverse instability of the line solitary water-waves. *Invent. Math.*, 184(2):257–388, 2011. Prépublication 2009.

[52] Guido Schneider and C. Eugene Wayne. The rigorous approximation of long-wavelength capillary-gravity waves. *Arch. Ration. Mech. Anal.*, 162(3):247–285, 2002.

[53] Ben Schweizer. On the three-dimensional Euler equations with a free boundary subject to surface tension. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 22(6):753–781, 2005.

[54] Jalal Shatah and Chongchun Zeng. A priori estimates for fluid interface problems. *Comm. Pure Appl. Math.*, 61(6):848–876, 2008.

[55] Luis Vega. Schrödinger equations: pointwise convergence to the initial data. *Proc. Amer. Math. Soc.*, 102(4):874–878, 1988.

[56] Xuecheng Wang. Global regularity for the 3d finite depth capillary water waves. arXiv:1611.05472, 2016.

[57] Sijue Wu. Well-posedness in Sobolev spaces of the full water wave problem in 2-D. *Invent. Math.*, 130(1):39–72, 1997.

[58] Sijue Wu. Well-posedness in Sobolev spaces of the full water wave problem in 3-D. *J. Amer. Math. Soc.*, 12(2):445–495, 1999.

[59] Sijue Wu. Almost global wellposedness of the 2-D full water wave problem. *Invent. Math.*, 177(1):45–135, 2009.

[60] Sijue Wu. Global wellposedness of the 3-D full water wave problem. *Invent. Math.*, 184(1):125–220, 2011.

[61] Hideaki Yosihara. Gravity waves on the free surface of an incompressible perfect fluid of finite depth. *Publ. Res. Inst. Math. Sci.*, 18(1):49–96, 1982.

[62] V. E. Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Journal of Applied Mechanics and Technical Physics*, 9(2):190–194, 1968.

[63] Vladimir E. Zakharov. Weakly nonlinear waves on the surface of an ideal finite depth fluid. In *Nonlinear waves and weak turbulence*, volume 182 of *Amer. Math. Soc. Transl. Ser. 2*, pages 167–197. Amer. Math. Soc., Providence, RI, 1998.

[64] Hui Zhi. Control of three dimensional water waves. arXiv:1712.06130, 2017.

[65] Hui Zhu. Propagation of singularities for gravity-capillary water waves. arXiv:1810.09339, 2018.

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