LOCAL WELL-POSEDNESS IN SOBOLEV SPACES FOR FIRST-ORDER BAROTROPIC CAUSAL RELATIVISTIC VISCOUS HYDRODYNAMICS

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Abstract. We study the theory of relativistic viscous hydrodynamics introduced in [15,58], which provided a causal and stable first-order theory of relativistic fluids with viscosity in the case of barotropic fluids. The local well-posedness of its equations of motion has been previously established in Gevrey spaces. Here, we improve this result by proving local well-posedness in Sobolev spaces.

Keywords: relativistic viscous fluids; causality; local well-posedness.

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1. Introduction

Relativistic fluid dynamics is widely used in many branches of physics, including high-energy nuclear physics [11], astrophysics [84], and cosmology [97]. Its power stems from conservation laws, such as the local conservation of energy and momentum, which allow one to investigate the macroscopic dynamics of conserved quantities without knowing the fate of the system's microscopic degrees of freedom. In other words, although the complete behavior of physical systems is ultimately determined by the dynamics of its microscopic constituents, one can bypass the usually intractable problem of solving the full microscopic dynamics and work instead within the scope of the the so-called *fluid approximation.* The latter is understood as a regime determined by energy scales where the system's microscopic constituents behave collectively as a continuum, which is then identified as a fluid [24]. While there remain questions about the details of how to fully derive relativistic fluid dynamics from an underlying microscopic theory [11, 17, 25, 27, 50, 85], and rigorous mathematical results in this direction are few [37, 90], the overwhelming success of the relativistic fluid dynamics more than justifies the importance of studying its mathematical properties. Furthermore, from a purely mathematical point of view, relativistic fluid dynamics has also been a fertile source of mathematical problems (see, e.g., [10, 20–22, 34, 84] and references therein).

The first works on relativistic fluids go back to the early days of relativity theory with the works of Einstein [36] and Schwarzchild [87]. The first general[1] mathematical treatment of relativistic fluids was done by Choquet-Bruhat [39] and Lichnerowicz [65]. Such works, as well as most of the studies in relativistic fluid dynamics since then, focused on perfect fluids, i.e., fluids where viscosity and heat dissipation are absent[2]. The equations describing relativistic perfect fluids are the well-known relativistic Euler equations.

There are, however, important situations in physics where the relativistic Euler equations are not appropriate, and a model of relativistic fluids *with viscosity* is needed. One such situation is in the study of the quark-gluon plasma, which is an exotic type of fluid forming in collisions of heavy ions performed at the Large Hadron Collider (LHC) at CERN and at the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory. Its discovery was named by the American Physical Society one of the 10 most important findings in physics in the last decade [1] and continues to be a source of scientific breakthroughs [6, 60]. For the quark-gluon plasma, it is well-attested that theoretical predictions do not match experimental data if viscosity is not taken into account [49, 85]. Another case, where viscosity is likely to play an important role, is in the study of gravitational waves produced by neutron star mergers, which have been detected by the Laser Interferometer Gravitational-Wave Observatory (LIGO) [2–5, 8]. Recent state-of-the-art numerical simulations [7, 88, 89] convincingly show that the post-merger gravitational wave signal is likely to be affected by viscous effects. Thus, one has two of the most cutting-edge experimental apparatuses in modern science (LHC and LIGO) producing data that *requires relativistic fluids with*

[1] By "general" we mean outside symmetry classes or beyond one spatial dimension. With symmetry or in $1+1$ dimensions, the equations of relativistic fluid dynamics studied by Choquet-Bruhat and Lichnerowicz reduce to equations for which earlier techniques could have been applied, although it seems difficult to locate in the literature specific applications of such known techniques to the equations of relativistic fluids under symmetry assumptions or in one spatial dimension.

[2] The literature on this topic is quite large and an appropriate review is beyond the scope of this work. See the literature cited in the first paragraph of this introduction and references therein for further information.

viscosity for its explanation[3]. But despite the importance of relativistic viscous fluids, many essential questions remain unanswered and very little is known about their mathematical properties.

Unlike the case of perfect fluids, it remains open what the best model for the description of relativistic viscous fluids is[4]. This is because it is challenging to construct theories of relativistic viscous fluids that are (i) causal, (ii) stable, and (iii) locally well-posed [84]. Causality is a fundamental property of relativity stating that no information propagates faster than the speed of light. Stability here means linear stability about constant equilibrium states, i.e., mode stability, which on physical grounds is expected to hold when viscous dissipation is present. Local well-posedness assures that the equations of motion admit a unique solution, a crucial property for physical models[5]. One requires (i) and (iii) to hold both in a fixed background and when the fluid equations are coupled to Einstein's equations, whereas (ii) is usually required only in Minkowski background (thus, all references to stability in what follows refer to the equations in Minkowski space).

The first theory of relativistic viscous fluids was introduced by Eckart [35], followed by a similar theory by Landau and Lifshitz [61]. While these theories can be viewed as the simplest generalization of the classical Navier-Stokes equations to the relativistic setting, they have been showed to be acausal and unstable [52, 81]. The Müeller-Israel-Stweart theory originally introduced in the references [54, 55, 72] is an attempt to overcome the acausality and instability of the Eckart and Landau-Lifshitz theories and is based on extended irreversible thermodynamics [57, 73]. In this formalism, viscous and dissipative contributions to the fluid's energy-momentum tensor are not given in terms of standard *hydrodynamic variables,* which are the fluid's velocity, energy density, baryon density, and quantities derived from these[6]. Rather, in extended thermodynamic theories, viscous and dissipative contributions are modeled by new variables, commonly referred to as *extended variables*, which satisfy further equations of motion. In the original works of Müller, Israel, and Stewart, such equations of motion were chosen in order to enforce the second-law of thermodynamics. In modern versions of the Müller-Israel-Stewart theory[7], the equations of motion are derived from microscopic theory or are based on effective theory arguments (see below for further remarks on the derivation of fluid equations from microscopit theory) [11, 27, 86]. Theories where viscous or dissipative effects are modeled solely by the hydrodynamics variables are

[3]Although it is not claimed that *all* the data generated in these experiments can only be explained with viscosity.

[4]It is interesting to notice that the generalization of the classical Navier-Stokes to a general Riemannian manifold is also somewhat problematic, and there are different possible choices for the equations, see [18].

[5]Local well-posedness is also important in the study of convergence of numerical schemes. The relation between local well-posedness and convergence is subtle and a discussion of this topic is outside the scope of this work. Interested readers can consult [47] and references therein. This is an important topic since many studies of realistic physical systems do rely on numerical computations.

[6]Given an equation of state, whose form depends on the nature of the fluid, all thermodynamic scalars (such as energy density, entropy, temperature, pressure, etc.) are related via the laws of thermodynamics and only two of them are independent. Absent phase transitions, all such relations are invertible and the choice of which two thermodynamic scalars are independent is a matter of convenience.

[7]Strictly speaking, these modern derivations do not exactly reproduce the original Müller-Israel-Stewart equations, but are close enough so that it has become common practice to still call them Müller-Israel-Stewart, although sometimes they are also referred by another names (re-summed [11] BRSSS or DNMR [27, 86]). All such theories are based on extended variables and behave very similarly when it comes to issues of stability and causality, so that it does not seem important to distinguish them here.

known as *first-order theories,* whereas those where such effects are modeled by extended variables are known as *second-order theories* [84].

The Müller-Israel-Stewart theory has been proved to be stable and its linearization about constant equilibrium states is causal [51, 77]. Furthermore, it has been extensively applied to the construction of successful phenomenological models of the quark-gluon plasma [49, 85]. Consequently, the Müller-Israel-Stewart theory is currently the most used theory for the description of relativistic viscous fluids. More recently, it has been proved that the Müller-Israel-Stewart theory with bulk viscosity[8] (but with no shear viscosity nor heat conduction) is locally well-posed and causal in the full nonlinear[9] regime, both in a fixed background and when the equations are coupled to Einstein's equations [14] (see [26, 38, 82] for earlier causality results also valid in the nonlinear regime but under strong symmetry assumptions or in $1 + 1$ dimensions). A similar causality and well-posedness result is valid in Gevrey spaces when shear viscosity is present [12].

Its great success nonetheless, it is far from clear whether the Müller-Israel-Stewart theory provides the most accurate description of relativistic viscous fluids over all scales where the fluid approximation is supposed to hold and viscous effects expected to be relevant. For instance, it is not known whether the Müller-Israel-Stewart equations can be applied to the study of neutron star mergers [7, 71]. Moreover, the mathematical foundations of the Müller-Israel-Stewart are for the most part lacking, with the aforementioned results [12, 14] being the only ones available in the literature. Finally, the Müller-Israel-Stewart equations do not seem capable to describe the dynamics of shock waves or more general types of fluid singularities [30, 44, 78]. In view of these limitations, there is a strong interest in searching for alternative theories of relativistic viscous fluids [58].

The instability results that ruled out the Eckart and Landau theories are in fact applicable to a large class of first-order theories [52]. Consequently, for a long time it was thought that first-order theories were intrinsically unstable (see discussions in [58, 84, 92–96]). Nevertheless, in recent years this perception has been shown to be overstated, with several different results showing the viability of first-order theories. In [28], causality and local well-posedness (in Gevrey spaces) of the Lichnerowicz theory has been established in the case of irrotational fluids with or without coupling to Einstein's equations, a result that has been slightly improved in [23]. The Lichnerowicz theory is a first-order theory introduced in [64] and which has led to interesting applications in cosmology [32, 33]. However, it remains open whether Lichnerowicz's theory is stable. In [13] a first-order theory of relativistic viscous *conformal* fluids has been introduced based on kinetic theory. Its stability, causality, and local well-posedness (in Gevrey spaces when the equations are coupled to Einstein's equations and in Sobolev spaces when the equations are considered in Minkowski background) has been proven in the works [13, 16, 29] , and applications relevant to the study of the quark-gluon plasma have also been developed [13]. In [41–43] (see also [40]) a first-order theory has been introduced for which stability holds in the fluid's rest frame. This leads to the possibility that

[8] As in the case of classical fluids, there are generally two types of viscosity in relativistic fluids, namely, shear viscosity and bulk viscosity. Also as in classical fluids, heat conduction is present in relativistic theories of non-perfect fluids. Since these are all phenomena associated with out-of-equilibrium physics, for simplicity we will henceforth refer to all of them simply as "viscous," making no distinction between viscosity and heat conduction effects.

[9] Talking about causality in the "nonlinear regime" is redundant in that the equations of motion are nonlinear. However, this language is sometimes used in the literature to make a contrast with earlier and more common causality results that apply only to the linearization of the equations about constant solutions.

such theory might be stable and causal, although it is known that stability in the rest frame is not enough to ensure stability in general [52]. Earlier first-order theories for which stability has also been established can be found in [95, 96]. Aside from all these results concerning first-order theories, further causality, stability, and local-wellposedness results have been established in the context of the so-called divergence-type theories [44, 45, 62, 66, 73, 79, 80], which constitute examples of second-order theories different than the Müller-Israel-Stewart theory (see also [59, 74, 83]). We also mention the so-called anisotropic hydrodynamics [9], which is a stable second-order theory that has been very successful in studies of the quark-gluon plasma, although to the best of our knowledge there has been no results showing causality or local well-posedenss for anisotropic hydrodynamics.

The previous discussion highlights not only the importance of investigating relativistic fluids with viscosity but also how its study is a very active field of research, with some of the most basic questions, namely, causality, stability, and local well-posedness, remaining largely open. This paper is concerned with the well-posedness of the Cauchy problem for the first-order theory of relativistic fluids defined by the energy-momentum tensor (2.1) below.

This energy-momentum tensor was introduced simultaneously in [15] and [58] using effective field theory arguments. In [15] a kinetic theory derivation (at zero chemical potential) was also given, while [53] discussed the necessary modifications that stem from the inclusion of a conserved current. Under the assumption of a barotropic equation of state (i.e., when the pressure is a function of the energy density only), the stability of the corresponding equations of motion has been established in these works, whereas in [15] causality and local well-posedness of the equations of motion has also been proven. Such local well-posedness has been established in Gevrey spaces with and without coupling to Einstein's equations[10]. Our goal in this manuscript is to improve this result by proving local well-posedness in Sobolev spaces. However, contrary to [15], here we do not consider coupling to Einstein's equations, restricting ourselves to the case where the evolution takes place in Minkowski space.

We finish this introductions with two explanatory remarks. First, the question of the correct theory of relativistic viscous fluids cannot be decided solely by considerations from microscopic theory. This is because the same underlying microscopic theory can give rise to different, inequivalent, fluid approximations depending on the chosen coarse-graining procedure [24, 27]. Second, above we referred to numerical simulations that show the importance of viscous effects in neutron star mergers [7]. We remark that these simulations do not numerically solve models relativistic fluids with viscosity, relying rather on estimates for the relevant transport scales and the size of gradients of the hydrodynamic fields determined in an inviscid evolution. Indeed, as hinted above, it is not yet known which, if any, of the current models of relativistic viscous fluids is appropriate to describe neutron star mergers.

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2. Equations of motion and statement of the results

The energy-momentum tensor that defines the first-order theory of relativistic viscous fluids studied here (introduced in [15, 58], see discussion in the introduction) is given by

$$\mathcal{T}_{\alpha\beta} = (\varepsilon + A_1)u_\alpha u_\beta + (P + A_2)\Pi_{\alpha\beta} - 2\eta\sigma_{\alpha\beta} + u_\alpha Q_\beta + u_\beta Q_\alpha, \tag{2.1}$$

[10] In fact, a slightly weaker statement has been proved in [15], but this does not change the overall theme discussed here nor the goal of this manuscript. See Remark 7.1.

where

$$A_1 = \chi_1 \frac{u^\alpha \nabla_\alpha \varepsilon}{\varepsilon + P} + \chi_2 \nabla_\alpha u^\alpha,$$
$$A_2 = \chi_3 \frac{u^\alpha \nabla_\alpha \varepsilon}{\varepsilon + P} + \chi_4 \nabla_\alpha u^\alpha,$$
$$Q_\alpha = \lambda\left(\frac{c_s^2}{\varepsilon + P} \Pi_\alpha^\mu \nabla_\mu \varepsilon + u^\mu \nabla_\mu u_\alpha\right),$$
$$\sigma_{\alpha\beta} = \frac{1}{2}\left(\Pi_\alpha^\mu \nabla_\mu u_\beta + \Pi_\beta^\mu \nabla_\mu u_\alpha - \frac{2}{3}\Pi_{\alpha\beta}\nabla_\mu u^\mu\right).$$

Here, ε is the fluid's energy density; P is the fluid's pressure, where we assume a barotropic equation of state, thus $P = P(\varepsilon)$; g is the spacetime metric[11]; u is the fluid's four-velocity, which is future-pointing and unit timelike with respect to g, so in particular u satisfies the constraint

$$g_{\alpha\beta} u^\alpha u^\beta = -1. \tag{2.2}$$

Notice that we are assuming the spacetime to be time-oriented as u is taken as a future-pointing vectorfield. In practice, we will work in Minkowski space with standard orientation; Π is the projection onto the space orthogonal to u, given by

$$\Pi_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta;$$

η, $\chi_1, \chi_2, \chi_3, \chi_4$, and λ are transport coefficients, which are known functions of ε and model the viscous effects in the fluid; and ∇ is the covariant derivative associated with the metric g. Indices are raised and lowered using the spacetime metric. We adopt the convention that lowercase Greek indices vary from 0 to 3, Latin indices vary from 1 to 3, and repeated indices are summed over their range. Expressions such as z_α, $w_{\alpha\beta}$, etc. represent the components of a vector or tensor with respect to a system of coordinates $\{x^\alpha\}_{\alpha=0}^3$ in spacetime, where the coordinates are always chosen so that $x^0 = t$ represents a time coordinate. We will consider the fluid dynamics in Minkowski background, so that the g is the Minkowski metric. We note for future reference that equation (2.2) implies

$$u^\alpha \nabla_\beta u_\alpha = 0. \tag{2.3}$$

The equations of motion are given by

$$\nabla_\alpha \mathcal{T}_\beta^\alpha = 0 \tag{2.4}$$

supplemented by the constraint (2.2).

We are now ready to state our main result, which is the following.

Theorem 2.1. *Let g be the Minkowski metric on $\mathbb{R} \times \mathbb{T}^3$, where $\mathbb{T}^3$ is the three-dimensional torus. Let $P, \eta, \chi_1, \chi_2, \chi_3, \chi_4, \lambda : (0, \infty) \to (0, \infty)$ be analytic functions satisfying $\lambda, \chi_1, \eta > 0$, $P > 0$, $c_s^2 := P' > 0$, and*

$$9\lambda^2\chi_2^2 c_s^4 + 6\lambda c_s^2 \left(\chi_1 (4\eta - 3\chi_4)(2\lambda + \chi_2) + 3\chi_2\chi_3 (\lambda + \chi_2)\right)$$
$$+ \left(\chi_1 (4\eta - 3\chi_4) + 3\chi_3 (\lambda + \chi_2)\right)^2 > 0,$$
$$\lambda \geq \eta,$$
$$3\chi_4 > 4\eta,$$

[11] By "metric" we always mean a "Lorentzian metric."

$$2\lambda\chi_1 \geq \lambda\chi_2 c_s^2 - \chi_1\left(\chi_4 - \frac{4\eta}{3}\right) + \lambda\chi_3 + \chi_3\chi_2,$$

$$\lambda\chi_1 + c_s^2\lambda\left(\chi_4 - \frac{4\eta}{3}\right) \geq c_s^2\lambda\chi_2 + \lambda\chi_3 + \chi_2\chi_3 - \chi_1\left(\chi_4 - \frac{4}{3}\eta\right) \geq 0.$$

Let $\varepsilon_{(0)} \in H^r(\mathbb{T}^3, \mathbb{R})$, $\varepsilon_{(1)} \in H^{r-1}(\mathbb{T}^3, \mathbb{R})$, $u_{(0)} \in H^r(\mathbb{T}^3, \mathbb{R}^3)$, and $u_{(1)} \in H^{r-1}(\mathbb{T}^3, \mathbb{R}^3)$ be given, where H^r is the Sobolev space and $r > 9/2$. Assume that $\varepsilon_{(0)} \geq C_0 > 0$ for some constant C_0.

Then, there exists a $T > 0$, a function

$$\varepsilon \in C^0([0,T], H^r(\mathbb{T}^3, \mathbb{R})) \cap C^1([0,T], H^{r-1}(\mathbb{T}^3, \mathbb{R})) \cap C^2([0,T], H^{r-2}(\mathbb{T}^3, \mathbb{R})),$$

and a vector field

$$u \in C^0([0,T], H^r(\mathbb{T}^3, \mathbb{R}^4)) \cap C^1([0,T], H^{r-1}(\mathbb{T}^3, \mathbb{R}^4)) \cap C^2([0,T], H^{r-2}(\mathbb{T}^3, \mathbb{R}^4))$$

such that equations (2.2) and (2.4) hold on $[0,T] \times \mathbb{T}^3$, and satisfy $\varepsilon(0,\cdot) = \varepsilon_{(0)}$, $\partial_t\varepsilon(0,\cdot) = \varepsilon_{(1)}$, $\mathcal{P}u(0,\cdot) = u_{(0)}$, and $\mathcal{P}\partial_t u(0,\cdot) = u_{(1)}$, where ∂_t is the derivative with respect to the first coordinate in $[0,T] \times \mathbb{T}^3$ and $\mathcal{P}$ is the canonical projection from the tangent bundle of $[0,T] \times \mathbb{T}^3$ onto the tangent bundle of $\mathbb{T}^3$. Moreover, (ε, u) is the unique solution with the stated properties.

We proceed to make some comments about the assumptions and conclusions of Theorem 2.1.

We note that in view of (2.2), it suffices to provide the components of u tangent to $\{t = 0\}$ as initial data; this explains the statement involving the projector $\mathcal{P}$ in Theorem 2.1. On the other hand, throughout the manuscript, we will consider systems of equations for the full four-velocity $u = (u^0, u^1, u^2, u^3)$. In these cases, we will always take the initial condition for u defined by (2.2) and (2.3) when (u^1, u^2, u^3) takes the values of the given initial data.

The quantity c_s^2 corresponds to the fluid's sound speed in the case of a perfect fluid. In the presence of viscosity, the fluid's sound speed is no longer given by c_s^2 (see section 7.2 for a description of the characteristic speeds of the system), but it is still convenient to introduce c_s^2. We work on $\mathbb{T}^3$ for simplicity, since using the domain of dependence property (proved in [15]) one can adapt the proof to $\mathbb{R}^3$. On the other hand, the assumption $\varepsilon_0 \geq C_0 > 0$ is essential. The equations can otherwise degenerate, resulting in a free-boundary dynamics, a problem that only quite recently was solved for the case of a perfect fluid [31, 70, 76] (see [46, 48, 56, 75, 91] for earlier work focusing on particular cases or a priori estimates).

The assumptions on $P, \eta, \chi_1, \chi_2, \chi_3, \chi_4$ and λ in Theorem 2.1 are precisely the conditions found in [15] that ensure the causality and stability of the equations of motion. Although these conditions are a bit cumbersome to write, it is not difficult to see that they are not empty. Moreover, given a specific choice of equation of state and transport coefficients, it is generally not difficult to verify whether such conditions are satisfied.

The idea behind the proof of Theorem 2.1 can be summarized as follows. First, we use (2.2) and (2.3) to decompose (2.4) in the directions parallel and orthogonal to u, as it is customary in both the cases of perfect and non-perfect relativistic fluids. Next, we construct new variables out of certain combinations of u, ε, and its derivatives, and rewrite the equations of motion in terms of these new variables. We then show that, under the hypotheses of Theorem 2.1, the principal symbol of the new system of equations can be diagonalized. This diagonalization procedure can be carried over to the equations of motion upon the introduction of suitable pseudodifferential operators. The pseudifferential calculus is needed

because the diagonalization of the principal symbol involves certain rational functions of the eigenvalues and of the determinant of the principal symbol. Due to the quasilinear nature of the problem, we have to deal with symbols of limited smoothness. Nevertheless, we are still able to obtain good energy estimates for a linearized version of the problem that can be used to set up a convergent iteration scheme, leading to existence and uniqueness of solutions to the new equations of motion we introduced. At this point, we need to show that this result gives rise to existence and uniqueness of solutions to the original equations of motion, i.e., (2.2) and (2.4). To do so, we need to derive yet another system of equations that ensures that the constraint (2.4) is satisfied. For this new system, solutions are obtained in a more restrictive class of functions, namely, Gevrey functions. Since these are dense in Sobolev spaces, we finally obtain existence and uniqueness for the original problem, in Sobolev spaces, by an approximation argument.

Before providing a proof of the main result, it is instructive to compare the structure of equations (2.4) with the energy-momentum tensor given by (2.1) with that of the equations obtained in the case of a perfect fluid, i.e., with the relativistic Euler equations. For this, we rewrite (2.1) as

$$\begin{aligned}\mathcal{T}_{\alpha\beta} = {}& \varepsilon u_\alpha u_\beta + P\Pi_{\alpha\beta} - \eta(\Pi^\mu_\alpha \nabla_\mu u_\beta + \Pi^\mu_\beta \nabla_\mu u_\alpha - \frac{2}{3}\Pi_{\alpha\beta}\nabla_\mu u^\mu) \\ & + \lambda u_\alpha(\frac{c_s^2}{\varepsilon + P}\Pi^\mu_\beta \nabla_\mu \varepsilon + u^\mu \nabla_\mu u_\beta) + \lambda u_\beta(\frac{c_s^2}{\varepsilon + P}\Pi^\mu_\alpha \nabla_\mu \varepsilon + u^\mu \nabla_\mu u_\alpha) \\ & + \chi_1 \frac{u^\mu \nabla_\mu \varepsilon}{\varepsilon + P} u_\alpha u_\beta + \chi_2 \nabla_\mu u^\mu u_\alpha u_\beta + \chi_3 \frac{u^\mu \nabla_\mu \varepsilon}{\varepsilon + P}\Pi_{\alpha\beta} + \chi_4 \nabla_\mu u^\mu \Pi_{\alpha\beta}.\end{aligned}$$

The first two terms on the RHS correspond to a perfect fluid, whereas all remaining terms correspond to viscous contributions, i.e., the energy-momentum tensor for a relativistic perfect fluid is obtained upon setting $\eta = \lambda = \chi_1 = \chi_2 = \chi_3 = \chi_4 = 0$ in the above expression. This highlights two important facts. First, the viscous dynamics is significantly more complext than the non-viscous one. Second, and more importantly, the terms multiplied by η, λ, χ_i, $i = 1, \dots, 3$, involve first derivatives of u and ϵ, whereas the first two terms on the RHS, which correspond to a perfect fluid, involve no derivatives of these variables. Therefore, the equation of motion obtained from (2.4) are of second-order when viscous effects are present but of first-order when they are absent. This means that the introduction of viscosity not only increases the complexity of the equations of motion as compared to those of a perfect fluid, but it also changes the order of the system. In particular, the principal part of the viscous system vanishes identically upon setting $\eta = \lambda = \chi_1 = \chi_2 = \chi_3 = \chi_4 = 0$. Consequently, one generally cannot obtain a solution to the perfect fluid equations by carrying out a proof in the viscous case and then setting $\eta = \lambda = \chi_1 = \chi_2 = \chi_3 = \chi_4 = 0$, as the time of existence for the viscous solutions will typically depend on these coefficients.

3. A new system of equations

In this section we derive a new system of equations that will allow us to establish Theorem 2.1. In order to do so, throughout this section, we assume to be given a sufficiently regular solution to equations (2.2) and (2.4).

We begin using (2.2) to decompose $\nabla_\alpha \mathcal{T}^\alpha_\beta$ in the directions orthogonal and parallel to u, so that equation (2.4) gives

$$u^\alpha \nabla_\alpha A_1 + \nabla_\alpha Q^\alpha + (\varepsilon + P + A_1 + A_2)\nabla_\alpha u^\alpha + u^\alpha \nabla_\alpha \varepsilon - 2\eta \sigma^{\alpha\beta}\sigma_{\alpha\beta}$$

$$+ Q_\alpha u^\beta \nabla_\beta u^\alpha = 0, \tag{3.1a}$$

$$\Pi^{\alpha\beta}\nabla_\beta A_2 + u^\beta \nabla_\beta Q^\alpha - 2\mathsf{\eta}\nabla_\beta \sigma^{\alpha\beta} + (\varepsilon + P + A_1 + A_2)u^\beta \nabla_\beta u^\alpha + c_s^2 \Pi^{\alpha\beta}\nabla_\beta \varepsilon - 2\sigma^{\alpha\beta}\nabla_\beta \mathsf{\eta}$$
$$+ 2u^\alpha \mathsf{\eta}\sigma^{\mu\nu}\sigma_{\mu\nu} + Q^\beta \nabla_\beta u^\alpha - u^\alpha Q^\beta u^\mu \nabla_\mu u_\beta + Q^\alpha \nabla_\beta u^\beta = 0. \tag{3.1b}$$

Define

$$\begin{aligned} S_\alpha{}^\beta &:= \Pi^\mu_\alpha \nabla_\mu u^\beta, \\ \mathsf{S}^\alpha &:= u^\mu \nabla_\mu u^\alpha, \\ \mathcal{V} &:= \frac{u^\mu \nabla_\mu \varepsilon}{\varepsilon + P}, \\ V^\alpha &:= \frac{\Pi^{\alpha\beta}\nabla_\beta \varepsilon}{\varepsilon + P}, \end{aligned}$$

so that

$$\chi_1 u^\alpha \nabla_\alpha \mathcal{V} + \lambda c_s^2 \nabla_\alpha V^\alpha + \lambda \nabla_\alpha \mathsf{S}^\alpha + \chi_2 u^\alpha \nabla_\alpha S_\nu{}^\nu + r_1 = 0, \tag{3.2a}$$

$$\chi_3 \Pi^{\mu\alpha}\nabla_\alpha \mathcal{V} + \lambda c_s^2 u^\alpha \nabla_\alpha V^\mu + \lambda u^\alpha \nabla_\alpha \mathsf{S}^\mu + B_\nu{}^{\mu\lambda\alpha}\nabla_\alpha S_\lambda{}^\nu + r_2 = 0, \tag{3.2b}$$

$$-\Pi^{\mu\alpha}\nabla_\alpha \mathcal{V} + u^\alpha \nabla_\alpha V^\mu + r_3 = 0, \tag{3.2c}$$

$$u^\alpha \nabla_\alpha S_\lambda{}^\nu - \Pi^\alpha_\lambda \nabla_\alpha \mathsf{S}^\nu + r_4 = 0, \tag{3.2d}$$

$$u^\alpha \nabla_\alpha \varepsilon + r_5 = 0, \tag{3.2e}$$

$$u^\mu \nabla_\mu u^\alpha + r_6 = 0, \tag{3.2f}$$

where

$$B_\nu{}^{\mu\lambda\alpha} := (\chi_4 + \frac{2\mathsf{\eta}}{3})\Pi^{\mu\alpha}\delta^\lambda_\nu - \mathsf{\eta}(\Pi^{\alpha\lambda}\delta^\mu_\nu + \Pi^{\mu\lambda}\delta^\alpha_\nu).$$

Above, r_i, $i = 1, \ldots, 6$ are analytic functions of $\mathcal{V}$, V^ν, S^ν, $S_\lambda{}^\nu$, ε, and u^α; no derivative of these quantities appears in the r_i's.

We now provide details on the derivation of (3.2). Equations (3.2a) and (3.2b) are equations (3.1a) and (3.1b), respectively; equations (3.2e) and (3.2f) are simply the definition of V and S^ν; equations (3.2c) and (3.2d) follow from contracting the identities

$$\begin{aligned} \nabla_\mu \nabla_\nu \varepsilon - \nabla_\nu \nabla_\mu \varepsilon &= 0, \\ \nabla_\mu \nabla_\nu u^\alpha - \nabla_\nu \nabla_\mu u^\alpha &= R_{\mu\nu}{}^\alpha{}_\lambda u^\lambda = 0, \end{aligned}$$

with u^μ and then with Π^ν_λ. We also used the identity

$$\nabla_\alpha u^\beta = -u_\alpha \mathsf{S}^\beta + S_\alpha{}^\beta.$$

We write equations (3.2) as a quasilinear first order system for the variable $\mathbf{\Psi} = (\mathcal{V}, V^\nu, \mathsf{S}^\nu, S_0{}^\nu, S_1{}^\nu, S_2{}^\nu, S_3{}^\nu, \varepsilon, u^\nu)$ as

$$\mathcal{A}^\alpha \nabla_\alpha \mathbf{\Psi} + \mathcal{R} = 0, \tag{3.3}$$

where $\mathcal{R} = (r_1, \dots, r_6)$ and $\mathcal{A}^\alpha$ is given by

$$\mathcal{A}^\alpha = \begin{bmatrix} \chi_1 u^\alpha & \lambda c_s^2 \delta^\alpha_\nu & \lambda \delta^\alpha_\nu & \chi_2 u^\alpha \delta^0_\nu & \chi_2 u^\alpha \delta^1_\nu & \chi_2 u^\alpha \delta^2_\nu & \chi_2 u^\alpha \delta^3_\nu & 0 & 0_{1\times 4} \\ \chi_3 \Pi^{\mu\alpha} & \lambda c_s^2 u^\alpha I_4 & \lambda u^\alpha I_4 & B_\nu{}^{\mu 0\alpha} & B_\nu{}^{\mu 1\alpha} & B_\nu{}^{\mu 2\alpha} & B_\nu{}^{\mu 3\alpha} & 0_{4\times 1} & 0_{4\times 4} \\ -\Pi^{\mu\alpha} & u^\alpha I_4 & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 1} & 0_{4\times 4} \\ 0_{4\times 1} & 0_{4\times 4} & -\Pi^\alpha_0 I_4 & u^\alpha I_4 & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 1} & 0_{4\times 4} \\ 0_{4\times 1} & 0_{4\times 4} & -\Pi^\alpha_1 I_4 & 0_{4\times 4} & u^\alpha I_4 & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 1} & 0_{4\times 4} \\ 0_{4\times 1} & 0_{4\times 4} & -\Pi^\alpha_2 I_4 & 0_{4\times 4} & 0_{4\times 4} & u^\alpha I_4 & 0_{4\times 4} & 0_{4\times 1} & 0_{4\times 4} \\ 0_{4\times 1} & 0_{4\times 4} & -\Pi^\alpha_3 I_4 & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & u^\alpha I_4 & 0_{4\times 1} & 0_{4\times 4} \\ 0 & 0_{1\times 4} & 0_{1\times 4} & 0_{1\times 4} & 0_{1\times 4} & 0_{1\times 4} & 0_{1\times 4} & u^\alpha & 0_{1\times 4} \\ 0_{4\times 1} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 1} & u^\alpha I_4 \end{bmatrix},$$

where $0_{m\times m}$ is the $m \times m$ zero matrix and $I_{m\times m}$ is the $m \times m$ identity matrix. Equation (3.3) is the main equation we will use to derive estimates.

4. Diagonalization

For everything that follows, we work under the assumptions of Theorem 2.1.

Remark 4.1 (Silent use of (2.2) and (2.3))**.** Throughout our computations, we will make successive use of equations (2.2) and (2.3) without explicitly mentioning them.

Proposition 4.2. *Let ξ be a timelike vector and assume that $\lambda, \chi_1, \eta > 0$, and*

$$\Delta_D := 9\lambda^2\chi_2^2 c_s^4 + 6\lambda c_s^2 \left(\chi_1 (4\eta - 3\chi_4)(2\lambda + \chi_2) + 3\chi_2\chi_3(\lambda + \chi_2)\right) + \left(\chi_1(4\eta - 3\chi_4) + 3\chi_3(\lambda + \chi_2)\right)^2 > 0, \tag{4.1}$$

$$\lambda \geq \eta, \tag{4.2}$$

$$3\chi_4 > 4\eta, \tag{4.3}$$

$$2\lambda\chi_1 \geq \lambda\chi_2 c_s^2 - \chi_1\left(\chi_4 - \frac{4\eta}{3}\right) + \lambda\chi_3 + \chi_3\chi_2, \tag{4.4}$$

$$\lambda\chi_1 + c_s^2\lambda\left(\chi_4 - \frac{4\eta}{3}\right) \geq c_s^2\lambda\chi_2 + \lambda\chi_3 + \chi_2\chi_3 - \chi_1\left(\chi_4 - \frac{4}{3}\eta\right) \geq 0. \tag{4.5}$$

Then:
(i) $\det(\mathcal{A}^\alpha \xi_\alpha) \neq 0$*;*
(ii) For any spacelike vector ζ, the eigenvalue problem $\mathcal{A}^\alpha(\zeta_\alpha + \Lambda\xi_\alpha)V = 0$ has only real eigenvalues Λ and a complete set of eigenvectors V.

Proof. Let Ξ_α be any co-vector and $\mathfrak{a} := u^\alpha \Xi_\alpha$, $\mathfrak{b}^\alpha := \Pi^{\alpha\beta}\Xi_\beta$, and $D_\nu{}^{\mu\lambda} := B_\nu{}^{\mu\lambda\alpha}\Xi_\alpha$. Also, consider the superscript μ labeling rows while the subscript ν labels columns. Then

$$\det(\Xi_\alpha \mathcal{A}^\alpha) = \det \begin{bmatrix} \chi_1\mathfrak{a} & \lambda c_s^2 \Xi_\nu & \lambda \Xi_\nu & \chi_2\mathfrak{a}\delta^0_\nu & \chi_2\mathfrak{a}\delta^1_\nu & \chi_2\mathfrak{a}\delta^2_\nu & \chi_2\mathfrak{a}\delta^3_\nu & 0 & 0_{1\times 4} \\ \chi_3\mathfrak{b}^\mu & \lambda c_s^2 \mathfrak{a} I_4 & \lambda\mathfrak{a} I_4 & D_\nu{}^{\mu 0} & D_\nu{}^{\mu 1} & D_\nu{}^{\mu 2} & D_\nu{}^{\mu 3} & 0_{4\times 1} & 0_{4\times 4} \\ -\mathfrak{b}^\mu & \mathfrak{a} I_4 & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 1} & 0_{4\times 4} \\ 0_{4\times 1} & 0_{4\times 4} & -\mathfrak{b}_0 I_4 & \mathfrak{a} I_4 & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 1} & 0_{4\times 4} \\ 0_{4\times 1} & 0_{4\times 4} & -\mathfrak{b}_1 I_4 & 0_{4\times 4} & \mathfrak{a} I_4 & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 1} & 0_{4\times 4} \\ 0_{4\times 1} & 0_{4\times 4} & -\mathfrak{b}_2 I_4 & 0_{4\times 4} & 0_{4\times 4} & \mathfrak{a} I_4 & 0_{4\times 4} & 0_{4\times 1} & 0_{4\times 4} \\ 0_{4\times 1} & 0_{4\times 4} & -\mathfrak{b}_3 I_4 & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & \mathfrak{a} I_4 & 0_{4\times 1} & 0_{4\times 4} \\ 0 & 0_{1\times 4} & 0_{1\times 4} & 0_{1\times 4} & 0_{1\times 4} & 0_{1\times 4} & 0_{1\times 4} & \mathfrak{a} & 0_{1\times 4} \\ 0_{4\times 1} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 1} & \mathfrak{a} I_4 \end{bmatrix}$$

$$
\begin{aligned}
=&\mathfrak{a}^5 \det \begin{bmatrix}
\chi_1\mathfrak{a} & \lambda c_s^2 \Xi_\nu & \lambda \Xi_\nu & \chi_2 \mathfrak{a}\delta^0_\nu & \chi_2\mathfrak{a}\delta^1_\nu & \chi_2\mathfrak{a}\delta^2_\nu & \chi_2\mathfrak{a}\delta^3_\nu \\
\chi_3\mathfrak{b}^\mu & \lambda c_s^2 \mathfrak{a} I_4 & \lambda\mathfrak{a} I_4 & D_\nu{}^{\mu 0} & D_\nu{}^{\mu 1} & D_\nu{}^{\mu 2} & D_\nu{}^{\mu 3} \\
-\mathfrak{b}^\mu & \mathfrak{a} I_4 & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\
0_{4\times 1} & 0_{4\times 4} & -\mathfrak{b}_0 I_4 & \mathfrak{a} I_4 & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\
0_{4\times 1} & 0_{4\times 4} & -\mathfrak{b}_1 I_4 & 0_{4\times 4} & \mathfrak{a} I_4 & 0_{4\times 4} & 0_{4\times 4} \\
0_{4\times 1} & 0_{4\times 4} & -\mathfrak{b}_2 I_4 & 0_{4\times 4} & 0_{4\times 4} & \mathfrak{a} I_4 & 0_{4\times 4} \\
0_{4\times 1} & 0_{4\times 4} & -\mathfrak{b}_3 I_4 & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & \mathfrak{a} I_4
\end{bmatrix} \\
=&\mathfrak{a}^{17} \det \begin{bmatrix}
\chi_1\mathfrak{a} & \lambda c_s^2 \Xi_\nu & \lambda \Xi_\nu + \chi_2 \mathfrak{b}_\nu \\
\chi_3 \mathfrak{a}\mathfrak{b}^\mu & \lambda c_s^2 \mathfrak{a}^2 I_4 & \lambda \mathfrak{a}^2 I_4 + D_\nu{}^{\mu\lambda}\mathfrak{b}_\lambda \\
-\mathfrak{b}^\mu & \mathfrak{a} I_4 & 0_{4\times 4}
\end{bmatrix} \\
=&\frac{\mathfrak{a}^{14}}{\chi_1^3} \det \begin{bmatrix}
(\lambda c_s^2 + \chi_3)\mathfrak{a}^2 I_4 & \lambda \mathfrak{a}^2 I_4 + D_\nu{}^{\mu\lambda}\mathfrak{b}_\lambda \\
\mathfrak{a}^2\chi_1 I_4 + \lambda c_s^2 \Xi_\nu \mathfrak{b}^\mu & (\lambda \Xi_\nu + \chi_2 \mathfrak{b}_\nu)\mathfrak{b}^\mu
\end{bmatrix} \\
=&\frac{\mathfrak{a}^{14}}{\chi_1^3} \det((\lambda c_s^2 + \chi_3)\mathfrak{a}^2(\lambda\Xi_\nu + \chi_2\mathfrak{b}_\nu)\mathfrak{b}^\mu - (\mathfrak{a}^2\chi_1\delta^\mu_\sigma + \lambda c_s^2 \Xi_\sigma \mathfrak{b}^\mu)(\lambda\mathfrak{a}^2\delta^\sigma_\nu + D_\nu{}^{\sigma\lambda}\mathfrak{b}_\lambda)) \\
=&\frac{\mathfrak{a}^{14}}{\chi_1^3} \det(\chi_1\mathfrak{a}^2(\lambda\mathfrak{a}^2 - \eta\mathfrak{b}^\alpha\mathfrak{b}_\alpha)I_4 + ((\chi_4 + \frac{2\eta}{3})(\mathfrak{a}^2\chi_1 + \lambda c_s^2\mathfrak{b}^\alpha\mathfrak{b}_\alpha) - (\lambda c_s^2 + \chi_3)\chi_2\mathfrak{a}^2)\mathfrak{b}^\mu\mathfrak{b}_\nu \\
&- (\mathfrak{a}^2(\chi_1\eta + \chi_3\lambda) + 2\lambda\eta c_s^2 \mathfrak{b}^\alpha\mathfrak{b}_\alpha)\mathfrak{b}^\mu\Xi_\nu) \\
=&\mathfrak{a}^{20}(\lambda\mathfrak{a}^2 - \eta\mathfrak{b}^\alpha\mathfrak{b}_\alpha)^3(\lambda(\chi_1\mathfrak{a}^2 - \chi_3\mathfrak{b}^\alpha\mathfrak{b}_\alpha)\mathfrak{a}^2 - \chi_2(\chi_3 + \lambda c_s^2)\mathfrak{a}^2\mathfrak{b}^\alpha\mathfrak{b}_\alpha \\
&+ (\chi_1\mathfrak{a}^2 + \lambda c_s^2)(\chi_4 - \frac{4\eta}{3}\mathfrak{b}^\beta\mathfrak{b}_\beta)\mathfrak{b}^\alpha\mathfrak{b}_\alpha) \\
=&\lambda^4\chi_1 \prod_{a=1,2,\pm}((u^\alpha\Xi_\alpha)^2 - \beta_a \Pi^{\alpha\beta}\Xi_\alpha\Xi_\beta)^{n_a},
\end{aligned}
\tag{4.6}
$$

where $n_1 = 10$, $n_2 = 3$, $n_\pm = 1$, $\beta_1 = 0$, and

$$
\begin{aligned}
\beta_2 &= \frac{\eta}{\lambda}, \\
\beta_\pm &= \frac{3\lambda\chi_2 c_s^2 + \chi_1(4\eta - 3\chi_4) + 3\chi_3(\lambda + \chi_2) \pm \sqrt{\Delta_D}}{6\lambda\chi_1}.
\end{aligned}
\tag{4.7}
$$

The Δ_D in (4.7) is defined in (4.1) and $\beta_\pm$ corresponds to two distinct real roots whenever (4.1) is observed. We made successive use of the formula

$$
\det \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} = \det(M_1)\det(M_4 - M_3 M_1^{-1} M_2),
$$

and also used the identity $\det(AI_4 + B\mathfrak{b}^\mu\mathfrak{b}_\nu + C\mathfrak{b}^\mu\Xi_\nu) = A^3(A + (B + C)\mathfrak{b}^\mu\mathfrak{b}_\mu)$, where $\mathfrak{b}^\mu\Xi_\mu = \mathfrak{b}^\mu\mathfrak{b}_\mu = \Pi^{\alpha\beta}\Xi_\alpha\Xi_\beta$.

We now verify condition *(i)*. Set $\zeta = 0$ in the above, so that $\Xi_\mu = \xi_\mu$, where $\xi^\alpha\xi_\alpha = -(u^\alpha\xi_\alpha)^2 + \Pi^{\alpha\beta}\xi_\alpha\xi_\beta < 0$, from (4.6) one obtains that $\det(A^\alpha\xi_\alpha) \neq 0$ if $\lambda, \chi_1 \neq 0$ and

$$
0 \leq \beta_a \leq 1,\ a = 1, 2, \pm. \tag{4.8}
$$

Since $\beta_1 = 0$, (4.8) is satisfied. As for β_2, (4.8) is satisfied whenever (4.2) is obeyed and $\lambda, \eta > 0$. Condition (4.1) and $\chi_1 > 0$ guarantee that $\beta_\pm$ are real and distinct with $\beta_- < \beta_+$,

while (4.3) sets $\beta_- > 0$ and (4.4) together with (4.5) assure $\beta_+ \leq 1$. Then, statement *(i)* in the Theorem is proved.

The eigenvalues in *(ii)* are the roots of (4.6) by setting $\Xi = \zeta + \Lambda\xi$. Reality of the eigenvalues Λ are obtained by studying the roots of the polynomials $(u^\alpha\Xi_\alpha)^2 - \beta\Pi^{\alpha\beta}\Xi_\alpha\Xi_\beta$ that appears in (4.6). The roots of $(u^\alpha\Xi_\alpha)^2 - \beta\Pi^{\alpha\beta}\Xi_\alpha\Xi_\beta = 0$ are, for each one of the β's,

$$\Lambda_\pm = (-u^\mu\zeta_\mu u^\nu\xi_\nu + \beta\Pi^{\mu\nu}\xi_\mu\zeta_\nu \pm \sqrt{\mathcal{W}})/((u^\mu\xi_\mu)^2(1-\beta) - \beta\xi^\mu\xi_\mu), \tag{4.9}$$

where

$$\begin{aligned}\mathcal{W} = \beta(((u^\mu\xi_\mu)^2 - \Pi^{\mu\nu}\xi_\mu\xi_\nu)(\Pi^{\alpha\beta}\zeta_\alpha\zeta_\beta - (u^\alpha\zeta_\alpha)^2) + (u^\mu\xi_\mu u^\nu\zeta_\nu + \Pi^{\mu\nu}\xi_\mu\zeta_\nu)^2 \\ + (1-\beta)(\Pi^{\mu\nu}\xi_\mu\xi_\nu\Pi^{\alpha\beta}\zeta_\alpha\zeta_\beta - (\Pi^{\mu\nu}\xi_\mu\zeta_\nu)^2)).\end{aligned}$$

We note that these roots are always real when $0 \leq \beta \leq 1$ because $\Pi_{\alpha\beta}\xi^\alpha\xi^\beta < (\xi_\alpha u^\alpha)^2$, $\Pi_{\alpha\beta}\zeta^\alpha\zeta^\beta > (\zeta_\alpha u^\alpha)^2$, and $(\Pi^{\mu\nu}\xi_\mu\zeta_\nu)^2 \leq \Pi^{\mu\nu}\xi_\mu\xi_\nu\Pi^{\alpha\beta}\zeta_\alpha\zeta_\beta$. Then, the conditions expressed in equations (4.1)–(4.5) give real eigenvalues Λ.

We now turn to the problem of the eigenvectors for each eigenvalue. We ended up with the root Λ_1 for $\beta_1 = 0$ with multiplicity 20, the roots $\Lambda_{2,\pm}$ for $\beta_2 = \eta/\lambda$ with multiplicity 3 each, and $\Lambda_{\pm,\pm}$ for $\beta_\pm$ given in (4.7) with multiplicity 1 each. The complete set of eigenvectors must contain 30 linearly independent eigenvectors.

The roots $\Lambda_{\pm,\pm}$ are obtained from (4.9) with $\beta = \beta_\pm$ and contains 4 linearly independent eigenvectors since they are 4 distinct eigenvalues $\Lambda_{\pm,\pm}$. The remaining 26 eigenvectors appears as follows:

$(u^\alpha\Xi_\alpha)^{20} = 0$ gives

$$\Lambda_1 = -\frac{u^\alpha\zeta_\alpha}{u^\beta\xi_\beta}.$$

There are 20 corresponding linearly independent eigenvectors given by

$$\begin{bmatrix} 0 \\ w_i^\nu \\ 0_{25\times1}\end{bmatrix}, \begin{bmatrix} 0_{25\times1} \\ 1 \\ 0_{4\times1}\end{bmatrix}, \begin{bmatrix} 0_{26\times1} \\ v_I^\nu\end{bmatrix}, \begin{bmatrix} 0_{9\times1} \\ f_0^\nu \\ f_1^\nu \\ f_2^\nu \\ f_2^\nu \\ 0_{5\times1}\end{bmatrix},$$

where v_I^ν, $I = 1,2,3,4$, are any 4 linearly independent vectors in $\mathbb{R}^4$, $w_i^\mu = \{u^\mu, w_2^\mu, w_3^\mu\}$ are the three linearly independent vectors of $\mathbb{R}^4$ that are orthogonal to $\zeta_\alpha + \Lambda_1\xi_\alpha$, and f_λ^ν totalizes 16 components that define the entries in the last vector. However, since these 16 components are constrained by the 4 equation $D_\nu{}^{\mu\lambda}f_\lambda^\nu = 0$, we end up with 12 independent entries. Then, $4+1+3+12 = 20$, which equals the multiplicity of the root Λ_1.

$((u^\alpha \Xi_\alpha) - \beta_2 \Pi^{\alpha\beta}\Xi_\alpha \Xi_\beta)^3 = 0$, where the roots $\Lambda_{2,\pm}$ are given by (4.9) with $\beta = \beta_2 = \frac{\mathfrak{n}}{\lambda}$. Each one of these 2 roots has multiplicity 3. The corresponding eigenvectors are

$$\begin{bmatrix} C_\pm \\ \frac{C_\pm}{\mathfrak{a}_\pm}(\mathfrak{b}_\pm)^\nu \\ E_\pm^\nu \\ \frac{(\mathfrak{b}_\pm)_0 E_\pm^\nu}{\mathfrak{a}_\pm} \\ \frac{(\mathfrak{b}_\pm)_1 E_\pm^\nu}{\mathfrak{a}_\pm} \\ \frac{(\mathfrak{b}_\pm)_2 E_\pm^\nu}{\mathfrak{a}_\pm} \\ \frac{(\mathfrak{b}_\pm)_3 E_\pm^\nu}{\mathfrak{a}_\pm} \\ 0_{5\times 1} \end{bmatrix},$$

where $\mathfrak{a}_\pm = u^\alpha(\zeta_\alpha + \Lambda_{2,\pm}\xi_\alpha)$, $(\mathfrak{b}_\pm)^\alpha = \Pi^{\alpha\beta}(\zeta_\beta + \Lambda_{2,\pm}\xi_\beta)$ (so that $\mathfrak{a}_\pm^2 = \beta_2(\mathfrak{b}_\pm)^\mu(\mathfrak{b}_\pm)_\mu$),

$$C_\pm = \frac{\mathfrak{n}(\Xi_\pm)_\alpha (E_\pm)^\alpha - (\chi_4 + \frac{2\mathfrak{n}}{3})(\mathfrak{b}_\pm)_\alpha (E_\pm)^\alpha}{(\lambda c_s^2 + \chi_3)\mathfrak{a}_\pm},$$

where $\Xi_\pm = \zeta + \Lambda_{2,\pm}\xi$, and $E_\pm$ obeys the following constraint

$$(\lambda(\Xi_\pm)_\alpha + \chi_2(\mathfrak{b}_\pm)_\alpha + \frac{\mathfrak{n}}{\lambda^2}(\lambda^2 c_s^2 + \chi_1 \mathfrak{n})(\mathfrak{n}(\Xi_\pm)_\alpha - (\chi_4 + \frac{2\mathfrak{n}}{3})(\mathfrak{b}_\pm)_\alpha)(E_\pm)^\alpha = 0.$$

Thus, the eigenvectors are written in terms of 3 independent components of $(E_\pm)^\mu$ for each root, giving a total of 6 eigenvectors.

□

From the above Proposition, we immediately obtain:

Corollary 4.3. *Assume that* $\lambda, \chi_1, \mathfrak{n} > 0$ *and that* (4.1), (4.2), (4.3), *and* (4.5) *hold. Then, the system* (3.3) *can be written as*

$$\nabla_0 \mathbf{\Psi} + \tilde{\mathcal{A}}^i \nabla_i \mathbf{\Psi} = \tilde{\mathcal{R}}, \tag{4.10}$$

where $\tilde{\mathcal{A}}^i = (\mathcal{A}^0)^{-1}\mathcal{A}^i$ *and* $\tilde{\mathcal{R}} = -(\mathcal{A}^0)^{-1}\mathcal{R}$, *and the eigenvalue problem* $(\tilde{\mathcal{A}}^i \zeta_i - \Lambda I)V = 0$ *possesses only real eigenvalues* Λ *and a set of complete eigenvectors* V.

5. Energy estimates

5.1. **Preliminaries.** We begin introducing some notation. Let $I = [0, T]$ for some $T > 0$. We use $\mathscr{K} : \mathbb{R}_+ \to \mathbb{R}_+$ to denote a continuous function which may vary from line to line. Similarly, $\mathscr{K}_I : \mathbb{R}_+ \to \mathbb{R}_+$ denotes a continuous function depending on I. Further, the notation $\mathfrak{R}$ always denotes a pseudodifferential operator (henceforth abbreviated ΨDO) whose mapping properties may vary from line to line. We denote the L^2 based Sobolev space of order r by H^r, with norm $\|\cdot\|_r$.

The quasilinear nature of our equations leads us to consider a pseudodifferential calculus for symbols with limited smoothness. Such a calculus can be found in [67–69], to which we will refer frequently. For the reader's convenience, we recall the definition of these symbols and the corresponding ΨDO on $\mathbb{R}^3$.

Definition 5.1 (ΨDO with limited smoothness, [69])**.** Let $k \in \mathbb{R}$ and $r > 3/2$. Define $\mathcal{S}^k_0(r,2)(\mathbb{R}^3) = \mathcal{S}^k_0(r,2)(\mathbb{R}^3,\mathbb{C})$ to be the space of all symbols $a : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C}$ such that for all spatial multi-indices $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$

$$|\partial^{\vec{\alpha}}_\zeta a(x,\zeta)| \le C_{\vec{\alpha}}(1+|\zeta|)^{k-|\vec{\alpha}|},$$
$$\|\partial^{\vec{\alpha}}_\zeta a(x,\zeta)\|_{H^r} \le C_{\vec{\alpha}}(1+|\zeta|)^{k-|\vec{\alpha}|}.$$

For a matrix-valued symbol $a : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C}^{h\times l}$ with $h, l \in \mathbb{N}$, we say $a \in \mathcal{S}^k_0(r,2)(\mathbb{R}^3,\mathbb{C}^{h\times l})$ if all the entries of a belong to $\mathcal{S}^k_0(r,2)(\mathbb{R}^3)$. The ΨDO $Op(a)$, associated with a symbol $a \in \mathcal{S}^k_0(r,2)(\mathbb{R}^3,\mathbb{C}^{h\times l})$ is defined by

$$Op(a)f(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^3} e^{\mathrm{i}x\cdot\zeta}a(x,\zeta)\hat{f}(\zeta)\,d\zeta$$

for $f \in \mathcal{S}(\mathbb{R}^3,\mathbb{C}^l)$, the space of Schwartz functions in $\mathbb{R}^3$, and $\mathrm{i} = \sqrt{-1}$.

Having defined symbols and ΨDO operators of limited smoothness in $\mathbb{R}^3$, we can use the coordinate invariance of the definition and standard arguments (see [69, Theorem 5.1, Corollary 5.2]) to obtain a ΨDO calculus on any smooth closed manifold. In particular, we obtain such a calculus on $\mathbb{T}^3$. We denote the class of symbols on $\mathbb{T}^3$ of order k with Sobolev regularity r by $\mathcal{S}^k_0(r,2)(\mathbb{T}^3)$ or simply $\mathcal{S}^k_0(r,2)$. Given $a \in \mathcal{S}^k_0(r,2)(\mathbb{T}^3)$, we denote the ΨDO associated with a by $Op(a)$ and the resulting space of kth order ΨDO's by $OP\mathcal{S}^k_0(r,2)$. We will not typically specify if the symbol is scalar or matrix valued since this will be clear from the context.

The (flat) Laplacian on $\mathbb{T}^3$ is denoted by Δ, and we define

$$\langle\nabla\rangle := (1-\Delta)^{\frac{1}{2}},$$

which is an element of $OP\mathcal{S}^1_0(r,2)$ for every $r \in \mathbb{R}$. Finally, we recall that

$$\|\cdot\|_k \simeq \|\langle\nabla\rangle^k \cdot\|_0.$$

Remark 5.2. In what follows, we will use Corollary 4.3. This Corollary follows from Proposition 4.2, which involved computing the principal symbol of (3.3) with $\partial_k \mapsto \zeta_k$. For the pseuddifferential calculus introduced above, one uses $\partial_k \mapsto \mathrm{i}\zeta_k$ instead. In view of the homogeneity of the symbols involved and multiplying and dividing by i when necessary, it is not difficult to make the two procedures compatible.

5.2. **Main estimates.** We consider the linear system naturally associated with (4.10). Given ψ, we define the operator $\mathcal{F}(\psi)$ by

$$\mathcal{F}(\psi)\mathbf{\Psi} = \partial_t\mathbf{\Psi} + \tilde{\mathcal{A}}^i(\psi)\nabla_i\mathbf{\Psi},$$

where $\tilde{\mathcal{A}}^i(\psi)$ corresponds to the matrix $\tilde{\mathcal{A}}^i = (\mathcal{A}^0)^{-1}\mathcal{A}^i$ of Corollary 4.3, but with the entries of the matrix computed using ψ. Similarly, letting $\tilde{\mathcal{R}}(\psi) = -(\mathcal{A}^0)^{-1}(r_1,\dots,r_6)^T$ correspond to $\tilde{\mathcal{R}}$ in Corollary 4.3 with the entries computed using ψ, we see that the first system (3.3), or, equivalently (4.10), can be written as

$$\mathcal{F}(\mathbf{\Psi})\mathbf{\Psi} = \tilde{\mathcal{R}}(\mathbf{\Psi}), \quad \mathbf{\Psi}(0) = \mathbf{\Psi}_0, \tag{5.1}$$

Remark 5.3. In what follows, we will often think of $\mathbf{\Psi}$ and ψ as maps from a time interval to a suitable function space.

Proposition 5.4. *Let $r > 9/2$, $I \subset \mathbb{R}$ and*

$$\mathbb{E}_1(I) := C(I; H^r) \cap C^1(I; H^{r-1}).$$

There exist increasing continuous functions $\tilde{M}, \omega : [0, \infty) \to (0, \infty)$ such that if $\mathbf{\Psi}, \psi \in C^\infty(I \times \mathbb{T}^3)$ satisfy

$$\mathcal{F}(\psi)\mathbf{\Psi} = \tilde{\mathcal{R}}(\psi), \text{ on } I \times \mathbb{T}^3, \tag{5.2}$$

then

$$\|\mathbf{\Psi}(t)\|_r^2 \leq \tilde{M}(\|\psi\|_{L^\infty(I;H^{r-1})}) e^{t\omega(\|\psi\|_{\mathbb{E}_1(I)})} \left[\|\mathbf{\Psi}_0\|_r^2 + \int_0^t \|\tilde{\mathcal{R}}(\psi(s))\|_r^2 \, ds \right] \tag{5.3}$$

for all $t \in I$, where $\mathbf{\Psi}_0 = \mathbf{\Psi}(0)$.

Proof. For $\zeta = \zeta_i dx^i \in T^*\mathbb{T}^3$, let $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}(\psi, \zeta) = \tilde{\mathcal{A}}^i(\psi)\zeta_i$ and $\mathfrak{U} = Op(\tilde{\mathcal{A}})$. From the results of Section 4, we see that there exist a matrix $\mathcal{S} = \mathcal{S}(\psi, \zeta)$ and a diagonal matrix $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}(\psi, \zeta)$ such that

$$\mathcal{S}\tilde{\mathcal{A}} = \tilde{\mathcal{D}}\mathcal{S}.$$

Set $\mathfrak{S} := Op(\mathcal{S})$ and $\tilde{\mathfrak{D}} := Op(\tilde{\mathcal{D}})$. From the expression for $\tilde{\mathcal{A}}^i(\psi)\zeta_i$, it is not difficult to see that all its entries belong to $\mathcal{S}_0^1(r, 2)$. Denote by $\Lambda_k = \Lambda_k(\psi, \zeta)$ all the distinct eigenvalues of $\tilde{\mathcal{A}}$. Noting that $\partial_\zeta^{\vec{\alpha}} \tilde{\mathcal{A}}(\psi, \zeta)$ is homogeneous of degree $1 - |\vec{\alpha}|$ for $|\vec{\alpha}| \leq 1$ and $\partial_\zeta^{\vec{\alpha}} \tilde{\mathcal{A}}(\psi, \zeta) = 0$ for $|\vec{\alpha}| > 1$, we infer that $\Lambda_k/|\zeta|$ is homogeneous in ζ of degree zero.

Because the map $[(\psi, \zeta) \mapsto \Lambda_k(\psi, \zeta)] \in C^\infty(H^r \times T^*\mathbb{T}^3, H^r)$, it follows that

$$\|\Lambda_k(\psi, \zeta)\|_r \leq C, \quad |\zeta| = 1$$

for some constant $C = C(\|\psi\|_r)$ depending on $\|\psi\|_r$. By the homogeneity of $\Lambda_k/|\zeta|$, we can conclude that

$$\|\Lambda_k(\psi, \zeta)\|_r \leq C(1 + |\zeta|),$$

for all ζ and some $C = C(\|\psi\|_r)$. Differentiating the characteristic polynomial of $\tilde{\mathcal{A}}$ with respect to ζ and using induction immediately yield

$$\|\partial_\zeta^{\vec{\alpha}} \Lambda_k(\psi, \zeta)\|_r \leq C_{\vec{\alpha}} (1 + |\zeta|)^{1 - |\vec{\alpha}|}, \tag{5.4}$$

for all ζ and some $C_{\vec{\alpha}} = C_{\vec{\alpha}}(\|\psi\|_r)$. This implies, by Sobolev embedding, that $\Lambda_k \in \mathcal{S}_0^1(r, 2)$ and therefore

$$\tilde{\mathfrak{D}} \in OP\mathcal{S}_0^1(r, 2).$$

The projection onto the eigenspace associated to the eigenvalue Λ_k is given by

$$P_k = P_k(\psi, \zeta) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_k} (z - \tilde{\mathcal{A}}(\psi, \zeta))^{-1} \, dz, \tag{5.5}$$

where γ_k is a smooth contour enclosing only one pole Λ_k. By properly choosing contours γ_k, we can always make the eigenvalues $\tilde{\Lambda}_i(z, \psi, \zeta)$ of $(z - \tilde{\mathcal{A}}(\psi, \zeta))^{-1}$ satisfy

$$\|\tilde{\Lambda}_i(z, \psi, \zeta)\|_r \leq C = C(\|\psi\|_r), \quad |\zeta| \leq 1, \, z \in \gamma_k$$

for all k. From the homogeneity of $\tilde{\mathcal{A}}$ and Λ_k, we infer that P_k is homogeneous of degree 0 in ζ. Combining with (5.4) and (5.5), we obtain that

$$\|P_k(\psi, \zeta)\|_{H^r} \leq C = C(\|\psi\|_r), \quad |\zeta| = 1.$$

In light of the homogeneity of $P_k(\psi,\cdot)$, this implies that for all ζ we have

$$\|P_k(\psi,\zeta)\|_{H^r} \leq C = C(\|\psi\|_r).$$

For a given pair of (ψ,ζ), we can always choose the contour γ_k in (5.5) to be fixed in a neighborhood of (ψ,ζ). Applying a similar argument to the ζ-derivatives of P_k and using the homogeneity of $\partial_\zeta^{\vec{\alpha}}\tilde{\mathcal{A}}$, direct computations lead to $P_k \in \mathcal{S}_0^0(r,2)$. This implies that

$$\mathcal{S} = \mathcal{S}(\psi,\zeta) \in \mathcal{S}_0^0(r,2) \tag{5.6}$$

and thus

$$\mathfrak{S} = \mathfrak{S}(\psi) \in OP\mathcal{S}_0^0(r,2)$$

with norm depending on $\|\psi\|_r$.

We can now invoke [69, Corollary 3.4] to conclude

$$\mathfrak{S}\mathfrak{U} = \tilde{\mathfrak{D}}\mathfrak{S} + \mathfrak{R}$$

with

$$\mathfrak{R} \in \mathcal{L}(H^s, H^s), \quad 1 - r < s \leq r - 2,$$

where $\mathcal{L}(\mathsf{X},\mathsf{Y})$ denotes the space of linear continuous maps between Banach spaces X and Y.

We write $\mathfrak{U} = \mathrm{i}\mathfrak{A}\langle\nabla\rangle$. Let $\mathcal{A} = \mathcal{A}(\zeta)$ denote the symbol of $\mathfrak{A}$, i.e. $\mathcal{A} = -\mathrm{i}\tilde{\mathcal{A}}/(1+|\zeta|^2)^{\frac{1}{2}}$. Therefore, $\mathfrak{A} \in OP\mathcal{S}_0^0(r,2)$. Then there exists a ΨDO $\mathfrak{D}$ with symbol $\mathcal{D} \in \mathcal{S}_0^0(r,2)$ such that

$$\mathcal{S}\mathcal{A} = \mathcal{D}\mathcal{S}$$

and thus

$$\mathfrak{S}\mathfrak{A} = \mathfrak{D}\mathfrak{S} + \mathfrak{R}$$

with

$$\mathfrak{R} \in \mathcal{L}(H^{s-1}, H^s), \quad 1 - r < s \leq r - 1. \tag{5.7}$$

We can thus rewrite (5.2) as

$$\partial_t \mathbf{\Psi} = \mathrm{i}\mathfrak{A}(\psi)\langle\nabla\rangle\mathbf{\Psi} + \tilde{\mathcal{R}}(\psi),$$

or

$$\partial_t \mathbf{\Psi} = \mathfrak{U}(\psi)\mathbf{\Psi} + \tilde{\mathcal{R}}(\psi).$$

Denote by $\mathcal{S}^*$ the conjugate transpose matrix of $\mathcal{S}$. We further set $\tilde{\mathfrak{S}} := Op(\mathcal{S}^*)$. Note that $\tilde{\mathfrak{S}} = \tilde{\mathfrak{S}}(\psi) \in OP\mathcal{S}_0^0(r,2)$. Since $\mathcal{S}$ is homogeneous of degree 0 in ζ, invoking the discussion in Section 4, we infer that

$$\mathbf{\Psi}^T\mathcal{S}^*(\psi,\zeta)\mathcal{S}(\psi,\zeta)\mathbf{\Psi} \geq C_0|\mathbf{\Psi}|^2$$

for some $C_0 = C_0(\|\psi\|_{L^\infty}) > 0$. Let $\mathcal{B} = \mathcal{B}(\psi,\zeta) = \sqrt{\mathcal{S}^*(\psi,\zeta)\mathcal{S}(\psi,\zeta) - \frac{C_0}{2}I}$ and $\mathfrak{B} = Op(\mathcal{B})$, where I is the identity matrix and, for a positive definite matrix A, $B = \sqrt{A}$ denotes its square-root matrix, i.e. $B^*B = A$. It is not difficult to see via the Cholesky algorithm that $\mathcal{B} \in \mathcal{S}_0^0(r,2)$. Putting $\tilde{\mathfrak{B}} = Op(\mathcal{B}^*) \in OP\mathcal{S}_0^0(r,2)$, it follows from [69, Corollaries 3.4 and 3.6] that

$$\mathfrak{R} = \tilde{\mathfrak{S}} \circ \mathfrak{S} - \frac{C_0}{2}I - \mathfrak{B}^*\mathfrak{B}$$

$$=[(\tilde{\mathfrak{S}}\circ\mathfrak{S}-\frac{C_0}{2}I)-\tilde{\mathfrak{B}}\circ\mathfrak{B}]+(\tilde{\mathfrak{B}}\circ\mathfrak{B}-\tilde{\mathfrak{B}}\mathfrak{B})+(\tilde{\mathfrak{B}}\mathfrak{B}-\mathfrak{B}^*\mathfrak{B})\in\mathcal{L}(H^{s-1},H^s) \quad (5.8)$$

for all $1-r<s<r$. Define

$$N_r(t):=\langle\nabla\rangle^r(\frac{C_0}{2}I+\mathfrak{B}^*\mathfrak{B})\langle\nabla\rangle^r.$$

It is an immediately conclusion from its definition that

$$(N_r(t)\mathbf{\Psi},\mathbf{\Psi})\geq\frac{C_0}{2}\|\mathbf{\Psi}\|_r^2. \quad (5.9)$$

We have

$$\begin{aligned}N_r=&\langle\nabla\rangle^r(\frac{C_0}{2}I+\mathfrak{B}^*\mathfrak{B}-\tilde{\mathfrak{S}}\circ\mathfrak{S})\langle\nabla\rangle^r+\langle\nabla\rangle^r(\tilde{\mathfrak{S}}\circ\mathfrak{S}-\tilde{\mathfrak{S}}\mathfrak{S})\langle\nabla\rangle^r\\&+\langle\nabla\rangle^r(\tilde{\mathfrak{S}}-\mathfrak{S}^*)\mathfrak{S}\langle\nabla\rangle^r+\langle\nabla\rangle^r\mathfrak{S}^*\mathfrak{S}\langle\nabla\rangle^r.\end{aligned}$$

It follows from [69, Corollary 3.4] that

$$\tilde{\mathfrak{S}}\circ\mathfrak{S}-\tilde{\mathfrak{S}}\mathfrak{S}\in\mathcal{L}(H^{s-1},H^s),\quad 1-r<s\leq r, \quad (5.10)$$

and from [69, Corollary 3.6] that

$$\tilde{\mathfrak{S}}-\mathfrak{S}^*\in\mathcal{L}(H^{s-1},H^s),\quad 1-r<s<r, \quad (5.11)$$

Next, we compute

$$\begin{aligned}\frac{d}{dt}(N_r\mathbf{\Psi},\mathbf{\Psi})&=(N_r\frac{d}{dt}\mathbf{\Psi},\mathbf{\Psi})+(N_r\mathbf{\Psi},\frac{d}{dt}\mathbf{\Psi})+(N_r'\mathbf{\Psi},\mathbf{\Psi})\\&=(N_r\mathfrak{U}\mathbf{\Psi},\mathbf{\Psi})+(N_r\tilde{\mathcal{R}},\mathbf{\Psi})+(N_r\mathbf{\Psi},\mathfrak{U}\mathbf{\Psi})+(N_r\mathbf{\Psi},\tilde{\mathcal{R}})+(N_r'\mathbf{\Psi},\mathbf{\Psi})\\&=((N_r\mathfrak{U}+\mathfrak{U}^*N_r)\mathbf{\Psi},\mathbf{\Psi})+(N_r\tilde{\mathcal{R}},\mathbf{\Psi})+(N_r\mathbf{\Psi},\tilde{\mathcal{R}})+(N_r'\mathbf{\Psi},\mathbf{\Psi}),\end{aligned}$$

where $'=\frac{d}{dt}$. We have

$$\begin{aligned}N_r\mathfrak{U}+\mathfrak{U}^*N_r=&[\langle\nabla\rangle^r(\frac{C_0}{2}I+\mathfrak{B}^*\mathfrak{B})\langle\nabla\rangle^r]\mathrm{i}\mathfrak{A}\langle\nabla\rangle\\&-\mathrm{i}\langle\nabla\rangle\mathfrak{A}^*[\langle\nabla\rangle^r(\frac{C_0}{2}I+\mathfrak{B}^*\mathfrak{B})\langle\nabla\rangle^r]\\=&\mathrm{i}[\langle\nabla\rangle^r(\frac{C_0}{2}I+\mathfrak{B}^*\mathfrak{B})\langle\nabla\rangle^r\mathfrak{A}\langle\nabla\rangle\\&-\langle\nabla\rangle\mathfrak{A}^*\langle\nabla\rangle^r(\frac{C_0}{2}I+\mathfrak{B}^*\mathfrak{B})\langle\nabla\rangle^r].\end{aligned}$$

Observe that $\langle\nabla\rangle^r\in OP\mathcal{S}^r_0(k,2)$ for any k. We can infer from (5.8), (5.10) and (5.11) that

$$\begin{aligned}N_r\mathfrak{U}=&\mathrm{i}N_r\mathfrak{A}\langle\nabla\rangle\\=&\mathrm{i}\langle\nabla\rangle^r\mathfrak{S}^*\mathfrak{S}\langle\nabla\rangle^r\mathfrak{A}\langle\nabla\rangle+\mathfrak{R},\end{aligned}$$

where $\mathfrak{R}=\mathfrak{R}(\psi)\in\mathcal{L}(H^r,H^{-r})$.

To estimate the first term in the second line, we first notice that [69, Corollary 3.4] implies

$$\mathfrak{b}=\mathfrak{b}(\psi):=[\langle\nabla\rangle^r,\mathfrak{A}(\psi)]\in\mathcal{L}(H^{r-1},H^0),$$

with its norm depending on $\|\psi\|_r$. Therefore, we have that

$$\langle\nabla\rangle^r\mathfrak{S}^*\mathfrak{S}\langle\nabla\rangle^r\mathfrak{A}\langle\nabla\rangle=\langle\nabla\rangle^r\mathfrak{S}^*\mathfrak{S}\mathfrak{A}\langle\nabla\rangle\langle\nabla\rangle^r+\mathfrak{R},$$

where $\mathfrak{R} = \mathfrak{R}(\psi) \in \mathcal{L}(H^r, H^{-r})$. Next, observe that by (5.7)

$$\begin{aligned}\mathfrak{S}\mathfrak{A}\langle\nabla\rangle &= \mathfrak{D}\mathfrak{S}\langle\nabla\rangle + \mathfrak{R}\\ &= \mathfrak{D}\langle\nabla\rangle\mathfrak{S} + \mathfrak{R},\end{aligned}$$

where in the second equality we used [69, Corollary 3.4]. We recall our convention that the operator $\mathfrak{R}$ may vary from line to line, both $\mathfrak{R}$'s in the last two equalities satisfy

$$\mathfrak{R} = \mathfrak{R}(\psi) \in \mathcal{L}(H^s, H^s) \quad \text{for all } -r+1 < s \le r-1.$$

Therefore,

$$N_r\mathfrak{U} = \mathrm{i}\langle\nabla\rangle^r\mathfrak{S}^*\mathfrak{D}\langle\nabla\rangle\mathfrak{S}\langle\nabla\rangle^r + \mathfrak{R},$$

where $\mathfrak{R} = \mathfrak{R}(\psi) \in \mathcal{L}(H^r, H^{-r})$ and its norm depends on $\|\psi\|_r$.

We can carry out a similar analysis for the term $\mathfrak{U}^*N_r$. More precisely, observe that

$$\begin{aligned}\mathfrak{U}^*N_r =& -\mathrm{i}\langle\nabla\rangle\mathfrak{A}^*\langle\nabla\rangle^r\mathfrak{S}^*\mathfrak{S}\langle\nabla\rangle^r\\ &-\mathrm{i}\langle\nabla\rangle\mathfrak{A}^*[\langle\nabla\rangle^r(\frac{C_0}{2}I + \mathfrak{B}^*\mathfrak{B} - \tilde{\mathfrak{S}}\circ\mathfrak{S})\langle\nabla\rangle^r\\ &+\langle\nabla\rangle^r(\tilde{\mathfrak{S}}\circ\mathfrak{S} - \tilde{\mathfrak{S}}\mathfrak{S})\langle\nabla\rangle^r]\\ &+\langle\nabla\rangle^r(\tilde{\mathfrak{S}} - \mathfrak{S}^*)\mathfrak{S}\langle\nabla\rangle^r.\end{aligned}$$

Using (5.8), (5.10), (5.11) and [69, Theorem 2.4], we infer that the last three terms on the right-hand side belong to $\mathcal{L}(H^r, H^{-r})$. Since

$$-\mathrm{i}\langle\nabla\rangle\mathfrak{A}^*\langle\nabla\rangle^r\mathfrak{S}^*\mathfrak{S}\langle\nabla\rangle^r = [\mathrm{i}\langle\nabla\rangle^r\mathfrak{S}^*\mathfrak{S}\langle\nabla\rangle^r\mathfrak{A}\langle\nabla\rangle]^*,$$

we conclude that

$$\begin{aligned}N_r\mathfrak{U} + \mathfrak{U}^*N_r =& \mathrm{i}[\langle\nabla\rangle^r\mathfrak{S}^*\mathfrak{D}\langle\nabla\rangle\mathfrak{S}\langle\nabla\rangle^r\\ &-\langle\nabla\rangle^r\mathfrak{S}^*\langle\nabla\rangle\mathfrak{D}^*\mathfrak{S}\langle\nabla\rangle^r] + \mathfrak{R}_0,\\ =& \mathrm{i}\langle\nabla\rangle^r\mathfrak{S}^*[\mathfrak{D}\langle\nabla\rangle - \langle\nabla\rangle\mathfrak{D}^*]\mathfrak{S}\langle\nabla\rangle^r + \mathfrak{R}_0,\end{aligned}$$

where $\mathfrak{R}_0 = \mathfrak{R}_0(\psi) \in \mathcal{L}(H^r, H^{-r})$ with norm depending on $\|\psi\|_r$. The term in the parenthesis is bounded in $\mathcal{L}(H^0)$ due to [69, Corollary 3.6]. Therefore,

$$\begin{aligned}\frac{d}{dt}(N_r\mathbf{\Psi}, \mathbf{\Psi}) =& \mathrm{i}(\langle\nabla\rangle^r[\mathfrak{S}^*\mathfrak{D}\langle\nabla\rangle\mathfrak{S} - \mathfrak{S}^*\langle\nabla\rangle\mathfrak{D}^*\mathfrak{S}]\langle\nabla\rangle^r\mathbf{\Psi}, \mathbf{\Psi})\\ &+ (\mathfrak{R}_0\mathbf{\Psi}, \mathbf{\Psi}) + (N_r\tilde{\mathcal{R}}, \mathbf{\Psi}) + (N_r\mathbf{\Psi}, \tilde{\mathcal{R}}) + (N_r'\mathbf{\Psi}, \mathbf{\Psi}).\end{aligned} \tag{5.12}$$

We also have

$$|\mathrm{i}(\langle\nabla\rangle^r[\mathfrak{S}^*\mathfrak{D}\langle\nabla\rangle\mathfrak{S} - \mathfrak{S}^*\langle\nabla\rangle\mathfrak{D}^*\mathfrak{S}]\langle\nabla\rangle^r\mathbf{\Psi}, \mathbf{\Psi})| \le C_1\|\mathbf{\Psi}\|_r^2,$$

$$|(\mathfrak{R}_0\mathbf{\Psi}, \mathbf{\Psi})| \le C_2\|\mathfrak{R}_0\mathbf{\Psi}\|_{-r}\|\mathbf{\Psi}\|_r \le C_2\|\mathbf{\Psi}\|_r^2,$$

$$|(N_r\tilde{\mathcal{R}}, \mathbf{\Psi})| + |(N_r\mathbf{\Psi}, \tilde{\mathcal{R}})| \le C_3\|\mathbf{\Psi}\|_r\|\tilde{\mathcal{R}}\|_r \le C_4\|\mathbf{\Psi}\|_r^2 + \frac{1}{2}\|\tilde{\mathcal{R}}\|_r^2.$$

Here the constants C_i all depend on $\|\psi\|_r$. To estimate the last term in (5.12), observe that

$$N'(t) = \langle\nabla\rangle^r\partial[\mathfrak{B}^*(\psi)\mathfrak{B}(\psi)]\psi'\langle\nabla\rangle^r,$$

where here ∂ stands for the Frechét derivative. From (5.5) and (5.6), it is not hard to see that

$$\partial\mathcal{B}(\psi)\psi' \in \mathcal{S}^0_0(r-1,2).$$

Hence [69, Theorem 2.3] implies that

$$\partial\mathfrak{B}(\psi)\psi' = Op(\partial\mathcal{B}(\psi)\psi') \in \mathcal{L}(H^0).$$

As $\partial\mathfrak{B}^*(\psi)\psi' = [\partial\mathfrak{B}(\psi)\psi']^*$, we immediate conclude that

$$\partial[\mathfrak{B}^*(\psi)\mathfrak{B}(\psi)]\psi' \in \mathcal{L}(H^0).$$

Now it follows that

$$|(N_r'\mathbf{\Psi}, \mathbf{\Psi})| \leq C_6\|N_r'\mathbf{\Psi}\|_{-r}\|\mathbf{\Psi}\|_r \leq C_6\|\mathbf{\Psi}\|_r^2,$$

where C_6 depends on $\|\psi\|_{\mathbb{E}_1(I)}$. In summary,

$$\frac{d}{dt}(N_r\mathbf{\Psi}, \mathbf{\Psi}) \leq C_7\|\mathbf{\Psi}\|_r^2 + C_5\|\tilde{\mathcal{R}}\|_r^2 \tag{5.13}$$

with $C_7 = C_7(\|\psi\|_{\mathbb{E}_1(I)})$. As a direct conclusion from (5.9) and Grönwall's inequality, we finally conclude

$$\|\mathbf{\Psi}(t)\|_r^2 \leq \tilde{M}e^{t\omega(\|\psi\|_{\mathbb{E}_1(I)})}\left[\|\mathbf{\Psi}_0\|_r^2 + \int_0^t \|\tilde{\mathcal{R}}(\psi(s))\|_r^2\, ds\right]$$

where $\tilde{M}$ is a constant argument depending on $\|\psi\|_{L^\infty}$, and thus, on $\|\psi\|_{r-1}$ by Sobolev embedding. □

6. Local existence and uniqueness

We will now use the energy estimate of Proposition 5.4 to establish local well-posedness for the system (3.3).

6.1. **Approximating sequence.** We take a sequence of smooth initial data $\mathbf{\Psi}_{0,n} \to \mathbf{\Psi}_0$ in H^r with $r > 9/2$. We inductively consider the problem

$$\mathcal{F}(\mathbf{\Psi}_{n-1})\mathbf{\Psi}_n = \tilde{\mathcal{R}}(\mathbf{\Psi}_{n-1}), \quad \mathbf{\Psi}_n(0) = \mathbf{\Psi}_{0,n}. \tag{6.1}$$

Let $\|\mathbf{\Psi}_0\|_r^2 \leq K$. We may assume

$$\|\mathbf{\Psi}_{0,n}\|_r^2 \leq K + 1. \tag{6.2}$$

Further, we define continuous functions $\mathscr{K}_i : \mathbb{R}_+ \to \mathbb{R}_+$ with $i = 1, 2$ such that

$$\|\mathfrak{A}(\psi)\|_{\mathcal{L}(H^{r-1})} \leq \mathscr{K}_1(\|\psi\|_r)$$

and

$$\|\tilde{\mathcal{R}}(\psi)\|_s \leq \mathscr{K}_2(\|\psi\|_s), \quad s = r-1, r.$$

We now make the following inductive assumption

$$H(n-1) : \|\mathbf{\Psi}_k\|_{C(I;H^r)} \leq \mathcal{C}_1 \text{ and } \|\partial_t\mathbf{\Psi}_k\|_{C(I;H^{r-1})} \leq \mathcal{C}_2 \text{ for } k = 1, 2, \cdots, n-1.$$

Note that it follows from $H(n-1)$ and (6.2) that by choosing T small enough, we have

$$\|\mathbf{\Psi}_k(t)\|_{r-1} \leq M, \quad k = 1, 2, \cdots, n-1 \text{ and } t \in [0, T]$$

for some sufficiently large uniform constant M independent of $\mathcal{C}_i$. Consequently, we can take the constant $\tilde{M}$ in (5.3) to be uniform in the ensuing iteration argument.

Furthermore, we choose $\mathcal{C}_i$ in $H(n-1)$ sufficiently large so that

$$\sqrt{\tilde{M}(2K+4)} \le \mathcal{C}_1$$

and

$$M'\mathscr{K}_1(\mathcal{C}_1)\mathcal{C}_1 + \mathscr{K}_2(\mathcal{C}_1) \le \mathcal{C}_2,$$

where $M' = \|\langle\nabla\rangle\|_{\mathcal{L}(H^r,H^{r-1})}$. Now we will use (5.3) to estimate

$$\begin{aligned}\|\mathbf{\Psi}_n(t)\|_r^2 \le& \tilde{M}e^{t\omega(\|\mathbf{\Psi}_{n-1}\|_{\mathbb{E}_1(I)})}[\|\mathbf{\Psi}_{0,n}\|_r^2 + \int_0^t \|\tilde{\mathcal{R}}(\mathbf{\Psi}_{n-1}(s))\|_r^2\,ds]\\ \le& \tilde{M}e^{t\omega(\mathcal{C}_1+\mathcal{C}_2)}[K+1+t\mathscr{K}_2(\mathcal{C}_1)],\end{aligned}$$

By choosing T small enough, we can control

$$\|\mathbf{\Psi}_n(t)\|_r^2 \le \tilde{M}(2K+4) \quad \text{for all } t\in[0,T],$$

which gives

$$\|\mathbf{\Psi}_n\|_{C(I;H^r)} \le \mathcal{C}_1.$$

Plugging this estimate into (6.1) we obtain

$$\|\partial_t\mathbf{\Psi}_n(t)\|_{r-1} \le M'\mathscr{K}_1(\mathcal{C}_1)\mathcal{C}_1 + \mathscr{K}_2(\mathcal{C}_1) \le \mathcal{C}_2.$$

This completes the verification of $H(n)$, and we conclude that

$$\|\mathbf{\Psi}_n\|_{\mathbb{E}_1(I)} \le \mathcal{C} \tag{6.3}$$

for all n and some $\mathcal{C} > 0$.

6.2. **Energy estimate for the difference of two solutions.** For $i = 1, 2$, we consider

$$\mathcal{F}(\psi_i)\tilde{w}_i = \tilde{\mathcal{R}}(\psi_i), \quad \tilde{w}_i(0) = \tilde{w}_{0,i}.$$

Set $\psi = \psi_2 - \psi_1$ and $\tilde{w} = \tilde{w}_2 - \tilde{w}_1$. Take the difference of the above two systems to obtain

$$\partial_t\tilde{w} = \mathfrak{U}(\psi_2)\tilde{w} + [\mathfrak{U}(\psi_1) - \mathfrak{U}(\psi_2)]\tilde{w}_1 + \tilde{\mathcal{R}}(\psi_2) - \tilde{\mathcal{R}}(\psi_1), \quad \tilde{w}(0) = \tilde{w}_{0,2} - \tilde{w}_{0,1}. \tag{6.4}$$

Let

$$\mathfrak{F} = [\mathfrak{U}(\psi_1) - \mathfrak{U}(\psi_2)]\tilde{w}_1 + \tilde{\mathcal{R}}(\psi_2) - \tilde{\mathcal{R}}(\psi_1)$$

and

$$\mathbb{E}_0(I) := C(I;H^{r-1}) \cap C^1(I;H^{r-2}).$$

By (5.3), we have

$$\|\tilde{w}(t)\|_{r-1}^2 \le \tilde{M}e^{t\omega}[\|\tilde{w}_{0,2} - \tilde{w}_{0,1}\|_{r-1}^2 + \int_0^t \|\mathfrak{F}(s)\|_{r-1}^2\,ds],$$

where $\tilde{M} = \tilde{M}(\|\psi_2\|_{r-2})$ and $\omega = \omega(\|\psi_2\|_{\mathbb{E}_0(I)})$. Estimate

$$\begin{aligned}\|[\mathfrak{U}(\psi_1) - \mathfrak{U}(\psi_2)]\tilde{w}_1\|_{r-1} \le& \int_0^1 \|\partial\mathfrak{U}(s\psi_1 + (1-s)\psi_2)(\psi)\tilde{w}_1\|_{r-1}ds\\ \le& \int_0^1 \|\partial\mathfrak{A}(s\psi_1 + (1-s)\psi_2)(\psi)\langle\nabla\rangle\tilde{w}_1\|_{r-1}ds\\ \le& \mathscr{K}(\|\psi_1\|_{r-1} + \|\psi_2\|_{r-1})\|\psi\|_{r-1}\|\tilde{w}_1\|_r.\end{aligned}$$

Similarly,

$$\|\tilde{\mathcal{R}}(\psi_2)-\tilde{\mathcal{R}}(\psi_1)\|_{r-1}\leq \mathscr{K}(\|\psi_1\|_{r-1}+\|\psi_2\|_{r-1})\|\psi\|_{r-1}.$$

This produces

$$\begin{aligned}\|\tilde{w}(t)\|^2_{r-1}\leq&\tilde{M}e^{t\omega}[\|\tilde{w}_{0,2}-\tilde{w}_{0,1}\|^2_{r-1}\\&+t(1+\|\tilde{w}_1\|^2_r)\mathscr{K}(\|\psi_1\|_{r-1}+\|\psi_2\|_{r-1})\|\psi\|^2_{r-1}].\end{aligned}\tag{6.5}$$

Using (6.4), we further have

$$\|\partial_t\tilde{w}\|_{r-2}\leq\mathscr{K}(\|\psi_2\|_{r-2})\|\tilde{w}\|_{r-1}+\|\mathfrak{F}\|_{r-2}.\tag{6.6}$$

6.3. **Convergence of the iterates.** Now we choose $\psi_2=\tilde{w}_1=\mathbf{\Psi}_{n-1}$, $\psi_1=\mathbf{\Psi}_{n-2}$ and $\tilde{w}_2=\mathbf{\Psi}_n$. Note that as in Section 6.1, the constant $\tilde{M}$ in (6.5) can be taken to be independent of n. Estimates (6.5) and (6.6) imply

$$\begin{aligned}&\|\mathbf{\Psi}_n-\mathbf{\Psi}_{n-1}\|_{\mathbb{E}_0(I)}\\\leq&\sqrt{\tilde{M}}e^{\frac{T}{2}\omega(\mathcal{C})}[\|\mathbf{\Psi}_{0,n}-\mathbf{\Psi}_{0,n-1}\|_{r-1}+\sqrt{T}(1+\mathcal{C})\mathscr{K}_I(\mathcal{C})\|\mathbf{\Psi}_{n-1}-\mathbf{\Psi}_{n-2}\|_{\mathbb{E}_0(I)}]\\&+\mathscr{K}_I(\mathcal{C})\sqrt{\tilde{M}}e^{\frac{T}{2}\omega(\mathcal{C})}[\|\mathbf{\Psi}_{0,n}-\mathbf{\Psi}_{0,n-1}\|_{r-1}+\sqrt{T}(1+\mathcal{C})\mathscr{K}_I(\mathcal{C})\|\mathbf{\Psi}_{n-1}-\mathbf{\Psi}_{n-2}\|_{\mathbb{E}_0(I)}]\\&+\mathscr{K}_I(\mathcal{C})\sup_{t\in I}\|\mathbf{\Psi}_{n-1}(t)-\mathbf{\Psi}_{n-2}(t)\|_{r-2}.\end{aligned}$$

In the last line, we can use (6.5) once more to obtain

$$\begin{aligned}&\sup_{t\in I}\|\mathbf{\Psi}_{n-1}(t)-\mathbf{\Psi}_{n-2}(t)\|_{r-1}\\\leq&\sqrt{\tilde{M}}e^{\frac{T}{2}\omega(\mathcal{C})}[\|\mathbf{\Psi}_{0,n-1}-\mathbf{\Psi}_{0,n-2}\|_{r-1}+\sqrt{T}(1+\mathcal{C})\mathscr{K}_I(\mathcal{C})\|\mathbf{\Psi}_{n-2}-\mathbf{\Psi}_{n-3}\|_{\mathbb{E}_0(I)}].\end{aligned}$$

We can choose T small and $(\mathbf{\Psi}_{0,n})$ in such a way that

$$(1+\mathscr{K}_I(\mathcal{C}))\sqrt{\tilde{M}}e^{\frac{T}{2}\omega(\mathcal{C})}(\|\mathbf{\Psi}_{n-1}(t)-\mathbf{\Psi}_{n-2}(t)\|_{r-1}+\|\mathbf{\Psi}_{0,n-1}-\mathbf{\Psi}_{0,n-2}\|_{r-1})\leq 2^{-n},$$

$$\sqrt{\tilde{M}}e^{\frac{T}{2}\omega(\mathcal{C})}\sqrt{T}(1+\mathcal{C})\mathscr{K}_I^2(\mathcal{C})\leq 1/16,$$

and

$$\sqrt{\tilde{M}}e^{\frac{T}{2}\omega(\mathcal{C})}\sqrt{T}(1+\mathcal{C})(\mathscr{K}_I^2(\mathcal{C})+\mathscr{K}_I(\mathcal{C}))\leq 1/4.$$

Set $a_n=\|\mathbf{\Psi}_n-\mathbf{\Psi}_{n-1}\|_{\mathbb{E}_0(I)}$. We then have

$$a_n\leq 2^{-n}+a_{n-1}/4+a_{n-2}/16.$$

Using induction, it follows that

$$a_n\leq\frac{s_n}{2^{2n-3}}+\frac{F_n}{2^{2n-4}}a_2+\frac{F_{n-1}}{2^{2n-2}}a_1,\tag{6.7}$$

where F_n is the $n-$th term of Fibonacci sequence (starting from 0) and

$$s_n=2^{n-3}+s_{n-1}+s_{n-2}.$$

Letting $b=(1-\sqrt{5})/2$, we obtain the following identities

$$\begin{aligned}s_n-bs_{n-1}&=2^{n-3}+(1-b)(s_{n-1}-bs_{n-2})\\b(s_{n-1}-bs_{n-2})&=2^{n-4}b+b(1-b)(s_{n-2}-bs_{n-3})\end{aligned}$$

$$\vdots$$
$$b^{n-3}(s_3 - bs_2) = b^{n-3} + b^{n-3}(1-b)(s_2 - bs_1).$$

We sum these expressions to conclude

$$s_n - b^{n-2}s_2 = \sum_{k=0}^{n-3} 2^k b^{n-3-k} + (1-b)(s_{n-1} - b^{n-2}s_1).$$

We carry out a similar computation and sum to obtain

$$\begin{aligned} s_n - (1-b)s_{n-1} =& \sum_{k=0}^{n-3} 2^k b^{n-3-k} + b^{n-2}s_2 - (1-b)b^{n-2}s_1 \\ (1-b)(s_{n-1} - (1-b)s_{n-2}) =& (1-b)\sum_{k=0}^{n-4} 2^k b^{n-4-k} + (1-b)b^{n-3}s_2 - (1-b)^2 b^{n-3}s_1 \\ & \vdots \\ (1-b)^{n-3}(s_3 - (1-b)s_2) =& (1-b)^{n-3} + (1-b)^{n-3}bs_2 - (1-b)^{n-2}bs_1. \end{aligned}$$

This yields

$$\begin{aligned} s_n - (1-b)^{n-2}s_2 =& [\sum_{k=0}^{n-3} 2^k b^{n-3-k} + (1-b)\sum_{k=0}^{n-4} 2^k b^{n-4-k} + \cdots + (1-b)^{n-3}] \\ & + s_2 \sum_{k=1}^{n-2} b^k (1-b)^{n-2-k} + s_1 \sum_{k=1}^{n-2} b^{n-1-k}(1-b)^k \end{aligned}$$

and thus

$$s_n \leq (n-2)2^{n-3} + F_n s_2 \leq (n-2)2^{n-3} + 2^{n-3}s_2,$$

where we have used that $(1-b)^k = F_{k+2} - F_{k+1}b$. Plugging this expression into (6.7) gives

$$a_n \leq \frac{n-2}{2^n} + \frac{s_2}{2^n} + \frac{a_2}{2^{n-4}} + \frac{a_1}{2^{n-2}}.$$

Therefore

$$\|\mathbf{\Psi}_n - \mathbf{\Psi}_{n+j}\|_{\mathbb{E}_0(I)} \leq \|\mathbf{\Psi}_n - \mathbf{\Psi}_{n+1}\|_{\mathbb{E}_0(I)} + \cdots \|\mathbf{\Psi}_{n+j-1} - \mathbf{\Psi}_{n+j}\|_{\mathbb{E}_0(I)}$$

can be made arbitrarily small by taking n large. We thus conclude that $\{\mathbf{\Psi}_n\}$ is Cauchy in $C(I; H^{r-1}) \cap C^1(I; H^{r-2})$ and therefore converges in this space. We denote the limit by $\mathbf{\Psi} \in C(I; H^{r-1}) \cap C^1(I; H^{r-2})$. We can let $n \to \infty$ in (6.1) and thus $\mathbf{\Psi}$ satisfies

$$\mathcal{F}(\mathbf{\Psi})\mathbf{\Psi} = \tilde{\mathcal{R}}(\mathbf{\Psi}), \quad \mathbf{\Psi}(0) = \mathbf{\Psi}_0.$$

Finally, it follows from (6.3) that

$$\|\mathbf{\Psi}(t)\|_r + \|\partial_t \mathbf{\Psi}(t)\|_{r-1} \leq \mathcal{C}, \quad t \in [0, T].$$

It remains to prove uniqueness. But this follows at once since we have an estimate for the difference of two solutions.

6.4. **Continuity of solution.** The proof of time-continuity of the solution with respect to the top norm follows a standard procedure: first, we prove weak continuity; then, we obtain strong continuity by showing continuity of the norm.

The weak continuity of the solution $\boldsymbol{\Psi}$ can be proved by a similar argument to that of quasilinear wave equations, since in that proof the structure of the equation is not necessary but only the convergence $\boldsymbol{\Psi}_n \to \boldsymbol{\Psi}$ in $C(I;H^{r-1}) \cap C^1(I;H^{r-2})$ and an estimate of the form (6.3) are used.

We put

$$\mathfrak{K}(t) = \sqrt{\frac{C_0}{2} I + (\mathfrak{B}(\boldsymbol{\Psi}(t)))^* \mathfrak{B}(\boldsymbol{\Psi}(t))}$$

and

$$\mathcal{A}_r(t) = \mathcal{A}_r(\boldsymbol{\Psi}(t)) = \mathfrak{K}(t)\langle\nabla\rangle^r.$$

Hence

$$N_r(t) = N_r(\boldsymbol{\Psi}(t)) = \mathcal{A}_r(\boldsymbol{\Psi}(t))^* \mathcal{A}_r(\boldsymbol{\Psi}(t)).$$

Recall that $\mathfrak{B} \in OP\mathcal{S}^0_0(r,2)$. From [69, Theorems 2.2 and 2.4], it follows that

$$\mathfrak{K}(t) \in \mathcal{L}(H^s), \quad -r < s < r-1. \tag{6.8}$$

Fix $t_0 \in [0,T]$. We will show that $\mathcal{A}_r(t_0)\boldsymbol{\Psi}(t)$ is weakly continuous in H^0. Given any $\epsilon > 0$ and $\phi \in H^0$, take a sequence of Schwartz's functions $\phi_j \to \phi$ in H^0. Then

$$\begin{aligned}&(\mathcal{A}_r(t_0)\boldsymbol{\Psi}(t) - \mathcal{A}_r(t_0)\boldsymbol{\Psi}_n(t), \phi)\\ =&(\mathcal{A}_r(t_0)\boldsymbol{\Psi}(t) - \mathcal{A}_r(t_0)\boldsymbol{\Psi}_n(t), \phi - \phi_j) + (\mathcal{A}_r(t_0)\boldsymbol{\Psi}(t) - \mathcal{A}_r(t_0)\boldsymbol{\Psi}_n(t), \phi_j)\end{aligned}$$

In view of (6.3), the first term on the RHS is bounded by

$$|(\mathcal{A}_r(t_0)\boldsymbol{\Psi}(t) - \mathcal{A}_r(t_0)\boldsymbol{\Psi}_n(t), \phi - \phi_j)| \leq \mathscr{K}(\mathcal{C})\|\phi - \phi_j\|_0,$$

so that this term can be made less than $\epsilon/2$ upon choosing j large enough. Next, fixing j in the second term, we have

$$\begin{aligned}&|(\mathcal{A}_r(t_0)\boldsymbol{\Psi}(t) - \mathcal{A}_r(t_0)\boldsymbol{\Psi}_n(t), \phi_j)|\\ =&|(\langle\nabla\rangle^{r-1}(\boldsymbol{\Psi}(t) - \boldsymbol{\Psi}_n(t)), \langle\nabla\rangle\mathfrak{K}^*(t)\phi_j)|\end{aligned}$$

Because $\boldsymbol{\Psi}_n \to \boldsymbol{\Psi}$ in $C(I;H^{r-1})$, invoking [69, Theorem 2.4] and (6.8), we obtain

$$|(\mathcal{A}_r(t_0)\boldsymbol{\Psi}(t) - \mathcal{A}_r(t_0)\boldsymbol{\Psi}_n(t), \phi_j)| < \epsilon/2$$

for all $n \geq n_0$ with some large enough n_0. Hence,

$$|(\mathcal{A}_r(t_0)\boldsymbol{\Psi}(t) - \mathcal{A}_r(t_0)\boldsymbol{\Psi}_n(t), \phi)| < \epsilon \quad \text{for all } n \geq n_0 \text{ and } t \in [0,T],$$

showing that $\mathcal{A}_r(t_0)\boldsymbol{\Psi}_n(t)$ converges to $\mathcal{A}_r(t_0)\boldsymbol{\Psi}(t)$ uniformly in t in the weak topology. Thus, $\mathcal{A}_r(t_0)\boldsymbol{\Psi}(t)$ is weakly continuous in t with respect to the norm of H^0.

We are now ready to show that $\boldsymbol{\Psi} \in C(I;H^r)$. In view of the weak continuity of $\boldsymbol{\Psi}(t)$, it suffices to demonstrate that the map

$$[t \mapsto \|\boldsymbol{\Psi}(t)\|_r] \quad \text{is continuous.}$$

Applying (5.13) to (5.1) and using(6.3), we conclude that

$$\frac{d}{dt}\|\mathcal{A}_r(t)\boldsymbol{\Psi}(t)\|_0^2 \leq \mathscr{K}(\mathcal{C}).$$

This implies that

$$\|\mathcal{A}_r(t)\mathbf{\Psi}(t)\|_0^2 =: Y(t) \quad \text{is Lipschitz continuous in } t. \tag{6.9}$$

Consider

$$\begin{aligned}&\|\mathcal{A}_r(t_0)\mathbf{\Psi}(t)\|_0^2 - \|\mathcal{A}_r(t_0)\mathbf{\Psi}(t_0)\|_0^2\\ =&(\|\mathcal{A}_r(t_0)\mathbf{\Psi}(t)\|_0^2 - \|\mathcal{A}_r(t)\mathbf{\Psi}(t)\|_0^2) + (\|\mathcal{A}_r(t)\mathbf{\Psi}(t)\|_0^2 - \|\mathcal{A}_r(t_0)\mathbf{\Psi}(t_0)\|_0^2).\end{aligned}$$

The first term on the RHS can be estimated as follows.

$$\begin{aligned}&|(\|\mathcal{A}_r(t_0)\mathbf{\Psi}(t)\|_0^2 - \|\mathcal{A}_r(t)\mathbf{\Psi}(t)\|_0^2)|\\ =&|\|\mathcal{A}_r(t_0)\mathbf{\Psi}(t)\|_0 - \|\mathcal{A}_r(t)\mathbf{\Psi}(t)\|_0|(\|\mathcal{A}_r(t_0)\mathbf{\Psi}(t)\|_0 + \|\mathcal{A}_r(t)\mathbf{\Psi}(t)\|_0)\\ \le&\mathscr{K}(\mathcal{C})\|(\mathfrak{K}(t) - \mathfrak{K}(t_0))\langle\nabla\rangle^r\mathbf{\Psi}(t)\|_0\\ \le&\mathscr{K}(\mathcal{C})\|\mathfrak{K}(t) - \mathfrak{K}(t_0)\|_{\mathcal{L}(H^0)}.\end{aligned}$$

As elements in $\mathcal{L}(H^0)$, it is not hard to check that $\mathfrak{K}(\mathbf{\Psi})$ depends continuously on $\|\mathbf{\Psi}\|_{r-1}$. Combining with (6.9), this observation shows that $[t \mapsto \|\mathcal{A}_r(t_0)\mathbf{\Psi}(t)\|_0]$ is continuous at t_0; and thus

$$\mathcal{A}_r(t_0)\mathbf{\Psi}(t) \text{ is continuous in } t \text{ at } t_0 \text{ w.r.t. } H^0.$$

Since t_0 is arbitrary, from

$$\|\mathbf{\Psi}(t) - \mathbf{\Psi}(t_0)\|_r^2 \le C\|\mathcal{A}_r(t_0)\mathbf{\Psi}(t) - \mathcal{A}_r(t_0)\mathbf{\Psi}(t_0)\|_0^2,$$

we obtain that $\mathbf{\Psi} \in C(I; H^r)$. Using this fact and equation (6.1), we immediately conclude

$$\mathbf{\Psi} \in C(I; H^r) \cap C^1(I, H^{r-1}).$$

7. Solution to the original system

It remains to prove that the solution $\mathbf{\Psi}$ that we obtained for the system (3.3) yields a solution to (2.2) and (2.4). The argument follows a known approximation argument by analytic functions, so the main task is to show that the analytic Cauchy problem for (2.2) and (2.4) can be solved. In fact, due to the compactness of $\mathbb{T}^3$ and some minor technical points related to a localization procedure, we work in a slightly larger space than that of analytic functions, namely, Gevrey spaces.

Remark 7.1. In the introduction, we alluded to a local well-posendess in Gevrey spaces for equations (2.2) and (2.4). However, what was actually established in [15] was the local well-posedness in Gevrey spaces for equations (3.1), i.e., the projection onto the spaces orthogonal and parallel to u of equations (2.2) and (2.4). It was not showed in [15] that a Gevrey solution to (3.1) implies a Gevrey solution to (2.2) and (2.4). Because we do need Gevrey solutions to (2.2) and (2.4) to carry out the aforementioned approximation argument, we cannot rely on [15] and, therefore, we establish the desired result here.

Remark 7.2. In practice (e.g., when implementing numerical simulations), physicists do not use equations (2.2) and (2.4), adopting instead (3.1) as the starting point. Therefore, for applications in physics, the result obtained in [15] (see previous Remark) is enough as far as only Gevrey regularity is concerned.

7.1. **The original equations in explicit form.** Equation (2.4) reads

$$(\varepsilon + A_1 + A_2 + P)(u^\mu \nabla_\nu u^\nu + u^\nu \nabla_\nu u^\mu) + u^\mu u^\nu \nabla_\nu(\varepsilon + A_1) + \Pi^{\mu\nu}\nabla_\nu(P + A_2)$$
$$- 2\sigma^{\mu\nu}\nabla_\nu \eta - 2\eta \nabla_\nu \sigma^{\mu\nu} + Q^\mu \nabla_\nu u^\nu + u^\nu \nabla_\nu Q^\mu + Q^\nu \nabla_\nu u^\mu + u^\mu \nabla_\nu Q^\nu = 0.$$

Applying $u^\mu \nabla_\mu$ twice to (2.2) produces:

$$u_\nu u^\alpha u^\beta \nabla_\alpha \nabla_\beta u^\nu + u^\alpha u^\beta (\nabla_\alpha u_\nu)(\nabla_\beta u^\nu) = 0.$$

We may rewrite the above equations to obtain the complete set of equations given by

$$u_\nu u^\alpha u^\beta \nabla_\alpha \nabla_\beta u^\nu = B(\partial u), \tag{7.1a}$$
$$B^{\mu\alpha\beta}\partial_\alpha \partial_\beta \varepsilon + B^{\mu\alpha\beta}_\nu \partial_\alpha \partial_\beta u^\nu = B^\mu(\partial\varepsilon, \partial u, \partial g), \tag{7.1b}$$

where

$$B^{\mu\alpha\beta} = \frac{\chi_1 u^\mu u^\alpha u^\beta + (\chi_3 + \lambda c_s^2) u^{(\alpha}\Pi^{\beta)\mu} + \lambda c_s^2 u^\mu \Pi^{\alpha\beta}}{(\varepsilon + P)}$$
$$B^{\mu\alpha\beta}_\nu = (\chi_2 + \lambda) u^\mu u^{(\alpha}\delta^{\beta)}_\nu + (\chi_4 - \frac{\eta}{3})\Pi^{\mu(\alpha}\delta^{\beta)}_\nu - \eta \Pi^{\alpha\beta}\delta^\mu_\nu + \lambda u^\alpha u^\beta \delta^\mu_\nu,$$

$B(\partial u)$ and $B^\mu(\partial\varepsilon, \partial u)$ are analytic functions of ε, u, and their first order derivatives and contain no term in second order derivatives of ε or u. By constructing the vector $U = (\varepsilon, u^\alpha) \in \mathbb{R}^5$, we may write the above equations in matrix form as $\mathfrak{m}^{\alpha\beta}\partial^2_{\alpha\beta} U = \mathcal{B}$, where $\mathcal{B} = (B, B_\mu) \in \mathbb{R}^5$ and

$$\mathfrak{m}^{\alpha\beta} = \begin{bmatrix} 0 & u^\alpha u^\beta u_\nu \\ B^{\mu\alpha\beta} & B^{\mu\alpha\beta}_\nu \end{bmatrix}.$$

7.2. **The characteristic determinant.** We will here compute the characteristic determinant of equations (7.1). Let ξ be an arbitrary co-vector in spacetime. To simplify the notation, denote $\mathfrak{a} := u^\mu \xi_\mu$ and $\mathfrak{b}^\mu := \Pi^{\mu\nu}\xi_\nu$, and

$$B^\mu := B^{\mu\alpha\beta}\xi_\alpha \xi_\beta = \frac{\chi_1 u^\mu \mathfrak{a}^2 + (\chi_3 + \lambda c_s^2)\mathfrak{a}\mathfrak{b}^\mu + \lambda c_s^2 u^\mu \mathfrak{b}^\alpha \mathfrak{b}_\alpha}{(\varepsilon + P)},$$
$$B^\mu_\nu := B^{\mu\alpha\beta}_\nu \xi_\alpha \xi_\beta = (\mathfrak{a}(\chi_2 + \lambda)u^\mu + (\chi_4 - \frac{\eta}{3})\mathfrak{b}^\mu)\xi_\nu + (\lambda \mathfrak{a}^2 - \eta \mathfrak{b}^\alpha \mathfrak{b}_\alpha)\delta^\mu_\nu.$$

1 Also, let us define

$$C \equiv [C^A_B]_{5\times 5} := \mathfrak{m}^{\alpha\beta}\xi_\alpha \xi_\beta = \begin{bmatrix} 0 & \mathfrak{a}^2 u_\nu \\ B^\mu & B^\mu_\nu \end{bmatrix},$$

where $A, B = \bar{1}, \mu = \bar{1}, 0, 1, 2, 3$, with $\bar{1}$ denoting the first line or column index. Then, $C^{\bar{1}}_{\bar{1}} = 0$, $C^{\bar{1}}_\nu = \mathfrak{a}^2 u_\nu$, $C^\mu_{\bar{1}} = B^\mu$, and $C^\mu_\nu = B^\mu_\nu$. Using the Levi-Civita symbol ϵ^{ABCDE} with $\epsilon^{\bar{1}0123} = \epsilon_{\bar{1}0123} = 1$, where $\epsilon^{ABCDE}\epsilon_{FGHIJ} = 5!\delta^A_{[F}\delta^B_G\delta^C_H\delta^D_I\delta^E_{J]}$ with the bracket $_{[\dots]}$ being the anti-symmetrization of the indexes $FGHI$, we obtain that

$$\det(\mathfrak{m}^{\alpha\beta}\xi_\alpha\xi_\beta) = \det(C) = \frac{\epsilon_{A_1A_2A_3A_4A_5}\epsilon^{B_1B_2B_3B_4B_5}}{5!} C^{A_1}_{B_1} C^{A_2}_{B_2} C^{A_3}_{B_3} C^{A_4}_{B_4} C^{A_5}_{B_5}$$
$$= C^{\bar{1}}_\nu \frac{\epsilon^{\nu B_2 B_3 B_4 B_5}\epsilon_{\bar{1}\mu_2\mu_3\mu_4\mu_5}}{4!} C^{\mu_2}_{B_2} C^{\mu_3}_{B_3} C^{\mu_4}_{B_4} C^{\mu_5}_{B_5} = -C^{\bar{1}}_\nu C^\mu_{\bar{1}} \frac{\epsilon^{\bar{1}\nu\nu_3\nu_4\nu_5}\epsilon_{\bar{1}\mu\mu_3\mu_4\mu_5}}{3!} C^{\mu_3}_{\nu_3} C^{\mu_4}_{\nu_4} C^{\mu_5}_{\nu_5}$$
$$= -\mathfrak{a}^2 u_\nu B^\mu \frac{\epsilon^{\bar{1}\nu\nu_2\nu_3\nu_4}\epsilon_{\bar{1}\mu\mu_2\mu_3\mu_4}}{3!} B^{\mu_2}_{\nu_2} B^{\mu_3}_{\nu_3} B^{\mu_4}_{\nu_4}$$

$$
\begin{aligned}
&= -\mathfrak{a}^2 u_\nu B^\mu (\lambda\mathfrak{a}^2 - \eta\mathfrak{b}^\alpha\mathfrak{b}_\alpha)^3 \frac{\epsilon^{\bar{1}\nu\mu_2\mu_3\mu_4}\epsilon_{\bar{1}\mu\mu_2\mu_3\mu_4}}{3!} \\
&- \mathfrak{a}^2 u_\nu B^\mu (\mathfrak{a}(\chi_2+\lambda)u^{\mu_2} + (\chi_4 - \frac{\eta}{3})\mathfrak{b}^{\mu_2})\xi_{\nu_2}(\lambda\mathfrak{a}^2 - \eta\mathfrak{b}^\alpha\mathfrak{b}_\alpha)^2 \frac{\epsilon^{\bar{1}\nu\nu_2\mu_3\mu_4}\epsilon_{\bar{1}\mu\mu_2\mu_3\mu_4}}{2!} \\
&= \frac{\mathfrak{a}^2}{\varepsilon+P}(\lambda\mathfrak{a}^2 - \eta\mathfrak{b}^\alpha\mathfrak{b}_\alpha)^2(\lambda(\chi_1\mathfrak{a}^2 - \chi_3\mathfrak{b}^\alpha\mathfrak{b}_\alpha)\mathfrak{a}^2 - \chi_2(\chi_3+\lambda c_s^2)\mathfrak{a}^2\mathfrak{b}^\alpha\mathfrak{b}_\alpha \\
&+ (\chi_1\mathfrak{a}^2 + \lambda c_s^2)(\chi_4 - \frac{4\eta}{3}\mathfrak{b}^\beta\mathfrak{b}_\beta)\mathfrak{b}^\alpha\mathfrak{b}_\alpha) \\
&= \frac{\lambda^3\chi_1}{\varepsilon+P}\prod_{a=1,2,\pm}((u^\alpha\xi_\alpha)^2 - \beta_a\Pi^{\alpha\beta}\xi_\alpha\xi_\beta)^{m_a},
\end{aligned}
$$

where $m_1 = 1$, $m_2 = 2$, and $m_\pm = 1$, while the β_a's are the same as the one obtained in (4.6).

We have already showed that conditions $\lambda, \eta, \chi_1 > 0$ together with (4.1)–(4.5) (which are the assumptions in Theorem 2.1) guarantee that $0 \le \beta_a \le 1$. Therefore, comparing $(u^\alpha\xi_\alpha)^2 - \beta_a\Pi^{\alpha\beta}\xi_\alpha\xi_\beta$ with the characteristics of an acoustical metric (see, e.g., [34]), we conclude that $\det(\mathfrak{m}^{\alpha\beta}\xi_\alpha\xi_\beta)$ is a product of hyperbolic polynomials.

Remark 7.3 (The system's characteristics)**.** Setting $\det(\mathfrak{m}^{\alpha\beta}\xi_\alpha\xi_\beta)$ equal to zero, we obtain the characteristics of the system (7.1). Not surprisingly, these are the same as the characteristics of equations (3.1) which have been computed in [15].

7.3. **Proof of Theorem 2.1.** . Let us group that unknowns ε, u^α in the 5-component vector $V = (\varepsilon, u^\alpha)$. To each component V^I we associate an index m_I, $I = 1, \dots, 5$, and to each one of the 5 equations (7.1a)-(7.1b) we associate an index n_J, in such a way that equations (7.1a)-(7.1b) can be written as

$$
h_I^J(\partial^{m_K - n_J - 1}V^K, \partial^{m_I - n_J})V^I + b^J(\partial^{m_K - n_J - 1}V^K) = 0, \tag{7.2}
$$

where $I, J = 1, \dots, 5$, $h_I^J(\partial^{m_K - n_J - 1}V^K, \partial^{m_I - n_J})$ is a homogeneous differential operator of order $m_I - n_J$ (which could possibly be zero) whose coefficients depend on at most $m_K - n_J - 1$ derivatives of V^K, $K = 1, \dots, 5$, and there is a sum over I in $h_I^J(\cdot)V^I$. The terms $b^J(\partial^{m_K - n_J - 1}V^K)$ also depend on at most $m_K - n_J - 1$ derivatives of V^K, $K = 1, \dots, 5$. The indices m_I and n_J are defined up to an overall additive constant, but the simplest choice to have equations (7.1a), (7.1b) written as (7.2) is $m_I = 2$, $n_J = 0$, for all $I, J = 1, \dots, 5$.

Using the fact that the characteristic determinant of (7.2) computed above is a product of hyperbolic polynomials, we conclude that (7.2) forms a Leray-Ohya system (see [29]). We can then apply theorems A.18 and A.23 of [29] (whose proofs can be found in [19, 63]) to conclude that equations (7.1) are locally well-posed in suitable[12] Gevrey spaces.

Remark 7.4. Theorems A.18 and A.23 are applicable to systems in $[0, T] \times \mathbb{R}^n$, whereas here we have $[0, T] \times \mathbb{T}^3$. Thus, one needs to carry out a localization and gluing argument before invoking these theorems. Such argument is possible due to the existence of a domain of dependence for solutions guaranteed by Theorem A.19 of [29]. The procedure is exactly the same as in [29] so we will not present it here.

[12] From the previously mentioned theorems, it is not difficult to see that one can be very precise about the quantitative properties of solutions, including the exact Gevrey regularity. Such details, however, are not important here for our argument.

Equation (7.1a) can be written as

$$u^\mu u^\nu \nabla_\mu \nabla_\nu (u_\alpha u^\alpha) = 0.$$

Viewing this as an equation for $u^\alpha u_\alpha$, we see that it forms a Leray-Ohya equation, so it admits unique solutions in Gevrey spaces. Therefore, we conclude that a solution to (7.2) satisfies $u^\alpha u_\alpha = -1$ provided that this condition holds initially, which is the by construction (see comment after the statement of Theorem 2.1).

The conclusion that $\mathbf{\Psi}$ yields a solution to (2.2) and (2.4) now follows from a known approximation argument, so we will be brief.

Consider the initial data $\mathcal{I} = (\varepsilon_{(0)}, \varepsilon_{(1)}, u_{(0)}, u_{(1)}) \in H^r$ for (2.2)-(2.4) and let $\mathcal{I}_k$ be a sequence of Gevrey regular data converging to $\mathcal{I}$ in H^r. For each k, let $V_k = (\varepsilon_k, u_k)$ be the Gevrey regular solution to (2.2)-(2.4) with data $\mathcal{I}_k$, whose existence is ensured by the foregoing discussion. In view of the way (3.3) was derived from (2.2)-(2.4), for each k, we obtain a Gevrey regular solution $\mathbf{\Psi}_k$ to (3.3), with $\mathbf{\Psi}_k$ defined in terms of V_k according to the definitions of Section 3.

Let $\mathbf{\Psi}_0$ be initial data for (3.3) constructed out of $\mathcal{I}$, i.e., we define $\mathbf{\Psi}_0$ in terms of $\mathcal{I}$ using the definitions of Section 3. This is possible since the entries of $\mathbf{\Psi}_0$ will be simple algebraic expressions in terms of $\mathcal{I}$.

Let $\mathbf{\Psi}$ be the solution to (3.3) with data $\mathbf{\Psi}_0$. Note that we do not assume that $\mathbf{\Psi}$ is given in terms of the original fluid variables via the relations of Section 3 since at this point we do not yet have a solution to (2.2)-(2.4) with data $\mathcal{I}$. In other words, the entries of $\mathbf{\Psi}$ are treated as independent variables; at this point the only relation between $\mathbf{\Psi}$ and the original system (2.2)-(2.4) is that $\mathbf{\Psi}_0$ is constructed out of $\mathcal{I}$.

The estimates for solutions to (3.3) derived in Section 5 combined with the estimates for the difference of solutions in Section 6.2, imply that as $\mathcal{I}_k \to \mathcal{I}$ in H^r, $\mathbf{\Psi}_k$ converges to $\mathbf{\Psi}$, and thus the solutions V_k to (2.2)-(2.4) converge to a limit V in H^r. Since $r > 9/2$, we can pass to the limit in the equations (2.2)-(2.4) satisfied by V_k to conclude that V solves (2.2)-(2.4) as well (and that $\mathbf{\Psi}$ is in fact given in terms of V by the same expressions that define $\mathbf{\Psi}_k$ in terms of V_k). By construction, V takes the data $\mathcal{I}$.

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