Towards a Unified Theory of Rational Number Arithmetic

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Abstract

To explain children's difficulties learning fraction arithmetic, Braithwaite, Pyke, and Siegler (2017) proposed FARRA, a theory of fraction arithmetic implemented as a computational model. The present study tested predictions of the theory in a new domain, decimal arithmetic, and investigated children's use of conceptual knowledge in that domain. Sixth and eighth grade children (N = 92) solved decimal arithmetic problems while thinking aloud and afterward explained solutions to decimal arithmetic problems. Consistent with the hypothesis that FARRA's theoretical assumptions would generalize to decimal arithmetic, results supported three predictions derived from the model: (1) accuracies on different types of problems paralleled the frequencies with which the problem types appeared in textbooks; (2) most errors involved overgeneralization of strategies that would be correct for problems with different operations or types of number; and (3) individual children displayed patterns of strategy use predicted by FARRA. We also hypothesized that during routine calculation, overt reliance on conceptual knowledge is most likely among children who lack confidence in their procedural knowledge. Consistent with this hypothesis, (4) many children displayed conceptual knowledge when explaining solutions but not while solving problems; (5) during problem-solving, children who more often overtly used conceptual knowledge also displayed doubt more often; and (6) problem solving accuracy was positively associated with displaying conceptual knowledge while explaining, but not with displaying conceptual knowledge while solving problems. We discuss implications of the results for rational number instruction and for the creation of a unified theory of rational number arithmetic.

Towards a Unified Theory of Rational Number Arithmetic

Rational number arithmetic is a foundational mathematical skill whose importance is difficult to overstate. It is pervasive in everyday tasks such as calculating the total cost of a $2.99 gallon of milk and two $2.75 loaves of bread, the number of 1½'×2' tiles needed to cover a 6'×12' floor, and the Celsius equivalent of 37 degrees Fahrenheit. It also is used frequently in all later taught areas of mathematics; in a variety of common occupations such as pharmacy, nursing, and modern factory work (Douglas & Attewell, 2017; Handel, 2016); and in all areas of physical, biological, and social science.

The vast majority of research on arithmetic development has focused on whole number arithmetic. However, as the above examples illustrate, many arithmetic tasks involve fractions or decimals. Unfortunately, rational number arithmetic presents large and persistent difficulty for many children (e.g., for fraction arithmetic: Byrnes & Wasik, 1991; Gabriel et al., 2013; Hansen et al., 2015; Hecht & Vagi, 2012; Mack, 1995; Newton, Willard, & Teufel, 2014; and Siegler, Thompson, & Schneider, 2011; for decimal arithmetic: Hiebert & Wearne, 1985; Hurst & Cordes, 2018; Kouba et al., 1988; Lortie-Forgues & Siegler, 2017; Ren & Gunderson, 2021; Rittle-Johnson & Koedinger, 2009). For example, in two studies that examined knowledge of all four arithmetic operations, sixth and eighth graders in Siegler and Pyke (2013) correctly answered only 52% of fraction arithmetic problems, and sixth graders in Tian, Braithwaite, and Siegler (2021) correctly answered only 57% of decimal arithmetic problems.

To account for children's performance—especially their difficulties—with fraction arithmetic, Braithwaite, Pyke, and Siegler (2017) proposed FARRA (Fraction Arithmetic Reflects Rules and Associations), a theory of fraction arithmetic implemented as a computational model. The model generated eight phenomena that had been previously observed in empirical studies of children's fraction arithmetic, providing initial evidence for the theory. FARRA also generated several novel empirical predictions that were supported in subsequent studies of children's fraction arithmetic (Braithwaite et al., 2018, 2019; Braithwaite & Siegler, 2018).

The present study extended this work in two ways. First, we hypothesized that the theoretical assumptions underlying FARRA would generalize beyond the domain of fraction arithmetic. To evaluate this hypothesis, we tested several predictions derived from FARRA in a new domain, decimal arithmetic. Second, we investigated children's use of conceptual knowledge, a type of knowledge that was not included in FARRA. We tested predictions based on the hypothesis that when solving routine decimal arithmetic problems, children overtly use conceptual knowledge mainly when in doubt about their procedural knowledge. By extending FARRA to a new domain and clarifying the role of conceptual knowledge in that domain, we hoped to lay empirical foundations for a comprehensive theory of rational number arithmetic.

The FARRA Theory of Fraction Arithmetic

Below, we describe some of FARRA's main predictions about fraction arithmetic, the theoretical assumptions on which the predictions were based, and empirical evidence supporting the predictions.

Prediction 1. Accuracies on different types of problem parallel the frequencies with which the problem types are encountered during practice.

FARRA assumes that children learn from the distributional characteristics of their environment, a mechanism known as statistical learning (Perruchet & Pacton, 2006; Saffran et al., 1996). In particular, the theory assumes that children associate correct procedures strongly with types of problems that are encountered frequently and weakly with types of problems that are encountered rarely. To test this assumption, Braithwaite et al. (2017; see also Braithwaite &

Siegler, 2018) analyzed the distributions of fraction arithmetic problems in commercial math textbooks. Certain problem types proved particularly rare and, as predicted by FARRA, children performed more poorly on those problem types than on more common ones. For example, equal-denominator multiplication problems (e.g., 3/5×1/5) accounted for only 1% of all fraction arithmetic problems in the textbooks; in contrast, unequal denominator multiplication problems (e.g., 3/5×1/4) accounted for 29% of the problems. Analogously, children were considerably less accurate on equal-denominator than on unequal-denominator fraction multiplication problems (e.g., 37% vs. 58% in Siegler and Pyke, 2013), despite the same procedure producing accurate performance on both.

Prediction 2. Most errors result from overgeneralization of strategies that would be appropriate for other arithmetic operations or types of number.

FARRA, like many theories of skill learning (e.g., Repair Theory, Brown & VanLehn, 1980; SimStudent, MacLellan et al., 2014), posits that students broadly generalize strategies to solve novel problems. Such generalization can be adaptive, as when a student, having learned to solve 2/4+1/4 by passing through the common denominator into the answer, uses the same strategy for 2/5+1/5. However, generalization can also generate errors, if a strategy that would be appropriate for some problems is used to solve a problem for which it is not appropriate. For example, a student might pass through a common denominator when multiplying fractions, as in 2/5×1/5 = 2/5. We refer to such errors as strategy overgeneralizations.

Within FARRA, strategy overgeneralization is the main source of children's errors. A computer model based on this assumption generated very similar specific errors to those displayed by children on a range of problems (Braithwaite et al., 2017). For 14 of the 16 problems presented to children in Siegler and Pyke (2013), the most common incorrect response generated by the model and by children was identical. The correlation between the frequencies of errors generated by the model and by children on each problem was $r = .878$, $p < .001$.

Prediction 3. Different children display distinct patterns of strategy use: consistent use of correct strategies, reliance on a single flawed strategy, or use of varied strategies.

FARRA assumes that individual differences in learning and decision-making parameters lead to individual differences in strategy use. This assumption is shared with Siegler's (1988a) Strategy Choice Model (a computational implementation of overlapping waves theory, Siegler, 1996), in which variation in a confidence criterion parameter generates variation in use of fact retrieval and counting strategies to solve whole number arithmetic problems. Braithwaite et al. (2019) demonstrated through computational simulations that when FARRA's free parameters were systematically varied, the model predicted several qualitatively distinct patterns of strategy use. One pattern was characterized by using a correct strategy on most or all problems. Other patterns were characterized by using one strategy—either the standard procedure for adding fractions or the standard procedure for multiplying fractions—on problems involving different arithmetic operations, resulting in correct answers on problems for which the strategy was appropriate and incorrect answers on problems for which the strategy was inappropriate. Another pattern was characterized by using varied strategies for each arithmetic operation. The predicted patterns all appeared, and jointly accounted for the performance of 90% of children in data from Siegler and Pyke (2013). Similar findings were obtained with a second dataset (Braithwaite et al., 2019).

Extending the Theory to Decimal Arithmetic

Although fractions and decimals are both rational number notations, fraction and decimal arithmetic involve quite different procedures. For example, to calculate 1/2+1/4, a child might convert the operands to a common denominator, then add the numerators while maintaining the common denominator (1/2+1/4 = 2/4 + 1/4 = 3/4). To calculate an equivalent decimal problem (0.5+0.25), the child might append a zero to the first addend and then add digits with the same place value (0.5+0.25 = 0.50+0.25 = 0.75). Such large differences between fraction and decimal arithmetic procedures raise the question of whether a single theory could account in a straightforward way for phenomena in both domains. However, FARRA's predictions were based not on assumptions about fractions in particular, but on general assumptions about statistical learning, generalization of strategies, and parametric variation among individuals. We therefore hypothesized that FARRA's predictions would also apply to decimal arithmetic.

Prediction 1. Accuracies on different types of decimal arithmetic problems parallel the frequencies with which the problem types are encountered during practice.

A recent study yielded evidence consistent with this prediction in the context of decimal arithmetic. Tian and colleagues (2021) found that the vast majority (95%) of decimal addition and subtraction problems in textbooks involved two decimal operands (DD), whereas most (61%) decimal multiplication and division problems involved a whole number and a decimal operand (WD). In all three datasets that were analyzed by Tian et al., children were more accurate on the more frequently presented type of problems. That is, with addition and subtraction, they were more accurate on DD problems, and with multiplication and division, they were more accurate on WD problems.

A limitation of Tian et al.'s (2021) findings is that the WD and DD problems might have differed in computational complexity. For example, the WD multiplication problems in their Study 4 (e.g., 4.5×2) involved fewer digits, and therefore required fewer single-digit whole number multiplication operations, than the DD multiplication problems in the same study (e.g., 4.74×1.5). Such differences could have affected children's accuracy. To address this confound, in the present study, accuracies on WD and DD addition and multiplication problems were assessed using stimuli that were controlled for computational complexity. We expected to replicate Tian et al.'s (2021) findings, namely higher accuracy on DD than WD addition problems and higher accuracy on WD than DD multiplication problems.

Prediction 2. Most decimal arithmetic errors result from overgeneralization of strategies that would be appropriate for other arithmetic operations or types of number.

Though decimal arithmetic involves different procedures than fraction arithmetic, it provides analogous opportunities for strategy overgeneralization. When adding decimals, children are taught to align the decimal points of the operands and then to place the decimal point in the answer at the same location as it appears in the operands (e.g., .4+.2 = .6). Overgeneralizing that strategy to multiplying decimals would lead to errors such as .4×.2 = .8. Similarly, when adding or multiplying whole numbers and when multiplying decimals, children are taught to align the rightmost digit of the operands (e.g., to calculate 6+32, align 6 and 2); overgeneralizing this procedure to adding decimals could generate errors such as 6+.32 = .38. These errors were the most common incorrect responses given to.4×.2 and 6+.32 by fifth to ninth graders in Hiebert and Wearne (1985). Similar decimal arithmetic errors were found in more recent studies with sixth and eighth graders (Lortie-Forgues & Siegler, 2017; Tian et al., 2021).

These findings provide preliminary evidence of strategy overgeneralization errors in decimal arithmetic. However, none of the above studies classified errors according to whether they involved strategy overgeneralization or reported the frequency of such errors in proportion to all errors. Also, both studies focused on children's answers rather than on processes used to generate the answers. The present study addressed these limitations by classifying strategies based on analysis of intermediate calculations as well as final answers, and by reporting percentages of strategy overgeneralization errors over a range of problems.

Prediction 3. Different children display distinct patterns of strategy use: consistent use of correct strategies, reliance on a single flawed strategy, or use of varied strategies.

Prediction 2 reflected the assumption that fraction and decimal arithmetic afford similar opportunities for confusion among strategies. We further hypothesized that parametric variation among individuals generates distinct strategy use patterns in decimal arithmetic, as in fraction arithmetic. We therefore predicted that strategy use patterns analogous to those found in fraction arithmetic would also appear in children's decimal arithmetic, and that most children would display one of these patterns. That is, we expected some children to use correct strategies consistently, some to use a single strategy—either an addition strategy or a multiplication strategy—on most or all problems, and some to display variable strategy use by using both addition and multiplication strategies for both addition and multiplication problems.

The Role of Conceptual Knowledge in Decimal Arithmetic

Mathematical competence includes procedural knowledge, defined as knowledge of step-by-step procedures for solving routine problems, and conceptual knowledge, defined as knowledge of concepts, principles, and relations (Byrnes & Wasik, 1991; Rittle-Johnson, 2017). Each type of knowledge is believed to support the other (Rittle-Johnson et al., 2001, 2015). Evidence for this view includes findings of positive correlations between individual differences in the two types of knowledge in both cross-sectional studies (Booth & Siegler, 2008; Gilmore et al., 2017) and longitudinal studies (Bailey et al., 2017; Schneider et al., 2011). Moreover, interventions designed to improve one type of knowledge often improve the other as well (Canobi, 2009; Fuchs et al., 2013; Fyfe et al., 2014; Siegler & Ramani, 2009).

The present study investigated children's use of conceptual knowledge to support procedural knowledge of decimal arithmetic. We hypothesized that when solving routine calculation problems, children who have relevant conceptual knowledge overtly use it mainly if they are unsure of their procedural knowledge. Put differently, children rarely display conceptual knowledge while solving routine calculation problems unless they doubt their procedural knowledge. This hypothesis is consistent with Siegler's (1988b, 1988a) Strategy Choice Model, in which low confidence in retrieved answers to arithmetic problems leads to use of (usually overt) backup strategies. However, this hypothesized relation between conceptual and procedural knowledge has not been investigated in the context of rational number arithmetic. We tested three predictions, described below, relating to this central hypothesis.

Prediction 4. Many children have relevant conceptual knowledge about decimal arithmetic but do not overtly use such knowledge when solving routine calculation problems.

Children regularly violate conceptual principles when solving decimal calculation problems (Hiebert & Wearne, 1985; Tian et al., 2021). For example, the common error 5.1+.46 = .97 violates the principle that a sum of positive numbers must be larger than both addends. Hiebert and Wearne (1985) accordingly argued that "students compute by relying solely on syntax-based rules; semantic knowledge has no effect on performance" (p. 175).

However, on decimal arithmetic tasks other than calculation, children often display substantial conceptual knowledge, at least for addition[1]. For example, when asked whether sums of a whole number and a decimal would be larger than the larger addend (e.g., "Is 6+0.182 > 6?"), sixth and eighth graders answered "yes" on 95% of trials (Lortie-Forgues & Siegler, 2017).

[1] Conceptual understanding of decimal multiplication and division appears to be more limited than for addition and subtraction (Lortie-Forgues & Siegler, 2017).

Similarly, when estimating decimal sums, children's estimates are rarely smaller than either addend (Braithwaite et al., 2018).

Our hypothesis suggests a way to reconcile these apparently discrepant findings. Children may have relevant conceptual knowledge, but not use it on routine calculation problems because they do not doubt their procedural knowledge. Certainty does not imply accuracy. Relying entirely on procedural knowledge could leave children open to errors resulting from flawed procedural knowledge—such as strategy overgeneralization errors—that children could, in principle, avoid if they applied their conceptual knowledge (Siegler et al., 2020).

To assess this possibility, children in the present study performed two tasks. The *calculation task*, on which children provided concurrent verbal protocols while solving problems, assessed spontaneous use of conceptual knowledge during routine calculation. The *explanation task*, on which children explained how they solved problems and why they used that approach, why they thought a solution that they identified as incorrect was incorrect, and why certain parts of standard correct algorithms made sense, assessed whether children have relevant conceptual knowledge, regardless of whether they use it during routine calculation. We predicted that many children would display conceptual knowledge on the explanation task, but that few would overtly use such knowledge during calculation.

Prediction 5. Doubt about one's procedural knowledge predicts overt use of conceptual knowledge during routine calculation in decimal arithmetic.

Another straightforward implication of our hypothesis regarding conceptual knowledge is that overt displays of doubt during routine calculation should be positively associated with overt use of conceptual knowledge in that context. Braithwaite and Sprague (submitted) obtained preliminary evidence consistent with this prediction for adults. Undergraduates solved 12 fraction and decimal arithmetic problems; trials were coded for whether students displayed doubt and whether they overtly applied conceptual knowledge. During calculation, individuals who more often showed doubt also more often overtly used conceptual knowledge. We expected to find similar relations between doubt and use of conceptual knowledge among sixth and eighth grade children in the present study.

Prediction 6. Overtly using conceptual knowledge during routine decimal arithmetic calculation does not predict accuracy.

Our hypothesis regarding conceptual knowledge also implies that stronger procedural knowledge should lead to decreases in doubt and use of conceptual knowledge during routine calculation, analogous to how stronger knowledge of arithmetic facts leads to increased confidence and reduced reliance on counting in the Strategy Choice Model (Siegler, 1988b, 1988a). Conversely, the hypothesis implies that children who use conceptual knowledge frequently often do so because of weak procedural knowledge. These children may perform better by using conceptual knowledge than they would without doing so, but may not outperform children who have strong procedural knowledge and thus do not use conceptual knowledge. Thus, using conceptual knowledge during routine calculation might not be associated with high accuracy. Consistent with this possibility, undergraduate students in Braithwaite and Sprague (submitted) showed no association between use of conceptual knowledge during calculation and calculation accuracy. The present study tested whether the same was true for children.

To summarize, we tested six predictions about children's decimal arithmetic (Table 1). Predictions 1-3 were based on the hypothesis that FARRA's theoretical assumptions would generalize to decimal arithmetic; Predictions 4-6 were based on the hypothesis that during decimal arithmetic calculation, children overtly use conceptual knowledge primarily when they doubt their procedural knowledge.

Table 1. Predictions About Children's Decimal Arithmetic Tested in the Present Study.

Number	Prediction
1	Accuracies on different types of decimal arithmetic problems parallel the frequencies with which the problem types are encountered during practice
2	Most decimal arithmetic errors result from overgeneralization of strategies that would be appropriate for other arithmetic operations or types of number
3	Different children display distinct patterns of strategy use: consistent use of correct strategies, reliance on a single flawed strategy, or use of varied strategies
4	Many children have conceptual knowledge of decimal arithmetic but do not overtly use such knowledge when solving routine calculation problems
5	Doubt about one's procedural knowledge predicts overt use of conceptual knowledge during routine calculation in decimal arithmetic
6	Overtly using conceptual knowledge during routine decimal arithmetic calculation does not predict accuracy

Method

Participants

Participants were 92 middle school students, 57 sixth graders (33 female, 24 male) and 35 eighth graders (26 female, 9 male). Children were recruited from two middle schools in Tallahassee, FL. In one school (N = 73), 36% of enrolled students were eligible for free or reduced-price lunch (FRPL); the student body was 59% Black, 31% White, 5% multiracial, 3% Hispanic, 2% Asian, and <1% other. In the other school (N = 19), 41% of enrolled students were FRPL-eligible; the student body was 54% Black, 37% White, 4% multiracial, 3% Hispanic, 1% Asian, and <1% other. Parental consent forms were distributed to all students in the classrooms of teachers who agreed to participate. All children whose parents consented and who provided assent were included in the study (two children did not assent). The study was approved by Florida State University's Institutional Review Board (#27505).

The planned sample size was 96. *A priori* power analysis indicated that this was the minimum sample that would guarantee 95% confidence intervals of no more than ±5% around the estimated proportion of children who displayed a given behavior, such as the proportion of children who used conceptual knowledge at least once during calculation or during explanation. Fieldwork was terminated prematurely because of school closures due to COVID-19. Because the size of the completed sample was close to the target, we proceeded with analysis.

Tasks and Materials

Calculation Task

Stimuli for the calculation task were 12 decimal arithmetic problems, including six addition problems (24.45+0.34, 12.3+5.6, 2.46+4.1, 0.826+0.12, 5.61+23, 0.415+52) and six multiplication problems (0.41×0.31, 2.4×1.2, 2.3×0.13, 0.31×2.1, 31×3.2, 14×0.21). The six problems for each arithmetic operation included two problems for each of three types of operand pairs: two decimals with equal numbers of decimal digits (DDE), two decimals with unequal numbers of decimal digits (DDU), and one whole number and one decimal (WD).

Addition problems were designed so that correct solutions would require either two single-digit whole number additions in the case of DDE and DDU operands (e.g., 12.3+5.6 requires adding 3+6 and 2+5), or one single-digit whole number addition in the case of WD operands (e.g., 5.61+23 requires adding 5+3). The smaller number of addition operations on the WD problems guaranteed that lower accuracy on them than on DDE and DDU addition problems could not reflect greater number of arithmetic operations. Multiplication problems were designed so that correct solutions would require two whole number multiplications of a two-digit number by a single-digit number (e.g., 31×3.2 requires multiplying 31×2 and 31×3). Again, higher accuracy on WD than DDE and DDU multiplication problems could not reflect greater numbers of arithmetic operations on WD multiplication problems.

Problems were presented in one of four orders, which were counterbalanced among participants. Problem order had an effect on accuracy but did not interact with any other factor in the analyses reported below. The main effect of problem order on accuracy is described in the Supplementary Materials, Part 3. Cronbach's alpha with this sample was .72.

Explanation Task

The explanation task included six trials, one for each combination of the two arithmetic operations and three trial types: *explain-solution, explain-error,* and *explain-algorithm*. Stimuli were three DDU addition problems (5.73+1.2, 6.15+2.1, 4.32+3.4) and three DDE multiplication problems (7.1×2.1, 5.1×3.1, 3.1×4.1). For explain-error trials, two solutions were created: a correct solution using the standard algorithm and an incorrect solution involving a common error. We focused on one type of operand pair for each operation, rather than including all three types, to maximize similarities among trials within each operation. We chose DDU for addition and DDE for multiplication because previous research (e.g., Tian et al., 2021) suggested that these problem types would be of intermediate difficulty within each operation.

Addition and multiplication problems were presented in separate blocks; whether addition or multiplication appeared first was counterbalanced among participants. Order of addition and multiplication did not affect accuracy on the tasks for which accuracy was calculated (explain-solution and explain-error). Within each block, trials appeared in a fixed order: explain-solution, explain-error, explain-algorithm.

Procedure

The problems for each task were presented in printed packets. The calculation task was presented first, followed by the explanation task. Children completed the tasks working individually with an experimenter. The experimenters were a Black man, a White woman, and a White man. Sessions were audio recorded. All materials and interviewer scripts are provided in the Supplementary Materials, Parts 1 and 2.

Calculation Task

Experimenters first trained children to think aloud using a script adapted from Fox et al. (2011). Children were instructed not to explain their thoughts, but only to say their thoughts aloud. Explaining one's thoughts can affect strategy choices (Kirk & Ashcraft, 2001; Thevenot et al., 2010), but there is no evidence that stating one's thoughts aloud does so. A meta-analysis by Fox et al. (2011) found that explaining affects accuracy, whereas thinking aloud does not. To preview our results, children's accuracies in the present study (addition: 79%, multiplication: 49%) were close to those of sixth graders in a previous study that used similar problems but did not require thinking aloud (addition: 75%, multiplication, 48%; Tian et al., 2021, Study 4).

Children then completed the 12 problems in the calculation task while thinking aloud. After finishing each problem, children rated their confidence in their solutions on a scale from 1 ("not confident at all") to 5 ("extremely confident"). Confidence ratings were collected to validate our measure of doubt, described below. Because our predictions related to doubt during calculation, rather than confidence after calculation, analyses of confidence are reported in the Supplementary Materials, Part 4, with the exception that the correlation between confidence and doubt and analysis of group differences in confidence are reported in the main text.

Children who did not think aloud during calculation (n = 2) were included in analyses of accuracy and strategies but excluded from analyses of doubt and conceptual knowledge use. One child skipped one trial due to experimenter error; this trial was excluded from all analyses.

Explanation Task

Explain-Solution Trials. Children were asked to solve a problem and explain how they solved it. They then were asked a question regarding a key step of their solution. For addition, the question was “How did you line up the numbers at the beginning?” For multiplication, the question was “How did you place the decimal point in the final answer?” For both addition and multiplication, the experimenter then asked why the child lined up the numbers or placed the decimal point as they did and whether there was any reason why it made sense to do it that way.

Explain-Error Trials. Children were shown a problem with two alternative solutions, which were said to be from other students, and told to circle the solution they thought was incorrect. Having done so, they were asked “Why do you think that solution is incorrect?” and “Is there any way you could know that that solution couldn’t possibly be correct, even if you didn’t know the right way to do it?”

Explain-Algorithm Trials. Children were shown a standard algorithm for either adding or multiplying decimals, together with a worked example. The experimenter read the algorithm and the corresponding steps of the worked example aloud, asking the child to follow along. Then, the experimenter asked a question about a key step. For addition, the question was “Why does it make sense to write down the decimals so that their decimal points line up before adding them?” For multiplication, the question was “Why does it make sense for the answer to have as many digits to the right of the decimal point as the total number of digits to the right of the decimal points in the two numbers you were multiplying?”

Five trials—two explain-solution trials, one explain-error trial, and two explain-algorithm trials—were excluded from all analyses because of experimenter error (n = 2) or because participants ran out of time before completing them (n = 3).

Outcome Measures

Accuracy

Answers on the calculation task and explain-solution trials of the explanation task were scored as correct if they were equal to the correct answer. Responses on explain-error trials were scored as correct if children correctly identified the incorrect solution. Accuracy was not scored for explain-algorithm trials of the explanation task.

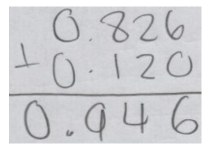
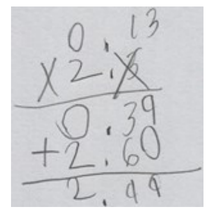
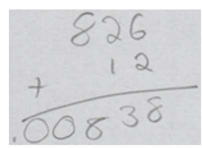
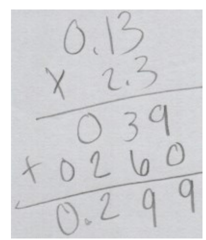
Strategies (Calculation Task)

Each calculation trial was classified as displaying an addition strategy, a multiplication/whole number strategy, both, or neither. These classifications were based on whether the answer was correct, how the operands were aligned in the child's written work, and where the child placed the decimal point in the answer[2]. Table 2 shows the criteria for each strategy classification and examples of addition and multiplication trials that received it. A trial was classified as displaying a strategy if it met any of the criteria for the strategy; trials that met criteria for both addition and multiplication/whole number strategies were classified as displaying both. We intended the multiplication/whole number strategy to reflect how the operands are aligned and how the decimal point is placed in the standard procedure for multiplying decimals; it was named "multiplication/whole number" because one of the criteria

[2] How the operands were aligned was not used as a basis for classifying strategies on DDE problems because for these problems, the addition and multiplication/whole number strategies would yield the same alignment of operands. Similarly, where the decimal point was placed in the answer was not used as a basis for classifying strategies on WD problems, because for these problems, the addition and multiplication/whole number strategies would yield the same decimal point placement.

for this strategy—writing down the operands so that their rightmost digits are aligned—is also consistent with standard procedures for adding and multiplying multidigit whole numbers.

Table 2. Criteria for strategy classifications and examples of trials that received each classification for two problems.

		Problem	
Strategy	Criteria	0.826+0.12	2.3×0.13
Addition	• Problem involves addition and answer is correct, OR • Operands written so digits with same place value are aligned, OR • Answer has as many decimal digits as maximum number of decimal digits in operands		
Multiplication/ Whole Number	• Problem involves multiplication and answer is correct, OR • Operands written so rightmost digits are aligned, OR • Answer has as many decimal digits as total number of decimal digits in operands		

Note. “Decimal digits” refers to digits to the right of the decimal point.

Doubt (Calculation Task)

Children were coded as displaying doubt on a given trial if they erased or crossed out any part of their written work, re-did any part of the problem, or verbally considered multiple ways of solving the problem. They also were coded as displaying doubt if they expressed uncertainty about how to solve the problem or about the correctness of their solution (or a belief that it was incorrect). The more detailed guideline used to code doubt is provided in the Supplementary Materials, Part 5. Trials were coded separately by two coders, who agreed on 85% of trials. Disagreements were resolved through discussion between the two coders and the first author.

As expected, proportion of trials on which participants displayed doubt was negatively correlated with average confidence rating for correctness of the solution, $r = -.37$, $t(88) = 3.8$, $p < .001$. However, doubt was uncorrelated with accuracy, $r = -.11$, $t(88) = 1.1$, $p = .29$. This null finding could reflect doubt leading children to reflect on how to solve the problem, with the reflection leading them sometimes, but not consistently, to find a correct solution.

Overt Use of Conceptual Knowledge

Participants' written work and transcripts of their think-aloud protocols and explanations were used to code overt application of conceptual knowledge on the calculation and explanation tasks. A trial was coded as displaying conceptual knowledge if a participant referred to or reasoned about (1) the magnitude of the answer (e.g., "158 is too big"), (2) place value (e.g., "put the one in the tenths place"), (3) fractions or percentages (e.g., "seven and one tenth as a fraction"), or (4) other knowledge deemed to be conceptual by coders (0% of calculation trials and 0.2% of explanation trials). The guideline for coding use of conceptual knowledge, and the frequencies with which different types of conceptual knowledge were displayed, are provided in the Supplementary Materials, Parts 6 and 7. In the main text, we report analyses of whether conceptual knowledge was used, without distinguishing among types of conceptual knowledge.

Trials were coded separately by two coders. The coders agreed about whether conceptual knowledge was displayed on 96% of calculation trials and 89% of explanation trials. Disagreements were resolved through discussion between the coders and the first author.

Analyses

Effects of problem features on accuracy, doubt, and conceptual knowledge use were analyzed using within-subjects *ANOVA*, with arithmetic operation (addition or multiplication) and operand pair type (DDE, DDU, or WD) as factors for the calculation task, and arithmetic operation and trial type (explain-solution, explain-error, or explain-algorithm) as factors for the explanation task. Relations between outcomes were analyzed using linear regression. Analyses were conducted in *R* (R Core Team, 2020) using the *ez* package (Lawrence, 2016). All significant effects are reported.

Results

Findings are organized around the predictions listed in Table 1. The first three predictions reflect extensions of the FARRA theory of fraction arithmetic to decimal arithmetic. The final three predictions concern the role of conceptual knowledge and doubt in decimal arithmetic, issues that were not addressed by FARRA.

Accuracy

Mean percent correct was 64% (*SD* = 22%) on the calculation task, 73% (*SD* = 30%) on explain-solution trials of the explanation task, and 86% (*SD* = 25%) on explain-error trials of the explanation task. In explain-solution trials and explain-error trials, correctness reflected the child stating the correct answer rather than a correct explanation. Accuracy was not scored for explain-algorithm trials.

Prediction 1. Accuracies on different types of decimal arithmetic problems parallel the frequencies with which the problem types are encountered during practice. Based on problem frequencies in textbooks, addition accuracy was predicted to be lower on WD problems (e.g., 0.415+52) than on DDE or DDU problems (e.g., 12.3+5.6, 2.46+4.1), whereas multiplication accuracy was expected to be higher on WD problems (e.g., 31×3.2) than on DDE or DDU problems (e.g., 0.41×0.31, 2.3×0.13). Accuracy was higher for addition than multiplication (79% vs. 49%), $F(1, 91) = 45.7$, $p < .001$, $\eta_g^2 = 0.14$, and on DDE (71%) than DDU (60%) or WD (62%) problems, $F(2, 182) = 8.7$, $p < .001$, $\eta_g^2 = 0.02$. Critically, operand type interacted with arithmetic operation, $F(2, 182) = 24.8$, $p < .001$, $\eta_g^2 = 0.04$, which allowed a straightforward test of the prediction.

To investigate the interaction, we conducted an *ANOVA* for each arithmetic operation, with type of operand pair as a factor. Type of operand pair affected accuracy for both addition, $F(2, 182) = 22.0$, $p < .001$, $\eta_g^2 = 0.10$, and multiplication, $F(2, 182) = 10.2$, $p < .001$, $\eta_g^2 = 0.03$. As predicted, addition accuracy was higher on DDE (93%) and DDU (79%) than WD (65%) problems, whereas multiplication accuracy was higher on WD (58%) than DDE (49%) or DDU (40%) problems. For each arithmetic operation, all pairwise comparisons between operand types, with a Holm correction for multiple comparisons, were significant, *p*s < .05.

Strategies (Calculation Task)

Prediction 2. Most decimal arithmetic errors result from overgeneralization of strategies that would be appropriate for other arithmetic operations or types of number. To test this prediction, errors were classified as strategy overgeneralization errors if either the operation was addition and the child displayed a multiplication/whole number strategy or the operation was multiplication and the child displayed an addition strategy. As predicted, most errors (70%) involved strategy overgeneralization, including most addition errors (72%) and most multiplication errors (69%). On addition problems, strategy overgeneralization errors usually resulted in adding digits with different place values, and occasionally also resulted in misplacing the decimal point in the answer (both types of error are illustrated in Table 2, second example under 0.826+0.12). On multiplication problems, strategy overgeneralization errors usually involved multiplying the digits correctly but misplacing the decimal point in the answer (Table 2, first example under 2.3×0.13). Most other errors (21% of errors, including 19% of addition errors and 22% of multiplication errors) involved using an appropriate strategy but executing it incorrectly, for example by committing whole number arithmetic errors or by failing to shift the second partial product one column to the left when multiplying.

Prediction 3. Different children display distinct patterns of strategy use in decimal arithmetic: consistent use of correct strategies, reliance on a single flawed strategy, or use of varied strategies. Children were classified using an adapted version of Braithwaite et al.'s (2019) criteria for classifying individual differences in fraction arithmetic strategy use. (1) A child was classified as consistently using correct strategies if the child displayed an appropriate strategy on at least 75% of trials. (2) A child was classified as relying on a single flawed strategy if the child displayed a single strategy, either addition or multiplication, on at least 75% of trials, for example if the child used an addition strategy on all addition problems and half or more of multiplication problems or vice versa. (3) A child was classified as using varied strategies if the child displayed each strategy (i.e., addition and multiplication) at least once both on addition problems and on multiplication problems. These criteria were applied in the order they appeared in this paragraph; for example, a child meeting the criteria for both using correct strategies and relying on a single flawed strategy would be classified as using correct strategies. Note that it was possible for one or more of the predicted patterns not to appear or for children not to meet any of the predicted patterns, for example if a child's strategies for all multiplication problems were inconsistent with the standard strategies for both addition and multiplication.

The predicted patterns all appeared, and they jointly accounted for nearly all children. Specifically, 47% of children were classified as consistently using correct strategies; 29% as relying on a single flawed strategy (25% addition and 4% multiplication), and 21% as using varied strategies. Only 3% of children did not match the criteria for any hypothesized pattern.

One-way *ANOVA*s were used to test for differences on the main outcome variables among children showing different strategy patterns. Children who did not match any pattern were excluded from these analyses. Accuracy and confidence on the calculation task were higher among children classified as using correct strategies (accuracy = 83%, confidence = 4.4) than among those who relied mainly on a single flawed strategy (accuracy = 53%, confidence = 3.8) or used varied strategies (accuracy = 41%, confidence = 3.8), $F(2, 86) = 106.9$, $p < .001$, $\eta_g^2 = .71$ for accuracy and $F(2, 86) = 12.2$, $p < .001$, $\eta_g^2 = .22$ for confidence. Doubt was displayed more often by children who relied mainly on a single flawed strategy (38%) than those who used correct strategies (29%) or varied strategies (29%), $F(2, 85) = 3.1$, $p = .050$, $\eta_g^2 = .07$. Use of conceptual knowledge on the calculation task did not differ among groups, but use of conceptual knowledge on the explanation task (including all three types of trials) was higher among children who consistently used correct strategies (49%) or relied mainly on a single flawed strategy (53%) than among those who used varied strategies (16%), $F(2, 85) = 10.8$, $p < .001$, $\eta_g^2 = .20$.

Doubt (Calculation Task)

Children displayed doubt during calculation on 32% (*SD* = 17%) of trials. Doubt was more common on multiplication problems (44%) than on addition problems (20%), $F(1, 89) = 63.3$, $p < .001$, $\eta_g^2 = .11$, on DDU problems (39%) than WD problems (34%), and on both DDU and WD problems than on DDE problems (24%), $F(2, 178) = 10.9$, $p < .001$, $\eta_g^2 = .03$.

Overt Use of Conceptual Knowledge

On the calculation task, children displayed conceptual knowledge on 6% (*SD* = 10%) of trials. They did so more often on WD than on the other problems (WD: 8%, DDE: 5%, DDU: 4%), $F(2, 178) = 3.6$, $p = .03$, $\eta_g^2 = .008$. An operation × type of operand pair interaction also was found, $F(2, 178) = 7.1$, $p = .001$, $\eta_g^2 = .02$. *ANOVA*s conducted separately for each operation found that for addition problems, children displayed conceptual knowledge more often on WD than on other problems (WD: 12%, DDE: 7%, DDU: 2%), $F(2, 178) = 8.1$, $p < .001$, $\eta_g^2 = .04$, whereas for multiplication problems, operand pair type did not affect frequency of displaying conceptual knowledge (WD: 4%, DDE: 4%, DDU: 6%), $p = .55$.

In contrast, on the explanation task, children displayed conceptual knowledge far more frequently, 43% (*SD* = 32%) of trials, especially on addition (50% vs. 36% of multiplication trials), $F(1, 87) = 14.5$, $p < .001$, $\eta_g^2 = .02$. Frequency of displaying conceptual knowledge did not differ on explain-solution, explain-error, and explain-algorithm trials, $p = .24$.

Prediction 4. Many children have conceptual knowledge of decimal arithmetic but do not overtly use such knowledge when solving routine calculation problems. Consistent with this prediction, 79% of children displayed conceptual knowledge on at least one trial of the explanation task, whereas only 36% of children did so on the calculation task. Nearly half of children (49%) displayed conceptual knowledge at least once on the explanation task but never did so on the calculation task. (30% displayed conceptual knowledge on both tasks, 16% on neither task, and 6% on the calculation task but not the explanation task.)

Particularly striking were instances in which a child committed errors on the calculation task that could have been avoided using conceptual knowledge that the child displayed on the explanation task. For example, on the calculation task, one child claimed 2.46+4.1 = .287; on the explanation task, the same child identified 6.15+2.1 = 6.36 as incorrect on the grounds that "you can just see that six plus two is eight." Similarly, another child claimed on the calculation task that 2.4×1.2 = 28.8; on the explanation task, the same child identified 5.1×3.1 = 158.1 as incorrect on the grounds that "Five times three is 15 ... You just multiply the first number and the second number as an estimate" (presumably referring to the whole number components).

Prediction 5. Doubt about one's procedural knowledge predicts overt use of conceptual knowledge during routine calculation in decimal arithmetic. This prediction was tested using linear regression, with proportion of calculation trials on which each child used conceptual knowledge as the dependent variable. The predictors were proportion of calculation trials on which doubt was displayed and proportion of explanation trials (including explain-solution, explain-error, and explain-algorithm trials) on which conceptual knowledge was displayed. Consistent with the prediction, children who more frequently displayed doubt during calculation also overtly used conceptual knowledge more often during calculation, $B = 0.15$, $\beta = 0.27$, $t(87) = 2.6$, $p = .010$. Use of conceptual knowledge on the explanation task also predicted use of conceptual knowledge on the calculation task, $B = 0.07$, $\beta = 0.24$, $t(87) = 2.4$, $p = .019$.

Prediction 6. Overtly using conceptual knowledge during routine decimal arithmetic calculation does not predict accuracy. This prediction was tested using linear regression, with calculation accuracy as the dependent variable. The predictors were proportion of calculation trials and proportion of explanation trials on which conceptual knowledge was displayed. Calculation accuracy was predicted by use of conceptual knowledge on the explanation task, $B = 0.22$, $\beta = 0.33$, $t(87) = 3.1$, $p = .003$, but not by its use on the calculation task, $B = -0.14$, $\beta = -0.06$, $t(87) = 0.6$, $p = .55$. Thus, more frequent use of conceptual knowledge in response to explicit conceptual prompts (on the explanation task) was associated with higher accuracy on the calculation task, but spontaneous, overt use of conceptual knowledge during calculation was not.

Summary of Findings

Predictions 1-3 (Table 1), which tested the hypothesis that FARRA's theoretical assumptions would generalize to decimal arithmetic, were all supported. Predictions 4-6 (Table 1), which tested the hypothesis that children use conceptual knowledge mainly if in doubt about their procedural knowledge, were also supported.

Discussion

Despite prolonged efforts to reform mathematics education, learners continue to struggle with rational number arithmetic. Decimal arithmetic is no exception. More than three decades ago, 272 seventh grade children[3] correctly answered 69% of a set of decimal addition and multiplication problems (Hiebert & Wearne, 1985). In the present study, sixth and eighth graders correctly answered 64% of a very similar set of decimal addition and multiplication problems. Other recent studies (Lortie-Forgues & Siegler, 2017; Tian et al., 2021) have reported similarly poor performance. Below, we examine what the present findings tell us about sources of difficulty in acquiring procedural and conceptual knowledge of decimal arithmetic, as well as implications of the findings for theories of arithmetic development.

Procedural Knowledge of Decimal Arithmetic

The present findings highlight several phenomena that appear critical for understanding and improving children's procedural knowledge of decimal arithmetic. First, the study replicated Tian et al.'s (2021) finding that compared to DD operands, WD operands result in lower accuracy for addition but higher accuracy for multiplication. This outcome was predicted based

[3] Hiebert and Wearne (1985) reported data from multiple grades and from multiple time points within each grade. Data reported here and subsequently are based on the data from seventh graders collected during spring semester.

on Tian et al.'s (2021) observation that WD addition problems are extremely rare in textbooks, whereas WD multiplication problems are common. Despite the apparent simplicity of WD addition problems, such as 0.415+52, many children answer such problems inaccurately.

In the present study, children were more accurate on addition than multiplication of decimals (79% vs. 49%). The same was true among sixth graders in Tian et al. (2021, Study 4: 75% vs. 48%), though not among seventh graders in Hiebert and Wearne (1985: 62% vs. 76%). This difference is largely attributable to lower multiplication accuracy in the two recent studies than in Hiebert and Wearne (1985), a phenomenon that appeared even on very similar items. For example, 42% of sixth and eighth graders in the present study correctly solved 0.32×2.1, whereas 79% of seventh graders in Hiebert & Wearne (1985) correctly solved .34×2.1. The present findings and those of Tian et al. (2021) suggest an urgent need to improve children's proficiency with decimal multiplication, especially for problems in which both multiplicands have a decimal component.

As predicted, a large majority (70%) of children's errors involved overgeneralization of strategies that would be appropriate for a different arithmetic operation or type of number. This difficulty may be exacerbated by textbooks typically presenting problems in blocks, where all problems within a block can be solved by the same procedure (Rohrer et al., 2019). This approach absolves children of the need to think about which procedure to use on each problem. Educators might provide more practice opportunities in which different problem types are interspersed, thus requiring children to decide which procedure to use on each problem. Interleaving practice problems has improved learning of algebra and geometry (Rohrer & Taylor, 2007; Taylor & Rohrer, 2010), and would likely do so for decimal arithmetic as well.

Finally, individual children showed qualitatively distinct patterns of strategy use. One such pattern was using a single flawed strategy on most or all problems, including problems where the strategy was inappropriate. Children who show such overgeneralization would likely benefit from encountering explanations of why their preferred strategy is not always appropriate, and from practice using other strategies on problems where the overgeneralized strategy is incorrect. Another pattern involved variable strategy use, in which children chose among multiple strategies in seemingly haphazard ways. To help children who show this pattern, educators might encourage (and scaffold) more reflective strategy choice, for example by asking children to state and justify their strategy choices before solving problems—a form of deliberate practice (Ericsson et al., 1993; Lehtinen et al., 2018). The small proportion of explanation trials on which these children displayed conceptual knowledge (16%) suggests that they might also profit more than other children from reinforcement of basic concepts.

Conceptual Knowledge of Decimal Arithmetic

The findings also highlight several issues regarding children's conceptual knowledge of decimal arithmetic. Below, we discuss how different tasks differentially elicit such conceptual knowledge, when and why conceptual knowledge is used during calculation, and relations between conceptual knowledge and accurate calculation.

How different tasks differentially elicit conceptual knowledge of decimal arithmetic

When asked to calculate answers to decimal arithmetic problems, children displayed conceptual knowledge on only 6% of trials. Only 36% of children did so on even one calculation trial. However, the same children displayed conceptual knowledge on 43% of explanation trials, and 79% of children did so on at least one explanation trial. Moreover, on the explanation task, many children displayed conceptual knowledge that, had they applied it, could have helped them avoid errors that they committed on the calculation task.

The fact that children rarely displayed conceptual knowledge in their think-aloud protocols on the calculation task is consistent with Hiebert and Wearne's (1985) claim that, when solving decimal arithmetic problems, children rarely apply conceptual knowledge (in their terms, "semantic knowledge"). In principle, infrequent use of conceptual knowledge during routine calculation could reflect children not having such knowledge. However, the present findings suggest instead that many children have relevant conceptual knowledge that they rarely overtly use during routine calculation.

These findings dovetail with previous results regarding children's understanding of decimal addition. When middle school children are asked to estimate sums of positive decimals or to compare such sums to the larger addend, they show understanding that the sum is larger than either addend (Braithwaite et al., 2018; Lortie-Forgues & Siegler, 2017). However, as shown here and previously, such children frequently violate this principle when calculating decimal sums. These findings appear to reflect children being less likely to apply the conceptual knowledge that they have to calculation than to other tasks, such as estimation or explanation.

Children may prefer to use procedures rather than concepts in the context of calculation, because they perceive procedures as faster, more reliable, or less effortful. Another reason may be that children receive extensive practice in using procedures without emphasis on the concepts that underlie them. In Perry (1991), fourth and fifth graders received instruction in mathematical equivalence that was based on principles only, procedures only, or procedures+principles. Principles-only instruction yielded better learning outcomes than either procedures only or procedures+principles instruction. Perry (1991) argued that "if children are provided with an easily accessible approach to solving a problem (in this case, instruction containing a procedure), they may never consider the rationale underlying their problem-solving actions" (p. 463). Similarly, McNeil (2007) hypothesized that greater practice with the standard calculation procedure of adding all numbers in the problem explains why, on mathematical equality problems, 9-year-olds more often than 7-year-olds err by adding all numbers in the problem or all numbers before the equal sign, leading them to answer "23" or "16" on "7+4+5 = __+7."

When and why conceptual knowledge is used during calculation

Given that children occasionally use conceptual knowledge during calculation, when and why do they do so? We proposed that in the context of calculation, children most often overtly use conceptual knowledge when they are unsure of their procedural knowledge. Consistent with this proposal, in the present study, overt use of conceptual knowledge during calculation was positively associated with displays of doubt.

Our proposal resembles a central claim of the Strategy Choice Model of whole number arithmetic (Siegler, 1988a, 1988b). In this model, when presented a problem, children attempt to retrieve the answer from memory. The retrieved answer is stated if it is accompanied by high confidence; otherwise, an answer is generated using a counting-based strategy. Analogously, in the present proposal, to solve decimal arithmetic problems, children are likely to rely entirely on procedures if they are confident that the procedures are correct. Otherwise, children may use conceptual knowledge to resolve their uncertainty.

If confidence affects strategy choices, then what factors determine confidence? In the Strategy Choice Model, confidence in a retrieved answer depends on the strength of its activation in memory. A similar mechanism could determine children's initial confidence in decimal arithmetic procedures. However, confidence in one's final answer could vary independently of initial confidence in the procedure that generated the answer. For example, a child might retrieve a procedure confidently yet still doubt their final answer because it is implausibly large or small. Thus, determinants of confidence in decimal arithmetic may be more complex than in simple whole number arithmetic, a possibility worth exploring in the future.

Relations between conceptual knowledge and accurate calculation

Use of conceptual knowledge during explanation was positively related to accuracy on the calculation task, similar to previous studies that have found positive relations between conceptual and procedural knowledge of decimal arithmetic (e.g., Lortie-Forgues & Siegler, 2017; Rittle-Johnson & Koedinger, 2009). However, overt use of conceptual knowledge during calculation was unrelated to calculation accuracy.

These apparently conflicting results may be reconciled by assuming that the relation between calculation accuracy and conceptual knowledge use during explanation was driven by previous learning rather than by online use of conceptual knowledge during calculation. Students with relevant conceptual knowledge may previously have used conceptual knowledge to solve problems. Doing so may have strengthened their procedural knowledge, eventually reducing their reliance on conceptual knowledge during calculation. However, the conceptual knowledge, would remain available for use when needed, for example on the explanation task in the present study. This explanation is again consistent with the Strategy Choice Model, which posits that solving arithmetic problems using counting strategies strengthens children's memory for arithmetic facts, thus reducing reliance on counting in the arithmetic context but not in contexts where counting remains useful (Siegler & Shipley, 1995).

Limitations of the Present Study

Several limitations of the present study deserve mention. First, children were coded as having used conceptual knowledge even when they displayed incorrect or inexact reasoning. For example, when solving 31×3.2, one child estimated the answer to be 63; the child was coded as using conceptual knowledge, due to reasoning about the magnitude of the answer, even though her estimate was inaccurate. Thus, the present findings permit drawing conclusions about whether children have or use conceptual knowledge, but not about the quality of the knowledge. Second, only overt displays of conceptual knowledge were coded. Children may have used conceptual knowledge in implicit, even unconscious, ways during execution of procedures. The present conclusions apply only to overt uses of conceptual knowledge, not to implicit ones. Third, only overt displays of doubt were coded. Future studies should employ more sensitive measures of doubt to test the conclusions of the present study. Fourth, although this study was intended to provide the basis for a model of rational number arithmetic, that model remains to be constructed; we are currently working to overcome this limitation.

Towards a Unified Theory of Arithmetic

Arithmetic development begins before formal education and continues at least through middle school. Most theories of this development have focused on small segments of the process, such as addition and subtraction of small whole numbers (Ashcraft, 1987; Aubin et al., 2017; Barrouillet & Thevenot, 2013; Chen & Campbell, 2018; Fayol & Thevenot, 2012; Shrager & Siegler, 1998; Siegler & Shipley, 1995; Siegler & Shrager, 1984; Widaman et al., 1989), multiplication and division of small whole numbers (Campbell & Graham, 1985; Rickard, 2005; Rickard et al., 1994; Siegler, 1988b; Verguts & Fias, 2005), subtraction of multi-digit whole numbers (Brown & VanLehn, 1980), decimal arithmetic (Hiebert & Wearne, 1985), and fraction arithmetic (Braithwaite et al., 2017). A few theories have incorporated additive and multiplicative operations on small whole numbers into a single model (Campbell, 1995; Campbell et al., 2006; Lebiere, 1999). However, no unified theoretical account of arithmetic development, including all operations with all types of numbers, exists.

The present study highlights four phenomena that should be explained by any such unified theory: parallels between accuracies and problem frequencies in textbooks, overgeneralization of strategies that are correct for other operations or types of numbers, distinct error patterns involving consistent use of a single flawed strategy versus variable use of multiple strategies, and positive correlations between conceptual and procedural knowledge. All four phenomena have now been observed in both decimal arithmetic and fraction arithmetic. All but the third have also been observed in whole number arithmetic. These phenomena suggest that a unified theory of fraction, decimal, and whole number arithmetic is possible.

Phenomenon 1. Parallels between accuracies and problem frequencies

The assumption that the likelihood of generating a correct response for a problem depends in part on how often similar problems have been encountered previously can explain not only children's lower accuracy on WD than DD addition problems in the present study, but also their lower accuracy on equal-denominator than unequal-denominator fraction multiplication problems in previous studies (Siegler et al., 2011; Siegler & Pyke, 2013). In each case, on problem types that are highly similar and can be solved by the same procedure (e.g., fraction multiplication with equal and unequal denominators), problems that appear less often in math textbooks are solved less accurately (Braithwaite et al., 2017; Tian et al., 2021).

Similar phenomena have been observed in whole number arithmetic. Parents present preschoolers with addition problems of the form N+1 more often than those of the form 1+N, and math textbooks present "tie" multiplication problems such as 4×4 more often than non-tie problems such as 5×4 (Siegler, 1988b; Siegler & Shrager, 1984). In both cases, children are more accurate on the more frequent problem type than on the less frequent one. Some theories have explained these phenomena in whole number arithmetic without relying on exposure frequency (e.g., Campbell, 1995). However, it remains to be seen whether such theories can also explain the analogous phenomena now known to exist in fraction and decimal arithmetic.

Phenomenon 2. Prevalence of overgeneralization errors

Overgeneralization of strategies to operations and types of numbers where they are inappropriate accounts for most of children's errors in decimal arithmetic, as shown in the present study, and in fraction arithmetic, as shown previously (Siegler et al., 2011; Siegler & Pyke, 2013). Errors resulting from overgeneralization of strategies that would be correct for other operations are also prevalent in whole number arithmetic, as when children claim that 4×8 = 12 (Graham & Campbell, 1992; Miller & Paredes, 1990). Similarly, when solving equivalence problems, children add all numbers in the problem both when doing so is appropriate (e.g., answering "23" on 7+4+5+7 = __) and when it is not (e.g., answering "23" on 7+4+5 = __+7; McNeil & Alibali, 2004, 2005). A third instance is inversion problems, where children who repeatedly see problems on which a+b−b = a overgeneralize to claim afterward that a+b−c = a (Siegler & Stern, 1998). Thus, overgeneralization of strategies provides a unifying explanation for a wide range of children's errors in whole number, fraction, and decimal arithmetic. Knowing not only what to do, but when to do it, is a persistent challenge for children throughout the course of development in arithmetic and other domains (Siegler, 1996).

Phenomenon 3. Distinct error patterns involving consistent use of a single flawed strategy versus variable use of multiple strategies

The individual differences in decimal arithmetic strategy use found in the present study were closely analogous to individual differences in fraction arithmetic found in previous research. Braithwaite et al. (2019) used computational simulations to explore possible causes of individual differences in fraction arithmetic strategy use. The simulations demonstrated that consistent use of a single strategy can result from an initial preference for that strategy becoming self-reinforcing and preventing other strategies from being considered. In contrast, variable strategy use can occur when slow learning rate or relatively random choice processes prevent a learner from forming strong strategy preferences at all. Similar mechanisms could generate the decimal arithmetic error patterns observed in the present study, though this possibility remains to be demonstrated computationally.

The similarities between individual difference patterns in decimal and fraction arithmetic also suggest new questions. First, do analogous individual difference patterns appear in whole number arithmetic; for example, do different children show different error patterns when borrowing across zeroes in multi-digit subtraction? Second, do individuals who display a given pattern with one type of arithmetic (e.g., fraction arithmetic) tend to display the analogous error pattern with other types of arithmetic (e.g., decimal arithmetic)? Affirmative answers to either or both of these questions would suggest that individual differences in learning parameters have similar effects in multiple areas of arithmetic.

Phenomenon 4. Positive correlations between conceptual and procedural knowledge

The positive correlations between conceptual knowledge and accurate use of calculation procedures in the present study join a long list of similar correlations previously found for decimal arithmetic (Lortie-Forgues & Siegler, 2017; Rittle-Johnson & Koedinger, 2009), fraction arithmetic (Bailey et al., 2017; Hansen et al., 2015), and whole number arithmetic (Booth & Siegler, 2008; Fuchs et al., 2010; Linsen et al., 2015). Such correlations are among the most robust phenomena in the literature on mathematical cognition (Rittle-Johnson et al., 2001, 2015). Experimental studies have established causal effects of conceptual interventions on procedural knowledge of arithmetic, again including decimal arithmetic (Rittle-Johnson & Koedinger, 2009), fraction arithmetic (Dyson et al., 2018; Fuchs et al., 2013, 2014), and whole number arithmetic (Booth & Siegler, 2008; Siegler & Ramani, 2009).

Despite the robustness of these relations, no cognitive process model of arithmetic has explained them. Such models typically focus on procedural knowledge; few even include conceptual knowledge (for exceptions, see Ohlsson & Rees, 1991; Shrager & Siegler, 1998). Researchers interested in these relations typically approach them using statistical models at the level of individual differences, often supplemented by verbal qualitative theories. The result is a disconnect between theories concerning relations between conceptual and procedural knowledge of arithmetic and theories that describe the cognitive processes involved in doing arithmetic.

One reason for this disconnect may be that different tasks are often used to assess conceptual and procedural knowledge, resulting in a lack of data about how they interact. The calculation task in the present study would be classified as a procedural task, and indeed, children relied mainly on procedural knowledge to complete it. However, concurrent think-aloud protocols revealed that children also displayed conceptual knowledge on some trials, suggesting that uncertainty about procedural knowledge may elicit recruitment of conceptual knowledge. These findings, while preliminary, demonstrate the feasibility of investigating interactions between conceptual and procedural knowledge in the context of a single task. Doing so may illuminate processes underlying relations between conceptual and procedural knowledge, creating a foundation for process models that can explain such relations.

Conclusion

Despite their differences, decimal and fraction arithmetic present similar challenges to children and evoke a number of similar phenomena, with the phenomena appearing to reflect similar learning mechanisms. Many of these challenges and learning mechanisms appear to be shared with whole number arithmetic as well. Future theories of arithmetic development should be designed to provide integrated accounts of different domains of arithmetic.

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