BASED QUASI-HEREDITARY ALGEBRAS

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Abstract. A notion of a split quasi-hereditary algebra has been defined by Cline, Parshall and Scott. Du and Rui describe a based approach to split quasi-hereditary algebras. We develop this approach further to show that over a complete local Noetherian ring, one can achieve even stronger basis properties. This is important for *'schurifying'* quasi-hereditary algebras as developed in our subsequent work. The schurification procedure associates to an algebra A a new algebra, which is the classical Schur algebra if A is a field. Schurification produces interesting new quasi-hereditary and cellular algebras. It is important to work over an integral domain of characteristic zero, taking into account a super-structure on the input algebra A. So we pay attention to super-structures on quasi-hereditary algebras and investigate a subtle *conforming* property of heredity data which is crucial to guarantee that the schurification of A is quasi-hereditary if so is A. We establish a Morita equivalence result which allows us to pass to basic quasi-hereditary algebras *preserving conformity*.

1. Introduction

Working over an arbitrary ground field, Cline, Parshall and Scott [**CPS**$_1$] axiomatized the notion of a *highest weight category* and defined *quasi-hereditary algebras*. However, it is important to be able to work more generally over a reasonable commutative ring. This was pursued in [**CPS**$_2$, **DuS**, **DuRu**, **Ro**]. In particular, if $\Bbbk$ is a Noetherian ground ring, a notion of a split quasi-hereditary algebra has been defined in [**CPS**$_2$], cf. also [**Ro**]. On the other hand, Du and Rui [**DuRu**] described a based approach to split quasi-hereditary algebras, showing that it is equivalent to that of [**CPS**$_2$] provided that $\Bbbk$ is Noetherian and local.

The goal of this paper is to develop Du and Rui's approach further to show that over a complete local Noetherian ring, we can achieve even stronger basis properties, see Definition 2.4. This is important for *'schurifying'* quasi-hereditary algebras as developed in [**KM**$_2$]. The schurification procedure associates to a $\Bbbk$-algebra A (with suitable subalgebra $\mathfrak{a}$) a new algebra $T^A_{\mathfrak{a}}(n,d)$, which is the classical Schur algebra if $A=\Bbbk$. Schurification often produces interesting new quasi-hereditary and cellular algebras which are important in representation theory of symmetric groups, Hecke algebras, classical Schur algebras, etc., see e.g. [**Tu**, **EK**$_1$, **EK**$_2$].

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It is clear from [**EK**$_1$, **EK**$_2$] that to define many interesting quasi-hereditary algebras, it is important to work over an integral domain of characteristic zero, taking into account a super-structure on the input algebra A. Therefore we pay attention to super-structures (as well as $\mathbb{Z}$-gradings) on quasi-hereditary algebras. We investigate a subtle *conforming* property of heredity data, see Definition 4.9. This is non-trivial only if the super-structure is non-trivial, and is crucial to guarantee that $T^A_{\mathfrak{a}}(n,d)$ is quasi-hereditary if A is quasi-hereditary.

We further establish some Morita equivalence results which sometimes allow us to pass to basic (or almost basic) quasi-hereditary algebras *preserving conformity*, see Theorem 4.13. This is crucial for studying decomposition numbers and other properties of $T^A_{\mathfrak{a}}(n,d)$, see for example [**KM**$_2$].

2. Based quasi-hereditary algebras

Throughout the paper $\Bbbk$ is always a commutative unital ring. Sometimes we will require more in which case this will be stated explicitly.

2.1. **Algebras and modules.** Let V be a *graded* $\Bbbk$*-supermodule*, i.e. V is endowed with a $\Bbbk$-module decomposition

$$V = \bigoplus_{n\in\mathbb{Z},\,\varepsilon\in\mathbb{Z}/2} V^n_\varepsilon.$$

We set $V^n := V^n_{\bar 0} \oplus V^n_{\bar 1}$ and $V_\varepsilon := \bigoplus_{n\in\mathbb{Z}} V^n_\varepsilon$. Then $V = \bigoplus_{n\in\mathbb{Z}} V^n$ is a grading, and $V = V_{\bar 0} \oplus V_{\bar 1}$ is a superstructure. For $v \in V_\varepsilon$, we write $\bar v := \varepsilon$. Of course, the grading and/or the superstructure could be trivial, for example we could have $V = V^0_{\bar 0}$.

An element $v \in V$ is called homogeneous if $v \in V^m_\varepsilon$ for some ε and m. We denote by V_{hom} the set of all non-zero homogeneous elements of V. For a subset $S \subseteq V_{\mathrm{hom}}$ and $\varepsilon \in \mathbb{Z}/2$ we denote

$$S_\varepsilon := S \cap V_\varepsilon. \tag{2.1}$$

A map $f : V \to W$ of graded $\Bbbk$-supermodules is called *homogeneous* if $f(V^m_\varepsilon) \subseteq W^m_\varepsilon$ for all m and ε. Let

$$R := \mathbb{Z}[q, q^{-1}][t]/(t^2 - 1), \tag{2.2}$$

and denote the image of t in the quotient ring by π, so that π^ε makes sense for $\varepsilon \in \mathbb{Z}/2$. For $v \in V^n_\varepsilon$, we write

$$\deg(v) := q^n \pi^\varepsilon. \tag{2.3}$$

For a free $\Bbbk$-module W of finite rank d, we write $d = \dim W$. A graded $\Bbbk$-supermodule V is free of finite rank if each V^n_ε is free of finite rank and we have $V^n = 0$ for almost all n. Let V be a free graded $\Bbbk$-supermodule of finite rank. A *homogeneous basis* of V is a $\Bbbk$-basis all of whose elements are homogeneous. The *graded dimension* of V is

$$\dim^q_\pi V := \sum_{n\in\mathbb{Z},\,\varepsilon\in\mathbb{Z}/2} (\dim V^n_\varepsilon) q^n \pi^\varepsilon \in R.$$

A (not necessarily unital) $\Bbbk$-algebra A is called a *graded* $\Bbbk$*-superalgebra*, if A is a graded $\Bbbk$-supermodule and $A^n_\varepsilon A^m_\delta \subseteq A^{n+m}_{\varepsilon+\delta}$ for all ε, δ and n, m. By a *graded*

A-*supermodule* we understand an A-module V which is a graded $\Bbbk$-supermodule and $A^n_\varepsilon V^m_\delta \subseteq V^{n+m}_{\varepsilon+\delta}$ for all ε, δ and n, m. We denote by A-mod the category of all finitely generated graded A-supermodules and homogeneous A-homomorphisms. All ideals, subalgebras, submodules, etc. are assumed to be homogeneous. In particular the Jacobson ideal $J(A)$ is the intersection of the annihilators of all graded simple A-supermodules.

Given a graded A-supermodule V, $n \in \mathbb{Z}$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, we denote by $q^n\pi^\varepsilon V$ the graded A-supermodule which is the same as V as an A-module but with $(q^n\pi^\varepsilon V)^m_\delta = V^{m-n}_{\delta+\varepsilon}$.

2.2. **Definition and first properties.** Let A be a graded $\Bbbk$-superalgebra, and I be a finite partially ordered set. A subset $\Omega \subseteq I$ is called an *upper set* if $i \in \Omega$ and $j \geq i$ imply $j \in \Omega$. Examples of upper sets are

$$I^{>i} := \{j \in I \mid j > i\} \quad \text{and} \quad I^{\geq i} := \{j \in I \mid j \geq i\}$$

for a fixed $i \in I$.

Definition 2.4. A *heredity data* on A consist of a partially ordered set I and finite sets $X = \bigsqcup_{i \in I} X(i)$ and $Y = \bigsqcup_{i\in I} Y(i)$ of non-zero homogeneous elements of A with distinguished *initial elements* $e^i \in X(i) \cap Y(i)$ for each $i \in I$. For $i \in I$ and $\Omega \subseteq I$, we set

$$\begin{aligned}
&Z(i) := X(i) \times Y(i),\ Z(\Omega) := \textstyle\bigsqcup_{j\in\Omega} Z(j),\\
&Z^{>i} := Z(I^{>i}),\ Z^{\geq i} := Z(I^{\geq i}),\\
&A(\Omega) := \operatorname{span}\{xy \mid (x, y) \in \Omega\},\\
&A^{>i} := A(Z^{>i}),\ A^{\geq i} := A(Z^{\geq i}).
\end{aligned}$$

We require that the following axioms hold:

(a) $B := \{xy \mid (x, y) \in Z\}$ is a basis of A;

(b) For all $i \in I$, $x \in X(i)$, $y \in Y(i)$ and $a \in A$, we have

$$ax \equiv \sum_{x'\in X(i)} l^x_{x'}(a)x' \pmod{A^{>i}} \quad \text{and} \quad ya \equiv \sum_{y'\in Y(i)} r^y_{y'}(a)y' \pmod{A^{>i}}$$

for some $l^x_{x'}(a), r^y_{y'}(a) \in \Bbbk$;

(c) For all $i \in I$, we have

$$xe_i = x,\ e_ix = \delta_{x,e_i}x,\ e_iy = y,\ ye_i = \delta_{y,e_i}y \qquad (x \in X(i),\ y \in Y(i));$$
$$e_jx = x \text{ or } 0,\ ye_j = y \text{ or } 0 \qquad (x \in X,\ y \in Y,\ j \in I).$$

If A is endowed with a heredity data I, X, Y, we call A *based quasi-hereditary (with respect to the poset I)*, and refer to B as a *heredity basis* of A.

Lemma 2.5. *If $\Omega \subseteq I$ is an upper set, then $A(\Omega)$ is the (two-sided) ideal generated by $\{e_i \mid i \in \Omega\}$.*

Proof. That $A(\Omega)$ is an ideal is clear from Definition 2.4(b). That $A(\Omega)$ contains the ideal generated by $\{e_i \mid i \in \Omega\}$ is now clear since $A(\Omega) \supseteq \{e_i \mid i \in \Omega\}$. The converse containment follows from $xy = xe_iy$ for $(x, y) \in Z(i)$, see Definition 2.4(c). □

Lemma 2.6. *Let $\Omega, \Theta \subseteq I$ be upper sets.*

(i) $A(\Omega) \subseteq A(\Theta)$ *if and only if* $\Omega \subseteq \Theta$;
(ii) $A(\Omega)A(\Theta) \subseteq A(\Omega) \cap A(\Theta) = A(\Omega \cap \Theta)$.

Proof. (i) If $\Omega \not\subseteq \Theta$ and $i \in \Omega \setminus \Theta$, it follows from Definition 2.4(a) that $xy \in A(\Omega) \setminus A(\Theta)$ for all $(x, y) \in Z(i)$, i.e. $A(\Omega) \not\subseteq A(\Theta)$. The converse is obvious.

(ii) As $A(\Omega)$, $A(\Theta)$ are ideals by Lemma 2.5, the containment $A(\Omega)A(\Theta) \subseteq A(\Omega) \cap A(\Theta)$ is clear. The equality $A(\Omega) \cap A(\Theta) = A(\Omega \cap \Theta)$ comes from Definition 2.4(a). □

Lemma 2.7. *Let* $x \in X(i)$, $y \in Y(i)$. *If* $j \not\leq i$, *then* $e_j x = y e_j = 0$.

Proof. As $e_j \in Y(j)$, we have by Definition 2.4(b) that $e_j x \in A^{\geq j}$. Since $x \notin A^{\geq j}$, we have that $e_j x \neq x$, so Definition 2.4(c) gives us $e_j x = 0$. The proof of $y e_j = 0$ is similar. □

Lemma 2.8. *For any* $i, j \in I$, *we have* $e_i e_j = \delta_{i,j} e_i$.

Proof. Since $e_i \in I$, the equality $e_i^2 = e_i$ comes from Definition 2.4(c). Let $i \neq j$. By Definition 2.4(c) again, we have that $e_i e_j$ is either e_j or 0 and on the other hand either e_i or 0. Since $e_i \neq e_j$ by Definition 2.4(a), we deduce that $e_i e_j = 0$. □

Let $i \in I$, $x \in X(i)$ and $y \in Y(i)$. By Definition 2.4(b),

$$\sum_{x' \in X(i)} l^x_{x'}(y) x' \equiv yx \equiv \sum_{y' \in Y(i)} r^y_{y'}(x) y' \pmod{A^{>i}}.$$

By Definition 2.4(c), we have $x' = x' e_i$ and $y' = e_i y'$, so taking into account Definition 2.4(a), we deduce that

$$yx \equiv f_i(y, x) e_i \pmod{A^{>i}} \tag{2.9}$$

for some $f_i(y, x) \in \Bbbk$. This defines a function $f_i : Y(i) \times X(i) \to \Bbbk$. Note that

$$f_i(e_i, e_i) = 1 \tag{2.10}$$

and

$$f_i(y, x) = 0 \quad \text{unless} \quad \deg(x)\deg(y) = 1. \tag{2.11}$$

Definition 2.12. [**DuRu**, 1.2.1] A graded $\Bbbk$-superalgebra A is called *standardly based* with respect to a finite poset I if it possesses a *standard basis*, i.e. a homogeneous basis of the form

$$\{b^i_{x,y} \mid i \in I,\ x \in X(i),\ y \in Y(i)\}$$

for some index sets $X(i), Y(i)$ such that, setting $A^{>i} := \text{span}\{b^j_{x,y} \mid j > i\}$, for all $a \in A$, $i \in I$, $x \in X(i)$, $y \in Y(i)$, we have

$$a b^i_{x,y} \equiv \sum_{x' \in X(i)} l^x_{x'}(a) b^i_{x',y} \pmod{A^{>i}},$$

$$b^i_{x,y} a \equiv \sum_{y' \in Y(i)} r^y_{y'}(a) b^i_{x,y'} \pmod{A^{>i}}$$

for some $l^x_{x'}(a) \in \Bbbk$ independent of y and $r^y_{y'}(a) \in \Bbbk$ independent of x.

By [**DuRu**, (1.2.3)],

$$b^i_{x,y} b^i_{x',y'} \equiv f_i(y,x') b^i_{x,y'} \pmod{A^{>i}}$$

for some $f_i(y,x') \in \Bbbk$. The standardly based algebra is called standardly full-based if the $\Bbbk$-span of the elements $f_i(y,x)$, with $x \in X(i)$, $y \in Y(i)$, is $\Bbbk$. The following is clear using (2.10):

Lemma 2.13. *If A is a based quasi-hereditary algebra then it is standardly full-based with $b^i_{x,y} = xy$ for all $i \in I$ and all $x \in X(i)$, $y \in Y(i)$.*

A homogeneous anti-involution τ on A is called *standard* (with respect to I, X, Y) if for all $i \in I$ there is a bijection $X(i) \xrightarrow{\sim} Y(i)$, $x \mapsto y(x)$ such that $y(e_i) = e_i$ and

$$\tau(x) = y(x). \tag{2.14}$$

For a standard anti-involution τ, we have

$$\tau(xy(x')) = x'y(x) \tag{2.15}$$

and $\tau(e_i) = e_i$ for all $i \in I$, $x, x' \in X(i)$. If τ is a standard anti-involution on A then $\{xy \mid (x,y) \in Z\}$ is a *cellular basis* of A with respect to τ, see [**DuRu**, (6.1.4)].

2.3. **Standard modules.** Throughout the subsection, A is a based quasi-hereditary $\Bbbk$-superalgebra with heredity data I, X, Y.

Fix $i \in I$ and upper sets $\Omega', \Omega \subseteq I$ such that $\Omega' \setminus \Omega = \{i\}$. For example we could take $\Omega' = I^{\geq i}$ and $\Omega = I^{>i}$. Denote

$$\tilde{A} := A/A(\Omega) \quad \text{and} \quad \tilde{a} := a + A(\Omega) \in \tilde{A} \qquad (a \in A).$$

By inflation, $\tilde{A}$-modules will be automatically considered as A-modules. The *standard module* $\Delta(i)$ and the *right standard module* $\Delta^{\mathrm{op}}(i)$ are defined as

$$\Delta(i) := \tilde{A}\tilde{e}_i \quad \text{and} \quad \Delta^{\mathrm{op}}(i) := \tilde{e}_i\tilde{A}. \tag{2.16}$$

By Definition 2.4, we have

$$\Delta(i) = \mathrm{span}\{\tilde{x} \mid x \in X(i)\} \quad \text{and} \quad \Delta^{\mathrm{op}}(i) = \mathrm{span}\{\tilde{y} \mid y \in Y(i)\},$$

so $\Delta(i)$ and $\Delta^{\mathrm{op}}(i)$ can be defined respectively as free $\Bbbk$-modules with bases $\{v_x \mid x \in X(i)\}$ and $\{w_y \mid y \in Y(i)\}$ and the actions

$$av_x = \sum_{x' \in X(i)} l^x_{x'}(a) v_{x'} \quad \text{and} \quad w_y a = \sum_{y' \in Y(i)} r^y_{y'}(a) w_{y'} \qquad (a \in A).$$

This implies in particular that the definition of $\Delta(i)$ and $\Delta^{\mathrm{op}}(i)$ does not depend on the choice of Ω and Ω' as long as $\Omega' \setminus \Omega = \{i\}$.

Note that $v_i := v_{e_i}$ is a cyclic generator of $\Delta(i)$ such that

$$e_i v_i = v_i \quad \text{and} \quad x v_i = v_x \qquad (x \in X(i)). \tag{2.17}$$

Moreover,

$$e_i v_x = 0 \qquad (x \in X(i) \setminus \{e_i\}). \tag{2.18}$$

Taking into account Lemma 2.7, we deduce that $e_j \Delta(i) \neq 0$ implies $j \leq i$. Similar statements hold for $\Delta^{\mathrm{op}}(i)$. We have

$$\mathrm{End}_A(\Delta(i)) \cong \mathrm{End}_{\tilde{A}}(\Delta(i)) \cong \mathrm{End}_{\tilde{A}}(\tilde{A}e_i, \Delta(i)) \cong e_i\Delta(i) \cong \Bbbk. \tag{2.19}$$

It follows from the definitions that as A-bimodules,

$$A(\Omega')/A(\Omega) \cong \Delta(i) \otimes_{\Bbbk} \Delta^{\mathrm{op}}(i). \tag{2.20}$$

Recalling (2.9), we have a bilinear pairing $(\cdot,\cdot)_i : \Delta(i) \times \Delta^{\mathrm{op}}(i) \to \Bbbk$ satisfying

$$(v_x, w_y)_i = f_i(y,x).$$

Lemma 2.21. *We have*

(i) $(v_i, w_i)_i = 1$;
(ii) $(av, w)_i = (v, wa)_i$ *for all* $v \in \Delta(i), w \in \Delta^{\mathrm{op}}(i), a \in A$.

Proof. (i) comes from (2.10).

(ii) We follow [**DuRu**, (2.3.1)]. Let $x \in X(i), y \in Y(i)$. We have

$$(av_x, w_y)_i = \sum_{x' \in X(i)} l^x_{x'}(a)(v_{x'}, w_y)_i = \sum_{x' \in X(i)} l^x_{x'}(a) f_i(y, x'),$$

$$(v_x, w_y a)_i = \sum_{y' \in Y(i)} r^y_{y'}(a)(v_x, w_{y'})_i = \sum_{y' \in Y(i)} r^y_{y'}(a) f_i(y', x).$$

On the other hand, by Definition 2.4(b) and (2.9), modulo $A(\Omega)$ we have

$$\sum_{y' \in Y(i)} r^y_{y'}(a) f_i(y', x) e_i \equiv \sum_{y' \in Y(i)} r^y_{y'}(a) y' x = (ya)x = y(ax) = \sum_{x' \in X(i)} l^x_{x'}(a) y x'$$
$$\equiv \sum_{x' \in X(i)} l^x_{x'}(a) f_i(y, x') e_i,$$

so

$$\sum_{y' \in Y(i)} r^y_{y'}(a) f_i(y', x) = \sum_{x' \in X(i)} l^x_{x'}(a) f_i(y, x') e_i,$$

completing the proof. ☐

By the lemma,

$$\operatorname{rad}\Delta(i) := \{v \in \Delta(i) \mid (v,w)_i = 0 \text{ for all } w \in \Delta^{\mathrm{op}}(i)\}$$

is a submodule of $\Delta(i)$.

Lemma 2.22. [**DuRu**, (2.4.1)] *Let* $\Bbbk$ *be a field. Then for each* $i \in I$ *we have that*

$$L(i) := \Delta(i)/\operatorname{rad}\Delta(i)$$

is an absolutely irreducible A*-module. Furthermore, ignoring grading and superstructure,* $\{L(i) \mid i \in I\}$ *is a complete and irredundant set of irreducible* A*-modules up to an isomorphism.*

By definition, the form $(\cdot,\cdot)_i$ is homogeneous, so $\operatorname{rad}\Delta(i)$ is a homogeneous submodule of $\Delta(i)$ and $L(i)$ is naturally a graded A-supermodule. We refer to the modules $L(i)$ as the *canonical irreducible* A*-modules.* From Lemma 2.22, we get:

Lemma 2.23. *Let* $\Bbbk$ *be a field. Then*

$$\{q^n \pi^\varepsilon L(i) \mid i \in I,\ n \in \mathbb{Z},\ \varepsilon \in \mathbb{Z}/2\}$$

is a complete and irredundant set of irreducible graded A*-supermodules up to a homogeneous isomorphism.*

Corollary 2.24. *Suppose that* $\Bbbk$ *is a local ring with the maximal ideal* $\mathfrak{m}$ *and the quotient field* $F = \Bbbk/\mathfrak{m}$. *Then:*

(i) $A/\mathfrak{m}A \cong A \otimes_{\Bbbk} F$ *is based quasi-hereditary* F*-superalgebra.*

(ii) *For each* $i \in I$, *denote the corresponding canonical irreducible* $A/\mathfrak{m}A$*-module by* $L_{A/\mathfrak{m}A}(i)$ *and denote by* $L_A(i)$ *the* A*-module obtained from* $L_{A/\mathfrak{m}A}(i)$ *by inflation. Then*

$$\{q^n \pi^{\varepsilon} L_A(i) \mid i \in I,\ n \in \mathbb{Z},\ \varepsilon \in \mathbb{Z}/2\}$$

is a complete and irredundant set of irreducible graded A*-supermodules up to a homogeneous isomorphism.*

If $\Bbbk$ is a local ring, we call A *basic* if the the modules $L_{A/\mathfrak{m}A}(i)$ are 1-dimensional as F-vector spaces, equivalently if the modules $L_A(i)$ are free of rank 1 as $\Bbbk$-modules.

Let $\Bbbk$ be a field. Recalling the ring R from (2.2), we can now consider *bigraded decomposition numbers*

$$d_{ij}(q,\pi) := \sum_{n\in\mathbb{Z},\, \varepsilon\in\mathbb{Z}/2} d_{ij}^{n,\varepsilon} q^n \pi^{\varepsilon} \in R \qquad (i,j \in I), \tag{2.25}$$

where

$$d_{ij}^{n,\varepsilon} := [\Delta(i) : q^n \pi^{\varepsilon} L(j)] \qquad (n \in \mathbb{Z},\, \varepsilon \in \mathbb{Z}/2). \tag{2.26}$$

Lemma 2.27. *For* $i, j \in I$, *we have* $d_{ii}(q,\pi) = 1$, *and* $d_{ij}(q,\pi) \neq 0$ *implies* $j \leq i$.

Proof. Denote

$$\hat{v}_i := v_i + \operatorname{rad} \Delta(i) \in \Delta(i)/\operatorname{rad} \Delta(i) = L(i).$$

Then $e_i \Delta(i) = \Bbbk \cdot v_i$ implies $e_i L(i) = \Bbbk \cdot \hat{v}_i$. Moreover, $e_j \Delta(i) \neq 0$ only if $j \leq i$ implies that $e_j L(i) \neq 0$ only if $j \leq i$. The result follows. □

3. Based quasi-hereditary versus split quasi-hereditary

Throughout the section we assume that A is unital. Our goal is to show that under reasonable assumptions on $\Bbbk$, the notion of based quasi-hereditary and split qusi-hereditary are the same.

3.1. **Based quasi-hereditary algebras are split quasi-hereditary.** Assume that $\Bbbk$ is noetherian and A is a graded $\Bbbk$-superalgebra, which is finitely generated projective as a $\Bbbk$-module. The following definition goes back to [**CPS**$_2$,**DuS**], but we follow the version of [**Ro**]:

Definition 3.1. A (homogeneous) ideal J of A is called an *indecomposable split heredity ideal* if the following conditions hold:

(1) A/J is projective as a $\Bbbk$-module;
(2) J is projective as a left A-module;
(3) J is idempotent, i.e. $J^2 = J$;
(4) $\operatorname{End}_A(J)$ is Morita equivalent to $\Bbbk$.

Definition 3.2. The graded $\Bbbk$-superalgebra A is *split quasi-hereditary* with respect to a finite partially ordered set I if for every upper set $\Omega \subseteq I$ there is an ideal $A(\Omega)$ in A such that

(1) if $\Omega \subseteq \Omega'$ are upper sets then $A(\Omega) \subseteq A(\Omega')$;

(2) if $\Omega \subseteq \Omega'$ are upper sets with $|\Omega' \setminus \Omega| = 1$, then $A(\Omega)/A(\Omega)$ is an indecomposable split heredity ideal in $A/A(\Omega)$.

Lemma 3.3. *Let $\Bbbk$ be noetherian. If A is based quasi-hereditary then it is split quasi-hereditary.*

Proof. By Lemma 2.5, we have the ideals $A(\Omega)$ which clearly satisfy Definition 3.2(1). Let $\Omega \subseteq \Omega'$ satisfy $\Omega' \setminus \Omega = \{i\}$. We need to check that the ideal $A(\Omega')/A(\Omega)$ in $A/A(\Omega)$ satisfies (1)–(4) of Definition 3.1. Note that

$$\{xy + A(\Omega') \mid (x, y) \in Z(I \setminus \Omega')\}$$

is a $\Bbbk$-basis of $A/A(\Omega')$, which gives (1). The property (2) follows from (2.20) and (2.16). The property (3) comes from the fact that $A(\Omega')/A(\Omega)$ is generated by the idempotent $e_i+A(\Omega)$. Finally, by (2.20) and (2.19), we have $\mathrm{End}_A(A(\Omega')/A(\Omega)) \cong M_m(\Bbbk)$, where $m = |Y(i)|$, which gives (4). □

3.2. **Split quasi-hereditary algebras are based quasi-hereditary.** In this subsection, we assume that the ground ring $\Bbbk$ is noetherian and local and that A is a split quasi-hereditary graded superalgebra. In particular, A is a free $\Bbbk$-module of finite rank and hence Noetherian.

In addition we assume that A is *semiperfect*, i.e. $A/J(A)$ is a left Artinian and homogeneous idempotents lift from $A/J(A)$ to A, cf. [**Da**, Definition 3.3]. By [**Da**, Theorem 3.5], this is equivalent to $A_{\bar{0}}^{0}$ being semiperfect (in the usual sense). So, as noted in [**CPS**$_2$, §1], A is semiperfect provided $\Bbbk$ is complete (local Noetherian). The proof of [**CPS**$_2$, (1.3)] now goes through to give:

Lemma 3.4. *Let A be semiperfect. If $J_1 \supseteq \cdots \supseteq J_t$ are idempotent ideals in A then there exist idempotents $f_1, \dots, f_t$ in A such that $J_r = Af_rA$ for all r and $f_rf_s = f_sf_r = f_r$ for all $r > s$.*

Proposition 3.5. *Assume that $\Bbbk$ is Noetherian and local and that A is a semiperfect graded $\Bbbk$-superalgebra. If A is split quasi-hereditary, then A is based quasi-hereditary.*

Proof. We may assume that $I = \{0, 1, \dots, \ell\}$ for some $\ell \in \mathbb{Z}_{>0}$ and $0 < 1 < \cdots < \ell$ is a total order refining the given partial order on I. Then $\Omega_i := \{i, i+1, \dots, \ell\}$ is an upper set for any $i \in I$, and we have a chain

$$I = \Omega_0 \supseteq \Omega_1 \supseteq \cdots \supseteq \Omega_\ell \supseteq \Omega_{\ell+1} := \varnothing$$

with $\Omega_i \backslash \Omega_{i+1} = \{i\}$ for $i \in I$. By Lemma 3.4 there exist idempotents $f_0, \dots, f_\ell$ such that $A(\Omega_i) = Af_iA$, and $f_if_j = f_jf_i = f_i$ whenever $i > j$. Define $e_\ell := f_\ell$ and $e_i := f_i - f_{i+1}$ for $i = 0, 1, \dots, \ell-1$. Then for all $i, j \in I$, we have $e_ie_j = \delta_{ij}e_i$, and $f_i = e_i + \cdots + e_\ell$.

Let $i \in I$, $\tilde{A} := A/A(\Omega_{i+1})$ and $\tilde{a} := a + A(\Omega_{i+1}) \in \tilde{A}$ for $a \in A$. It follows from Definition 3.1(1) that $\tilde{A}$ is projective as a $\Bbbk$-module. Moreover, $A(\Omega_i)/A(\Omega_{i+1})$ is projective as an $\tilde{A}$-module. Since

$$A(\Omega_i)/A(\Omega_{i+1}) = \tilde{A}\tilde{f}_i\tilde{A} = \tilde{A}\tilde{e}_i\tilde{A}$$

by the previous paragraph, [**DR**$_1$, Statement 7] implies that the multiplication map

$$m : \tilde{A}\tilde{e}_i \otimes_{\tilde{e}_i\tilde{A}\tilde{e}_i} \tilde{e}_i\tilde{A} \to \tilde{A}\tilde{e}_i\tilde{A}$$

is an isomorphism of $\tilde{A}$-bimodules. By [**Ro**, Lemma 4.5, Proposition 4.7], we have that

$$\tilde{e}_i\tilde{A}\tilde{e}_i \cong \mathrm{End}_{\tilde{A}}(\tilde{A}\tilde{e}_i)^{\mathrm{op}} = \mathrm{End}_A(\tilde{A}\tilde{e}_i)^{\mathrm{op}} \cong \Bbbk,$$

so $\tilde{e}_i\tilde{A}\tilde{e}_i = \Bbbk\tilde{e}_i$, and $A(\Omega_i)/A(\Omega_{i+1}) \cong \tilde{A}\tilde{e}_i \otimes_\Bbbk \tilde{e}_i\tilde{A}$.

The left $\tilde{A}$-module $\tilde{A}\tilde{e}_i$ is projective as an $\tilde{A}$-module, hence projective as a $\Bbbk$-module. Writing $e_* := 1 - e_0 - \cdots - e_\ell$, we have

$$\tilde{A}\tilde{e}_i = \tilde{e}_0\tilde{A}\tilde{e}_i \oplus \cdots \oplus \tilde{e}_\ell\tilde{A}\tilde{e}_i \oplus \tilde{e}_*\tilde{A}\tilde{e}_i.$$

Each of the summands above is projective as a $\Bbbk$-module, hence is free as a $\Bbbk$-module since $\Bbbk$ is local. Then there exists a set of elements $X(i) \subset A_{\mathrm{hom}}$ such that:

- $e_i \in X(i)$;
- $\{\tilde{x} \mid x \in X(i)\}$ is a $\Bbbk$-basis for $\tilde{A}\tilde{e}_i$;
- For all $x \in X(i)$, we have $x = e_t x e_i$ for some $t \in \{0, \dots, \ell, *\}$.

In similar fashion we may choose a set of elements $Y(i) \subset A_{\mathrm{hom}}$ such that:

- $e_i \in Y(i)$;
- $\{\tilde{y} \mid y \in Y(i)\}$ is a $\Bbbk$-basis for $\tilde{e}_i\tilde{A}$;
- For all $y \in Y(i)$, we have $y = e_i y e_t$ for some $t \in \{0, \dots, \ell, *\}$.

Since m is an isomorphism, $\{\tilde{x}\tilde{y} \mid x \in X(i),\ y \in Y(i)\}$ is a $\Bbbk$-basis for $\tilde{A}\tilde{e}_i\tilde{A} = A(\Omega_i)/A(\Omega_{i+1})$, for all $i \in I$, which implies that $\{xy \mid i \in I,\ x \in X(i),\ y \in Y(i)\}$ is a basis for A. The remaining conditions of Definition 2.4 are now easily checked. For example, $e_i x = \delta_{x,e_i} x$ for $x \in X(i)$ follows from $\tilde{e}_i\tilde{A}\tilde{e}_i \cong \Bbbk\tilde{e}_i$. Thus $\{I,\ \bigsqcup_i X(i),\ \bigsqcup_i Y(i)\}$ constitutes based quasi-hereditary data for A. □

4. Further properties

Let A be a based quasi-hereditary $\Bbbk$-superalgebra with heredity data I, X, Y.

4.1. **Involution and idempotent truncation.** If $e \in A$ is a homogeneous idempotent, we consider the idempotent truncation $\bar{A} := eAe$, and denote $\bar{a} := eae \in \bar{A}$ for $a \in A$. We say that e is *adapted* (with respect to the given heredity data) if for all $i \in I$ there exist subsets $\bar{X}(i) \subseteq X(i)$ and $\bar{Y}(i) \subseteq Y(i)$ such that for all $(x, y) \in Z(i)$ we have:

$$ex = \begin{cases} x & \text{if } x \in \bar{X}(i), \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad ye = \begin{cases} y & \text{if } y \in \bar{Y}(i), \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Setting

$$\bar{I} := \{i \in I \mid \bar{X}(i) \neq \emptyset \neq \bar{Y}(i)\}, \tag{4.2}$$

the *e-truncation of B* is defined to be

$$\bar{B} := \{xy \mid i \in \bar{I}, x \in \bar{X}(i), y \in \bar{Y}(i)\}. \tag{4.3}$$

We say that e is *strongly adapted* if it is adapted and $ee_i = e_i e = e_i$ for all $i \in \bar{I}$.

Lemma 4.4. *Let $e \in A$ be an adapted idempotent.*

(i) *The e-truncation $\bar{B}$ is a standard basis of $\bar{A}$ in the sense of Definition 2.12.*

(ii) *If τ is a standard anti-involution of A such that $\tau(e) = e$, then $\bar{B}$ is a cellular basis of $\bar{A}$ with respect to the restriction $\tau|_{\bar{A}}$.*

(iii) *If* e *is strongly adapted then* $\bar{A}$ *is based quasi-hereditary with heredity data* $\bar{I}$, $\bar{X} := \bigsqcup_{i\in\bar{I}} \bar{X}(i)$, $\bar{Y} := \bigsqcup_{i\in\bar{I}} \bar{Y}(i)$.

Proof. (i) follows from $xy = exye$. To check (ii) one needs to observe that $ex = x$ if and only if $y(x)e = y(x)$, and so $\bar{Y}(i) = \{y(x) \mid x \in \bar{X}(i)\}$. Part (iii) is clear. □

Remark 4.5. Let $e \in A$ be an adapted idempotent. For $i \in I$, consider the $\bar{A}$-module $\bar{\Delta}(i) := e\Delta(i)$. If τ is a standard anti-involution of A with $\tau(e) = e$ then by Lemma 4.4(ii), $\bar{A}$ is cellular and $\{\bar{\Delta}(i) \mid i \in \bar{I}\}$ are the cell modules for $\bar{A}$. If e is strongly adapted then by Lemma 4.4(iii), $\bar{A}$ is quasi-hereditary and $\{\bar{\Delta}(i) \mid i \in \bar{I}\}$ are the standard modules for $\bar{A}$.

Remark 4.6. Given a cellular algebra $\bar{A}$ with cellular basis $\bar{B}$ and a subalgebra $\bar{\mathfrak{a}} \subseteq \bar{A}_{\bar{0}}$, is there a based quasi-hereditary algebra A with heredity basis B, a standard anti-involution τ and τ-invariant adapted idempotent e such that $\bar{A} = eAe$, $\bar{\mathfrak{a}} = e\mathfrak{a}e$, and $\bar{B}$ is the e-truncation of B? We do not know if this converse of Lemma 4.4(ii) always holds true. This question seems to be related to problems studied in [**Ro**, **DR**$_2$, **Ko**, **Aus**].

Lemma 4.7. *Let* $\Bbbk$ *be a field, and* $e \in A$ *be an adapted idempotent.*

(i) $eL(i) = 0$ *if and only if* $e\Delta(i) \subseteq \operatorname{rad}\Delta(i)$.
(ii) $eL(i) = 0$ *if and only if* $yex \in A^{>i}$ *for all* $x \in X(i)$ *and* $y \in Y(i)$.
(iii) $eL(i) = 0$ *if and only if* $yx \in A^{>i}$ *for all* $x \in \bar{X}(i)$ *and* $y \in \bar{Y}(i)$.
(iv) $eL(i) = 0$ *for all* $i \in I \setminus \bar{I}$.

Proof. Part (i) is clear. By part (i), $eL(i) = 0$ if and only if $ev_x \in \operatorname{rad}\Delta(i)$ for all $x \in X(i)$. Recalling the definition of the form $(\cdot,\cdot)_i$, this is equivalent to $yex \in A^{>i}$, proving part (ii). Pari (iii) follows from part (ii) since $ex = \delta_{\{x\in\bar{X}\}}x$ and $ye = \delta_{\{y\in\bar{Y}\}}y$. Finally, if $i \in I \setminus \bar{I}$ then $\bar{X}(i) = \varnothing$ or $\bar{Y}(i) = \varnothing$ (or both). So part (iv) follows from part (iii). □

Corollary 4.8. *Let* $\Bbbk$ *be a field, and* $e \in A$ *be an adapted idempotent. Then there exists a subset* $\bar{I}' \subseteq \bar{I}$ *such that* $\{eL(i) \mid i \in \bar{I}'$ *is a complete and irredundant set of irreducible* $\bar{A}$*-modules up to isomorphism.*

4.2. **Conformity.** We now turn to more subtle additional properties of heredity data, which have to do with the super-structure. Recalling (2.1), we have sets $B_\varepsilon, X(i)_\varepsilon, Y_\varepsilon$ etc.

Definition 4.9. Suppose that $\mathfrak{a} \subseteq A_{\bar{0}}$ is a subalgebra. The heredity data I, X, Y of A is $\mathfrak{a}$-*conforming* if $I, X_{\bar{0}}, Y_{\bar{0}}$ is a heredity data for $\mathfrak{a}$.

If the heredity data I, X, Y of A is $\mathfrak{a}$-conforming then $\mathfrak{a}$ is recovered as follows:

$$\mathfrak{a} = \operatorname{span}(xy \mid i \in I,\ x \in X(i)_{\bar{0}},\ y \in Y(i)_{\bar{0}}).$$

So sometimes we will just speak of a *conforming heredity data*. Even though in some sense $\mathfrak{a}$ is redundant in the definition of conormity, it is often convenient to use it. For example, in [**KM**$_1$], we will construct generalized Schur algebras $T^A_{\mathfrak{a}}(n, d)$, which will only depend on A and $\mathfrak{a}$, but not on I, X, Y.

Recall that we have standard A-modules $\Delta(i)$ and simple A-modules $L(i)$ (if $\Bbbk$ is a field). If the heredity data I, X, Y of A is $\mathfrak{a}$-conforming then by definition $\mathfrak{a}$ is also based quasi-hereditary and has its own standard $\mathfrak{a}$-modules $\Delta_{\mathfrak{a}}(i)$ and simple $\mathfrak{a}$-modules $L_{\mathfrak{a}}(i)$ (if $\Bbbk$ is a field).

We describe an additional property which implies conformity. This property is readily checked in some important examples and will be preserved under formation of the generalized Schur algebra $T^A_{\mathfrak{a}}(n,d)$. The following is easy to see:

Lemma 4.10. *Suppose that A possesses a $(\mathbb{Z}/2\times\mathbb{Z}/2)$-grading $A=\bigoplus_{\varepsilon,\delta\in\mathbb{Z}/2}A_{\varepsilon,\delta}$ such that the following conditions hold:*

(1) *$A_{\varepsilon,\delta}A_{\varepsilon',\delta'}\subseteq A_{\varepsilon+\varepsilon',\delta+\delta'}$ for all $\varepsilon,\delta,\varepsilon',\delta'\in\mathbb{Z}/2$;*
(2) *For all $\varepsilon\in\mathbb{Z}/2$, we have $A_\varepsilon=\bigoplus_{\varepsilon'+\varepsilon''=\varepsilon}A_{\varepsilon',\varepsilon''}$.*
(3) *$X_\varepsilon\subseteq A_{\varepsilon,\bar 0}$ and $Y_\varepsilon\subseteq A_{\bar 0,\varepsilon}$ for all $\varepsilon\in\mathbb{Z}/2$.*

Then the heredity data I,X,Y is $\mathfrak{a}$-conforming for $\mathfrak{a}=A_{\bar 0,\bar 0}$.

4.3. **Morita equivalence.** Throughout the section, we assume that $\Bbbk$ is local. We also assume that A is a unital based quasi-hereditary graded $\Bbbk$-superalgebra with heredity data I,X,Y which is $\mathfrak{a}$-conforming for a unital subalgebra $\mathfrak{a}$, in particular, $I,X_{\bar 0},Y_{\bar 0}$ is a heredity data for $\mathfrak{a}$ and $1_{\mathfrak{a}}=1_A$.

Our goal is to find an idempotent $f\in\mathfrak{a}$ such that $\bar A:=fAf$ is based quasi-hereditary with $\bar{\mathfrak{a}}$-conforming hereditary data, where $\bar{\mathfrak{a}}:=f\mathfrak{a}f$ is basic and the functors

$$\mathcal{F}_A: A\text{-mod}\to\bar A\text{-mod},\ V\mapsto fV\quad\text{and}\quad \mathcal{F}_{\mathfrak{a}}:\mathfrak{a}\text{-mod}\to\bar{\mathfrak{a}}\text{-mod},\ V\mapsto fV$$

are equivalences of categories, such that

$$\begin{aligned}\mathcal{F}_A(L_A(i))&\cong L_{\bar A}(i), & \mathcal{F}_A(\Delta_A(i))&\cong\Delta_{\bar A}(i),\\ \mathcal{F}_{\mathfrak{a}}(L_{\mathfrak{a}}(i))&\cong L_{\bar{\mathfrak{a}}}(i), & \mathcal{F}_{\mathfrak{a}}(\Delta_{\mathfrak{a}}(i))&\cong\Delta_{\bar{\mathfrak{a}}}(i).\end{aligned}$$

The first step allows us to reduce to the situation where $\sum_{i\in I}e_i=1_A=1_{\mathfrak{a}}$:

Lemma 4.11. *Let $e:=\sum_{i\in I}e_i$. Then $\bar A:=eAe$ is based quasi-hereditary with $\bar{\mathfrak{a}}$-conforming hereditary data, where $\bar{\mathfrak{a}}:=e\mathfrak{a}e$ and the functors*

$$\mathcal{F}_A: A\text{-mod}\to\bar A\text{-mod},\ V\mapsto eV\quad\textit{and}\quad \mathcal{F}_{\mathfrak{a}}:\mathfrak{a}\text{-mod}\to\bar{\mathfrak{a}}\text{-mod},\ V\mapsto eV$$

are equivalences of categories, such that

$$\begin{aligned}\mathcal{F}_A(L_A(i))&\cong L_{\bar A}(i), & \mathcal{F}_A(\Delta_A(i))&\cong\Delta_{\bar A}(i),\\ \mathcal{F}_{\mathfrak{a}}(L_{\mathfrak{a}}(i))&\cong L_{\bar{\mathfrak{a}}}(i), & \mathcal{F}_{\mathfrak{a}}(\Delta_{\mathfrak{a}}(i))&\cong\Delta_{\bar{\mathfrak{a}}}(i).\end{aligned}$$

Proof. This follows using Lemma 4.4 since e is strongly adapted. □

Lemma 4.12. *There exists an $\mathfrak{a}$-conforming heredity data I,X',Y' for A with the same ideals $A(\Omega)$ and $\mathfrak{a}(\Omega)$, and such that the new initial elements $\{e_i'\mid i\in I\}$ are primitive idempotents in $\mathfrak{a}$ satisfying $e_ie_i'=e_i'=e_i'e_i$ and $e_i'\equiv e_i\pmod{\mathfrak{a}^{>i}}$ for all $i\in I$.*

Proof. Let $i\in I$. Set $\tilde{\mathfrak{a}}:=\mathfrak{a}/\mathfrak{a}^{>i}$ and $\tilde a:=a+\mathfrak{a}^{>i}\in\tilde{\mathfrak{a}}$ for $a\in\mathfrak{a}$. Then $\tilde e_i$ is a primitive idempotent in $\tilde{\mathfrak{a}}$ since $\mathrm{End}_{\tilde{\mathfrak{a}}}(\tilde{\mathfrak{a}}\tilde e_i)\cong\tilde e_i\tilde{\mathfrak{a}}\tilde e_i\cong\Bbbk$ is local. So if $e_i=e_i^1+\dots+e_i^r$ is a sum of orthogonal primitive idempotents in $\mathfrak{a}$ then there is exactly one t with $1\le t\le r$ and $\tilde e_i=\tilde e_i^t$. We set $e_i':=e_i^t$. Note that $e_ie_i'=e_i'=e_i'e_i$, hence $e_i'e_j'=0$ for $i\neq j$.

Let Ω be an upper set of I. It easily follows that $A(\Omega)$, which by Lemma 2.5 is the ideal of A generated by $\sum_{i\in\Omega} e_i$, is also generated by $\sum_{i\in\Omega} e'_i$. Similarly, $\mathfrak{a}(\Omega)$ is the ideal of $\mathfrak{a}$ generated by $\sum_{i\in\Omega} e'_i$.

We have that $\mathfrak{a}^{\geq i}/\mathfrak{a}^{>i}$ is projective as an $\tilde{\mathfrak{a}}$-module, $\mathfrak{a}^{\geq i}/\mathfrak{a}^{>i} = \tilde{\mathfrak{a}}\tilde{e}_i\tilde{\mathfrak{a}} = \tilde{\mathfrak{a}}\tilde{e}'_i\tilde{\mathfrak{a}}$ and $\tilde{e}'_i\tilde{\mathfrak{a}}\tilde{e}'_i = \tilde{e}_i\tilde{\mathfrak{a}}\tilde{e}_i \cong \mathbb{k}$. So [**DR**$_1$, Statement 7] implies that the multiplication map

$$m : \tilde{\mathfrak{a}}\tilde{e}'_i \otimes_{\mathbb{k}} \tilde{e}'_i\tilde{\mathfrak{a}} \to \tilde{\mathfrak{a}}\tilde{e}'_i\tilde{\mathfrak{a}}$$

is an isomorphism of $\tilde{\mathfrak{a}}$-bimodules. By definition, $\tilde{\mathfrak{a}}\tilde{e}'_i = \tilde{\mathfrak{a}}\tilde{e}_i$ has $\mathbb{k}$-basis $\{\tilde{x} \mid x \in X(i)_{\bar{0}}\}$, $\tilde{A}_{\bar{1}}\tilde{e}'_i = (\tilde{A}\tilde{e}'_i)_{\bar{1}} = (\tilde{A}\tilde{e}_i)_{\bar{1}}$ has $\mathbb{k}$-basis $\{\tilde{x} \mid x \in X(i)_{\bar{1}}\}$, and $\tilde{A}\tilde{e}'_i = \tilde{A}\tilde{e}_i = \tilde{\mathfrak{a}}\tilde{e}'_i \oplus \tilde{A}_{\bar{1}}\tilde{e}'_i$ as $\mathbb{k}$-modules. Let

$$e'_* := 1_A - \sum_{i\in I} e'_i.$$

Since $1_A = 1_{\mathfrak{a}}$, we have $e'_* \in \mathfrak{a}$. Note that

$$\tilde{\mathfrak{a}}\tilde{e}'_i = \bigoplus_{j\in I\sqcup\{*\}} \tilde{e}'_j\tilde{\mathfrak{a}}\tilde{e}'_i \quad \text{and} \quad \tilde{A}_{\bar{1}}\tilde{e}'_i = \bigoplus_{j\in I\sqcup\{*\}} \tilde{e}'_j\tilde{A}_{\bar{1}}\tilde{e}'_i.$$

Each of the summands above is projective, hence free, as a $\mathbb{k}$-module. So there exists a set of elements $X'(i) = X'(i)_{\bar{0}} \sqcup X'(i)_{\bar{1}}$ such that:

- $e'_i \in X(i)_{\bar{0}}$;
- $\{\tilde{x} \mid x \in X'(i)_{\bar{0}}\}$ is a $\mathbb{k}$-basis for $\tilde{\mathfrak{a}}\tilde{e}'_i$ and $\{\tilde{x} \mid x \in X'(i)_{\bar{1}}\}$ is a $\mathbb{k}$-basis for $\tilde{A}_{\bar{1}}\tilde{e}'_i$;
- For all $x \in X'(i)$, we have $x = e'_j x e'_i$ for some $j \in I \sqcup \{*\}$.

In similar fashion we may choose a set of elements $Y'(i) = Y'(i)_{\bar{0}} \sqcup Y'(i)_{\bar{1}}$ such that:

- $e'_i \in Y'(i)_{\bar{0}}$;
- $\{\tilde{y} \mid y \in Y'(i)_{\bar{0}}\}$ is a $\mathbb{k}$-basis for $\tilde{e}'_i\tilde{\mathfrak{a}}$ and $\{\tilde{y} \mid y \in Y'(i)_{\bar{1}}\}$ is a $\mathbb{k}$-basis for $\tilde{e}'_i\tilde{A}_{\bar{1}}$;
- For all $y \in Y'(i)$, we have $y = e'_i y e'_j$ for some $j \in I \sqcup \{*\}$.

Since m is an isomorphism, $\{\tilde{x}\tilde{y} \mid x \in X'(i),\ y \in Y'(i)\}$ is a $\mathbb{k}$-basis for $\tilde{A}\tilde{e}'_i\tilde{A} = A^{\geq i}/A^{>i}$ and $\{\tilde{x}\tilde{y} \mid x \in X'(i)_{\bar{0}},\ y \in Y'(i)_{\bar{0}}\}$ is a $\mathbb{k}$-basis for $\tilde{\mathfrak{a}}\tilde{e}'_i\tilde{\mathfrak{a}} = \mathfrak{a}^{\geq i}/\mathfrak{a}^{>i}$. Doing this for all $i \in I$, we deduce that $\{xy \mid i \in I,\ x \in X'(i), y \in Y'(i)\}$ is a basis for A and $\{xy \mid i \in I,\ x \in X'(i)_{\bar{0}}, y \in Y'(i)_{\bar{0}}\}$ is a basis for $\mathfrak{a}$. The remaining conditions of Definitions 2.4 and 4.9 are now easily checked. Thus $\{I,\ \bigsqcup_i X'(i),\ \bigsqcup_i Y'(i)\}$ is an $\mathfrak{a}$-conforming heredity data for A. □

In Lemma 4.12, we have obtained the condition that all the heredity ideals $A(\Omega)$ are the same for the two heredity bases coming from (I, X, Y) and (I, X', Y'). This implies that the standard modules $\Delta_A(i)$ and hence the simple modules $L_A(i)$ are unchanged when we pass from (I, X, Y) and (I, X', Y'). The similar statement holds for $\Delta_{\mathfrak{a}}(i)$ and $L_{\mathfrak{a}}(i)$.

For a strongly adapted idempotent $e \in A$, recall the notation $\bar{X}(i), \bar{Y}(i)$ from (4.1), (4.2). These will be applied for the idempotent f appearing in the following theorem:

Theorem 4.13. *Let $\mathbb{k}$ be local and A be a unital based quasi-hereditary graded $\mathbb{k}$-superalgebra with $\mathfrak{a}$-conforming heredity data (I, X, Y) for a unital subalgebra $\mathfrak{a}$.*

Then there exists an $\mathfrak{a}$-conforming heredity data (I, X', Y') with the same ideals $A(\Omega)$ and $\mathfrak{a}(\Omega)$ and such that the new initial elements $\{e'_i \mid i \in I\}$ are primitive idempotents in $\mathfrak{a}$ satisfying $e_i e'_i = e'_i = e'_i e_i$ and $e'_i \equiv e_i \pmod{\mathfrak{a}^{>i}}$ for all $i \in I$. Moreover, setting $f := \sum_{i\in I} e'_i$, we have:

(i) *f is strongly adapted with respect to (I, X', Y'), so that $\bar{A}$ is based quasi-hereditary with heredity data $(I, \bar{X}', \bar{Y}')$.*
(ii) *$(I, \bar{X}', \bar{Y}')$ is $\bar{\mathfrak{a}}$-conforming;*
(iii) *$\bar{\mathfrak{a}}$ is basic and if $A_{\bar{1}} \subset J(A)$ then $\bar{A}$ is a basic as well;*
(iv) *The functors*

$$\mathcal{F}_A : A\text{-mod} \to \bar{A}\text{-mod},\ V \mapsto fV \quad \textit{and} \quad \mathcal{F}_{\mathfrak{a}} : \mathfrak{a}\text{-mod} \to \bar{\mathfrak{a}}\text{-mod},\ V \mapsto fV$$

are equivalences of categories, such that

$$\mathcal{F}_A(L_A(i)) \cong L_{\bar{A}}(i), \qquad \mathcal{F}_A(\Delta_A(i)) \cong \Delta_{\bar{A}}(i),$$
$$\mathcal{F}_{\mathfrak{a}}(L_{\mathfrak{a}}(i)) \cong L_{\bar{\mathfrak{a}}}(i), \qquad \mathcal{F}_{\mathfrak{a}}(\Delta_{\mathfrak{a}}(i)) \cong \Delta_{\bar{\mathfrak{a}}}(i).$$

Proof. Let $e = \sum_{i\in I} e_i$. By Lemma 4.11, the algebra eAe satisfies the assumptions of Lemma 4.12. The application of that lemma yields a conforming heredity data (I, X'', Y'') in eAe with initial elements $\{e''_i \mid i \in I\}$. To extend it to the needed heredity data (I, X', Y') for A define

$$X' := X'' \sqcup \bigsqcup_{i\in I} \{x e''_i \mid x \in X(i) \text{ with } ex = 0\},$$
$$Y' := Y'' \sqcup \bigsqcup_{i\in I} \{e''_i y \mid y \in Y(i) \text{ with } ye = 0\}.$$

It is easy to see that this new heredity data with initial elements $e'_i = e''_i$ satisfies the required conditions. □

4.4. **Examples.** Our two main examples of based quasi-hereditary algebras are the classical *Schur algebra* $S(n, d)$ and the *extended zigzag algebra* Z.

The classical Schur algebra with trivial grading and superalgebra structures has the basis $\{Y^\lambda_{S,T}\}$ of codeterminants constructed in [**Gr**$_1$]. It is essentially checked in [**Gr**$_1$] that $S(n, d)$ with the codeterminant basis is a based quasi-hereditary algebra with perfect heredity data and standard anti-involution. So is the extended zigzag algebra, which we define next.

Given $n \geq d$, let $\lambda = (1^d)$, and let T^λ be the λ-tableau with the entry r in the rth row. Define

$$e := Y^\lambda_{T^\lambda, T^\lambda} = \xi_{1\cdots d, 1\cdots d}.$$

Then e is an adapted idempotent, and $eS(n, d)e \cong \Bbbk\mathfrak{S}_d$. Thus

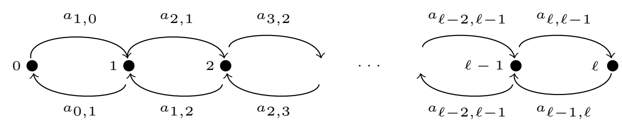
$$\{eY^\lambda_{S,T}e \mid eY^\lambda_{S,T}e \neq 0\}$$

defines a cellular basis for $\mathfrak{S}_d$, known as a *Murphy basis.*

Fix $\ell \geq 1$ and set

$$I := \{0, 1, \dots, \ell\}, \quad J := \{0, \dots, \ell - 1\}.$$

Let Γ be the quiver with vertex set I and arrows $\{a_{j,j+1}, a_{j+1,j} \mid j \in J\}$ as in the picture:



The *extended zigzag algebra* Z is the path algebra $\Bbbk\Gamma$ modulo the following relations:

(i) All paths of length three or greater are zero.
(ii) All paths of length two that are not cycles are zero.
(iii) All length-two cycles based at the same vertex are equivalent.
(iv) $a_{\ell,\ell-1}a_{\ell-1,\ell} = 0$.

Length zero paths yield the standard idempotents $\{e_i \mid i \in I\}$ with $e_i a_{i,j} e_j = a_{i,j}$ for all admissible i, j. The algebra Z is graded by the path length: $Z = Z^0 \oplus Z^1 \oplus Z^2$. We also consider Z as a superalgebra with $Z_{\bar{0}} = Z^0 \oplus Z^2$ and $Z_{\bar{1}} = Z^1$.

Define

$$c_j := a_{j,j+1}a_{j+1,j} \qquad (j \in J).$$

The algebra Z has an anti-involution τ with

$$\tau(e_i) = e_i, \quad \tau(a_{ij}) = a_{ji}, \quad \tau(c_j) = c_j.$$

We consider the total order on I given by $0 < 1 < \cdots < \ell$. For $i \in I$, we set

$$X(i) := \begin{cases} \{e_i, a_{i-1,i}\} & \text{if } i > 0, \\ \{e_0\} & \text{if } i = 0, \end{cases} \qquad Y(i) := \begin{cases} \{e_i, a_{i,i-1}\} & \text{if } i > 0, \\ \{e_0\} & \text{if } i = 0. \end{cases}$$

With respect to this data we have:

Lemma 4.14. *The graded superalgebra Z is a basic based quasi-hereditary with perfect heredity data and standard anti-involution τ.*

Proof. This is well-known and easy to check. □

Note that

$$B_{\bar{1}} = \{a_{j,j+1}, a_{j+1,j} \mid j \in J\}, \quad B_{\bar{0}} = \{e_i \mid i \in I\} \sqcup \{c_j \mid j \in J\}.$$

Let $e := e_0 + \cdots + e_{\ell-1} \in Z$. Note that e is an adapted idempotent, and $\tau(e) = e$ so the *zigzag algebra* $\overline{Z} := eZe \subset Z$ is a cellular algebra with involution $\tau|_{\overline{Z}}$, and cellular basis

$$\overline{B} = \{xy \mid i \in I, x \in \overline{X}(i), y \in \overline{Y}(i)\},$$

where $\overline{X}(\ell) = \{a_{\ell-1,\ell}\}$, $\overline{Y}(\ell) = \{a_{\ell,\ell-1}\}$, and $\overline{X}(i) = X(i)$, $\overline{Y}(i) = Y(i)$ for all $i \in J$.

Note that, when $\Bbbk$ is a field, we have $eL(\ell) = 0$, and $eL(j) = L(j)$ for all $j \in J$, so the standard $\overline{Z}$-modules are $\{\bar{\Delta}(i) = e\Delta(i) \mid i \in I\}$, and the simple $\overline{Z}$-modules are $\{\bar{L}(j) = eL(j) \mid j \in J\}$. The following lemma is easily checked.

Lemma 4.15. *Let $\Bbbk$ be a field. Let $i \in I$, and $j \in I$ (resp. $j \in J$). Then the graded decomposition numbers for standard Z-modules (resp. $\overline{Z}$-modules) are given by*

$$d_{i,j} = \delta_{i,j} + \delta_{i-1,j} q\pi.$$

For integers n, m, consider the matrix superalgebra $M_{n|m}(\Bbbk)$ of rank $n|m$, with entries in $\Bbbk$. For $r, s \in [1, n+m]$, let $E_{r,s}$ be the matrix with 1 in the (r,s)-th component, and zeros elsewhere. We have

$$\overline{E}_{r,s} := \begin{cases} \bar{0} & \text{if } r, s \leq n \text{ or } r, s > n, \\ \bar{1} & \text{otherwise.} \end{cases}$$

and

$$\deg(E_{r,s}) = r - s.$$

Then $B := \{E_{r,s} \mid r, s \in [1, n+m]\}$ constitutes a homogeneous basis for $M_{n|m}(\Bbbk)$.

Now, let $I = \{\bullet\}$ be the singleton set, and define:

$$e_\bullet := E_{1,1}, \quad X(\bullet) := \{E_{r,1} \mid r \in [1, n+m]\}, \quad Y(\bullet) := \{E_{1,s} \mid s \in [1, n+m]\}.$$

Then (I, X, Y) constitutes conforming heredity data for $M_{n|m}(\Bbbk)$ with heredity basis B.

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