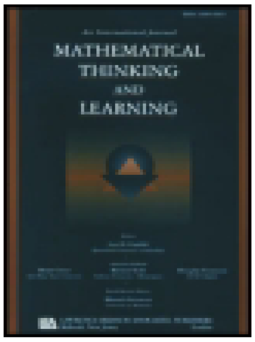

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Integrating math and science content through covariational reasoning: the case of gravity

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ABSTRACT
Integrating mathematics content into science usually plays a supporting role, where students use their existing mathematical knowledge for solving science tasks without exhibiting any new mathematical meanings during the process. To help students explore the reciprocal relationship between math and science, we designed an instructional module that prompted them to reason covariationally about the quantities involved in the phenomenon of the gravitational force. The results of a whole-class design experiment with sixth-grade students showed that covariational reasoning supported students' understanding of the phenomenon of gravity. Also, the examination of the phenomenon of gravity provided a constructive space for students to construct meanings about co-varying quantities. Specifically, students reasoned about the change in the magnitudes and values of mass, distance, and gravity as those changed simultaneously as well as the multiplicative change of these quantities as they changed in relation to each other. They also reasoned multivariationally illustrating that they coordinated mass and distance working together to define the gravitational force. Their interactions with the design, which included the tool, tasks, representations, and questioning, showed to be a structuring factor in the formation and reorganization of meanings that students exhibited. Thus, this study illustrates the type of design activity that provided a constructive space for students' forms of covariational reasoning in the context of gravity. This design can be used to develop other STEM modules that integrate scientific phenomena with covariational reasoning through technology.

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Covariational reasoning; quantitative reasoning; STEM integration; technology; design experiment; gravity

Introduction

While investigating the reasons behind the high STEM attrition rate in the United States, Christensen et al. (2014) found that, in schools, STEM subjects are often introduced in discrete and uninspiring ways and as a result, students often find the content matter of the subjects difficult and unrelated to other disciplines and their regular life. To provide students with a connected, meaningful, and relevant platform to build their higher scientific conceptions, Roschelle et al. (2007) recommended an in-depth understanding of the mathematics embedded in topics of science. In line with this call, various initiatives identified the commonalities among the math and science practices and the ways that these can help learners discover crosscutting themes connecting the different disciplines. Two examples of these efforts are the K-12 Framework for Science Education (National Research Council, 2012) and the Next Generation Science Standards (National Research Council, 2013), which explicitly state practices and crosscutting concepts between math and science. Another example is Mayes and Koballa (2012) table illustrating the connections between the math and science practices that studies can use to design integrated lessons.

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Previous studies examining students' knowledge in math and science as they engage in integrated lessons show a lack of evidence to support positive impacts on mathematics outcomes (Becker & Park, 2011; Honey et al., 2014). In discussing STEM integration, English (2016) called for a more balanced focus on each of the disciplines, especially the discipline of mathematics which is usually under-represented. Indeed, often mathematics content plays a service role in science, where students use their existing mathematical knowledge for solving science tasks without developing any new mathematical meanings during the process (Honey et al., 2014; Tytler et al., 2019). Even when studies claim the development of mathematical ideas in integrative activities, they do not illustrate the *reciprocal* relationship between mathematics and STEM, by exemplifying "the way in which mathematics can influence and contribute to the understanding of the ideas and concepts of other STEM disciplines" (Fitzallen, 2015, p. 241).

As a result, despite the efforts for illustrating connections between math and science and designing learning experiences for fostering those connections, there is limited research on how we can best achieve that. A handful of studies (e.g., Becker & Park, 2011; Honey et al., 2014) examining success in math and science integration focus on quantitative analyses of students' assessment scores in those disciplines to examine the success of different forms of integration. But this focus provides limited information of the forms of integrated reasoning that students may exhibit, or the type of activity that can foster these integrated math and science forms of reasoning.

To foster students' integrated forms of reasoning where both math and science content are honored, we designed an instructional module that prompted them to reason covariationally about the quantities involved in the phenomenon of the gravitational force. Our goal was to illustrate this *reciprocal* relationship (Fitzallen, 2015) of mathematics and science by examining the ways in which students' covariational reasoning could influence and contribute to their understanding of the gravitational force. In this paper, we describe the forms of reasoning that students exhibited as they interacted with our tasks, tools, representations, and questioning and discuss the emerging activity that constituted a productive space for content integration and helped students express interdisciplinary reasoning.

The current state of teaching and learning gravity

Gravity is the force of attraction that exists between any two objects. According to Newton's law of universal gravitation, the force of gravity between two bodies is proportional to the product of the masses of the two bodies (m1 and m2), and inversely proportional to the square of the distance (d) between their centers of mass. This law is written with the formula $F = G\frac{m_1 \times m_2}{d^2}$, where G is called the Gravitational Constant.

The Framework for K-12 Science Education (National Research Council, 2012) suggests that the concept of gravitational force should be introduced early in schooling as a force between any two objects that depend on the objects' masses and the distance between them. Students are often introduced to gravity by examining the "falling" of objects. Later in upper elementary and middle school they are introduced to the above formula and are asked to find the values of gravity given the values of mass and distance. The focus is on the cause and effect relationship between masses, distance, and gravity.

Despite these efforts, research shows that students develop a variety of naïve understandings about gravity. These include students believing that gravity is an unseen force acting only on heavy objects (Kavanagh & Sneider, 2007), only happens when there is air (Andersson, 1990; Bar, 1989; Bar et al., 1997), is affected by temperature and is stronger at great distances (Treagust & Smith, 1989), or only applies to falling objects and not to objects at rest or objects thrown up in the air (Watts, 1982). Many of these ideas held by young students continue in high school and are also held even by elementary school teachers (Kavanagh & Sneider, 2007; Kruger et al., 1990).

These naïve understandings show that the current experiences that students have in their schooling do not support them to construct relationships between gravity, the distance between two objects, and the masses of these objects. To support students' learning of these ideas, we began with the assumption that the examination of gravity through a quantitative and covariational reasoning lens could help students actively construct meaning about these mathematical relationships and avoid the misconceptions typically reported in the literature.

Examining gravity through a quantitative and covariational reasoning lens

We followed a *quantitative reasoning* approach (P. Thompson, 1989; Thompson, 1994) to discuss the forms of reasoning that are possible about the relationships that underlie the concept of gravity. A *quantity*, as described by P. Thompson (1989) and Thompson (1994), is a conceived attribute of an object or phenomenon that is measurable. Reasoning quantitatively involves analyzing "a situation into a quantitative structure – a network of quantities and quantitative relationships" (Thompson, 1993, p. 1). The phenomenon of gravity involves conceiving four quantities that are measurable: the mass of object 1, the mass of object 2, the distance between the two objects, and the gravity they exert. These four quantities in relation to one another constitute a quantitative relationship.

Mentally imagining two quantities' values (magnitudes) changing simultaneously is what research refers to as *covariational reasoning* (Carlson et al., 2002; Confrey & Smith, 1994). For instance, as we increase the mass of one of the two objects, gravity is also increased simultaneously. Previous research on covariational reasoning showed that it could help students in developing deep and meaningful understandings of various mathematical ideas, such as rate of change (e.g., H. L. Johnson, 2012; Saldanha & Thompson, 1998) and linear relationships (e.g., Ellis, 2011). Although many studies have been conducted to investigate the role of covariational reasoning in helping students to understand specific math topics, there is a limited number of studies that examine students' covariational reasoning within the context of science in ways that illustrate the reciprocal relationship between the two disciplines. However, what is available shows this potential. The results of a prior study in which students examined the covarying quantities underlying the science of the greenhouse effect showed that students developed sophisticated forms of reasoning about both the greenhouse effect and covariation (Basu & Panorkou, 2019). Similarly, in a study with preservice mathematics teachers, Gonzalez (2019) found that the teachers' covariational reasoning supported their understanding of a simple energy balance model for global warming, and this also shaped their understanding of the connection between CO_2 pollution and global warming.

These findings showed the potential of covariational reasoning for supporting the construction of this reciprocal relationship between math and science. In the current state of teaching and learning gravity, there are rarely any discussions about how the quantities change simultaneously. To put it another way, there is a difference between reasoning about the cause and effect relationship, for instance, the mass of the objects affects gravity, and reasoning covariationally about the relationship, for example, the gravity is changing as the masses or distance of the two objects are changing. This study of simultaneous change that covariational reasoning exhibits presented a promising route for supporting students' understanding of the phenomenon of gravity in more depth. We also hoped that reasoning covariationally about the relationships between mass, distance, and gravity could provide a constructive environment for students to reason mathematically, such as constructing meanings about the rate of change of the quantities or distinguishing between linear and nonlinear relationships.

Possible forms of covariational reasoning

Covariational reasoning can be numerical or non-numerical. As Thompson (1993) argued, the conception of a quantity is not necessarily numerical, and the individual may conceive a quantity if they recognize that it is measurable, whether or not they have carried out that measurement. To illustrate, Thompson (1994) distinguished between quantitative operations that are non-numerical and have to do with comprehending the situation and numerical operations that have to do with the arithmetic operations of evaluation. For instance, the child can conceive gravity as a quantity dependent on the masses of the two objects and their distance and also construct relationships between those quantities (e.g., as the distance between two objects increases, the gravity decreases) without assigning any numerical values to the quantities involved.

Reasoning covariationally using numerical values was first described by Confrey and Smith (1994) who explained that the notion of covariation involves moving between successive values of one variable and coordinating with the movement between successive values of another variable. For example, when the mass of object 1 increases from 100 kg to 200 kg (while distance remains constant), the gravity increases from 83.4 nN to 166.8 nN.

Saldanha and Thompson (1998) extended this idea to emphasize a continuous perspective of covariation, or that "if either quantity has different values at different times, it changed from one to another by assuming all intermediate values" (p. 2). In other words, this mental activity involves coordinating the two quantities with the awareness that both quantities have a value at every moment in time. For example, as the mass of one object grows continuously from 100 kg to 200 kg taking all intermediate values between 100 and 200, the gravity decreases continuously from 83.4 nN to 166.8 nN, taking all intermediate values between 83.4 and 166.8.

Building on both Confrey and Smith (1994) as well as Saldanha and Thompson (1998) work, Castillo-Garsow et al. (2013) proposed two contrasting images of change: *chunky* and *smooth*. The chunky images of change involve imagining change as generated by equal-sized chunks while smooth images of change involve imagining a change as a continuous progression. An example of chunky thinking is imagining how the quantities of the distance between two objects and gravity change according to incremental changes in the distance. In chunky thinking, one does not imagine that change happens within the chunk. In contrast, an example of smooth thinking is imagining how the distance between two objects and their gravity change as the distance changes continuously.

Note that when we refer to the coordination of two quantities, we mean the mental activity of coupling the two quantities, "so that, on one's understanding, a multiplicative object is formed of two" (Saldanha & Thompson, 1998, p. 299). Different studies have defined coordination in various ways, such as the work of H. L. Johnson (2015a) in which coordination was defined as involving smooth images of change. In this study, we consider the coordination of quantities to include both chunky and smooth images of change. In other words, we also characterize as coordination of quantities statements that coordinate the amount of change in gravity with changes in the value of mass or distance by a constant amount each time, for example, reasoning that as the mass increases by 1 m, the gravity increases by 3nN.

In a study exploring students' reasoning with functions, Carlson et al. (2002) proposed a framework that describes five mental actions of covariational reasoning that develop to reason about rate of change. In particular, these mental actions involve coordinating the change of one variable with changes in the other variable (MA1), coordinating the direction of change of one variable with changes in the other variable (MA2), coordinating the amount of change of one variable with changes in the other variable (MA3), coordinating the average rate-of-change of the function with uniform increments of change in the input variable (MA4), and finally coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function (MA5).

More recently, Thompson and Carlson (2017) revised this framework to include the work of Castillo-Garsow et al. (2013) on smooth and chunky images of change. Specifically, Thompson and Carlson (2017) described six levels of covariational reasoning that begin with the absence of a mental image of variables varying together (No coordination), envisioning two variables' values varying but asynchronously (Precoordination of values), forming a gross image of quantities' values varying together but still not envisioning that individual values of quantities go together (Gross coordination of values), coordinating the values of one variable with values of another variable (Coordination of values), envisioning changes in one variable's values happening simultaneously with changes in another variable's value and envisioning this variation as a chunky continuous variation (Chunky continuous covariation), and envisioning changes in one variables' values as happening simultaneously with changes in another variable's value and envisioning this variation smoothly and continuously (smooth continuous covariation). In this study, we utilize this framework to characterize students' covariational reasoning in the gravity context.

In addition to Thompson and Carlson (2017) framework of covariational reasoning, we also draw on Corley et al.'s (2012) construct of *co-splitting* to describe a specific form of covariational reasoning which establishes a ratio relationship between two quantities, such as any multiplicative change in one quantity is coordinated with the same multiplicative change in the other quantity in the same direction (increasing or decreasing). In a previous study (Basu et al., 2020), we found that in expressing the direct proportional relationship between the mass of an object and gravity, sixth-grade students conceived that as mass is changed multiplicatively, the gravity is also changed multiplicatively by the same factor. They also reasoned about the inversely proportional relationship between distance and gravity, such as conceiving that as the distance is doubled, the gravity becomes four times smaller than before. This form of covariational reasoning may provide a foundation for developing students' reasoning about ratio and proportion.

Research questions

To develop a learning opportunity for students to explore the reciprocal relationship between covariational reasoning and the science of gravity, we considered the important role of design in our efforts. Following a radical constructivist perspective of learning (Von Glasersfeld, 1995), we believe that every individual builds their knowledge through their own perceptual experience. We considered that although it is not possible to gain direct access to the student's knowledge or perception of experience, it is possible to observe their interactions and reasoning as they engage in learning activity and make inferences about their construction and reorganization of meanings from a researcher's perspective. We use the term *meaning* to refer to "the space of implications that the current understanding mobilizes – actions or schemes that the current understanding implies, that the current understanding brings to mind with little effort" (Thompson et al., 2014, p. 12). We identify and characterize these meanings via their forms of covariational reasoning as presented in the theoretical framework previously. We use the term *reorganization* (Piaget, 2001) of students' meanings to make humble inferences about their reflections and projections of particular meanings about the quantities of the phenomenon of gravity and their relationships to a higher conceptual level where these initial meanings become part of a more coherent whole.

We also considered that because knowledge is dynamically constructed through constructive activity, we can provide new designs and examine how students' forms of covariational reasoning in the context of gravity could be perturbed. In other words, we aimed to understand how students' meanings about varying quantities could be shaped and reorganized as students engage in activity with our design. We conceptualize activity with our design to include students' interactions with the tools, the task design, the representations, and the questioning that provided a space for students' construction or reorganization of meanings. As Noss and Hoyles (1996) state, "by analyzing shifting rather than inert states of understanding (or more exactly, performance), we have a greater chance of gaining insight into what is known, how that knowledge is mobilized, and the meanings which an individual constructs as new knowledge comes into being" (p. 77). Based on the above, this study explored the following research questions:

(1) What forms of covariational reasoning did students exhibit as a result of their engagement with our specific design in the context of gravity?
(2) What type of activity with our design provided a space for students' construction and reorganization of meanings about quantities and their relationships in the context of gravity?

Design framework and conjectures

While some studies showed the benefit of using physical manipulatives for studying the change in co-varying quantities, such as manipulating physical gears or dragging a ladder down a wall (Carlson et al., 2002; Ellis, 2011), other studies discussed the affordances of technological tools over the physical for illustrating the change in progress (Castillo-Garsow, 2012) as well as the ability to reverse change. Examples include using dynamic geometry software, such as Geometer's Sketchpad and Geogebra, or the Desmos graphic calculator for students to study the coordinated change in quantities (Ellis, 2011; Ellis et al., 2015; Stevens & Moore, 2016). Although each of these studies explored different quantitative relationships, the common element is that digital tools offer students the opportunity for direct and dynamic manipulation of quantities and a prompt to explore their coordinated and smooth change.

Similarly, research in science education has shown that the nature of the design that students engage with, which includes the mathematical tools and representations as well as questioning, shapes the depth of their science learning (e.g., Gweon et al., 2016; Sengupta & Wilensky, 2011). These studies identified the role of dynamic simulations as well as multi-agent-based models for helping students' investigation of science explorations and encouraging them to make claims about how factors of a scientific event affect another variable over time. Consequently, for our design, we considered the studies in both science and in covariational reasoning to design dynamic situations using digital tools in which students would not only engage with different quantities working together but would also be prompted to reason about those quantities covariationally. The following paragraphs describe our design and our initial conjectures for engaging students in specific forms of reasoning.

To design tasks that illustrate a change in the quantities related to gravity dynamically, we used the 'Gravity Force Lab' simulation from PhET (https://phet.colorado.edu), a research-based compilation of simulations for teaching and learning science. The PhET simulations were developed to actively engage students with the content and learning from that engagement (Wieman et al., 2008). The Gravity Force Lab simulation (Figure 1) is an open-exploratory model of a situation in which two people are pulling two objects (blue and red circles). The sizes of these objects represent their masses,

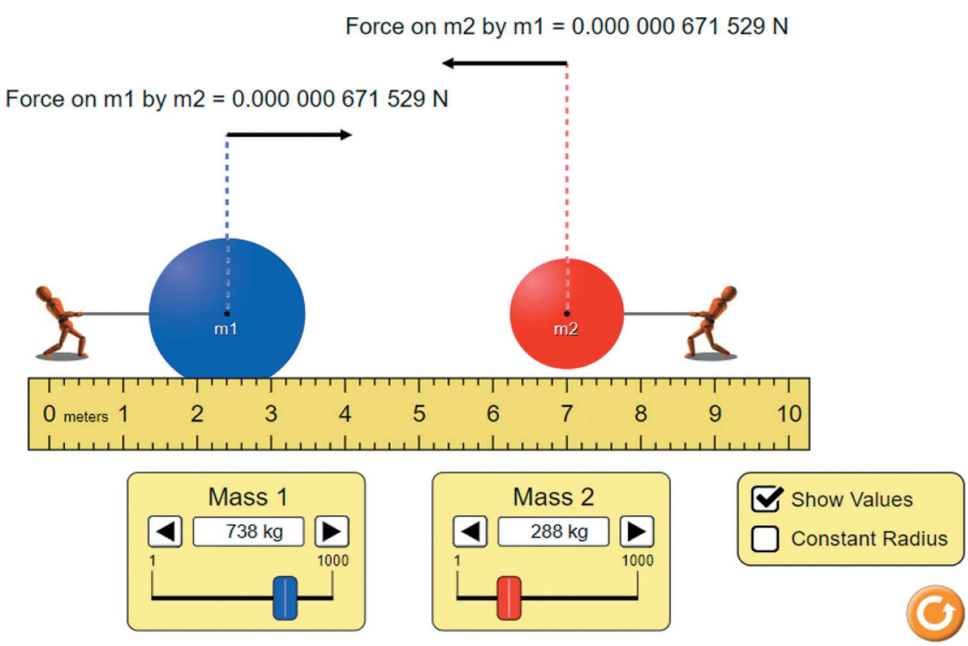

Figure 1. The Gravity Force Lab simulation; PhET Interactive Simulations, University of Colorado Boulder. URL: http://phet.colorado.edu/.

which can be adjusted using the sliders in the simulation. Gravity is represented by the lengths of the arrows on top of the objects. For example, if the mass of one of the objects gets bigger, the size of the object is made larger and the arrows representing gravity become longer. The distance between the two objects can also be changed by moving the objects away from or closer to each other. As the objects move closer to each other, the lengths of the arrows representing gravity become longer. At the bottom right of the screen, there is a “Show Values” check box that the user can select to make the gravity values appear or disappear. Our conjecture was that by experimenting with the Gravity Force Lab simulation and receiving instant feedback from the environment, students will identify the quantities that are involved in the phenomenon of gravity and construct quantitative relationships. We asked the students to freely explore the simulation and then engage in three sets of tasks.

For the first set of tasks, we asked the students to uncheck the “Show Values” checkbox in the simulation to hide the numerical values. Following Thompson’s (1993, 1994) research on non-numeric quantitative reasoning, the conjecture behind this choice was that the representation of mass, distance, and gravity as magnitudes would enable students to focus on the quantitative operations and relationships, as opposed to numeric operations and numbers as mere counts. The tasks in this set involved asking the students to manipulate the mass and distance in the simulation and reason about the change in gravity. Our conjecture was that in this first set of tasks students would reason covariationally by coordinating the change in the quantities’ magnitudes. Starting from generic questions, such as “What relationships have you found?” and “What is the relationship between the mass of one object and gravity?”, we moved to more specific questions, such as “What happens to gravity, as I increase/decrease the distance between the two objects?”, “What happens to gravity as I increase/decrease the mass of one object?”, and “How can I make the gravity bigger/smaller?”

For the second set of tasks, our goal was to examine students’ reasoning about the change in gravity when one of the quantities (mass or distance) changed multiplicatively (Basu et al., 2020; Corley et al., 2012). First, we asked them to make conjectures by imagining what would happen to gravity if the mass of one object, the masses of both objects, or the distance changes. Then, we asked students to check the “Show Values” box and search “for confirming-disconfirming evidence” (Wilensky & Reisman, 2006, p. 172) to either accept or refine their initial conjecture. Through this process, we anticipated that they would confirm or reorganize their meanings to include this co-splitting construct.

In the final set of tasks, we aimed to explore how students would reason about those relationships outside of the simulation environment through tables and graphs. The goal was to examine any reorganizations of their meanings as students engaged with these other representations. We asked students to use the simulation to collect data about the quantities in tables and also graph the mass versus gravity and the distance versus gravity relationships. Connecting dynamic representations of relationships with the graphing of those relationships was found to advance students’ conceptions of graphs of functions as a representation of coordinated change (e.g., Ellis et al., 2018). While we considered showing the developing graph or the quantities varying as distances on the coordinate plane while students manipulated the simulation similar to other studies (e.g., Frank, 2016; H. L. Johnson, 2015b), it was not possible to modify this simulation to include that feature. Therefore, we asked students to use the data to draw the graph on paper, hoping that they would also illustrate smooth images of change as they imagined the trace of the graph passing through every value for the quantities. To examine students’ reasoning about these representations, we included questions such as “What does the table/graph show?”, “Do you see any patterns in the table/graph?”, “What relationships does the table/graph show?,” and “Compare the mass versus gravity graph with the distance versus gravity graph. What do you observe?”

It is worth mentioning here that although the simulation presents the values for gravity using newtons (N) we asked students to convert to nanonewtons (nN) by having a third column in the tables. The reason behind this choice was that in a previous study with the same simulation (Basu et al., 2020) students illustrated a difficulty in reasoning about gravity using the correct place value (in terms of billionths). Therefore, in this study, we used the raw data of the simulation as a learning opportunity to discuss the need for unit conversions and asked students to convert to a unit that would be easier to work with, especially in graphing.

Methods

This paper describes a whole-class design experiment (Cobb et al., 2003) from a larger design research project aiming to engineer particular forms of covariational reasoning within the context of gravity and study those forms within the activity in which they were generated. The purpose of the design experiment was to build theories of student learning as well as to examine ways of supporting these forms of learning. The experiment is conjecture-driven, in the sense that the research team formed some initial conjectures about the means of supporting a particular form of learning (see Design framework and conjectures), and these conjectures were open for modification as the experiment unfolded.

Research setting and participants

A sixth-grade math class and their teacher from a school from the Northeast of the United States volunteered in this study. The district of the participating school has about 59% Hispanic or African Americans students and about the same percentage of students were classified as economically disadvantaged. Our partner school is also a low performing school, where only 28% of students met or exceeded expectations schoolwide in the average Partnership for Assessment of Readiness for College and Careers (PARCC) results (2017 to 2018).

Prior to the experiment, we made some conjectures about the students' prior knowledge of gravity and covariation. Based on the fifth-grade standard 5-PS2-1 of the Next Generation Science Standards (National Research Council, 2013), we expected that these students had some knowledge about gravity as the force that pulls objects toward the center of the Earth and that they could also express some cause and effect relationships between mass, distance, and gravity. In terms of covariational reasoning, in this grade, they are expected to begin to reason about relationships between varying quantities (Common Core State Standards for Mathematics 6.EE.9 and 6.RP.A.3) (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). At the first session of the experiment, we asked them what they knew about gravity. A student responded that "It's the force that is pulling you down so you can stay on the ground." Others talked about a sideways pull. Two students talked about the relationship with mass. One argued that "there's also another term for gravity, called zero mass. That means that you have no force going down." The other agreed saying that "the more they have, gravity also." None of the students discussed any relationship with distance.

The design experiment consisted of 5 one-period (45–50 minutes) sessions over 5 days. While the teacher took responsibility for the classroom instruction, three researchers sat with three pairs of students, respectively, aiming to create "a small-scale version of a learning ecology so that it can be studied in depth and detail" (Cobb et al., 2003, p. 9). The teacher nominated those students because they were more verbally expressive. We asked students to work in pairs so that they worked with a class partner with whom they felt comfortable and also because the pair interaction and discussion would act as a window for the researcher to interpret the activity. Although data were gathered from all the students in the class and the class discussions, in this paper we focus on the analysis of the three pairs of students for whom we were able to study the progression of their reasoning in detail and discuss the possible construction and reorganization of their meanings.

Data analysis

The unit of analysis is students' diverse forms of reasoning as they engaged in activity with our design (task design, tools, representations, and questioning) that integrates covariation with the phenomenon of gravity. In particular, the analysis purposely aims to capture the reflexive relation between a particular form of reasoning and the nature of our design. The analysis was guided by the orienting questions that Simon (2018) suggested for studying mathematics concept learning and instructional design (Table 1).

As the experiment unfolded, we conducted an ongoing analysis based on students' reasoning and revised our tasks and questioning accordingly during each session. This paper focuses mainly on the retrospective analysis (Cobb et al., 2003), which was conducted at the end of the experiment. During the retrospective analysis, the data here were analyzed in two stages. First, we worked through the data chronologically and identified students' episodes. We consider an episode to illustrate a single form of reasoning in the activity and discourse of students. We characterized students' meanings about covarying quantities based on the theoretical framework we constructed from our exploration of the different forms of covariational reasoning in the literature (Table 2).

As Table 2 illustrates, we characterized students' forms of reasoning based on the mental action framework of covariational reasoning by Thompson and Carlson (2017). We also looked for evidence of reasoning that coordinates the multiplicative change of one variable with a change in the other variable (Corley et al., 2012), especially for the second set of tasks that were designed having that notion in mind. While we had this theoretical framework as a base to guide our characterizations, we were also open to any new forms of covariational reasoning that students may exhibit that were not documented by these previous studies.

Three broad qualitatively different categories of meanings about quantities and their relationships were identified to be distinguishing the episodes, namely a) a gross coordination of quantities' magnitudes, b) a coordination of the multiplicative change of the quantities' values, and c) a coordination of values and partial chunky continuous covariation. Within each broad category, we then identified the qualitatively different ways that students expressed those meanings, investigated any reorganizations, and also examined the regularities and patterns in the ways that the students acted, interacted, and reasoned with particular forms of activity. The outcome of this analysis was a set of diverse ways that students expressed these meanings and the means that supported them in doing so.

Results

We present how students reasoned about quantities and their relationships in each of the three sets of tasks and make inferences about their construction and reorganization of meanings, aiming to address the research question *(1)*. In addition, to address the research question *(2)*, we describe the ways in which students' activity within the tools, tasks, representations, and questioning they encountered has supported their ability to build or reorganize their meanings. The above is described by presenting a set of episodes from the data of the three pairs we interviewed.

Table 1. Orienting questions guiding the analysis.

Forms of Covariational Reasoning (Research Question 1)	Type of Activity with our Design (Research Question 2)
• What forms of covariational reasoning did the students exhibit? What does this show about the students' construction or reorganization of meanings about covarying quantities at this point? • How can we characterize students' meanings about covarying quantities as they progressed from one form of reasoning to the next?	• How do the task, tool, representation, and questioning engage students in covariational reasoning? • What is the nature of students' covariational reasoning in each task or sequence of tasks? • How is students' covariational reasoning changed, modified, and refined in each task, question, and representation, or after a sequence of tasks, questions, and representations? What does this show about the students' construction or reorganization of meanings about covarying quantities at this point? • How did the task, questioning, and representation sequence afford and constrain students' construction and reorganization of meanings about covarying quantities?

Table 2. Theoretical framework that guides the analysis.

Forms of Covariational Reasoning (Research Question 1)	Type of Activity with our Design (Research Question 2)
Levels of covariational reasoning (Thompson & Carlson, 2017) • No coordination: No image of variables varying together. • Precoordination of values: Envisioning two variables varying but asynchronously. • Gross coordination: Forming a gross image of quantities' values (magnitudes) varying together. • Coordination of values: Coordinating the values of one variable with values of another variable with the anticipation of creating a discrete collection of pairs. • Chunky continuous covariation: Envisioning both quantities varying simultaneously and with chunky continuous variation. • Smooth continuous covariation: Envisioning both quantities varying simultaneously as well as smoothly and continuously.Co-splitting (Corley et al., 2012) • Coordinating the multiplicative change of one variable with change in the other variable.	First set of tasks: Hiding numerical values • Examining change based on magnitudes. • What relationships have you found? • What is the relationship between the mass of one object and gravity? • What happens to gravity as I increase/decrease the distance between the two objects? • What happens to gravity as I increase/decrease the mass of one object? • How can I make the gravity bigger/smaller?Second set of tasks: Showing numerical values • Imagining what would happen to gravity if the mass of one object, the masses of both objects, or the distance changes and then acting it out on the simulation to confirm or reorganize their reasoning.Third set of tasks: Relationships in tables and graphs • Using the simulation to collect data in a table and then using those tables to graph the relationships. • What patterns do you see in the table/graph? • What relationships do you see in the table/graph? • Compare the mass versus gravity graph with the distance versus gravity graph. What do you observe?

Gross coordination of quantities' magnitudes

At the beginning of the design experiment, students were asked to explore the interface of the simulation and describe the relationships between the different quantities. As aforementioned, we asked them to uncheck the tool "Show Values" of gravity to help them move beyond numerical computations and focus on the quantities and the relationships between them. First, the students dragged the mass slider of one object and examined the relationship between mass and gravity. During this exploration, Korbi [Pair A] noticed that "the mass and gravity they have a relationship" adding that "the greater the mass, the greater the gravity." Likewise, Izza [Pair B] stated that "I think the more the mass, the more the gravity." Daniel [Pair B] talked about the change in the masses of both objects, arguing that "The bigger the masses get, the more force of gravity it has for each other." Similar to Korbi, Izza, and Daniel, all students showed that they perceive the mass (of either body) and gravity as two quantities that co-vary and reasoned about the direction of change of these quantities. At this stage, students did not show evidence that they envisioned individual values of quantities changing together rather they illustrated a loose link between the overall changes in the two quantities' magnitudes. Therefore, we interpret this reasoning to exhibit a gross coordination of values (magnitudes) as per the Thompson and Carlson (2017) framework.

As aforementioned, one of the benefits of technology is the ability to reverse change. By increasing and decreasing the mass, students observed how the gravity arrows grew bigger and smaller, respectively (Figure 2). Their generalizations illustrated that they coordinated the direction of change in mass with the direction of change in gravity and acknowledged that the change in mass can be an increase or a decrease. For instance, Alexa [Pair C] stated that "the larger the mass, the more gravity it's going to exert or the more force it is going to exert on the other object. The less mass, the less of a force it exerts." Similarly, Daniel [Pair B] explained that "if they [the mass balls] get smaller, the force of gravity gets weaker – the bigger the mass, the more force of gravity there is."

Next, students manipulated the distance between the two objects by dragging them toward and away from each other and observed the change in the gravity arrows (Figure 3). As they dragged the objects closer and further from each other all students reasoned covariationally about the relationship between distance and mass. For instance, Daniel [Pair B] argued that "the farther away they [the objects] are, the less the force of gravity." Similar to Daniel, students showed that he could coordinate

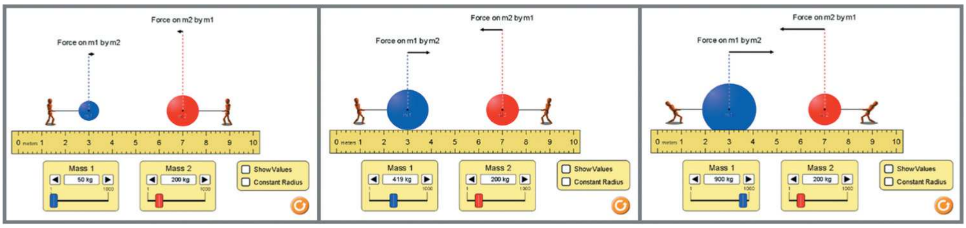

Figure 2. Illustrating the change in gravity in the length of the arrows as the mass of one object is changing.

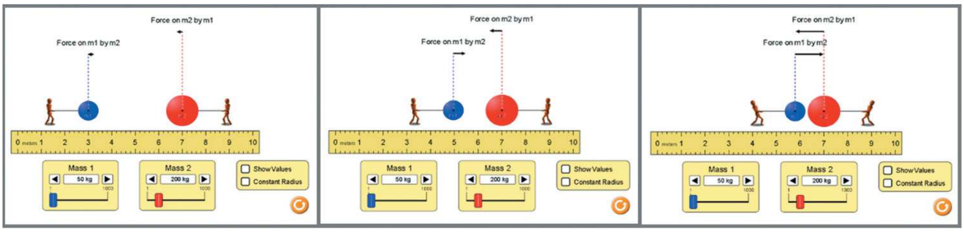

Figure 3. Illustrating the change in gravity in the length of the arrows as the distance between the two objects is changing.

the direction of change in distance with the direction of change in mass, illustrating a gross coordination of magnitudes as per Thompson and Carlson (2017) similar to their reasoning before about mass. Also similar to mass, the simulation allowed students to also reverse the change and this led students to coordinate the change in gravity with both an increase and a decrease in distance. For instance, Korbi [Pair A] reasoned that "When the distance is smaller, that is when the gravity is bigger; when the distance is further, then the gravity is less stronger." Alexa's [Pair C] explanation is also an example of this reasoning:

Alexa: When the distance decreases, the force on the mass of the object gets larger. But as you increase the distance, the force gets smaller . . . the larger the distance, the more spread apart they are, the less force being applied. The closer they are, the more force.

[Excerpt 1, Pair C]

We also noticed that students' use of language alone might illustrate diverse images of change, therefore we were looking for further evidence in the data in order to characterize them. For instance, in Excerpt 1, Alexa's use of the language "as you increase the distance, the force gets smaller" might show that she perceives distance and gravity varying simultaneously. However, her previous statement in the same excerpt "When the distance decreases, the force on the mass of the object gets larger" shows that she might be considering a cause and effect relationship between distance and gravity, where the distance decreases first and then this has an effect on the force. One might interpret that this reasoning is illustrating a precoordination of magnitudes (Thompson & Carlson, 2017) because the student envisions the two quantities changing asynchronously. This is an example of a case where language alone does not provide sufficient evidence for interpreting students' image of change. Therefore, we had to look at students' overall activity to make our inferences.

Additionally, according to Castillo-Garsow et al. (2013), a characterization of a smooth image of change is the perception of a situation as continuous motion in progress. In Alexa's case, we noticed that when Alexa was describing the relationship between distance and gravity, she was illustrating her thinking by moving the objects in the simulation so that the distance was smoothly increasing and decreasing between them, showing that she perceived this change to be continuous and smooth. By interacting with the smooth tools of the simulation, all students perceived this continuous motion in progress and this might have led them to a construction of smooth images of change. We use the term 'might' here because there is the possibility that their actions in the simulation looked smooth because of the tools regardless of if their conceptions were smooth or chunky. For this reason, we could not characterize their reasoning as illustrating smooth continuous covariation as per Thompson and Carlson (2017). Nevertheless, these experiences could be beneficial for supporting students' development of this form of reasoning.

In contrast to other studies that focus on the covariation of only two quantities (e.g., Carlson et al., 2002, 2001; Ellis et al., 2015; Frank, 2016; H. L. Johnson, 2015b), the exploration of scientific phenomena usually involves more complex relationships with more than two variables. To prompt the students to construct a relationship relating all three quantities, we asked, "What is the relationship between mass, distance, and gravity?" In response to this question, Alexa [Pair C] stated, "As the distance increases, the force decreases. But then as you increase the mass, the force increases." Alexa showed that she considered both the change in mass and the change in distance as resulting in a change in gravity. Daniel [Pair B] also reasoned in a similar way:

Researcher: What's the relationship between the mass, distance, and gravity?

Daniel: The farther the distance is, the weaker the force of gravity is. And the closer they are, the stronger it is. And that also is the same to size, if the size gets smaller, the weaker it is. And if it gets bigger, they get stronger.

[Excerpt 2, Pair B]

Daniel explicitly described the changes in gravity when the mass and distance increased and decreased. Similar to Daniel and Alexa, although students identified that both a change in the mass and distance cause a simultaneous change in gravity, they did not provide any evidence that they could coordinate all three quantities (mass, distance, and gravity) at the same time. Therefore, in the next task, we asked students to discuss different ways to increase gravity in order to support them in coordinating all three quantities. The following excerpt presents George's response [Pair C]:

George: If you increase the mass of this one [dragging mass slider 2], of both of them [dragging mass slider 1], the larger, the force is bigger. As you decrease them [dragging them both smaller], it's not, the gravity on them, both of them, is not as big as if it were larger than that. And the distance also, cause if you have both of them [masses of objects] like small, it [the gravity] would still be big if you had them closer together [dragging the objects closer] than farther away.

[Excerpt 3, Pair C]

George's reasoning in the last sentence above shows that he coordinated the overall changes in three quantities' magnitudes simultaneously, illustrating a gross coordination of three quantities' magnitudes. His statement that even if both masses are small, the gravity will still be big if they are closer together, shows that he reorganized his previous meaning of the two relationships into one unified construct where all three quantities work together. Similar to George, we noticed that this prompt encouraged the students to exhibit multivariational reasoning as they coordinated multiple quantities changing simultaneously.

Activity fostering students' gross coordination of magnitudes

Table 3 presents an overview of students' forms of reasoning that illustrate gross coordination of magnitudes and the activity with our design that might have supported those forms of reasoning.

Specifically, this first set of non-numeric tasks provided a space for students to reason about the quantities and their relationships by studying the change in their magnitudes without actually measuring the objects or assigning any numerical values to the quantities involved. We would argue that these experiences supported students in reorganizing their prior meanings about gravity (see Research setting and participants) into more sophisticated meanings. Specifically, by experimenting with the simulation, all pairs noticed that there are three quantities varying as part of the gravitational force phenomenon: mass, distance, and gravity. Their reasoning also illustrated a gross image of

Table 3. Overview of students' gross coordination of magnitudes.

Forms of Covariational Reasoning (RQ 1)	Type of Activity with our Design (RQ 2)
• Gross coordination of two quantities' magnitudes ○ e.g., "the farther away they are, the less the force of gravity." • Gross coordination of two quantities' magnitudes and reasoning about both increase and decrease. ○ e.g., "When the distance decreases, the force on the mass of the object gets larger. But as you increase the distance, the force gets smaller." • Gross coordination of three quantities' magnitudes ○ e.g., "if you have both of them like small [masses], it [the gravity] would still be big if you had them closer together [dragging the objects closer] than farther away."	• Students examined change based on the smooth growth of two quantities' magnitudes and illustrated a gross coordination as they were asked: ○ What relationships have you found? ○ What is the relationship between the mass of one object and gravity? ○ What happens to gravity as I increase/decrease the distance between the two objects? ○ What happens to gravity as I increase/decrease the mass of one object? • The smooth dragging of sliders and the growing circles for mass and growing arrows for gravity might have supported students in constructing smooth images of change of those quantities. • As they changed and reversed change in both directions they reasoned about the relationships of the quantities as mass and distance increased and decreased. • In responding to "How can I make the gravity bigger/smaller?" they illustrated a gross coordination of three quantities' magnitudes.

quantities' magnitudes varying together, what Thompson and Carlson would characterize as gross coordination of values.

The dynamic tools and models of the simulation (e.g., dragging sliders, growing circles for mass, growing arrows for gravity) allowed students to change the mass and distance of the objects continuously and perceive the continuous and smooth change in gravity as the gravity arrows grew bigger or smaller. This might have also supported their construction of smooth images of change about these quantities. By being able to both increase and decrease the masses and the distance, they were able to observe the change in magnitudes in both directions and coordinate the change of gravity with the change of mass and distance both increasing and decreasing. Finally, the open-ended question "How can I make gravity bigger/smaller?" gave them the agency to reorganize their meanings from having an image of two separate relationships (mass-gravity and distance-gravity) into a gross image of all three quantities' magnitudes varying together, thus exhibiting multivariational forms of reasoning.

Coordination of the multiplicative change of the quantities' values

For the second set of tasks, we asked students to activate the tool "Show Values" of gravity to help them reason numerically about the quantities as they change with respect to each other. First, students were asked to imagine the change in gravity if they doubled, tripled, or halved the mass of one object and then act it out in the simulation to verify or reorganize their reasoning. All students made correct conjectures about the change in gravity and showed evidence that they coordinated the multiplicative change in mass with the change in gravity. For instance, Alexa [Pair C] reasoned, "I thought when I doubled it, the gravity would be doubled," arguing that her conjecture was correct. She also discussed her conjecture about tripling the mass, stating, "When it said what do you think will happen when you triple the mass of one of the objects, I thought the gravity would have tripled. And the gravity in fact did triple." Similar to Alexa, through this process, students reorganized their meanings of gross coordination of magnitudes into conceptually-higher meanings that included a coordination of the multiplicative change of the two quantities. For instance, Daniel [Pair B] argued that "The gravity of the two objects is tripled if we triple one of the object's mass" and "The force of gravity gets cut in half because we halve the mass of one of the objects."

By forming conjectures and getting feedback from the simulation, students began seeing a pattern. For instance, Rico [Pair A] argued that "The force will be multiplied by 4 [if mass is multiplied by 4]. And if you multiply the mass by 2, the force will be multiplied by 2. If you multiplied by 3, the force will be multiplied by 3." When they were asked to imagine the change in gravity when the mass became 100 times bigger, Alexa argued that "It will become 100 times bigger." From this exploration, they were able to reorganize these meanings about multiplicative change into a generalization that when the mass of one object is changed multiplicatively, then gravity also changes by the same factor, a construct defined by Corley et al. (2012) as co-splitting. Korbi's [Pair A] response illustrates this type of reasoning:

Researcher: What relation do you see between mass and gravity?

Korbi: That each time the mass increased, the gravity also increased by that number. So then, if mass 1 increased, in this case by 3, it triples, so then gravity also tripled, by 3.

[Excerpt 4, Pair A]

Reasoning about the multiplicative change of quantities can be very powerful because it illustrates that students do not focus on individual values of quantities changing together but rather consider the quantity as a variable that can take any value. Korbi's reasoning in Excerpt 4 "that each time" the mass changed the gravitational force also changed by that number along with his example illustrates that he does not focus on specific values but rather provides a relationship that works for any multiplicative change in gravity. This reasoning could be foundational for developing students' smooth continuous covariation which is the highest level of the Thompson and Carlson (2017) framework.

Students were then asked to form a conjecture about the change in gravity when both the masses of the two objects are doubled or tripled. For doubling both masses, Alexa [Pair C] imagined that the gravity "will be quadrupled … because it doubles for each." However, when she was asked to explain the change in gravity when both masses tripled, she argued that it would be six times bigger:

Alexa: I thought it would become six times bigger, because three times two is six.

Researcher: But what happened when you tried it?

Alexa: It became nine times bigger.

Researcher: How does that change what you thought was going on?

Alexa: Because I thought it would have been three times two is six, but it was three times three is nine.

[Excerpt 5, Pair C]

The excerpt above is an example of a student reorganizing their reasoning as they tried to explain the behavior of the simulation that was different from their conjecture. The feedback from the simulation led Alexa to reorganize her meanings about finding gravity from doubling the multiplicative change (in this case, tripling twice) into multiplying each of the quantities with that change ("three times three is nine"). Rico [Pair A] explained this reorganization more clearly. First, he made the same conjecture, arguing that "Because if you triple twice, then, three times two is six." However, after receiving feedback from the simulation he realized that it becomes nine times larger and this led to a reorganization of his thinking to explain the simulation's behavior. He argued that this happens because "It will be tripled after you changed one mass [pointing at mass 1] and it will be tripled when you tripled the other mass [pointing at mass 2]." By asking them to explain why the gravity became nine times bigger, students were led to reconsider their reasoning, reorganize their thinking, and search for another explanation, leading them to shift from additive to multiplicative forms of reasoning. The conversation below is another example of this reorganization:

Researcher: If you triple both the masses of the objects, what happens to the gravity?

Daniel: They got times 9.

Researcher: Why times 9?

Daniel: I think it's going by multiplication now because before I said it was 2 times 2 equals 4, so it did quadruple, and now it's 3, so 3 times 3 equals 9.

Researcher: What did you do?

Daniel: I multiplied by 9 . . . Because I tripled this [mass 2] and I tripled this [mass 1] and 3 [pointing at mass 2] times 3 [pointing at mass 1] equals 9.

Researcher: Do you agree, Izza?

Izza: Yes, because I tripled both of them [pointing at the mass sliders of the two objects]. And since they are both tripled and I got them 375 [gravity]. And they should, I think what Daniel said would not the addition but the multiplication. So, since I tripled both of them, so 3 times 3 equals 9.

[Excerpt 6, Pair B]

Similar to Daniel, Izza, Alexa, and Rico, students were able to coordinate the multiplicative change in gravity when both masses changed and identify the proportional relationship between mass and gravity by forming conjectures and getting feedback from the simulation. Students used this understanding to reason about the change of gravity if both masses were quadrupled, arguing for instance, that gravity "will be multiplied by 16" [Rico, Pair A].

Next, we asked the students to think about how gravity would change if they doubled the distance between the two bodies. For example, Alexa [Pair C] argued that gravity "in this case, it is divided by 4" while Daniel [Pair B] noticed that "it gets 4 times smaller." Students then used this understanding to reason about the change in gravity when the distance is halved. The excerpt below illustrates Daniel's [Pair B] reasoning:

Researcher: What happens if we halve the distance between the two objects?

Daniel: If we double the distance, it got 4 times smaller. So, if we halve the distance, it should be 8 times bigger, no 4 times bigger, wait. Yeah, it should be 8 times bigger.

Researcher: Can you try it?

Daniel: I'm changing my opinion. OK, let's see. If it's 4 inches away, it should be 2 [moving the objects closer and measuring the distance using the ruler on the screen]. Alright you said it, how big it gets. Let's see. 166–167.

Izza: That is really close.

Daniel: OK, we got it. It was 4 times bigger.

[Excerpt 7, Pair B]

The excerpt above shows how Daniel needed to reorganize his meaning as he observed that the gravity only became four times bigger instead of eight times bigger as he conjectured. Similar to Daniel, we noticed that students struggled in defining the multiplicative change in gravity when distance was halved. Although they knew that the two quantities have an inversely proportional relationship, they struggled in finding the pattern in the change of the two quantities. The feedback from the simulation was useful in helping them modify their conjectures but we have limited information about whether students actually understood why the specific change was happening and indeed reorganized their meanings. When Daniel was asked to explain why gravity changed that way, he referred back to his gross image of quantities' values varying, stating, "The farther away it gets, the weaker the gravity is."

Another example is Rico who argued that "it will be multiplied by 4, each ball will be multiplied by 2" because "we are moving these balls 2 meters apart, and 2 times 2 is 4." In other words, he was multiplying the additive change in the distance of each object with the number of objects moving. He later noticed that this explanation did not work for other values of distance. For example, when we asked him to halve the distance when it is 6 m to 3 m he thought that gravity will become 6 times smaller because "they are moving 3 meters, there are two balls moving, so we have to multiply by 2," in other words multiplying the additive change with the number of objects moving. When the simulation showed that gravity became four times bigger, he could not provide an explanation of why this was happening.

Activity fostering students' coordination of the multiplicative change of the quantities' values

Table 4 presents an overview of the students' forms of reasoning about the multiplicative change of the quantities and the activity with our design that might have supported those forms. Specifically, this second set of tasks included three types of explorations: changing the mass of one object multiplicatively, changing the masses of both objects multiplicatively, and changing the distance between the objects multiplicatively. The sequence of tasks in the first exploration started from simple doubling and tripling the mass of one object and then moved to halving. This sequence supported students in identifying a pattern and reorganize their meanings about the mass versus gravity relationship to include the generalization that if the mass of one object changes by a factor, then the gravity changes by the same factor, characterizing in that way the direct proportional relationship between mass and gravity.

By asking them to predict what would happen to gravity if they multiplicatively changed the masses of both objects, students were encouraged to put their reasoning in harms' way, test their conjecture, and re-organize their meanings about gravity to include a coordination of the multiplicative change of the quantities. The feedback they received from the simulation led them to reflect on their thinking to explain the simulation's behavior. Questioning such as, "How does that change what you thought was going on?" led them to reorganize their meanings from describing the change in gravity by adding the change in the two masses into multiplicative forms of reasoning.

As students explored the multiplicative change in distance, we noticed that while students were able to predict the multiplicative change of gravity when distance was doubled before acting it out on the simulation, they struggled to explain why gravity would become four times larger when the distance is halved. Students could not find a pattern that would relate the change in distance with the inverse change in gravity. Possibly this is because of the nature of the inversely proportional relationship which is harder to identify from two sets of values. It can also be a limitation of the design. Specifically, we would like to examine whether asking them to triple or quadruple the distance before halving it might have yielded different forms of reasoning.

Table 4. An overview of students' coordination of the multiplicative change.

Forms of Covariational Reasoning (RQ 1)	Type of Activity with our Design (RQ 2)
• Coordination of the multiplicative change of mass and gravity ○ e.g., "That each time the mass increased, the gravity also increased by that number." • Construction of smooth images of change ○ e.g., "that each time" one changes multiplicatively, the other changes by the same factor. • Coordination of the multiplicative change of distance and gravity ○ e.g., "For every time you double the distance between the two objects, it becomes four times smaller of what it was."	• The progression of tasks beginning first with doubling, then tripling, and then halving masses helped students to see the pattern and reorganize their meanings about the relationship to include the co-splitting generalization. • By imagining the change in gravity and making a conjecture before acting it out they put their prior understandings into harm's way. • By receiving feedback from the simulation and trying to explain the simulation's behavior they reorganized their meanings from additive into multiplicative forms when reasoning about the change in gravity when both masses are changed multiplicatively. • In responding to "How does that change what you thought was going on?" they were able to reflect on their reasoning and articulate in words the reorganization of their meanings.

Coordination of values and partial chunky continuous covariation

In the third set of tasks, we asked students to use the simulation to collect data and represent the relationships of mass versus gravity and distance versus gravity in tables in graphs. The goal was to examine whether these different representations would support new reorganizations of the students' meanings. First, we asked the students to manipulate the model in the simulation so that the mass of one object is 1 kg, the mass of the other object is 200 kg, and the distance between them is 2 m. Then, they were asked to complete Table 5 and describe any patterns they notice.

To describe the table, students focused on the increments of change in mass to reason about the amount of change of gravity. The excerpt below illustrates this reasoning from Alexa [Pair C]:

Researcher: What pattern did you see in that table [mass and gravity]?

Alexa: That for every one kilogram added to the mass, there was three nanonewtons added to the force.

[Excerpt 8, Pair C]

Alexa's reasoning involved coordinating the uniform increments in the values of gravity with the uniform increments in the values of mass, illustrating coordination of values as per the Thompson and Carlson (2017) framework. Although her language shows that she acknowledged that this rule works for any value of gravity, we do not have evidence that she envisioned the mass and gravity changing during the intermediate values. Therefore, we interpret her reasoning to illustrate a chunky image of change. Similar to Alexa, all other students were able to reason in this way to explain the relationship in the table. Aiming to support them to reorganize this chunky reasoning to also include a more continuous representation of change, we then asked them to imagine what the graph would look like based on this relationship. Rico and Korbi [Pair A] argued that it would be a straight line:

Researcher: If you have to plot it, can you tell me how the graph will look like?

Rico: A straight line ... up by 3.

Korbi: Straight line.

Researcher: Why straight though?

Korbi: Because it is consistent. 3, plus 6, 6 plus 3 is 9, 9 plus 3.

Researcher: Why is it straight again?

Rico: Because it is going up by same number each time. In math, we do things like this and it would be straight line.

[Excerpt 9, Pair A]

Table 5. Daniel's completed table representing the relationship between mass and gravity.

Mass of one object in kg	Gravity Force in N	Gravity Force in nN
1	0.000 000 003 N	3 nN
2	0.000 000 006 N	6 nN
3	0.000 000 009 N	9 nN
4	0.000 000 012 N	12 nN
5	0.000 000 015 N	15 nN
6	0.000 000 018 N	18 nN
10	0.000 000 030 N	30 nN
100	0.000 000 300 N	300 nN

The excerpt above shows that both students agreed that the graph for mass and gravity is a straight line because the value of gravity increases by a constant amount each time the value of mass increases by 3. Similar to Pair A, all students used the uniform amount of change in both quantities to reason about why the graph would be a straight line. Students then plotted the ordered pairs and then connected them to form a smooth straight line similar to Rico's graph in Figure 4. Only one student plotted the pairs without connecting them. However, this is not strong evidence that they had a smooth image of change as one might argue that students might have connected the points simply because of a convention they learned in class.

Similar to Rico's graph, students' drawings were smooth and were not restricted only to the plotting of the specific ordered pairs of the table. For instance, his graph starts from (0,0) which is not part of the table of ordered pairs. Figure 4 also shows that Rico viewed the values of each quantity to be extended infinitely. His line is also extended outside of the graph paper showing that he might have perceived the graph not as a static image but as something more continuous. Consequently, we might interpret students' reasoning and actions to illustrate a reorganization of meanings from a coordination of values into a partial chunky continuous covariation (Thompson & Carlson, 2017). We say "partial" because their image through graphing entailed intermediate values but we do not have evidence that they had an image of the quantities actually having those values.

Similar to mass, we asked students to explore the distance versus gravity relationship in tables and graphs. We first asked them to manipulate the model in the simulation so that the mass of each object was set to 50 kg and the objects were 2 m apart, and then we asked them to change the distance between the two bodies by 1 meter and determine the amount of change in gravity to complete Table 6.

When we asked the students if they saw a pattern in Table 6, we noticed that the students were having difficulties in identifying a numeric relationship, for example, reasoning that as the distance between the two objects increases by 1 meter the gravity force is approximately halved, or that as the

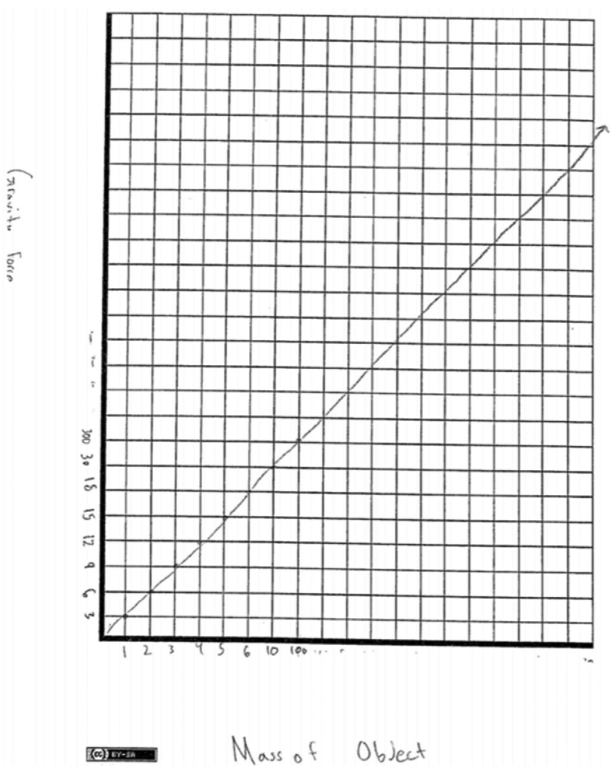

Figure 4. Rico's graph representing mass versus gravity.

Table 6. Alexa's completed table representing the relationship between distance and gravity.

x Distance between two objects in m	Gravity Force in N	y Gravity Force in nN
2	0.000 000 041 N	41 nN
3	0.000 000 018 N	18 nN
4	0.000 000 010 N	10 nN
5	0.000 000 006 N	6 nN
6	0.000 000 004 N	4 nN
10	0.000 000 001 N	1 nN

> by 8 distance

distance is doubled, the gravity force becomes approximately four times smaller. The following conversation with Alexa [Pair C] illustrates this struggle:

Alexa: I don't like the other table [Table 6].

Researcher: Why don't you like the other table?

Alexa: Because it was more complicated.

Researcher: How was it more complicated? What was the relationship?

Alexa: For every … I forgot. Ten or five … I know it was something with ten and five. It was like for every time you double two or every time you double something, it was something else. I know that.

[Excerpt 10, Pair C]

Alexa's language "for every time you double something, it was something else" shows that she conceived the two quantities covarying but she could not express this relationship numerically as she could not find a rule. Note that during the simulation exploration she was able to state that "it is divided by 4" but she could not recall or discover this relationship in the table. After prompted by the researcher to state what happens to gravity if distance is doubled, Alexa looked at the table and argued that "For every time you double the distance between the two objects, it becomes four times smaller of what it was." However, similar to Alexa, all students had difficulty in reasoning about the relationship between distance and gravity in the table numerically although they had reasoned about it in previous activities. There are two possible reasons for this difficulty. First, students might have been searching for an additive pattern in the increase of the values of gravity, similar to their experience with mass which was the activity that preceded this one. Second, as illustrated in Table 6, the rounding of the real data in the simulation caused the values not to show a perfect relationship (e.g., the value of gravity as 18nN is *approximately* half of the previous value of 41nN but not *exactly* half) and this might have impeded students from seeing a multiplicative pattern in the values of gravity.

When they were asked to state what the graph would look like, students argued that it would not be a straight line. The explanation below from Korbi [Pair A] illustrates this form of reasoning.

Korbi: I do not think it will look like straight …. because, in order to get straight, it must have a certain pattern. This one doesn't have, I don't see any pattern. There are some odd numbers, and there are some even. So, I don't think there will be any straight line … it has to have something, then add that, then add that by, so let's say it starts by 1, then add that by 4, then add that by 4 again and again and again. It has to be constant … well if it is like the same pattern, you know like add 2, add 2, add 2, add 2, then it will be like straight. If it is like, add 2, then add 4, then add 8, then it will be curved up.

[Excerpt 11, Pair A]

Similar to Korbi, the students were looking for an arithmetic pattern to describe the change in the values of gravity, but they could not find a constant change. This might again be the result of the presentation of raw data in tables, which do not illustrate perfect relationships. Although in the previous activities they were able to identify that by doubling distance, the gravity becomes four times smaller, they could not coordinate this multiplicative change or the amount of change of the two quantities by looking at the table. When Daniel [Pair B] was asked to tell a relationship that he saw in the table, he stated, "the distance gets bigger while the force gets less," illustrating a gross coordination of values.

We then asked them to use the values of Table 6 to construct the distance versus gravity graph and compare it with the mass versus gravity graph, aiming to explore whether this comparison would trigger any new reorganizations of students' meanings. All students connected the ordered pairs to form a curve but not all students drew this curve as smooth as the straight line they formed in the other graph. This was probably because they have not worked with curves as much as they had with linear relationships. Another possible reason might be that they were thinking in chunks about this relationship. The conversation that follows gives more insight into their conception of the relationship.

By comparing the two relationships graphically, Alexa [Pair C] described the graphs as "the other graph was a linear kind of graph [Figure 4], and this graph is curved [Figure 5]." When they were asked to explain why the nature of the two graphs is different, students reasoned in terms of the uniform increments of mass and gravity, and the uneven increments of distance and gravity. For instance, Daniel [Pair B] argued that "This one [mass versus gravity graph] has a pattern of multiples of 3, this one [distance versus gravity graph] doesn't have a pattern." Similar to the tables, we noticed that it was difficult for students to identify the pattern in the distance versus gravity graph and this might have

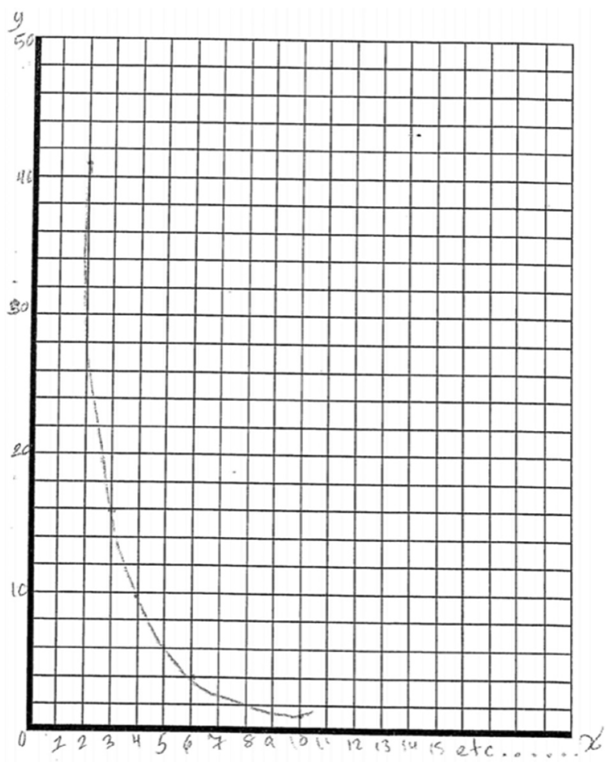

Figure 5. Alexa's graph representing distance versus gravity (she defined x and y in her table as distance and gravity respectively).

been the result of showing a more chunky image of change in their graphs. When we asked Daniel to elaborate, he stated,

Daniel: No, no pattern.

Researcher: Why not?

Daniel: Because there's not a set number of patterns. The only thing I see is like the bigger this is, the lower this is. That's all I see.

Researcher: What's bigger? Can you state that in complete sentence?

Daniel: The larger the distance between the two objects gets, the less the gravitational force gets.

[Excerpt 12, Pair B]

We interpret Daniel's reasoning to show a gross coordination of values because although he formed a gross image of the two quantities' values varying together, he does not show that he envisions individual values of distance and gravity changing together. Korbi [Pair A] illustrated this same struggle:

Korbi: I think there is [a pattern] but I just don't know what it is. Each time, I know each time I decrease, increase this [showing the distance], I know this [showing the gravity] must be decreasing by the same rate.

Researcher: It is decreasing by the same?

Korbi: Rate, as distance is increasing. There is a connection, I know. But I just don't, you know.

[Excerpt 13, Pair A]

Similar to Korbi and Daniel, students stated that although they saw that as distance is increasing, gravity is decreasing, they could not find a pattern to explain this change in gravity. When we asked them to compare the relationships in the two graphs Daniel looked at the graphs and argued, "If the mass of this gets bigger, then the gravitational force gets bigger. But if the distance gets bigger, the gravitational force gets less," showing that he perceived the two quantities in each graph varying together and possibly as a smooth image of change, but could not reason in terms of chunks or specific values. In a similar manner, all students described the graphs by reasoning about both quantities in each graph, showing that they perceived the graph as representing two quantities simultaneously.

Activity fostering students' coordination of values and partial chunky continuous covariation

Table 7 presents an overview of students' forms of reasoning about the coordination of values and partial chunky continuous covariation and the activity with our design that might have supported those forms.

In the third set of tasks, we aimed to use students' exploration of the relationships in tables and graphs to examine different reorganizations of meanings that these representations might have helped them to construct. By using the simulation to collect data in a table, students' focus shifted to the amount of change of gravity as mass was changing in uniform increments, illustrating a coordination of values (Thompson & Carlson, 2017) that was not evident in previous activities. During the graph activity, most students' graphs illustrated a smooth image of change about the relationship between mass and gravity. However, their responses illustrated both smooth and chunky images of change, which we interpreted as showing a partial chunky continuous covariation (Thompson & Carlson, 2017). This might be the result of using the chunky nature of the table to plot a continuous graph, an activity that could have helped them bridge the two images. It might also be the result of our questioning. When students were asked whether they saw a pattern in the tables and graphs, they

Table 7. Overview of students' coordination of values and partial chunky continuous covariation.

Forms of Covariational Reasoning (RQ 1)	Type of Activity with our Design (RQ 2)
• Coordination of the change in the values of mass and gravity ○ e.g., "That for every one kilogram added to the mass, there was three nanonewtons added to the force." • Partial chunky continuous covariation by reasoning that a graph would be a straight line because the value of gravity increases by a constant amount each time the value of mass increases by 3. ○ e.g., "Because it is consistent." • Illustrating smooth images of change in graphing. ○ e.g., Smooth drawings, extending the graph beyond the given ordered pairs. • Gross coordination of the values of distance and gravity ○ e.g., "If the mass of this gets bigger, then the gravitational force gets bigger. But if the distance gets bigger, the gravitational force gets less."	• In examining data in a table, students' focus shifted to the change in the quantities' values, illustrating more chunky images of change. • In responding to the question "What relationships do you see in the table/graph?" students illustrated a gross image of the two quantities changing together. • In responding to the question "Do you see any patterns in the table/graph?" students coordinated the change in the values of mass with the change in the values of gravity. The raw data gathered in the table that did not show a perfect relationship might have impeded them from coordinating the change in the values of gravity with the change in the values of distance. • By comparing the two graphs, they distinguished between a linear graph which has a consistent pattern and a curve which does not have a linear consistent pattern they were looking for.

reasoned about the simultaneous change in the values of the two quantities and in terms of chunks. But, when students were asked to state the relationship they saw in the graph, they illustrated a perception of a smoother image of change of the two quantities' values.

Students' experiences with the distance versus gravity graph was a bit different. Although students focused on the change of the values of the two quantities during the table exploration, they struggled in identifying the change in the values of gravity as the values of distance changed uniformly. We were also surprised that students did not connect the multiplicative change generalizations they formed during the simulation exploration to reason about the same relationships in these new representations. This might be the result of students searching for an additive pattern in the increase of the values of gravity as the values of distance changed additively, which would be a similar type of relationship as with mass. It might also be a limitation of our sequence of tasks. Possibly students might have made those connections if the exploration of the distance versus gravity relationship in tables and graphs came after the exploration of the same relationship in the simulation. We also acknowledge that this might be a result of the design, which did not show a clear pattern in the raw data. Despite not finding a pattern, students described the graph by pointing to the direction of change of the two quantities, illustrating a gross coordination of values. We also noticed that their graphs were not as smooth as the ones they constructed for the mass versus gravity relationship. This might be the result of not being able to coordinate the uniform change in values of gravity in relation to the change in the values of distance.

Conclusion: Designing a learning space for integrated math and science reasoning

This study explored students' integrated math and science forms of reasoning as they explored the covarying quantities involved in the science phenomenon of gravity. The findings showed that students' forms of reasoning integrated both mathematics and science as one unified construct, avoiding disconnected disciplinary learning that other studies have previously noted (Honey et al., 2014; Tytler et al., 2019). Students' constructions of meanings cannot be categorized as illustrating math or science reasoning, rather they exhibit the reciprocal relationship (Fitzallen, 2015) between the two disciplines.

Our findings showed how covariational reasoning can influence and contribute to the understanding of scientific ideas. Specifically, the examination of gravity through a quantitative and covariational reasoning lens showed to have helped students avoid the naïve conceptions of gravity reported in the literature. By examining the quantities that are changing and also how they are changing together, students reorganized the initial meanings they had about gravity (see Research setting and participants) into higher-order meanings that included an understanding of the distance between two objects and masses of the objects as factors that influence gravity. They also actively constructed mathematical relationships between these quantities that helped them examine the phenomenon in more depth than it is typically taught.

At the same time, the results illustrated how the context of science can provide a constructive space for students to reason covariationally about quantities. In particular, students illustrated meanings aligned to gross coordination of values and coordination of values in terms of the Thompson and Carlson (2017) framework. They also illustrated a form of partial chunky continuous covariation when their smooth graphs showed that students had an image of change that entailed intermediate values, but we do not have evidence that they had an image of the quantities actually having those values. Specifically, they reasoned that the graph representing this consistent change in tables would be a straight line and they also extended the graph beyond the given ordered pairs, but they reasoned about the quantities in chunks.

Currently, the Thompson and Carlson (2017) framework presents smooth and continuous covariation as the top level in their progression; however, students showed traces of smooth images of change in their actions throughout the experiment. More research is needed to examine students' actual perceptions of change as they engage with continuous motion in the simulation and smooth graphs as well as the possibility of early forms of continuous and smooth images of change when students engage with these dynamic representations. We would argue that the closest evidence we have of students' continuous and smooth images was their reasoning about coordinating the multiplicative change of the quantities as they showed an understanding that the quantities can take any value. This might be the kind of reasoning that could support students' smooth continuous covariation. These are opportunities for future studies to further explore these connections and offer recommendations for enriching the Thompson and Carlson (2017) framework accordingly.

The students in this study also exhibited forms of multivariational reasoning that are often neglected by other studies which usually focus on two covarying quantities. Asking students to express different ways that they can change the amount of gravity in the simulation led them to reorganize their meanings from isolated images of the two relationships (i.e., mass versus gravity, distance versus gravity) into a unified image of three quantities working together. Other science phenomena similar to gravity, which usually involve more than two quantities varying, can also provide a constructive space for these explorations and reorganizations.

The role of the design for providing a constructive space for students' development and reorganization of these meanings was also illustrated in this study. By exploring the simulation, students reorganized their initial meanings about gravity into a gross coordination of the magnitudes of mass, distance, and gravity. The focus on magnitudes and not numbers in the early tasks helped students to mentally coordinate the change in the magnitude of mass or distance with the change in gravity and also imagine this change *in progress* (Castillo-Garsow et al., 2013) and possibly construct smooth images of change. We use the term "possibly" here because more research is needed to explore the connection between students' smooth actions in the simulation and their construction of smooth and chunky images of change during those actions. Having the ability to change the quantities and also reverse those changes helped them to reason about the relationships when the quantities were increasing and decreasing. By imaging the multiplicative change in quantities and then experimenting with the simulation, students formulated conjectures that they tested and revised. The feedback provided by the simulation helped them to search for new explanations to describe the simulation's behavior and reorganize their meanings from additive to multiplicative forms of reasoning.

Students reorganized their meanings to include a coordination of values during the work with tables, where the representations focused on uniform increments of change in the values of the two quantities. In describing the relationships in graphs, students illustrated both chunky and smooth images of change, which we characterized as partial chunky continuous covariation. By examining the relationships in tables and graphs, students distinguished between the straight-line graph of mass versus gravity and the curvy graph of distance versus gravity, arguing that the first relationship is changing uniformly but that the second is not. We also found that students were not able to find a pattern to coordinate the change in the values of gravity with the change in the values of distance. Future studies can explore whether this was a result of a design limitation and whether activities that connect dynamic experiences working with a simulation to the nature of a developing graph (similar to other studies, e.g., Frank, 2016; H. L. Johnson, 2015b) might help students exhibit different forms of reasoning.

The findings showed that the way that the quantities and their relationships were modeled in the simulation, tables, and graphs as well as the way that probing questions were formulated shaped the nature of students' covariational reasoning. Therefore, we acknowledge that other forms of covariational reasoning might have been possible through different activities that would have provided a constructive space for these other forms. This study aims to initiate more questions about the nature of students' covariational reasoning and how it can be used to develop a conceptual understanding of scientific phenomena and provide a bridge for students to explore the reciprocal relationship between math and science learning. Other studies can build on these findings to design and study other STEM modules that integrate scientific phenomena with covariational reasoning through technology.

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