



## Strong convergence of a fully discrete finite element method for a class of semilinear stochastic partial differential equations with multiplicative noise\*

Xiaobing Feng

*Department of Mathematics, The University of Tennessee,  
Knoxville, TN 37996, U.S.A.<sup>1)</sup>*

*Email: xfeng@math.utk.edu*

Yukun Li

*Department of Mathematics, University of Central Florida,  
Orlando, FL 32816, U.S.A.<sup>2)</sup>*

*Email: yukun.li@ucf.edu*

Yi Zhang

*Department of Mathematics and Statistics, The University of North Carolina at Greensboro,  
Greensboro, NC 27402, U.S.A.*

*Email: y\_zhang7@uncg.edu*

### Abstract

This paper develops and analyzes a fully discrete finite element method for a class of semilinear stochastic partial differential equations (SPDEs) with multiplicative noise. The nonlinearity in the diffusion term of the SPDEs is assumed to be globally Lipschitz and the nonlinearity in the drift term is only assumed to satisfy a one-sided Lipschitz condition. These assumptions are the same ones as used in [18] where numerical methods for general nonlinear stochastic ordinary differential equations (SODEs) under “minimum assumptions” were studied. As a result, the semilinear SPDEs considered in this paper are a direct generalization of the SODEs considered in [18]. There are several difficulties which need to be overcome for this generalization. First, obviously the spatial discretization, which does not appear in the SODE case, adds an extra layer of difficulty. It turns out a spatial discretization must be designed to guarantee certain properties for the numerical scheme and its stiffness matrix. In this paper we use a finite element interpolation technique to discretize the nonlinear drift term. Second, in order to prove the strong convergence of the proposed fully discrete finite element method, stability estimates for higher order moments of the  $H^1$ -seminorm of the numerical solution must be established, which are difficult and delicate. A judicious combination of the properties of the drift and diffusion terms and a nontrivial technique borrowed from [28] is used in this paper to achieve the goal. Finally, stability estimates for the second and higher order moments of the  $L^2$ -norm of the numerical solution are also difficult to obtain due to the fact that the mass matrix may not be diagonally dominant. This is done by utilizing the interpolation theory and the higher moment estimates for the  $H^1$ -seminorm of the numerical solution. After overcoming these difficulties, it is proved that the proposed fully discrete finite element method is convergent in strong norms with nearly optimal rates of convergence. Numerical experiment results are also presented to validate the theoretical results and to demonstrate the efficiency of the proposed numerical method.

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## 1. Introduction

We consider the following initial-boundary value problem for general semilinear stochastic partial differential equations (SPDEs) with function-type multiplicative noise:

$$du = [\Delta u + f(u)] dt + g(u) dW(t), \quad \text{in } \mathcal{D} \times (0, T), \quad (1.1)$$

$$\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial \mathcal{D} \times (0, T), \quad (1.2)$$

$$u(\cdot, 0) = u_0(\cdot), \quad \text{in } \mathcal{D}. \quad (1.3)$$

Here  $\mathcal{D} \subset \mathbf{R}^d$  ( $d = 1, 2, 3$ ) is an open bounded domain with smooth boundary,  $W : \Omega \times (0, T) \rightarrow \mathbf{R}$  denotes the standard Wiener process on the filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$ , and  $f, g \in C^1$  are two given functions and  $f(u)$  takes the form

$$f(u) = c_0 u - c_1 u^3 - c_2 u^5 - c_3 u^7 - \dots, \quad (1.4)$$

where  $c_i \geq 0, i = 0, 1, 2, \dots$ . For the sake of clarity, we only consider the case  $f(u) = u - u^q$  in this paper, where  $q \geq 3$  is an odd integer (it is trivial when  $f(u) = c_0 u$ ). We remark that similar results still hold for the general nonlinear function  $f(u)$  in (1.4), and when  $f(u) = \frac{1}{\epsilon^2}(u - u^3)$ , (1.1) is known as the stochastic Allen-Cahn equation with function-type multiplicative noise and interaction length  $\epsilon$  (see [28]). We also assume that  $g$  is globally Lipschitz and satisfies the growth condition, that is, there exist constants  $\kappa_1 > 0$  and  $C > 0$  such that

$$|g(a) - g(b)| \leq \kappa_1 |a - b|, \quad (1.5)$$

$$|g(a)|^2 \leq C(1 + a^2). \quad (1.6)$$

By (1.6), we get

$$|g(a)a| \leq C(1 + a^2). \quad (1.7)$$

Under the above assumptions for the drift term and the diffusion term, it can be proved in [15] that there exists a unique strong variational solution  $u$  such that

$$\begin{aligned} (u(t), \phi) &= (u(0), \phi) - \int_0^t (\nabla u(s), \nabla \phi) ds + \int_0^t (f(u(s)), \phi) ds \\ &\quad + \int_0^t (g(u), \phi) dW(s) \quad \forall \phi \in H^1(\mathcal{D}) \end{aligned} \quad (1.8)$$

holds  $\mathbb{P}$ -almost surely. Moreover, when the initial condition  $u_0$  is sufficiently smooth, the following stability estimate for the strong solution  $u$  holds:

$$\sup_{t \in [0, T]} \mathbb{E} \left[ \|u(t)\|_{H^2}^{2q} \right] + \sup_{t \in [0, T]} \mathbb{E} \left[ \|u(t)\|_{L^{4q-2}}^{4q-2} \right] \leq C, \quad (1.9)$$

where  $q$  is the exponent in the nonlinear term of  $f(u) = u - u^q$ .

Clearly, when the  $\Delta u$  term in (1.1) is dropped, the PDE reduces to a stochastic ODE. A convergence theory for numerical approximations for this stochastic ODE was established



long ago (see [29] and [30]) under the global Lipschitz assumptions on  $f$  and  $g$ . Later, the convergence was proved in [18] under a weaker condition on  $f$  known as a one-sided Lipschitz condition in the sense that there exists a constant  $\mu > 0$  such that

$$(a - b, f(a) - f(b)) \leq \mu(a - b)^2 \quad \forall a, b \in \mathbb{R}. \quad (1.10)$$

The optimal rate of convergence was also obtained in [18] under an extra assumption that  $f$  behaves like a polynomial. The one-sided Lipschitz condition is widely used and it has broad applications [6, 7, 13, 14, 33].

We also note that numerical approximations of the SPDE (1.1) with various special drift terms and/or diffusion terms have been extensively investigated in the literature (see [10, 11, 28, 32]). In particular, we mention that the case  $f(u) = u - u^3$ ,  $g(u), g'(u), g''(u)$  are bounded and  $g(u)$  is global Lipschitz continuous was studied in [28], the high moments of the  $H^1$ -norm of the numerical solution were proved to be stable, and a nearly optimal strong convergence rate was established. A specially designed discretization is used for  $f(u) = u - u^3$ , and it is not trivial to extend the idea to the case when  $f(u) = u - u^q$  where  $q > 3$ . Besides the variational approach used in the above references, a few other results were obtained for the corresponding stochastic ODEs (see [19, 20]) and the stochastic PDEs (see [2, 5, 16, 17, 22–24, 27]) based on the semigroup approach. Specifically, in [2, 5, 23, 24, 27], additive noises were considered for the stochastic Allen-Cahn equation; in [2], the fully discrete exponential Euler and fully discrete implicit Euler approximations were proved to diverge for stochastic Allen-Cahn equation, which indirectly justifies the usefulness of the proposed interpolation scheme of this paper; in [16], the multiplicative noise was considered and the convergence of the explicit and implicit Euler schemes was established when the drift term grows at most linearly; in [17], the convergence without order was established for the super-linearly growing drift term for a tamed Euler time-stepping scheme; in [22], the first convergence result with a rate was obtained for the one-sided Lipschitz continuity nonlinearity using a nonlinearities-stopped approximation.

The goal of this paper is to generalize the numerical SODE theory in [18] to the SPDE case. Specifically, we want to design a fully discrete finite element method for problem (1.1)–(1.3) which can be proved to be stable and convergent with optimal rates in strong norms under “minimum” assumptions on nonlinear functions  $f$  and  $g$  as those used in [18]. We recall that the “minimum” assumptions refer to that  $g$  is assumed to be global Lipschitz, and  $f$  satisfies the one-sided Lipschitz condition (1.10) and it behaves like a polynomial. To the best of our knowledge, such a goal has yet been achieved before in the literature.

The remainder of this paper is organized as follows. In Section 2, we establish several Hölder continuity properties (in different norms) for the SPDE solution  $u$  and for the composite function  $f(u)$ . These properties play an important role in our error analysis. In Section 3, we first present our fully discrete finite element method for problem (1.1)–(1.3), which consists of an Euler-type scheme for time discretization and a nonstandard finite element method for spatial discretization. The novelty of our spatial discretization is to approximate the nonlinear function  $f$  by its finite element interpolation in the scheme. We then establish several key properties for the numerical solution, among them are the stability of the second and higher order moments of its  $H^1$ -seminorm and the stability of the second and higher order moments of its  $L^2$ -norm. We note that the proofs of the stability of these higher order moments are quite involved, and they require some special techniques and rely on the structure of the proposed numerical method. For example, the diagonal dominance property of the stiffness matrix is needed to show the stability of the second and higher order moments of the  $H^1$ -seminorm of the numerical solution,

however, the mass matrix may not be diagonally dominant. To circumvent this difficulty, we use the stability of the second and higher order moments of the  $H^1$ -seminorm of the numerical solution and the interpolation theory to get the desired  $L^2$ -norm stability. Finally, in this section we prove nearly optimal order error estimates for the numerical solution by utilizing the stability of higher order moments of the  $L^2$ -norm and  $H^1$ -seminorm of the numerical solution. We like to emphasize that only sub-optimal order error estimates could be obtained should the stability of higher order moments of the  $H^1$ -seminorm of the numerical solution were not known, see [32] where the special case  $f(u) = u - u^3$  was considered. In Section 4, we present several numerical experiments to validate our theoretical results, especially to verify the stability of numerical solution using different initial conditions  $u_0$  and different functions  $f$  and  $g$ . As a special case, the stochastic Allen-Cahn equation with function-type multiplicative noise is also tested.

## 2. Preliminaries and properties of the SPDE solution

Throughout this paper, we shall use  $C$  to denote a generic constant, and we take the standard Sobolev notations in [3]. When it is the whole domain  $\mathcal{D}$ ,  $\|\cdot\|_{H^k}$  and  $\|\cdot\|_{L^p}$  are used to simplify  $\|\cdot\|_{H^k(\mathcal{D})}$  and  $\|\cdot\|_{L^p(\mathcal{D})}$  respectively, and  $(\cdot, \cdot)$  is used to denote the standard inner product of  $L^2(\mathcal{D})$ .  $\mathbb{E}[\cdot]$  denotes the expectation operator on the filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$ .

In this section, we first derive the Hölder continuity in time for the strong solution  $u$  with respect to the spatial  $H^1$ -seminorm and for the composite function  $f(u)$  with respect to the spatial  $L^2$ -norm. Both results will play a key role in the error analysis (see Subsection 3.4). The time derivatives of  $\nabla u$  and the composite function  $f(u)$  do not exist in the stochastic case, so these Hölder continuity results will substitute for the differentiability of  $\nabla u$  and  $f(u)$  with respect to time in the error analysis.

**Lemma 2.1.** *Let  $u$  be the strong solution to problem (1.8). Then for any  $s, t \in [0, T]$  with  $s < t$ , we have*

$$\mathbb{E}[\|\nabla(u(t) - u(s))\|_{L^2}^2] + \mathbb{E}\left[\int_s^t \|\Delta(u(\zeta) - u(s))\|_{L^2}^2 d\zeta\right] \leq C_1(t - s),$$

where

$$C_1 = C \left( \sup_{s \leq \zeta \leq t} \mathbb{E}[\|\Delta u(\zeta)\|_{L^2}^2] + \sup_{s \leq \zeta \leq t} \mathbb{E}[\|u(\zeta)\|_{L^{2q}}^{2q}] + \sup_{s \leq \zeta \leq t} \mathbb{E}[\|u(\zeta)\|_{L^2}^2] \right).$$

*Proof.* Applying Itô's formula (see [21, 31]) to the functional  $\Phi(u(\cdot)) := \|\nabla u(\cdot) - \nabla u(s)\|_{L^2}^2$  with fixed  $s \in [0, T)$  and using integration by parts, we get

$$\begin{aligned} \|\nabla u(t) - \nabla u(s)\|_{L^2}^2 &= -2 \int_s^t (\Delta u(\zeta) - \Delta u(s), \Delta u(\zeta)) d\zeta \\ &\quad - 2 \int_s^t (\Delta u(\zeta) - \Delta u(s), f(u(\zeta))) d\zeta \\ &\quad - 2 \int_s^t (\Delta u(\zeta) - \Delta u(s), g(u(\zeta))) dW(\zeta) + \int_s^t \|\nabla g(u(\zeta))\|^2 d\zeta. \end{aligned} \tag{2.1}$$

The expectation of the first term on the right-hand side of (2.1) can be bounded by the Cauchy-Schwarz inequality as follows

$$\begin{aligned}
& -2\mathbb{E} \left[ \int_s^t (\Delta u(\zeta) - \Delta u(s), \Delta u(\zeta)) d\zeta \right] \\
&= -2\mathbb{E} \left[ \int_s^t \|\Delta u(\zeta) - \Delta u(s)\|_{L^2}^2 d\zeta + \int_s^t (\Delta u(\zeta) - \Delta u(s), \Delta u(s)) d\zeta \right] \\
&\leq -\frac{3}{2}\mathbb{E} \left[ \int_s^t \|\Delta u(\zeta) - \Delta u(s)\|_{L^2}^2 d\zeta \right] + \mathbb{E} [\|\Delta u(s)\|_{L^2}^2] (t-s).
\end{aligned} \tag{2.2}$$

The expectation of the second term on the right-hand side of (2.1) can be bounded by

$$\begin{aligned}
& -2\mathbb{E} \left[ \int_s^t (\Delta u(\zeta) - \Delta u(s), f(u(\zeta))) d\zeta \right] \\
&\leq \mathbb{E} \left[ \frac{1}{2} \int_s^t \|\Delta u(\zeta) - \Delta u(s)\|_{L^2}^2 d\zeta + 2 \int_s^t \|f(u(\zeta))\|_{L^2}^2 d\zeta \right] \\
&\leq \frac{1}{2}\mathbb{E} \left[ \int_s^t \|\Delta u(\zeta) - \Delta u(s)\|_{L^2}^2 d\zeta \right] + C \sup_{s \leq \zeta \leq t} \mathbb{E} [\|u(\zeta)\|_{L^{2q}}^{2q}] (t-s) \\
&\quad + C \sup_{s \leq \zeta \leq t} \mathbb{E} [\|u(\zeta)\|_{L^2}^2] (t-s).
\end{aligned} \tag{2.3}$$

Next we bound the expectation of the fourth term on the right-hand side of (2.1) as follows

$$\mathbb{E} \left[ \int_s^t \|\nabla g(u(\zeta))\|^2 d\zeta \right] \leq C \sup_{s \leq \zeta \leq t} \mathbb{E} [\|\nabla u(\zeta)\|_{L^2}^2] (t-s). \tag{2.4}$$

Then Lemma 2.1 follows from (2.1)–(2.4) and the fact that the expectation of the third term on the right-hand side of (2.1) is zero.

Next we prove the Hölder continuity result for the nonlinear term  $f(u(t)) - f(u(s))$  with respect to the spatial  $L^2$ -norm.

**Lemma 2.2.** *Let  $u$  be the strong solution to problem (1.8). Then for any  $s, t \in [0, T]$  with  $s < t$ , we have*

$$\mathbb{E} [\|f(u(t)) - f(u(s))\|_{L^2}^2] \leq C_2(t-s),$$

where

$$C_2 = C \left( 1 + \sup_{s \leq \zeta \leq t} \mathbb{E} [\|\Delta u(\zeta)\|_{L^2}^2] + \sup_{s \leq \zeta \leq t} \mathbb{E} [\|u(\zeta)\|_{L^{4q-2}}^{4q-2}] \right).$$

*Proof.* Applying Ito's formula (see [21, 31]) to  $\Phi(u(\cdot)) := \|f(u(\cdot)) - f(u(s))\|_{L^2}^2$  with fixed  $s \in [0, T)$ , we obtain

$$\begin{aligned}
\|f(u(t)) - f(u(s))\|_{L^2}^2 &= 2 \int_s^t \int_{\mathcal{D}} (f(u(\zeta)) - f(u(s))) f'(u(\zeta)) \\
&\quad \times [\Delta u(\zeta) + f(u(\zeta))] dx d\zeta \\
&\quad + 2 \int_s^t \int_{\mathcal{D}} (f(u(\zeta)) - f(u(s))) f'(u(\zeta)) g(u(\zeta)) dx dW(\zeta) \\
&\quad + \int_s^t \int_{\mathcal{D}} (f(u(\zeta)) - f(u(s))) f''(u(\zeta)) |g(u(\zeta))|^2 dx d\zeta \\
&\quad + \int_s^t \int_{\mathcal{D}} [f'(u(\zeta))]^2 |g(u(\zeta))|^2 dx d\zeta.
\end{aligned} \tag{2.5}$$

Taking the expectation on both sides, the first term on the right-hand side of (2.5) can be bounded by

$$\begin{aligned} & \mathbb{E} \left[ 2 \int_s^t \int_{\mathcal{D}} (f(u(\zeta)) - f(u(s))) f'(u(\zeta)) \times [\Delta u(\zeta) + f(u(\zeta))] dx d\zeta \right] \quad (2.6) \\ & \leq C(t-s) \times \left( \sup_{s \leq \zeta \leq t} \mathbb{E} [\|u(\zeta)\|_{L^2}^2] + \sup_{s \leq \zeta \leq t} \mathbb{E} [\|u(\zeta)\|_{L^{2q}}^{2q}] + \sup_{s \leq \zeta \leq t} \mathbb{E} [\|u(\zeta)\|_{L^{4q-2}}^{4q-2}] \right. \\ & \quad \left. + \sup_{s \leq \zeta \leq t} \mathbb{E} [\|\Delta u(\zeta)\|_{L^2}^2] + \sup_{s \leq \zeta \leq t} \mathbb{E} [\|u(\zeta)\|_{L^2}^2] + \sup_{s \leq \zeta \leq t} \mathbb{E} [\|u(\zeta)\|_{L^{2q}}^{2q}] \right). \end{aligned}$$

The expectation of the second term on the right-hand side of (2.5) is zero since it is a martingale.

The third term on the right-hand side of (2.5) can be bounded by

$$\begin{aligned} & \mathbb{E} \left[ \int_s^t \int_{\mathcal{D}} (f(u(\zeta)) - f(u(s))) f''(u(\zeta)) |g(u(\zeta))|^2 dx d\zeta \right] \quad (2.7) \\ & \leq C(t-s) \times \left( C + \sup_{s \leq \zeta \leq t} \mathbb{E} [\|u(\zeta)\|_{L^{2q}}^{2q}] + \sup_{s \leq \zeta \leq t} \mathbb{E} [\|u(\zeta)\|_{L^{4q-2}}^{4q-2}] \right. \\ & \quad \left. + C + \sup_{s \leq \zeta \leq t} \mathbb{E} [\|u(\zeta)\|_{L^4}^4] \right). \end{aligned}$$

The fourth term on the right-hand side of (2.5) can be bounded by the upper bound in (2.7). Then (2.5) can be written as

$$\begin{aligned} \mathbb{E} [\|f(u(t)) - f(u(s))\|_{L^2}^2] & \leq C(t-s) \times \left( 1 + \sup_{s \leq \zeta \leq t} \mathbb{E} [\|\Delta u(\zeta)\|_{L^2}^2] + \right. \\ & \quad \left. + \sup_{s \leq \zeta \leq t} \mathbb{E} [\|u(\zeta)\|_{L^{4q-2}}^{4q-2}] \right). \quad (2.8) \end{aligned}$$

The proof is complete.

**Remark 2.1.** For the diffusion term, the global Lipschitz condition, which is stronger than the one-sided Lipschitz condition, is needed as in the SODE case. Using the  $C^1$  assumption and the global Lipschitz assumption, we can derive that the derivative of the diffusion term is bounded by the Lipschitz constant  $\kappa$ , i.e.,  $|g'(u)| \leq \kappa$ , but the diffusion term itself may not be bounded. For instance,  $g(u) = u$ ,  $g(u) = \sqrt{u^2 + 1}$ , etc. Note these two assumptions are consistent with the SODE case in [18], and they are also the conditions to guarantee the well-posedness of the strong SODE solution [18].

### 3. Fully discrete finite element approximation

#### 3.1. Formulation of the finite element method

In this section, we first construct a fully discrete finite element method for problem (1.1)–(1.3). we then establish several stability properties for the numerical solution including the stability of higher order moments for its  $H^1$ -seminorm and  $L^2$ -norm. Finally, we derive optimal order error estimates in strong norms for the numerical solution using the stability estimates.

Let  $t_n = n\tau$  ( $n = 0, 1, \dots, N$ ) be a uniform partition of  $[0, T]$  and  $\mathcal{T}_h$  be the triangulation of  $\mathcal{D}$  satisfying the following assumption [35]:

$$\frac{1}{d(d-1)} \sum_{K \supset E} |\kappa_E^K| \cot \theta_E^K \geq 0, \quad (3.1)$$

where  $E$  denotes the edge of simplex  $K$ . It was proved in [35] that the stiffness matrix for the Poisson equation with zero Dirichlet boundary is an  $M$ -matrix if and only if this assumption holds for all edges. The stiffness matrix is diagonally dominant if the Neumann boundary condition is considered. Note this assumption is just the Delaunay triangulation when  $d = 2$ . In 3D, the notations in the assumption (3.1) are as follows:  $a_i$  ( $1 \leq i \leq d+1$ ) denote the vertices of  $K$ ,  $E = E_{ij}$  the edge connecting two vertices  $a_i$  and  $a_j$ ,  $F_i$  the  $(d-1)$ -dimensional simplex opposite to the vertex  $a_i$ ,  $\theta_{ij}^K$  or  $\theta_E^K$  the angle between the faces  $F_i$  and  $F_j$ ,  $\kappa_E^K = F_i \cap F_j$ , the  $(d-2)$ -dimensional simplex opposite to the edge  $E = E_{ij}$ . See Figure 3.1 below.

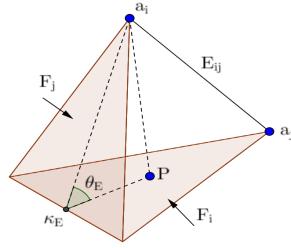


Fig. 3.1.: 3D triangulation.

Consider the  $\mathcal{P}_1$ -Lagrangian finite element space

$$V_h = \{v_h \in H^1(\mathcal{D}) : v_h|_K \in \mathcal{P}_1(K) \quad \forall K \in \mathcal{T}_h\}, \quad (3.2)$$

where  $\mathcal{P}_1$  denotes the space of all linear polynomials. Then the finite element approximation of (1.8) is to seek an  $\mathcal{F}_{t_n}$  adapted  $V_h$ -valued process  $\{u_h^n\}_{n=1}^N$  such that it holds  $\mathbb{P}$ -almost surely that

$$\begin{aligned} & (u_h^{n+1}, v_h) + \tau(\nabla u_h^{n+1}, \nabla v_h) \\ &= (u_h^n, v_h) + \tau(I_h f^{n+1}, v_h) + (g(u_h^n), v_h) \bar{\Delta} W_{n+1} \quad \forall v_h \in V_h, \end{aligned} \quad (3.3)$$

where  $f^{n+1} := u_h^{n+1} - (u_h^{n+1})^q$ ,  $\bar{\Delta} W_{n+1} = W(t_{n+1}) - W(t_n) \sim \mathcal{N}(0, \tau)$ , and  $I_h$  is the standard nodal value interpolation operator  $I_h : C(\bar{\mathcal{D}}) \rightarrow V_h$ , i.e.,

$$I_h v := \sum_{i=1}^{N_h} v(a_i) \varphi_i, \quad (3.4)$$

where  $N_h$  denotes the number of vertices of  $\mathcal{T}_h$ , and  $\varphi_i$  denotes the nodal basis function of  $V_h$  corresponding to the vertex  $a_i$ . The initial condition is chosen by  $u_h^0 = P_h u_0$  where  $P_h : L^2(\mathcal{D}) \rightarrow V_h$  is the  $L^2$ -projection operator defined by

$$(P_h w, v_h) = (w, v_h) \quad v_h \in V_h.$$

For all  $w \in H^s(\mathcal{D})$  for  $s > \frac{3}{2}$ , the following well-known error estimate results can be found in [3, 8]:

$$\|w - P_h w\|_{L^2} + h \|\nabla(w - P_h w)\|_{L^2} \leq Ch^{\min\{2, s\}} \|w\|_{H^s}, \quad (3.5)$$

$$\|w - P_h w\|_{L^\infty} \leq Ch^{2-\frac{d}{2}} \|w\|_{H^2}. \quad (3.6)$$

Finally, given  $v_h \in V_h$ , we define the discrete Laplace operator  $\Delta_h : V_h \rightarrow V_h$  by

$$(\Delta_h v_h, w_h) = -(\nabla v_h, \nabla w_h) \quad \forall w_h \in V_h. \quad (3.7)$$

### 3.2. Stability estimates for the $p$ -th moment of the $H^1$ -seminorm of $u_h^n$

First we shall prove the second moment discrete  $H^1$ -seminorm stability result, which is necessary to establish the corresponding higher moment stability result.

**Theorem 3.1.** *Suppose the mesh assumption in (3.1) holds. Then we have*

$$\begin{aligned} \sup_{0 \leq n \leq N} \mathbb{E} [\|\nabla u_h^n\|_{L^2}^2] + \frac{1}{4} \sum_{n=0}^{N-1} \mathbb{E} [\|\nabla (u_h^{n+1} - u_h^n)\|_{L^2}^2] \\ + \tau \sum_{n=0}^{N-1} \mathbb{E} [\|\Delta_h u_h^{n+1}\|_{L^2}^2] \leq C. \end{aligned} \quad (3.8)$$

*Proof.* Testing (3.3) with  $-\Delta_h u_h^{n+1}$ . Then

$$\begin{aligned} (u_h^{n+1} - u_h^n, -\Delta_h u_h^{n+1}) + \tau (\nabla u_h^{n+1}, -\nabla \Delta_h u_h^{n+1}) \\ = \tau (I_h f^{n+1}, -\Delta_h u_h^{n+1}) + (g(u_h^n), -\Delta_h u_h^{n+1}) \bar{\Delta} W_{n+1}. \end{aligned} \quad (3.9)$$

Using the definition of the discrete Laplace operator, we get

$$\begin{aligned} (u_h^{n+1} - u_h^n, -\Delta_h u_h^{n+1}) &= \frac{1}{2} \|\nabla u_h^{n+1}\|_{L^2}^2 - \frac{1}{2} \|\nabla u_h^n\|_{L^2}^2 \\ &\quad + \frac{1}{2} \|\nabla (u_h^{n+1} - u_h^n)\|_{L^2}^2, \end{aligned} \quad (3.10)$$

$$\tau (\nabla u_h^{n+1}, -\nabla \Delta_h u_h^{n+1}) = \tau \|\Delta_h u_h^{n+1}\|_{L^2}^2, \quad (3.11)$$

$$\begin{aligned} \mathbb{E} [(g(u_h^n), -\Delta_h u_h^{n+1}) \bar{\Delta} W_{n+1}] &= \mathbb{E} [(\nabla (P_h g(u_h^n)), \nabla (u_h^{n+1} - u_h^n)) \bar{\Delta} W_{n+1}] \\ &\leq C \tau \mathbb{E} [\|\nabla u_h^n\|_{L^2}^2] + \frac{1}{4} \mathbb{E} [\|\nabla (u_h^{n+1} - u_h^n)\|_{L^2}^2], \end{aligned} \quad (3.12)$$

where the stability in the  $H^1$ -seminorm of the  $L^2$  projection (see [1]) is used in the inequality of (3.12).

The crucial part is to bound the first term on the right-hand side of (3.9) since it cannot be treated as a bad term, which aligns with the continuous case. Denote  $u_i = u_h^{n+1}(a_i)$ , and then

$$\begin{aligned} \tau (I_h f^{n+1}, -\Delta_h u_h^{n+1}) &= \tau \|\nabla u_h^{n+1}\|_{L^2}^2 - \tau (\nabla \sum_{i=1}^{N_h} u_i^q \varphi_i, \nabla \sum_{j=1}^{N_h} u_j \varphi_j) \\ &= \tau \|\nabla u_h^{n+1}\|_{L^2}^2 - \tau \sum_{i,j=1}^{N_h} (u_i^q \nabla \varphi_i, u_j \nabla \varphi_j) \\ &= \tau \|\nabla u_h^{n+1}\|_{L^2}^2 - \tau \sum_{i,j=1}^{N_h} b_{ij} (\nabla \varphi_i, \nabla \varphi_j), \end{aligned} \quad (3.13)$$

where  $b_{ij} = u_i^q u_j$ .

Using Young's inequality when  $i \neq j$ , we have

$$|b_{ij}| \leq \frac{q}{q+1} u_i^{q+1} + \frac{1}{q+1} u_j^{q+1}. \quad (3.14)$$

Besides, since the stiffness matrix is diagonally dominant, [and then](#)

$$\begin{aligned}
 -\tau \sum_{i,j=1}^{N_h} b_{ij} (\nabla \varphi_i, \nabla \varphi_j) &\leq -\tau \sum_{k=1}^{N_h} b_{kk} [(\nabla \varphi_k, \nabla \varphi_k) - \frac{q}{q+1} \sum_{\substack{i=1, \\ i \neq k}}^{N_h} |(\nabla \varphi_i, \nabla \varphi_k)|] \\
 &\quad - \frac{1}{q+1} \sum_{\substack{j=1, \\ j \neq k}}^{N_h} |(\nabla \varphi_k, \nabla \varphi_j)| \\
 &\leq -\tau \sum_{k=1}^{N_h} b_{kk} [(\nabla \varphi_k, \nabla \varphi_k) - \sum_{\substack{i=1, \\ i \neq k}}^{N_h} (\nabla \varphi_i, \nabla \varphi_k)] \\
 &\leq 0.
 \end{aligned} \tag{3.15}$$

Then we have

$$\tau (I_h f^{n+1}, -\Delta_h u_h^{n+1}) \leq \tau \|\nabla u_h^{n+1}\|_{L^2}^2. \tag{3.16}$$

Combining (3.9)–(3.12) and (3.16), and taking the summation, we have

$$\begin{aligned}
 \frac{1}{2} \mathbb{E} [\|\nabla u_h^\ell\|_{L^2}^2] + \frac{1}{4} \sum_{n=0}^{\ell-1} \mathbb{E} [\|\nabla (u_h^{n+1} - u_h^n)\|_{L^2}^2] + \tau \sum_{n=0}^{\ell-1} \mathbb{E} [\|\Delta_h u_h^{n+1}\|_{L^2}^2] \\
 \leq C \tau \sum_{n=0}^{\ell-1} \mathbb{E} [\|\nabla u_h^n\|_{L^2}^2].
 \end{aligned} \tag{3.17}$$

Using Gronwall's inequality, we obtain (3.8).

Before we establish the error estimates, we need to prove the stability of the higher order moments for the  $H^1$ -seminorm of the numerical solution.

**Theorem 3.2.** *Suppose the mesh assumption in (3.1) holds. [Then](#) for any  $p \geq 2$ ,*

$$\sup_{0 \leq n \leq N} \mathbb{E} [\|\nabla u_h^n\|_{L^2}^p] \leq C.$$

*Proof.* The proof is divided into three steps. In Step 1, we establish the bound for  $\mathbb{E} \|\nabla u_h^\ell\|_{L^2}^4$ . In Step 2, we give the bound for  $\mathbb{E} \|\nabla u_h^\ell\|_{L^2}^p$ , where  $p = 2^r$  and  $r$  is an arbitrary positive integer. In Step 3, we obtain the bound for  $\mathbb{E} \|\nabla u_h^\ell\|_{L^2}^p$ , where  $p$  is an arbitrary real number and  $p \geq 2$ .

**Step 1.** Based on (3.9)–(3.16), we have

$$\begin{aligned}
 \frac{1}{2} \|\nabla u_h^{n+1}\|_{L^2}^2 - \frac{1}{2} \|\nabla u_h^n\|_{L^2}^2 + \frac{1}{2} \|\nabla (u_h^{n+1} - u_h^n)\|_{L^2}^2 + \tau \|\Delta_h u_h^{n+1}\|_{L^2}^2 \\
 - (g(u_h^n), -\Delta_h u_h^{n+1}) \bar{\Delta} W_{n+1} \leq \tau \|\nabla u_h^{n+1}\|_{L^2}^2.
 \end{aligned} \tag{3.18}$$

[Note](#) the following identity

$$\|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{1}{2} \|\nabla u_h^n\|_{L^2}^2 = \frac{3}{4} (\|\nabla u_h^{n+1}\|_{L^2}^2 + \|\nabla u_h^n\|_{L^2}^2) + \frac{1}{4} (\|\nabla u_h^{n+1}\|_{L^2}^2 - \|\nabla u_h^n\|_{L^2}^2). \tag{3.19}$$

Multiplying (3.18) by  $\|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|\nabla u_h^n\|_{L^2}^2$ , we obtain

$$\begin{aligned}
 & \frac{3}{8}(\|\nabla u_h^{n+1}\|_{L^2}^4 - \|\nabla u_h^n\|_{L^2}^4) + \frac{1}{8}(\|\nabla u_h^{n+1}\|_{L^2}^2 - \|\nabla u_h^n\|_{L^2}^2)^2 \\
 & + \left(\frac{1}{2}\|\nabla(u_h^{n+1} - u_h^n)\|_{L^2}^2 + \tau\|\Delta_h u_h^{n+1}\|_{L^2}^2\right)(\|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|\nabla u_h^n\|_{L^2}^2) \\
 & \leq \tau\|\nabla u_h^{n+1}\|_{L^2}^2(\|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|\nabla u_h^n\|_{L^2}^2) \\
 & + (g(u_h^n), -\Delta_h u_h^{n+1}) \bar{\Delta} W_{n+1}(\|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|\nabla u_h^n\|_{L^2}^2).
 \end{aligned} \tag{3.20}$$

The first term on the right-hand side of (3.20) can be written as

$$\begin{aligned}
 & \tau\|\nabla u_h^{n+1}\|_{L^2}^2(\|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|\nabla u_h^n\|_{L^2}^2) \\
 & = \tau\|\nabla u_h^{n+1}\|_{L^2}^2\left(\frac{3}{2}\|\nabla u_h^{n+1}\|_{L^2}^2 - \frac{1}{2}(\|\nabla u_h^{n+1}\|_{L^2}^2 - \|\nabla u_h^n\|_{L^2}^2)\right) \\
 & \leq C\tau\|\nabla u_h^{n+1}\|_{L^2}^4 + \theta_1(\|\nabla u_h^{n+1}\|_{L^2}^2 - \|\nabla u_h^n\|_{L^2}^2)^2,
 \end{aligned} \tag{3.21}$$

where  $\theta_1 > 0$  will be determined later.

The second term on the right-hand side of (3.20) can be written as

$$\begin{aligned}
 & (g(u_h^n), -\Delta_h u_h^{n+1}) \bar{\Delta} W_{n+1}(\|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|\nabla u_h^n\|_{L^2}^2) \\
 & = (\nabla P_h g(u_h^n), \nabla u_h^{n+1}) \bar{\Delta} W_{n+1}(\|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|\nabla u_h^n\|_{L^2}^2) \\
 & = ((\nabla P_h g(u_h^n), \nabla u_h^{n+1}) - (\nabla P_h g(u_h^n), \nabla u_h^n)) \bar{\Delta} W_{n+1} \\
 & \quad + (\nabla P_h g(u_h^n), \nabla u_h^n) \bar{\Delta} W_{n+1}(\|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|\nabla u_h^n\|_{L^2}^2) \\
 & \leq \left(\frac{1}{4}\|\nabla u_h^{n+1} - \nabla u_h^n\|_{L^2}^2 + C\|\nabla u_h^n\|_{L^2}^2\right) (\bar{\Delta} W_{n+1})^2 \\
 & \quad + (\nabla P_h g(u_h^n), \nabla u_h^n) \bar{\Delta} W_{n+1}(\|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|\nabla u_h^n\|_{L^2}^2).
 \end{aligned} \tag{3.22}$$

For the right-hand side of (3.22), using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
 & C\|\nabla u_h^n\|_{L^2}^2(\bar{\Delta} W_{n+1})^2(\|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|\nabla u_h^n\|_{L^2}^2) \\
 & = C\|\nabla u_h^n\|_{L^2}^2(\bar{\Delta} W_{n+1})^2(\|\nabla u_h^{n+1}\|_{L^2}^2 - \|\nabla u_h^n\|_{L^2}^2 + \frac{3}{2}\|\nabla u_h^n\|_{L^2}^2) \\
 & \leq \theta_2(\|\nabla u_h^{n+1}\|_{L^2}^2 - \|\nabla u_h^n\|_{L^2}^2)^2 + C\|\nabla u_h^n\|_{L^2}^4(\bar{\Delta} W_{n+1})^4 \\
 & \quad + C\|\nabla u_h^n\|_{L^2}^4(\bar{\Delta} W_{n+1})^2,
 \end{aligned} \tag{3.23}$$

where  $\theta_2 > 0$  will be determined later. Similarly, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
 & (\nabla P_h g(u_h^n), \nabla u_h^n) \bar{\Delta} W_{n+1}(\|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|\nabla u_h^n\|_{L^2}^2) \\
 & = (\nabla P_h g(u_h^n), \nabla u_h^n) \bar{\Delta} W_{n+1}(\|\nabla u_h^{n+1}\|_{L^2}^2 - \|\nabla u_h^n\|_{L^2}^2 + \frac{3}{2}\|\nabla u_h^n\|_{L^2}^2) \\
 & \leq \theta_3(\|\nabla u_h^{n+1}\|_{L^2}^2 - \|\nabla u_h^n\|_{L^2}^2)^2 + C\|\nabla u_h^n\|_{L^2}^4(\bar{\Delta} W_{n+1})^2 \\
 & \quad + \frac{3}{2}(\nabla P_h g(u_h^n), \nabla u_h^n) \bar{\Delta} W_{n+1}\|\nabla u_h^n\|_{L^2}^2,
 \end{aligned} \tag{3.24}$$

where  $\theta_3 > 0$  will be determined later.

Choosing  $\theta_1, \theta_2, \theta_3$  such that  $\theta_1 + \theta_2 + \theta_3 \leq \frac{1}{16}$ , and then taking the summation over  $n$  from 0 to  $\ell - 1$  and taking the expectation on both sides of (3.20), we obtain

$$\begin{aligned}
 & \frac{3}{8} \mathbb{E} [\|\nabla u_h^\ell\|_{L^2}^4] + \frac{1}{16} \sum_{n=0}^{\ell-1} \mathbb{E} [(\|\nabla u_h^{n+1}\|_{L^2}^2 - \|\nabla u_h^n\|_{L^2}^2)^2] \\
 & + \sum_{n=0}^{\ell-1} \mathbb{E} \left[ \left( \frac{1}{4} \|\nabla(u_h^{n+1} - u_h^n)\|_{L^2}^2 + \tau \|\Delta_h u_h^{n+1}\|_{L^2}^2 \right) (\|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{1}{2} \|\nabla u_h^n\|_{L^2}^2) \right] \\
 & \leq C\tau \sum_{n=0}^{\ell-1} \mathbb{E} [\|\nabla u_h^{n+1}\|_{L^2}^4] + \frac{3}{8} \mathbb{E} [\|\nabla u_h^0\|_{L^2}^4] + C\tau^2 \sum_{n=0}^{\ell-1} \mathbb{E} [\|\nabla u_h^n\|_{L^2}^4] \\
 & + C\tau \sum_{n=0}^{\ell-1} \mathbb{E} [\|\nabla u_h^n\|_{L^2}^4].
 \end{aligned} \tag{3.25}$$

When restricting  $\tau \leq C$ , we have

$$\begin{aligned}
 & \frac{1}{4} \mathbb{E} [\|\nabla u_h^\ell\|_{L^2}^4] + \frac{1}{16} \sum_{n=0}^{\ell-1} \mathbb{E} [(\|\nabla u_h^{n+1}\|_{L^2}^2 - \|\nabla u_h^n\|_{L^2}^2)^2] \\
 & + \sum_{n=0}^{\ell-1} \mathbb{E} \left[ \left( \frac{1}{4} \|\nabla(u_h^{n+1} - u_h^n)\|_{L^2}^2 + \tau \|\Delta_h u_h^{n+1}\|_{L^2}^2 \right) (\|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{1}{2} \|\nabla u_h^n\|_{L^2}^2) \right] \\
 & \leq C\tau \sum_{n=0}^{\ell-1} \mathbb{E} [\|\nabla u_h^n\|_{L^2}^4] + \frac{3}{8} \mathbb{E} [\|\nabla u_h^0\|_{L^2}^4].
 \end{aligned} \tag{3.26}$$

Using Gronwall's inequality, we obtain

$$\begin{aligned}
 & \frac{1}{4} \mathbb{E} [\|\nabla u_h^\ell\|_{L^2}^4] + \frac{1}{16} \sum_{n=0}^{\ell-1} \mathbb{E} [(\|\nabla u_h^{n+1}\|_{L^2}^2 - \|\nabla u_h^n\|_{L^2}^2)^2] \\
 & + \sum_{n=0}^{\ell-1} \mathbb{E} \left[ \left( \frac{1}{4} \|\nabla(u_h^{n+1} - u_h^n)\|_{L^2}^2 + \tau \|\Delta_h u_h^{n+1}\|_{L^2}^2 \right) (\|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{1}{2} \|\nabla u_h^n\|_{L^2}^2) \right. \\
 & \left. + \frac{1}{2} \|\nabla u_h^n\|_{L^2}^2 \right] \leq C.
 \end{aligned} \tag{3.27}$$

**Step 2.** Similar to Step 1, using (3.20)–(3.24), we have

$$\begin{aligned}
 & \frac{3}{8} (\|\nabla u_h^{n+1}\|_{L^2}^4 - \|\nabla u_h^n\|_{L^2}^4) + \frac{1}{16} (\|\nabla u_h^{n+1}\|_{L^2}^2 - \|\nabla u_h^n\|_{L^2}^2)^2 \\
 & + \left( \frac{1}{4} \|\nabla(u_h^{n+1} - u_h^n)\|_{L^2}^2 + \tau \|\Delta_h u_h^{n+1}\|_{L^2}^2 \right) (\|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{1}{2} \|\nabla u_h^n\|_{L^2}^2) \\
 & \leq C\tau \|\nabla u_h^{n+1}\|_{L^2}^4 + C \|\nabla u_h^n\|_{L^2}^4 (\bar{\Delta}W_{n+1})^4 + C \|\nabla u_h^n\|_{L^2}^4 (\bar{\Delta}W_{n+1})^2 \\
 & + C \|\nabla u_h^n\|_{L^2}^4 \bar{\Delta}W_{n+1}.
 \end{aligned} \tag{3.28}$$

Proceed similarly as in Step 1, multiplying (3.28) with  $\|\nabla u_h^{n+1}\|_{L^2}^4 + \frac{1}{2} \|\nabla u_h^n\|_{L^2}^4$ , we can obtain the 8-th moment of the  $H^1$ -seminorm stability result of the numerical solution. Then repeating this process, the  $2^r$ -th moment of the  $H^1$ -seminorm stability result of the numerical solution can be obtained.

**Step 3.** Suppose  $2^{r-1} \leq p \leq 2^r$ , and then by Young's inequality, we have

$$\mathbb{E} [\|\nabla u_h^\ell\|_{L^2}^p] \leq \mathbb{E} [\|\nabla u_h^\ell\|_{L^2}^{2^r}] + C < \infty, \quad (3.29)$$

where the second inequality follows from the results of Step 2. The proof is complete.

### 3.3. Stability estimates for the $p$ -th moment of the $L^2$ -norm of $u_h^n$

Since the mass matrix may not be the diagonally dominated matrix, we cannot use the above idea to prove the  $L^2$  stability. Instead, we prove the stability results by utilizing the above established results. The following results hold when  $q \geq 3$  is the odd integer in 2D case, and when  $q = 3$  or  $q = 5$  in 3D case.

**Theorem 3.3.** *Under the mesh assumption in (3.1), there holds*

$$\begin{aligned} \sup_{0 \leq n \leq N} \mathbb{E} [\|u_h^n\|_{L^2}^2] + \sum_{n=0}^{N-1} \mathbb{E} [\|(u_h^{n+1} - u_h^n)\|_{L^2}^2] + \tau \sum_{n=0}^{N-1} \mathbb{E} [\|\nabla u_h^{n+1}\|_{L^2}^2] \\ + \frac{\tau}{2} \sum_{n=0}^{N-1} \mathbb{E} [\|u_h^{n+1}\|_{L^{q+1}}^{q+1}] \leq C. \end{aligned}$$

*Proof.* Testing (3.3) with  $u_h^{n+1}$  yields

$$(u_h^{n+1} - u_h^n, u_h^{n+1}) + \tau(\nabla u_h^{n+1}, \nabla u_h^{n+1}) = \tau(I_h f^{n+1}, u_h^{n+1}) + (g(u_h^n), u_h^{n+1}) \bar{\Delta} W_{n+1}. \quad (3.30)$$

We can easily prove the following inequalities:

$$\begin{aligned} (u_h^{n+1} - u_h^n, u_h^{n+1}) &= \frac{1}{2} \|u_h^{n+1}\|_{L^2}^2 - \frac{1}{2} \|u_h^n\|_{L^2}^2 + \frac{1}{2} \|u_h^{n+1} - u_h^n\|_{L^2}^2, \\ \mathbb{E}[(g(u_h^n), u_h^{n+1}) \bar{\Delta} W_{n+1}] &= \mathbb{E}[(g(u_h^n), (u_h^{n+1} - u_h^n)) \bar{\Delta} W_{n+1}] \\ &\leq C\tau + C\tau \mathbb{E}[\|u_h^n\|_{L^2}^2] + \frac{1}{4} \mathbb{E}[\|u_h^{n+1} - u_h^n\|_{L^2}^2], \end{aligned}$$

where (1.6) is used in the inequality above.

We have the following standard interpolation result and the inverse inequality (see [8]):

$$\|v - I_h v\|_{L^{\frac{q+1}{q}}(K)} \leq C h_K \|\nabla v\|_{L^{\frac{q+1}{q}}(K)}, \quad (3.31)$$

$$\|v\|_{L^{q+1}(K)}^{q+1} \leq \frac{C}{h_K^{d \cdot \frac{q-1}{2}}} \|v\|_{L^2(K)}^{q+1}. \quad (3.32)$$

Using (3.31)–(3.32), and Young's inequality, we have

$$\begin{aligned}
\tau(I_h f^{n+1}, u_h^{n+1}) &= \tau(f^{n+1}, u_h^{n+1}) - \tau(f^{n+1} - I_h f^{n+1}, u_h^{n+1}) \\
&\leq \tau \|u_h^{n+1}\|_{L^2}^2 - \tau \|u_h^{n+1}\|_{L^{q+1}}^{q+1} \\
&\quad + C\tau \|f^{n+1} - I_h f^{n+1}\|_{L^{\frac{q+1}{q}}}^{\frac{q+1}{q}} + \frac{\tau}{4} \|u_h^{n+1}\|_{L^{q+1}}^{q+1} \\
&\leq \tau \|u_h^{n+1}\|_{L^2}^2 - \tau \|u_h^{n+1}\|_{L^{q+1}}^{q+1} \\
&\quad + C\tau \sum_{K \in \mathcal{T}_h} h_K^{\frac{q+1}{q}} \left( (u_h^{n+1})^{\frac{q^2-1}{q}}, (\nabla u_h^{n+1})^{\frac{q+1}{q}} \right)_K + \frac{\tau}{4} \|u_h^{n+1}\|_{L^{q+1}}^{q+1} \\
&\leq \tau \|u_h^{n+1}\|_{L^2}^2 - \frac{\tau}{2} \|u_h^{n+1}\|_{L^{q+1}}^{q+1} + C\tau \sum_{K \in \mathcal{T}_h} h_K^{q+1} \|\nabla u_h^{n+1}\|_{L^{q+1}(K)}^{q+1} \\
&\leq \tau \|u_h^{n+1}\|_{L^2}^2 - \frac{\tau}{2} \|u_h^{n+1}\|_{L^{q+1}}^{q+1} + C\tau \sum_{K \in \mathcal{T}_h} h_K^{q+1-d\frac{q-1}{2}} \|\nabla u_h^{n+1}\|_{L^2(K)}^{q+1}.
\end{aligned} \tag{3.33}$$

**Note** when  $d = 2$ ,  $q + 1 - d\frac{q-1}{2} \geq 0$  if  $q \geq 0$ , and when  $d = 3$ ,  $q + 1 - d\frac{q-1}{2} \geq 0$  if  $q \leq 5$ . Using the above inequalities, Theorem 3.2, taking summation over  $n$  from 0 to  $\ell - 1$ , and taking expectation on both sides of (3.30), we obtain

$$\begin{aligned}
&\frac{1}{4} \mathbb{E} [\|u_h^\ell\|_{L^2}^2] + \frac{1}{4} \sum_{n=0}^{\ell-1} \mathbb{E} [\|(u_h^{n+1} - u_h^n)\|_{L^2}^2] + \tau \sum_{n=0}^{\ell-1} \mathbb{E} [\|\nabla u_h^{n+1}\|_{L^2}^2] \\
&\quad + \frac{\tau}{2} \sum_{n=0}^{\ell-1} \mathbb{E} [\|u_h^{n+1}\|_{L^{q+1}}^{q+1}] \\
&\leq \tau \sum_{n=0}^{\ell-1} \mathbb{E} [\|u_h^n\|_{L^2}^2] + C\tau \sum_{n=0}^{\ell-1} \mathbb{E} [\|\nabla u_h^{n+1}\|_{L^2}^{q+1}] + C \\
&\leq \tau \sum_{n=0}^{\ell-1} \mathbb{E} [\|u_h^n\|_{L^2}^2] + C,
\end{aligned} \tag{3.34}$$

where Theorem 3.2 is used in the last inequality.

The conclusion is a direct result by using Gronwall's inequality.

To obtain the error estimates results, we need to establish a higher moment discrete  $L^2$  stability result for the numerical solution  $u_h$ .

**Theorem 3.4.** Suppose the mesh assumption in (3.1) holds. Then there holds for any  $p \geq 2$ ,

$$\sup_{0 \leq \ell \leq N} \mathbb{E} [\|u_h^\ell\|_{L^2}^p] \leq C.$$

*Proof.* The proof is divided into three steps. In Step 1, we give the bound for  $\mathbb{E} \|u_h^\ell\|_{L^2}^4$ . In Step 2, we give the bound for  $\mathbb{E} \|u_h^\ell\|_{L^2}^p$ , where  $p = 2^r$  and  $r$  is an arbitrary positive integer. In Step 3, we give the bound for  $\mathbb{E} \|u_h^\ell\|_{L^2}^p$ , where  $p$  is an arbitrary real number and  $p \geq 2$ .

*Step 1.* Based on (3.30) and (3.33), we have

$$\begin{aligned}
&\frac{1}{2} \|u_h^{n+1}\|_{L^2}^2 - \frac{1}{2} \|u_h^n\|_{L^2}^2 + \frac{1}{2} \|u_h^{n+1} - u_h^n\|_{L^2}^2 + \tau \|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{\tau}{2} \|u_h^{n+1}\|_{L^{q+1}}^{q+1} \\
&\leq \tau \|u_h^{n+1}\|_{L^2}^2 + C\tau \|\nabla u_h^{n+1}\|_{L^2}^{q+1} + (g(u_h^n), u_h^{n+1}) \bar{\Delta} W_{n+1}.
\end{aligned} \tag{3.35}$$

**Note** the following identity

$$\|u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|u_h^n\|_{L^2}^2 = \frac{3}{4}(\|u_h^{n+1}\|_{L^2}^2 + \|u_h^n\|_{L^2}^2) + \frac{1}{4}(\|u_h^{n+1}\|_{L^2}^2 - \|u_h^n\|_{L^2}^2). \quad (3.36)$$

Multiplying (3.35) with  $\|u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|u_h^n\|_{L^2}^2$ , we obtain

$$\begin{aligned} & \frac{3}{8}(\|u_h^{n+1}\|_{L^2}^4 - \|u_h^n\|_{L^2}^4) + \frac{1}{8}(\|u_h^{n+1}\|_{L^2}^2 - \|u_h^n\|_{L^2}^2)^2 + \left(\frac{1}{2}\|(u_h^{n+1} - u_h^n)\|_{L^2}^2\right. \\ & \quad \left. + \tau\|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{\tau}{2}\|u_h^{n+1}\|_{L^{q+1}}^{q+1}\right)(\|u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|u_h^n\|_{L^2}^2) \\ & \leq (\tau\|u_h^{n+1}\|_{L^2}^2 + C\tau\|\nabla u_h^{n+1}\|_{L^2}^{q+1})(\|u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|u_h^n\|_{L^2}^2) \\ & \quad + (g(u_h^n), u_h^{n+1})\bar{\Delta}W_{n+1}(\|u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|u_h^n\|_{L^2}^2). \end{aligned} \quad (3.37)$$

The first term on the right-hand side of (3.37) can be written as

$$\begin{aligned} & (\tau\|u_h^{n+1}\|_{L^2}^2 + C\tau\|\nabla u_h^{n+1}\|_{L^2}^{q+1})(\|u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|u_h^n\|_{L^2}^2) \\ & \leq \tau\|u_h^{n+1}\|_{L^2}^2\left(\frac{3}{2}\|u_h^{n+1}\|_{L^2}^2 - \frac{1}{2}(\|u_h^{n+1}\|_{L^2}^2 - \|u_h^n\|_{L^2}^2)\right) \\ & \quad + C\tau\|\nabla u_h^{n+1}\|_{L^2}^{2(q+1)} + \tau\|u_h^{n+1}\|_{L^2}^4 + \tau(\|u_h^{n+1}\|_{L^2}^2 - \|u_h^n\|_{L^2}^2)^2 \\ & \leq C\tau\|u_h^{n+1}\|_{L^2}^4 + C\tau\|\nabla u_h^{n+1}\|_{L^2}^{2(q+1)} + \theta_1(\|u_h^{n+1}\|_{L^2}^2 - \|u_h^n\|_{L^2}^2)^2, \end{aligned} \quad (3.38)$$

where  $\theta_1 > 0$  will be determined later.

The second term on the right-hand side of (3.37) can be written as

$$\begin{aligned} & (g(u_h^n), u_h^{n+1})\bar{\Delta}W_{n+1}(\|u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|u_h^n\|_{L^2}^2) \\ & = (g(u_h^n), u_h^{n+1} - u_h^n + u_h^n)\bar{\Delta}W_{n+1}(\|u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|u_h^n\|_{L^2}^2) \\ & \leq \left(\frac{1}{4}\|u_h^{n+1} - u_h^n\|_{L^2}^2 + C(1 + \|u_h^n\|_{L^2}^2)(\bar{\Delta}W_{n+1})^2\right. \\ & \quad \left. + (g(u_h^n), u_h^n)\bar{\Delta}W_{n+1}\right)(\|u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|u_h^n\|_{L^2}^2). \end{aligned} \quad (3.39)$$

For the second term on the right-hand side of (3.39), using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} & C(1 + \|u_h^n\|_{L^2}^2)(\bar{\Delta}W_{n+1})^2(\|u_h^{n+1}\|_{L^2}^2 + \frac{1}{2}\|u_h^n\|_{L^2}^2) \\ & = C(1 + \|u_h^n\|_{L^2}^2)(\bar{\Delta}W_{n+1})^2(\|u_h^{n+1}\|_{L^2}^2 - \|u_h^n\|_{L^2}^2 + \frac{3}{2}\|u_h^n\|_{L^2}^2) \\ & \leq \theta_2(\|u_h^{n+1}\|_{L^2}^2 - \|u_h^n\|_{L^2}^2)^2 + (C + C\|u_h^n\|_{L^2}^4)(\bar{\Delta}W_{n+1})^4 \\ & \quad + C\|u_h^n\|_{L^2}^4(\bar{\Delta}W_{n+1})^2 + C\|u_h^n\|_{L^2}^2(\bar{\Delta}W_{n+1})^2, \end{aligned} \quad (3.40)$$

where  $\theta_2 > 0$  will be determined later. Using (1.7), the third term on the right-hand side of

(3.39) can be bounded by

$$\begin{aligned}
 & (g(u_h^n), u_h^n) \bar{\Delta} W_{n+1} (\|u_h^{n+1}\|_{L^2}^2 + \frac{1}{2} \|u_h^n\|_{L^2}^2) \\
 &= (g(u_h^n), u_h^n) \bar{\Delta} W_{n+1} (\|u_h^{n+1}\|_{L^2}^2 - \|u_h^n\|_{L^2}^2 + \frac{3}{2} \|u_h^n\|_{L^2}^2) \\
 &\leq \theta_3 (\|u_h^{n+1}\|_{L^2}^2 - \|u_h^n\|_{L^2}^2)^2 + (C + C \|u_h^n\|_{L^2}^4) (\bar{\Delta} W_{n+1})^2 \\
 &\quad + \frac{3}{2} (g(u_h^n), u_h^n) \|u_h^n\|_{L^2}^2 \bar{\Delta} W_{n+1},
 \end{aligned} \tag{3.41}$$

where  $\theta_3 > 0$  will be determined later.

Choosing  $\theta_1, \theta_2, \theta_3$  such that  $\theta_1 + \theta_2 + \theta_3 \leq \frac{1}{16}$ , and then taking the summation over  $n$  from 0 to  $\ell - 1$  and taking the expectation on both sides of (3.37), we obtain

$$\begin{aligned}
 & \frac{3}{8} \mathbb{E} [\|u_h^\ell\|_{L^2}^4] + \frac{1}{16} \sum_{n=0}^{\ell-1} \mathbb{E} [(\|u_h^{n+1}\|_{L^2}^2 - \|u_h^n\|_{L^2}^2)^2] + \sum_{n=0}^{\ell-1} \mathbb{E} \left[ \left( \frac{1}{4} \|u_h^{n+1} - u_h^n\|_{L^2}^2 \right. \right. \\
 & \quad \left. \left. + \tau \|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{\tau}{2} \|u_h^{n+1}\|_{L^{q+1}}^{q+1} \right) (\|u_h^{n+1}\|_{L^2}^2 + \frac{1}{2} \|u_h^n\|_{L^2}^2) \right] \\
 &\leq C \tau \sum_{n=0}^{\ell-1} \mathbb{E} [\|u_h^{n+1}\|_{L^2}^4] + C \tau \sum_{n=0}^{\ell-1} \mathbb{E} [\|\nabla u_h^{n+1}\|_{L^2}^{2(q+1)}] + \frac{3}{8} \mathbb{E} [\|u_h^0\|_{L^2}^4] \\
 &\quad + C \tau \sum_{n=0}^{\ell-1} \mathbb{E} [\|u_h^n\|_{L^2}^4] + C.
 \end{aligned} \tag{3.42}$$

When  $\tau \leq C$ , we have

$$\begin{aligned}
 & \frac{1}{4} \mathbb{E} [\|u_h^\ell\|_{L^2}^4] + \frac{1}{16} \sum_{n=0}^{\ell-1} \mathbb{E} [(\|u_h^{n+1}\|_{L^2}^2 - \|u_h^n\|_{L^2}^2)^2] + \sum_{n=0}^{\ell-1} \mathbb{E} \left[ \left( \frac{1}{4} \|u_h^{n+1} - u_h^n\|_{L^2}^2 \right. \right. \\
 & \quad \left. \left. + \tau \|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{\tau}{2} \|u_h^{n+1}\|_{L^4}^4 \right) (\|u_h^{n+1}\|_{L^2}^2 + \frac{1}{2} \|u_h^n\|_{L^2}^2) \right] \\
 &\leq C \tau \sum_{n=0}^{\ell-1} \mathbb{E} [\|u_h^n\|_{L^2}^4] + C \tau \sum_{n=0}^{\ell-1} \mathbb{E} [\|\nabla u_h^{n+1}\|_{L^2}^{2(q+1)}] + \frac{3}{8} \mathbb{E} [\|u_h^0\|_{L^2}^4] + C.
 \end{aligned} \tag{3.43}$$

Using Gronwall's inequality, we obtain

$$\begin{aligned}
 & \frac{1}{4} \mathbb{E} [\|u_h^\ell\|_{L^2}^4] + \frac{1}{16} \sum_{n=0}^{\ell-1} \mathbb{E} [(\|u_h^{n+1}\|_{L^2}^2 - \|u_h^n\|_{L^2}^2)^2] + \sum_{n=0}^{\ell-1} \mathbb{E} \left[ \left( \frac{1}{4} \|u_h^{n+1} - u_h^n\|_{L^2}^2 \right. \right. \\
 & \quad \left. \left. + \tau \|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{\tau}{2} \|u_h^{n+1}\|_{L^4}^4 \right) (\|u_h^{n+1}\|_{L^2}^2 + \frac{1}{2} \|u_h^n\|_{L^2}^2) \right] \leq C.
 \end{aligned} \tag{3.44}$$

*Step 2.* Similar to Step 1, using (3.37)–(3.41), we have

$$\begin{aligned}
 & \frac{3}{8} (\|u_h^{n+1}\|_{L^2}^4 - \|u_h^n\|_{L^2}^4) + \frac{1}{16} (\|u_h^{n+1}\|_{L^2}^2 - \|u_h^n\|_{L^2}^2)^2 \\
 & \quad + \left( \frac{1}{4} \|u_h^{n+1} - u_h^n\|_{L^2}^2 + \tau \|\nabla u_h^{n+1}\|_{L^2}^2 + \frac{\tau}{2} \|u_h^{n+1}\|_{L^4}^4 \right) (\|u_h^{n+1}\|_{L^2}^2 + \frac{1}{2} \|u_h^n\|_{L^2}^2) \\
 &\leq C \tau \|u_h^{n+1}\|_{L^2}^4 + C \tau \|\nabla u_h^{n+1}\|_{L^2}^{2(q+1)} + (C + C \|u_h^n\|_{L^2}^4) (\bar{\Delta} W_{n+1})^4 \\
 &\quad + (C + C \|u_h^n\|_{L^2}^4) (\bar{\Delta} W_{n+1})^2 + (g(u_h^n), u_h^n) \|u_h^n\|_{L^2}^2 \bar{\Delta} W_{n+1}.
 \end{aligned} \tag{3.45}$$

Similar to Step 1, multiplying (3.45) with  $\|u_h^{n+1}\|_{L^2}^4 + \frac{1}{2}\|u_h^n\|_{L^2}^4$ , we can obtain the 8-th moment of the  $L^2$  stability result of the discrete solution. Then repeating this process, the  $2^r$ -th moment of the  $L^2$  stability result of the discrete solution can be obtained.

*Step 3.* Suppose  $2^{r-1} \leq p \leq 2^r$ , and then by Young's inequality, we have

$$\begin{aligned} \mathbb{E} [\|u_h^\ell\|_{L^2}^p] &\leq \mathbb{E} [\|u_h^\ell\|_{L^2}^{2^r}] + C \\ &\leq C, \end{aligned} \quad (3.46)$$

where the second inequality uses Step 2. The proof is complete.

### 3.4. Error estimates

Let  $e^n = u(t_n) - u_h^n$  ( $n = 0, 1, 2, \dots, N$ ). In the following theorem, the  $L^2$  projection is used in the proof of the error estimates and the strong convergence rate is given.

**Theorem 3.5.** *Let  $u$  and  $\{u_h^n\}_{n=1}^N$  denote respectively the solutions of problem (1.8) and scheme (3.3). Then, under the condition (1.9), there holds*

$$\sup_{0 \leq n \leq N} \mathbb{E} [\|e^n\|_{L^2}^2] + \mathbb{E} \left[ \tau \sum_{n=1}^N \|\nabla e^n\|_{L^2}^2 \right] \leq C\tau + Ch^2 |\ln h|^2.$$

*Proof.* We write  $e^n = \eta^n + \xi^n$  where

$$\eta^n := u(t_n) - P_h u(t_n) \quad \text{and} \quad \xi^n := P_h u(t_n) - u_h^n, \quad n = 0, 1, 2, \dots, N.$$

It follows from (1.8) that for all  $t_n$  ( $n \geq 0$ ) there holds  $\mathbb{P}$ -almost surely

$$\begin{aligned} &(u(t_{n+1}), v_h) - (u(t_n), v_h) + \int_{t_n}^{t_{n+1}} (\nabla u(s), \nabla v_h) ds \\ &= \int_{t_n}^{t_{n+1}} (f(u(s)), v_h) ds + \int_{t_n}^{t_{n+1}} (g(u(s)), v_h) dW(s) \quad \forall v_h \in V_h. \end{aligned} \quad (3.47)$$

Subtracting (3.3) from (3.47) and setting  $v_h = \xi^{n+1}$ , the following error equation holds  $\mathbb{P}$ -almost surely,

$$\begin{aligned} &(\xi^{n+1} - \xi^n, \xi^{n+1}) = -(\eta^{n+1} - \eta^n, \xi^{n+1}) - \int_{t_n}^{t_{n+1}} (\nabla u(s) - \nabla u_h^{n+1}, \nabla \xi^{n+1}) ds \\ &\quad + \int_{t_n}^{t_{n+1}} (f(u(s)) - I_h f^{n+1}, \xi^{n+1}) ds \\ &\quad + \int_{t_n}^{t_{n+1}} ((g(u(s)) - g(u_h^n)), \xi^{n+1}) dW(s), \\ &:= T_1 + T_2 + T_3 + T_4. \end{aligned} \quad (3.48)$$

The left-hand side of (3.48) can be handled by

$$\begin{aligned} \mathbb{E} [(\xi^{n+1} - \xi^n, \xi^{n+1})] &= \frac{1}{2} \mathbb{E} [\|\xi^{n+1}\|_{L^2}^2 - \|\xi^n\|_{L^2}^2] \\ &\quad + \frac{1}{2} \mathbb{E} [\|\xi^{n+1} - \xi^n\|_{L^2}^2]. \end{aligned} \quad (3.49)$$

Next, we bound the right-hand side of (3.48). First, since  $P_h$  is the  $L^2$ -projection operator, we have  $\mathbb{E}[T_1] = 0$ .

For the second term on the right-hand side of (3.48), using the Hölder continuity in Lemma 2.1, we have

$$\begin{aligned}
 \mathbb{E}[T_2] &= -\mathbb{E} \left[ \int_{t_n}^{t_{n+1}} (\nabla u(s) - \nabla u(t_{n+1}), \nabla \xi^{n+1}) ds \right] \\
 &\quad - \mathbb{E} \left[ \int_{t_n}^{t_{n+1}} (\nabla \eta^{n+1} + \nabla \xi^{n+1}, \nabla \xi^{n+1}) ds \right] \\
 &\leq C \mathbb{E} \left[ \int_{t_n}^{t_{n+1}} \|\nabla u(s) - \nabla u(t_{n+1})\|_{L^2}^2 ds \right] \\
 &\quad - \frac{3}{4} \mathbb{E} [\|\nabla \xi^{n+1}\|_{L^2}^2] \tau + C \mathbb{E} [\|\nabla \eta^{n+1}\|_{L^2}^2] \tau \\
 &\leq C \tau^2 + C \mathbb{E} [\|\nabla \eta^{n+1}\|_{L^2}^2] \tau - \frac{3}{4} \mathbb{E} [\|\nabla \xi^{n+1}\|_{L^2}^2] \tau.
 \end{aligned} \tag{3.50}$$

In order to estimate the third term on the right-hand side of (3.48), we write

$$\begin{aligned}
 (f(u(s)) - I_h f^{n+1}, \xi^{n+1}) &= (f(u(s)) - f(u(t_{n+1})), \xi^{n+1}) \\
 &\quad + (f(u(t_{n+1})) - f(P_h u(t_{n+1})), \xi^{n+1}) \\
 &\quad + (f(P_h u(t_{n+1})) - f^{n+1}, \xi^{n+1}) \\
 &\quad + (f^{n+1} - I_h f^{n+1}, \xi^{n+1}).
 \end{aligned} \tag{3.51}$$

Using the Hölder continuity in Lemma 2.2, we obtain

$$\begin{aligned}
 \mathbb{E} [(f(u(s)) - f(u(t_{n+1})), \xi^{n+1})] &\leq C \mathbb{E} [\|f(u(s)) - f(u(t_{n+1}))\|_{L^2}^2] + \mathbb{E} [\|\xi^{n+1}\|_{L^2}^2] \\
 &\leq C \tau + \mathbb{E} [\|\xi^{n+1}\|_{L^2}^2].
 \end{aligned} \tag{3.52}$$

Next, using properties of the projection, we have

$$\begin{aligned}
 \mathbb{E} [(f(u(t_{n+1})) - f(P_h u(t_{n+1})), \xi^{n+1})] &= -\mathbb{E} \left[ \left( \eta^{n+1} \left( \sum_{i=0}^{q-1} (u(t_{n+1}))^i (P_h u(t_{n+1}))^{q-1-i} - 1 \right), \xi^{n+1} \right) \right] \\
 &\leq C \mathbb{E} \left[ \left\| \sum_{i=0}^{q-1} (u(t_{n+1}))^i (P_h u(t_{n+1}))^{q-1-i} - 1 \right\|_{L^\infty}^2 \times \|\eta^{n+1}\|_{L^2}^2 \right] + \mathbb{E} [\|\xi^{n+1}\|_{L^2}^2] \\
 &\leq C \left( \mathbb{E} \left[ \left( \|P_h u(t_{n+1})\|_{L^\infty}^{2q} + \|u(t_{n+1})\|_{L^\infty}^{2q} + |D|^{\frac{2q}{q-1}} \right) \right] \right)^{\frac{q-1}{q}} \\
 &\quad \times \left( \mathbb{E} [\|\eta^{n+1}\|_{L^2}^{2q}] \right)^{\frac{1}{q}} + \mathbb{E} [\|\xi^{n+1}\|_{L^2}^2] \\
 &\leq C \left( \mathbb{E} [\|\eta^{n+1}\|_{L^2}^{2q}] \right)^{\frac{1}{q}} + \mathbb{E} [\|\xi^{n+1}\|_{L^2}^2].
 \end{aligned} \tag{3.53}$$

The third term on the right-hand side of (3.51) can be bounded by

$$\mathbb{E} [(f(P_h u(t_{n+1})) - f^{n+1}, \xi^{n+1})] \leq \mathbb{E} [\|\xi^{n+1}\|_{L^2}^2]. \tag{3.54}$$

Using Theorem 3.2, properties of the interpolation operator, the inverse inequality, and the fact that  $u_h^{n+1}$  is a piecewise linear polynomial, the fourth term on the right-hand side of (3.51) can be handled by

$$\begin{aligned}
 & \mathbb{E}[(f^{n+1} - I_h f^{n+1}, \xi^{n+1})] \\
 & \leq \mathbb{E}\left[Ch^2 \sum_{K \in \mathcal{T}_h} \|q(u_h^{n+1})^{q-1} \nabla u_h^{n+1}\|_{L^2(K)}^2\right] + \mathbb{E}[\|\xi^{n+1}\|_{L^2}^2] \\
 & \leq \mathbb{E}\left[Ch^2 \sum_{K \in \mathcal{T}_h} \left(\|u_h^{n+1}\|_{L^\infty(K)}^{2(q-1)} \|\nabla u_h^{n+1}\|_{L^2(K)}^2\right)\right] + \mathbb{E}[\|\xi^{n+1}\|_{L^2}^2] \\
 & \leq \mathbb{E}\left[Ch^2 |\ln h|^2 \sum_{K \in \mathcal{T}_h} \left((\|\nabla u_h^{n+1}\|_{L^2(K)}^{2(q-1)} + \|u_h^{n+1}\|_{L^2(K)}^{2(q-1)}) \|\nabla u_h^{n+1}\|_{L^2(K)}^2\right)\right] \\
 & \quad + \mathbb{E}[\|\xi^{n+1}\|_{L^2}^2] \\
 & \leq \mathbb{E}\left[Ch^2 |\ln h|^2 \sum_{K \in \mathcal{T}_h} \left(\|\nabla u_h^{n+1}\|_{L^2(K)}^{2q} + \|u_h^{n+1}\|_{L^2(K)}^{2q}\right)\right] + \mathbb{E}[\|\xi^{n+1}\|_{L^2}^2] \\
 & \leq \mathbb{E}\left[Ch^2 |\ln h|^2 (\|u_h^{n+1}\|_{L^2}^{2q} + \|\nabla u_h^{n+1}\|_{L^2}^{2q})\right] + \mathbb{E}[\|\xi^{n+1}\|_{L^2}^2] \\
 & \leq Ch^2 |\ln h|^2 + \mathbb{E}[\|\xi^{n+1}\|_{L^2}^2].
 \end{aligned} \tag{3.55}$$

Combining (3.52)–(3.55) to obtain

$$\mathbb{E}[T_3] \leq C\tau^2 + Ch^2 |\ln h|^2 \tau + C\mathbb{E}[\|\xi^{n+1}\|_{L^2}^2] \tau + C\left(\mathbb{E}[\|\eta^{n+1}\|_{L^2}^{2q}]\right)^{\frac{1}{q}} \tau. \tag{3.56}$$

By the martingale property, the Itô isometry, the Hölder continuity of  $u$  and the global Lipschitz condition (1.5), we have

$$\begin{aligned}
 \mathbb{E}[T_4] & \leq \frac{1}{2}\mathbb{E}[\|\xi^{n+1} - \xi^n\|_{L^2}^2] + \frac{1}{2}\mathbb{E}\left[\int_{t_n}^{t_{n+1}} \|g(u(s)) - g(u_h^n)\|_{L^2}^2 ds\right] \\
 & \leq \frac{1}{2}\mathbb{E}[\|\xi^{n+1} - \xi^n\|_{L^2}^2] + C\mathbb{E}\left[\int_{t_n}^{t_{n+1}} \|u(s) - u_h^n\|_{L^2}^2 ds\right] \\
 & \leq \frac{1}{2}\mathbb{E}[\|\xi^{n+1} - \xi^n\|_{L^2}^2] + C\mathbb{E}\left[\int_{t_n}^{t_{n+1}} \|u(s) - u(t_n)\|_{L^2}^2 ds\right] \\
 & \quad + C\mathbb{E}[\|\eta^n + \xi^n\|_{L^2}^2] \tau \\
 & \leq \frac{1}{2}\mathbb{E}[\|\xi^{n+1} - \xi^n\|_{L^2}^2] + C\tau^2 + C\mathbb{E}[\|\eta^n\|_{L^2}^2] \tau \\
 & \quad + C\mathbb{E}[\|\xi^n\|_{L^2}^2] \tau.
 \end{aligned} \tag{3.57}$$

Taking the expectation on (3.48) and combining estimates (3.49)–(3.57), summing over  $n = 0, 1, 2, \dots, \ell - 1$  with  $1 \leq \ell \leq N$ , and using the properties of the  $L^2$  projection and the regularity assumption, we obtain

$$\begin{aligned}
 & \frac{1}{4}\mathbb{E}[\|\xi^\ell\|_{L^2}^2] + \frac{1}{4}\mathbb{E}\left[\tau \sum_{n=1}^{\ell} \|\nabla \xi^n\|_{L^2}^2\right] \\
 & \leq \frac{1}{2}\mathbb{E}[\|\xi^0\|_{L^2}^2] + C\mathbb{E}\left[\tau \sum_{n=0}^{\ell-1} \|\xi^n\|_{L^2}^2\right] + C\tau + Ch^2 |\ln h|^2.
 \end{aligned} \tag{3.58}$$

Finally, the assertion of the theorem follows from (3.58), the discrete Gronwall's inequality, the  $L^2$ -projection properties, the fact that  $\xi^0 = 0$  and the triangle inequality. The proof is complete.

The following strong stability result is a direct corollary of Theorem 3.5.

**Corollary 3.1.** *Suppose the mesh assumption in (3.1) holds and  $h^2|\ln h|^2 \leq C\tau$ , and then*

$$\mathbb{E} \left[ \sup_{0 \leq n \leq N} (\nabla \xi^n, \nabla \xi^n) \right] \leq C.$$

*Proof.* For each sample point,

$$\sup_{0 \leq n \leq N} (\nabla \xi^n, \nabla \xi^n) \leq \sum_{n=0}^N (\nabla \xi^n, \nabla \xi^n). \quad (3.59)$$

When  $h^2|\ln h|^2 \leq C\tau$ , taking the expectation on both sides of (3.59), and using Theorem 3.5, we obtain

$$\mathbb{E} \left[ \sup_{0 \leq n \leq N} (\nabla \xi^n, \nabla \xi^n) \right] \leq C + C \frac{h^2 |\ln h|^2}{\tau} \leq C.$$

**Remark 3.1.** (a) *Note that the elliptic projection cannot be used due to the first term  $T_1$  in (3.48). In [28], it is  $C\tau + Ch^2$  since  $L^2$  projection is used there.*

(b) *For the diffusion term, We need  $g(u) \in C^1$  and  $g(u)$  to be Lipschitz continuous, which are the same assumptions as in stochastic ODE case. [18] The analysis in [28] requires two extra conditions:  $g(u)$  and  $g''(u)$  are bounded. Note that  $g(u) = u$ ,  $g(u) = \sqrt{u^2 + 1}$  or some others satisfy the assumptions in this paper, but they do not satisfy the assumptions in [28].*

## 4. Numerical experiments

In this section, we present several two dimensional numerical examples to gauge the performance of the proposed stochastic finite element scheme for the stochastic partial differential equations satisfying the proposed assumptions for the nonlinear term and the diffusion term. Test 1 is designed to demonstrate the error orders with respect to mesh size  $h$  for small and big noises; Test 2 is designed to demonstrate the stability results and evolution of the stochastic Allen-Cahn equation, which is a special case of the SPDE in this paper; Test 3 is designed to demonstrate the stability results of the SPDE with a different initial condition; Test 4 is designed to demonstrate the stability results of the SPDE with a different nonlinear term; Test 5 is designed to demonstrate the stability results of the SPDE with a different diffusion term. The square domain  $\mathcal{D} = [-1, 1]^2$ , and 10000 sample points are used in these tests.

**Test 1** In this test, we consider the following smooth initial condition

$$u_0(x, y) = \tanh \left( \frac{x^2 + y^2 - 0.6^2}{\sqrt{2}\epsilon} \right), \quad (4.1)$$

where  $\epsilon = 0.2$ . Time step size  $\tau = 1 \times 10^{-6}$  is used in this Test 1. The nonlinear term is  $f(u) = \frac{1}{\epsilon^2}(u - u^3)$ , and the diffusion term is  $g(u) = \delta u$ . Table 4.1 shows the following three types

of errors  $\{\sup_{0 \leq n \leq N} \mathbb{E}[\|e^n\|_{L^2(\mathcal{D})}^2]\}^{\frac{1}{2}}$ ,  $\{\mathbb{E}[\sup_{0 \leq n \leq N} \|e^n\|_{L^2(\mathcal{D})}^2]\}^{\frac{1}{2}}$ , and  $\{\mathbb{E}[\sum_{n=1}^N \tau \|\nabla e^n\|_{L^2(\mathcal{D})}^2]\}^{\frac{1}{2}}$  respectively, and the rates of convergence. The noise intensity is  $\delta = 1$ . In the table, we use  $L^\infty \mathbb{E} L^2$ ,  $\mathbb{E} L^\infty L^2$  and  $\mathbb{E} L^2 H^1$  to denote these three types of errors respectively.

	$L^\infty \mathbb{E} L^2$ error	order	$\mathbb{E} L^\infty L^2$ error	order	$\mathbb{E} L^2 H^1$ error	order
$h = 0.5\sqrt{2}$	0.2909	—	0.2900	—	2.2386	—
$h = 0.25\sqrt{2}$	0.0759	1.9384	0.0757	1.9377	1.1401	0.9734
$h = 0.125\sqrt{2}$	0.0202	1.9097	0.0201	1.9131	0.5919	0.9457
$h = 0.0625\sqrt{2}$	0.0051	1.9858	0.0051	1.9786	0.2996	0.9823

Table 4.1:: Spatial errors and convergence rates of Test 1:  $\epsilon = 0.2$ ,  $\tau = 1 \times 10^{-6}$ ,  $\delta = 1$ .

Table 4.2 shows the errors  $L^\infty \mathbb{E} L^2$ ,  $\mathbb{E} L^\infty L^2$  and  $\mathbb{E} L^2 H^1$  respectively, and the rates of convergence at final time  $T = 2^{-5}$ . The noise intensity  $\delta = 50$ .

	$L^\infty \mathbb{E} L^2$ error	order	$\mathbb{E} L^\infty L^2$ error	order	$\mathbb{E} L^2 H^1$ error	order
$h = 0.5\sqrt{2}$	0.3391	—	0.2979	—	2.2723	—
$h = 0.25\sqrt{2}$	0.0885	1.9380	0.0778	1.9370	1.1573	0.9734
$h = 0.125\sqrt{2}$	0.0235	1.9130	0.0206	1.9171	0.6008	0.9458
$h = 0.0625\sqrt{2}$	0.0060	1.9696	0.0053	1.9586	0.3042	0.9819

Table 4.2:: Spatial errors and convergence rates of Test 1:  $\epsilon = 0.2$ ,  $\tau = 1 \times 10^{-6}$ ,  $\delta = 50$ .

From these two tables, we observe that the error orders of  $L^\infty \mathbb{E} L^2$  and  $\mathbb{E} L^\infty L^2$  are 2, and the error order of  $\mathbb{E} L^2 H^1$  is 1. Besides, the error orders keep the same when the noise intensity increases.

In the following tests,  $\mathbb{E} L^2$  and  $\mathbb{E} H^1$  are used to denote  $\mathbb{E} \|u_h^n\|_{L^2}^2$  and  $\mathbb{E} \|\nabla u_h^n\|_{L^2}^2$  respectively.

**Test 2** In this test, we consider the following initial condition

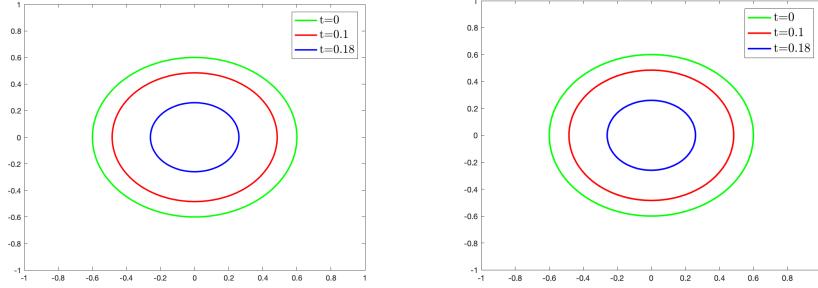
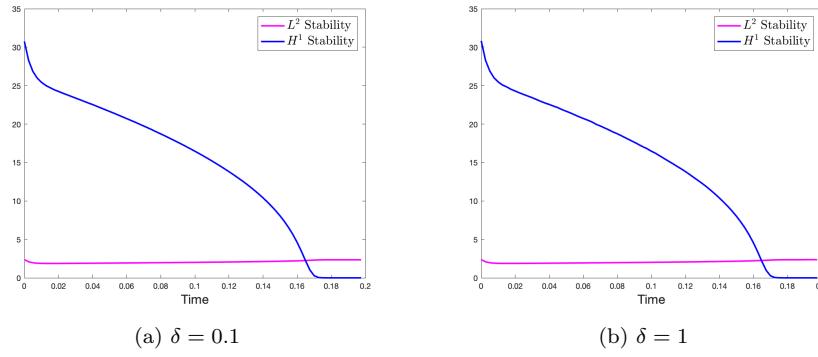
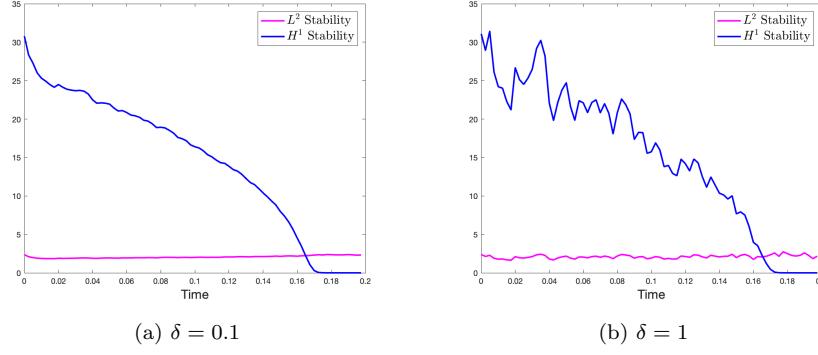
$$u_0(x, y) = \tanh\left(\frac{\sqrt{x^2 + y^2} - 0.6}{\sqrt{2}\epsilon}\right). \quad (4.2)$$

The nonlinear term is  $f(u) = \frac{1}{\epsilon^2}(u - u^3)$ , and the diffusion term is  $g(u) = \delta u$ , which corresponds to the stochastic Allen-Cahn equation. More tests related to the Allen-Cahn equation can be found in [9, 11, 12, 26, 34]. Figure 4.1 shows the evolution of the zero-level sets of the solutions under different intensity of the noise. We observe that although the circle may shrink or dilate (depending on the sign of the diffusion term), the average zero-level sets shrink for smaller and bigger noises. Figure 4.2 shows the  $\mathbb{E} L^2$  and  $\mathbb{E} H^1$  stability results at each time step, which verifies the results in Theorems 3.1 and 3.3. We also observe that they are both bounded. Figure 4.3 shows the  $\mathbb{E} L^2$  and  $\mathbb{E} H^1$  stability results for one sample point. From the graphs, these stability results seem to be bounded although they are not sample-wise decreasing.

**Test 3** In this test, we consider the following initial condition

$$u_0(x, y) = \tanh\left(\frac{1}{\sqrt{2}\epsilon}(\sqrt{x^2/0.04 + y^2/0.36} - 1)(\sqrt{x^2/0.36 + y^2/0.04} - 1)\right). \quad (4.3)$$

The nonlinear term is  $f(u) = \frac{1}{\epsilon^2}(u - u^3)$ , and the diffusion term is  $g(u) = \delta u$ . Figure 4.4 shows the  $\mathbb{E} L^2$  and  $\mathbb{E} H^1$  stability results at each time step, which verifies the results in Theorems 3.1

Fig. 4.1.: Zero level sets of the solutions:  $\tau = 1 \times 10^{-4}$ ,  $h = 0.008$ ,  $\epsilon = 0.01$ .Fig. 4.2.: Stability results (average):  $\tau = 2.5 \times 10^{-3}$ ,  $\epsilon = 0.1$ , and  $h = 0.04$ .Fig. 4.3.: Stability results (one sample point):  $\tau = 2.5 \times 10^{-3}$ ,  $\epsilon = 0.1$ , and  $h = 0.04$ .

and 3.3. Figure 4.5 shows the  $\mathbb{E}L^2$  and  $\mathbb{E}H^1$  stability results for one sample point. Similarly, these stability results are bounded although they are not sample-wise decreasing.

**Test 4** In this test, we consider the initial condition in (4.1) with  $\epsilon = 0.5$ . The nonlinear term is  $f(u) = \frac{1}{\epsilon^2}(u - u^{11})$ , and the diffusion term is  $g(u) = \delta u$ . Figure 4.6 shows the  $\mathbb{E}L^2$

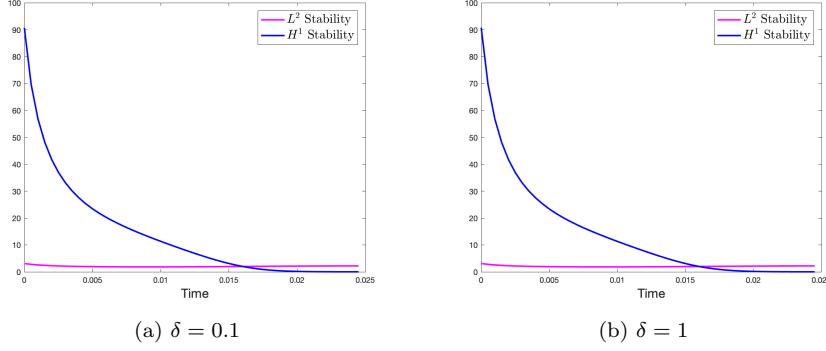


Fig. 4.4.: Stability results:  $\tau = 5 \times 10^{-4}$ ,  $\epsilon = 0.1$ , and  $h = 0.04$ .

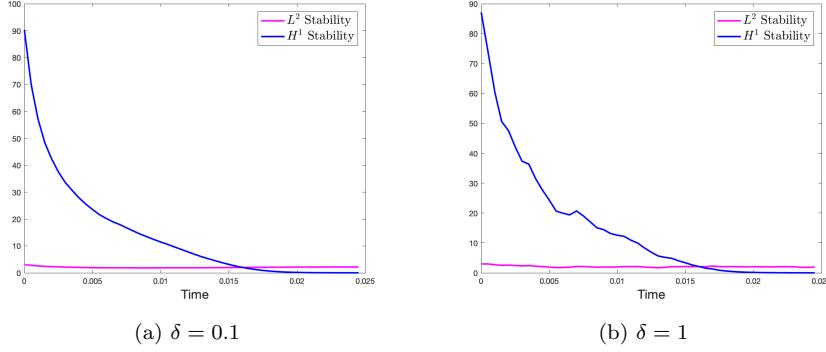


Fig. 4.5.: Stability results (one sample point):  $\tau = 5 \times 10^{-4}$ ,  $\epsilon = 0.1$ , and  $h = 0.04$ .

and  $\mathbb{E}H^1$  stability results at each time step, which verifies the results in Theorems 3.1 and 3.3. Figure 4.7 shows the  $\mathbb{E}L^2$  and  $\mathbb{E}H^1$  stability results for one sample point. We can observe some oscillations especially when  $\delta$  is big.

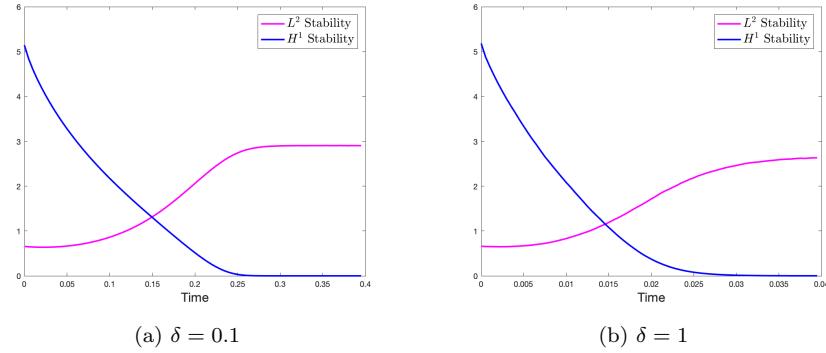


Fig. 4.6.: Stability results:  $\tau = 5 \times 10^{-3}$ ,  $\epsilon = 0.5$ , and  $h = 0.04$ .

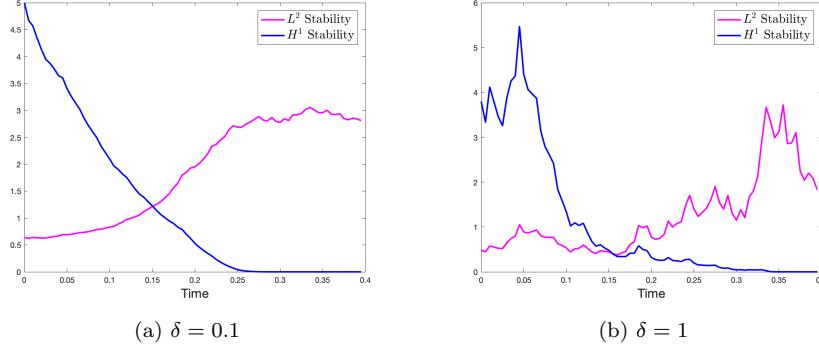


Fig. 4.7.: Stability results (one sample point):  $\tau = 5 \times 10^{-3}$ ,  $\epsilon = 0.5$ , and  $h = 0.04$ .

**Test 5** In this test, we consider the initial condition in (4.1) with  $\epsilon = 0.5$ . The nonlinear term is  $f(u) = \frac{1}{\epsilon^2}(u - u^3)$ , and the diffusion term is  $g(u) = \delta \sqrt{u^2 + 1}$ . Figure 4.8 shows the  $\mathbb{E}L^2$  and  $\mathbb{E}H^1$  stability results at each time step, which verifies the results in Theorems 3.1 and 3.3. Figure 4.9 shows the  $\mathbb{E}L^2$  and  $\mathbb{E}H^1$  stability results for one sample point. We can observe big oscillations of the  $\mathbb{E}L^2$  stability when  $\delta = 1$ .

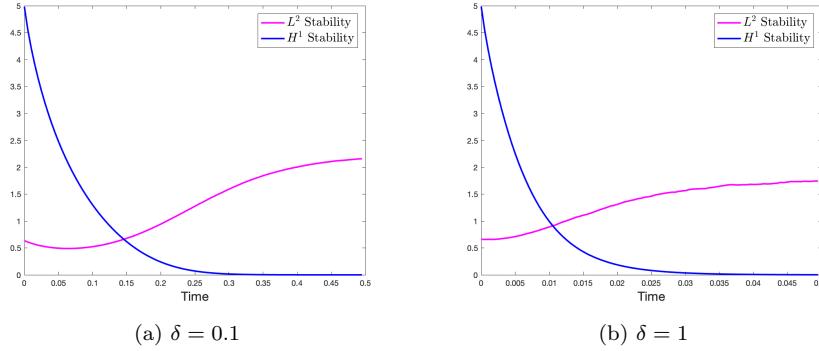


Fig. 4.8.: Stability results:  $\tau = 5 \times 10^{-3}$ ,  $\epsilon = 0.5$ , and  $h = 0.04$ .

**Test 6** In this test, we consider a random initial condition in Figure 4.10. The nonlinear term is  $f(u) = \frac{1}{\epsilon^2}(u - u^3)$ , and the diffusion term is  $g(u) = \delta u$ . Figure 4.11 shows the  $\mathbb{E}L^2$  and  $\mathbb{E}H^1$  stability results at each time step, which verifies the results in Theorems 3.1 and 3.3. Figure 4.12 shows the  $L^2$  and  $H^1$  stability results for one sample point. We observe that the  $L^2$  and  $H^1$  stability are all bounded, and they decrease very fast in the phase coarsening process.

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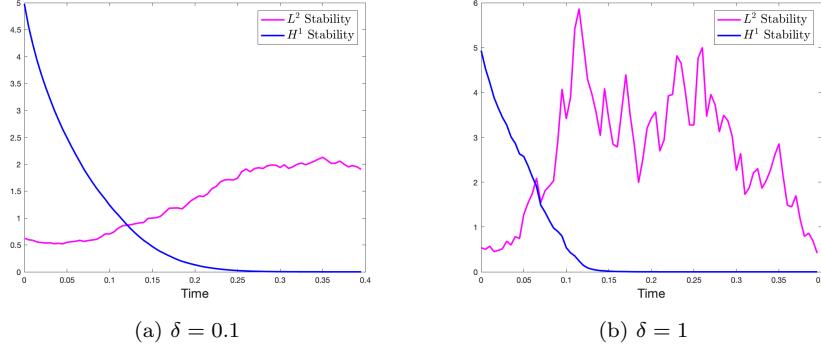


Fig. 4.9.: Stability results (one sample point):  $\tau = 5 \times 10^{-3}$ ,  $\epsilon = 0.5$ , and  $h = 0.04$ .

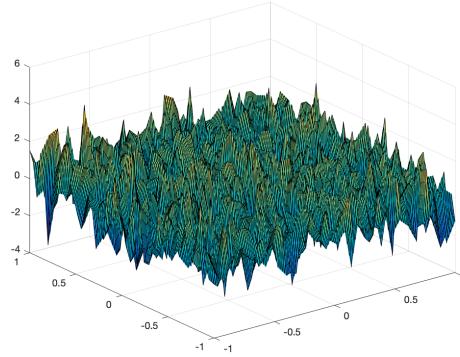


Fig. 4.10.: A random initial condition.

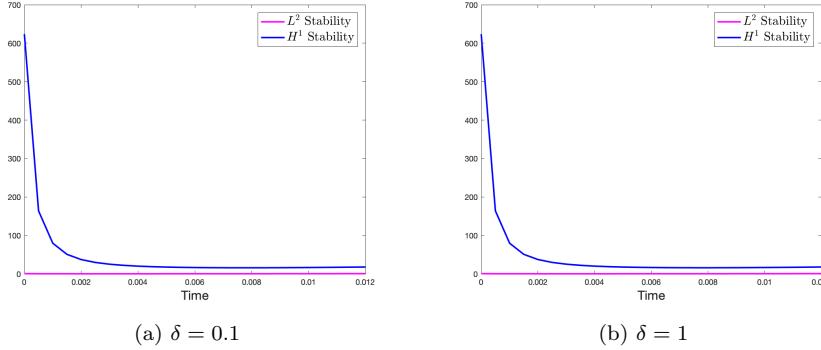


Fig. 4.11.: Stability results:  $\tau = 5 \times 10^{-4}$ ,  $\epsilon = 0.1$ , and  $h = 0.04$ .

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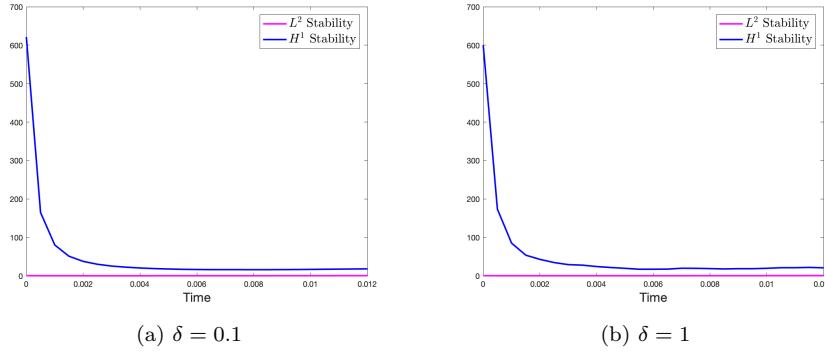


Fig. 4.12.: Stability results (one sample point):  $\tau = 5 \times 10^{-4}$ ,  $\epsilon = 0.1$ , and  $h = 0.04$ .

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