Bounded Derivations on Uniform Roe Algebras

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Abstract

We show that if $C_u^*(X)$ is a uniform Roe algebra associated to a bounded geometry metric space X, then all bounded derivations on $C_u^*(X)$ are inner.

1 Introduction

Let A be a C^*-algebra. A *derivation* of A is a linear map $\delta : A \to A$ satisfying $\delta(ab) = a\delta(b) + \delta(a)b$. In this paper, we always assume that our derivations are defined on all of A, and are thus bounded by a fundamental result of Sakai [9]. A derivation δ of A is *inner* if there exists d in the multiplier algebra $M(A)$ of A such that $\delta(a) = ad - da$ for all $a \in A$. Let us say that a C^*-algebra A *only has inner derivations* if all (bounded) derivations are inner.

Motivated by the needs of mathematical physics and the study of one-parameter automorphism groups, it is interesting to study whether all derivations are inner for a particular C^*-algebra. In the 1970s, a complete solution to this problem was obtained in the separable case via the work of several authors. The definitive result was obtained by Akemann and Pedersen [1] (see also Elliott [6], which contains a closely related result). These authors showed that a separable C^*-algebra only has inner derivations if and only if it isomorphic to a C^*-algebra of the form

$$C \oplus \bigoplus_{i \in I} S_i, \tag{1}$$

where C is continuous trace (possibly zero), and each S_i is simple (possibly zero). Thus in particular, all separable commutative, and all separable simple, C^*-algebras only have inner derivations. However, one might reasonably say that most separable C^*-algebras admit non-inner derivations.

For non-separable C^*-algebras the picture is murkier. It is well-known that there are non-separable C^*-algebras that are not of the form in line (1) and that only have inner derivations: perhaps most famously, Sakai [10] has shown this for all von Neumann algebras. See also for example [6, page 123] for some examples that are not von Neumann algebras, nor of the form in line (1), and that only have inner derivations.

Our goal in this paper is to give a new class of examples that only have inner derivations: uniform Roe algebras. Uniform Roe algebras are a well-studied class of non-separable C^*-algebras associated to metric spaces; see Section 2 below for basic definitions. They were originally introduced for index-theoretic purposes, but are now studied for their own sake as a bridge between C^*-algebra theory and coarse geometry, as well as having interesting applications to single operator theory and mathematical physics, amongst other things. Due to the presence of $\ell^\infty(X)$ as a diagonal[1] MASA, they have a somewhat von Neumann algebraic flavor, but are von Neumann algebras only in the trivial finite-dimensional case. They are also essentially never of the form in line (1). Moreover, in many ways they are quite tractable as C^*-algebras, often having good regularity properties such as nuclearity.

Here is our main theorem.

Theorem 1.1. *Uniform Roe algebras associated to bounded geometry metric spaces only have inner derivations.*

The key ingredients in the proof are: a basic form of a 'reduction of cocycles' argument used by Sinclair and Smith [11] in their study of Hochschild cohomology of von-Neumann algebras; and recent applications of Ramsey-theoretic ideas to the study of uniform Roe algebras by Braga and Farah [3].

We conclude this introduction by noting that the fact that all derivations on A are inner can be restated as saying that the first Hochschild cohomology group $H^1(A, A)$ vanishes. For A a uniform Roe algebra, it is then natural to ask if all the higher groups $H^n(A, A)$ vanish. See [11] for a survey of this problem in the case that A is a von Neumann algebra.

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[1]In the sense of Kumjian: see [8]

result in the special case that $C_u^*(X)$ is nuclear, and via a more complicated method. We are grateful to Roger Smith for suggesting we think about the 'reduction of cocycle method', which allowed us to both generalize the main result, and simplify its proof. Both authors were partially supported by NSF grants DMS-1564281 and DMS-1901522.

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2 Definitions and background results

In this section, we recall some basic definitions, as well as a classical result of Kadison stating that all derivations on a C^*-algebra are spatially implemented.

Inner products are linear in the first variable. For a Hilbert space $\mathcal{H}$ we denote the space of bounded operators on $\mathcal{H}$ by $\mathscr{B}(\mathcal{H})$, and the space of compact operators by $\mathscr{K}(\mathcal{H})$. The commutator of $a, b \in \mathscr{B}(\mathcal{H})$ is denoted by $[a, b] := ab - ba$.

The Hilbert space of square-summable sequences on a set X is denoted $\ell^2(X)$, and the canonical basis of $\ell^2(X)$ will be denoted $(\vartheta_x)_{x\in X}$ (we reserve δ for derivations). For $a \in \mathscr{B}(\ell^2(X))$ we define its matrix entries by

$$a_{xy} := \langle a\vartheta_y, \vartheta_x\rangle .$$

Definition 2.1 (propagation, uniform Roe algebra)**.** Let X be a metric space and $r \geq 0$. An operator $a \in \mathscr{B}(\ell^2(X))$ *has propagation at most* r if $a_{xy} = 0$ whenever $d(x, y) > r$ for all $(x, y) \in X \times X$. In this case, we write $\mathrm{prop}(a) \leq r$. The set of all operators with propagation at most r is denoted $\mathbb{C}_u^r[X]$. We define

$$\mathbb{C}_u[X] := \{a \in \mathscr{B}(\ell^2(X)) : \mathrm{prop}(a) < \infty\};$$

it is not difficult to see that this is a $*$-algebra. The *uniform Roe algebra*, denoted $C_u^*(X)$, is defined to be the norm closure of $\mathbb{C}_u[X]$.

Definition 2.2 (ϵ-r-*approximated*)**.** Let X be a metric space. Given $\epsilon > 0$ and $r > 0$, an operator $a \in \mathscr{B}(\ell^2(X))$ can be ϵ-r-*approximated* if there exists a $b \in \mathbb{C}_u^r[X]$ such that $\|a - b\| \leq \epsilon$.

We will exclusively be interested in uniform Roe algebras associated to bounded geometry metric spaces as in the next definition.

Definition 2.3 (bounded geometry)**.** A metric space X is said to have *bounded geometry* if for every $r \geq 0$ there exists an $N_r \in \mathbb{N}$ such that for all $x \in X$, the ball of radius r about x has at most N_r elements.

Finally in this section, we recall a general fact about derivations.

Definition 2.4 (spatial derivation)**.** Let $A \subseteq \mathscr{B}(\mathcal{H})$ be a concrete C^*-algebra. A derivation δ of A is *spatial* if there is a bounded operator $d \in \mathscr{B}(\mathcal{H})$ such that $\delta(a) = [a, d]$.

The following is due to Kadison [7, Theorem 4].

Theorem 2.5. *Let $A \subseteq \mathscr{B}(\mathcal{H})$ be a concrete C^*-algebra. Then every derivation on A is spatial.* □

Note that a uniform Roe algebra $C^*_u(X)$ always contains the compact operators on $\ell^2(X)$. For a concrete C^*-algebra $A \subseteq \mathscr{B}(\mathcal{H})$ containing the compact operators $\mathscr{K}(\mathcal{H})$, there are simpler proofs of Theorem 2.5 available: see for example [5, Corollary 3.4 and Remark on page 284].

3 Averaging operators over amenable groups

In this section, we summarize some facts we need about averaging operators on a Hilbert space over an amenable group. Most of this material seems likely to be well-known; however, we could not find convenient references for the facts we wanted, so provide most details here.

Let G be a discrete (possibly uncountable) group. If A is a complex Banach space, we let $\ell^\infty(G, A)$ denote the Banach space of bounded functions from G to A equipped with the supremum norm; in the case $A = \mathbb{C}$, we just write $\ell^\infty(G)$. We also equip $\ell^\infty(G, A)$ with the right-action of G defined for $a \in \ell^\infty(G, A)$ and $h, g \in G$ by

$$(ag)(h) := a(hg^{-1}).$$

If Z is any set, a function $\phi : \ell^\infty(G, A) \to Z$ is *invariant* if $\phi(ag) = \phi(a)$ for all $a \in \ell^\infty(G, A)$ and $g \in G$. Recall that G is amenable if there exists an *invariant mean* on $\ell^\infty(G)$, i.e. an invariant function $\phi : \ell^\infty(G) \to \mathbb{C}$ that is also a state.

Fix now an invariant mean on $\ell^\infty(G)$, which we denote[2] by

$$a \mapsto \int_G a(g) \,\mathrm{d}\,\mu(g).$$

[2]The integral notation is meant to be suggestive, but we do not need to, and will not, assign any specific meaning to the 'measure' μ.

Let now B be a complex Banach space with dual B^*. We may upgrade an invariant mean on $\ell^\infty(G)$ to an invariant contractive linear map $\ell^\infty(G, B^*) \to B^*$ in the following way. Let $b \in B$, $g \in G$, and $a \in \ell^\infty(G, B^*)$, and write $\langle b, a(g)\rangle$ for the pairing between b and $a(g)$. Then the map

$$G \to \mathbb{C}, \quad g \mapsto \langle b, a(g)\rangle$$

is bounded, and so we may apply the invariant mean to get a complex number

$$\int_G \langle b, a(g)\rangle \,\mathrm{d}\,\mu(g).$$

It is not difficult to check that the map

$$B \to \mathbb{C}, \quad b \mapsto \int_G \langle b, a(g)\rangle \,\mathrm{d}\,\mu(g)$$

is a bounded linear functional on B. We write $\int_G a(g)\,\mathrm{d}\,\mu(g)$ for this bounded linear functional.

The following lemma is straightforward. We leave the details to the reader.

Lemma 3.1. *With notation as above, the map*

$$\ell^\infty(G, B^*) \to B^*, \quad a \mapsto \int_G a(g)\,\mathrm{d}\,\mu(g)$$

is uniquely determined by the condition

$$\left\langle b, \int_G a(g)\,\mathrm{d}\,\mu(g)\right\rangle = \int_G \langle b, a(g)\rangle \,\mathrm{d}\,\mu(g) \tag{2}$$

for $b \in B$ and $a \in \ell^\infty(G, B^*)$. It is contractive, linear, invariant, and acts as the identity on constant functions. □

We will apply this machinery in the case that $B = \mathcal{L}^1(\ell^2(X))$ is the trace class operators on $\ell^2(X)$. In this case, the dual B^* canonically identifies with $\mathscr{B}(\ell^2(X))$: indeed, if Tr is the canonical trace $\mathcal{L}^1(\ell^2(X))$, $b \in \mathcal{L}^1(\ell^2(X))$, and $a \in \mathscr{B}(\ell^2(X))$, then the pairing inducing this duality isomorphism is defined by

$$\langle b, a\rangle := \mathrm{Tr}(ba). \tag{3}$$

We will need some basic lemmas. The first can be deduced very quickly from the theory of conditional expectations (see for example [4, Lemma 1.5.10]); we instead give a slightly longer naive proof.

Lemma 3.2. *With notation as above, for any $a \in \ell^\infty(G, \mathscr{B}(\ell^2(X)))$ and $c \in \mathscr{B}(\ell^2(X))$, we have that*

$$c\int_G a(g)\,\mathrm{d}\,\mu(g) = \int_G ca(g)\,\mathrm{d}\,\mu(g) \quad \textit{and} \quad \int_G a(g)\,\mathrm{d}\,\mu(g)c = \int_G a(g)c\,\mathrm{d}\,\mu(g)$$

Proof. Using lines (2) and (3), for any $b \in \mathcal{L}^1(\ell^2(X))$, we have

$$\begin{aligned}\Big\langle b, c\int_G a(g)\,\mathrm{d}\,\mu(g)\Big\rangle &= \mathrm{Tr}\Big(bc\int_G a(g)\,\mathrm{d}\,\mu(g)\Big) = \Big\langle bc, \int_G a(g)\,\mathrm{d}\,\mu(g)\Big\rangle \\ &= \int_G \langle bc, a(g)\rangle\,\mathrm{d}\,\mu(g) = \int_G \mathrm{Tr}(bca(g))\,\mathrm{d}\,\mu(g) \\ &= \int_G \langle b, ca(g)\rangle\,\mathrm{d}\,\mu(g) = \Big\langle b, \int_G ca(g)\,\mathrm{d}\,\mu(g)\Big\rangle.\end{aligned}$$

As $b \in \mathcal{L}^1(\ell^2(X))$ was arbitrary, this implies that $c\int_G a(g)\,\mathrm{d}\,\mu(g) = \int_G ca(g)\,\mathrm{d}\,\mu(g)$. The other case is similar, using also the trace identity $\mathrm{Tr}(cd) = \mathrm{Tr}(dc)$, which is valid whenever either c or d is trace class. □

The next lemma says that our averaging process behaves well with respect to propagation. Again, we proceed naively; the key point of the lemma is that the collection of operators in $\mathscr{B}(\ell^2(X))$ that have propagation at most r is weak-$*$ closed for the weak-$*$ topology inherited from the pairing with $\mathcal{L}^1(\ell^2(X))$.

Lemma 3.3. *With notation as above, if $r \geq 0$ and $a \in \ell^\infty(G, \mathscr{B}(\ell^2(X)))$ is such that the propagation of each $a(g)$ is at most r, then the propagation of $\int_G a(g)\,\mathrm{d}\,\mu(g)$ is also at most r.*

Proof. Let $e_{xy} \in \mathcal{L}^1(\ell^2(X))$ be the standard matrix unit. Then one computes using line (3) above that for any $a \in \mathscr{B}(\ell^2(X))$,

$$\langle e_{yx}, a\rangle = \mathrm{Tr}(e_{yx}a) = a_{xy}. \tag{4}$$

Using lines (2) and (4), we see that

$$\Big\langle e_{yx}, \int_G a(g)\,\mathrm{d}\,\mu(g)\Big\rangle = \int_G \langle e_{yx}, a(g)\rangle\,\mathrm{d}\,\mu(g) = \int_G a(g)_{xy}\,\mathrm{d}\,\mu(g),$$

where the last expression means the image of the function

$$G \to \mathbb{C}, \quad g \mapsto a(g)_{xy}$$

under the invariant mean. If $d(x,y) > r$, we have that $a(g)_{xy} = 0$ for all $g \in G$, and therefore that $\int_G a(g)_{xy} \, \mathrm{d}\,\mu(g) = 0$. Hence by the above computation,

$$d(x,y) > r \quad \text{implies} \quad \left\langle e_{yx}, \int_G a(g) \, \mathrm{d}\,\mu(g) \right\rangle = 0.$$

Using line (4), this says that $\int_G a(g) \, \mathrm{d}\,\mu(g)$ has propagation at most r, so we are done. $\square$

Lemma 3.4. *With notation as above, say that there is a unitary representation $g \mapsto u_g$ of G on $\ell^2(X)$. For any fixed $d \in \mathscr{B}(\ell^2(X))$, define $a \in \ell^\infty(G, \mathscr{B}(\ell^2(X)))$ by $a(g) := u_g^* d u_g$. Then $\int_G a(g) \, \mathrm{d}\,\mu(g)$ is in the commutant of the set $\{u_g \mid g \in G\}$.*

Proof. Let $h \in G$. Then by Lemma 3.2,

$$u_h \int_G u_g^* d u_g \, \mathrm{d}\,\mu(g) = \int_G u_h u_g^* d u_g \, \mathrm{d}\,\mu(g) = \int_G u_{gh^{-1}}^* d u_g \, \mathrm{d}\,\mu(g).$$

Making the 'change of variables' $k = gh^{-1}$ and using right-invariance of the map $a \mapsto \int_G a(g) \, \mathrm{d}\,\mu(g)$, this equals

$$\int_G (u_k)^* d u_{kh} \, \mathrm{d}\,\mu(k) = \int_G u_k^* d u_k u_h \, \mathrm{d}\,\mu(k).$$

Using Lemma 3.2 again we get $\int_G u_k^* d u_k u_h \, \mathrm{d}\,\mu(k) = \int_G u_k^* d u_k \, \mathrm{d}\,\mu(k) u_h$, so are done. $\square$

4 A result of Braga and Farah

In this section, we present a result of Braga and Farah from [3, Lemma 4.9]: see Proposition 4.2 below. Although 'only' a lemma, this result is quite substantial. On the referee's suggestion, we include a proof to keep our paper self-contained. Our argument does not contain anything new over the original one, although we have arranged it differently for the sake of variety.

To state the result, let $\mathbb{D} := \{z \in \mathbb{C} \mid |z| \leq 1\}$ denote the closed unit disk in the complex plane. Let I be a countably infinite set, and let $\mathbb{D}^I$ denote as usual the space of all I-indexed tuples $\lambda := (\lambda_i)_{i \in I}$ with each $\lambda_i \in \mathbb{D}$. We fix this notation throughout this section.

Definition 4.1 (symmetrically summable)**.** A sequence $(a_i)_{i\in I}$ is *symmetrically summable* if for all $\lambda \in \mathbb{D}^I$, the sum $\sum_{i\in I} \lambda_i a_i$ converges in the strong operator topology to an element of $C_u^*(X)$. If (a_i) is symmetrically summable and $\lambda = (\lambda_i)$ is in $\mathbb{D}^I$, we write a_λ for the operator $\sum_{i\in I} \lambda_i a_i$.

Here is the statement we need, which is essentially[3] a special case of [3, Lemma 4.9].

Proposition 4.2. Let (a_i) be a symmetrically summable collection of operators in $C_u^*(X)$. Then for any $\epsilon > 0$ there exists $r > 0$ such that for all $\lambda \in \mathbb{D}^I$, the operator a_λ is ϵ-r-approximated.

The content of the result is the order of quantifiers: the point is that given an $\epsilon > 0$ there is an $r > 0$ that works for all the a_λ at once. The proof of Proposition 4.2 proceeds via an application of the Baire category theorem to the following sets.

Definition 4.3. Say (a_i) is symmetrically summable, and for any $\epsilon, r > 0$ define

$$U_{\epsilon,r} := \{\lambda \in \mathbb{D}^I \mid a_\lambda \text{ can be } \epsilon\text{-}r\text{-approximated}\}.$$

Note that the hypothesis of Theorem 4.2 says that for any $\epsilon > 0$,

$$\mathbb{D}^I = \bigcup_{r=1}^{\infty} U_{\epsilon,r}, \tag{5}$$

while the conclusion of Theorem 4.2 says that for any $\epsilon > 0$ there exists r such that $\mathbb{D}^I = U_{\epsilon,r}$.

We equip $\mathbb{D}^I$ with the product topology, which is compact (by Tychonoff's theorem) and metrizable (as I is countable), so in particular a space to which the Baire category theorem applies.

We will first show that the sets in Definition 4.3 are closed for any symmetrically summable (a_i). Then we will show that if (a_i) does not satisfy the conclusion of Theorem 4.2, there is $\epsilon > 0$ such that for all $r > 0$, $U_{r,\epsilon}$ is nowhere dense in $\mathbb{D}^I$. As we have the union in line (5), this contradicts the Baire category theorem and we will be done.

[3]There are two differences with [3, Lemma 4.9]. The first is that Braga and Farah allow index sets I of some other cardinalities, and some non-metrizable coarse spaces. For simplicity, and as we only need that case, we assume I is countable and X is metrizable here. The second is that the result of [3, Lemma 4.9] is only stated for finite rank operators a_i. However, the same proof establishes the result without using that assumption, so we state the stronger version here.

We now embark on the proof that $U_{\epsilon,r}$ is closed. We will need two preliminary lemmas.

Lemma 4.4. *(i) If a is a bounded operator on $\ell^2(X)$ such that for all finite rank projections p in $\ell^\infty(X)$ the product ap can be ϵ-r-approximated, then a itself can be ϵ-r-approximated.*

(ii) Say a is a bounded operator on $\ell^2(X)$ and $\epsilon, r > 0$ are such that for all $\delta > 0$, a can be $(\epsilon+\delta)$-r-approximated. Then a can be ϵ-r-approximated.

Proof. (i) Let J be the net of all finite rank projections in $\ell^\infty(X)$, equipped with the usual operator ordering. For each $p \in J$, choose $b_p \in \mathbb{C}_u^r[X]$ such that $\|ap - b_p\| \leq \epsilon$. Then the net $(b_p)_{p\in J}$ is norm bounded, so has a weak operator topology convergent subnet, say $(b_p)_{p\in J'}$, converging to some bounded operator b on $\ell^2(X)$. Note moreover that $\lim_{p\in J'} p$ equals the identity in the weak operator topology, and so $\lim_{p\in J'} ap = a$ and $\lim_{p\in J'}(ap - b_p) = a - b$ in the weak operator topology.

Now, as weak operator topology limits do not increase norms, we see that

$$\|a - b\| \leq \limsup_{p\in J'} \|ap - b_p\| \leq \epsilon.$$

Hence to complete the proof, it suffices to show that b is in fact in $\mathbb{C}_u^r[X]$. Indeed, for each $(x, y) \in X \times X$, the function taking a bounded operator c on $\ell^2(X)$ to its matrix entry c_{xy} is weak operator topology continuous. Hence if $d(x, y) > r$ then

$$b_{xy} = \lim_{p\in J'} \big((b_p)_{xy}\big) = 0$$

and so b is in $\mathbb{C}_u^r[X]$ as desired.

(ii) For each n, let $b_n \in \mathbb{C}_u^r[X]$ be such that $\|a - b_n\| \leq \epsilon + 1/n$. As in the previous part, there is a subnet $(b_{n_j})_{j\in J}$ of the sequence (b_n) that converges to some $b \in \mathbb{C}_u^r[X]$ in the weak operator topology. As weak operator topology limits cannot increase norms, we see that

$$\|a - b\| \leq \limsup_{j\in J} \|a - b_{n_j}\| \leq \limsup_{j\in J}(\epsilon + 1/n_j) = \epsilon,$$

which shows that a can be ϵ-r-approximated as claimed.

□

Lemma 4.5. *Say $(x_i)_{i\in I}$ is a collection in a Banach space such that $\sum_i \lambda_i x_i$ converges in norm for all $(\lambda_i) \in \mathbb{D}^I$. Then for any $\delta > 0$ there exists a finite subset F of I such that for all $(\lambda_i) \in \mathbb{D}^I$*

$$\left\| \sum_{i\in I\setminus F} \lambda_i x_i \right\| < \delta.$$

Proof. For notational convenience, identify I with $\mathbb{N}$, so we are just dealing with a sequence (x_n). Assume for contradiction that there exists $\delta > 0$ such that for all N there exists $(\lambda_n) \in \mathbb{D}^{\mathbb{N}}$ such that

$$\left\| \sum_{n>N} \lambda_n x_n \right\| \geq \delta.$$

We will inductively define sequences $(\lambda^{(m)})_{m=1}^{\infty}$ of points in $\mathbb{D}^{\mathbb{N}}$ and $N_1 < M_1 < N_2 < M_2 < \cdots$ of natural numbers such that for all m,

$$\left\| \sum_{n=N_m+1}^{M_m} \lambda_n^{(m)} x_n \right\| \geq \delta/2.$$

Indeed, let $m = 1$, and let N_1 and $\lambda^{(1)}$ be such that

$$\left\| \sum_{n>N_1} \lambda_n^{(1)} x_n \right\| \geq \delta.$$

As $\sum_{n>N_1} \lambda_n^{(1)} x_n$ is norm convergent, there exists $M_1 > N_1$ such that

$$\left\| \sum_{n>M_1} \lambda_n^{(1)} x_n \right\| \leq \delta/2$$

(such exists by our convergence assumption). Now, having chosen $N_1 < M_1 < N_2 < \cdots < M_m$, let us choose $N_{m+1} > M_m$ and $(\lambda)^{(m+1)}$ so that

$$\left\| \sum_{n>N_{m+1}} \lambda_n^{(m+1)} x_n \right\| \geq \delta,$$

and choose $M_{m+1} > N_{m+1}$ such that

$$\left\| \sum_{n > M_{m+1}} \lambda_n^{(m+1)} x_n \right\| \leq \delta/2.$$

Then the constructed sequences have the desired properties.

Now, define a new sequence $\lambda \in \mathbb{D}^{\mathbb{N}}$ by the formula

$$\lambda_n := \begin{cases} \lambda_n^{(m)}, & N_m < n \leq M_m \\ 0, & \text{otherwise.} \end{cases}$$

Then $\sum_{n=1}^{\infty} \lambda_n x_n$ converges in norm. In particular, it is Cauchy. This implies that for all suitably large m, $\| \sum_{n=N_m+1}^{M_m} \lambda_n x_n \| < \delta/2$, which contradicts the properties of our construction. $\square$

Lemma 4.6. *Say (a_i) is a symmetrically summable collection. Then for any $\epsilon, r > 0$ the set $U_{\epsilon,r}$ of Definition 4.3 is closed.*

Proof. Assume for contradiction that for some $\epsilon, r > 0$, $U_{\epsilon,r}$ is not closed. Then there exists some $\lambda \in \overline{U_{\epsilon,r}} \setminus U_{\epsilon,r}$. As $\lambda \notin U_{\epsilon,r}$, we have that a_λ cannot be ϵ-r-approximated. Using (the contrapositive of) Lemma 4.4, part (i), there exists a finite rank projection $p \in \ell^\infty(X)$ such that $a_\lambda p$ cannot be ϵ-r-approximated.

Now, for any $\eta \in \mathbb{D}^I$, the sum $\sum_{i \in I} \eta_i a_i$ defining a_η is strongly convergent. As p is finite rank, this implies that the sum $\sum_{i \in I} \eta_i a_i p$ is norm convergent. Hence using Lemma 4.5, for any $\delta > 0$ there exists a finite subset F of I such that

$$\left\| \sum_{i \in I \setminus F} \eta_i a_i p \right\| < \delta \tag{6}$$

for all $\eta \in \mathbb{D}^I$ (and in particular for $\eta = \lambda$).

As F is finite, the set

$$\left\{ \eta \in \mathbb{D}^I \;\middle|\; |F| \max_{i \in F} \|a_i\| |\eta_i - \lambda_i| < \delta \text{ for all } i \in F \right\} \tag{7}$$

is an open neighbourhood of λ for the product topology. As λ is in the closure of $U_{\epsilon,r}$, the set in line (7) thus contains some $\theta \in U_{\epsilon,r}$. Hence in particular $a_\theta p$ can be ϵ-r-approximated, so there is $b \in \mathbb{C}_u^r[X]$ be such that $\|a_\theta p - b\| \leq \epsilon$. Note that

$$\begin{aligned}\|a_\lambda p - b\| &\leq \|a_\theta p - b\| + \|a_\lambda p - a_\theta p\| \\ &\leq \|a_\theta p - b\| + \left\| \sum_{i\in F} (\lambda_i - \theta_i) a_i p \right\| + \left\| \sum_{i \in I \setminus F} \theta_i a_i p \right\| + \left\| \sum_{i\in I\setminus F} \lambda_i a_i p \right\|.\end{aligned}$$

The first term on the bottom line is bounded above by ϵ by choice of b, the second is bounded above by δ using that θ is in the set in line (7), and the third and fourth terms are bounded above by δ using the estimate in line (6) (which is valid for all elements η of $\mathbb{D}^I$).

Now, we have shown that for arbitrary $\delta > 0$, we have found $b \in \mathbb{C}_u^r[X]$ such that $\|a_\lambda p - b\| \leq \epsilon + 3\delta$. Using Lemma 4.4, part (ii), this implies that $a_\lambda p$ can be ϵ-r-approximated. This contradicts our assumption in the first paragraph, so we are done. □

Now we turn to showing that if the conclusion of Theorem 4.2 is false, then for suitably small $\epsilon > 0$, all the sets $U_{\epsilon,r}$ of Definition 4.3 are nowhere dense in $\mathbb{D}^I$. We need another two preliminary lemmas.

Lemma 4.7. *If K is a norm-compact subset of $C_u^*(X)$ then for any $\epsilon > 0$ there exists $r > 0$ such that all operators in K can be ϵ-r-approximated.*

Proof. We choose a finite subset $\{a_1, ..., a_n\} \subseteq K$ such that every point of K is within $\epsilon/2$ of an element of $\{a_1, ..., a_n\}$. As each a_i is in $C_u^*(X)$, it can be $\epsilon/2$-r_i-approximated for some r_i. Is then straightforward to see that $r = \max\{r_1, ..., r_n\}$ has the desired property. □

Lemma 4.8. *Let (a_i) be a symmetrically summable collection that does not satisfy the conclusion of Proposition 4.2. Then there is an $\epsilon > 0$ so that for all $r > 0$ and all finite subsets F of I there exists $(\lambda_i) \in \mathbb{D}^I$ such that $\sum_{i\in I\setminus F} \lambda_i a_i$ cannot be ϵ-r approximated.*

Proof. Let (a_i) be as in the statement. Then there exists $\delta > 0$ such that for all $r > 0$ there exists $\lambda \in \mathbb{D}^I$ such that a is not δ-r-approximable. Assume for contradiction that the conclusion of the lemma fails. Then there exists $s > 0$ and a finite subset F of I such that for all $(\lambda_i) \in \mathbb{D}^I$ we have that $\sum_{i\in I\setminus F} \lambda_i a_i$ is $\delta/2$-s-approximated. As F is finite, the set

$$K := \left\{ \sum_{i\in F} \lambda_i a_i \;\middle|\; \lambda \in \mathbb{D}^I \right\}$$

is norm-compact. Hence Lemma 4.7 gives $t > 0$ such that every element of K can be $\delta/2$-t-approximated. Now, for arbitrary $\lambda \in \mathbb{D}^I$,

$$a_\lambda = \sum_{i \in F} \lambda_i a_i + \sum_{i \in I \setminus F} \lambda_i a_i;$$

as the first term above can be $\delta/2$-s-approximated, and as the second can be $\delta/2$-t-approximated, this implies that a_λ can be δ-$\max\{s,t\}$-approximated. As λ was arbitrary, this contradicts the first sentence in the proof, and we are done. $\square$

As already noted after the statement of Proposition 4.2, the following lemma completes the proof of the proposition.

Lemma 4.9. *Say (a_i) is a symmetrically summable collection that does not satisfy the conclusion of Proposition 4.2. Then there is $\epsilon > 0$ such that for each $r > 0$ the set $U_{\epsilon,r}$ of Definition 4.3 is nowhere dense in $\mathbb{D}^I$.*

Proof. Let $\epsilon' > 0$ have the property from Lemma 4.8. We claim that $\epsilon := \epsilon'/2$ has the property required for this lemma. Assume for contradiction that for some $r > 0$, $U_{\epsilon,r}$ is not nowhere dense. Lemma 4.6 implies that $U_{\epsilon,r}$ is closed, and so it contains a point λ in its interior. Then by definition of the product topology there exists a finite set $F \subseteq I$ and $\delta > 0$ such that the set

$$V := \{\eta \in \mathbb{D}^I \mid |\eta_i - \lambda_i| < \delta \text{ for all } i \in F\} \tag{8}$$

is contained in $U_{\epsilon,r}$.

Note that the element $\sum_{i \in F} \lambda_i a_i$ is in $C^*_u(X)$ by assumption, so can be ϵ-s-approximated for some s; let $b_\lambda \in \mathbb{C}^s_u[X]$ be such that $\|\sum_{i \in F} \lambda_i a_i - b_\lambda\| \leq \epsilon$. On the other hand, Lemma 4.8 gives us $\eta \in \mathbb{D}^I$ so that $\sum_{i \in I \setminus F} \eta_i a_i$ cannot be ϵ'-$\max\{r,s\}$-approximated. We may further assume that $\eta_i = 0$ for $i \in F$. Define $\theta \in \mathbb{D}^I$ by

$$\theta_i := \begin{cases} \lambda_i & i \in F \\ \eta_i & i \notin F \end{cases}$$

Then θ is clearly in the set V of line (8), and so a_θ is ϵ-r-approximated. Let then $b_\theta \in \mathbb{C}^r_u[X]$ be such that $\|a_\theta - b_\theta\| \leq \epsilon$. We then see that

$$\|a_\eta - (b_\theta - b_\lambda)\| \leq \|a_\eta - a_\theta + b_\lambda\| + \|a_\theta - b_\theta\| \leq \left\| b_\lambda - \sum_{i \in F} \lambda_i a_i \right\| + \|a_\theta - b_\theta\|$$

The terms on the right are each less than ϵ by choice of b_λ and b_θ, and so $\|a_\eta - (b_\theta - b_\lambda)\| \leq 2\epsilon = \epsilon'$. As $b_\lambda + b_\theta$ has propagation at most $\max\{r, s\}$, this contradicts the assumption that a_η cannot be ϵ'-$\max\{r, s\}$-approximated, so we are done. □

5 Proof of the main result

In this section, we prove Theorem 1.1.

Proof of Theorem 1.1. Let $\delta : C_u^*(X) \to C_u^*(X)$ be a derivation. Theorem 2.5 implies that δ is spatially implemented, so there is $d \in \mathscr{B}(\ell^2(X))$ such that $\delta(a) = [a, d]$ for all $a \in C_u^*(X)$. We will show that d is in $C_u^*(X)$.

Let $\mathcal{U}$ be the unitary group of $\ell^\infty(X)$, equipped with the discrete topology. As $\mathcal{U}$ is abelian, it is amenable (see for example [2, Theorem G.2.1]), and so we may fix a right-invariant mean on $\ell^\infty(\mathcal{U})$. As in Lemma 3.1 above, this allows us to build a right-invariant, contractive, linear map

$$\ell^\infty(\mathcal{U}, \mathscr{B}(\ell^2(X))) \to \mathscr{B}(\ell^2(X)), \quad a \mapsto \int_{\mathcal{U}} a(u)\, \mathrm{d}\, \mu(u). \tag{9}$$

We apply this to the bounded function

$$\mathcal{U} \to \mathscr{B}(\ell^2(X)), \quad u \mapsto u^* d u$$

to get a bounded operator

$$d' := \int_{\mathcal{U}} u^* d u \, \mathrm{d}\, \mu(u) \in \mathscr{B}(\ell^2(X)).$$

Using Lemma 3.4 applied to the identity representation of $\mathcal{U}$, d' is in the commutant of $\mathcal{U}$. As $\mathcal{U}$ spans $\ell^\infty(X)$, and as $\ell^\infty(X)$ is maximal abelian in $\mathscr{B}(\ell^2(X))$, this implies that d' is in $\ell^\infty(X)$. To show that d is in $C_u^*(X)$, it therefore suffices to show that $h := d - d'$ is in $C_u^*(X)$.

Continuing, let $p_x \in \mathscr{B}(\ell^2(X))$ be the rank one projection onto the span of the Dirac mass at x. For an element f of the unit ball of $\ell^\infty(X)$ (considered as a multiplication operator on $\ell^2(X)$), write f as a strongly convergent sum

$$f = \sum_{x \in X} f(x) p_x.$$

Then using strong continuity of subtraction, and separate strong continuity of multiplication on bounded sets,

$$[f,d] = \Big[\sum_{x\in X} f(x)p_x, d\Big] = \sum_{x\in X} f(x)[p_x,d].$$

On the other hand, by the assumption that δ is a derivation on $C^*_u(X)$, $[f,d]$ is in $C^*_u(X)$ for all $f \in \ell^\infty(X)$. It follows that if we set $I = X$, and if for each $x \in X$ we set $a_x := [p_x, d]$, then the collection $(a_x)_{x\in X}$ satisfies the assumptions of Proposition 4.2. Hence, for every $\epsilon > 0$ there exists $r > 0$ such that for every f in the unit ball of $\ell^\infty(X)$, the operator $[f,d]$ can be ϵ-r-approximated. In particular, using that any $u \in \mathcal{U}$ has propagation zero and norm one, for any $\epsilon > 0$ there exists $r > 0$ such that $d - u^*du = u^*[u,d]$ can be ϵ-r-approximated.

For each $u \in \mathcal{U}$, we can therefore choose $a(u)$ of propagation at most r such that $b(u) := d - u^*du - a(u)$ has norm at most ϵ. Note that the functions $a : u \mapsto a(u)$ and $b : u \mapsto b(u)$ are in $\ell^\infty(\mathcal{U}, \mathscr{B}(\ell^2(X)))$. Hence we may consider their images under the map in line (9). Using that the map in line (9) is linear and acts as the identity on constant functions (see Lemma 3.1), we see that

$$\begin{aligned}\int_{\mathcal{U}} a(u)\,\mathrm{d}\,\mu(u) + \int_{\mathcal{U}} b(u)\,\mathrm{d}\,\mu(u) = \int_{\mathcal{U}} d - u^*du\,\mathrm{d}\,\mu(u) = d - \int_{\mathcal{U}} u^*du\,\mathrm{d}\,\mu(u) \\ = d - d' = h. \qquad (10)\end{aligned}$$

On the other hand, $\int_{\mathcal{U}} a(u)\,\mathrm{d}\,\mu(u)$ has propagation at most r by Lemma 3.3, and $\int_{\mathcal{U}} b(u)\,\mathrm{d}\,\mu(u)$ has norm at most ϵ as the map in line (9) is contractive (see Lemma 3.1). In particular, line (10) writes h as a sum of an element of $C^*_u(X)$, and an element of norm at most ϵ. As ϵ was arbitrary, h is in $C^*_u(X)$, and we are done. □

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