LAGRANGIAN COBORDISMS AND LEGENDRIAN INVARIANTS IN KNOT FLOER HOMOLOGY

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Abstract. We prove that the LOSS and GRID invariants of Legendrian links in knot Floer homology behave in certain functorial ways with respect to decomposable Lagrangian cobordisms in the symplectization of the standard contact structure on $\mathbb{R}^3$. Our results give new, computable, and effective obstructions to the existence of such cobordisms.

1. Introduction

Let ξ_{std} be the standard contact structure on $\mathbb{R}^3$, given by the kernel of the 1-form

$$\alpha_{\text{std}} = dz - y\,dx.$$

A difficult problem in contact and symplectic geometry, which has attracted a great deal of attention in recent years, is to decide, given two Legendrian links

$$\Lambda_-, \Lambda_+ \subset (\mathbb{R}^3, \xi_{\text{std}}),$$

whether there exists an exact Lagrangian cobordism from Λ_- to Λ_+ in the symplectization

$$(\mathbb{R}_t \times \mathbb{R}^3, d(e^t \alpha_{\text{std}})).$$

In the smooth category, any two links are cobordant in $\mathbb{R}\times\mathbb{R}^3$, and the challenge is to determine the minimum genus among such cobordisms. The opposite is true in the Lagrangian setting, where the existence of an exact Lagrangian cobordism is constrained but its genus is completely determined by the classical Thurston–Bennequin and rotation numbers of the Legendrian links at the ends. Indeed, Chantraine showed in [Cha10] that if L is an exact Lagrangian cobordism from Λ_- to Λ_+, then

$$tb(\Lambda_+) - tb(\Lambda_-) = -\chi(L) \text{ and } r(\Lambda_+) = r(\Lambda_-). \tag{1}$$

An important goal, therefore, is to develop obstructions to the existence of exact Lagrangian cobordisms that are *effective*, meaning that they can obstruct such cobordisms where smooth topology and the classical invariants do not.

In this article, we restrict our attention to *decomposable* Lagrangian cobordisms, which are those that can be obtained as compositions of elementary cobordisms associated to Legendrian isotopies, pinches, and births, as shown in Figure 2. Decomposable cobordisms are exact, and constitute most known examples of exact Lagrangian cobordisms (see the discussion at the

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end of Section 2.1). It is open whether all connected exact Lagrangian cobordisms between non-empty Legendrian links are decomposable.[1]

Our main result, described in Sections 1.1-1.3, is that knot Floer homology provides effective obstructions to decomposable Lagrangian cobordisms. Symplectic Field Theory also furnishes various obstructions to exact Lagrangian cobordisms; see [EHK16, CNS16, Pan17, CDGG15, ST13]. One advantage of our knot Floer obstructions is that they are generally much easier to compute than those coming from SFT. Moreover, we show that knot Floer homology obstructs decomposable cobordisms in cases where the SFT invariants do not (and vice versa).

As discussed in Section 1.4, there is an existing body of work [BS18a, BS18b, GJ19] showing that knot Floer homology effectively obstructs Lagrangian cobordisms of genus-zero in various settings. Ours is the first result that shows that knot Floer homology can effectively obstruct Lagrangian cobordisms of *positive* genus.

1.1. **Obstructions.** In [OSzT08], Ozsváth, Szabó, and Thurston used the combinatorial grid diagram formulation of knot Floer homology [MOS09] to define invariants of Legendrian links in $(\mathbb{R}^3, \xi_{\text{std}})$. These so-called GRID invariants assign to such a Legendrian link Λ two elements in the *hat* flavor of the knot Floer homology of $\Lambda \subset -S^3$,[2]

$$\widehat{\lambda}^+(\Lambda), \widehat{\lambda}^-(\Lambda) \in \widehat{\mathrm{HFK}}(-S^3, \Lambda),$$ [3]

which depend only on the Legendrian isotopy class of Λ. These elements are *effective* invariants in that they can distinguish Legendrian links that are not isotopic but have the same classical invariants (see [NOT08], for example), and are combinatorially computable.

Remark 1.1. The Maslov (or Alexander) gradings of the classes $\widehat{\lambda}^\pm(\Lambda)$ recover the Thurston–Bennequin and rotation numbers of Λ (see Section 2.2).

We prove that the GRID invariants are well-behaved under decomposable Lagrangian cobordisms. As explained in Section 1.3, this provides effective obstructions to such cobordisms.

Theorem 1.2. *Suppose Λ_-, Λ_+ are Legendrian links in $(\mathbb{R}^3, \xi_{\text{std}})$ such that either*

- $\widehat{\lambda}^+(\Lambda_+) = 0$ *and* $\widehat{\lambda}^+(\Lambda_-) \neq 0$*, or*
- $\widehat{\lambda}^-(\Lambda_+) = 0$ *and* $\widehat{\lambda}^-(\Lambda_-) \neq 0$*.*

Then there is no decomposable Lagrangian cobordism from Λ_- to Λ_+.

Theorem 1.2 has the following corollaries.

Corollary 1.3. *Suppose Λ is a Legendrian link in $(\mathbb{R}^3, \xi_{\text{std}})$ such that either*

$$\widehat{\lambda}^+(\Lambda) = 0 \text{ or } \widehat{\lambda}^-(\Lambda) = 0.$$

Then there is no decomposable Lagrangian filling of Λ.

[1] Lin gives examples of Lagrangian caps in [Lin16], and these cannot be decomposable.

[2] The GRID invariant is defined for knots in S^3, but a Legendrian knot in $(\mathbb{R}^3, \xi_{\text{std}})$ can be viewed naturally as a Legendrian in the standard contact structure on S^3. We follow the conventions of [OSzT08] and view these invariants as living in $\widehat{\mathrm{HFK}}(S^3, m(\Lambda))$, which we identify with $\widehat{\mathrm{HFK}}(-S^3, \Lambda)$.

[3] There are also versions of the GRID invariants in the more general *minus* flavor.

As explained in Section 3.3, this follows from the fact that the GRID invariants are nonzero for the $tb = -1$ Legendrian unknot.

Corollary 1.4. *Suppose Λ_-, Λ_+ are Legendrian links in $(\mathbb{R}^3, \xi_{\text{std}})$ such that either*

- *$\widehat{\lambda}^+(\Lambda_-) \neq 0$ and Λ_+ is the positive stabilization of a Legendrian link, or*
- *$\widehat{\lambda}^-(\Lambda_-) \neq 0$ and Λ_+ is the negative stabilization of a Legendrian link.*

Then there is no decomposable Lagrangian cobordism from Λ_- to Λ_+.

This follows immediately from the fact (see Proposition 2.2) that the elements $\widehat{\lambda}^+$ and $\widehat{\lambda}^-$ vanish for positively and negatively stabilized Legendrian links, respectively.

In [LOSSz09], Lisca, Ozsváth, Stipsicz, and Szabó used open book decompositions to define a knot Floer invariant of Legendrian *knots* in any closed contact 3-manifold. For a Legendrian knot $\Lambda \subset (\mathbb{R}^3, \xi_{\text{std}})$, their so-called LOSS invariant also takes the form of an element

$$\widehat{\mathfrak{L}}(\Lambda) \in \widehat{\mathrm{HFK}}(-S^3, \Lambda).$$

Although the LOSS invariant is not algorithmically computable, Baldwin, Vela-Vick, and Vértesi proved in [BVV13] that it agrees with the GRID invariants, for Legendrian knots in $(\mathbb{R}^3, \xi_{\text{std}})$. More precisely, given such a knot Λ, there are isomorphisms

$$\phi_\pm : \widehat{\mathrm{HFK}}(-S^3, \Lambda) \to \widehat{\mathrm{HFK}}(-S^3, \pm\Lambda)$$

such that

$$\phi_\pm(\widehat{\lambda}^\pm(\Lambda)) = \widehat{\mathfrak{L}}(\pm\Lambda).$$

This gives corresponding versions of Theorem 1.2 and its corollaries for the LOSS invariant.

1.2. **Proof.** Theorem 1.2 follows from a similar result for the *tilde* version of the GRID invariants. We explain this below after providing a bit of additional background on the construction of the GRID invariants (see Section 2.2 for details).

A Legendrian link $\Lambda \subset (\mathbb{R}^3, \xi_{\text{std}})$ can be represented by a grid diagram $\mathbb{G}$. This grid diagram determines a combinatorially computable, bigraded chain complex whose *grid homology* agrees with the knot Floer homology of $\Lambda \subset -S^3$,

$$\widehat{\mathrm{GH}}(\mathbb{G}) \cong \widehat{\mathrm{HFK}}(-S^3, \Lambda).$$

There are two canonical cycles in this grid chain complex, representing elements

$$\widehat{\lambda}^+(\mathbb{G}), \widehat{\lambda}^-(\mathbb{G}) \in \widehat{\mathrm{GH}}(\mathbb{G}).$$

The *hat* version of the GRID invariants discussed previously are defined by

$$\widehat{\lambda}^\pm(\Lambda) := \widehat{\lambda}^\pm(\mathbb{G}).$$

A specialization of this chain complex gives rise to the *tilde* version of grid homology, which agrees with the *tilde* flavor of knot Floer homology, and is related to the *hat* flavor by

$$\widetilde{\mathrm{GH}}(\mathbb{G}) \cong \widehat{\mathrm{HFK}}(-S^3, \Lambda) \otimes V^{\otimes |\mathbb{G}| - |\Lambda|},$$

where

$$V = \mathbb{F}_{0,0} \oplus \mathbb{F}_{-1,-1}$$

is the two-dimensional vector space supported in the Maslov–Alexander bigradings indicated by the subscripts; $|\mathbb{G}|$ is the grid number of $\mathbb{G}$; and $|\Lambda|$ is the number of components of Λ. There are two canonical elements in this version of grid homology as well,

$$\widetilde{\lambda}^+(\mathbb{G}), \widetilde{\lambda}^-(\mathbb{G}) \in \widetilde{\mathrm{GH}}(\mathbb{G}),$$

which we refer to as the *tilde* version of the GRID invariants. Moreover, there is an injection

$$\widehat{\mathrm{GH}}(\mathbb{G}) \hookrightarrow \widetilde{\mathrm{GH}}(\mathbb{G})$$

that sends $\widehat{\lambda}^\pm(\mathbb{G})$ to $\widetilde{\lambda}^\pm(\mathbb{G})$ [NOT08]. In particular,

$$\widehat{\lambda}^\pm(\Lambda) = \widehat{\lambda}^\pm(\mathbb{G}) = 0 \text{ iff } \widetilde{\lambda}^\pm(\mathbb{G}) = 0.$$

Theorem 1.2 therefore follows immediately from our main technical result below, which states that the *tilde* versions of the GRID invariants satisfy a weak functoriality under decomposable Lagrangian cobordisms.

Theorem 1.5. *Suppose Λ_-, Λ_+ are Legendrian links in $(\mathbb{R}^3, \xi_{\mathrm{std}})$ with grid representatives $\mathbb{G}_-, \mathbb{G}_+$, respectively. Suppose there exists a decomposable Lagrangian cobordism L from Λ_- to Λ_+. Then there is a homomorphism*

$$\Phi_L : \widetilde{\mathrm{GH}}(\mathbb{G}_+) \to \widetilde{\mathrm{GH}}(\mathbb{G}_-)$$[4]

such that

$$\Phi_L(\widetilde{\lambda}^\pm(\mathbb{G}_+)) = \widetilde{\lambda}^\pm(\mathbb{G}_-).$$

This map has Maslov–Alexander bidegree

$$(\chi(L), \frac{1}{2}(\chi(L) + |\Lambda_-| - |\Lambda_+|)),$$

where $|\Lambda_\pm|$ is the number of components of $\Lambda_\pm$.

Recall that a decomposable cobordism L as in the theorem can be described as a composition of elementary cobordisms associated with Legendrian isotopies, pinches, and births. To prove Theorem 1.5, we define combinatorially computable maps on the *tilde* version of grid homology for each of these elementary cobordisms (the maps corresponding to Legendrian isotopies were defined in [OSzT08]), and show that these elementary maps preserve the *tilde* GRID invariant. We then define Φ_L to be the appropriate composition of these elementary maps.

In [Juh16, Zem19], Juhász and Zemke independently proved that decorated link cobordisms between pointed links induce well-defined maps on knot Floer homology. (They defined these maps differently, but showed in [JZ19] that their definitions agree for the *tilde* flavor of HFK.) A grid diagram naturally specifies a pointed link, and the sequence of grid moves corresponding to a decomposition of a Lagrangian cobordism L from Λ_- to Λ_+ into elementary pieces specifies a decorated cobordism between pointed copies of $\Lambda_\pm$. We believe that the map Φ_L agrees with the *functorial* map of Juhász–Zemke associated to this decorated cobordism, but do not prove this here.

[4]The notation Φ_L is a slight abuse of notation as we do not prove that this map depends only on L.

1.3. **Effectiveness.** In Section 4, we give several examples that show that Theorem 1.2 can be used to obstruct decomposable Lagrangian cobordisms where the classical invariants and smooth topology do not. In particular, we prove the following in Section 4.3:

Theorem 1.6. *For each $g \in \mathbb{Z}_{\geq 0}$, there are Legendrian knots $\Lambda_-, \Lambda_+ \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ such that*

- *there is a smooth cobordism of genus g in $\mathbb{R} \times \mathbb{R}^3$ between Λ_- and Λ_+,*
- *$tb(\Lambda_+) - tb(\Lambda_-) = 2g$ and $r(\Lambda_+) = r(\Lambda_-)$,*
- *$\widehat{\lambda}^+(\Lambda_+) = 0$ and $\widehat{\lambda}^+(\Lambda_-) \neq 0$.*

The last item implies that there is no decomposable Lagrangian cobordism from Λ_- to Λ_+.

As alluded to above, Symplectic Field Theory [EGH00] also provides effective obstructions to Lagrangian cobordisms. The most-studied such SFT obstruction comes from the Chekanov–Eliashberg DGA [Che02, Eli98], which assigns to a Legendrian knot $\Lambda \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ a differential graded algebra $(\mathcal{A}_\Lambda, \partial_\Lambda)$, which is an invariant of the Legendrian isotopy class of Λ, up to stable tame isomorphism. This DGA is said to be *trivial* if it is stable tame isomorphic to a DGA in which the unit is a boundary. Ekholm, Honda, and Kálmán proved in [EHK16] that an exact Lagrangian cobordism from Λ_- to Λ_+ induces a DGA morphism

$$(\mathcal{A}_{\Lambda_+}, \partial_{\Lambda_+}) \to (\mathcal{A}_{\Lambda_-}, \partial_{\Lambda_-}).$$

Therefore, if the first DGA is trivial and the second is nontrivial then there cannot exist such a cobordism. It can be difficult to determine whether these DGAs are trivial, meaning that this obstruction can be hard to apply in practice. By contrast, there is a simple algorithm to decide whether the GRID invariants vanish and apply Theorem 1.2.

Another advantage of the GRID invariants is that the elements $\widehat{\lambda}^+$ and $\widehat{\lambda}^-$ are preserved by negative and positive Legendrian stabilization, respectively. This implies, for example, that for any pair Λ_-, Λ_+ of Legendrian knots as in Theorem 1.6, the GRID invariants also obstruct the existence of a decomposable Lagrangian cobordism from any negative stabilization of Λ_- to any negative stabilization of Λ_+. By contrast, the Chekanov–Eliashberg DGA is trivial for stabilized knots, and therefore cannot obstruct such cobordisms.

We should point out that there are also examples for which the DGA obstructs decomposable Lagrangian cobordisms where the GRID invariants do not (see Section 4.4).

1.4. **Antecedents.** As mentioned above, there are a few prior works that use knot Floer homology to obstruct genus zero Lagrangian cobordisms; such cobordisms are called *Lagrangian concordances*, and are automatically exact.

In [BS18a], for instance, Baldwin and Sivek defined an invariant of Legendrian knots in arbitrary closed contact 3-manifolds using monopole knot homology, and showed that it satisfies functoriality with respect to Lagrangian concordances in symplectizations of such manifolds. They then proved in [BS18b] that there is an isomorphism between monopole knot homology and knot Floer homology that identifies their Legendrian invariant with the LOSS invariant. This implies that the LOSS invariant is well-behaved with respect to Lagrangian concordances, and, in particular, reproduces Theorem 1.2 for concordances between knots in the symplectization of $(\mathbb{R}^3, \xi_{\mathrm{std}})$, without the assumption of decomposability.

The other notable result in this area is due to Golla and Juhász, who proved in [GJ19] that the LOSS invariant satisfies functoriality with respect to *regular* Lagrangian concordances in Weinstein cobordisms between closed contact 3-manifolds (which include symplectizations). More precisely, they showed that the functorial map (of Juhász–Zemke)

$$\widehat{\mathrm{HFK}}(-Y_+, \Lambda_+) \to \widehat{\mathrm{HFK}}(-Y_-, \Lambda_-)$$

associated to a decorated regular Lagrangian concordance L in a Weinstein cobordism W,

$$(W, L) : (Y_-, \Lambda_-) \to (Y_+, \Lambda_+),$$

sends $\widehat{\mathfrak{L}}(\Lambda_+)$ to $\widehat{\mathfrak{L}}(\Lambda_-)$, for decorations consisting of two parallel arcs that partition the cylinder into disks. We note that regular cobordisms are exact and, in the symplectization of $(\mathbb{R}^3, \xi_{\mathrm{std}})$, include decomposable cobordisms [CET19]; in brief,

$$\{\text{decomposable}\} \subseteq \{\text{regular}\} \subseteq \{\text{exact}\},$$

and it is open whether any of these inclusions are proper. The Golla–Juhász result therefore recovers Theorem 1.2 for concordances between knots in the symplectization of $(\mathbb{R}^3, \xi_{\mathrm{std}})$, with the *potentially* weaker assumption of regularity.

As noted previously, what most differentiates the results in this paper from those in previous works is that ours apply to positive genus Lagrangian cobordisms as well as to concordances. The table below summarizes the different settings in which the various knot Floer obstructions to Lagrangian cobordism are known to hold.

	[BS18a, BS18b]	[GJ19]	Present paper
For L in symplectization of $(\mathbb{R}^3, \xi_{\mathrm{std}})$	✓	✓	✓
For L in any symplectization	✓	✓	✗
For L in any Weinstein cobordism	✗	✓	✗
For any decomposable L	✓	✓	✓
For any regular L	✓	✓	✗
For any exact L	✓	✗	✗
For $g(L) = 0$	✓	✓	✓
For $g(L) > 0$	✗	✗	✓

TABLE 1. The settings in which various knot Floer obstructions to Lagrangian cobordisms have been established.

1.5. **Organization.** Section 2 consists of background material. In Section 3, we prove Theorem 1.5, which, as described above, implies Theorem 1.2. In Section 4, we show via examples that our obstructions are effective, proving Theorem 1.6.

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2. Background

In this section, we provide some background on Legendrian knots, Lagrangian cobordisms, knot Floer homology, and the GRID invariants.

2.1. **Legendrian knots and Lagrangian cobordisms.** Let $\xi_{\mathrm{std}} = \ker(\alpha_{\mathrm{std}})$ be the standard contact structure on $\mathbb{R}^3$, where

$$\alpha_{\mathrm{std}} = dz - y dx,$$

as in the introduction. Recall that a smooth link $\Lambda \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ is called *Legendrian* if

$$T_p\Lambda \subset (\xi_{\mathrm{std}})_p \text{ for all } p \in \Lambda.$$

We will primarily study Legendrian links up to Legendrian isotopy (and will frequently blur the distinction between Legendrian links and Legendrian link types). Furthermore, our Legendrian links will generally be oriented but we will often suppress the orientation from the notation.

We will typically represent a Legendrian link by its *front diagram*, which is its projection to the xz-plane, as illustrated in Figure 1. Note that a Legendrian link is completely determined by its front diagram; in particular, the crossing information is encoded in the slopes of the strands in the diagram (strands with more negative slope pass over strands with less negative slope). Front diagrams for Legendrian isotopic links are related by a sequence of Legendrian planar isotopies and Legendrian Reidemeister moves, shown in the first three diagrams of Figure 2.

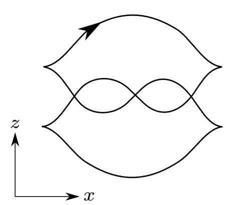


Figure 1. A front diagram for a Legendrian representative of the right-handed trefoil with $(tb, r) = (1, 0)$. The positive y-axis points into the page.

There are two *classical* Legendrian isotopy class invariants: the Thurston–Bennequin number tb and rotation number r. These can be computed from a front diagram by

$$tb = \mathrm{wr} - \frac{1}{2}(c_+ + c_-) \text{ and } r = \frac{1}{2}(c_- - c_+),$$

where wr denotes the writhe of the diagram, and c_+ and c_- denote the number of upward and downward pointing cusps in the oriented diagram. Two important operations on Legendrian

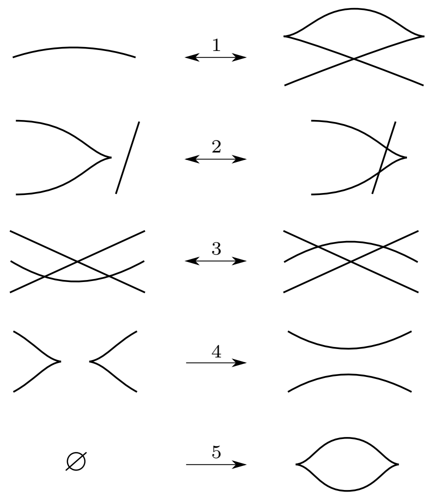


FIGURE 2. The moves on front diagrams corresponding to elementary cobordisms; horizontal and vertical reflections of these moves are also allowed. Apart from these moves, there are also planar isotopies that preserve left and right cusps and do not introduce any vertical tangencies. Moves 1-3 are the Legendrian Reidemeister moves; move 4 is called a pinch; move 5 is a birth. Note that moves 4 and 5 are directed.

isotopy classes are positive and negative *Legendrian stabilization*. These operations are defined locally in terms of front diagrams as in Figure 3. In particular, given a Legendrian link L, its positive and negative stabilizations $S_+(L)$ and $S_-(L)$ are obtained by adding downward and upward pointing cusps, respectively, as shown in the figure. Note that

$$tb(S_\pm(L)) = tb(L) - 1 \text{ and } r(S_\pm(L)) = r(L) \pm 1. \tag{2}$$

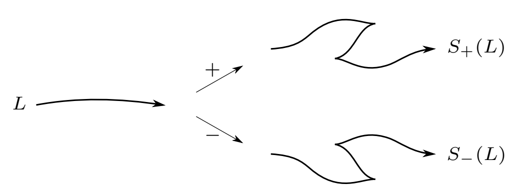


FIGURE 3. The positive and negative Legendrian stabilizations of a Legendrian link.

Recall that the *symplectization* of $(\mathbb{R}^3, \xi_{\mathrm{std}})$ is the symplectic 4-manifold

$$(\mathbb{R}_t \times \mathbb{R}^3, d(e^t \alpha_{\mathrm{std}})),$$

and that an embedded surface L in the symplectization is called *Lagrangian* if

$$d(e^t \alpha_{\mathrm{std}})|_L \equiv 0.$$

Suppose Λ_-, Λ_+ are two oriented Legendrian links in $(\mathbb{R}^3, \xi_{\text{std}})$. A *Lagrangian cobordism from* Λ_- *to* Λ_+ is an oriented, embedded Lagrangian surface L in the symplectization such that

$$L \cap ([-S, S] \times \mathbb{R}^3)$$

is compact for any $S > 0$, and

$$L \cap ((-\infty, -T) \times \mathbb{R}^3) = (-\infty, -T) \times \Lambda_-,$$
$$L \cap ((T, \infty) \times \mathbb{R}^3) = (T, \infty) \times \Lambda_+$$

for some $T > 0$. This Lagrangian is said to be *exact* if there exists a function $f : L \to \mathbb{R}$ that is constant on the cylindrical ends and satisfies

$$(e^t \alpha_{\text{std}})|_L = df.$$

A Lagrangian cobordism of genus zero is called a *Lagrangian concordance*, and is automatically exact. To see this, note that the curve $\{T\} \times \Lambda_+$ generates $H_1(L)$ for some $T > 0$. To prove $(e^t \alpha_{\text{std}})|_L$ is exact, it then suffices to show that $e^t \alpha_{\text{std}}$ is trivial when restricted to this curve, but that is immediate from the fact that Λ_+ is Legendrian with respect to α_{std}. As mentioned in the introduction, Chaintraine proved in [Cha10] that the existence of a Lagrangian cobordism L from Λ_- to Λ_+ implies that

$$tb(\Lambda_+) - tb(\Lambda_-) = -\chi(L) \text{ and } r(\Lambda_+) = r(\Lambda_-).\tag{3}$$ [5]

In particular, Lagrangian cobordisms (even concordances [Cha15]) are directed.

By work of Bourgeois, Sabloff, and Traynor [BST15], Chantraine [Cha10], Dimitroglou Rizell [Dim16], and Ekholm, Honda, and Kálmán [EHK16], there exists an elementary exact Lagrangian cobordism from Λ_- to Λ_+ whenever Λ_+ is obtained from Λ_- via Legendrian isotopy, a pinch, or a birth, as illustrated in Figure 2. Note that for a pinch, it is Λ_- that in fact looks as if it has been obtained from pinching Λ_+. Topologically, these elementary cobordisms are annuli, saddles, and cups, respectively. Any composition of elementary cobordisms yields an exact Lagrangian cobordism, and an exact Lagrangian cobordism is called *decomposable* if it is isotopic through exact Lagrangians to such a composition [Cha12]. As mentioned in the introduction, it is open whether every exact Lagrangian cobordism is decomposable.

2.2. **Knot Floer homology and the GRID invariants.** We begin by reviewing the grid diagram formulation of knot Floer homology, following the conventions in [OSSz15]. See also [MOS09, MOSzT07].

A *grid diagram* $\mathbb{G}$ is an $n \times n$ grid of squares together with sets

$$\mathbb{O} = \{O_1, \dots, O_n\} \text{ and } \mathbb{X} = \{X_1, \dots, X_n\}$$

of markings in the squares such that each row and column of $\mathbb{G}$ contains exactly one O marking and one X marking (we omit the subscripts indexing these markings when convienient); n is called the *grid number* of $\mathbb{G}$. We will think of $\mathbb{G}$ as a torus by identifying its top and bottom sides and its left and right sides in the standard way, so that the horizontal grid lines become horizontal circles and the vertical grid lines become vertical circles, as indicated in Figure 4.

[5] He proved this for cobordisms between Legendrian *knots*, but the proof extends immediately to links.

A grid diagram specifies an oriented link in $\mathbb{R}^3$, obtained as the union of vertical segments from the Xs to the Os in each column with horizontal segments from the Os to the Xs in each row, such that vertical segments pass over horizontal ones, as shown in Figure 4. Conversely, every oriented link in $\mathbb{R}^3$ can be represented by a grid diagram in this way.

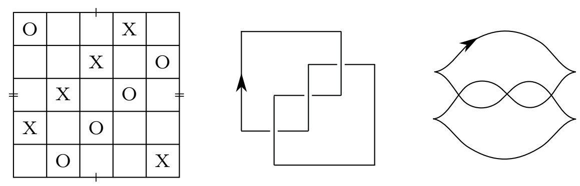


FIGURE 4. A grid diagram $\mathbb{G}$ for the right-handed trefoil L, and the corresponding front diagram for a Legendrian representative Λ of $m(L)$, obtained by changing all crossings in the link diagram and rotating 45 degrees clockwise.

Suppose $\mathbb{G}$ is a grid diagram as above, representing an oriented link L. (The use of L for links will only occur in this subsection, and hence should not cause confusion with Lagrangian cobordisms.) Let

$$\alpha = \{\alpha_1, \dots, \alpha_n\} \text{ and } \beta = \{\beta_1, \dots, \beta_n\}$$

denote the vertical and horizontal circles of $\mathbb{G}$, respectively. The *minus* flavor of the grid chain complex,

$$(\mathrm{GC}^-(\mathbb{G}), \partial^-),$$

is generated by one-to-one correspondences between the vertical and horizontal circles. Equivalently, a generator is a set of n intersection points between these circles where each intersection point in the set belongs to exactly one α circle and one β circle. Letting $\mathbf{S}(\mathbb{G})$ denote the set of generators, $\mathrm{GC}^-(\mathbb{G})$ is defined to be the free $\mathbb{F}[U_1, \dots, U_n]$-module generated by the elements of $\mathbf{S}(\mathbb{G})$, where each U_i is a formal variable corresponding to the marking O_i and $\mathbb{F}$ is the 2-element field.

Given $\mathbf{x}, \mathbf{y} \in \mathbf{S}(\mathbb{G})$, let $\mathrm{Rect}_{\mathbb{G}}(\mathbf{x}, \mathbf{y})$ be the set of rectangles in $\mathbb{G}$ with the following properties. $\mathrm{Rect}_{\mathbb{G}}(\mathbf{x}, \mathbf{y})$ is empty unless $\mathbf{x}$ and $\mathbf{y}$ coincide in exactly $n-2$ intersection points. An element $r \in \mathrm{Rect}_{\mathbb{G}}(\mathbf{x}, \mathbf{y})$ is an embedded rectangle in the toroidal grid whose edges are arcs contained in the vertical and horizontal circles, and whose four corners are points in $\mathbf{x} \cup \mathbf{y}$. Moreover, we require that, with respect to the induced orientation on ∂r, every vertical edge of r is directed from a point in $\mathbf{y}$ to a point in $\mathbf{x}$, and vice versa for horizontal edges; that is,

$$\partial(\partial_\alpha(r)) = \mathbf{x} - \mathbf{y} \text{ and } \partial(\partial_\beta(r)) = \mathbf{y} - \mathbf{x}.$$

(The astute reader may have noticed that this does not seem to line up with the usual convention in Lagrangian Floer homology, but we are in fact computing Heegaard Floer homology for $(-\mathbb{T}^2, \alpha, \beta)$.) If $\mathrm{Rect}_{\mathbb{G}}(\mathbf{x}, \mathbf{y})$ is non-empty then it contains exactly two rectangles, as illustrated in Figure 5. Let $\mathrm{Rect}^{o}{}_{\mathbb{G}}(\mathbf{x}, \mathbf{y})$ denote the subset consisting of $r \in \mathrm{Rect}_{\mathbb{G}}(\mathbf{x}, \mathbf{y})$ with

$$r \cap \mathbb{X} = \mathrm{Int}(r) \cap \mathbf{x} = \emptyset.$$

The differential

$$\partial^-_{\mathbb{G}} : \mathrm{GC}^-(\mathbb{G}) \to \mathrm{GC}^-(\mathbb{G})$$

is the $\mathbb{F}[U_1, \dots, U_n]$-module endomorphism defined on $\mathbf{S}(\mathbb{G})$ by

$$\partial^-_{\mathbb{G}}(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbf{S}(\mathbb{G})} \sum_{r \in \mathrm{Rect}^o{}_{\mathbb{G}}(\mathbf{x},\mathbf{y})} U_1^{O_1(r)} \cdots U_n^{O_n(r)} \cdot \mathbf{y},$$

where $O_i(r)$ denotes the number of times the marking O_i appears in r.

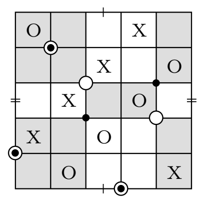


FIGURE 5. A grid diagram $\mathbb{G}$. The generator $\mathbf{x}$ comprises the black intersection points while $\mathbf{y}$ comprises the white intersection points. $\mathrm{Rect}_{\mathbb{G}}(\mathbf{x},\mathbf{y})$ contains the two shaded rectangles shown on this torus, while $\mathrm{Rect}^o{}_{\mathbb{G}}(\mathbf{x},\mathbf{y})$ contains only the smaller of the two.

This complex is equipped with two gradings, the *Maslov* grading and the *Alexander*, defined as follows. Consider the partial ordering on points in $\mathbb{R}^2$ given by

$$(p_1, p_2) < (q_1, q_2)$$

if $p_1 < q_1$ and $p_2 < q_2$. Given two sets P and Q consisting of finitely many points in $\mathbb{R}^2$, let

$$\mathcal{I}(P,Q) = \#\{(p,q) \in P \times Q \mid p < q\}.$$

We symmetrize this quantity by defining

$$\mathcal{J}(P,Q) = \frac{\mathcal{I}(P,Q) + \mathcal{I}(Q,P)}{2}.$$

A generator $\mathbf{x} \in \mathbf{S}(\mathbb{G})$ can be viewed as a finite set of points in $\mathbb{R}^2$, as can the marking sets $\mathbb{X}$ and $\mathbb{O}$. It therefore makes sense to define

$$M_{\mathbb{O}}(\mathbf{x}) = \mathcal{J}(\mathbf{x},\mathbf{x}) - 2\mathcal{J}(\mathbf{x},\mathbb{O}) + \mathcal{J}(\mathbb{O},\mathbb{O}) + 1, \tag{4}$$

$$M_{\mathbb{X}}(\mathbf{x}) = \mathcal{J}(\mathbf{x},\mathbf{x}) - 2\mathcal{J}(\mathbf{x},\mathbb{X}) + \mathcal{J}(\mathbb{X},\mathbb{X}) + 1. \tag{5}$$

The Maslov and Alexander gradings of a generator $\mathbf{x}$ are then given by

$$M(\mathbf{x}) = M_{\mathbb{O}}(\mathbf{x}),$$

$$A(\mathbf{x}) = \frac{1}{2}(M_{\mathbb{O}}(\mathbf{x}) - M_{\mathbb{X}}(\mathbf{x})) - \Big(\frac{n - |L|}{2}\Big),$$

where $|L|$ is the number of components of the link L. It follows that for $\mathbf{x}, \mathbf{y} \in \mathbf{S}(\mathbb{G})$ and $r \in \mathrm{Rect}_{\mathbb{G}}(\mathbf{x},\mathbf{y})$, the *relative* Maslov and Alexander gradings of these generators are given by

$$M(\mathbf{x}) - M(\mathbf{y}) = 1 - 2\#(r \cap \mathbb{O}) + 2\#(\mathrm{Int}(r) \cap \mathbf{x}), \tag{6}$$

$$A(\mathbf{x}) - A(\mathbf{y}) = \#(r \cap \mathbb{X}) - \#(r \cap \mathbb{O}). \tag{7}$$

These gradings are extended to gradings on the complex $\mathrm{GC}^-(\mathbb{G})$ by the rule that multiplication by any of the U_i lowers Maslov grading by 2 and Alexander grading by 1. Note that the differential $\partial_{\mathbb{G}}^-$ lowers the Maslov grading by 1 and preserves the Alexander grading.

The *grid homology* of the grid diagram $\mathbb{G}$ is the Maslov–Alexander bigraded $\mathbb{F}[U_1, \dots, U_n]$-module denoted by

$$\mathrm{GH}^-(\mathbb{G}) = H_*(\mathrm{GC}^-(\mathbb{G}), \partial_{\mathbb{G}}^-).$$

The grid diagram $\mathbb{G}$ is actually a multi-pointed Heegaard diagram for the link $L \subset S^3$, and the grid chain complex agrees with the *minus* version of the corresponding knot Floer chain complex. Therefore,

$$\mathrm{GH}^-(\mathbb{G}) \cong \mathrm{HFK}^-(S^3, L).$$

Suppose L has ℓ components. Label the markings so that $O_1, \dots, O_\ell$ belong to the ℓ different components of L. Setting the corresponding U_i equal to zero on the chain level results in a chain complex

$$(\widehat{\mathrm{GC}}(\mathbb{G}), \widehat{\partial}_{\mathbb{G}}) = (\mathrm{GC}^-(\mathbb{G})/(U_1 = \cdots = U_\ell = 0), \partial_{\mathbb{G}}^-)$$

whose homology agrees with the *hat* flavor of knot Floer homology,

$$\widehat{\mathrm{GH}}(\mathbb{G}) = H_*(\widehat{\mathrm{GC}}(\mathbb{G}), \widehat{\partial}_{\mathbb{G}}) \cong \widehat{\mathrm{HFK}}(S^3, L).$$

Setting *all* of the U_i to zero yields the *tilde* version of the grid complex,

$$(\widetilde{\mathrm{GC}}(\mathbb{G}), \widetilde{\partial}_{\mathbb{G}}) = (\mathrm{GC}^-(\mathbb{G})/(U_1 = \cdots = U_n = 0), \partial_{\mathbb{G}}^-),$$

whose homology agrees with the *tilde* version of knot Floer homology, and is related to the *hat* flavor by

$$\widetilde{\mathrm{GH}}(\mathbb{G}) = H_*(\widetilde{\mathrm{GC}}(\mathbb{G}), \widetilde{\partial}_{\mathbb{G}}) \cong \widehat{\mathrm{HFK}}(S^3, L) \otimes V^{\otimes n-\ell}, \tag{8}$$

where

$$V = \mathbb{F}_{0,0} \oplus \mathbb{F}_{-1,-1}$$

is the two-dimensional vector space supported in Maslov–Alexander bigradings $(0, 0)$ and $(-1, -1)$. Finally, the quotient map

$$j : \widehat{\mathrm{GC}}(\mathbb{G}) \to \widetilde{\mathrm{GC}}(\mathbb{G})$$

induces an injection

$$j_* : \widehat{\mathrm{GH}}(\mathbb{G}) \to \widetilde{\mathrm{GH}}(\mathbb{G})$$

on homology [NOT08].

Suppose $\mathbb{G}$ is a grid diagram representing L. By changing all crossings in the associated link diagram, rotating 45 degrees clockwise, smoothing the top and bottom pointing corners, and turning the left and right pointing corners into cusps, we obtain a front diagram for a Legendrian representative Λ of $m(L)$, as indicated in Figure 4. We say that a grid diagram $\mathbb{G}$ *represents* a Legendrian link Λ if the front diagram obtained from $\mathbb{G}$ in the manner above is isotopic to the front diagram for Λ. Every Legendrian link in $(\mathbb{R}^3, \xi_{\mathrm{std}})$ can be represented by a grid diagram in this way.

Suppose $\mathbb{G}$ represents the smooth link L and a Legendrian representative Λ of $m(L)$ as above. As shown in [OSzT08], there are two canonical cycles

$$\mathbf{x}^+(\mathbb{G}), \mathbf{x}^-(\mathbb{G}) \in \mathrm{GC}^-(\mathbb{G})$$

consisting of the intersection points to the immediate upper right and lower left, respectively, of the markings in $\mathbb{X}$. These two generators give rise to cycles in the *hat* and *tilde* complexes as well, which we denote in the same way. The Maslov and Alexander gradings of these cycles are given by

$$M(\mathbf{x}^\pm(\mathbb{G})) = tb(\Lambda) \mp r(\Lambda) + 1, \tag{9}$$

$$A(\mathbf{x}^\pm(\mathbb{G})) = \frac{1}{2}(tb(\Lambda) \mp r(\Lambda) + |\Lambda|), \tag{10}$$

where $|\Lambda|$ is the number of components of Λ. In particular, the gradings of the two generators recover $tb(\Lambda)$ and $r(\Lambda)$. The *hat* and *tilde* versions of the GRID invariants of the Legendrian link Λ are then defined [OSzT08] by

$$\widehat{\lambda}^\pm(\Lambda) = \widehat{\lambda}^\pm(\mathbb{G}) := [\mathbf{x}^\pm(\mathbb{G})] \in \widehat{\mathrm{GH}}(\mathbb{G}) \cong \widehat{\mathrm{HFK}}(S^3, L) \cong \widehat{\mathrm{HFK}}(-S^3, \Lambda)$$

and

$$\widetilde{\lambda}^\pm(\mathbb{G}) := [\mathbf{x}^\pm(\mathbb{G})] \in \widetilde{\mathrm{GH}}(\mathbb{G}).$$

In particular,

$$\widetilde{\lambda}^\pm(\mathbb{G}) = j_*(\widehat{\lambda}^\pm(\mathbb{G})),$$

which implies that

$$\widehat{\lambda}^\pm(\Lambda) = \widehat{\lambda}^\pm(\mathbb{G}) = 0 \text{ iff } \widetilde{\lambda}^\pm(\mathbb{G}) = 0 \tag{11}$$

since j_* is injective.[6]

Ozsváth, Szabó, and Thurston proved that $\widehat{\lambda}^\pm(\mathbb{G})$ are invariants of the Legendrian isotopy class of Λ. Specifically, if $\mathbb{G}_0$ and $\mathbb{G}_1$ are grid diagrams representing Legendrian isotopic links then there is an isomorphism [OSzT08, Theorem 1.1]

$$\widehat{\mathrm{GH}}(\mathbb{G}_0) \to \widehat{\mathrm{GH}}(\mathbb{G}_1)$$

of Maslov–Alexander bidegree $(0,0)$ that sends $\widehat{\lambda}^\pm(\mathbb{G}_0)$ to $\widehat{\lambda}^\pm(\mathbb{G}_1)$. This map is defined combinatorially, in terms of chain maps on the grid complex associated to grid diagram versions of the Legendrian Reidemeister moves. Their argument also gives rise to the following statement for the *tilde* flavor of the GRID invariants.

Proposition 2.1. *If $\mathbb{G}_0$ and $\mathbb{G}_1$ are grid diagrams representing Legendrian isotopic links then there is a homomorphism*

$$\widetilde{\mathrm{GH}}(\mathbb{G}_0) \to \widetilde{\mathrm{GH}}(\mathbb{G}_1)$$

that sends $\widetilde{\lambda}^\pm(\mathbb{G}_0)$ to $\widetilde{\lambda}^\pm(\mathbb{G}_1)$. This map has Maslov–Alexander bidegree $(0,0)$.

Note that the homomorphism above may not be an isomorphism. The GRID invariants also behave as follows under stabilization [OSzT08, Theorem 1.3].

[6] Unlike for the *hat* flavor of the GRID invariants, we do not denote $\widetilde{\lambda}^\pm(\mathbb{G})$ by $\widetilde{\lambda}^\pm(\Lambda)$ since these classes (and the group $\widetilde{\mathrm{GH}}(\mathbb{G})$) depend not only on the Legendrian link Λ but also on the grid number n, as in (8).

Proposition 2.2. *Suppose $\mathbb{G}$ is a grid representative of a Legendrian link Λ, and that $\mathbb{G}_\pm$ are grid representatives of the positive and negative Legendrian stabilizations $S_\pm(\Lambda)$, respectively. Then*

$$\widehat{\lambda}^+(\mathbb{G}_+) = \widehat{\lambda}^-(\mathbb{G}_-) = 0,$$

and

$$\widehat{\lambda}^+(\mathbb{G}_-) = 0 \text{ iff } \widehat{\lambda}^+(\mathbb{G}) = 0 \text{ and } \widehat{\lambda}^-(\mathbb{G}_+) = 0 \text{ iff } \widehat{\lambda}^-(\mathbb{G}) = 0.$$

The analogous statement holds for the tilde *invariants, by* (11).

3. Proofs of main results

To define the map Φ_L in Theorem 1.5 associated to a decomposable Lagrangian cobordism L, we first define maps associated to Legendrian isotopies, pinches, and births, as discussed in the introduction. The maps associated to Legendrian isotopies were defined previously by Ozsváth, Szabó, and Thurston in [OSzT08], and are described in Proposition 2.1, so we will restrict our attention below to the maps associated to pinches and births.[7]

3.1. Pinches.

Proposition 3.1. *Suppose Λ_+ is obtained from Λ_- via a pinch move. For any grid diagrams $\mathbb{G}_+$ and $\mathbb{G}_-$ representing Λ_+ and Λ_-, respectively, there is a homomorphism*

$$\Phi : \widetilde{\mathrm{GH}}(\mathbb{G}_+) \to \widetilde{\mathrm{GH}}(\mathbb{G}_-)$$

that sends $\widetilde{\lambda}^\pm(\mathbb{G}_+)$ to $\widetilde{\lambda}^\pm(\mathbb{G}_-)$. This map has Maslov–Alexander bidegree

$$\begin{cases} (-1,0), & \text{if } |\Lambda_-| = |\Lambda_+| + 1, \\ (-1,-1), & \text{if } |\Lambda_-| = |\Lambda_+| - 1, \end{cases}$$

where $|\Lambda_\pm|$ is the number of components of $\Lambda_\pm$.

Proof. By Proposition 2.1, it suffices to show that there exist *some* grid diagrams $\mathbb{G}_+$ and $\mathbb{G}_-$ representing links Legendrian isotopic to Λ_+ and Λ_-, respectively, for which the conclusions of Proposition 3.1 hold. For this, note that there are grid diagrams $\mathbb{G}_\pm$ representing $\Lambda_\pm$ that are identical except for the positions of two markings in adjacent rows, as shown in Figure 6. Since L is an *oriented* cobordism, these two special markings must either both be Xs, which we refer to as Case I, or both be Os, which we refer to as Case II.

We may combine the grid diagrams $\mathbb{G}_-$ and $\mathbb{G}_+$ into a single toroidal diagram, as shown in Figure 7, which we will refer to as the *combined diagram*. From this perspective, the markings in $\mathbb{X}, \mathbb{O}$ are fixed and $\mathbb{G}_-$ and $\mathbb{G}_+$ differ in a single horizontal circle. We denote these differing horizontal circles by β and γ, as shown in Figure 7. Let a and b be the intersection points of β with γ shown in the figure. Below, we define the map Φ for each of Cases I and II.

[7] They also defined pinch maps, but did not study them in the Legendrian/Lagrangian setting—only in the smooth setting. They did not define the birth maps.

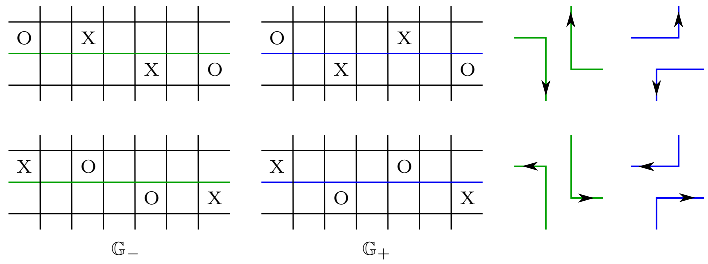


FIGURE 6. The grid diagrams for $\mathbb{G}_\pm$ corresponding to a pinch move; Case I on the top, Case II on the bottom.

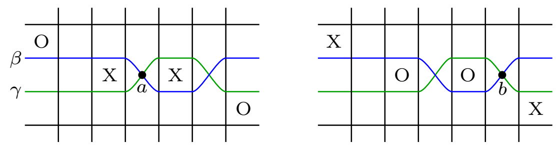


FIGURE 7. The grid diagrams $\mathbb{G}_\pm$ combined; Case I on the left, Case II on the right.

3.1.1. *Case I.* For $\mathbf{x} \in \mathbf{S}(\mathbb{G}_+)$ and $\mathbf{y} \in \mathbf{S}(\mathbb{G}_-)$, let $\mathrm{Pent}(\mathbf{x}, \mathbf{y})$ be the space of pentagons in the combined diagram with the following properties:

- $\mathrm{Pent}(\mathbf{x}, \mathbf{y})$ is empty unless $\mathbf{x}$ and $\mathbf{y}$ coincide in exactly $n-2$ intersection points, where n is the grid number of $\mathbb{G}_\pm$.
- An element $p \in \mathrm{Pent}(\mathbf{x}, \mathbf{y})$ is an embedded pentagon in the toroidal diagram whose edges are arcs contained in the vertical and horizontal circles, and whose five corners are points in $\mathbf{x} \cup \mathbf{y} \cup \{a\}$.
- We require that, with respect to the induced orientation on ∂p, the boundary of this pentagon may be traversed as follows: start at the point in $\mathbf{x}$ on β and proceed along an arc of β until arriving at a; next, proceed along an arc of γ until arriving at a point in $\mathbf{y}$; next, follow an arc of a vertical circle until arriving at a point in $\mathbf{x}$; next, proceed along an arc of a horizontal circle until arriving at a point in $\mathbf{y}$; finally, follow an arc of a vertical circle back to the initial point in $\mathbf{x}$.

See Figure 8 for such pentagons. Let $\mathrm{Pent}^{\mathrm{o}}(\mathbf{x}, \mathbf{y})$ be the subset consisting of $p \in \mathrm{Pent}(\mathbf{x}, \mathbf{y})$ with

$$p \cap \mathbb{O} = p \cap \mathbb{X} = \mathrm{Int}(p) \cap \mathbf{x} = \emptyset.$$

Let

$$\phi : \widetilde{\mathrm{GC}}(\mathbb{G}_+) \to \widetilde{\mathrm{GC}}(\mathbb{G}_-)$$

be the linear map defined on generators by counting such pentagons,

$$\phi(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbf{S}(\mathbb{G}_-)} \sum_{p \in \mathrm{Pent}^{\mathrm{o}}(\mathbf{x}, \mathbf{y})} \mathbf{y}.$$

Lemma 3.2. *ϕ is a chain map.*

Proof. To show that ϕ is a chain map, we must prove the equality of coefficients,

$$\langle(\widetilde{\partial}_{\mathbb{G}_-} \circ \phi)(\mathbf{x}), \mathbf{y}\rangle = \langle(\phi \circ \widetilde{\partial}_{\mathbb{G}_+})(\mathbf{x}), \mathbf{y}\rangle,$$

for every pair of generators $\mathbf{x} \in \mathbf{S}(\mathbb{G}_+)$ and $\mathbf{y} \in \mathbf{S}(\mathbb{G}_-)$. The coefficients on the left and right count concatenations of rectangles and pentagons from $\mathbf{x}$ to $\mathbf{y}$ of the forms $p * r$ and $r * p$, respectively, where p is a pentagon of the sort used to define ϕ, and r is a rectangle of the sort used to define the differentials. Every domain in the combined diagram that decomposes as the juxtaposition of a rectangle and pentagon in this way admits exactly one other such decomposition, exactly as in the proof of commutation invariance for grid homology [MOSzT07, Lemma 3.1]. In particular, the concatenations of pentagons and rectangles contributing to the coefficients above cancel in pairs, proving the lemma. □

Lemma 3.3. *ϕ sends $\mathbf{x}^{\pm}(\mathbb{G}_+)$ to $\mathbf{x}^{\pm}(\mathbb{G}_-)$.*

Proof. There is a unique pentagon contributing to each of $\phi(\mathbf{x}^+(\mathbb{G}_+))$ and $\phi(\mathbf{x}^-(\mathbb{G}_+))$, shown in Figure 8, that certifies that

$$\phi(\mathbf{x}^{\pm}(\mathbb{G}_+)) = \mathbf{x}^{\pm}(\mathbb{G}_-).$$

(All other potential pentagons are blocked by X markings, because all but the two components of $\mathbf{x}^{\pm}(\mathbb{G}_+)$ shown in the figure are to the upper right or lower left, respectively, of X markings; and the pentagons shown are clearly the unique pentagons with two corners at these two components of $\mathbf{x}^{\pm}(\mathbb{G}_+)$.)

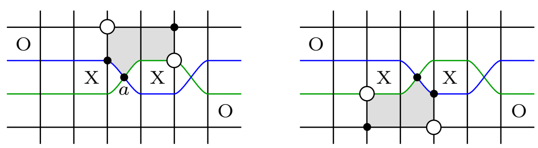


FIGURE 8. Left, the pentagon certifying that the map ϕ sends $\mathbf{x}^+(\mathbb{G}_+)$ in black to $\mathbf{x}^+(\mathbb{G}_-)$ in white. Right, the pentagon from $\mathbf{x}^-(\mathbb{G}_+)$ to $\mathbf{x}^-(\mathbb{G}_-)$.

□

Lemma 3.4. *ϕ is homogeneous of Maslov–Alexander bidegree*

$$\begin{cases} (-1,0), & \text{if } |\Lambda_-| = |\Lambda_+| + 1, \\ (-1,-1), & \text{if } |\Lambda_-| = |\Lambda_+| - 1. \end{cases}$$

Proof. This is a straightforward calculation from the definitions of the Maslov and Alexander gradings in (4) and (5), and the map ϕ; see the proofs of [OSSz15, Lemma 5.3.1] and [Won17, Lemma 6.6], for example. □

Remark 3.5. If one could show that ϕ is homogeneous (say, using the relative grading formulas in (6) and (7)), the bidegree of ϕ would be determined by the fact that this map sends $\mathbf{x}^+(\mathbb{G}_+)$ to $\mathbf{x}^+(\mathbb{G}_-)$, and the lemma would follow immediately from (9) and (10), together with the facts that

$$tb(\Lambda_-) = tb(\Lambda_+) - 1,$$
$$r(\Lambda_-) = r(\Lambda_+).$$

3.1.2. *Case II.* For $\mathbf{x} \in \mathbf{S}(\mathbb{G}_+)$ and $\mathbf{y} \in \mathbf{S}(\mathbb{G}_-)$, let $\mathrm{Tri}(\mathbf{x}, \mathbf{y})$ be the space of triangles in the combined diagram with the following properties:

- $\mathrm{Tri}(\mathbf{x}, \mathbf{y})$ is empty unless $\mathbf{x}$ and $\mathbf{y}$ coincide in exactly $n-1$ intersection points.
- An element $p \in \mathrm{Tri}(\mathbf{x}, \mathbf{y})$ is an embedded triangle in the torus whose edges are arcs contained in the vertical and horizontal circles, and whose three corners are points in $\mathbf{x} \cup \mathbf{y} \cup \{b\}$.
- We require that, with respect to the induced orientation on ∂p, the boundary of this triangle may be traversed as follows: start at the point in $\mathbf{x}$ on β and proceed along an arc of β until arriving at b; next, proceed along an arc of γ until arriving at a point in $\mathbf{y}$; finally, follow an arc of a vertical circle back to the initial point in $\mathbf{x}$.

See Figure 9 for such triangles. Note that all such triangles automatically satisfy

$$p \cap \mathbb{X} = \mathrm{Int}(p) \cap \mathbf{x} = \emptyset.$$

Let $\mathrm{Tri}^{\circ}(\mathbf{x}, \mathbf{y})$ be the subset consisting of $p \in \mathrm{Tri}(\mathbf{x}, \mathbf{y})$ with $p \cap \mathbb{O} = \emptyset$. Let

$$\phi : \widetilde{\mathrm{GC}}(\mathbb{G}_+) \to \widetilde{\mathrm{GC}}(\mathbb{G}_-)$$

be the linear map defined on generators by counting such triangles,

$$\phi(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbf{S}(\mathbb{G}_-)} \sum_{p \in \mathrm{Tri}^{\circ}(\mathbf{x}, \mathbf{y})} \mathbf{y}.$$

Lemma 3.6. *ϕ is a chain map.*

Proof. This follows from an argument identical to that in the proof of Lemma 3.2, except that here we consider canceling concatenations of rectangles with triangles rather than pentagons. See the proof of [Won17, Lemma 3.4] for details in this case. □

Lemma 3.7. *ϕ sends $\mathbf{x}^{\pm}(\mathbb{G}_+)$ to $\mathbf{x}^{\pm}(\mathbb{G}_-)$.*

Proof. There is a unique triangle contributing to each of $\phi(\mathbf{x}^+(\mathbb{G}_+))$ and $\phi(\mathbf{x}^-(\mathbb{G}_+))$, shown in Figure 9, that certifies that

$$\phi(\mathbf{x}^{\pm}(\mathbb{G}_+)) = \mathbf{x}^{\pm}(\mathbb{G}_-).$$

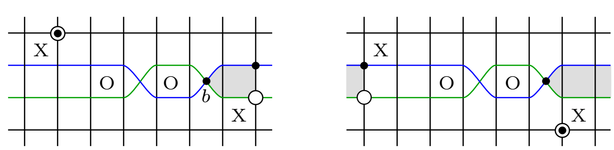


FIGURE 9. Left, the triangle certifying that the map ϕ sends $\mathbf{x}^+(\mathbb{G}_+)$ in black to $\mathbf{x}^+(\mathbb{G}_-)$ in white. Right, the triangle from $\mathbf{x}^-(\mathbb{G}_+)$ to $\mathbf{x}^-(\mathbb{G}_-)$.

□

Lemma 3.8. *ϕ is homogeneous of Maslov–Alexander bidegree*

$$\begin{cases} (-1, 0), & \text{if } |\Lambda_-| = |\Lambda_+| + 1, \\ (-1, -1), & \text{if } |\Lambda_-| = |\Lambda_+| - 1. \end{cases}$$

Proof. As with Lemma 3.4, this is a straightforward calculation from the definitions of these gradings in (4) and (5), and the map ϕ; see the proof of [Won17, Lemma 6.6] for details. □

The map Φ induced by ϕ therefore satisfies the conclusions of Proposition 3.1. □

3.2. **Births.**

Proposition 3.9. *Suppose Λ_+ is obtained from Λ_- via a birth move. For any grid diagrams $\mathbb{G}_+$ and $\mathbb{G}_-$ representing Λ_+ and Λ_-, respectively, there is a homomorphism*

$$\Phi : \widetilde{\mathrm{GH}}(\mathbb{G}_+) \to \widetilde{\mathrm{GH}}(\mathbb{G}_-)$$

that sends $\widetilde{\lambda}^{\pm}(\mathbb{G}_+)$ to $\widetilde{\lambda}^{\pm}(\mathbb{G}_-)$. This map has Maslov–Alexander bidegree $(1,0)$.

Proof. By Proposition 2.1, it suffices to show that there exist *some* grid diagrams $\mathbb{G}_+$ and $\mathbb{G}_-$ representing links Legendrian isotopic to Λ_+ and Λ_-, respectively, for which the conclusions of Proposition 3.9 hold. For this, let $\mathbb{G}_-$ be any grid diagram representing Λ_-, with marking sets $\mathbb{X}, \mathbb{O}$. Fix a marking $X_1 \in \mathbb{X}$. Let $\mathbb{G}_+$ be the grid diagram obtained from $\mathbb{G}_-$ by inserting two rows and two columns to the immediate bottom right of X_1, with four new markings X_2, X_3, O_2, O_3, as shown in Figure 10. Let a and b be the intersection points between the new vertical and horizontal circles indicated in the figure. Note that $\mathbb{G}_+$ represents the disjoint union of Λ_- with the $tb = -1$ Legendrian unknot, which is Legendrian isotopic to Λ_+.

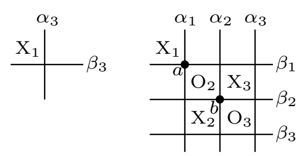


FIGURE 10. Left, part of a grid diagram $\mathbb{G}_-$ for Λ_-. Right, the corresponding part of the grid diagram $\mathbb{G}_+$ for Λ_+.

The generating set $\mathbf{S}(\mathbb{G}_+)$ can be expressed as a disjoint union,

$$\mathbf{S}(\mathbb{G}_+) = \mathrm{AB} \cup \mathrm{AN} \cup \mathrm{NB} \cup \mathrm{NN},$$

where

- AB consists of $\mathbf{x} \in \mathbf{S}(\mathbb{G}_+)$ with $a, b \in \mathbf{x}$,
- AN consists of $\mathbf{x} \in \mathbf{S}(\mathbb{G}_+)$ with $a \in \mathbf{x}$ but $b \notin \mathbf{x}$,
- NB consists of $\mathbf{x} \in \mathbf{S}(\mathbb{G}_+)$ with $a \notin \mathbf{x}$ but $b \in \mathbf{x}$,
- NN consists of $\mathbf{x} \in \mathbf{S}(\mathbb{G}_+)$ with $a \notin \mathbf{x}$ and $b \notin \mathbf{x}$.

This induces a decomposition of the vector space $\widetilde{\mathrm{GC}}(\mathbb{G}_+)$ as a direct sum,

$$\widetilde{\mathrm{GC}}(\mathbb{G}_+) = \widetilde{\mathrm{AB}} \oplus \widetilde{\mathrm{AN}} \oplus \widetilde{\mathrm{NB}} \oplus \widetilde{\mathrm{NN}},$$

where these summands are the vector spaces generated by the corresponding subsets of $\mathbf{S}(\mathbb{G}_+)$. Note that we have a sequence of subcomplexes,

$$\widetilde{\mathrm{NB}} \subset \widetilde{\mathrm{AB}} \oplus \widetilde{\mathrm{NB}} \subset \widetilde{\mathrm{GC}}(\mathbb{G}_+).$$

This follows immediately from the observation that any rectangle either starting at b or terminating at a must pass through one of the new markings or X_1 (and therefore does not contribute to the differential). Let

$$(\widetilde{\mathrm{AB}}, \widetilde{\partial}_{\mathrm{AB}})$$

be the quotient complex of $\widetilde{\mathrm{AB}} \oplus \widetilde{\mathrm{NB}}$ by $\widetilde{\mathrm{NB}}$. In other words, for $\mathbf{x}, \mathbf{y} \in \mathrm{AB}$, the coefficient

$$\langle \widetilde{\partial}_{\mathrm{AB}}(\mathbf{x}), \mathbf{y} \rangle = \langle \widetilde{\partial}_{\mathbb{G}_+}(\mathbf{x}), \mathbf{y} \rangle$$

counts the number of rectangles in $\mathrm{Rect}^{o}{}_{\mathbb{G}_+}(\mathbf{x}, \mathbf{y})$, as usual.

Note that there is a bijection between generators in AB and generators in $\mathbf{S}(\mathbb{G}_-)$, given by

$$\mathbf{x} \mapsto \mathbf{x} \smallsetminus \{a, b\}.$$

This bijection extends linearly to an isomorphism of chain complexes,

$$e : \widetilde{\mathrm{AB}} \to \widetilde{\mathrm{GC}}(\mathbb{G}_-),$$

since for $\mathbf{x}, \mathbf{y} \in \mathrm{AB}$ there is also a natural bijection

$$\mathrm{Rect}_{\mathbb{G}_+}(\mathbf{x}, \mathbf{y}) \to \mathrm{Rect}_{\mathbb{G}_-}(\mathbf{x} \smallsetminus \{a, b\}, \mathbf{y} \smallsetminus \{a, b\}),$$

which identifies rectangles avoiding the O and X markings in $\mathbb{G}_+$ with rectangles avoiding the O and X markings in $\mathbb{G}_-$. Moreover, it follows readily from (6) and (7), together with this bijection of rectangles, that e is homogeneous with respect to the Maslov–Alexander bigrading.

For $\mathbf{x} \in \mathrm{NB}$ and $\mathbf{y} \in \mathrm{AB}$, let

$$\mathrm{Rect}_{\mathrm{AB}}(\mathbf{x}, \mathbf{y}) \subset \mathrm{Rect}_{\mathbb{G}_+}(\mathbf{x}, \mathbf{y})$$

be the subset consisting of rectangles p satisfying

- $p \cap (\mathbb{O} \cup \{O_2, O_3\}) = \{O_2, O_3\}$,
- $p \cap (\mathbb{X} \cup \{X_2, X_3\}) = \{X_2, X_3\}$,
- $\mathrm{Int}(p) \cap \mathbf{x} = \mathrm{Int}(p) \cap \mathbf{y} = \{b\}$.

Let

$$\psi : \widetilde{\mathrm{NB}} \to \widetilde{\mathrm{AB}}$$

be the linear map defined on generators by counting such rectangles,

$$\psi(\mathbf{x}) = \sum_{\mathbf{y} \in \mathrm{AB}} \sum_{p \in \mathrm{Rect}_{\mathrm{AB}}(\mathbf{x}, \mathbf{y})} \mathbf{y}.$$

Let

$$\Pi : \widetilde{\mathrm{GC}}(\mathbb{G}_+) \to \widetilde{\mathrm{NB}}$$

be projection onto the summand $\widetilde{\mathrm{NB}}$, and define

$$\phi : \widetilde{\mathrm{GC}}(\mathbb{G}_+) \to \widetilde{\mathrm{GC}}(\mathbb{G}_-)$$

to be the linear map given as the composition

$$\phi = e \circ \psi \circ \Pi.$$

Lemma 3.10. *ϕ is a chain map.*

Proof. Since e is a chain map, it suffices to prove that $\psi \circ \Pi$ is a chain map. Note that both

$$\widetilde{\partial}_{\mathrm{AB}} \circ (\psi \circ \Pi) \text{ and } (\psi \circ \Pi) \circ \widetilde{\partial}_{\mathbb{G}_+}$$

vanish for generators $\mathbf{x} \notin \mathrm{AB} \cup \mathrm{NB}$. The first vanishes on such generators $\mathbf{x}$ because $\Pi(\mathbf{x}) = 0$. The second vanishes on such generators $\mathbf{x}$ because

$$(\Pi \circ \widetilde{\partial}_{\mathbb{G}_+})(\mathbf{x}) = 0.$$

Indeed, for every $\mathbf{y} \in \mathrm{NB}$, the coefficient

$$\langle \widetilde{\partial}_{\mathbb{G}_+}(\mathbf{x}), \mathbf{y} \rangle = 0$$

since every rectangle from a generator not containing b to a generator containing b must pass through a marking. To prove that ϕ is a chain map, it therefore suffices to prove the equality of coefficients

$$\langle (\widetilde{\partial}_{\mathrm{AB}} \circ \psi \circ \Pi)(\mathbf{x}), \mathbf{y} \rangle = \langle (\psi \circ \Pi \circ \widetilde{\partial}_{\mathbb{G}_+})(\mathbf{x}), \mathbf{y} \rangle \tag{12}$$

for every pair of generators $\mathbf{x} \in \mathrm{AB} \cup \mathrm{NB}$ and $\mathbf{y} \in \mathrm{AB}$. These coefficients on the left and right count concatenations of rectangles from $\mathbf{x}$ to $\mathbf{y}$ of the forms $p * r$ and $r * p$, respectively, where p is a rectangle of the sort used to define ψ, and r is a rectangle of the sort used to define the differentials. For $\mathbf{x} \in \mathrm{NB}$, every domain in $\mathbb{G}_+$ that decomposes as the juxtaposition of the form $p * r$ admits exactly one other decomposition into rectangles p and r, of the form $r * p$, exactly as in the proof that the grid differential squares to zero [MOSzT07, Proposition 2.10]. Figure 11 depicts all possible such domains where p and r share a corner; there are also other trivial canceling pairs where p and r do not share a corner, which we omit.

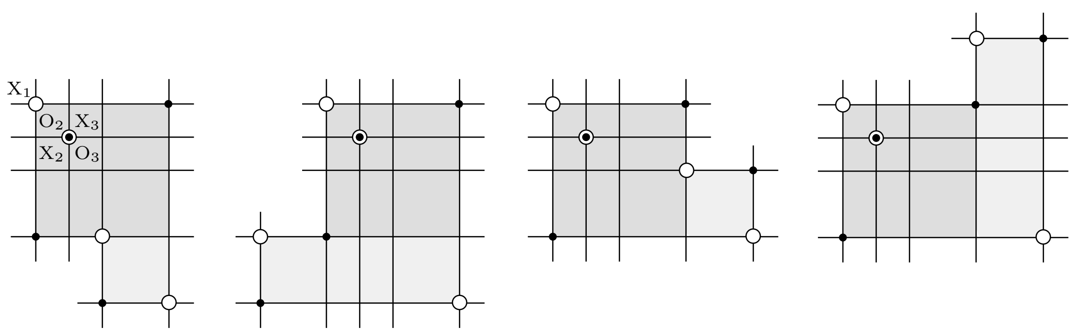


FIGURE 11. The domains corresponding to the juxtapositions of the form $p*r$, with p dark and r light, which contribute to the coefficient on the left of (12). Here, $\mathbf{x} \in \mathrm{NB}$ is shown in black and $\mathbf{y} \in \mathrm{AB}$ in white. Note that each of these domains admits one other decomposition of the form $r * p$, contributing to the coefficient on the right of (12).

There are additional domains that decompose as a juxtaposition of the form $r * p$ (and not as $p * r$), but these cancel in pairs as well. Figure 12 depicts the possible such domains where r and p share a corner; as above, there are also trivial canceling pairs where p and r do not share a corner, and we omit these.

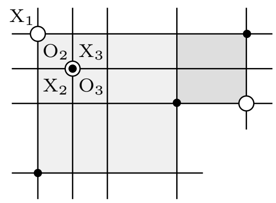


FIGURE 12. The additional domains corresponding to juxtapositions of the form $r * p$, with r dark and p light, which contribute to the coefficient on the right of (12), with $\mathbf{x} \in$ NB in black and $\mathbf{y} \in$ AB in white. Note that each of these domains admits one other cancelling decomposition of the form $r * p$.

In particular, the concatenations of rectangles contributing to the coefficients above cancel in pairs, proving the lemma in this case. For $\mathbf{x} \in$ AB, the first coefficient is zero since $\Pi(\mathbf{x}) = 0$, and there are exactly two concatenations of the form $r*p$ contributing to the second coefficient. These two cancelling concatenations correspond to vertical and horizontal annular domains of width 2, as shown in Figure 13. $\square$

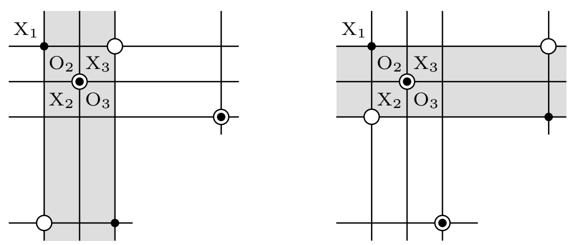


FIGURE 13. The vertical and horizontal annular domains corresponding to the two cancelling concatenations of the form $r * p$ for $\mathbf{x} \in$ AB shown in black and $\mathbf{y} \in$ AB shown in white.

Lemma 3.11. *ϕ sends $\mathbf{x}^{\pm}(\mathbb{G}_+)$ to $\mathbf{x}^{\pm}(\mathbb{G}_-)$.*

Proof. Note that $\mathbf{x}^{\pm}(\mathbb{G}_+) \in$ NB, so

$$\phi(\mathbf{x}^{\pm}(\mathbb{G}_+)) = e(\psi(\mathbf{x}^{\pm}(\mathbb{G}_+))).$$

There is a unique rectangle contributing to each of $\psi(\mathbf{x}^{+}(\mathbb{G}_+))$ and $\psi(\mathbf{x}^{-}(\mathbb{G}_+))$, as shown in Figure 14. It is then clear from the figure that

$$e(\psi(\mathbf{x}^{\pm}(\mathbb{G}_+))) = \mathbf{x}^{\pm}(\mathbb{G}_-).$$

$\square$

Lemma 3.12. *ϕ is homogeneous of Maslov–Alexander bidegree* $(1, 0)$.

Proof. It is clear from the definition of ψ, together with (6) and (7), that ψ is homogeneous. Since e and Π are also homogeneous, the same is true of ϕ. The bidegree of ϕ is then determined

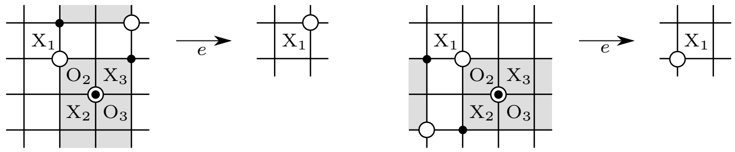


FIGURE 14. Left, $\mathbf{x}^+(\mathbb{G}_+)$ in black and its image under ψ in white. The latter is then sent to $\mathbf{x}^+(\mathbb{G}_-)$ by e as indicated by the arrow. Right, the corresponding pictures for $\mathbf{x}^-(\mathbb{G}_+)$ and its image under ϕ.

by the fact that this map sends $\mathbf{x}^+(\mathbb{G}_+)$ to $\mathbf{x}^+(\mathbb{G}_-)$, and the lemma follows immediately from (9) and (10), together with the facts that

$$\begin{aligned} tb(\Lambda_-) &= tb(\Lambda_+) + 1, \\ r(\Lambda_-) &= r(\Lambda_+), \\ |\Lambda_-| &= |\Lambda_+| - 1. \end{aligned}$$

□

The map Φ induced by ϕ therefore satisfies the conclusions of Proposition 3.9. □

3.3. **Putting it together.**

Proof of Theorem 1.5. Suppose L is a decomposable Lagrangian cobordism from Λ_- to Λ_+. Then there is a sequence

$$\Lambda_- = \Lambda_0, \dots, \Lambda_m = \Lambda_+$$

of Legendrian links such that for each i, Λ_{i+1} is obtained from Λ_i by either Legendrian isotopy, a pinch, or a birth. Let $\mathbb{G}_i$ be a grid diagram representing Λ_i. Then Proposition 2.1, 3.1, or 3.9 provides a map

$$\Phi_i : \widetilde{\mathrm{GH}}(\mathbb{G}_{i+1}) \to \widetilde{\mathrm{GH}}(\mathbb{G}_i)$$

that sends $\widetilde{\lambda}^{\pm}(\mathbb{G}_{i+1})$ to $\widetilde{\lambda}^{\pm}(\mathbb{G}_i)$, for each $i = 0, \dots, m-1$. Note that the bidegree of each Φ_i may be expressed simply as

$$(\chi(L_i), \frac{1}{2}(\chi(L_i) + |\Lambda_i| - |\Lambda_{i+1}|)),$$

where L_i is the corresponding elementary cobordism from Λ_i to Λ_{i+1}. We define the map

$$\Phi_L := \Phi_0 \circ \cdots \circ \Phi_{m-1} : \widetilde{\mathrm{GH}}(\Lambda_+) \to \widetilde{\mathrm{GH}}(\Lambda_-).$$

This map then sends $\widetilde{\lambda}^{\pm}(\mathbb{G}_+)$ to $\widetilde{\lambda}^{\pm}(\mathbb{G}_-)$ and has bidegree

$$(\chi(L), \frac{1}{2}(\chi(L) + |\Lambda_-| - |\Lambda_+|)),$$

as desired. □

Proof of Theorem 1.2. As explained in the introduction, this theorem follows from Theorem 1.5 and the fact that

$$\widehat{\lambda}^{\pm}(\mathbb{G}) = 0 \text{ iff } \widetilde{\lambda}^{\pm}(\mathbb{G}) = 0$$

for any grid diagram $\mathbb{G}$, as in (11). □

Proof of Corollary 1.3. A decomposable Lagrangian filling of Λ is a decomposable Lagrangian cobordism from the empty link to Λ, and can thus be described as a composition of elementary cobordisms starting with a birth. The rest of the filling is therefore a decomposable Lagrangian cobordism from the $tb = -1$ Legendrian unknot Λ_U to Λ. The corollary then follows from Theorem 1.2 combined with the fact that $\widehat{\lambda}^{\pm}(\Lambda_U) \neq 0$. □

Proof of Corollary 1.4. This follows immediately from Theorem 1.2 and the fact that $\widehat{\lambda}^{\pm}$ vanishes for positive and negative Legendrian stabilizations, respectively, as in Proposition 2.2. □

4. Examples

In this section, we illustrate the effectiveness of Theorem 1.2 via examples, proving Theorem 1.6 along the way.

4.1. **Examples of genus zero.** Let K denote either one of the (oriented) smooth knot types given by $m(10_{132})$ or $m(12n_{200})$. In [NOT08, Section 3], Ng, Ozsváth, and Thurston describe two Legendrian representatives Λ_0 and Λ_1 of K with

$$tb(\Lambda_0) = tb(\Lambda_1) = -1 \text{ and } r(\Lambda_0) = r(\Lambda_1) = 0, \tag{13}$$

but

$$\widehat{\lambda}^{+}(\Lambda_0) = 0 \text{ and } \widehat{\lambda}^{+}(\Lambda_1) \neq 0. \tag{14}$$

It follows that Λ_0 and Λ_1 are not Legendrian isotopic despite having the same classical invariants, by [OSzT08]. These are among the smallest crossing examples known that demonstrate the effectiveness of the GRID invariants in obstructing Legendrian isotopy. Ng, Ozsváth, and Thurston further observed, using an argument by Ng and Traynor from the proof of [NT04, Proposition 5.9], that these Legendrians are orientation reversals of one another,

$$\Lambda_0 = -\Lambda_1.$$

(Note that $m(10_{132})$ and $m(12n_{200})$ are reversible.) Thus, [OSzT08, Proposition 1.2] implies that

$$\widehat{\lambda}^{-}(\Lambda_0) \neq 0 \text{ and } \widehat{\lambda}^{-}(\Lambda_1) = 0 \tag{15}$$

as well. Combining (14) and (15) with Theorem 1.2, we obtain the following.

Proposition 4.1. *There is no decomposable Lagrangian concordance from Λ_0 to Λ_1 or from Λ_1 to Λ_0.*

Proof. The first is obstructed by $\widehat{\lambda}^{-}$, the second by $\widehat{\lambda}^{+}$. □

Note that the Thurston–Bennequin and rotation numbers do not obstruct the existence of decomposable Lagrangian concordances between Λ_0 and Λ_1 via (3).

Remark 4.2. One of the two directions in Proposition 4.1 is proven by Baldwin and Sivek [BS18b] and independently Golla and Juhász in [GJ19, Proposition 1.7]. In particular, they use the equivalence between $\widehat{\lambda}^+$ and $\widehat{\mathfrak{L}}$ to obstruct a decomposable Lagrangian concordance from Λ_1 to Λ_0.

4.2. **Examples of genus one.** Let K denote one of $m(10_{132})$ or $m(12n_{200})$, and let Λ_0 and Λ_1 be the Legendrian representatives of K discussed above. Front diagrams for these Legendrians are given in [NOT08, Figures 2 and 3]. By modifying these front diagrams for Λ_0 and Λ_1 first by a Legendrian Reidemeister I move, and then by adding a positive clasp, as in Figure 15, we obtain new Legendrian knots Λ_0' and Λ_1', respectively, whose front diagrams are shown in Figure 16. These two Legendrian knots belong to the smooth knot type

$$K' = \begin{cases} m(12n_{199}), & \text{if } K = m(10_{132}), \\ m(14n_{5047}), & \text{if } K = m(12n_{200}). \end{cases}$$

(The knot types were found using the program Knotscape by Hoste and Thistlethwaite [HT99].)

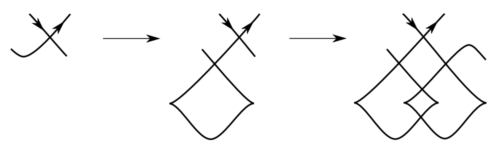


FIGURE 15. A local modification of the front diagram of Λ_i. The first move is a Legendrian Reidemeister I move; the second introduces a positive clasp.

Observe that there is a smooth cobordism of genus one between K and K' since the latter is obtained from the former via the addition of a positive clasp. Moreover, it is easy to see that the local modification in Figure 15 increases the Thurston–Bennequin number by 2 and preserves the rotation number. Combined with (13), this implies that for any $i, j \in \{0, 1\}$,

$$tb(\Lambda_i') = tb(\Lambda_j) + 2 \text{ and } r(\Lambda_i') = r(\Lambda_j).$$

In particular, the classical invariants and smooth topology do not obstruct the existence of a decomposable genus one Lagrangian cobordism from Λ_0 to Λ_1' or from Λ_1 to Λ_0'.

However, a direct computer calculation using the program [MQR+19],[8] applied to the diagrams in Figure 16, shows that

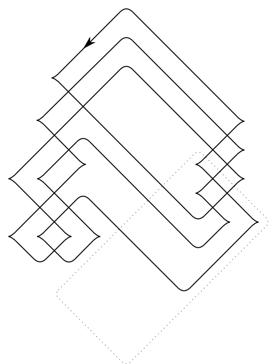
$$\widehat{\lambda}^+(\Lambda_0') = 0 \text{ and } \widehat{\lambda}^+(\Lambda_1') \neq 0.$$

The argument used by Ng, Ozsváth, and Thurston to show that $\Lambda_0 = -\Lambda_1$ shows that Λ_0' and Λ_1' are also orientation reversals of one another, which then implies that

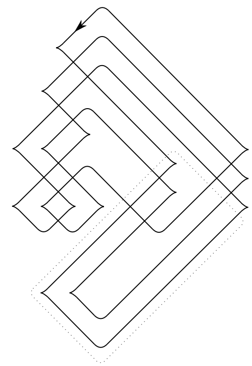
$$\widehat{\lambda}^-(\Lambda_0') \neq 0 \text{ and } \widehat{\lambda}^-(\Lambda_1') = 0.$$

These calculations, combined with Theorem 1.2, lead immediately to the following.

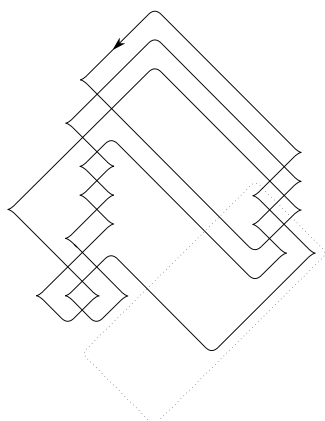
[8] This program is a newer version of the program written by Ng, Ozsváth, and Thurston [NOT07], with minor bug fixes and improvements in computational efficiency.



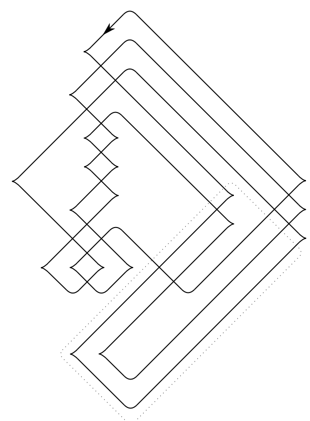
$X = \{13, 6, 11, 7, 1, 2, 3, 4, 10, 12, 8, 9, 5\}$
$O = \{8, 12, 3, 4, 6, 5, 1, 2, 13, 9, 11, 7, 10\}$



$X = \{13, 8, 11, 9, 3, 4, 5, 6, 10, 2, 7, 12, 1\}$
$O = \{10, 12, 5, 6, 8, 7, 3, 4, 1, 9, 13, 2, 11\}$



$X = \{15, 8, 13, 9, 6, 7, 1, 2, 3, 4, 12, 14, 10, 11, 5\}$
$O = \{10, 14, 3, 7, 8, 4, 6, 5, 1, 2, 15, 11, 13, 9, 12\}$



$X = \{15, 10, 13, 11, 8, 9, 3, 4, 5, 6, 12, 2, 7, 14, 1\}$
$O = \{12, 14, 5, 9, 10, 6, 8, 7, 3, 4, 1, 11, 15, 2, 13\}$

FIGURE 16. Top, two Legendrian representatives of $m(12n_{199})$. Bottom, two Legendrian representatives of $m(14n_{5047})$. In each case, Λ_0' is shown on the left and Λ_1' on the right. Note that Λ_0' and Λ_1' differ only in the dashed boxes. We have also included the XO coordinates of the corresponding grid diagrams.

Proposition 4.3. *There is no decomposable Lagrangian cobordism from Λ_0 to Λ_1' or from Λ_1 to Λ_0'.*

Proof. The first is obstructed by $\widehat{\lambda}^-$, the second by $\widehat{\lambda}^+$. □

4.3. **An infinite family.** Let K be one of the knot types $m(10_{145})$, $m(10_{161})$, or $12n_{591}$. In [CN13, Proposition 6], Chongchitmate and Ng provide two Legendrian representatives Λ_0 and Λ_1 of K (denoted by L_1 and L_3 there) satisfying

$$tb(\Lambda_0) = tb(\Lambda_1) + 2 \text{ and } r(\Lambda_0) = r(\Lambda_1) \tag{16}$$

and

$$\widehat{\lambda}^+(\Lambda_1) \neq 0 \text{ and } \widehat{\lambda}^-(\Lambda_1) \neq 0. \tag{17}$$

Fix any positive crossing in the front diagram for Λ_0. Let Λ_0' be the Legendrian knot representing the smooth knot type K' obtained from Λ_0 by first performing a Legendrian Reidemeister I move near this crossing, and then adding m positive clasps, as in Figure 17. It is easy to see that this local modification increases the Thurston–Bennequin number by $2m$ and preserves rotation number,

$$tb(\Lambda_0') = tb(\Lambda_0) + 2m \text{ and } r(\Lambda_0') = r(\Lambda_0). \tag{18}$$

This implies, by (2) combined with (16) and (18), that

$$tb(S_+(S_-(\Lambda_0'))) = tb(\Lambda_1) + 2m \text{ and } r(S_+(S_-(\Lambda_0'))) = r(\Lambda_1). \tag{19}$$

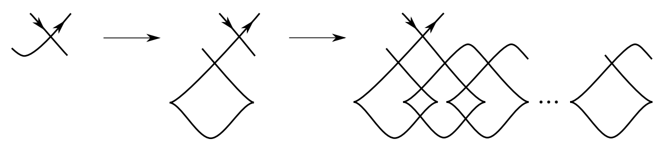


FIGURE 17. A modification of the front diagram of Λ_0 near a positive crossing. The first move is a Legendrian Reidemeister I move; the second introduces m positive clasps.

With this, we may now prove Theorem 1.6.

Proof of Theorem 1.6. Let us adopt the notation from above. Take $g = m$ and let

$$\Lambda_- = \Lambda_1 \text{ and } \Lambda_+ = S_+(S_-(\Lambda_0')).$$

The first is a Legendrian representative of K, the second of K', and there is a smooth genus g cobordism from K to K' since K' is obtained from K by adding g clasps, fulfilling the first bullet point of the theorem. The second bullet point is fulfilled by (19). Finally,

$$\widehat{\lambda}^+(\Lambda_+) = 0$$

by Proposition 2.2 since Λ_+ is a positive Legendrian stabilization, and

$$\widehat{\lambda}^+(\Lambda_-) \neq 0$$

by (17), fulfilling the third bullet point of the theorem. □

4.4. **DGA versus GRID.** We assume below that the reader is familiar with the Chekanov–Eliashberg DGA; for a survey, see [EN18].

As mentioned in the introduction, an exact Lagrangian cobordism from Λ_- to Λ_+ induces a DGA morphism [EHK16]

$$(\mathcal{A}_{\Lambda_+}, \partial_{\Lambda_+}) \to (\mathcal{A}_{\Lambda_-}, \partial_{\Lambda_-}),$$

so if the first DGA is trivial while the second is not then there cannot be such a cobordism. The DGA is trivial for stabilized Legendrians [Che02], so, as noted in Section 1.3, this functoriality cannot obstruct decomposable Lagrangian cobordisms between stabilizations of the examples in Sections 4.1-4.3, while the GRID invariants can.

We mentioned in Section 1.3 that there are also cases in which the DGA obstruction applies where the GRID obstruction does not. For example, there is a Legendrian representative Λ_- of the figure eight with

$$tb(\Lambda_-) = -3 \text{ and } r(\Lambda_-) = 0,$$

such that $(\mathcal{A}_{\Lambda_-}, \partial_{\Lambda_-})$ admits an *augmentation*, and is therefore nontrivial (see e.g. [CN13]). This example was pointed out to the authors by Steven Sivek. Now, let Λ_+ be the Legendrian representative of the right-handed trefoil with

$$tb(\Lambda_+) = -1 \text{ and } r(\Lambda_+) = 0,$$

obtained by stabilizing the $tb = 1$ representative twice, once with each sign, so that $(\mathcal{A}_{\Lambda_+}, \partial_{\Lambda_+})$ is trivial. These DGAs therefore obstruct an exact Lagrangian cobordism from Λ_- to Λ_+. There is a smooth genus one cobordism from the figure eight to the right-handed trefoil, as indicated in Figure 18, so the classical invariants and smooth topology do not obstruct such a cobordism. On the other hand, the figure eight has trivial Heegaard Floer tau-invariant, so that

$$tb(\Lambda_-) + |r(\Lambda_-)| < 2\tau(\Lambda_-) - 1.$$

Since the figure eight is also *thin*, this implies that the GRID invariants of Λ_- vanish [NOT08, Proposition 3.4], and therefore do not obstruct a decomposable Lagrangian cobordism from Λ_- to Λ_+ via Theorem 1.2.

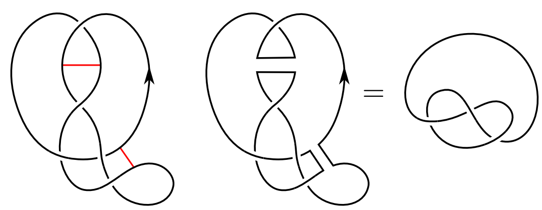


FIGURE 18. Two oriented band moves certifying the existence of a genus one cobordism between the figure eight knot and the right-handed trefoil.

Finally, in the interest of completeness, we remark that the DGAs of the Legendrian representatives of $m(10_{132})$, $m(12n_{200})$, $m(10_{145})$, $m(10_{161})$, and $12n_{591}$, which served as examples of Λ_- in Sections 4.1-4.3 admit no augmentations and therefore no linearized Legendrian contact homologies. Indeed, Rutherford showed in [Rut06] that the Kauffman polynomial bound on $tb(\Lambda)$ is sharp if and only if $(\mathcal{A}_\Lambda, \partial_\Lambda)$ admits an augmentation, and one can check on KnotInfo [CL] that the Kauffman bounds are not sharp for the Legendrians above. This does not completely rule out the possibility that their DGAs are nontrivial (and could therefore perhaps obstruct the Lagrangian cobordisms we are considering), but it eliminates the most tractable approach to proving nontriviality. Pan proved in [Pan17, Theorem 1.6] that if there exists an exact Lagrangian cobordism (with Maslov number 0) from Λ_- to Λ_+, then the number of graded augmentations of Λ_- up to a certain equivalence is less than or equal to the number of graded augmentations of Λ_+ up to equivalence. The fact that the Legendrians above admit no augmentations at all also rules out the possibility of applying Pan's more refined obstruction in these examples.

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