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Arithmeticity, superrigidity, and totally geodesic submanifolds

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Abstract

Let Γ be a lattice in $\mathrm{SO}_0(n,1)$. We prove that if the associated locally symmetric space contains infinitely many maximal totally geodesic subspaces of dimension at least 2, then Γ is arithmetic. This answers a question of Reid for hyperbolic n-manifolds and, independently, McMullen for hyperbolic 3-manifolds. We prove these results by proving a superrigidity theorem for certain representations of such lattices. The proof of our superrigidity theorem uses results on equidistribution from homogeneous dynamics, and our main result also admits a formulation in that language.

1. Introduction

In this paper, a totally geodesic subspace of a finite volume hyperbolic manifold or orbifold will always mean a properly immersed, topologically closed, totally geodesic subspace. A totally geodesic subspace is *maximal* if it is not properly contained in another proper totally geodesic subspace. The main result of this paper is the following.

Theorem 1.1. *Let Γ be a lattice in $\mathrm{SO}_0(n,1)$. If the associated locally symmetric space contains infinitely many maximal totally geodesic subspaces of dimension at least* 2, *then Γ is arithmetic.*

This answers a question, first posed informally by Alan Reid in the mid-2000s. Independently, Curtis McMullen asked whether Theorem 1.1 is true in the setting of hyperbolic 3-manifolds (see [12, Qn. 7.6] or [28, Qn. 8.2]). Theorem 1.1 is also motivated in part by a question of Gromov and Piatetski-Shapiro [17, Qn. 0.4]. In a prior paper with J.-F. Lafont, the last three authors proved that a large class of nonarithmetic hyperbolic n-manifolds, including all the hybrids constructed by Gromov and Piatetski-Shapiro, have only finitely many maximal totally geodesic submanifolds [14]. This provided the first known

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examples of hyperbolic n-manifolds, $n \geq 3$, for which the collection of totally geodesic hypersurfaces is finite and nonempty. The case when M is a closed hyperbolic 3-manifold was very recently and independently proved by Margulis and Mohammadi [25]. Their proof and ours both use a superrigidity theorem to prove arithmeticity, but the superrigidity theorems and their proofs are quite different.

We now briefly give some applications of Theorem 1.1 and its proof. First, combining Theorem 1.1 with a theorem of Reid [34] we obtain the following.

Corollary 1.2. *Let K be a knot in S^3 such that $S^3 \smallsetminus K$ admits a complete hyperbolic structure. Then $S^3 \smallsetminus K$ contains infinitely many immersed totally geodesic surfaces if and only if K is the figure-eight knot.*

Combining Theorem 1.1 with results of Benoist–Oh [4, Thm. 10.1], Lee–Oh [19, Thm. 1.9(3)], and the classification of arithmetic hyperbolic n-manifolds (e.g., see [29]), we also obtain the following.

Corollary 1.3.

(1) *If M is a geometrically finite hyperbolic 3-manifold containing infinitely many totally geodesic surfaces with finite area, then M has finite volume and $\pi_1(M)$ is arithmetic.*
(2) *If M is a convex cocompact hyperbolic n-manifold containing infinitely many maximal totally geodesic surfaces with finite area, then M is compact and $\pi_1(M)$ is arithmetic.*
(3) *If $n \geq 4$ is even and M is a finite volume hyperbolic n-manifold, then M is arithmetic if and only if it contains infinitely many totally geodesic hypersurfaces.*

For convex cocompact acylindrical 3-manifolds, this result already follows from work of McMullen–Mohammadi–Oh [26], [27] and Theorem 1.1. See Section 5.2 for discussion of (3) in odd dimensions.

Methods analogous to those used in the proof of Theorem 1.1 can also be used to show the following.

Theorem 1.4. *Let M be a cusped hyperbolic 3-manifold of finite volume with at least one torus cusp, and let N be a hyperbolic 3-manifold obtained by Dehn filling on some nonempty subset of the torus cusps of M. Then only finitely many totally geodesic surfaces in N are isotopic to the image of a totally geodesic surface in M.*

If either M or N is nonarithmetic, then this simply follows from Theorem 1.1. However, there are examples where M and N are both arithmetic and some totally geodesic surface in M remains totally geodesic in N, and hence Theorem 1.1 is not relevant. See Section 5.1 for the proof of Theorem 1.4, discussion, and examples.

Our approach to proving Theorem 1.1 is inspired by the Margulis superrigidity and arithmeticity theorems [22], [23]. The superrigidity theorem gives criteria for when a representation of Γ extends to a representation of the ambient Lie group G. Arithmeticity is then deduced using these criteria to control the representations of Γ one obtains by varying embeddings of the adjoint trace field of Γ into other local fields. See Section 3.2 for more discussion. A famous example employing this strategy is the proof by Margulis of arithmeticity of lattices with *dense commensurator* [22]. This theorem also holds in rank one and is the full converse to a theorem of Borel [5]. Margulis proved this by classifying representations of lattices that extend to representations of some dense subgroup of G contained in the commensurator.

Relating dense commensurators of arithmetic lattices back to the existence of infinitely many totally geodesic submanifolds, one can easily observe

ARITHMETIC GEODESIC SUBMANIFOLD DICHOTOMY. *For any dimension $1 \leq k \leq n-1$, an arithmetic hyperbolic n-manifold either contains no codimension k geodesic submanifolds, or it contains infinitely many and they are everywhere dense.*

This observation is one of the motivations for the question answered by Theorem 1.1 and was perhaps first made precise in dimension 3 by Maclachlan–Reid and Reid [20], [35], who also exhibited the first hyperbolic 3-manifolds with no totally geodesic surfaces. Note that an analogous statement holds for any arithmetic locally symmetric space. See [14] for further discussion and examples.

Our proof of Theorem 1.1 rests on two key points:

(1) From certain homomorphisms $\rho : \Gamma \to H$, we construct a good measure on a fiber bundle over G/Γ that is invariant under a proper noncompact connected simple subgroup $W < G$. This is accomplished in Section 3.
(2) We prove a superrigidity theorem showing that the measure constructed in (1) allows us to extend ρ, provided that H satisfies an additional *compatibility* condition. This is proved in Section 4.

In the standard language of superrigidity and its proofs, one can view (1) as the analogue for constructing a boundary map and (2) as the analogue for using the boundary map to show that the representation ρ extends.

We now discuss each of these steps briefly and begin by stating a version of Theorem 1.1 in language from homogeneous dynamics. We consider a proper noncompact connected closed simple subgroup $W < G = \mathrm{SO}_0(n,1)$. Then W is isomorphic to $\mathrm{SO}_0(m,1)$ for some $1 < m < n$. We have a W-action on G/Γ, and results of Ratner classify the W-invariant ergodic measures for this action [32], [33]. We say that a measure ν on G/Γ has *proper support* if its support is a proper closed subset.

THEOREM 1.5. *If there exists an infinite sequence* $\{\mu_i\}$ *of* W*-invariant, ergodic measures with proper support for which Haar measure on* G/Γ *is a weak-*$*$ *limit of the* μ_i, *then* Γ *is arithmetic.*

We show in Proposition 3.1 that Theorem 1.5 implies Theorem 1.1.

In proving arithmeticity we are given a local field k of characteristic zero, a connected semisimple adjoint k-algebraic group $\mathbf{H}$ with k-points $\mathbf{H}(k)$, and a representation $\rho : \Gamma \to \mathbf{H}(k)$. We consider a certain irreducible representation of $\mathbf{H}(k)$ on a finite dimensional k-vector space V and the associated projective space $\mathbb{P}(V)$. We then use the hypotheses of either Theorem 1.1 or Theorem 1.5 to build a W-invariant ergodic measure on the bundle $(G \times \mathbb{P}(V))/\Gamma$ that projects to Haar measure on G/Γ.

We now state the superrigidity theorem that finishes the proof from the existence of such a measure. This requires an additional technical assumption on the pair k and $\mathbf{H}$. Let P be a minimal parabolic subgroup of G and U its unipotent radical. A pair consisting of a local field k and a k-algebraic group $\mathbf{H}$ is said to be *compatible* with G if for every nontrivial k-subgroup $\mathbf{J} < \mathbf{H}$ and any continuous homomorphism $\tau : P \to N_{\mathbf{H}}(\mathbf{J})/\mathbf{J}(k)$, where $N_{\mathbf{H}}(\mathbf{J})$ is the normalizer of $\mathbf{J}$ in $\mathbf{H}$, we have that the Zariski closure of $\tau(U')$ coincides with the Zariski closure of $\tau(U)$ for every nontrivial subgroup $U' < U$ (see Section 3.4).

THEOREM 1.6. *Let* G *be* $\mathrm{SO}_0(n,1)$ *for* $n \geq 3$, $W < G$ *be a noncompact simple subgroup, and* $\Gamma < G$ *be a lattice. Suppose that* k *is a local field and* $\mathbf{H}$ *is a connected* k*-algebraic group such that the pair consisting of* k *and* $\mathbf{H}$ *is compatible with* G. *Finally, let* $\rho : \Gamma \to \mathbf{H}(k)$ *be a homomorphism with unbounded, Zariski dense image. If there exist a* k*-rational faithful irreducible representation* $\mathbf{H} \to \mathrm{SL}(V)$ *on a* k*-vector space* V *and a* W*-invariant measure* ν *on* $(G \times \mathbb{P}(V))/\Gamma$ *that projects to Haar measure on* G/Γ, *then* ρ *extends to a continuous homomorphism from* G *to* $\mathbf{H}(k)$.

Remark 1.7. We state the theorem for $G = \mathrm{SO}_0(n,1)$ for simplicity, but the same theorem holds, with practically the same proof, for every connected simple $\mathbb{R}$-rank one Lie group. In particular, there is an analogue of Theorem 1.6 for lattices in $\mathrm{SU}(n,1)$.

Understanding invariant measures for dynamical systems that are not homogeneous plays an important role in other recent results in rigidity theory. For example, see work of Brown, Hurtado, and the second author on Zimmer's conjecture [7], [8]. In that context, Theorem 1.6 can be thought of as classifying invariant measures in a nonhomogeneous setting. Indeed, Theorem 1.6 shows that either there is no extension of ρ and hence no such W-invariant measures exist, or there is a simple classification of all invariant measures on the projective bundle.

We note in closing that Theorem 1.6 can be reformulated in several equivalent ways. There is also an analogous *superrigidity for cocycles* that follows from the same proof, and which provides some partial technical results towards questions raised by results of Zimmer and Bader–Furman–Sauer [40], [2].

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2. Fixed Notation

We first fix some notation that will be used throughout our paper. Let $\mathbf{G}_0$ denote $\mathrm{SO}(n,1)$ for $n \geq 3$, considered as a real algebraic group. We let G be the connected component of the identity in $\mathbf{G}_0(\mathbb{R})$, i.e., $G = \mathrm{SO}_0(n,1)$. Set $K = \mathrm{SO}(n) < G$ and identify $K\backslash G$ with hyperbolic n-space. For a noncompact almost simple subgroup $W < G$, fix a maximal $\mathbb{R}$-split torus $A < W$. Since W and G are both $\mathbb{R}$-rank one, A is also a maximal $\mathbb{R}$-split torus of G. Fix a maximal unipotent subgroup U of G normalized by A, and let M be the compact factor of the Levi decomposition of the connected component of the identity in the centralizer of A. Then $P = MAU$ is the *Langlands decomposition* of the maximal parabolic subgroup of G associated with the pair (A, U). Set $U' = W \cap U$, and note that it is a maximal unipotent subgroup of W.

Now, fix a lattice $\Gamma < G$. When considering the action of Γ on G, we always consider the right action, $g \cdot \gamma = g\gamma^{-1}$, and $X_\Gamma = K\backslash G/\Gamma$ will denote the corresponding locally symmetric space. Let ℓ be the trace field of Γ, that is the subfield of $\mathbb{R}$ generated by all elements of the form $\mathrm{Tr}(\mathrm{Ad}(\gamma))$ for $\gamma \in \Gamma$, where Ad denotes the adjoint representation. Denote the inclusion of ℓ in $\mathbb{R}$ by $w : \ell \to \mathbb{R}$. By work of Vinberg [39], there exist an ℓ-group $\mathbf{G}$ and an $\mathbb{R}$-isogeny $\mathbf{G} \to \mathbf{G}_0$ such that the image of $\mathbf{G}(\ell)$ in $\mathbf{G}_0(\mathbb{R})$ contains a finite index subgroup of Γ. Passing to this finite index subgroup, we will assume throughout that Γ is contained in the image of $\mathbf{G}(\ell)$. By [39], ℓ is the minimal field of definition of Γ. Moreover, it follows from work of Selberg [37], Calabi [9], Raghunathan [31], and Garland [15] that ℓ is in fact a number field.

3. Finding invariant measures and arithmeticity

In this section we show how Theorem 1.6 implies Theorem 1.1. We show in Section 3.1 that the hypotheses of Theorem 1.5 are implied by the hypotheses of Theorem 1.1, Section 3.2 recalls the overall strategy of deducing arithmeticity from superrigidity, Section 3.3 finds the measure ν from the hypotheses of Theorem 1.6 using the hypotheses of Theorem 1.5, and finally Section 3.4 shows that all the target groups considered for proving arithmeticity are compatible. In particular, this section reduces Theorem 1.1 to Theorem 1.5 and Theorem 1.5 to Theorem 1.6.

3.1. *Geodesic submanifolds and properly supported measures.* Recall that a finite measure μ on G/Γ is called *homogeneous* if there is a closed subgroup $S < G$ such that μ is Haar measure on a closed S-orbit in G/Γ. Such a homogeneous measure is said to be W-ergodic when W is a closed subgroup of S under which μ is ergodic. In this case, the support of the measure is said to be a *W-ergodic homogeneous subspace* of G/Γ. For $1 < m \leq n$, we let $W_m \leq G$ be the standard embedding of $\mathrm{SO}_0(m, 1)$ into G. The entirety of this subsection is devoted to proving the following proposition.

PROPOSITION 3.1. *For the real hyperbolic space $X_\Gamma = K\backslash G/\Gamma$, the following are equivalent*:

(1) *X_Γ contains infinitely many maximal totally geodesic subspaces of dimension two or higher*;
(2) *for some $1 < m < n$, there exists an infinite sequence $\{\mu_i\}$ of W_m-invariant, ergodic measures with proper support for which Haar measure on G/Γ is a weak-$*$ limit of the μ_i*;
(3) *for some $1 < m < n$, there exists an infinite sequence of homogeneous, W_m-ergodic measures $\{\mu_i\}$ for which Haar measure on G/Γ is a weak-$*$ limit of the μ_i.*

That (3) implies (2) is clear, and the reverse implication is a theorem of Ratner [32], [33] (see also Einsiedler [13]). It therefore suffices to show that (1) and (3) are equivalent. Throughout this section we let $\pi : G/\Gamma \to X_\Gamma$ be the natural projection.

We start by clarifying the relationship between totally geodesic subspaces and homogeneous measures. We first recall that a subspace Z of a hyperbolic n-orbifold X is totally geodesic if it is properly immersed and if one (hence any) lift to a map of orbifold universal covers $\widetilde{Z} \to \widetilde{X}$ is a totally geodesic isometric embedding of hyperbolic m-space in hyperbolic n-space for some $m \leq n$. In particular, totally geodesic subspaces are by definition connected.

LEMMA 3.2. *Fix $1 < m \leq n$ and $X_\Gamma = K\backslash G/\Gamma$. Then the following hold:*

(1) *Let $S \leq G$ be a closed subgroup containing W_m and $h \in G$ be such that $Sh\Gamma/\Gamma \subset G/\Gamma$ is a closed S-orbit. Then the subspace $Z = \pi(Sh\Gamma/\Gamma)$ of X_Γ is a closed totally geodesic m'-dimensional subspace for some $m' \geq m$ and, up to normalization, the m'-volume of Z is the push-forward of the corresponding homogeneous measure on G/Γ.*

(2) *Furthermore, under the assumption above, $m' = n$ if and only if $S = G$ and $m' = m$ if and only if all unipotent elements of S are contained in W_m. In the latter case S is a subgroup of the normalizer N_m of W_m in G, and $N_m h\Gamma/\Gamma \subset G/\Gamma$ is also closed with projection $\pi(N_m h\Gamma/\Gamma) = Z$.*

(3) *Conversely, every m-dimensional closed totally geodesic subspace Z in X_Γ has finite m-volume and, moreover, $Z = \pi(Sh\Gamma/\Gamma)$ for some closed intermediate subgroup $W_m \leq S \leq N_m$ and some homogeneous, W_m-ergodic subspace $Sh\Gamma/\Gamma \subset G/\Gamma$.*

Proof. We start by observing that for each m', the image of $W_{m'}$ in $K\backslash G$, namely $K\backslash KW_{m'}$, is an m'-dimensional closed totally geodesic subspace. As G acts transitively on the collection of m'-dimensional closed totally geodesic subspaces of $K\backslash G$, any such subspace is of the form $K\backslash KW_{m'}g$ for some $g \in G$. Since $N_{m'}$ is contained in $KW_{m'}$, every intermediate subgroup $W_{m'} \leq S_1 \leq N_{m'}$ has the property that $K\backslash KS_1g = K\backslash KW_{m'}g$ is an m'-dimensional closed totally geodesic subspace of $K\backslash G$, and the push-forward of the volume form on the Lie group S_1 is its m'-volume. Conversely, $N_{m'}$ is exactly the stabilizer of $K\backslash KW_{m'}$ in G. Thus if $K\backslash KS_1g$ is an m'-dimensional closed totally geodesic subspace of $K\backslash G$, then $W_{m'} \leq S_1 \leq N_{m'}$.

We now prove Part (1). Let $S \leq G$ be a closed subgroup that contains W_m, and let $h\Gamma/\Gamma \in G/\Gamma$ be a point whose S-orbit is closed. Denote by $S^+ \leq S$ the closed normal subgroup generated by unipotent elements in S. Then S^+ is a connected semisimple subgroup of G that contains W_m, hence it is conjugate to $W_{m'}$ for some $m' \geq m$. In fact, if C_m denotes the centralizer of W_m in G, it is straightforward to see that there exists $g \in C_m$ such that $W_{m'} = (S^+)^g = gS^+g^{-1}$. We fix such a g and set $S_1 = S^g$.

Since S^+ is normal in S, $W_{m'}$ is normal in S_1, and thus $W_{m'} \leq S_1 \leq N_{m'}$. From the fact that $C_m < K$ and $g \in C_m$, we get that

$$KS = Kg^{-1}S_1g = KS_1g.$$

Since the projection π is proper, $Z = \pi(Sh\Gamma/\Gamma)$ is closed, and since

$$Z = K\backslash KSh\Gamma/\Gamma = K\backslash KS_1gh\Gamma/\Gamma \subset X_\Gamma,$$

we conclude that Z is m'-dimensional and totally geodesic, since it is the image of $K\backslash KS_1gh$ under the projection $K\backslash G \to X_\Gamma$. Consequently, the m'-volume on Z is the push forward of the S_1-volume form. As the S_1-volume form is the g^{-1}-conjugate of the S-volume form, the m'-volume of Z is the push forward of Haar measure on $Sh\Gamma/\Gamma$. This completes the proof of Part (1).

We now prove Part (2). Clearly, if $S = G$ then $m' = n$, and if $m' = n$, then $S = G$, as S contains a conjugate of $G = W_{m'}$. Thus we discuss the second part of the statement. If all unipotent elements of S are contained in W_m, then $S^+ \leq W_m$, but $W_m \leq S^+$ by hypothesis, so we conclude that $W_m = S^+$ and $m' = m$. Conversely, if $m' = m$, then since $W_m \leq S^+ = W_{m'}^{g^{-1}}$, we conclude that $S^+ = W_m$ and all unipotent elements of S are contained in W_m. In this case, W_m is normal in S and thus $W_m \leq S \leq N_m$. Since W_m is cocompact in N_m, S is as well, and therefore the fact that $Sh\Gamma/\Gamma$ is closed in G/Γ implies that the same holds for $N_m h\Gamma/\Gamma$. Moreover, from the chain of equations

$$K\backslash KW_m h = K\backslash KSh = K\backslash KN_m h,$$

we conclude that $\pi(N_m h\Gamma/\Gamma) = Z$ in X_Γ. This proves Part (2).

Before turning to Part (3), we discuss m-dimensional closed totally geodesic immersed submanifolds of n-dimensional Riemannian manifolds in general. Given such a pair $M \subset N$, we let $F(N)$ be the oriented orthonormal frame bundle of N and we let $F_N(M)$ be the subbundle where the fiber over each point $x \in M$ is the subset of frames in $F(N)_x$ whose first m vectors are tangent to M. We note that $F_N(M)$ is a principal $\mathrm{S}(\mathrm{O}(m) \times \mathrm{O}(n-m))$-bundle over M and it is closed in $F(N)$, which itself is a principal $\mathrm{SO}(n)$-bundle over N. This construction is natural under covering maps.

Identifying G with $F(K\backslash G)$, one checks easily that N_m gets identified with $F_{K\backslash G}(K\backslash KW_m)$ and thus for every $g \in G$, $N_m g$ gets identified with $F_{K\backslash G}(K\backslash KW_m g)$. In accordance with the identification of G with $F(K\backslash G)$ we identify G/Γ with $F(X_\Gamma)$. For an m-dimensional closed totally geodesic subspace $Z \subset X_\Gamma$, the subbundle $F_{X_\Gamma}(Z)$ gets identified with a closed subset of G/Γ. Finding $g \in G$ such that $K\backslash KW_m g \subset K\backslash G$ projects to Z under the natural map $K\backslash G \to X_\Gamma$, we conclude by naturality under covering maps that $F_{X_\Gamma}(Z)$ is identified with $N_m g\Gamma/\Gamma$. In particular, the latter is a closed subset of G/Γ whose image under π is Z.

We are now in a position to prove Part (3). Let $Z \subset X_\Gamma$ be an m-dimensional closed totally geodesic subspace. The fact that it has a finite measure is well known; see [16, Prop. 3.4] for a recent reference. By the discussion above there exists $g \in G$ such that $N_m g\Gamma/\Gamma$ is a closed subset of G/Γ whose image under π is Z. Note that $N_m g\Gamma/\Gamma$ has a finite volume, as it is a compact extension of Z. Since $W_m \leq N_m$, $N_m g\Gamma/\Gamma$ is W_m-invariant, though it might not be W_m-ergodic. Fix a W_m-ergodic measure μ in its ergodic decomposition, and let S be the stabilizer of μ in N_m. Clearly, $W_m \leq S \leq N_m$ and μ is S-homogeneous by Ratner's Theorem. Let $h \in G$ be such that μ is Haar measure on the S-orbit $Sh\Gamma/\Gamma$. It follows from Part (1) of the lemma that $\pi(Sh\Gamma/\Gamma)$ is a closed totally geodesic subspace of X_Γ of dimension m or higher. Since this image is contained in the m-dimensional closed totally geodesic subspace Z, it must coincide with it. This completes the proof of the lemma. □

We will also use the following theorem, which follows from combining the existence of compact cores for hyperbolic manifolds with work of Dani–Margulis [11, Thm. 6.1] and Mozes–Shah [30].

THEOREM 3.3. *Suppose that $W < G$ is a closed connected semisimple subgroup that is generated by unipotent elements. Let $\{\mu_i\}$ be a sequence of homogeneous, W-ergodic probability measures on G/Γ that weak-$*$ converges in the space of all finite Radon measures to a measure μ. Then μ is a homogeneous, W-ergodic probability measure on G/Γ and there exist a sequence $\{g_i\}$ in G and a natural number i_0 such that, for each $i \geq i_0$, the measure $g_i\mu$ is a homogeneous, W-ergodic probability measure on G/Γ whose support contains the support of μ_i.*

Proof. We first claim that μ is not the zero measure. This is trivial if G/Γ is compact. As this claim is conjugation invariant, we will assume, as we may, that $W = W_m$ for some $m > 1$. Using [14, Lem. 5.13] we fix a compact set C_1 in $X_\Gamma = K\backslash G/\Gamma$ that meets every closed totally geodesic subspace of dimension at least 2. Then, choose a compact set C_2 that contains C_1 in its interior and consider its preimage in G/Γ, i.e., the compact subset $F = \pi^{-1}(C_2)$ of G/Γ. Fix a 1-parameter unipotent subgroup $\{u_t\}$ in W, and set $\epsilon = 1/2$. Applying [11, Thm. 6.1] we find a compact subset $F' \subset G/\Gamma$ such that

$$(1) \qquad \frac{1}{T}\int_0^T \chi_{F'}(u_t y)dt = \frac{1}{T}\lambda\left\{t \in [0,T] \mid u_t y \in F'\right\} \geq \frac{1}{2}$$

for every $y \in F$ and every $T \geq 0$, where $\chi_{F'}$ is the characteristic function of F' and λ is the Lebesgue measure on $\mathbb{R}$. The claim will follow once we show that $\mu_i(F') \geq 1/2$ for every i, thus $\mu(F') \geq 1/2$. We now fix i and show that indeed $\mu_i(F') \geq 1/2$.

By Lemma 3.2(1), $\pi_*\mu_i$ is the unit renormalization of the volume measure associated with a closed totally geodesic subspace of dimension at least $m \geq 2$. This subspace intersects C_1 nontrivially, by the choice of C_1, thus it intersects C_2 in an open set. It follows that $\mu_i(F) = \pi_*\mu_i(C_2) > 0$, as $\pi_*\mu_i$ is proportional to a volume measure. We note that μ_i is $\{u_t\}$-ergodic by the Howe–Moore theorem, and we let $y \in F$ be a $\{u_t\}$-generic point with respect to μ_i. Applying the Birkhoff ergodic theorem to the function $\chi_{F'}$ we conclude by Equation (1) that indeed $\mu_i(F') \geq 1/2$. This proves the claim.

Using [30, Cor. 1.3] we therefore conclude that μ is a homogeneous, S-ergodic probability measure on G/Γ, where S is the subgroup of the stabilizer of μ generated by unipotent elements. The group S is not unipotent, as it contains W, therefore it must be semisimple. We conclude by the Howe–Moore theorem that μ is W-ergodic.

Let $Y_i = \mathrm{supp}(\mu_i)$ and $Y = \mathrm{supp}(\mu)$ in G/Γ, and fix one dimensional unipotent subgroups $U_1, \dots, U_k < W$ that generate W. Note that μ_i is U_j-ergodic for every i and every $1 \le j \le k$ by the Howe–Moore theorem. Thus, for any fixed i, the subset Y_i' consisting of the points in Y_i that are U_j-generic for every $1 \le j \le k$ is of full μ_i-measure, hence it is dense in Y_i. We fix a point $y \in Y$. As $\mu = \lim \mu_i$, we can find a sequence of points $\{y_i\}$ converging to y such that y_i is in Y_i for every i. By deforming such a sequence, using the fact that Y_i' is dense in Y_i, we find and fix a sequence $\{y_i'\}$ converging to y such that y_i' is in Y_i' for every i. We fix a sequence $\{g_i\}$ in G such that $\lim g_i = e$ and $g_i y = y_i'$ for all i.

By [30, Thm. 1.1], we conclude that for every $1 \le j \le k$, there exists an i_j such that $Y_i \subset g_i Y$ and μ is $U_j^{g_i^{-1}}$-invariant for every $i \ge i_j$. Let $i_0 = \max\{i_j\}$, and fix $i \ge i_0$. As $g_i Y = \mathrm{supp}(g_i \mu)$, we conclude that $\mathrm{supp}(\mu_i) \subseteq \mathrm{supp}(g_i \mu)$. It remains to show that $g_i \mu$ is W-ergodic or, equivalently, that μ is $W^{g_i^{-1}}$-ergodic. Since μ is $U_j^{g_i^{-1}}$-invariant for every $1 \le j \le k$, it is $W^{g_i^{-1}}$-invariant. It follows that $W^{g_i^{-1}}$ is contained in S. Then $W^{g_i^{-1}}$-ergodicity follows from the Howe–Moore theorem and S-ergodicity. This completes the proof. □

Proof of Proposition 3.1. As noted immediately after the statement of the proposition, it is enough to show that (1) and (3) are equivalent. We begin with the easier implication, namely that (3) implies (1). Fix $1 < m < n$, and suppose that $\{\mu_i\}$ is a sequence of homogeneous, W_m-ergodic measures for which Haar measure on G/Γ is a weak-$*$ limit. Let $\bar{\mu}_i = \pi_* \mu_i$ be the push-forward measures. By Lemma 3.2(1), each measure $\bar{\mu}_i$ is supported on a closed totally geodesic subspace of X_Γ, and we let Z_i be a maximal totally geodesic subspace of X_Γ containing it. Since Haar measure on G/Γ is by hypothesis a weak-$*$ limit of the sequence $\{\mu_i\}$, its push-forward is supported on the closure of $\bigcup Z_i$. Since the push forward of Haar measure on G/Γ is the volume form on X_Γ, it follows that $\bigcup Z_i$ is dense in X_Γ and hence the sequence $\{Z_i\}$ consists of infinitely many maximal totally geodesic subspaces. This implies (1).

Next we show that (1) implies (3). Assume that there exists an infinite sequence $\{Z_i\}$ of distinct closed maximal totally geodesic subspaces of X_Γ. By passing to a subsequence we assume that they all have the same dimension m for some $1 < m < n$. By Lemma 3.2(3), each Z_i is the image under $\pi : G/\Gamma \to X_\Gamma$ of a homogeneous, W_m-invariant subspace of G/Γ, which we denote by Y_i. Furthermore, each Y_i is the support of a homogeneous,

W_m-ergodic probability measure μ_i with stabilizer S_i containing W_m. Passing to a further subsequence, we can assume that the μ_i weak-$*$ converge to a measure μ. Note that Theorem 3.3 implies that μ is a homogeneous, W_m-ergodic probability measure on G/Γ. We now want to show that μ is Haar measure on G/Γ.

Assume for contradiction that μ is not Haar measure on G/Γ. If S denotes the stabilizer of μ, which contains W_m, then $Y_\infty = \mathrm{supp}(\mu) = Sh\Gamma/\Gamma$ for some $h \in G$. By Theorem 3.3 there exist a sequence $\{g_i\}$ in G and a natural number i_0 such that for each $i \geq i_0$, $g_i Y_\infty$ is a homogeneous, W_m-invariant subspace of G/Γ that contains Y_i. Once again passing to a subsequence we assume that this holds for every $i \geq 1$.

Fix any $i \in \mathbb{N}$. Applying Lemma 3.2(1) to $g_i Y_\infty$, we see that $\pi(g_i Y_\infty)$ is a closed totally geodesic subspace. Our assumption that μ is not Haar measure along with the case $m' = n$ in Lemma 3.2(2) implies that $\pi(g_i Y_\infty) \subset X_\Gamma$ is a proper subspace. Since $Y_i \subseteq g_i Y_\infty$, we deduce that $Z_i \subseteq \pi(g_i Y_\infty)$ and hence $Z_i = \pi(g_i Y_\infty)$ by maximality. In particular, $\dim(\pi(g_i Y_\infty)) = m$.

Note that S^{g_i} is the stabilizer of the measure $g_i\mu$ and hence $W_m \leq S^{g_i}$. By the case $m' = m$ in Lemma 3.2(2) we conclude that the subgroup of S^{g_i} generated by unipotent elements is W_m and that $W_m \leq S^{g_i} \leq N_m$. Since $W_m \leq S$, we also have that $W_m^{g_i} \leq S^{g_i}$, and since the subgroup of N_m generated by unipotents is exactly W_m, we see that $W_m = W_m^{g_i}$. Therefore $g_i \in N_m$. Applying Lemma 3.2(1) to the closed S^{g_i}-orbit

$$S^{g_i} g_i h\Gamma/\Gamma = g_i Sh\Gamma/\Gamma = g_i Y_\infty,$$

the N_m-orbit $N_m g_i h\Gamma/\Gamma$ is also closed in G/Γ and we have $Z_i = \pi(N_m g_i h\Gamma/\Gamma)$. However, $g_i \in N_m$, thus $N_m g_i h\Gamma/\Gamma = N_m h\Gamma/\Gamma$ is independent of i. We conclude that $Z_i = \pi(N_m h\Gamma/\Gamma)$ is independent of i, contradicting the assumption that the spaces Z_i are all distinct. This contradiction concludes the proof that (1) implies (3). □

3.2. *The proofs of Theorems* 1.1 *and* 1.5. We now explain how to prove Theorems 1.1 and 1.5 given Theorem 1.6. This closely follows Margulis's proof of arithmeticity from superrigidity. For more details, see [24, Ch. IX] or [41, Ch. 6].

We are given a lattice $\Gamma < G$ and want to show that it is arithmetic. As in Section 2, we consider Γ as a subgroup of $\mathbf{G}(\ell)$, where ℓ is the adjoint trace field of Γ, embedded in $\mathbb{R}$ via $w : \ell \to \mathbb{R}$. Consider the collection S of all places of ℓ, that is, the equivalence classes of dense embeddings of ℓ into a local field. For $v \in S$, ℓ_v will denote the corresponding local field. In particular, we have the aforementioned $w \in S$ and $w : \ell \to \ell_w = \mathbb{R}$. Considering the various embeddings $\Gamma \to \mathbf{G}(\ell) \to \mathbf{G}(\ell_v)$ for all $v \in S$, it is standard that Γ is arithmetic if and only if the image of Γ in $\mathbf{G}(\ell_v)$ is precompact for every $v \neq w$.

We let $\mathbf{H}$ be the adjoint group associated with $\mathbf{G}$ and claim that for $v \in S$, $v \neq w$, the corresponding homomorphism $\Gamma \to \mathbf{G}(\ell_v) \to \mathbf{H}(\ell_v)$ cannot be extended to $\mathbf{G}(\ell_w) \simeq G \to \mathbf{H}(\ell_v)$. By [24, Rem. 1.8.2(III)], such an extension gives rise to a continuous field embedding $\ell_w \to \ell_v$, and this field embedding clearly agrees with $v : \ell \to \ell_v$ on the set of elements of the form $\mathrm{Tr}(\mathrm{Ad}(\gamma))$ for $\gamma \in \Gamma$. As ℓ is generated by the above set, we get that $\ell_w \to \ell_v$ extends v, which contradicts the assumption that $v \neq w$. To be precise, in [24, Rem. 1.8.2(III)] the target group is assumed absolutely simple, which is not necessarily the case for $\mathbf{H}$. This can be remedied by passing to a certain finite field extension ℓ'_v/ℓ_v, considering the corresponding homomorphism $\mathbf{G}(\ell_v) \to \mathbf{H}(\ell_v) \to \mathbf{H}(\ell'_v)$, taking the restriction of scalars of $\mathbf{H}$ from ℓ'_v back to ℓ_v, then projecting to a simple factor. This procedure replaces the target group $\mathbf{H}$ with an absolutely simple ℓ_v-group, thus proving our claim by the argument presented above. In summary, we prove that Γ is arithmetic by showing that its image in $\mathbf{G}(\ell_v)$ is precompact, and we do that by proving that if this is not the case, then $\Gamma \to \mathbf{H}(\ell_v)$ must extend to G. Note that the failure of precompactness of the image of Γ in $\mathbf{G}(\ell_v)$ implies the same holds for the image of Γ in $\mathbf{H}(\ell_v)$, as the map $\mathbf{G} \to \mathbf{H}$ is a finite isogeny.

The existence of the desired extension $G \to \mathbf{H}(\ell_v)$ will follow from Theorem 1.6 once we verify its various assumptions in the specific settings of Proposition 3.1. In this setting, in Section 3.3 we will produce an ℓ_v-vector space V, endowed with a faithful irreducible representation of $\mathbf{H}(\ell_v)$, and a W-invariant measure on $(G \times \mathbb{P}(V))/\Gamma$, as required in Theorem 1.6. Our proof will be complete once we show that the pair consisting of ℓ_v and $\mathbf{H}$ is compatible with G. This will be done in Section 3.4.

3.3. *Lifting measures to the projective bundle.* Let ℓ be the number field and $\mathbf{G}$ the ℓ-algebraic group associated with Γ as in Section 2, and let $\mathbf{H}$ be the corresponding adjoint ℓ-group. In this subsection we let $k = \ell_v$ be any local completion of ℓ for which the natural inclusion $\rho' : \Gamma \to \mathbf{G}(\ell) \to \mathbf{G}(k)$ is not precompact. Consider the representation $\rho : \Gamma \to \mathbf{G}(k) \to \mathbf{H}(k)$ whose image is also not precompact. In this subsection we assume the hypotheses of Theorem 1.1. By (1) $\Leftrightarrow$ (2) in Proposition 3.1, the hypotheses of Theorem 1.5 hold. That is, for some $1 < m < n$, we have an infinite sequence of homogeneous, W_m-ergodic measures $\{\mu_i\}$ for which Haar measure on G/Γ is a weak-$*$ limit of the μ_i. Passing to a subsequence, we assume, as we may, that μ_i actually converges to Haar measure. As m is fixed, we set $W = W_m$. This subsection is then devoted to proving the following.

PROPOSITION 3.4. *Under the hypotheses of Theorem 1.5 or Theorem 1.1, there is a k-rational faithful irreducible representation $\mathbf{H} \to \mathrm{SL}(V)$ on a k-vector space V and a W-invariant measure on $(G \times \mathbb{P}(V))/\Gamma$ that projects to Haar measure on G/Γ.*

Proof. We retain all notation from prior subsections. We note that, as $\mathbf{H}$ is semisimple, each of its k-rational representation is into $\mathrm{SL}(V) < \mathrm{GL}(V)$. We will construct a faithful irreducible representation supporting a W-invariant measure as required. By (2) $\Leftrightarrow$ (3) in Proposition 3.1 we can assume that the W-invariant, ergodic measures μ_i on G/Γ converging to Haar measure are in fact homogeneous. Thus, for every i there exists a closed subgroup $S_i' < G$ such that μ_i is Haar measure on a closed S_i'-orbit in G/Γ.

By Lemma 3.2, for every i, $S_i' \leq N_{m'}$ for some $m \leq m' < n$, thus S_i' is not Zariski dense in G. Fixing $x_i \in G$ that project to a base point in this orbit, we denote this orbit by $S_i' x_i \Gamma/\Gamma$ and rewrite it as $x_i S_i \Gamma/\Gamma$, where $x_i^{-1} S_i x_i = S_i'$.

Let Γ_i' be the stabilizer in S_i' of the image of x_i in G/Γ and then set $\Gamma_i = x_i \Gamma_i' x_i^{-1}$. Thus $\Gamma_i = S_i \cap \Gamma$ and Γ_i is a lattice in S_i. We let $\mathbf{L}_i$ be the Zariski closure of $\rho(\Gamma_i)$. As S_i is not Zariski dense, neither is Γ_i, and we get that $\mathbf{L}_i < \mathbf{H}$ is a proper k-subgroup such that $\rho(\Gamma_i) < \mathbf{L}_i(k)$. We pass to a subsequence such that $\dim(\mathbf{L}_i)$ is constant and denote this constant by d.

We first assume that $\mathbf{H}$ is k-simple, which is automatic when $G \neq \mathrm{SO}_0(3,1)$, and we note that in this case faithfulness of a k-linear representation is equivalent to its nontriviality. We will consider the semisimple case at the end of the proof, where faithfulness will require an additional argument.

Consider the d^{th} exterior power $\wedge^d(\mathrm{Ad}) : \mathbf{H}(k) \to \mathrm{GL}(\bigwedge^d \mathfrak{h})$ of the adjoint representation of $\mathbf{H}(k)$ on its Lie algebra $\mathfrak{h}$. The Lie algebra $\mathfrak{l}_i$ of $\mathbf{L}_i(k)$ determines a line l_i in $\bigwedge^d \mathfrak{h}$. Since the stabilizer of l_i in $\mathbf{H}(k)$ is the normalizer of $\mathbf{L}_i(k)$ and hence a proper subgroup, this line is never $\mathbf{H}(k)$-invariant. Therefore each l_i projects nontrivially to some nontrivial irreducible summand of $\wedge^d(\mathrm{Ad}) : \mathbf{H}(k) \to \mathrm{GL}(\wedge^d \mathfrak{h})$. Since only finitely many irreducible representations can occur, one such irreducible representation V occurs infinitely often. Passing to a further subsequence, we obtain an irreducible subrepresentation V onto which each l_i projects nontrivially. The point stabilizer of l_i contains the image of $\mathbf{L}_i(k)$, and hence it contains $\rho(\Gamma_i)$.

Given the closed W-invariant subset $x_i(S_i/\Gamma_i)$, note that l_i determines an invariant line bundle over $x_i(S_i/\Gamma_i)$ and therefore defines a measurable section

$$s^i : x_i(S_i/\Gamma_i) \to (G \times \mathbb{P}(V))/\Gamma_i.$$

Identifying Haar measure on $x_i(S_i/\Gamma_i)$ with μ_i we define $\nu_i = s^i_* \mu_i$. We then construct a W-ergodic, W-invariant measure on $(G \times \mathbb{P}(V))/\Gamma$ by taking ν to be any ergodic component of any weak-$*$ limit of the ν_i on $(G \times \mathbb{P}(V))/\Gamma$. Since the μ_i converge to Haar measure on G/Γ and projection commutes with taking weak-$*$ limits, this implies that ν projects to Haar measure on G/Γ and completes the proof when $\mathbf{H}$ is simple.

For $G = \mathrm{SO}_0(3,1)$, the group $\mathbf{H}(k)$ need not be k-simple due to the exceptional isomorphism $\mathrm{PO}(4,k) \cong \mathrm{PGL}(2,k) \times \mathrm{PGL}(2,k)$. We therefore must find an irreducible representation V on which $\mathbf{H}(k)$ acts faithfully and for which the above construction yields the necessary invariant measure. To this end, we need to consider cases when

$$\mathbf{H}(k) = \mathrm{PGL}_2(k) \times \mathrm{PGL}_2(k),$$

where k is $\mathbb{R}$, $\mathbb{C}$, or a nonarchimedean local field of characteristic zero.

Notice that Γ is Zariski dense in the almost simple group $\mathbf{G}(\mathbb{R})$ and the groups Γ_i have proper Zariski closure. In particular, $\Gamma_i < \Gamma$ is not normal, and it follows from injectivity of ρ that $\rho(\Gamma_i)$ is not contained in a direct factor of $\mathbf{H}(k)$. As the Zariski closure of Γ_i is almost simple in $\mathbf{G}(\ell)$ and ρ is given by a field embedding, we conclude that $\rho(\Gamma_i)$ is contained in a conjugate of the diagonal subgroup $\Delta(\mathrm{PGL}_2(k))$ of $\mathrm{PGL}_2(k) \times \mathrm{PGL}_2(k)$ for all i.

We take the adjoint representation of $\mathrm{PGL}_2(k) \times \mathrm{PGL}_2(k)$ on k^6 and the diagonal three dimensional subspace $\Delta(k^3) < k^3 \oplus k^3$ stabilized by $\Delta(\mathrm{PSL}_2(k))$. A computation shows that $\bigwedge^3(k^3 \oplus k^3)$ splits as a direct sum of four irreducible representations of $\mathrm{PGL}_2(k) \times \mathrm{PGL}_2(k)$, two that are trivial and two that are isomorphic to the faithful representation $V(3,3)$ on $k^3 \otimes k^3$. One also checks that $\bigwedge^3(\Delta(k^3))$ projects nontrivially to each $V(3,3)$ (in fact, to all four summands). Taking $V = V(3,3)$ and arguing as above, we also produce a W-invariant measure on $(G \times \mathbb{P}(V))/\Gamma$ when $G = \mathrm{SO}_0(3,1)$. This completes the proof. □

3.4. *Compatibility.* Let G, U, and $P = MAU$ be as defined in Section 2. Let k be a local field and $\mathbf{H}$ a k-algebraic group. Recall that the pair consisting of k and $\mathbf{H}$ is *compatible* with G if for every nontrivial k-subgroup $\mathbf{J} < \mathbf{H}$ and any continuous homomorphism $\tau : P \to N_{\mathbf{H}}(\mathbf{J})/\mathbf{J}(k)$, where $N_{\mathbf{H}}(\mathbf{J})$ is the normalizer of $\mathbf{J}$ in $\mathbf{H}$, we have that the Zariski closure of $\tau(U')$ coincides with the Zariski closure of $\tau(U)$ for every nontrivial subgroup $U' < U$.

Note that if the pair $(k', \mathbf{H})$ is compatible, where k'/k is a finite field extension, then the pair $(k, \mathbf{H})$ is also compatible. Indeed, letting $\mathbf{J} < \mathbf{H}$ be a k-subgroup and $\tau : P \to N_{\mathbf{H}}(\mathbf{J})/\mathbf{J}(k)$ be a continuous homomorphism, composing τ with the homomorphism $N_{\mathbf{H}}(\mathbf{J})/\mathbf{J}(k) \to N_{\mathbf{H}}(\mathbf{J})/\mathbf{J}(k')$ defines a continuous homomorphism $\tau' : P \to N_{\mathbf{H}}(\mathbf{J})/\mathbf{J}(k')$. Compatibility of $(k', \mathbf{H})$ implies that the Zariski closure of $\tau(U')$ coincides with the Zariski closure of $\tau(U)$ for every nontrivial subgroup $U' < U$.

We note also that compatibility of $(k, \mathbf{H})$ follows immediately if $U < \ker \tau$ for every τ as above. This is automatically the case when k is nonarchimedean, since then the group $N_{\mathbf{H}}(\mathbf{J})/\mathbf{J}(k)$ is totally disconnected whereas U is connected.

LEMMA 3.5. *We retain the notation of Section 2 and let $\mathbf{H}$ be the adjoint group of $\mathbf{G}(k)$ for k a local field and $\ell \to k$ a field embedding. Then the pair consisting of k and $\mathbf{H}$ is compatible with G.*

Proof. As mentioned above, we can and hence will assume that k is archimedean and, passing to a finite extension, in fact assume that $k = \mathbb{C}$. In particular, we get that $\mathbf{H}$ is split and we identify it with $\mathrm{PO}(n+1,\mathbb{C})$. In what follows, we identify algebraic subgroups of $\mathbf{H}$ with their complex points.

Assume we have a nontrivial algebraic subgroup $\mathbf{J} < \mathrm{PO}(n+1,\mathbb{C})$ and a continuous homomorphism $\tau : P \to \mathbf{N}/\mathbf{J}$, where we denote $\mathbf{N} = \mathbf{N_H}(\mathbf{J})$. Fix a nontrivial subgroup $U' < U$. Letting $\mathbf{U}, \mathbf{U}', \mathbf{M}, \mathbf{A}$ and $\mathbf{P}$ be the Zariski closures of the images of U, U', M, A and P in $\mathbf{N}/\mathbf{J}$ correspondingly, we must show that $\mathbf{U}' = \mathbf{U}$. We can obviously assume that $\tau|_U$ is nontrivial.

We first make an observation for later use. Since U is the derived subgroup of the solvable group AU, $\mathbf{U}$ is contained in the derived subgroup of the solvable group $\mathbf{AU}$. It follows that $\mathbf{U}$ is a nontrivial unipotent group.

It is convenient to identify U with the additive group $\mathbb{R}^{n-1}$ and MA with its conformal group. In particular, we identify M with $\mathrm{SO}(n-1,\mathbb{R})$ and A with the group of homotheties $\mathbb{R}^*$. Using transitivity of the action of $\mathrm{SO}(n-1,\mathbb{R})$ on $\mathbb{P}^{n-2}(\mathbb{R})$, one easily checks that every non-central subgroup of P contains U. We conclude that τ has finite kernel.

We claim that $n = 3$. To prove this, we will assume that $n \geq 4$ and argue to show a contradiction.

As $\tau(M)$ is locally isomorphic to the compact group $\mathrm{SO}(n-1,\mathbb{R})$, we see that $\mathbf{M}$ is locally isomorphic to $\mathrm{SO}(n-1,\mathbb{C})$. Thus $\mathbf{M}$ is almost simple and it normalizes the solvable group $\mathbf{AU}$, as M normalizes the solvable group AU in P. Therefore $\mathbf{M}$ intersects the group $\mathbf{AU}$ almost trivially. As AU is not nilpotent, we get that $\mathbf{AU}$ is not unipotent, thus $\operatorname{rank} \mathbf{AU} \geq 1$. We conclude that $\operatorname{rank} \mathbf{P} \geq \operatorname{rank} \mathbf{M} + 1$. We note also that $\operatorname{rank} \mathbf{P}/\mathbf{U} = \operatorname{rank} \mathbf{P}$, as $\mathbf{U}$ is unipotent.

We let $\widetilde{\mathbf{U}}$ and $\widetilde{\mathbf{P}}$ be the corresponding preimages of $\mathbf{U}$ and $\mathbf{P}$ in $\mathbf{N}$ under the map $\mathbf{N} \to \mathbf{N}/\mathbf{J}$. From the sequence of inequalities

$$\begin{aligned}
\operatorname{rank} \mathbf{H} &\geq \operatorname{rank} \widetilde{\mathbf{P}} \geq \operatorname{rank} \widetilde{\mathbf{P}} - \operatorname{rank} \widetilde{\mathbf{U}} \\
&= \operatorname{rank} \widetilde{\mathbf{P}}/\widetilde{\mathbf{U}} = \operatorname{rank} \mathbf{P}/\mathbf{U} = \operatorname{rank} \mathbf{P} \\
&\geq \operatorname{rank} \mathbf{M} + 1 = \operatorname{rank} \mathrm{SO}(n-1,\mathbb{C}) + 1 \\
&= \operatorname{rank} \mathrm{SO}(n+1,\mathbb{C}) = \operatorname{rank} \mathbf{H}
\end{aligned}$$

we deduce that $\operatorname{rank} \widetilde{\mathbf{U}} = 0$.

Next we consider the identity component $\mathbf{J}^0$ of $\mathbf{J}$ and the identity component $\widetilde{\mathbf{U}}^0$ of $\widetilde{\mathbf{U}}$. We note that $\mathbf{J}^0 \leq \widetilde{\mathbf{U}}^0$ and that this is a proper inclusion by nontriviality of the connected group $\mathbf{U}$. As $\operatorname{rank} \widetilde{\mathbf{U}} = 0$, we deduce that $\widetilde{\mathbf{U}}^0$ is a unipotent subgroup of $\mathbf{N}$. We conclude that both $\mathbf{J}^0$ and $\widetilde{\mathbf{U}}^0$ are normal unipotent subgroups of $\widetilde{\mathbf{P}}$ with $\mathbf{J}^0 \lneq \widetilde{\mathbf{U}}^0$.

By [6, §3], we see that $\widetilde{\mathbf{P}}$ is contained in a maximal parabolic subgroup $\mathbf{Q} < \mathbf{H}$, as it contains a nontrivial normal unipotent subgroup. Note that the maximal parabolic subgroups of $\mathrm{SO}(n+1,\mathbb{C})$ are the stabilizers of isotropic subspaces of $\mathbb{C}^{n+1}$. For a k-dimensional isotropic subspace, the semisimple part $\mathbf{S} < \mathbf{Q}$ of the Levi subgroup is locally isomorphic to $\mathrm{SL}_k(\mathbb{C}) \times \mathrm{SO}(n+1-2k,\mathbb{C})$, noting that $2k \leq n+1$. Alternatively, this can be seen by removing a node from the Dynkin diagram associated with $\mathrm{SO}(n+1,\mathbb{C})$. Almost-simplicity of $\mathbf{M} < \mathbf{P} = \widetilde{\mathbf{P}}/\mathbf{J}$ implies that $\widetilde{\mathbf{P}}$ contains a group $\mathbf{M}'$ locally isomorphic to $\mathbf{M}$. Thus $\mathbf{M}' < \mathbf{Q}$, and upon conjugating $\mathbf{S}$ we can assume that $\mathbf{M}' \leq \mathbf{S}$. We conclude that $k = 1$ and $\mathbf{S}$ is locally isomorphic to $\mathrm{SO}(n-1,\mathbb{C})$. In particular, $\mathbf{S} = \mathbf{M}' < \widetilde{\mathbf{P}}$.

We denote the unipotent radical of $\mathbf{Q}$ by $\mathbf{R}$ and note that it has no proper, nontrivial $\mathbf{S}$-normalized subgroups. Indeed, this follows from the transitivity of the action of $\mathrm{PO}(n+1,\mathbb{C})$ on $\mathbb{P}^{n-2}(\mathbb{C})$. As $\mathbf{S} < \widetilde{\mathbf{P}}$ and $\mathbf{R}$ consists of all unipotent elements of $\mathbf{Q}$, we get that $\mathbf{J}^0 \lneq \widetilde{\mathbf{U}}^0$ are unipotent subgroups of $\mathbf{R}$ that are normalized by $\mathbf{S}$. We conclude that $\mathbf{J}^0$ is trivial and $\widetilde{\mathbf{U}}^0 = \mathbf{R}$.

As $\widetilde{\mathbf{P}} \leq \mathbf{Q}$ contains both $\mathbf{R}$ and $\mathbf{S}$, we see that $\widetilde{\mathbf{P}} = \mathbf{Q}$. As $\mathbf{J}^0$ is trivial, we obtain that $\mathbf{J}$ is a finite normal subgroup of $\mathbf{Q}$. However, $\mathbf{Q}$ is a parabolic subgroup of the adjoint group $\mathbf{H}$, and hence it contains no nontrivial finite normal subgroup. This implies that $\mathbf{J}$ is trivial, which gives the desired contradiction to the assumption $n \geq 4$.

We thus have $n = 3$. That is, we have

$$\mathbf{H} = \mathrm{PO}(4,\mathbb{C}) \simeq \mathrm{PGL}(2,\mathbb{C}) \times \mathrm{PGL}(2,\mathbb{C}).$$

We will assume $\mathbf{U}' \lneq \mathbf{U}$ and derive a contradiction. By almost injectivity of τ, $\mathbf{U}'$ is a nontrivial unipotent subgroup, and it follows that $\mathbf{U} \leq \mathbf{N}/\mathbf{J}$ is at least two dimensional. We thus can find a two dimensional unipotent subgroup $\mathbf{V} \leq \mathbf{N}$. Note that $\mathbf{H}$ has no three dimensional unipotent subgroup. It follows that $\mathbf{J}$ has no nontrivial unipotent subgroup, thus $\mathbf{V}$ is the unipotent radical of $\mathbf{VJ}$. As both $\mathbf{V}$ and $\mathbf{J}$ are normal in $\mathbf{VJ}$ and they have trivial intersection, it follows that they commute. We note that all two dimensional unipotent subgroups of $\mathbf{H}$ are conjugate and these are all unipotent radicals of Borel subgroups. It follows that $\mathbf{N}_{\mathbf{H}}(\mathbf{V})$ is a Borel subgroup $\mathbf{B} < \mathbf{H}$. As $\mathbf{J}$ normalizes $\mathbf{V}$, $\mathbf{J} < \mathbf{B}$. Up to conjugation, we may assume that $\mathbf{B}$ is the standard Borel subgroup of $\mathrm{PGL}(2,\mathbb{C}) \times \mathrm{PGL}(2,\mathbb{C})$, and it is easy to check that its unipotent radical is its own centralizer. It follows that $\mathbf{J} < \mathbf{V}$. This forces $\mathbf{J}$ to be trivial, as it has no unipotent subgroup. This gives the desired contradiction and thus finishes the proof. □

Remark 3.6. Note that $\mathrm{PO}(n+1,\mathbb{C})$ can also be viewed as an algebraic group over $\mathbb{R}$ and for $k = \mathbb{R}$, it is *not* compatible with $G = \mathrm{SO}_0(n,1)$. In the proof of arithmeticity, a Galois conjugate isomorphic to $\mathrm{PO}(n+1,\mathbb{C})$ is naturally given the real Zariski topology, but for the purposes of proving Theorem 1.6 we may instead consider it in the complex Zariski topology.

4. Proof of Theorem 1.6

Throughout this section, we assume that we have a W-invariant measure ν on the bundle $(G \times \mathbb{P}(V))/\Gamma$ that projects to Haar measure on G/Γ.

4.1. *From measures to measurable maps to varieties.* This subsection converts our W-invariant measure into a measurable Γ-equivariant map between varieties.

PROPOSITION 4.1. *Under the hypotheses of Theorem 1.6, there exist a proper k-algebraic subgroup $\mathbf{L} < \mathbf{H}$ and a measurable W-invariant, Γ-equivariant map $\phi : G \to \mathbf{H}/\mathbf{L}(k)$. We can also view ϕ as a measurable Γ-equivariant map from $W\backslash G$ to $\mathbf{H}/\mathbf{L}(k)$.*

Proof. Via disintegration, the W-invariant measure ν on $(G \times \mathbb{P}(V))/\Gamma$ yields a W-invariant, Γ-equivariant measurable map

$$\tilde{\phi} : G \to \mathcal{P}(\mathbb{P}(V)),$$

where $\mathcal{P}(\mathbb{P}(V))$ is the space of probability measures on $\mathbb{P}(V)$. By [41, Cor. 3.2.12 and Thm. 3.2.4], the image of this map lies in a single $\mathbf{H}(k)$-orbit that can be identified with $\mathbf{H}(k)/\tilde{L}$ for $\tilde{L}$ a compact extension of the k-points of a k-algebraic subgroup of $\mathbf{H}(k)$. Therefore we obatin a Γ-equivariant map $W\backslash G \to \mathbf{H}(k)/\tilde{L}$.

We claim that $\tilde{L}$ is not compact. If it were, we could find an $\mathbf{H}(k)$-invariant metric on $\mathbf{H}(k)/\tilde{L}$, but by [3, Cor. 6.7] the action of Γ on $W\backslash G$ is metrically ergodic (see [3, Def. 6.5] for the definition) and thus the map $W\backslash G \to \mathbf{H}(k)/\tilde{L}$ would be essentially constant with Γ-invariant image. This would contradict the assumption that $\rho : \Gamma \to \mathbf{H}(k)$ is unbounded, hence $\tilde{L}$ cannot be compact.

Let $\mathbf{L}$ be the Zariski closure of $\tilde{L}$. Then [41, Prop. 3.2.15] implies that $\mathbf{L}$ is a proper k-subgroup of $\mathbf{H}$. We are then done by composing $G \to \mathbf{H}(k)/\tilde{L}$ with the natural map $\mathbf{H}(k)/\tilde{L} \to \mathbf{H}/\mathbf{L}(k)$. □

Remark 4.2. One can also prove the group $\tilde{L}$ is noncompact by showing nontriviality of the Lyapunov spectrum of the W-action on $(G\times\mathbb{P}(V))/\Gamma$ using [24, Thm. V.5.15]. This is delicate when $\Gamma < G$ is nonuniform, relying on its weak cocompactness and integrability of the standard cocycle $\alpha : G\times G/\Gamma \to \Gamma$.

Note that the subgroup $\mathbf{L}$ of $\mathbf{H}$ might be a normal (even trivial) subgroup, or, when $\mathbf{H}$ is semisimple but not simple, it might consist of a nontrivial factor group. In the latter case the $\mathbf{H}$ action on $\mathbf{H}/\mathbf{L}$ is not effective. However, these caveats do not effect our proof.

4.2. *Algebraic representations.* In this subsection, we introduce the ideas from the work of Bader and Furman [1] used in the proof of our superrigidity theorem.

Let k be a local field, fix a k-algebraic group $\mathbf{H}$, and let $H = \mathbf{H}(k)$ denote the k-points of $\mathbf{H}$. To start, let G be a locally compact second countable group, $\Gamma < G$ be a lattice, and $\rho : \Gamma \to H$ be a Zariski dense representation.

Given a closed subgroup $T < G$, a *T-algebraic representation of G* consists of the following:

- a k-algebraic group $\mathbf{I}$;
- a k-$(\mathbf{H} \times \mathbf{I})$-algebraic variety $\mathbf{V}$, which is considered as a left $\mathbf{H}$-space and a right $\mathbf{I}$-space on which the $\mathbf{I}$-action is faithful;
- a Zariski dense homomorphism $\tau : T \to \mathbf{I}(k)$;
- an *algebraic representation* of G on $\mathbf{V}$, i.e., an almost-everywhere defined measurable map $\phi : G \to \mathbf{V}(k)$ such that

$$\phi(tg\gamma^{-1}) = \rho(\gamma)\phi(g)\tau(t)^{-1}$$

for every $\gamma \in \Gamma$, every $t \in T$, and almost every $g \in G$.

We denote the data for a T-algebraic representation of G by $\mathbf{I}_{\mathbf{V}}$, $\tau_{\mathbf{V}}$, and $\phi_{\mathbf{V}}$.

A T-algebraic representation is called *coset T-algebraic* when $\mathbf{V}$ is the coset space $\mathbf{H}/\mathbf{J}$ for some k-algebraic subgroup $\mathbf{J}$ of $\mathbf{H}$, and $\mathbf{I}$ is a k-subgroup of $N_{\mathbf{H}}(\mathbf{J})/\mathbf{J}$, where $N_{\mathbf{H}}(\mathbf{J})$ denotes the normalizer of $\mathbf{J}$ in $\mathbf{H}$. Given another T-algebraic representation $\mathbf{U}$, let $\mathbf{I}_{\mathbf{U},\mathbf{V}}$ be the Zariski closure of $(\tau_{\mathbf{U}} \times \tau_{\mathbf{V}})(T)$ in $\mathbf{I}_{\mathbf{U}} \times \mathbf{I}_{\mathbf{V}}$. Then a *morphism* $\pi : \mathbf{U} \to \mathbf{V}$ is an $(\mathbf{H} \times \mathbf{I}_{\mathbf{U},\mathbf{V}})$-equivariant k-regular morphism such that $\phi_{\mathbf{V}}$ agrees almost everywhere with $\pi \circ \phi_{\mathbf{U}}$. Recall that an *initial object* in a category is an object that has exactly one morphism to all other objects in the category. The proof of our superrigidity theorem uses the following.

THEOREM 4.3 ([1, Thm. 4.3]). *The collection of T-algebraic representations of G forms a category. If the T-action on G/Γ is weakly mixing, then this category has an initial object and this initial object is a coset T-algebraic representation.*

An initial object is characterized by the fact that $\mathbf{J}$ is the minimal subgroup, up to conjugacy, that can arise as a stabilizer in any coset T-algebraic representation in the category.

Though not stated explicitly, the following is also implicit in [1]. Given two subgroups S and T of G, we say that their initial objects $\phi_S : G \to \mathbf{V}(k)$ and $\phi_T : G \to \mathbf{W}(k)$ *have the same map* if $\mathbf{V} = \mathbf{W}$ as k-varieties and if ϕ_S and ϕ_T agree away from a set of measure zero.

LEMMA 4.4. *Assume that the action of T on G/Γ is weakly mixing. Then initial objects for T and $N_G(T)$ have the same map. Moreover, the initial object for T and for the iterated normalizer $N_G(N_G(\cdots(N_G(T))\cdots))$ have the same map.*

Proof. For the first claim, the forward direction is the content of [1, Thm. 4.6]. For the backward direction, if $\phi : G \to \mathbf{H}/\mathbf{J}(k)$ is an initial object in the category of $N_G(T)$-algebraic representations with associated homomorphism τ from T to $N_{\mathbf{H}}(\mathbf{J})/\mathbf{J}(k)$, then ϕ and $\tau|_T$ form a T-algebraic representation and this representation must be initial by minimality. Indeed, otherwise another application of the forward direction contradicts minimality of $\mathbf{J}$. The second claim follows immediately from the first. □

4.3. *From measurable maps to extension of homomorphisms.* We now complete the proof of Theorem 1.6. More specifically, we show that the existence of the map $\phi : G \to \mathbf{H}/\mathbf{L}(k)$ from Proposition 4.1 implies that the representation ρ of Γ extends to G.

Proof of Theorem 1.6. Observe that the action of G on G/Γ is mixing by the Howe–Moore theorem. In particular, the action of each noncompact subgroup of G is weakly mixing on G/Γ. This allows us to freely apply the discussion and results of Section 4.2 regarding T-algebraic representations of G for an arbitrary noncompact closed subgroup T of G.

Recall our setting from Section 2, and first consider $T = U' = U \cap W$, which is noncompact. Given an initial object in the category of U'-algebraic representations of G, Theorem 4.3 implies that there is a k-algebraic subgroup $\mathbf{J}$ of $\mathbf{H}$ such that this object is a measurable map $\Psi : G \to \mathbf{H}/\mathbf{J}(k)$ that is $(U' \times \Gamma)$-equivariant for a continuous homomorphism $\tau : U' \to N_{\mathbf{H}}(\mathbf{J})/\mathbf{J}(k)$. Since U' is normal in U and U is normal in P, Lemma 4.4 implies that τ extends to a continuous homomorphism $\tau : P \to N_{\mathbf{H}}(\mathbf{J})/\mathbf{J}(k)$ making the map Ψ an initial object in the category of P-algebraic representations of G.

We claim that $\mathbf{J}$ is trivial. Assume this is not the case. Since the pair consisting of k and $\mathbf{H}$ is compatible with G, we know that the Zariski closure of $\tau(U')$ coincides with the Zariski closure of $\tau(U)$. We note that the W-invariant map ϕ is also U'-invariant, as $U' < W$, thus it factors via $\Psi : G \to \mathbf{H}/\mathbf{J}(k)$ and via

$$G \to \mathbf{H}/\mathbf{J}(k) \to (\mathbf{H}/\mathbf{J})/\overline{\tau(U')}(k) = (\mathbf{H}/\mathbf{J})/\overline{\tau(U)}(k)$$

by U'-invariance, where $\overline{\tau(U')}$ and $\overline{\tau(U)}$ are the Zariski closures. Then, Ψ is U-equivariant, so the latter composed map is U-invariant, and it follows that ϕ is also U-invariant. Since ϕ is also W-invariant and $\langle U, W\rangle = G$, we obtain that $\phi : G \to \mathbf{H}/\mathbf{L}(k)$ is an essentially constant Γ-equivariant map, hence $\rho(\Gamma)$ has a fixed point on $\mathbf{H}/\mathbf{L}(k)$. This is impossible since $\rho(\Gamma)$ is Zariski dense in

$\mathbf{H}$ and $\mathbf{L}$ is a proper algebraic subgroup of the connected adjoint group $\mathbf{H}$. We conclude that $\mathbf{J}$ is indeed trivial.

Since $\mathbf{J}$ is trivial and $A < P$, we view τ as a morphism $\tau : P \to \mathbf{H}$ and Ψ as a $(P \times \Gamma)$-equivariant map $\Psi : G \to \mathbf{H}(k)$. In particular, Ψ is A-equivariant via the homomorphism $\tau|_A$, and therefore it must be an initial object for the category of A-algebraic representations by Lemma 4.4. Once again, Lemma 4.4 implies that $\tau|_A$ extends to a homomorphism $\tau' : N_G(A) \to \mathbf{H}(k)$ for which Ψ is $N_G(A)$-equivariant, where $N_G(A)$ is the normalizer of A in G.

Note that $N_G(A)$ contains a Weyl element w for A and thus $\langle P, N_G(A)\rangle = G$. Since Ψ is equivariant for both P and $N_G(A)$, using [1, Prop. 5.1] and following the end of the proof of [1, Thm. 1.3], we deduce that $\rho : \Gamma \to \mathbf{H}(k)$ extends to a continuous homomorphism $\widehat{\rho} : G \to \mathbf{H}(k)$. This proves the theorem. □

5. Theorem 1.4 and final remarks

In this section, we adapt the proof of Theorem 1.1 to prove Theorem 1.4, then make some final remarks and ask some questions related to our main results.

5.1. *The proof of Theorem 1.4*. Let M and N be connected, orientable hyperbolic 3-manifolds of finite volume, and suppose that N is obtained by Dehn filling on a nonempty subset of the torus cusps of M. If $\Gamma = \pi_1(M)$ and $\Lambda = \pi_1(N)$, the map $M \to N$ induced by the filling determines a surjective homomorphism $\rho : \Gamma \to \Lambda$. Since Γ and Λ are naturally lattices in $\mathrm{SO}_0(3,1)$, we can consider ρ as a homomorphism from Γ to $\mathrm{SO}_0(3,1)$ with $\rho(\Gamma)$ isomorphic to Λ. Note that ρ has nontrivial kernel.

If either of M or N is nonarithmetic, then Theorem 1.1 immediately implies that M and N contain only finitely many totally geodesic surfaces. However, Theorem 1.1 is not applicable with both M and N are arithmetic. Before giving the proof of Theorem 1.4, we give an example to show that the theorem is indeed nontrivial.

Example 5.1. Let N be the complement in S^3 of the 3-chain link, which is also called 6_1^3 in the Rolfsen tables [36]. Then N is arithmetic [21, §9.2]. Moreover, N is obtained from trivial Dehn filling on one component of the four component arithmetic link complement given in [38, Ex. 6.8.10], also known as L12n2210. See Figure 1. Using symmetries of the link diagrams, one sees that there are totally geodesic 4-punctured spheres in M that fill to become totally geodesic 3-punctured spheres in N. Therefore, the collection of totally geodesic surfaces in M that fill to a totally geodesic surface in N is nonempty.

One can easily find other examples of this nature. We now prove Theorem 1.4.

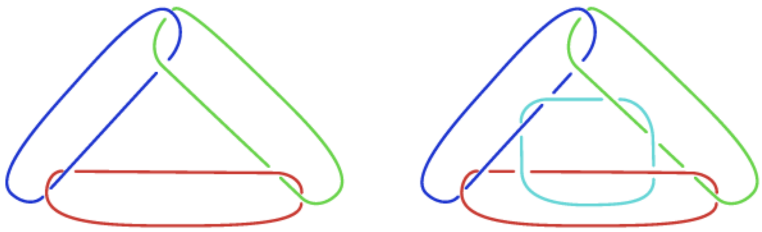

Figure 1. The arithmetic links 6_1^3 and L12n2210

Proof of Theorem 1.4. Given M and N as in the statement of the theorem, let $\rho : \Gamma \to \mathrm{SO}_0(3,1)$ be the representation defined above. We will prove that if infinitely many totally geodesic surfaces in N are the images of totally geodesic surfaces in M, then ρ must extend to a homomorphism $\mathrm{SO}_0(3,1) \to \mathrm{PO}_0(3,1)$. Since ρ has a nontrivial kernel, this is impossible. We let G denote $\mathrm{SO}_0(3,1)$ containing Γ as a lattice and H denote $\mathrm{PO}_0(3,1)$ as the target for ρ. Note that ρ has unbounded and Zariski dense image.

To apply Theorem 1.6, we take $W = \mathrm{SO}_0(2,1)$ and then must produce an H-representation V and a W-invariant measure ν on the bundle $(G \times \mathbb{P}(V))/\Gamma$ that projects to Haar measure on G/Γ. Then Theorem 1.6 implies that ρ extends to a representation of G, which gives the desired contradiction.

As in the proof of Theorem 1.1, we produce this measure by finding an invariant line bundle over each closed W-orbit in G/Γ. Let V be a nontrivial, faithful, irreducible summand of the third exterior power of $\mathfrak{so}(3,1)$ with the adjoint action of H. Let $\{\Delta_i\}$ be Fuchsian subgroups of Γ associated with totally geodesic surfaces of M that remain totally geodesic under Dehn filling. Then $\rho(\Delta_i)$ is a Fuchsian subgroup of $\Lambda = \rho(\Gamma)$, and hence it is contained in a subgroup W_i of H conjugate to the standard embedding of $\mathrm{Isom}^+(\mathbb{H}^2)$ in $\mathrm{SO}_0(3,1)$. Moreover, W_i stabilizes a line in V under the adjoint action, and the construction of ν proceeds exactly as in the proof of Proposition 3.4. Thus Theorem 1.6 applies and the proof is complete. □

Remark 5.2. We also note that it is frequently the case that a π_1-injective surface in a 3-manifold remains π_1-injective under Dehn filling (e.g., see [10]). Therefore, infinitely many totally geodesic surfaces in M may descend to π_1-injective surfaces in N. Our results say that these surfaces are very rarely totally geodesic.

5.2. *Final remarks and questions.* We begin by noting that every known construction of a nonarithmetic hyperbolic n-manifold for $n \geq 4$ contains a totally geodesic hypersurface. Theorem 1.1 implies that the set of such hypersurfaces is always finite.

QUESTION 5.3. *For each $n \geq 4$ and $1 \leq k < n-1$, does there exist a nonarithmetic hyperbolic n-manifold for which the set of totally geodesic subspaces of codimension k is empty?*

Answering this question in the positive will require a genuinely new construction of hyperbolic manifolds. Perhaps more tractable is

QUESTION 5.4. *For each $m \geq 1$, is there a hyperbolic 3-manifold containing exactly m totally geodesic surfaces?*[1]

Finally, we ask about asymptotic properties of our results:

QUESTION 5.5. *Let $H_{n,m}(v)$ be the number of lattices $\Gamma < \mathrm{SO}_0(n,1)$ such that $\mathbb{H}^n/\Gamma$ contains exactly m totally geodesic hypersurfaces and $\mathrm{vol}(\mathbb{H}^n/\Gamma) < v$. What is the growth type of $H_{n,m}(v)$ as a function of v?*

Remark 5.6. In part (3) of Corollary 1.3, we note that having infinitely many totally geodesic hypersurfaces gives a geometric characterization of arithmeticity in even dimensions. For $n \neq 3, 7$ odd, there is a similar statement. In this case, every arithmetic hyperbolic manifold contains maximal totally geodesic submanifolds of codimension 1 or 2 (see [29]), hence having infinitely many such submanifolds again characterizes arithmeticity. There are arithmetic and nonarithmetic hyperbolic 3-manifolds that contain no totally geodesic surfaces, so such a characterization is not possible; see [14, §6.1] for discussion and examples. For $n = 7$, one must classify the geodesic submanifolds of the arithmetic manifolds arising from triality; for those arithmetic manifolds not arising from that construction, the situation is the same as for other odd dimensions greater than 3.

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