Journal of Mathematical Physics

The Dirichlet isospectral problem for trapezoids

Cite as: J. Math. Phys. **62**, 051511 (2021); https://doi.org/10.1063/5.0036384
Submitted: 04 November 2020 . Accepted: 26 April 2021 . Published Online: 24 May 2021

Hamid Hezari, Z. Lu, and J. Rowlett

View Online Export Citation CrossMark

ARTICLES YOU MAY BE INTERESTED IN

Faber–Krahn inequalities for Schrödinger operators with point and with Coulomb interactions
Journal of Mathematical Physics **62**, 012105 (2021); https://doi.org/10.1063/5.0014360

Polynomial tau-functions of the KP, BKP, and the s-component KP hierarchies
Journal of Mathematical Physics **62**, 021702 (2021); https://doi.org/10.1063/5.0013017

Particle–hole symmetries in condensed matter
Journal of Mathematical Physics **62**, 021101 (2021); https://doi.org/10.1063/5.0035358

Sign up for Alerts
Journal of Mathematical Physics
Receive the latest research updates

AIP Publishing

© 2021 Author(s).

Journal of Mathematical Physics ARTICLE scitation.org/journal/jmp

The Dirichlet isospectral problem for trapezoids

Cite as: J. Math. Phys. 62, 051511 (2021); doi: 10.1063/5.0036384
Submitted: 4 November 2020 • Accepted: 26 April 2021 •
Published Online: 24 May 2021

View Online Export Citation CrossMark

Hamid Hezari,[1,a)] Z. Lu,[1,b)] and J. Rowlett[2,c)]

AFFILIATIONS

[1] Department of Mathematics, University of California, 510J Rowland Hall, Irvine, California 92697-3875, USA
[2] Mathematics Department, Chalmers University and the University of Gothenburg, 41296 Gothenburg, Sweden

[a)] Electronic mail: hezari@uci.edu
[b)] Electronic mail: zlu@uci.edu
[c)] Author to whom correspondence should be addressed: julie.rowlett@chalmers.se

ABSTRACT

We show that non-obtuse trapezoids are uniquely determined by their Dirichlet Laplace spectrum. This extends our previous result [Hezari *et al.*, Ann. Henri Poincare **18**(12), 3759–3792 (2017)], which was only concerned with the Neumann Laplace spectrum.

© 2021 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/). https://doi.org/10.1063/5.0036384

I. INTRODUCTION

Kac popularized the isospectral problem for planar domains with a paper[17] titled "Can one hear the shape of a drum?" For a bounded, connected domain Ω in $\mathbb{R}^2$, we define $\Delta_\Omega^{\mathcal{B}}$ to be the Laplace operator on Ω with the boundary condition $\mathcal{B}$, where $\mathcal{B}$ is either Dirichlet or Neumann. We consider the Laplace eigenvalue equation

$$\Delta_\Omega^{\mathcal{B}} u = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = \lambda u, \quad \mathcal{B}(u) = 0 \quad \text{on the boundary of } \Omega.$$

The eigenvalues form a discrete subset of $[0,\infty)$, $0 \le \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$. If one takes the Dirichlet boundary condition, requiring the function u to vanish at the boundary, then the set of eigenvalues, known as the spectrum of Ω, is in bijection with the resonant frequencies a drum would produce if Ω were its drumhead. With a perfect ear, one could hear all these frequencies and therefore know the spectrum. Kac's question mathematically means the following: *If two such domains are isospectral, then are they isometric?* Gordon, Webb, and Wolpert answered Kac's question in the negative[6,7] (see also Ref. 3 for an accessible presentation). All the known counterexamples to date consist of non-convex polygons.

On the other hand, in certain settings, this isospectral question can have a positive answer. There are many types of positive results. One question is whether any domain is spectrally unique (up to rigid motions) among a very large class of domains. In this direction, Kac proved that disks can be heard among all domains. He used the heat trace invariants to prove that the area and perimeter of a domain are determined by its spectrum, so by the isoperimetric inequality, disks are spectrally determined. Watanabe[29] proved that there are certain nearly circular oval domains that are spectrally unique. Recently, Hezari and Zelditch[13] showed that one can hear the shape of nearly circular ellipses among all smooth domains. A weaker inverse spectral problem is to find domains that are locally spectrally unique, meaning that they can be heard among nearby domains in a certain topology. Marvizi and Melrose[21] constructed a two-parameter family of planar domains that are locally spectrally unique in the C^∞ topology. The two-parameter family consists of domains that are defined by elliptic integrals and that resemble ellipses but are not ellipses. For more on positive inverse spectral problems, we refer the readers to the surveys in Refs. 4 and 32.

The notion of spectral rigidity of a domain Ω is even weaker than local spectral uniqueness. It means that any one-parameter family of isospectral domains containing Ω and staying within a limited class must be trivial, i.e., made out of rigid motions. In this setting, Popov and Topalov[24] recently showed that ellipses are spectrally rigid within the class of analytic domains with the two axial symmetries of an ellipse. In a recent article,[13] using a length spectral rigidity theorem of de Simoi, Kaloshin, and Wei,[27] Hezari and Zeldtich proved that nearly circular

© Author(s) 2021

domains with one axis of symmetry are spectrally rigid among such domains. The final setting is infinitesimal spectral rigidity of a given domain Ω, which requires that the first variation of any non-trivial one-parameter family of isospectral domains containing Ω and staying within a class vanishes. It was proved in Ref. 12 that ellipses are infinitesimally spectral rigid among smooth domains with the axial symmetries of an ellipse.

Another interesting setting, into which our result fits, is when one tries to show that the Laplace spectrum map is one-to-one in a relatively small class of domains. The class of domains is either infinite dimensional, in which case normally a generic property is added to simplify an otherwise difficult problem, or finite dimensional where no genericity assumption is imposed. In the former setting, Zelditch[31] proved that generic analytic domains with an axial symmetry are spectrally distinguishable from each other. The few inverse problems to date that consider a finite dimensional class of planar domains are about polygonal domains. The first result of this type is due to Durso,[5] who proved that the shape of a triangle can be heard among other triangles. Using the spectral invariants obtained from the short time asymptotic expansion of the heat trace, any two triangles that are isospectral must have the same area and perimeter. Since triangles depend on three independent parameters, to obtain her result, Durso used another spectral invariant, namely, the wave trace. She demonstrated that the length of the shortest closed geodesic in a triangular domain is also a spectral invariant. More recently, Grieser and Maronna[8] realized that if one used an additional spectral invariant from the heat trace, then this together with the area and perimeter uniquely determines the triangle; that is a much simpler proof.

After triangles, it is natural to consider quadrilaterals. For rectangles, it is straightforward to prove that if two rectangles are isospectral, then they are congruent. In fact, one only requires the first two Dirichlet eigenvalues to prove this. For parallelograms, it is also a straightforward argument using the first three heat trace invariants as in Ref. 20 to prove that isospectral parallelograms are congruent. The next natural generalization is to trapezoids. In this case, one can prove that the geometric information that can be extracted from the heat trace is insufficient to prove that isospectral trapezoids are congruent. It is therefore necessary to use the wave trace in the spirit of Ref. 5, which is a much more delicate matter. The wave trace is a tempered distribution that is a spectral invariant. To use the wave trace in an isospectral problem, one studies the times at which the wave trace is singular. For smoothly bounded domains, Ref. 9 showed that the set of positive times at which the wave trace is singular is contained in the set of lengths of closed geodesics; this is known as *the Poisson relation*. Once the boundary is no longer smooth, this relation is only known to hold in certain geometric settings. Here, we rely on Refs. 15 and 30 to obtain the Poisson relation. However, the Poisson relation is only a containment. To be able to state that the length of a certain closed geodesic is a spectral invariant, one must study the singularity in the wave trace at time equal to that length. Hence, exploiting this technique requires not only careful study of the wave trace but also detailed information on the closed geodesics in the domain.

The study of closed geodesics in polygonal domains has quite a long history, which to the best of our knowledge was initiated by Fagnano in 1775. Fagnano proved that the orthic triangle (also called the Fagnano triangle) is the shortest closed geodesic inside an acute triangle. Two centuries passed before Schwartz demonstrated the existence of closed geodesics in certain obtuse triangular domains.[25,26] We refer to the survey article of Gutkin[11] for what is known about the existence and distribution of closed geodesics in polygonal domains.

Our main result is the following theorem:

Theorem 1. *Let $T_1, T_2 \subset \mathbb{R}^2$ be two non-obtuse trapezoidal domains. Then, if the spectra of the Euclidean Laplacian with the Dirichlet boundary condition on T_1 and T_2 coincide, the trapezoids are congruent, that is, equivalent up to a rigid motion of the plane.*

In Ref. 14, we proved this result when the Neumann boundary condition is considered. One of the key elements of our proof was the use of the singularity in the wave trace that is produced by the orbit $2b$, the orbit that bounces between the top two vertices of the trapezoid (see Fig. 4). In the Neumann case, we were able to compute and use the leading term of the singularity expansion of the wave trace at $t = 2b$. In the Dirichlet case, this singularity has a lower order, and the computation of its leading term is much more complicated. In this paper, we avoid doing such computations and accomplish the Dirichlet case by carefully studying other periodic orbits. In fact, our proof includes the Neumann case as well and is not special to the Dirichlet case. Our proof still uses several key results from Ref. 14 on the wave trace singularity expansions associated with certain diffractive orbits.

A. Organization of this paper

In Sec. II, we present the heat trace invariants and their use and limitations in the determination of a trapezoid. Section III introduces the wave trace and the Poisson relation for polygons. We also define the order of a wave trace singularity. In Sec. IV, we specialize to the case of a trapezoid and study the important periodic orbits that we need for our argument, together with a thorough analysis of their singularity contribution. The proof of our main theorem is given in Sec. V.

II. HEAT TRACE INVARIANTS OF TRAPEZOIDS

In this section, we present a small collection of geometric spectral invariants that can be obtained through the asymptotic behavior of the heat trace as $t \downarrow 0$. For the sake of completeness and to set the notation, we define the parameters of a trapezoid.

Definition 1. A trapezoid is a convex quadrilateral that has two parallel sides of lengths b and B with $B \geq b$. The side of length B is called the base. The two angles α and β adjacent to the base are called base angles. A trapezoid is called non-obtuse if the base angles satisfy

© Author(s) 2021

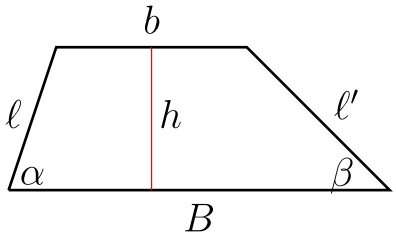

FIG. 1. A non-obtuse trapezoid and its parameters.

$$0 < \beta \le \alpha \le \frac{\pi}{2}.$$

We make this assumption throughout this paper. If $\alpha = \beta$, then we say the trapezoid is isosceles. The other two sides of the trapezoid are known as legs *of lengths ℓ and ℓ', respectively. The distance between two parallel sides is called the height. See Fig. 1 for a picture of a trapezoid.*

We consider a trapezoid T as in Fig. 1. We use b, B, ℓ, and ℓ' to denote the lengths of the shorter and longer parallel sides, respectively, and the lengths of the two legs of the trapezoid. Abusing notation, we may also use these to denote the corresponding edges.

Definition 2. Any quantity that is uniquely determined by the spectrum is known as a spectral invariant. Colloquially and in the spirit of Ref. 17, we say that "X can be heard" if the quantity X is a spectral invariant.

A. The heat trace

Let $\{\lambda_k\}_{k\ge1}$ denote the eigenvalues. We define

$$\operatorname{Tr} e^{-t\Delta_\Omega^{\mathcal{B}}} = \sum_{k\ge1} e^{-t\lambda_k}$$

to be the trace of the heat kernel, which is the Schwartz kernel of the fundamental solution to the heat equation. The heat trace, which is a spectral invariant, is an analytic function for $\Re(t) > 0$. It is well known in this setting (see Refs. 2, 17, and 22) that the heat trace on a polygonal domain Ω admits an asymptotic expansion as $t \downarrow 0$,

$$\operatorname{Tr} e^{-t\Delta_\Omega^{\mathcal{B}}} = \frac{|\Omega|}{4\pi t} \pm \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \sum_{k=1}^{n} \frac{\pi^2 - \theta_k^2}{24\pi\theta_k} + O(e^{-c/t}), \quad t \downarrow 0. \tag{1}$$

Above, $|\Omega|$ and $|\partial\Omega|$ denote, respectively, the area and perimeter of the domain Ω, and θ_k are the interior angles. In the second term, we choose the minus sign when $\mathcal{B}$ = Diriclet and the plus sign when $\mathcal{B}$ = Neumann. The constant $c > 0$ has been estimated in Ref. 2. Since the angles of a trapezoid are α, $\pi - \alpha$, β, and $\pi - \beta$, we therefore have the following proposition:

Proposition 1. For a trapezoidal domain, the area A, perimeter L, and the angle invariant q defined by

$$q := F(\alpha) + F(\beta), \quad F(x) := \frac{1}{x(\pi - x)} \tag{2}$$

are spectral invariants.

Proposition 2. The spectrum determines whether or not a trapezoid is a rectangle. If one trapezoid is a rectangle and is isospectral to another trapezoid, then that trapezoid is also a rectangle, and the two rectangles are congruent.

Proof. Note that if we rewrite the angle invariant as

$$q = \frac{1}{\pi}\left(\frac{1}{\alpha} + \frac{1}{\pi - \alpha} + \frac{1}{\beta} + \frac{1}{\pi - \beta}\right),$$

then by the "arithmetic mean–harmonic mean" inequality, we have

© Author(s) 2021

$$q \geq \frac{8}{\pi^2},$$

and the equality holds if and only if the trapezoid is actually a rectangle. If two trapezoids are isospectral, then they have the same value of q. Hence, either they are both rectangles or neither is a rectangle. If they are both rectangles and they are isospectral, then the dimensions of the rectangles can be obtained by the first two eigenvalues. These uniquely determine the rectangle up to congruency, that is, up to rigid motions of the plane. □

Since the moduli space of trapezoids is four dimensional, the three heat trace invariants introduced above cannot determine the shape of the trapezoids. To extract more information from the spectrum, we turn to the wave trace.

III. POISSON RELATION AND SINGULARITIES OF THE WAVE TRACE FOR POLYGONS

In this section, we study the singularities of the wave trace of the Laplacian on a polygon. The wave trace is the trace of the wave propagator, also known as the trace of the wave group, and can be written as

$$w_\Omega^{\mathcal{B}}(t) := \operatorname{Tr} e^{it\sqrt{\Delta_\Omega^{\mathcal{B}}}} = \sum_{k\geq 1} e^{it\sqrt{\lambda_k}}.$$

The wave trace is only well-defined when paired with a Schwartz class test function; it is a tempered distribution by Weyl's law. The connection between the wave trace and geodesic trajectories comes from the fact that the singularities of the wave operator propagate along geodesic trajectories. For smoothly bounded domains, the time at which the wave trace is singular is contained in the set of lengths of generalized broken periodic geodesics[23] (Theorem 5.4.6); see also Refs. 1 and 9. Propagation of singularities of the wave operator in a polygonal domain is more difficult to study because of diffraction phenomena that may occur at the vertices.

One way to study the wave trace on a polygonal domain is to double it and create an associated *Euclidean surface with conical singularities*, or ESCS as in Ref. 16. An ESCS is a compact manifold with finitely many conical singularities that is locally flat away from the conical points, and near the conical points, it is isometric to a neighborhood of the vertex of a Euclidean cone. It was shown separately by Hillairet[15] and Wunsch[30] that the positive singular support of the wave trace for the Friedrichs extension of the Laplacian on an ESCS is contained in the set of lengths of periodic geodesics on the ESCS. On an ESCS, the conical points are separated into two groups. A conical point on an ESCS is *non-diffractive* if its angle is equal to $\frac{2\pi}{n}$ for some positive integer, n; otherwise, it is called *diffractive*.

Definition 3. A closed geodesic in a polygonal domain is a geodesic trajectory that forms a closed, piecewise linear curve that bounces off the edges according to the equal angle law. We say that a closed geodesic is conical if it meets at least one vertex. When a geodesic trajectory meets a vertex, if the interior angle at a vertex is of the form $\frac{\pi}{N}$ for any positive integer $N > 1$, then noting that the upper half space is an N-fold covering, the angle at which the trajectory leaves that vertex is computed according to the equal angle law on the upper half space. In particular, if the angle is α, then when N is even, the trajectory returns along its incoming path; if N is odd, the outgoing trajectory is at an angle $\frac{\pi}{N} - \alpha$. However, when the interior angle at a corner is not of the form $\frac{\pi}{N}$ for any positive integer $N > 1$, Keller's democratic law of diffraction states that a billiard trajectory that hits the corner departs that corner in every direction.[19] Such corners are called diffractive. The other corners are non-diffractive. Similarly, geodesics are classified as diffractive if they meet at least one diffractive corner; otherwise, they are non-diffractive.

By our definitions, all non-conical closed geodesics are non-diffractive geodesics. However, because non-diffractive geodesics may pass through vertices with angles of the form $\frac{\pi}{N}$, not all non-diffractive closed geodesics are non-conical.

For a polygonal domain $\Omega \subset \mathbb{R}^2$, define

$$\mathcal{L}sp(\Omega) := \{\text{lengths of closed diffractive or non} - \text{diffractive geodesics in } \Omega\}.$$

Let

$$\operatorname{SingSupp} w_\Omega^{\mathcal{B}}(t) := \text{the singular support of } w_\Omega^{\mathcal{B}}(t).$$

The utility of the length spectrum follows from the Poisson relation.

Theorem 2 (Poisson relation for polygons Refs. 5, 15, and 30). *For a polygonal domain Ω, we have*

$$\operatorname{SingSupp} w_\Omega^{\mathcal{B}}(t) \subset \{0\} \cup \pm\mathcal{L}sp(\Omega).$$

This holds for both the Dirichlet and Neumann boundary conditions.

© Author(s) 2021

By a compactness argument, one can prove the following lemma:

Lemma 1. There are no accumulation points in $\mathcal{L}sp(\Omega)$ for any polynomial Ω.

In other words, the length spectrum of a polygon is a discrete set in $[0, \infty)$. However, this lemma also follows immediately from a much stronger result of Katok.

Theorem 3 (Ref. 18). *The counting function of the lengths of geodesics starting and ending at vertices of a polygonal domain is of sub-exponential growth.*

By the Poisson relation together with Lemma 1, the singularities of the wave trace are discrete. Consequently, we can enumerate the singularities and, for example, speak about the shortest or the second shortest positive singularity if they exist. It is also possible to find test functions that are supported in a neighborhood of one and only one singular time. This allows us to define the order of a singularity.

Definition 4. Suppose $t_0 > 0$ is in the singular support of $w_\Omega^{\mathcal{B}}(t)$. Let $\hat{\rho}(t)$ be a cutoff function supported in a neighborhood of t_0 such that $\hat{\rho} \equiv 1$ near t_0. Assume that

$$\operatorname{Supp} \hat{\rho} \cap \operatorname{SingSupp} w_\Omega^{\mathcal{B}} = \{t_0\}.$$

We define the frequency domain contribution of the singularity t_0 *by*

$$I_{t_0,\mathcal{B}}(k) := \int_{\mathbb{R}} \hat{\rho}(t) e^{-ikt} \operatorname{Tr} e^{it\sqrt{\Delta_\Omega^{\mathcal{B}}}} dt.$$

We say that a singularity t_0 is of order $a \in \mathbb{R}$ if

$$I_{t_0,\mathcal{B}}(k) = ck^a + o(k^a), \quad k \to \infty.$$

Above, c is a constant that depends on the microlocal germ of the domain near the closed geodesics of length t_0. We say t_0 is at most of order a if

$$I_{t_0,\mathcal{B}}(k) = O(k^a).$$

Remark 1. We note that near a singularity at $t = t_0$ of order a as defined above, the wave trace belongs to $H^{-s}(\mathbb{R})$ for all $s > a + \frac{1}{2}$ but does not belong to $H^{-s}(\mathbb{R})$ for $s = a + \frac{1}{2}$. Moreover, the notions in the preceding definition are independent of the choice of $\hat{\rho}$.

To use the Poisson relation, we investigate the shortest closed geodesics in trapezoids. Parallel families of closed geodesics play a central role.

IV. CLOSED GEODESICS INSIDE A TRAPEZOID AND THEIR SINGULARITY CONTRIBUTION

We start this section by recalling some standard facts about periodic orbits inside a polygon from the work of Gutkin (see Refs. 10 and 11). It is important to note that in the dynamical system literature, periodic orbits inside a polygon refer to our non-conical closed geodesics, i.e., closed geodesics that do not hit any vertices of the polygon. In addition, generalized periodic orbits refer to what we call geometric conical geodesics, which are precisely limits of non-conical closed geodesics. Diffractive periodic orbits that are not geometric (see, for example, Figs. 4 and 5) are not considered in the purely dynamical systems references but are of great interest in partial differential equations (PDE) because of their contribution to the singularities of solutions to the wave equation.

We start by defining prime periodic orbits.

Definition 5. A non-conical periodic orbit is called prime if it is not a multiple of another one.

We then have the following classification:

Proposition 3 (Gutkin Ref. 11, Corollary 1). Let g be a prime non-conical closed geodesic of period n in a polygon P. Here, period refers to the number of times the orbit meets the edges.

1. *If n is even, then g is contained in a band of parallel periodic orbits, all of the same length. Let S be the maximal connected band containing g. Then, S is a closed flat cylinder. Each of the boundary circles of S is a conical geodesic of P.*
2. *If n is odd, then the orbit g is isolated; every trajectory g' starting close to g and parallel to g comes back after n reflections to the same edge, at the same distance from g and in the same direction but on the opposite of g. The maximal connected band S of periodic orbits parallel to g^2 (the second iteration of g) and containing g^2 is a flat Möbius band, and g is the middle circle of S. The boundary circle of S is a conical geodesic.*

© Author(s) 2021

Remark 2. Given a singularity t_0 in the wave trace, there are only finitely many periodic orbits of length t_0, modulo parallel families when the impact number is even. These finitely many periodic orbits are disjoint from each other in phase space. Therefore, using microlocal cutoff operators, given a periodic orbit g, we define $I_{g,\mathcal{B}}$ (instead of $I_{t_0,\mathcal{B}}$), which now only contains the wave trace singularity contributed by g, not the other periodic orbits of length t_0 (should they exist). The order of the wave trace singularity contributed by g is then specified in Definition 4 by replacing $I_{t_0,\mathcal{B}}$ with $I_{g,\mathcal{B}}$.

This proposition makes an important distinction between the prime periodic billiard orbits of odd and even periods. The former are isolated; the latter form periodic cylinders and hence are never isolated. Cylinders of periodic orbits cause a larger singularity in the wave trace. Let us now discuss some examples.

A. Important examples of closed geodesics inside a trapezoid

Here, we list only examples of closed geodesics inside a trapezoid that are key to our argument of the main theorem. See Fig. 2. We postpone the study of their wave trace singularity contributions to Sec. IV B.

The $2h$ family. It consists of the bouncing ball orbits parallel to the height of the trapezoid. It is a cylinder of periodic orbits of order $n = 2$. The area it sweeps is $2hb$ [see Fig. 2(a)].

The Fagnano orbit (or l_F) and its double. This orbit is also called the *orthic* triangle and is the triangle that joins the feet of the altitudes of the extended triangle of the trapezoid [Figs. 2(b) and 3]. The Fagnano orbit exists only if the extended triangle is acute and if the height of the trapezoid is not too short. In fact, one can easily see that this condition on h is

$$h \geq \max\{B \sin\alpha \cos\alpha, B \sin\beta \cos\beta\} = B \sin\beta \cos\beta. \tag{3}$$

The last equality happens because the extended triangle of the trapezoid being acute requires that $\alpha + \beta \geq \frac{\pi}{2}$, which implies that $\sin 2\alpha \leq \sin 2\beta$. The length of the Fagnano orbit is given by

$$l_F = 2B \sin\alpha \sin\beta.$$

In the special case $\alpha = \frac{\pi}{2}$, the Fagnano orbit becomes degenerate and collapses into the $2h_\alpha$ orbit.

The doubled Fagnano orbit is a closed geodesic that belongs to a one-parameter family of closed geodesics forming a flat Möbius strip. We represent this family by its length, which is $2l_F$ (see Fig. 4).

The $2b$ orbit and its multiples. There is a closed geodesic that we identify with its length, $2b$, created by bouncing along the top side of the trapezoid [see Fig. 2(c)]. We also call the mth multiple of this orbit the $2mb$ orbit. These are diffractive periodic orbits and produce mild singularities in the wave trace. The larger the m, the milder the singularity. In our previous paper,[14] we took advantage of the contribution of the singularity $2b$ to prove that the Neumann spectrum determines a trapezoid among other trapezoids.

The height $2h_\alpha$. It corresponds to the height of the extended triangle of the trapezoid from the larger base angle α [see Fig. 2(d)]. The length of the orbit is given by

$$2h_\alpha = 2B \sin\beta.$$

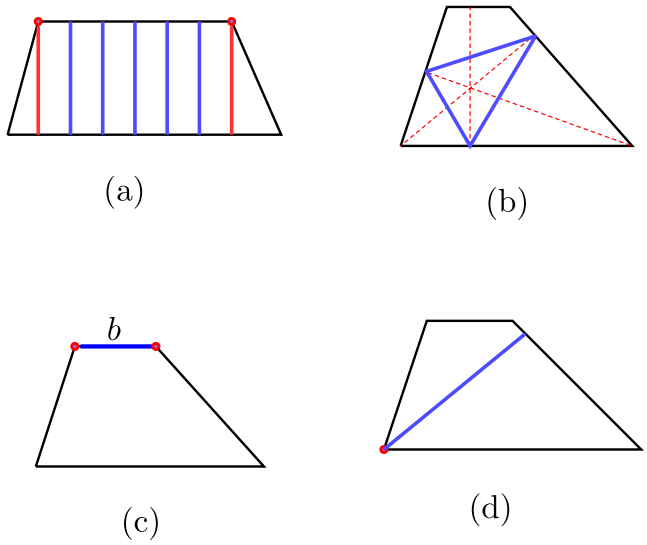

FIG. 2. (a) The $2h$ family. (b) The l_F orbit. (c) The $2b$ orbit. (d) The $2h_\alpha$ orbit.

© Author(s) 2021

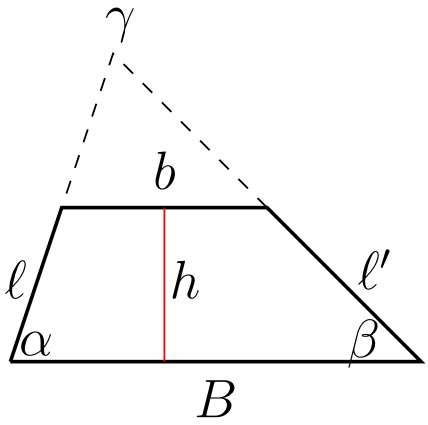

FIG. 3. A trapezoid T and its extended triangle $\widehat{T}$.

This orbit is diffractive unless $\alpha = \frac{\pi}{2}, \frac{\pi}{3}$, or $\frac{\pi}{4}$. This is because when $\alpha = \frac{\pi}{N}$, $N \geq 5$, the height h_α is not inside the extended triangle as in this case $\gamma > \pi/2$. When $\alpha = \frac{\pi}{3}$ or $\alpha = \frac{\pi}{4}$, we have $\gamma \geq \alpha \geq \beta$; therefore,

$$2h < 2h_\alpha. \tag{4}$$

In the case $\alpha = \frac{\pi}{2}$, the orbit $2h_\alpha$ is an isolated non-diffractive obit. The orbit $2h_\alpha$ is not always isolated. In fact, in an isosceles trapezoid (i.e., $\alpha = \beta$) and only in this case, this orbit belongs to a cylinder of periodic orbits. See Fig. 5 and the next definition.

The following family of closed geodesics exists in both trapezoids and triangles when the base angles satisfy a certain relationship.

Definition 6. Consider a trapezoid or a triangle when $m\alpha = n\beta \leq \pi/2$, with $m \leq n$, and m and n are co-prime positive integers. Then, the orbit $2h_\alpha$ belongs to a parallel family of periodic orbits called $C_{m,n}$ that contains $2h_\alpha$ as a boundary component. These families were introduced in Ref. 28. In particular, the family exists for $m = n = 1$, that is, for isosceles trapezoids and triangles (see Fig. 5).

B. Contributions of singularities in the wave trace

Here, we show that certain closed geodesics and families of closed geodesics contribute singularities to the wave trace, and we determine the order of these singularities. We begin with the mildest singularities. The following proposition was proved in Ref. 14 only in the case $m = 1$. We will need that the mth iterations of the $2b$ orbit produce very mild singularities and hence can be ruled out later in the proof of our main theorem.

Proposition 4. Let T be a trapezoid that is not a rectangle. Let $m \in \mathbb{N}$. Let $2mb$ denote the $2b$ orbit in phase space traversed m times; see Fig. 2(c). Then,

1. *If $\alpha \neq \frac{\pi}{2}$,*
$$I_{2mb,B}(k) = O(k^{-m}).$$
2. *If $\alpha = \frac{\pi}{2}$,*
$$I_{2mb,B}(k) = O(k^{-m/2}).$$

In other words, the order of the singularity contributed by the $2mb$ periodic orbit is at most $-\frac{m}{2}$.

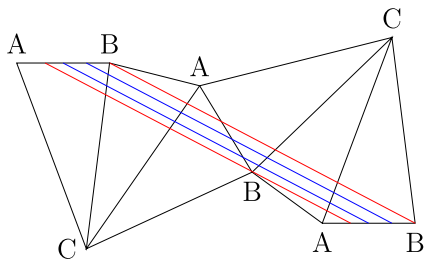

FIG. 4. The $2l_F$ family of closed geodesics, unfolded.

© Author(s) 2021

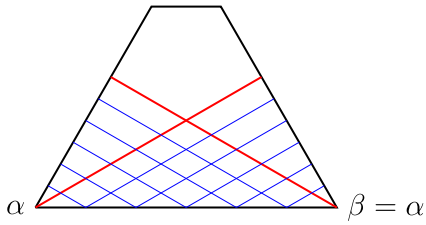

FIG. 5. Unobstructed $C_{1,1}$ family.

Proof. The proof is identical to the case $m = 1$ provided in Ref. 14 (see Sec. IV and in particular pp. 3774–3775). The only change that has to be made is that in Eq. (4.1) of Ref. 14, we have to let $n = 2m$ if $\alpha \neq \frac{\pi}{2}$ and $n = m$ if $\alpha = \frac{\pi}{2}$ and follow the same argument assuming throughout that the number of diffractions is n. □

Next, we study the singularity at time ℓ_F for the wave trace of a trapezoid in which the Fagnano triangle exists. In fact, we need a more general statement on non-conical periodic orbits with an odd number of reflections. As we discussed earlier, by Refs. 10 and 11, such periodic orbits are automatically isolated. The following proposition is a quick corollary of Ref. 9 (see Theorems 1 and 2). Our only contribution is to show that such periodic orbits are non-degenerate; that is, we show that the constant $c_0(g_0)$ in Proposition 5 is nonzero.

Proposition 5. Let T be a trapezoid (or in general a polygon). Let g_0 be a periodic non-conical geodesic of length l with an odd period. Then, as $k \to +\infty$, we have an asymptotic expansion of the form

$$I_{g_0,B}(k) = e^{-ikl}\sum_{j=0}^{\infty} c_j(g_0)k^{-j}.$$

The constant $c_0(g_0)$ is nonzero. Hence, the order of the wave trace singularity contribution from g_0 is 0.

Consequently, we have the following corollary:

Corollary 1. Let T be a trapezoid. Suppose the Fagnano triangle lies in T and is non-diffractive as in Fig. 2(b). Then, the order of the wave trace singularity from the orbit l_F at time $2B\sin\alpha\ \sin\beta$ is 0.

Proof of Proposition 5. By Ref. 9 (Theorems 1 and 2), we obtain the proposition and its corollary immediately if we can prove that such orbits are non-degenerate. To show this, we must verify that the linearized Poincaré map P_{g_0} has no eigenvalues equal to one, or equivalently, $\det(I - P_{g_0}) \neq 0$. In fact, we will prove that

$$\det(I - P_{g_0}) = 4. \tag{5}$$

The Poincaré map of a closed geodesic g_0 of period n is defined as follows. Let x_0 be a point of reflection of g_0 on an edge AB and θ_0 be the angle that g_0 makes with AB in the counterclockwise direction. Now, for (x, θ) near (x_0, θ_0), we define $f(x, \theta)$ to be the point (x', θ') in the phase space of the boundary of T that is obtained by following the trajectory g that starts at point (x, θ) and reflects precisely n times on the boundary. In other words, f is the nth iterate of the billiard map. The linearized Poincaré map P_{g_0} is the linearization (Jacobian) of f at (x_0, θ_0). To calculate P_{g_0}, we first unfold the trapezoid along the geodesic g as in Fig. 6. The top edge BA is obtained from the bottom edge AB after n reflections along the impact edges of g; note that n is odd.

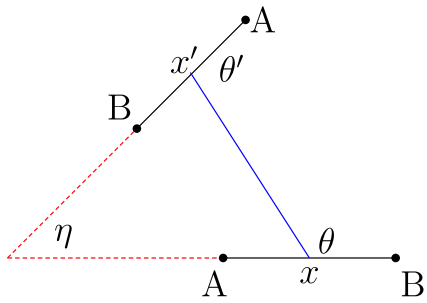

FIG. 6. The Poincaré map is the Jacobian of the map f. For a point (x, θ) in phase space, $f(x, \theta)$ is the point (x', θ') in phase space obtained by following the geodesic from the point x in the direction of θ and reflecting n times on the boundary, landing at the top edge BA corresponding to AB reflected n times.

© Author(s) 2021

It is clear from Fig. 6 that $\theta'(x,\theta) = \pi + \eta - \theta$. Since (x_0, θ_0) is a periodic orbit, we must have $\theta'(x_0,\theta_0) = \theta_0$. This shows that we must have $\pi + \eta = 2\theta_0$, and therefore,

$$\theta'(x,\theta) = \pi + \eta - \theta = 2\theta_0 - \theta.$$

One can actually calculate $x'(x,\theta)$ precisely; however, we shall only need $x'(x,\theta_0)$. Thus, for this computation we are only concerned with an orbit g that is parallel to g_0. Since by Proposition 3 (2), every trajectory g starting close to g_0 and parallel to g_0 comes back after n reflections to the same edge, at the same distance from g_0 and in the same direction but on the opposite of g_0, we obtain

$$x'(x,\theta_0) = x_0 - (x - x_0) = 2x_0 - x.$$

Hence,

$$P_{g_0} = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial \theta} \\ \frac{\partial \theta'}{\partial x} & \frac{\partial \theta'}{\partial \theta} \end{bmatrix} = \begin{bmatrix} -1 & \frac{\partial x'}{\partial \theta} \\ 0 & -1 \end{bmatrix},$$

and claim (5) follows. □

Next, we study the singularity associated with the height $2h_\alpha$. This was largely done by Durso.[5] The results of Ref. 5 concern triangles, but since the height h_α does not visit the top edge of the trapezoid, they also apply to trapezoids.

Proposition 6 (Ref. 5). Let T be a trapezoid (or a triangle) that is not isosceles. Suppose the height h_α lies inside T. Let $2h_\alpha$ denote the periodic orbit in phase space that traverses the height and returns.

1. *If α is diffractive,* i.e., $\alpha \neq \frac{\pi}{N}$, $N = 2, 3, 4$, *then*

$$I_{2h_\alpha,\mathcal{B}}(k) = c_0(h_\alpha) e^{-2ikh_\alpha} k^{-\frac{1}{2}} + O\left(k^{-\frac{3}{2}}\right),$$

   where $c_0(h_\alpha) \neq 0$. In particular, the order of the singularity contribution from the $2h_\alpha$ orbit is $-\frac{1}{2}$.
2. *If $\alpha = \frac{\pi}{2}$, then*

$$I_{2h_\alpha,\mathcal{B}}(k) = e^{-2ikh_\alpha} \sum_{j=0}^{\infty} d_j(h_\alpha) k^{-j},$$

   with $d_0(h_\alpha) \neq 0$. Thus, in this case, the order of the singularity is 0.

See Remark 4 for the isosceles case.

Remark 3. Two comments are in order. Since in the cases $\alpha = \frac{\pi}{3}, \frac{\pi}{4}$, we have $h < h_\alpha$, we will not need to investigate the singularity type of h_α. We also note that the different singularity behavior in the non-diffractive case $\alpha = \frac{\pi}{2}$ above is not surprising, as it can be understood as the limit of Fagnano orbits collapsed into a bouncing ball as $\alpha \to \frac{\pi}{2}^-$.

Finally, we investigate the contributions of the $2h$ family in arbitrary trapezoids and the $2h_\alpha$ family in isosceles trapezoids.

Proposition 7 (Ref. 14). Let T be a trapezoid. Then, the frequency domain contribution of the $2h$ family is given by

$$I_{2h,\mathcal{B}}(k) = \frac{e^{i\pi/4} e^{-2ihk}}{\sqrt{4\pi h}} A(R) k^{\frac{1}{2}} + o\left(k^{\frac{1}{2}}\right),$$

where $A(R) = hb$ is the area of the inner rectangle of T. In particular, the order of the wave trace singularity from this family is $\frac{1}{2}$.

Remark 4. The same result holds for the $C_{1,1}$ family in isosceles trapezoids, if the family is unobstructed as in Fig. 5, but h must be replaced by h_α and $A(R)$ by half of the area that the $C_{1,1}$ family sweeps. The proof is identical to the proof we provided in Ref. 14; hence, we omit it. The key point is that the geometrically diffractive orbits lying on the boundary of the $C_{1,1}$ family each go through only one diffractive corner, and hence, the result of Ref. 16 regarding such families on ESCS can be used in the poof of Ref. 14.

V. SPECTRAL UNIQUENESS OF A TRAPEZOID

Before we present the proof of our main theorem, let us state some simple facts (the following five propositions) that will facilitate our argument. We begin by recalling a statement from our previous work[14] that specifies the length of the shortest closed geodesic in a trapezoid. Since the proof is quite short, we include it for the convenience of the reader.

© Author(s) 2021

Proposition 8 (Ref. 14). The length of the shortest closed geodesic in a trapezoid is either 2h or 2b.

Proof. Any closed diffractive or non-diffractive geodesic that starts from the top edge (including the corners) and is transversal (i.e., not tangent) to the top edge must be of length strictly larger than $2h$ unless the closed geodesic also runs between the two parallel sides and is a member of the $2h$ family. Furthermore, any closed geodesic that touches the left and right edges (including the corners) must be of length larger than $2b$ unless it is the $2b$ orbit. If a geodesic touches the bottom edge and the right edge (respectively, left edge), then it must also visit the top edge or the left edge (respectively, right edge), and hence, its length is larger than $2h$ or $2b$. □

Proposition 9. If we exclude the lengths of periodic orbits that lie entirely on the top edge of a trapezoid T from its length spectrum, then the shortest periodic orbit is the 2h family or the Fagnano orbit l_F.

Proof. Since $2h$ is the shortest orbit, other than the $2mb$ orbits, that touches the top edge, the proposition follows quickly from the following two claims:

1. If the l_F orbit lies inside a trapezoid, then it must be the shortest geodesic that does not touch the top edge of the trapezoid [see Fig. 7(a)].
2. If the l_F orbit does not exist in the trapezoid or if it goes through a diffraction as in Figs. 7(b) and 7(c), then the $2h$ family is the shortest periodic orbit among all periodic orbits except possibly some $2mb$ orbits.

Note that if an orbit does not visit the top edge of the trapezoid, then it must be an orbit of the extended triangle $\hat{T}$. Hence, the first claim follows from the classical result of Fagnano (see also Ref. 5). For the second statement, any such periodic orbit must touch both b and B; hence, it is at least as long as $2h$ by the triangle inequality since the shortest path between b and B has length h. □

Proposition 10. Let T be a trapezoid. Suppose the l_F orbit exists inside T. Then, $2h_\alpha < 2l_F$.

Proof. Since the l_F orbit exists, we must have $\alpha + \beta \geq \frac{\pi}{2}$, in particular, $\alpha \geq \frac{\pi}{4}$. It is then obvious that

$$2h_\alpha = 2B \sin\beta < 4B \sin\alpha \sin\beta = 2l_F.$$

□

The next statement provides a useful lower bound for the length of conical periodic orbits and hence for families of periodic orbits.

Proposition 11. Any conical period orbit inside a non-rectangular trapezoid T that is not a 2mb orbit has length ≥ 2h or $2h_\alpha$, and the equality occurs if and only if the orbit belongs to the 2h family or if it is the $2h_\alpha$ orbit or belongs to the $C_{1,1}$ family when T is isosceles, respectively. In particular, any family of periodic orbits has length ≥ 2h or $2h_\alpha$.

Proof. Clearly, if a conical periodic orbit goes through one of the bottom vertices and does not touch the top edge, its length is at least $2h_\alpha$. If it passes through a bottom vertex and touches the top edge, it is longer than or equal to $2h$. If it goes through one of the top vertices, it is either a $2mb$ orbit or it must be transversal to the top edge and consequently be at least $2h$ long. The equality cases are all obvious. The second statement follows immediately because any boundary circle of a family of periodic orbits is conical. □

Proposition 12. If two trapezoids are isospectral and have the same height, then they are isometric. If two trapezoids are isospectral and have the same ℓ_F and h_α, then they are the same.

Proof. If two trapezoids are isospectral, then they have the same heat trace invariants. Consequently, they have the same area, perimeter, and angle invariant. If, in addition, they have the same height, then it was proved in Ref. 20 that they are isometric. Now, let us assume that two trapezoids have the same ℓ_F and h_α. Hence, we obtain that for trapezoids T_1 and T_2,

$$2B_1 \sin\beta_1 = 2B_2 \sin\beta_2, \quad 2B_1 \sin\alpha_1 \sin\beta_1 = 2B_2 \sin\alpha_2 \sin\beta_2.$$

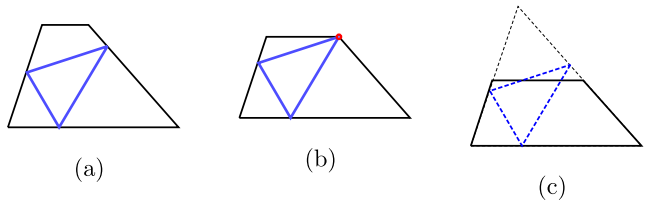

FIG. 7. (a) l_F orbit exists inside T. (b) l_F orbit is diffractive. (c) l_F does not lie in T.

© Author(s) 2021

Thus, $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. This further implies that $B_1 = B_2$. Since the trapezoids have the same perimeters,

$$B_1 + b_1 + h_1(\csc\alpha_1 + \csc\beta_1) = B_1 + b_2 + h_2(\csc\alpha_1 + \csc\beta_1).$$

Using

$$b_1 = B_1 - h_1(\cot\alpha_1 + \cot\beta_1), \quad b_2 = B_2 - h_2(\cot\alpha_1 + \cot\beta_1)$$

and

$$\csc\alpha_1 + \csc\beta_1 > \cot\alpha_1 + \cot\beta_1,$$

we obtain that $h_1 = h_2$. We then obtain that $b_1 = b_2$, and therefore, the trapezoids are isometric. □

We have now demonstrated everything that we need to give the proof of our main result.

Proof of Theorem 1. Assume that two non-obtuse trapezoids T_1 and T_2 are isospectral. We denote the parameters of T_1 and T_2 by $\alpha_1, \beta_1, b_1, B_1, h_1$ and $\alpha_2, \beta_2, b_2, B_2, h_2$, respectively. Since by Proposition 2, rectangles are spectrally unique among trapezoids, we assume that the trapezoids are not rectangles. We begin scanning the positive real line for wave trace singularities (Table I). By Proposition 8, the first singularity is either $2b$ or $2h$ or both. If the order is $\frac{1}{2}$, then by Propositions 7 and 4, the singularity must have a contribution from the $2h$ family. Then, the trapezoids have the same height, and by Proposition 12, they must be isometric. If the order of the first singularity is at most $-\frac{1}{2}$, it must be from the $2b$ orbit and $2b < 2h$. We assume this is the case and move on to the next singularity. By Proposition 9, after jumping over singularities of order at most $-\frac{m}{2}$ created by the $2mb$ orbits, we arrive at $t = 2h$ or $t = l_F$. If the order is $\frac{1}{2}$, there must be a contribution to the singularity from the $2h$ family, and we are done again by Proposition 12. Hence, from now on, we assume that $l_F(T_1) = l_F(T_2)$.

We then investigate the smallest singularity of order at least $-\frac{1}{2}$, call it t_0, in the open interval $(l_F, 2l_F)$. Note that t_0 cannot be $2b$ because in this case, $2b < l_F$, and it cannot be $2mb$, $m \geq 2$, because their orders are ≤ -1 by Proposition 4. We know, however, that t_0 must be the length of a conical periodic orbit or of a non-conical one with an odd number of reflections (recall that if the number of reflections is even, then the orbit belongs to a family that always contains a conical orbit on its boundary). By Proposition 11, if the singularity t_0 is from a conical orbit, it must be caused by the orbit $2h_\alpha$ or $2h$. If it is non-conical with an odd number of reflections, the order of the singularity must be 0 by Proposition 5. Note that multiple (but only finitely many) isolated non-conical orbits may have the same length, but their singularity contributions can never cancel out or add to become a singularity of order $-\frac{1}{2}$, which is the order of $2h_\alpha$, or $\frac{1}{2}$, which is of $2h$ (or $2h_\alpha$ when T is isosceles). This is because their k-expansions contain only integer powers of k. Note also that, although when $\alpha = \frac{\pi}{2}$, the order of the singularity at $2h_\alpha$ is 0, but in this case, $2h_\alpha = l_F$, so it does not belong to the open interval $(l_F, 2l_F)$. In short, the k-expansion of isolated non-conical orbits completely distinguishes them from the $2h_\alpha$ and $2h$ orbits; therefore, we skip them if we encounter them. Hence, the next singularity of nonzero order but at least $-\frac{1}{2}$, call this time t_1, that occurs in the interval $(l_F, 2l_F)$ must be either from $2h_\alpha$ or $2h$. If the order of singularity at t_1 is $-\frac{1}{2}$, we know that it must come from $2h_\alpha$; therefore, $2h_{\alpha_1} = 2h_{\alpha_2}$, which implies that $T_1 = T_2$ by Proposition 12. If the order of t_1 is $\frac{1}{2}$, then one of the following cases happens:

1. $2h_1 = 2h_2$.
2. $2h_1 = 2h_{\alpha_2}$, and T_2 is an isosceles trapezoid, i.e., $\alpha_2 = \beta_2$.

If case (1) holds, we are done again by Proposition 12. Hence, suppose case (2) holds. Since the $2h_1$ singularity of T_1 is observed first, this requires that

$$2h_1 < 2h_{\alpha_1}. \tag{6}$$

Since in this case, we have

$$2h_1 = 2h_{\alpha_2} = 2B_2 \sin\beta_2 = 2B_2 \sin\alpha_2,$$

we obtain from (6) that

$$B_2 \sin\beta_2 < B_1 \sin\beta_1.$$

Consequently, since $\ell_F(T_1) = \ell_F(T_2)$, we have

$$B_1 \sin\alpha_1 \sin\beta_1 = B_2 \sin\alpha_2 \sin\beta_2 < B_1 \sin\alpha_2 \sin\beta_1.$$

TABLE I. This table summarizes the important periodic orbits for our inverse problem.

Orbit	$2mb$	ℓ_F	$2h_\alpha$	$2h$	Non-conical with odd order
Order	$\leq -m/2$	0	$-\frac{1}{2}$, if $\alpha \neq \beta$, α diffractive 0, if $\alpha = \pi/2$ $\frac{1}{2}$, if $\alpha = \beta$	$\frac{1}{2}$	0
Proposition	4	5	6	7	5

© Author(s) 2021

From this, we obtain $\sin\alpha_1 < \sin\alpha_2$, which implies $\alpha_1 < \alpha_2 = \beta_2$. However, we also have by the heat trace the same angle invariants $q = F(\alpha) + F(\beta)$; see Proposition 1. Since $F(x) = \frac{1}{x(\pi - x)}$ is a strictly decreasing function on the interval $(0, \frac{\pi}{2}]$, we have

$$F(\alpha_1) + F(\beta_1) = 2F(\alpha_2) < 2F(\alpha_1).$$

This shows that $F(\beta_1) < F(\alpha_1)$, but $\beta_1 \leq \alpha_1$, which contradicts that F is decreasing. Therefore, case (2) cannot happen.

The final case of concern is when no singularities of order $\neq 0$ and at least $-\frac{1}{2}$ occur in the interval $(l_F, 2l_F)$. Since by Proposition 10, we have $2h_\alpha < 2l_F$ and since $2h_\alpha \geq l_F$ with equality only if $\alpha = \frac{\pi}{2}$, this scenario happens only if $\alpha_1 = \frac{\pi}{2}$ and $\alpha_2 = \frac{\pi}{2}$. However, then the angle invariant determines that $\beta_1 = \beta_2$, which, in turn, implies that $T_1 = T_2$ using the other heat trace invariants, i.e., the area and perimeter. □

Remark 5. In our proof, we never considered the $2h_\alpha$ orbit in the non-diffractive cases $\alpha = \frac{\pi}{3}, \frac{\pi}{4}$. This is because by (4), in these cases, $2h < 2h_\alpha$, so one would observe the singularity $2h$ sooner than $2h_\alpha$. We also did not study the obstructed $C_{1,1}$ families for the same reason.

ACKNOWLEDGMENTS

H.H. was supported by the Simons Foundation Collaboration Grant (No. 638398). Z.L. was supported by the National Science Foundation (Grant No. DMS-19-08513). J.R. was supported by the Swedish Research Council (Grant No. 2018-03402). H.H. and J.R. would like to acknowledge support of the National Science Foundation (Grant No. DMS-1440140) and are grateful for the opportunity provided to work together at the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2019 semester. The authors are grateful to the anonymous reviewer for comments that improved the quality and clarity of the final manuscript.

The authors declare that they have no conflict of interest.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

REFERENCES

[1] K. G. Andersson and R. B. Melrose, "The propagation of singularities along gliding rays," Inventiones Math. **41**(3), 197–232 (1977).
[2] M. van den Berg and S. Srisatkunarajah, "Heat equation for a region in R^2 with a polygonal boundary," J. London Math. Soc. **s2-37**(1), 119–127 (1988).
[3] S. J. Chapman, "Drums that sound the same," Am. Math. Mon. **102**, 124–138 (1995).
[4] K. Datchev and H. Hezari, *Inverse problems and applications: inside out. II*, Math. Sci. Res. Inst. Publ. (Cambridge Univ. Press, Cambridge, 2013), Vol. 60, pp. 455–485.
[5] C. Durso, "On the inverse spectral problem for polygonal domains," Ph.D. thesis, Massachusetts Institute of Technology, ProQuest LLC, Ann Arbor, MI, 1988.
[6] C. Gordon, D. Webb, and S. Wolpert, "Isospectral plane domains and surfaces via Riemannian orbifolds," Inventiones Math. **110**(1), 1–22 (1992).
[7] C. Gordon, D. L. Webb, and S. Wolpert, "One cannot hear the shape of a drum," Bull. Am. Math. Soc. **27**(1), 134–138 (1992).
[8] D. Grieser and S. Maronna, "Hearing the shape of a triangle," Not. Am. Math. Soc. **60**(11), 1440–1447 (2013).
[9] V. Guillemin and R. Melrose, "The Poisson summation formula for manifolds with boundary," Adv. Math. **32**(3), 204–232 (1979).
[10] E. Gutkin, "Billiards in polygons," Physica D **19**(3), 311–333 (1986).
[11] E. Gutkin, "Billiards in polygons: Survey of recent results," J. Stat. Phys. **83**(1-2), 7–26 (1996).
[12] H. Hezari and S. Zelditch, "c^∞ spectral rigidity of the ellipse," Anal. PDE **5**(5), 1105–1132 (2012).
[13] H. Hezari and S. Zelditch, "One can hear the shape of ellipses of small eccentricity," arXiv:1907.03882 (2019).
[14] H. Hezari, Z. Lu, and J. Rowlett, "The Neumann isospectral problem for trapezoids," Ann. Henri Poincare **18**(12), 3759–3792 (2017).
[15] L. Hillairet, "Formule de trace sur une surface euclidienne à singularités coniques," C. R. Math. **335**(12), 1047–1052 (2002).
[16] L. Hillairet, "Contribution of periodic diffractive geodesics," J. Funct. Anal. **226**(1), 48–89 (2005).
[17] M. Kac, "Can one hear the shape of a drum?," Am. Math. Mon. **73**(4), 1–23 (1966).
[18] A. Katok, "The growth rate for the number of singular and periodic orbits for a polygonal billiard," Commun. Math. Phys. **111**(1), 151–160 (1987).
[19] J. B. Keller, "A geometrical theory of diffraction," in *Calculus of Variations and Its Applications, Proceedings of Symposia in Applied Mathematics Vol. 8* (McGraw-Hill Book Co., Inc., New York-Toronto-London, 1958), pp. 27–52.
[20] Z. Lu and J. Rowlett, "The sound of symmetry," Am. Math. Mon. **122**(9), 815–835 (2015).
[21] S. Marvizi and R. Melrose, "Spectral invariants of convex planar regions," J. Differ. Geom. **17**(3), 475–502 (1982).
[22] J. H. P. McKean and I. M. Singer, "Curvature and the eigenvalues of the laplacian," J. Differ. Geom. **1**(1), 43–69 (1967).
[23] V. M. Petkov and L. N. Stoyanov, *Geometry of Reflecting Rays and Inverse Spectral Problems*, Pure and Applied Mathematics (John Wiley & Sons, Ltd., Chichester, 1992).
[24] G. Popov and P. Topalov, "From K.A.M. tori to isospectral invariants and spectral rigidity of billiard tables," arXiv:1602.03155v2 (2019).
[25] R. E. Schwartz, "Obtuse triangular billiards I. Near the (2, 3, 6) triangle," Exp. Math. **15**(2), 161–182 (2006).
[26] R. E. Schwartz, "Obtuse triangular billiards II. One hundred degrees worth of periodic trajectories," Exp. Math. **18**(2), 137–171 (2009).
[27] J. de Simoi, V. Kaloshin, and Q. Wei, "Dynamical spectral rigidity among $\mathbb{Z}_2$-symmetric strictly convex domains close to a circle," Ann. Math. **186**(1), 277–314 (2017).
[28] Y. B. Vorobets, G. A. Galʹperin, and A. M. Stëpin, "Periodic billiard trajectories in polygons: Generation mechanisms," Usp. Mat. Nauk **47**(3), 285 (1992).

© Author(s) 2021

[29]K. Watanabe, "Plane domains which are spectrally determined. II," J. Inequal. Appl. **7**(1), 25–47 (2002).
[30]J. Wunsch, "A Poisson relation for conic manifolds," Math. Res. Lett. **9**(6), 813–828 (2002).
[31]S. Zelditch, "Inverse spectral problem for analytic domains. II. $\mathbb{Z}_2$-symmetric domains," Ann. Math. **170**(1), 205–269 (2009).
[32]S. Zelditch, "Survey on the inverse spectral problem," ICCM Not. **2**(2), 1–20 (2014).

© Author(s) 2021