

THE THRESHOLD FOR THE SQUARE OF A HAMILTON CYCLE

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ABSTRACT. Resolving a conjecture of Kühn and Osthus from 2012, we show that $p = 1/\sqrt{n}$ is the threshold for the random graph $G_{n,p}$ to contain the square of a Hamilton cycle.

1. INTRODUCTION

Understanding thresholds for various properties of interest has been central to the study of random graphs since its initiation by Erdős and Rényi [4], and thresholds for containment of (copies of) specific graphs in the random graphs $G_{n,p}$ and $G_{n,m}$ have been the subject of some of the most powerful work in the area. (See e.g. [3, 11, 10], to which we also refer for threshold basics.)

Hamilton cycles in random graphs in particular are the subject of an extensive literature, with, to begin, the question of when they appear posed in [4] and answered in [18, 14, 2, 1]; see [9] for a thorough account. Here we consider a related question first raised by Kühn and Osthus [15]: *when does the square of a Hamilton cycle appear in the random graph?* (The k th power of a graph G is the graph on $V(G)$ with two vertices joined iff their distance in G is at most k .)

For this discussion we write $\mathcal{H}_n^k$ for the k th power of an n -vertex cycle (so a Hamilton cycle of K_n). The expected number of copies of $\mathcal{H}_n^k$ in $G_{n,p}$ is $(n-1)!p^{kn}/2$, implying that the threshold for appearance of $\mathcal{H}_n^k$ in $G_{n,p}$ (henceforth simply “threshold for $\mathcal{H}_n^k$ ”) is at least $n^{-1/k}$. (We follow a standard abuse in using “the” threshold for an order of magnitude rather than a specific value.) For $k=1$, it was famously shown by Pósa [18] that the threshold for a Hamilton cycle is $\log n/n$ —this is driven not by expectation considerations, but by the need to avoid isolated vertices—while for $k \geq 3$, it follows from a general result of Riordan [19], based on the second-moment method, that the threshold for $\mathcal{H}_n^k$ is $n^{-1/k}$.

The case $k=2$ has proved more stubborn: here there is no obvious analogue of isolated vertices pushing the threshold above $n^{-1/2}$, but, unlike for larger k , the second-moment method yields only weak upper bounds. Kühn and Osthus [15] conjectured that $n^{-1/2}$ is correct and showed that the threshold is $n^{-1/2+o(1)}$, a bound subsequently improved to $(\log n)^4 n^{-1/2}$ by Nenadov and Škorić [17]; to $(\log n)^3 n^{-1/2}$ by Fischer, Škorić, Steger and Trujić [5]; and to $(\log n)^2 n^{-1/2}$ in unpublished work of Montgomery (see [9]). Here we resolve the question, proving the conjecture of [15]:

Theorem 1.1. *There is a universal K such that for $p \geq K/\sqrt{n}$,*

$$\mathbb{P}(G_{n,p} \text{ contains the square of a Hamilton cycle}) \rightarrow 1 \text{ as } n \rightarrow \infty.$$

While the aforementioned attempts are all rooted in the notion of ‘absorption’ introduced in [20], the proof of Theorem 1.1 takes a different approach, based on the recent resolution, by Frankston and the

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present authors [6], of Talagrand’s relaxation [21] of the ‘expectation threshold’ conjecture of [12]. We will say (not quite following [6]) that a hypergraph $\mathcal{G}$ on a finite vertex set V is q -spread if

$$|\mathcal{G} \cap \langle I \rangle| \leq q^{|I|} |\mathcal{G}| \quad (1)$$

for each $I \subseteq V$, where $\langle I \rangle$ is the increasing family generated by I ; in this language, the main result of [6] says that there is a fixed C such that if a hypergraph $\mathcal{G}$ with edges of size at most ℓ is q -spread, then a $(Cq \log \ell)$ -random subset of V is likely to contain some edge of $\mathcal{G}$.

Applied to the hypergraph $\mathcal{G}$ consisting of all copies of $\mathcal{H}_n^2$ (which is q -spread with $q \sim \sqrt{e/n}$; see below), the result of [6] says that the threshold for $\mathcal{H}_n^2$ is at most $\log n / \sqrt{n}$. A key point in our proof of Theorem 1.1, which eliminates the offending $\log n$, is the observation that large “local spreads” $(|\mathcal{G} \cap \langle I \rangle| / |\mathcal{G}|)^{1/|I|}$ are relatively rare, a typical value being more like $1/n$ than $1/\sqrt{n}$.

Formally, we prove the following slight weakening of Theorem 1.1.

Theorem 1.2. *For each $\varepsilon > 0$ there is a K such that for $p \geq K/\sqrt{n}$,*

$$\mathbb{P}(G_{n,p} \text{ contains the square of a Hamilton cycle}) \geq 1 - \varepsilon$$

for sufficiently large n .

Getting Theorem 1.1 from this just requires applying the machinery of Friedgut [7, 8] to say that the property of containing $\mathcal{H}_n^2$ has a sharp threshold. We omit this by now routine step (and the relevant definitions), and refer the reader to (e.g.) [16] for a similar argument.

Though there seems little hope of proving such a statement along the present lines, it is natural to guess that the above expectation considerations drive the threshold more precisely, namely:

Conjecture 1.3. *For fixed $\varepsilon > 0$ and $p > (1 + \varepsilon)\sqrt{e/n}$,*

$$\mathbb{P}(G_{n,p} \text{ contains the square of a Hamilton cycle}) \rightarrow 1 \text{ as } n \rightarrow \infty.$$

The proof of Theorem 1.2 is given in Section 3, with some basic calculations supporting the argument provided in Section 2.

2. PRELIMINARIES

We will use $\mathcal{M}$ for $E(K_n)$ and from now on write $\mathcal{H}$ for $\mathcal{H}_n^2$. As above, $\mathcal{G}$ is the $(2n)$ -uniform hypergraph on vertex set $\mathcal{M}$ consisting of all copies of $\mathcal{H}$ in K_n . Thus $|\mathcal{G}| = (n-1)!/2$, and it is not hard to see that $\mathcal{G}$ is q -spread with $q = [2/(n-1)!]^{1/(2n)} \sim \sqrt{e/n}$, meaning (recall)

$$|\mathcal{G} \cap \langle I \rangle| \leq q^{|I|} |\mathcal{G}| \quad \forall I \subseteq \mathcal{M}. \quad (2)$$

The next two statements implement the basic idea mentioned above, that large values of $|\mathcal{G} \cap \langle I \rangle| / |\mathcal{G}|$ are rare.

Proposition 2.1. *For an $I \subseteq \mathcal{M}$ with $\ell \leq n/3$ edges and c components,*

$$|\mathcal{G} \cap \langle I \rangle| \leq (16)^\ell \left(n - \left\lceil \frac{\ell + c}{2} \right\rceil - 1 \right)!$$

Proposition 2.2. *For an $F \subseteq \mathcal{H}$ of size h , the number of subgraphs of F with ℓ edges and c components is at most*

$$(8e)^\ell \binom{2h}{c}.$$

Proof of Proposition 2.1. Let $I_1, \dots, I_c$ be the components of I and $v = |V(I)|$ (where $V(E)$ is the set of vertices used by $E \subseteq \mathcal{M}$). The upper bound on ℓ implies that no I_j can “wrap around,” so $|E(I_j)| \leq 2|V(I_j)| - 3$ for each j and

$$\ell \leq 2v - 3c. \quad (3)$$

We first designate a root vertex v_j for each I_j and order $V(I_j)$ by some $\prec_j$ that begins with v_j and in which each $v \neq v_j$ appears later than at least one of its neighbors. We may then bound $|\mathcal{G} \cap \langle I \rangle|$ as follows.

To specify a $J (\in \mathcal{G})$ containing I , we first specify a cyclic permutation of $\{v_1, \dots, v_c\} \cup (V(K_n) \setminus V(I))$. By (3), the number of ways to do this (namely, $(n - v + c - 1)!$) is at most $(n - \lceil \frac{\ell+c}{2} \rceil - 1)!$

We then extend to a full cyclic ordering of $V(K_n)$ (thus determining J) by inserting, for $j = 1, \dots, c$, the vertices of $V(I_j) \setminus \{v_j\}$ in the order $\prec_j$. This allows at most four places to insert each vertex (since one of its neighbours has been inserted before it and the edge joining them must belong to J), so the number of possibilities here is less than $4^v \leq (16)^\ell$, and the proposition follows. $\square$

Proof of Proposition 2.2. We need the following standard bound, which follows from the fact (e.g. [13, p. 396, Ex.11]) that the infinite Δ -branching rooted tree contains precisely

$$\frac{\binom{\Delta v}{v}}{(\Delta - 1)v + 1} \leq (e\Delta)^{v-1}$$

rooted subtrees with v vertices.

Lemma 2.3. *For a graph G of maximum degree Δ , the number of connected, h -edge subgraphs of G containing a given vertex is less than $(e\Delta)^h$.* $\square$

To specify a subgraph J of F as in Proposition 2.2, we proceed as follows. We first choose root vertices $v_1, \dots, v_c$ for the components, say $J_1, \dots, J_c$, of J , the number of possibilities for this being at most $\binom{2h}{c}$. We then choose the sizes, say $\ell_1, \dots, \ell_c$, of $J_1, \dots, J_c$; here the number of possibilities is at most $\binom{\ell-1}{c-1}$ (the number of c -compositions of ℓ , that is, positive integer solutions of $\ell_1 + \dots + \ell_c = \ell$). Finally, we specify for each i a connected J_i of size ℓ_i rooted at v_i , which according to Lemma 2.3 can be done in at most $\prod (4e)^{\ell_i} = (4e)^\ell$ ways. Combining these estimates (with the crude $\binom{\ell-1}{c-1} < 2^\ell$) yields the bound in the proposition. $\square$

3. PROOF OF THE MAIN RESULT

Recall that $\mathcal{M} = E(K_n)$ and $\mathcal{G}$ is the hypergraph of copies of $\mathcal{H} = \mathcal{H}_n^2$ in K_n , and set $m = |\mathcal{M}| (= \binom{n}{2})$.

For $S \in \mathcal{G}$ and $X \subseteq \mathcal{M}$, an (S, X) -fragment is a set of the form $J \setminus X$ with $J \in \mathcal{G}$ contained in $S \cup X$. Our main point, Lemma 3.1, says that for a suitably large w , most pairs (S, W) with $S \in \mathcal{G}$ and $W \in \binom{\mathcal{M}}{w}$ admit small fragments. (We will later need the usual easy transfer of the present discussion to a “binomial” W .)

Set $k = 4\sqrt{n}$ and (for S, X as above) call the pair (S, X) *good* if some (S, X) -fragment has size at most k , and *bad* otherwise. In what follows we will always assume $S, J \in \mathcal{G}$ and $W \in \binom{\mathcal{M}}{w}$, where w will be $Cn^{3/2}$ for some large constant C .

Lemma 3.1. *There is a fixed C_0 such that for all $C \geq C_0$ and $n \in \mathbb{N}$, with $w = Cn^{3/2}$,*

$$|\{(S, W) : (S, W) \text{ is bad}\}| \leq 2C^{-k/3} |\mathcal{G}| \binom{m}{w}. \quad (4)$$

Proof. We may of course assume n is large, since values below any fixed n_0 can be handled trivially by adjusting C_0 . It is enough to show

$$|\{(S, W) : (S, W) \text{ is bad}, |W \cap S| = t\}| \leq 2C^{-k/3} |\mathcal{G}| \binom{2n}{t} \binom{m-2n}{w-t} \quad (5)$$

for $t \in \{0, \dots, 2n\}$, since summing over t then gives (4).

Now aiming for (5), we fix t , set $w' = w - t$, and bound the number of bad (S, W) 's with $|W \cap S| = t$ (so $|W \setminus S| = w'$ and $|W \cup S| = w' + 2n$). Call $Z \in \binom{\mathcal{M}}{w'+2n}$ *pathological* if

$$|\{S \subseteq Z : (S, Z \setminus S) \text{ is bad}\}| > C^{-k/3} |\mathcal{G}| \binom{m-2n}{w'} / \binom{m}{w'+2n} = C^{-k/3} |\mathcal{G}| \binom{w'+2n}{2n} / \binom{m}{2n},$$

and, when $|S \cup X| = w' + 2n$, say (S, X) is pathological if $X \cup S$ is. We bound the nonpathological and pathological parts of (5) separately.

Nonpathological contributions. We claim that the number of nonpathological (S, W) 's in (5) is less than

$$C^{-k/3} |\mathcal{G}| \binom{2n}{t} \binom{m-2n}{w'}. \quad (6)$$

To see this we specify (S, W) by specifying first $Z := S \cup W$, then S , and then W . The number of possibilities for Z is at most

$$\binom{m}{w'+2n},$$

while, since (S, W) bad implies $(S, Z \setminus S)$ bad (and Z is nonpathological), the number of possibilities for S given Z is at most

$$C^{-k/3} |\mathcal{G}| \binom{m-2n}{w'} / \binom{m}{w'+2n};$$

and of course the number of possibilities for W given Z and S is at most $\binom{2n}{t}$. So we have (6).

Pathological contributions. The main point here is the following estimate. (Recall $S, J \in \mathcal{G}$.)

Claim 3.2. *For a given S, Y chosen uniformly from $\binom{\mathcal{M} \setminus S}{w'}$, and large enough C ,*

$$\mathbb{E}[|\{J \subseteq Y \cup S : |J \cap S| \geq k\}|] \leq C^{-2k/3} |\mathcal{G}| \binom{w'+2n}{2n} / \binom{m}{2n}. \quad (7)$$

This is proved below. Assuming for the moment it is true, we show that the number of pathological (S, W) 's in (5) is (for C as in the claim) less than

$$C^{-k/3} |\mathcal{G}| \binom{2n}{t} \binom{m-2n}{w'}. \quad (8)$$

To see this we think of choosing $(S, W \cap S)$ —which can be done in at most $|\mathcal{G}| \binom{2n}{t}$ ways—and then $W \setminus S$. For the latter, notice that (S, W) bad means that *every* $J \subseteq S \cup W$ has $|J \cap S| \geq k$; so, since (S, W) is pathological,

$$|\{J \subseteq S \cup (W \setminus S) : |J \cap S| \geq k\}| \geq C^{-k/3} |\mathcal{G}| \binom{w'+2n}{2n} / \binom{m}{2n}.$$

But then Claim 3.2 (with Markov's Inequality) says the number of possibilities for $W \setminus S$ is at most

$$C^{-k/3} \binom{m-2n}{w'}.$$

Thus we have (8) and combining with (6) completes the proof of Lemma 3.1. $\square$

Proof of Claim 3.2. With f_i the fraction of J 's (in $\mathcal{G}$) with $|J \cap S| = i$, the left-hand side of (7) is

$$\sum_{i \geq k} |\mathcal{G}| f_i \binom{w'}{2n-i} / \binom{m-2n}{2n-i},$$

so it is enough to show

$$f_i \left[\binom{w'}{2n-i} / \binom{m-2n}{2n-i} \right] \left[\binom{w'+2n}{2n} / \binom{m}{2n} \right]^{-1} = e^{O(i)} C^{-i}, \quad (9)$$

where—here and below—implied constants do not depend on C . The terms other than f_i on left-hand side of (9) reduce to

$$\frac{(w')_{2n-i}}{(w'+2n)_{2n-i}} \cdot \frac{(m)_{2n-i}}{(m-2n)_{2n-i}} \cdot \frac{(m-2n+i)_i}{(w'+i)_i} = e^{O(i)} C^{-i} n^{i/2}$$

(we omit the routine calculation, just noting that $\sqrt{n} = O(i)$ since $i \geq k$), so for (9) we just need

$$f_i \leq e^{O(i)} n^{-i/2}. \quad (10)$$

For $n/3 \leq i \leq 2n$, this follows from the fact that $\mathcal{G}$ is q -spread with $q \sim \sqrt{e/n}$ (see (2)), which gives

$$f_i \leq \binom{2n}{i} q^i = e^{O(i)} n^{-i/2}.$$

For $k \leq i \leq n/3$, Propositions 2.1 and 2.2 (with Stirling's formula) give

$$f_i \leq |\mathcal{G}|^{-1} (128e)^i \sum_{c=1}^i \binom{4n}{c} \left(n - \left\lceil \frac{i+c}{2} \right\rceil - 1 \right)! = e^{O(i)} n^{-i/2} \sum_{c=1}^i (\sqrt{n}/c)^c = e^{O(i)} n^{-i/2},$$

where at the end we use $(a/x)^x \leq e^{a/e}$ and $i \geq k$. $\square$

Proof of Theorem 1.2. We prove this for $K = 3C_0 + C$, with C_0 as in Lemma 3.1 and C a suitable function of ε (essentially $1/\varepsilon$). Let $p_0 = 3C_0/\sqrt{n}$, $p_1 = C/\sqrt{n}$ and $p = p_0 + p_1 - p_0 p_1 < K/\sqrt{n}$. We generate $G_{n,p}$ in two rounds, as $W_0 \cup W_1$, where W_0, W_1 are independent with W_ν distributed as G_{n,p_ν} (and W_1 chosen after W_0 , at which point we are really interested in $W_1 \setminus W_0$).

Call W_0 *successful* if

$$|\{S : (S, W_0) \text{ is bad}\}| \leq |\mathcal{G}|/2.$$

We first observe that W_0 is (very) likely to be successful: standard concentration estimates give (say)

$$\mathbb{P}(|W_0| < C_0 n^{3/2}) = \exp[-n^{3/2}],$$

and Lemma 3.1 gives

$$\mathbb{P}(W_0 \text{ unsuccessful} \mid |W_0| \geq C_0 n^{3/2}) < 4C_0^{-k/3};$$

in particular W_0 is successful with probability $1 - o(1)$.

Suppose now that W_0 is successful. For each S with (S, W_0) good, let $\chi(S, W_0)$ be some k -element subset of S containing an (S, W_0) -fragment, and let $\mathcal{R}$ be the k -uniform (*multi*)hypergraph

$$\{\chi(S, W_0) : (S, W_0) \text{ is good}\}. \quad (11)$$

To finish we use the second moment method to show that W_1 is reasonably likely to contain a member of $\mathcal{R}$. Setting

$$X = |\{A \in \mathcal{R} : A \subseteq W_1\}|,$$

we have

$$\mu := \mathbb{E}X = |\mathcal{R}| p_1^k$$

and

$$\text{Var}(X) \leq p_1^{2k} \sum \{p_1^{-|A \cap B|} : A, B \in \mathcal{R}, A \cap B \neq \emptyset\}. \quad (12)$$

For $R \in \mathcal{R}$ and $1 \leq i \leq k$, Propositions 2.1 and 2.2 (with Stirling) give

$$\begin{aligned} |\{A \in \mathcal{R} : |A \cap R| = i\}| &\leq \sum_{I \subseteq R, |I|=i} |\mathcal{R} \cap \langle I \rangle| \leq \sum_{I \subseteq R, |I|=i} |\mathcal{G} \cap \langle I \rangle| \\ &= e^{O(i)} \sum_{1 \leq c \leq i} \binom{2k}{c} \left(n - \left\lceil \frac{i+c}{2} \right\rceil - 1\right)! = e^{O(i)} n^{-i/2} |\mathcal{G}|; \end{aligned}$$

so (recall W_0 successful means $|\mathcal{R}| \geq |\mathcal{G}|/2$) the sum in (12) is at most

$$2|\mathcal{R}|^2 p_1^{2k} \sum_{i=1}^k e^{O(i)} p_1^{-i} n^{-i/2} = O(\mu^2/C)$$

for large enough C (where, again, the implied constant doesn't depend on C). Thus, finally, Chebyshev's Inequality gives

$$\mathbb{P}(X = 0) \leq \text{Var}(X)/\mu^2 = O(1/C),$$

and we are done. $\square$

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