Contemporary Mathematics

Spectral gaps on complete Riemannian manifolds

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This paper is dedicated to Professor B.Y. Chen on his 75th Birthday

Abstract. In this short note, we survey some basic results related to the New Weyl criterion for the essential spectrum. We then use the language of Gromov-Hausdorff convergence to prove a spectral gap theorem.

1. Introduction

Let X be a complete noncompact Riemannian manifold of dimension n and denote by Δ the Laplacian acting on smooth functions with compact support $\mathcal{C}_0^\infty(X)$. It is well known that the self-adjoint extension of Δ on $L^2(X)$ exists, and is a unique nonpositive definite and densely defined linear operator.

The spectrum of $-\Delta$, denoted by $\sigma(-\Delta)$, consists of all $\lambda \in \mathbb{C}$ for which $\Delta + \lambda I$ fails to be invertible. The essential spectrum of $-\Delta$, $\sigma_{\rm ess}(-\Delta)$, consists of the cluster points in the spectrum and of isolated eigenvalues of infinite multiplicity. The pure point spectrum is defined by

$$\sigma_{\rm pp}(-\Delta) = \sigma(-\Delta)\backslash\sigma_{\rm ess}(-\Delta).$$

The spectral structure of a noncompact complete manifold is in general more complex than in the compact case. For a compact Riemannian manifold, by the Hodge Theorem, all spectral points of the Laplacian belong to the pure point spectrum. However, interestingly enough, while for most compact manifolds it is impossible to accurately compute the pure point spectrum, for a complete noncompact Riemannian manifold it is possible to locate the essential spectrum of the Laplacian in a large class of manifolds.

In this note, we will first survey some major results about the essential spectrum of a complete Riemannian manifold. Then we will use the language of Gromov-Haudorff to discuss a spectra gap phenomenon similar to a recent result of Schoen-Tran [**25**].

2. The New Weyl criterion

Donnelly pioneered the study of the essential spectrum using the Weyl criterion. In 1981, he proved the following result [**11**]

Key words and phrases. Essential spectrum, spectral gap, Laplacian, Gromov-Hausdorff convergence.

The third author is partially supported by the NSF grant DMS-19-08513.

The second author is partially supported by the CNPq-Brazil Doctorate Abroad scholarship.

THEOREM 2.1. *Let $\sigma > 0$ be a positive number. If there exists an infinite dimensional subspace G in the domain of Δ such that*

$$\|\Delta u + \lambda u\|_{L^2} \leq \sigma \|u\|_{L^2} \tag{1}$$

for all $u \in G$, then

$$\sigma_{\rm ess}(-\Delta) \cap (\lambda - \sigma, \lambda + \sigma) \neq \emptyset.$$

The functions u are referred to as the *approximate eigenfunctions* corresponding to the eigenvalue λ. The above criterion is simple to apply and has directed the study of the essential spectrum of the Laplacian for the last three decades. A related result of the above is as follows: let u be a nonzero smooth function with compact support. If (1) is satisfied, then

$$\sigma(-\Delta) \cap (\lambda - \sigma, \lambda + \sigma) \neq \emptyset.$$

Using this criterion, it is not hard to prove that $\mathbb{R}^n$ has essential spectrum $[0, \infty)$. Moreover, with additional assumptions on the curvature and geometry of the manifold we can locate the essential spectrum (see for example [**8**,**12**,**14**,**15**,**18**,**29**]) by comparing the manifold to the n-dimensional Euclidean space.

The main difficulty in applying Donnelly's Weyl criterion stems from the fact that it requires canonical smooth functions on a manifold. There are many canonical smooth functions on a manifold, including the heat kernel and the Green's function. However, on a general manifold we do not have an explicit expression for these functions. It is possible to give upper and / or lower bounds for these functions, but those require another canonical function, namely the distance function on the manifold. Due to the presence of cut-loci, the distance function is in general not smooth. It is however Lipschitz and locally L^1 (cf. Cheeger [**6**, Chap 4], also see [**21**, **28**]). Thus in order to use the Weyl criterion, we must be in a setting where the distance function is smooth, or the manifold has a pole.

In [**4**] the following example is given which illustrates the non-regular nature of the distance function. Take $M = S^1 \times (-\infty, \infty)$, letting (θ, x) be the coordinates. Then the radial r function which gives the distance of (θ, x) to the point $(0, 0)$ is

$$r(\theta, x) = \sqrt{x^2 + (\min(\theta, 2\pi - \theta))^2}.$$

A straightforward computation gives

$$\Delta r = -\frac{2\pi}{\sqrt{x^2 + \pi^2}} \delta_{\{\theta = \pi\}} + \text{ a smooth function,}$$

where $\delta_{\{\theta=\pi\}}$ is the Delta function along the submanifold $\{\theta = \pi\}$. Therefore Δr is not locally L^2.

As was observed in [**28**] and [**4**], although in most problems the ideal space to work with is the L^2 function space, in comparison to L^q spaces, this is not the case when considering the spectrum of the Laplacian. On a Riemannian manifold, most of the approximate eigenfunctions we can write out explicitly must be related to the distance function. As the above example illustrates however, the Laplacian of the distance function is locally bounded in L^1, but not in L^2, thus making it easier to compute the L^1 spectrum of the Laplacian instead of the L^2 spectrum.

The failure of the L^2 integrability of the Laplacian of the distance function was one of the main difficulties in applying the classical criterion above. In fact, it was not possible to prove that the L^2 essential spectrum of the Laplacian on a manifold with nonnegative Ricci curvature is $[0, \infty)$ by directly computing the L^2 spectrum via the classical Weyl criterion. Around 1999, two papers addressed this difficulty. Donnelly [**12**] proved that the essential spectrum of the Laplacian is $[0,\infty)$ over manifolds with nonnegative Ricci curvature and maximal volume growth. Under his assumptions, the cone of the manifold at infinity is locally Euclidean and the result follows from spectrum continuity. On the other hand, J-P. Wang [**28**], by employing the seminal theorem of K. T. Sturm [**27**], removed the maximal volume growth condition. Wang's result confirmed the conjecture that the spectrum of manifolds with nonnegative Ricci curvature is $[0,\infty)$. In [**21**], Lu-Zhou gave a technical generalization of Wang's result which includes the case of manifolds of finite volume. We further elaborate on Sturm's result and its consequences in Section 3.

What was lacking from these results, was the direct relationship between the L^2 spectrum and L^p spaces. In [**4**], we found such a link.

THEOREM 2.2. *Let X be a Riemannian manifold and let Δ be the Laplacian. Assume that for $\lambda \in \mathbb{R}^+$, there exists a nonzero function u in the domain of Δ such that*

$$\|u\|_{L^\infty} \cdot \|\Delta u + \lambda u\|_{L^1} \leq \delta \|u\|_{L^2}^2 \tag{2}$$

for some positive number $\delta > 0$. Then

$$\sigma(-\Delta) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset,$$

where

$$\varepsilon = \min(1, (\lambda + 2)\delta^{1/3}).$$

Moreover,

$$\sigma_{\text{ess}}(-\Delta) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset,$$

if for any compact subset K of X, there exists a nonzero function u in the domain of Δ satisfying (2) *whose support is outside K.*

Using this theorem, we are able to compute the spectrum of a Ricci nonnegative manifold directly, using functions constructed from the distance function. The above theorem also allowed us to find the most general conditions possible so that the spectrum of the Laplacian on functions is maximal, in other words it is $[0,\infty)$.

THEOREM 2.3. *Let X be a complete noncompact Riemannian manifold. Take a fixed point x_o, and let $r(x) = d(x, x_o)$ be the radial distance to x_o. Assume that the radial Ricci curvature away from x_o is asymptotically nonnegative, in other words, there exists a continuous positive function $\delta(r)$ on $\mathbb{R}^+$ such that*

(i). $\lim_{r\to\infty} \delta(r) = 0$ *and*

(ii). $\mathrm{Ric}_X(\partial r, \partial r) \geq -(n-1)\delta(r)$ *away from the cut-locus of x_o.*

If the volume of the manifold is finite we additionally assume that its volume does not decay exponentially at x_o. Then the L^2 spectrum of the Laplacian is $[0,\infty)$.

3. On a theorem of Sturm

In 1993, Sturm [**27**] proved an interesting result, which relates the L^2 spectrum to the L^p spectrum of the Laplacian.

The L^p spectrum, denoted by $\sigma_p(-\Delta)$ for $p \geq 1$, is defined as the set of complex numbers λ such that the operator $\Delta + \lambda I$ fails to be invertible on the space $L^p(X)$. Note that unlike the case of L^2 spectrum, the L^p spectrum may contain complex numbers. For example, over real hyperbolic space, for $p \neq 2$ the L^p spectrum is a parabolic region in complex plane depending on p and the order of the form [**9**].

We say that a noncompact Riemannian manifold has uniformly subexponential volume growth, if for any $\varepsilon > 0$, there is a constant $C = C(\varepsilon)$, depending only on ε such that

$$\mathrm{vol}(B_x(R)) \leq C(\varepsilon)\,\mathrm{vol}(B_x(1))e^{\varepsilon R}.$$

for any $x \in X$, and $R \geq 1$, where $B_x(R)$ denotes the geodesic ball of radius R centered at x.

It should be noted that a manifold with finite volume does *not* necessarily have uniformly subexponential volume growth. For example, let X be a noncompact hyperbolic space with finite volume. Assume that X has a cusp at infinity. Then for $x_n \to \infty$, $\mathrm{vol}(B_{x_n}(1))$ has exponential decay with respect to the distance to a fixed point x_o. On the other hand, if X has uniformly subexponential volume growth, then for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$\mathrm{vol}(B_{x_n}(R)) \leq C(\varepsilon)\ \mathrm{vol}(B_{x_n}(1))e^{\varepsilon R}$$

for any $R > 1$. If we take $R = d(x_n, x_o) + 2$, then we have

$$\mathrm{vol}(B_{x_o}(1)) \leq \mathrm{vol}(B_{x_n}(R)) \leq C(\varepsilon)\ \mathrm{vol}(B_{x_n}(1))e^{\varepsilon d(x_n,x_o)}$$

which is a contradiction.

Theorem 3.1 (Sturm). *Assume that X is a complete Riemannian manifold of uniformly subexponential volume growth. Assume furthermore that the Ricci curvature of X has a lower bound. Then for any $p \geq 1$, $\sigma_p(-\Delta) = \sigma_2(-\Delta)$, that is, all the L^p spectra coincide.*

Using the above result, J-P. Wang [**28**] proved that

Theorem 3.2. *Let X be a complete Riemannian manifold and denote by $r(x)$ the radial distance to a fixed point $x_o \in X$. Suppose that*

$$\mathrm{Ric}_X(x) \geq -\frac{\delta}{r(x)^2},$$

where $\delta = \delta(n) > 0$ is a constant that depends only on the dimension n. Then, for all $p \geq 1$, $\sigma_p(-\Delta) = [0, \infty)$.

In [**21**], the following generalization of Wang's result has been proved, also by applying Sturm's Theorem

Theorem 3.3. *Let X be a complete Riemannian manifold and again denote by $r(x)$ the radial distance to a fixed point $x_o \in X$. Let $\varepsilon(r)$ be a positive function such that $\varepsilon(r) \to 0$ as $r \to \infty$. Assume that*

$$\mathrm{Ric}_X(x) \geq -\varepsilon(r(x)),$$

Then for all $p \geq 1$, $\sigma_p(-\Delta) = [0, \infty)$.

One can compare the consequences of Sturm's Theorem about the spectrum to those of Theorem 2.2. When we assume that the volume has uniformly subexponential growth and the Ricci curvature is bounded below, then we can get information about all of the L^p spectra. On the other hand, by applying Theorem 2.2, we can obtain stronger information for just the L^2 spectrum without any assumptions on the smooothness of the distance function.

4. Spectral continuity and a gap theorem

In the previous sections we have studied results that allow us to find large sets of noncompact manifolds whose essential spectrum is a connected subset of the real line. There are however many known cases where the essential spectrum has an arbitrary number of gaps [**1**, **19**, **20**, **22**, **25**]. In the last part of the paper we are interested in further exploring this set of manifolds. We will first turn our attention to spectral continuity and then study the evolution of the spectrum of a manifold under Gromov-Hausdorff convergence. We will then use these ideas to prove the existence of gaps in the essential spectrum of a periodic manifold, which is close in spirit to a recent result by Schoen and Tran [**25**].

The first natural case to consider is the evolution of eigenvalues under the continuous deformation of a manifold or its Riemannian metric. Dodziuk proved the following result in [**10**].

Theorem 4.1 (Dodziuk). *Let X be a compact manifold and let g_t be a family of Riemannian metrics on X. Assume that*

$$g_t \to g$$

in the $\mathcal{C}^0$ topology. Then the spectrum (eigenvalues) of g_t converges to the spectrum of g.

In the recent paper [**5**] (also see [**23**] for related results), we generalized spectral continuity to the case when the quadratic forms of two self-adjoint operators are ε-close.

Let $\mathcal{H}$ be a Hilbert space with two inner products $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_1$. We consider two densely defined nonnegative operators H_0 and H_1 on $\mathcal{H}$ that are self-adjoint with respect to the inner products $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_1$ respectively. Let Q_0, Q_1 be their respective quadratic forms and denote the two norms on $\mathcal{H}$ by $\|\cdot\|_0$ and $\|\cdot\|_1$. Note that both Q_0 and Q_1 are nonnegative.

We denote the domain of the Friedrichs extension of H_0 and H_1 by $\mathfrak{Dom}(H_0)$ and $\mathfrak{Dom}(H_1)$ respectively. We assume that there exists a dense subspace $\mathcal{C} \subset \mathcal{H}$ such that $\mathcal{C}$ is contained in $\mathfrak{Dom}(H_0) \cap \mathfrak{Dom}(H_1)$ (in the case of the Laplacian, $\mathcal{C}$ will be the space of smooth functions/forms with compact support).

Definition 1. We say that the operators H_0, H_1 are ε-close, if there exists a positive constant $0 < \varepsilon < 1$ such that for all $u \in \mathcal{C}$ the following two inequalities hold

$$(1-\varepsilon)\,\|u\|_0^2 \le \|u\|_1^2 \le (1+\varepsilon)\,\|u\|_0^2; \tag{3}$$

$$(1-\varepsilon)\,Q_0(u,u) \le Q_1(u,u) \le (1+\varepsilon)\,Q_0(u,u). \tag{4}$$

We note that if H_0, H_1 are ε-close, then for any $u, v \in \mathcal{C}$

$$|(u,v)_1 - (u,v)_0| \le \varepsilon(\|u\|_0\,\|v\|_0); \tag{5}$$

$$|Q_1(u,v) - Q_0(u,v)| \le \varepsilon\,[Q_0(u,u)\,Q_0(v,v)]^{1/2}. \tag{6}$$

Moreover, it can be shown that the resolvents of the two operators are also ε close.

In [**5**] we showed that two ε-close operators have nearby spectra. This result has an important application in the context of the Laplacian over a Riemannian manifold with two ε-close metrics over it. In particular, it allows us to prove the following theorem which holds even in the noncompact case, thus generalizing the result of Dodziuk.

THEOREM 4.2. *Let X^n be an orientable manifold, and let g_0, g_1 be two smooth complete Riemannian metrics on X that are ε-close for some $0 < \varepsilon < 1/2$.*

Fix $A > 0$. Then for any $\lambda \in \sigma(k, \Delta_1) \cap [0, A]$

$$\mathrm{dist}(\lambda, \sigma(k, \Delta_0)) < c(A, n)\, \varepsilon^{\frac{1}{3}}$$

for some constant $c(A)$ depending only on A. A similar result holds for the essential spectra of the operators. In particular,

$$d_{\mathfrak{h}}(\sigma(k, \Delta_1), \sigma(k, \Delta_0)) = o(1),$$

where $o(1) \to 0$, as $\varepsilon \to 0$.

In the above theorem $d_{\mathfrak{h}}$ denotes the pointed Gromov-Hausdorff distance between the spectra as subsets of the real line, and with a common fixed point $\{-1\}$.

In the setting of a family of compact Riemannian manifolds which is convergent in the Gromov-Hausdorff sense, we have the following important results due to Fukaya [**16**] and Cheeger-Colding [**7**].

THEOREM 4.3 (Fukaya). *Let X_t be a family of compact Riemannian manifolds which is Gromov-Hausdorff convergent to a compact metric space X. We assume that X is not a point. Assume that the curvatures of the manifolds X_t are uniformly bounded. Then the eigenvalues of X_t converge to those of X.*

Cheeger and Colding generalized the above theorem and proved

THEOREM 4.4 (Cheeger-Colding). *Let X_t be a family of compact Riemannian manifolds which is Gromov-Hausdorff convergent to a compact metric space X. We assume that X is not a point. Assume that the Ricci curvatures of the manifolds X_t are uniformly bounded below. Then the eigenvalues of X_t converge to those of X.*

There is no known common generalization of the results of Dodziuk and Fukaya-Cheeger-Colding. In this paper we will study a special case, which will allow us to find manifolds with gaps in the L^2 essential spectrum.

Let $(X_1, g_1), (X_2, g_2)$ be two complete Riemannian manifolds. Let $x_1 \in X_1$ and $x_2 \in X_2$ be two fixed points on the manifolds respectively.

Let

$$N = S^{n-1} \times (-2, 2)$$

be the product manifold equipped with the metric $g_N = \epsilon^2 g_0$, where g_0 is the standard product metric.

For any $\varepsilon > 0$, we construct the manifold X_ε by glueing the three manifolds X_1, X_2, N in the following way.

Let $f_1 : S^{n-1} \times (-2, -1) \to X_1$ be the function

$$f_1(\theta, t) = \exp_{x_1}(-t\varepsilon\theta),$$

where $\exp_{x_1}$ is the exponential map from $T_{x_1}X_1 \to X_1$. In particular $\exp_{x_1}(0) = x_1$. Similarly, let $f_2 : S^{n-1} \times (1, 2) \to X_2$ be the function

$$f_2(\theta, t) = \exp_{x_2}(t\varepsilon\theta),$$

where $\exp_{x_2}$ is the exponential map from $T_{x_2}X_2 \to X_2$. In particular $\exp_{x_2}(0) = x_2$.

It is clear that f_i $(i = 1, 2)$ are diffeomorphisms between their domains and ranges. Let X_ε denote the composite manifold defined by (X_1, X_2, N, f_1, f_2), such that

$$X_\varepsilon = (X_1 \backslash B_{x_1}(\varepsilon)) \cup (X_2 \backslash B_{x_2}(\varepsilon)) \cup N/\sim, \tag{7}$$

where we identify f_i with their images respectively for $i = 1, 2$. Roughly speaking, X_ε is constructed from X_1, X_2 by removing two balls of radius ε and adding a neck connecting them.

Abusing notation, we will identify g_i with $f_i^*(g_i)$ for $i = 1, 2$ on the sets where they are defined.

We construct the metric g_ε on X_ε as follows. Outside the neck region N, we do not change the metric, i.e., $g_\varepsilon = g_i$ on $X_i \setminus N$. On N, let ρ_0, ρ_1, ρ_2 be a partition of unity for $\mathbb{R}$ in the following sense. ρ_i are nonnegative smooth functions with compact support. ρ_0 is identically 1 on $[-1, 1]$ and $\mathrm{supp}(\rho_0) \subset [-2, 2]$; ρ_1 is identically 1 on $(-\infty, -2]$ and ρ_2 is identically 1 on $[2, \infty)$, and

$$\rho_0 + \rho_1 + \rho_2 = 1.$$

We define

$$g_\varepsilon = \rho_0 g_N + \rho_1 g_1 + \rho_2 g_2.$$

We use M to denote the manifold obtained by taking $X_1 = X_2 = \mathbb{R}^n$, $x_1 = x_2 = 0 \in \mathbb{R}^n$ and $\varepsilon = 1$. Let $x_0 = (1, 0) \in N$ be a fixed middle point of M. In order words, M consists of two copies of $\mathbb{R}^n$ joint by a tube of radius 1 and length 2.

We prove that

Proposition 4.1. *Let X_1, X_2 be two compact Riemannian manifolds. Using the above notations, we have*

(1) *$(X_\varepsilon, g_\varepsilon)$ is Gromov-Hausdorff convergent to the metric space X_0, which is the union $X_1 \cup X_2$ with x_1 identified with x_2 ;*
(2) *Let x_0 be a reference point in the middle of the neck N. The pointed Riemannian manifolds $(X_\varepsilon, \varepsilon^{-2} g_\varepsilon, x_0)$ are Gromov-Hausdorff convergent to (M, x_0).*

Proof. For (1), consider the Gromov-Hausdorff approximations $\varphi : X_\varepsilon \to X_0$ and $\psi : X_0 \to X_\epsilon$ defined by

$$\varphi(x) = \begin{cases} x & \text{if } x \in X_1 \cup X_2 \\ x_1 = x_2 & \text{if } x \in N/(X_1 \cup X_2) \end{cases}$$

and

$$\psi(x) = \begin{cases} x & \text{if } x \neq x_1 \text{ and } x \neq x_2 \\ x_0 & \text{if } x = x_1 = x_2 \end{cases}$$

It's easy to see that

$$|d_{X_0}(\varphi(x), \varphi(y)) - d_{X_\varepsilon}(x, y)| \leq 4\varepsilon, |d_{X_\varepsilon}(\psi(a), \psi(b)) - d_{X_0}(a, b)| \leq 4\varepsilon$$

$\forall x, y \in X_\varepsilon, a, b, \in X_0$, where L_ε is the length of the neck N. As $\lim_{x \to 0} L_\varepsilon = 0$, X_ε is Gromov-Hausdorff convergent to X_0.

Note that in this setting the collar region (where the cylinder is glued to the manifold) shrinks to a point.

For (2), it is easy to see convergence on the neck $N\setminus(X_1\cup X_2)$, so let's consider the congergence $(X_i,\varepsilon^{-2}g_i)\to\mathbb{R}^n$. Let $\delta>0$ be less than the injectivity radius of (X_i,g_i) at x_i. We know that

$$d(\exp_{x_i})_x = I + o(\delta^2),$$

where $\exp_{x_i}$ is the exponential map with respect to the metric g_i, $d(x,x_i)<\delta$.

Given $R>0$, choose $\varepsilon<\frac{\delta}{R}$ so that the ball $B_{x_i}(R)\subset(X_i,\varepsilon^{-2}g_i)$ is a subset of $B_{x_i}(\delta)\subset(X_i,g_i)$, and therefore

$$|d(exp_{x_i})^*g - I| = o(\delta^2)$$

in $B_{x_i}(R)\subset(X_i,\varepsilon^{-2}g_i)$, which proves the convergence.

□

Lemma 1. *The Sobolev constants for both $(X_\varepsilon,g_\varepsilon)$ and $(X_\varepsilon,\varepsilon^{-2}g_\varepsilon)$ are uniformly bounded.*

Proof. The limit of $(X_\varepsilon,\varepsilon^{-2}g_\varepsilon,x_\varepsilon)$ is the space M, which is obtained by connecting two copies of $\mathbb{R}^n$ by a neck with fixed size. By continuity, in order to prove the uniform bound for the Sobolev space, it suffices to prove the Sobolev inequality on the limiting space.

Let f be a smooth function of compact support on the limiting space M. As before, we can write

$$f = f_1 + f_2 + f_3$$

where f_1, f_2 has their support within one copy of $\mathbb{R}^n$ and f_3 has its support within a fixed geodesic ball. Since on Euclidean space we have uniform Sobolev constants, we have

$$\left(\int |f_i|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le C\int|\nabla f_i|^2$$

for $i=1,2$. On the other hand, we have the usual Sobolev inequality on a compact manifold, thus we have

$$\left(\int |f_3|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le C'\int|\nabla f_3|^2$$

with a possibly different Sobolev constant C'. Combining the above we get

$$\left(\int |f_i|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le max(C,C')\sum_{i=1}^{3}\int|\nabla f_i|^2 \le C\left(\int|\nabla f|^2+|f|^2\right).$$

Since the Sobolev constant is independent of scaling, we have also proved the existence of a uniform Sobolev constant for $(X_\varepsilon,g_\varepsilon)$. □

In what follows we prove the main technical theorem of this paper.

Theorem 4.5. *Let X_1,X_2 be two compact Riemannian manifolds and take $\lambda\notin\mathrm{Spec}(X_1)\cup\mathrm{Spec}(X_2)$. Consider the manifold $(X_\varepsilon,g_\varepsilon)$ defined above. Set $2\delta = dist(\lambda,\mathrm{Spec}(X_1)\cup\mathrm{Spec}(X_2))$ and take $\lambda'\in(\lambda-\delta,\lambda+\delta)$. Then, for any $\varepsilon>0$ small enough, $\lambda'\notin\mathrm{Spec}(X_\varepsilon)$.*

Proof. We will prove the theorem by contradiction. Assume that for any $\varepsilon>0$, there is a $\lambda_\varepsilon\in(\lambda-\delta,\lambda+\delta)$ such that λ_ε is an eigenvalue of X_ε. Let f_ε be the corresponding eigenfunction. Then

$$\Delta_\varepsilon f_\varepsilon + \lambda_\varepsilon f_\varepsilon = 0.$$

By a standard Moser iteration argument using the uniform Sobolev inequality, we have

$$\|f_\varepsilon\|^2_{L^\infty} \le C \int_{X_\varepsilon} \|f_\varepsilon\|^2.$$

As a result, if we normalize the L^2 norm of f_ε to be 1 and given that $\mathrm{vol}(X_\varepsilon)$ is uniformly bounded, we get that the f_ε are uniformly bounded in L^∞. At the same time must have a sequence ε_i such that $\lambda_{\varepsilon_i} \to \lambda_0 \in [\lambda-\delta, \lambda+\delta]$. By passing to a subsequence if necessary, we must have $f_{\varepsilon_i} \to \xi$, converging to an L^2 function ξ on X_0. Let $\xi_i = \xi|_{X_i \setminus \{x_i\}}$ for $i = 1, 2$. By the above argument, since ξ is bounded, it follows that each ξ_i extends to a smooth function on X_i. Moreover, we have

$$\Delta \xi_i + \lambda_0 \xi_i = 0$$

for $i = 1, 2$ with $\lambda_0 \in [\lambda-\delta, \lambda+\delta]$. Since each of the f_ε is bounded, with L^2 norm equal to 1 for all ε, at least one of the ξ_i is not zero. This contradicts the assumption on λ_ε. □

Remark 1. The above theorem can be interpreted as a spectral continuity result: let $\lambda_k(X_\varepsilon)$ be the k-th eigenvalue of X_ε. Then a subsequence $\lambda_k(X_{\varepsilon_i})$ is convergent to the corresponding eigenvalue of the limit space.

Remark 2. Since the Ricci curvature of X_ε has no lower bound, Theorem 4.5 is not a special case of the theorem of Cheeger-Colding. It is neither a special case of the theorem of Dodziuk because the limit space is singular.

Let X be a fixed compact manifold. Let $x_1, x_2 \in X$ two distinct points. Construct a metric space by first making $\mathbb{Z}$ copies of X, and labelling them X_j. Then glue the point x_2 of X_j onto the point x_1 of X_{j+1} for each $j \in \mathbb{Z}$. Denote by M the metric space obtained through this gluing process.

Definition 2. A smoothing X_eps of M is a smooth manifold constructed as in (7) at each x_1, x_2. Oviously, under the Gromov-Hausdorff convergence, we have

$$\lim_{\varepsilon \to 0} X_\varepsilon = M.$$

Similar to to Lemma 1, we have

Lemma 2. *The Sobolev constant for X_ε is independent to ε.*

Using Theorem 4.5 we can prove the following.

Theorem 4.6. *Let X_ε be a smoothing of M constructed in a similar process as in the beginning of the section. That is, X_ε is smooth and the Gromov-Hausdorff limit of X_ε is M. Then, for ε small enough, the essential spectrum of X_ε has gaps.*

Proof. Note that since X_ε is a periodic manifold, its spectrum must coincide with its essential spectrum.

Our proof is a generalization of the method used in the proof of Theorem 4.5. Let $\lambda_\varepsilon \in \mathrm{Spec}(X_\varepsilon, g_\varepsilon)$. We shall prove that a subsequence λ_{ε_i} should be convergent to $\lambda_0 \in \mathrm{Spec}(X)$. Fix $\delta > 0$. Let f_ε be the approximating eigenfunction by the Weyl criterion such that

$$\|\Delta f_\varepsilon + \lambda_\varepsilon f_\varepsilon\|_{L^2} \le \delta \|f_\varepsilon\|_{L^2}.$$

It is not difficult to see that there exist λ'_ε such that $\lambda_\varepsilon - \lambda'_\varepsilon = o(1)$ and for which we can find a function f'_ε that is the Dirichlet eigenfunction corresponding to λ'_ε on the support of f_ε. That is,

$$\Delta f'_\varepsilon + \lambda'_\varepsilon f'_\varepsilon = 0.$$

We normalize f'_ε so that the L^2 norm of f'_ε on K is 1. By using the uniform Sobolev inequality, we can prove that the f'_ε are uniformly bounded.

Let y_ε be the maximum point of f_ε. By the periodic property of X_ε, by translating if necessary, we may assume that y_ε is within a fixed copy of X. Normalizing f'_ε so that the maximum of it is 1. Then a subsequence of f'_ε will be convergent to a *non-zero* function f_0 on the copy X which, by elliptic estimates, must be smooth on the smooth part of M. Since f is also bounded, f must be an eigenfunction of X. Therefore $\lambda_0 \in \mathrm{Spec}(X)$.

□

Remark 3. It should be noted that the convergence rate depends on λ. For larger λ the rate of convergence may in fact be slower. It would therefore be interesting to know whether there is an infinite number of gaps in the spectrum of X_ε for ε small enough.

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