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ABSTRACT

We construct dynamical many-black-hole spacetimes with well-controlled asymptotic behavior as solutions of the Einstein vacuum equation with positive cosmological constant. We accomplish this by gluing Schwarzschild-de Sitter or Kerr-de Sitter black hole metrics into neighborhoods of points on the future conformal boundary of de Sitter space, under certain balance conditions on the black hole parameters. We give a self-contained treatment of solving the Einstein equation directly for the metric, given the scattering data we encounter at the future conformal boundary. The main step in the construction is the solution of a linear divergence equation for trace-free symmetric 2-tensors; this is closely related to Friedrich's analysis of scattering problems for the Einstein equation on asymptotically simple spacetimes.

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1. Introduction

A vacuum spacetime with cosmological constant $\Lambda \in \mathbb{R}$ is a 4-manifold M equipped with a Lorentzian metric g of signature $(- + + +)$ satisfying the Einstein vacuum equation

$$\text{Ric}(g) - \Lambda g = 0. \quad (1.1)$$

The Majumdar–Papapetrou [1, 2] spacetime is an explicit solution for the coupled Einstein–Maxwell system¹ in $\Lambda = 0$ describing several *extremely charged* black holes; a similar construction for $\Lambda > 0$ was given by Kastor and Traschen [3]. We will demonstrate how to construct *vacuum* spacetimes which, for late times, describe dynamical many-black-hole spacetimes *with precisely controlled asymptotic structure* using a gluing method. Our construction applies in the case $\Lambda > 0$, which is consistent with the ΛCDM model currently favored in cosmology [4, 5].

The simplest solution of (1.1) is de Sitter space

$$M^\circ = (-\pi/2, \pi/2)_s \times \mathbb{S}^3, \quad g_{\text{dS}} = (3/\Lambda) \cos^{-2}(s) (-ds^2 + g_{\mathbb{S}^3}),$$

where $g_{\mathbb{S}^3}$ is the standard metric on the 3-sphere; this describes an exponentially expanding (as $s \rightarrow \pi/2$) universe. The metric g_{dS} is asymptotically simple [6]: the

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¹This means that the right hand side of (1.1) is no longer 0, but related to the energy-momentum tensor of an electromagnetic field satisfying Maxwell's equation.

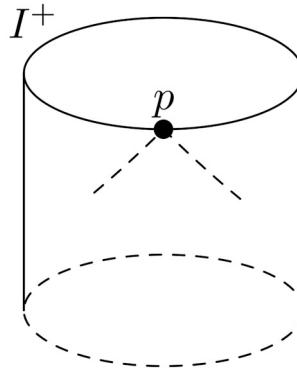


Figure 1. The (partial) conformal compactification M of de Sitter space, a point p on its future conformal boundary I^+ , and a piece of the backwards light cone from p .

conformal multiple $\cos^2(s)g_{\text{ds}}$ extends smoothly to a Lorentzian metric on the partial compactification

$$M = (-\pi/2, \pi/2] \times \mathbb{S}^3.$$

(M°, g_{ds}) is geodesically complete, so future timelike observers in M° can only tend to ∂M but never reach it; one calls ∂M *future timelike infinity*, or the *future conformal boundary* of de Sitter space, often also denoted I^+ . Since images of null-geodesics are conformally invariant, the backward light cone from a point $p \in \partial M$ is a null hypersurface inside M° and known as the *cosmological horizon* associated with p . See Figure 1.

The simplest black hole solution of (1.1) is the Schwarzschild-de Sitter (SdS) solution, recalled below. It depends on a mass parameter $m \in \mathbb{R}$ and can be thought of as describing a black hole tending to some fixed but arbitrary point p in I^+ ; it is defined in a neighborhood of p . Our main result gives a sufficient condition under which one can glue several SdS black holes into de Sitter space:

Theorem 1.1. *Let $N \in \mathbb{N}$. For $i = 1, \dots, N$, fix points $p_i \in \partial M = \mathbb{S}^3 \subset \mathbb{R}^4$ and (subextremal) masses $0 < m_i < (3\Lambda)^{-1/2}$ such that the balance condition*

$$\sum_{i=1}^N m_i p_i = 0 \in \mathbb{R}^4. \quad (1.2)$$

holds. Then there exists a metric g solving the Einstein vacuum equation (1.1) in a neighborhood of ∂M with the following properties:

- (1) *in a neighborhood of p_i , g is isometric to a Schwarzschild-de Sitter black hole metric with mass m_i , containing future affine complete event and cosmological horizons;*
- (2) *outside a small neighborhood of $\{p_1, \dots, p_N\}$, $\cos^2(s)g$ is smooth down to $s = \pi/2$, and asymptotic to the rescaled de Sitter metric $\cos^2(s)g_{\text{ds}}$ at the rate $\cos^3(s)$.*

See Figure 2. When $N \geq 2$, and all masses are sufficiently small in absolute value, we show that the cosmological horizons of different black holes intersect in the maximal globally hyperbolic development of g ; see the end of §3.3. Note that upon replacing s by $-s$, we glue SdS black holes, with *past* affine complete horizons, into a neighborhood of *past*

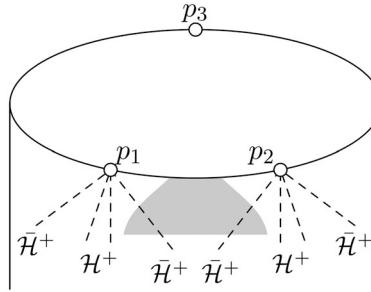


Figure 2. Illustration of Theorem 1.1. We glue SdS black holes into neighborhoods of the points p_i ; only two black holes are shown here. The dashed lines labeled $\bar{\mathcal{H}}^+$ are the cosmological horizons of the individual black holes, while the dashed lines labeled \mathcal{H}^+ (not drawn Penrose-diagrammatically) indicate their event horizons. (The two $\bar{\mathcal{H}}^+$ lines tending to p_1 are really a single $(0, \infty)_{t_*} \times \mathbb{S}^2$, forming one connected horizon, but for visualization purposes we needed to reduce dimension of the sphere by 2.) The gray region indicates the region where the metric is not isometric to some SdS metric.

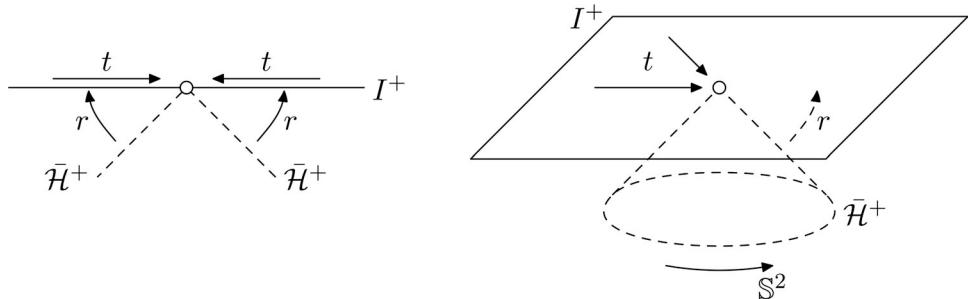


Figure 3. The Schwarzschild-de Sitter metric glued into de Sitter space. We only show the cosmological horizon and the cosmological region $r > r_+$ where r is timelike. On the right is the same picture, but we show an additional spatial dimension, thus showcasing the connectedness of $\bar{\mathcal{H}}^+$.

conformal infinity of de Sitter space; this provides interesting settings in which to (numerically) study the interaction of black holes in de Sitter space under forward evolution.

Recall here that for subextremal mass parameters $m \in (0, (3\Lambda)^{-1/2})$, the SdS metric is

$$\mathbb{R}_t \times (r_-, r_+)_r \times \mathbb{S}^2, \quad g_m = -\mu_m(r)dt^2 + \mu_m(r)^{-1}dr^2 + r^2g_{\mathbb{S}^2},$$

where $\mu_m(r) = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}$, and $0 < r_- < r_+$ are the unique positive real roots of μ_m . After a suitable coordinate change, one can extend g_m beyond the event horizon $r = r_-$ and beyond the cosmological horizon $r = r_+$ to a metric g_m on a larger manifold

$$M_m^\circ = \mathbb{R}_{t_*} \times (0, \infty)_r \times \mathbb{S}^2.$$

One can identify the piece $t_* > 0$ of M_m° with a subset of de Sitter space M° in such a way that the SdS cosmological horizon and the backward light cone from a point $p \in \partial M$ coincide in a neighborhood of p ; denote the resulting metric by $g_{p, m}$. This metric is in fact conformally smooth down to ∂M away from the singular point p , with $r \rightarrow \infty$ corresponding to $s \rightarrow \pi/2$. See Figure 3. Conclusion (1) in Theorem 1.1 is then the statement that $g = g_{p_i, m_i}$ near p_i .

The precise result, [Theorem 3.2](#), is more general: the masses m_i are allowed to be any real number, and we then glue the far end $r \gg 1$ of M_m° into de Sitter space. (For sub-extremal masses as in [Theorem 1.1](#), one can then ‘fill in’ the rest of the SdS black hole.) We also prove the necessity of the balance condition under certain asymptotic assumptions on g ; see [Theorem 3.4](#).

We prove a similar result for gluing Kerr-de Sitter (KdS) black holes into de Sitter space. The KdS family of metrics [7] depends on two parameters, m (mass) and \mathbf{a} (specific angular momentum). For the purpose of gluing KdS metrics into de Sitter space, we also keep track of the point on I^+ to which the black hole tends, and the orientation of its axis of rotation. We can then glue any finite number of KdS black holes into de Sitter space under two balance conditions: the first condition is similar to (1.2) but now involving the effective mass $m_{\text{eff}} = m/(1 + \Lambda \mathbf{a}^2/3)^2$, and the second condition requires the effective angular momenta $m_{\text{eff}} \mathbf{a}$ (taking into account the black hole locations and axes of rotation) to sum up to 0, see [Definition 4.3](#).

1.1. Gluing in general relativity

Most gluing constructions in general relativity operate on the level of (noncharacteristic) initial data sets. Recall that an initial data set for the Einstein vacuum [Equation \(1.1\)](#) is a 3-manifold Σ together with a Riemannian metric γ and a symmetric 2-tensor k on Σ satisfying the constraint equations

$$R_\gamma + (\text{tr}_\gamma k)^2 - |k|_\gamma^2 = 2\Lambda, \quad \delta_\gamma k + \text{dtr}_\gamma k = 0; \quad (1.3)$$

here R_γ is the scalar curvature, and δ_γ is the negative divergence (the adjoint of the symmetric gradient). Given (Σ, γ, k) , there exists a unique maximal globally hyperbolic development (M, g) , with g solving (1.1), and an embedding $\Sigma \hookrightarrow M$ such that the images of γ and k are the metric and second fundamental form of Σ [8, 9].

Brill–Lindquist [10] explicitly constructed initial data containing any number N of (charged) Einstein–Rosen bridges at arbitrarily chosen points in \mathbb{R}^3 and with arbitrary mass parameters; the resulting set of initial data has one distinguished asymptotically flat (AF) region and N AF regions on the other side of the wormholes. Misner [11] (and Lindquist [12] in the Einstein–Maxwell case) showed how, with a careful choice of parameters, one can identify all but two AF regions, and for just two points even create a spacetime with one AF end and a wormhole connecting two ‘points’. These constructions are global and rigid, the main tool being superpositions of shifted and scaled versions of $1/|x|$; this is also the case for Majumdar–Papapetrou and Kastor–Traschen spacetimes.

The starting point for localized gluing is Corvino’s work [13] on the gluing of the large end of Schwarzschild data ($\Lambda = 0$) to a given time-symmetric AF initial data set on \mathbb{R}^3 ; in this case, the constraint equations become $k = 0$ and $R_\gamma = 0$, and key to the localized gluing is the underdetermined nature of the scalar curvature operator $\gamma \mapsto R_\gamma$ (more precisely, the overdetermined nature of the adjoint of its linearization). The assumption of time-symmetry was removed by Corvino–Schoen [14] by allowing the AF end to be equal to Kerr initial data.

Chrūściel–Delay [15] extended the methods of Corvino–Schoen and also refined wormhole constructions by Isenberg–Mazzeo–Pollack [16, 17]. In [15, §4], they

constructed time-symmetric data containing any number of Schwarzschild black holes (meaning: neighborhoods of the neck region of the Riemannian Schwarzschild metric), placed at a collection of points in \mathbb{R}^3 which is symmetric around $0 \in \mathbb{R}^3$. (This assumption is loosely related to the balance condition (1.2).) The same authors also construct many-Kerr initial data [18, §8.9], again under a parity condition. In both papers, the smallness required for solving the nonlinear constraint equations comes from taking the black hole masses to be small compared to the distance of the points. Chruściel–Mazzeo [19] show that the maximally globally hyperbolic development of suitable many-Schwarzschild initial data has past-complete \mathcal{I}^+ , and the black hole region has several connected components. Their arguments use Friedrich’s stability result [20] and direct geometric arguments, a description of the *global* structure of the resulting spacetime being far beyond the reach of hyperbolic PDE theory at this point.

Chruściel–Isenberg–Pollack [21, 22] give sharp results on gluing in compact subsets of initial data sets, and also discuss the case $\Lambda > 0$ as well as matter models coupled to the Einstein equation; see also [23]. Carlotto–Schoen [24] gave another striking extension of this method, producing asymptotically flat initial data which are nontrivial (γ Euclidean, $k = 0$) only in arbitrary (noncompact!) *cones* in \mathbb{R}^3 .

We also mention Cortier’s work [25] on gluing exact Kerr–de Sitter ends to solutions with asymptotically KdS ends, generalizing from the Kottler–Schwarzschild–de Sitter case studied in [26]. The latter results are very different from Theorem 1.1 as they concern the periodic ends of the level set $t = 0$ in the maximal analytic extension of SdS and KdS spacetimes (called Delaunay ends in the SdS case). One can construct many-black-hole initial data sets, with a *finite* number of black holes, from [25, 26] by identifying two isometric (in particular, sufficiently far apart) copies of the fundamental domain of the maximally extended SdS or KdS data set glued in near spatial infinity. In the case of exact SdS or KdS data sets, the resulting spacetime is a quotient of the maximal analytic extension by a suitable discrete translational symmetry; in particular, the future conformal boundary has several connected components, each of which is an interval times \mathbb{S}^2 .

1.2. Scattering problems on asymptotically simple spacetimes; gluing in de Sitter space

As discovered by Friedrich [27], the ‘constraint equations’ at the conformal boundary of an asymptotically simple spacetime² with $\Lambda > 0$ simplify dramatically compared to (1.3). Indeed, fixing a Riemannian 3-manifold (S, h) , the degrees of freedom are two scalar functions as well as a symmetric 2-tensor k on S satisfying the *linear* equations

$$\text{tr}_h k = 0, \quad \delta_h k = 0. \quad (1.4)$$

Given these data, one can construct an asymptotically simple solution (M, g) of (1.1) so that $S \partial M$ and h is the restriction to ∂M of a suitable conformal multiple of g ; the tensor k is equal to certain components of the rescaled Weyl tensor of g at ∂M . (We remark here that Dafermos–Holzegel–Rodnianski [28] gave a scattering construction of black holes

²A manifold M with boundary, and a metric g on M° satisfying (1.1) such that, for a boundary defining function τ , the ‘unphysical metric’ $\tau^2 g$ is a smooth Lorentzian metric on M , with ∂M spacelike when $\Lambda > 0$.

settling down exponentially fast to a subextremal Kerr metric by solving a characteristic Cauchy problem ‘backwards’ with cosmological constant $\Lambda = 0$; see also [29, 30].)

The linear nature of the constraints (1.4) suggest a simple way of gluing pieces of asymptotically simple spacetimes into de Sitter space. Indeed, on Riemannian manifolds (S, h) of dimension 3 and higher, the divergence operator on trace-free symmetric 2-tensors is underdetermined, and one can solve the divergence equation

$$\delta_h k' = f, \quad \text{tr}_h k' = 0, \quad (1.5)$$

in such a way that the support of k' is contained in a small neighborhood of $\text{supp} f$. This requires that there is no obstruction, i.e. f must be orthogonal to the cokernel—the space of conformal Killing vector fields. Solvability then follows from a general result by Delay [31]. Thus, naively gluing many SdS black holes into neighborhoods of points $p_1, \dots, p_N \in I^+ = \mathbb{S}^3$ via a partition of unity, the constraints (1.4) will typically be violated for the induced data k , and with $h = g_{\mathbb{S}^3}$; one can, however, correct k by a 2-tensor k' supported away from the points p_i assuming the obstruction vanishes for $f = -\delta_h k$, which precisely leads to the balance condition (1.2); see §3.1.

We use the gluing problem as an opportunity to give a self-contained treatment of the scattering problem—the construction of a spacetime solution of (1.1) from asymptotic data at I^+ —in this specific context. Rather than using Friedrich’s conformal Einstein field equations, see [27, §2], in which one solves for quantities derived from the metric tensor, we directly construct the metric as a Lorentzian *0-metric* (uniformly degenerate metric), following the terminology of Mazzeo–Melrose [32]; see §3.2–3.3. As demonstrated by Vasy [33, Theorem 5.5], solutions of linear wave equations on a spacetime with asymptotically de Sitter type 0-metrics can be constructed from scattering data in Taylor series at I^+ using regular-singular point ODE methods; the remaining error, which vanishes to all orders at I^+ , is solved away by solving a wave equation with such essentially trivial forcing. (Similar constructions are fairly standard in the Riemannian context on conformally compact or asymptotically hyperbolic metrics, see e.g. Fefferman–Graham [34, 35] and Graham–Zworski [36].)

In our gluing problem, this approach does not work directly. Indeed, calling the naively glued metric from the previous paragraph g_0 , the leading order term of the resulting error $\text{Ric}(g_0) - \Lambda g_0$ is of size $\mathcal{O}(\tau^4)$ as a 2-tensor expressed in terms of $d\tau/\tau$ and sections of $T^*\mathbb{S}^3/\tau$ (and supported away from the points p_i) where $\tau = \cos s$ is a boundary defining function of M ; the degenerate nature of (the linearization of) the Einstein vacuum equation prevents us from solving this error away using a metric correction of the same size. Instead, we need to use a metric correction of size $\mathcal{O}(\tau^3)$ which does not produce any τ^3 error terms (i.e. lies in the kernel of the indicial operator of the linearization of (1.1)); in order for it to solve away the τ^4 error, one needs to solve an equation of the form (1.5).

To continue the construction, we use the now fairly precise glued metric, called g^0 , as a background metric in a generalized harmonic gauge, similarly to [37], and solve the gauge-fixed Einstein equation (see Definition 3.12), first in Taylor series in §3.2 (similarly now to the scalar wave equation case discussed in [33]), and then nonlinearly by solving a quasilinear wave equation with rapidly decaying (at ∂M) forcing in §3.3. We show that the resulting metric solves the Einstein vacuum equation by using the usual argument based on the second Bianchi identity and the propagation of the gauge condition. In this final step, the

sufficiently rapid vanishing of the gauge condition, a 1-form on spacetime, at ∂M replaces the vanishing of the Cauchy data of this 1-form in the standard short-time theory.

Our gluing method is very flexible. For instance, one can glue any number of KdS black holes into the upper half space model $[0, 1)_{\tilde{\tau}} \times \mathbb{R}\tilde{x}^3$, $g_{\text{dS}} = \tilde{\tau}^{-2}(-d\tilde{\tau}^2 + d\tilde{x}^2)$, of de Sitter space without any balance conditions if one allows for the solution to be sufficiently large at spatial infinity; in this case, we can of course only guarantee the existence of the nonlinear solution of (1.1) in a neighborhood of $\tilde{\tau} = 0$ which may shrink as $|\tilde{x}| \rightarrow \infty$. More generally, one can glue any metrics suitably asymptotic to de Sitter space into de Sitter space. See §3.5 for more on this. In particular, one may be able to glue several *dynamical* KdS black holes together once their behavior is understood globally; see the work [38] by the author with Vasy for the stability of the KdS exterior, and Schlu's ongoing project [39, 40] (building on his prior [41]) on the stability of the cosmological region.

We remark that, as another application of our approach, the polyhomogeneous formal solutions of (1.1) constructed by Fefferman–Graham can be corrected to true (asymptotically de Sitter like) solutions near the future conformal boundary; see Remark 3.17. This was previously shown by Rodnianski–Shlapentokh–Rothman [42].

Remark 1.2. We expect our methods to generalize in a straightforward manner to all higher dimensions, including to odd-dimensional spacetimes to which neither Friedrich's analysis nor the extensions by Anderson and Chruściel apply [43, 44]. For general results on the future stability of cosmological spacetimes, see Ringström [45].

Remark 1.3. It would be interesting to perform similar gluing constructions for Einstein–matter systems such as the Einstein–Maxwell equations, thus generalizing the family of Kastor–Traschen spacetimes. See also [46].

1.3. Outline of the paper

In §2, we recall relevant aspects of 0-analysis, i.e. the analysis of 0-metrics and associated uniformly degenerate differential operators. In §3, we present the details of the gluing construction for multi-SdS spacetimes; in §4, we extend this to the KdS case. Throughout the paper, the cosmological constant will be a fixed number

$$\Lambda > 0.$$

2. Analysis of uniformly degenerate metrics

We recall natural vector bundles associated with uniformly degenerate geometries in §2.1 and describe de Sitter space from this point of view; associated differential operators are discussed in §2.2. In §2.3, we discuss the case of the Einstein vacuum equations in detail.

2.1. Rescaled tangent and cotangent bundles; de sitter space

Let M be a smooth $(n + 1)$ -dimensional manifold with boundary $\partial M \neq \emptyset$; the space of smooth vector fields on M is denoted $\mathcal{V}(M) = C^\infty(M; TM)$. Following Mazzeo–Melrose [32], we define the space

$$\mathcal{V}_0(M) := \{V \in \mathcal{V}(M) : V(p) = 0 \ \forall \ p \in \partial M\}$$

of *0-vector fields* (or *uniformly degenerate vector fields*); this is a Lie subalgebra of $\mathcal{V}(M)$. If $\tau \in \mathcal{C}^\infty(M)$ denotes a boundary defining function, i.e. $\partial M = \tau^{-1}(0)$ and $d\tau \neq 0$ on ∂M , then $\mathcal{V}_0(M) = \tau\mathcal{V}(M)$. In local coordinates $[0, \infty)_\tau \times \mathbb{R}_x^n$, the space $\mathcal{V}_0(M)$ is the $\mathcal{C}^\infty(M)$ -span of the $n+1$ vector fields

$$\tau\partial_\tau, \ \tau\partial_{x^i}, \ i = 1, \dots, n.$$

Together, these provide a smooth frame of a vector bundle 0TM , called *0-tangent bundle*, which is *nondegenerate* down to $\tau=0$. Thus, for $z \in M$, there is a natural map ${}^0T_z M \rightarrow T_z M$ which is an isomorphism for $z \in M^\circ$. A section $V \in \mathcal{C}^\infty(M; {}^0TM)$ restricts to a smooth vector field on M° which extends smoothly to a vector field on M . This identifies $\mathcal{V}_0(M) = \mathcal{C}^\infty(M; {}^0TM)$.

The dual bundle of 0TM is called the *0-cotangent bundle* ${}^0T^*M$. In local coordinates near ∂M , a local frame is given by

$$\frac{d\tau}{\tau}, \ \frac{dx^i}{\tau}, \ i = 1, \dots, n.$$

These are smooth and nonvanishing down to $\tau=0$.

Definition 2.1. A *Lorentzian 0-metric* (or *uniformly degenerate Lorentzian metric*) g on M of class \mathcal{C}^∞ is a smooth section $g \in \mathcal{C}^\infty(M; S^2 {}^0T^*M)$ which has signature $(n, 1)$ at every point of M .

In local coordinates, a smooth Lorentzian 0-metric can be written as

$$g = \tau^{-2} \left(g_{00} d\tau^2 + 2 \sum_{i=1}^n g_{0i} d\tau \otimes_s dx^i + \sum_{i,j=1}^n g_{ij} dx^i \otimes_s dx^j \right),$$

with the $g_{\mu\nu}$ smooth functions of (τ, x) , and $(g_{\mu\nu})_{\mu, \nu=0, \dots, n}$ having signature $(n, 1)$. Note that $\tau^2 g \in \mathcal{C}^\infty(M; S^2 T^*M)$ is a smooth Lorentzian metric on M in the usual sense. In particular, the class of metrics g for which ∂M is spacelike for the metric $\tau^2 g$ is well-defined, and independent of the choice of boundary defining function τ ; we shall only be concerned with such metrics in the present paper. The Riemannian metric induced on ∂M by $\tau^2 g$ does depend on τ , but its conformal class is well-defined.

The prime example for us is the de Sitter spacetime in $3+1$ dimensions, with cosmological constant $\Lambda > 0$. It can be defined as the cylinder³

$$M = [-\pi/2, \pi/2]_s \times \mathbb{S}^3, \quad g_{ds} = \frac{3}{\Lambda} \cdot \frac{-ds^2 + g_{\mathbb{S}^3}}{\cos^2 s}, \quad (2.1a)$$

whose interior is conformally diffeomorphic to a slab inside the Einstein universe $(\mathbb{R}_s \times \mathbb{S}^3, -ds^2 + g_{\mathbb{S}^3})$; here $g_{\mathbb{S}^3}$ is the standard metric on \mathbb{S}^3 . The metric g_{ds} is a solution of the Einstein vacuum equation (1.1). To see that g_{ds} has the required form near $s = \pi/2$, let us take $\tau = \cos s$ near $s = \pi/2$; then

$$g_{ds} = (3/\Lambda) \tau^{-2} \left(-(1 - \tau^2)^{-1} d\tau^2 + g_{\mathbb{S}^3} \right) \quad \text{on } [0, \infty)_\tau \times \mathbb{S}^3_\psi. \quad (2.1b)$$

³Just this one time, we also include the past conformal boundary.

Note that $\tau^2 g_{\text{dS}}|_{\partial M} = (3/\Lambda) g_{\mathbb{S}^3}$ is a Riemannian metric, thus $\partial M = \mathbb{S}^3 \sqcup \mathbb{S}^3$ is spacelike with respect to g_{dS} .

Other forms of the de Sitter metric are useful for calculations. Regarding \mathbb{S}^3 as the unit sphere $\mathbb{S}^3 \subset \mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$, we define the map

$$[0, 1) \times \mathbb{R}^3 \ni (\tilde{\tau}, \tilde{x}) \mapsto (\tau, \psi) \in [0, \infty) \times \mathbb{S}^3, \\ \tau = \left(\left(\frac{1 - (\tilde{\tau}^2 - |\tilde{x}|^2)}{2\tilde{\tau}} \right)^2 + 1 \right)^{-1/2}, \quad \psi = \frac{\tau}{\tilde{\tau}} \left(\frac{1 + \tilde{\tau}^2 - |\tilde{x}|^2}{2}, \tilde{x} \right) \in \mathbb{S}^3 \quad (2.2a)$$

from part of the upper half space into de Sitter space (2.1b); here $|\cdot|$ is the Euclidean norm. The de Sitter metric then takes the form

$$g_{\text{dS}} = \frac{3}{\Lambda} \cdot \frac{-d\tilde{\tau}^2 + d\tilde{x}^2}{\tilde{\tau}^2} \quad \text{on } M_u := [0, \infty)_{\tilde{\tau}} \times \mathbb{R}\tilde{x}^3. \quad (2.2b)$$

See [47, §6.1] for these and related calculations (in particular, relating both (2.1b) and (2.2b) to the one-sheeted hyperboloid in $(1 + (n + 1))$ -dimensional Minkowski space which is isometric to global de Sitter space); they imply that the map (2.2a) composed with $(\tau, \psi) \mapsto (s, \psi)$, $s = \arccos \tau$ in the coordinates (2.1a), extends analytically to a map $[0, \infty)_{\tilde{\tau}} \times \mathbb{R}\tilde{x}^3 \rightarrow M$ whose image is the complement of the backward causal cone from the point $(-1, 0) \in \mathbb{S}^3$ at $s = \pi/2$; see [47, Figure 7].

Finally, introducing polar coordinates $\tilde{x} = \tilde{R}\tilde{\omega}$, $\tilde{R} = |\tilde{x}| \geq 0$, $\tilde{\omega} \in \mathbb{S}^2$, and putting

$$(t, r, \omega) = \left(-\frac{1}{2} \sqrt{\Lambda/3} \log (\tilde{R}^2 - \tilde{\tau}^2), \sqrt{\Lambda/3} \tilde{\tau}^{-1} \tilde{R}, \tilde{\omega} \right) \quad (2.3a)$$

in the *cosmological region* $\tilde{R} > \tilde{\tau}$,⁴ we have

$$g_{\text{dS}} = -\left(\frac{\Lambda r^2}{3} - 1 \right)^{-1} dr^2 + \left(\frac{\Lambda r^2}{3} - 1 \right) dt^2 + r^2 g_{\mathbb{S}^2}. \quad (2.3b)$$

This is a smooth 0-metric on a compactification of $(\sqrt{3/\Lambda}, \infty)_r \times \mathbb{R}_t \times \mathbb{S}^2_\omega$; indeed, letting $\tau_s = r^{-1}$, and defining

$$M_s := \left[0, \sqrt{\Lambda/3} \right]_{\tau_s} \times \mathbb{R}_t \times \mathbb{S}^2_\omega, \quad (2.3c)$$

we have

$$g_{\text{dS}} = \tau_s^{-2} \left(-\left(\Lambda/3 - \tau_s^2 \right)^{-1} d\tau_s^2 + \left(\Lambda/3 - \tau_s^2 \right) dt^2 + g_{\mathbb{S}^2} \right) \in \mathcal{C}^\infty(M_s; S^2 T^* M_s). \quad (2.3d)$$

The metric induced on $\tau_s = 0$ (factoring out the overall scalar factor $\Lambda/3$) is

$$h_s := (\Lambda/3) \tau_s^2 g_{\text{dS}}|_{\partial M} = \frac{\Lambda^2}{9} dt^2 + \frac{\Lambda}{3} g_{\mathbb{S}^2}. \quad (2.4)$$

We remark that τ , $\tilde{\tau}$, and τ_s are equivalent defining functions on the overlaps of the various coordinate charts.

⁴This region, $r > \sqrt{3/\Lambda}$, is the interior of the complement of the static region $r < \sqrt{3/\Lambda}$ (i.e. $\tilde{R} < \tilde{\tau}$). In the cosmological region, r is a time function whereas t has spacelike differential; by contrast, the static region is foliated by the spacelike level sets of t (which the Killing vector field ∂_t is orthogonal to).

2.2. Differential operators, function spaces

Geometric operators associated with a 0-metric g on an $(n + 1)$ -dimensional manifold M are examples of *0-differential operators*. Concretely, using abstract index notation, we shall in particular deal with the wave operator on bundles,

$$\square_g u = -u_{;\kappa}^\kappa, \quad (\square_g \dot{g})_{\mu\nu} = -\dot{g}_{\mu\nu;\kappa}^\kappa,$$

the divergence and (trace-free) symmetric gradient

$$\begin{aligned} \delta_g \omega &= -\omega_{\kappa;\kappa}^\kappa, & (\delta_g \dot{g})_\mu &= -\dot{g}_{\mu\kappa;\kappa}^\kappa, & (\delta_g^* \omega)_{\mu\nu} &= \frac{1}{2}(\omega_{\mu;\nu} + \omega_{\nu;\mu}), \\ \delta_{g,0}^* \omega &:= \delta_g^* \omega + \frac{1}{n+1} g \delta_g \omega, \end{aligned}$$

as well as the ‘trace reversal⁵ operator’ on 2-tensors,

$$\mathbb{G}_g \dot{g} := \dot{g} - \frac{1}{2} g (\text{tr}_g \dot{g}).$$

We define the space $\text{Diff}_0^m(M)$ of m -th order 0-differential operators to consist of all locally finite linear combinations of up to m -fold products of 0-vector fields. Then $\square_g \in \text{Diff}_0^2(M)$ for the scalar wave operator, $\square_g \in \text{Diff}_0^2(M; S^{2-0} T^* M)$ for the tensor wave operator acting on symmetric 2-tensors, $\delta_g^* \in \text{Diff}_0^1(M; {}^0 T^* M, S^{2-0} T^* M)$, and so forth. For instance, for the metric (2.2b) in $3 + 1$ dimensions, we have

$$3\Lambda^{-1} \square_g = (\tilde{\tau} \partial_{\tilde{\tau}})^2 - 3\tilde{\tau} \partial_{\tilde{\tau}} + \tilde{\tau}^2 \Delta_{\tilde{x}}, \quad \Delta_{\tilde{x}} := - \sum_{i=1}^n \partial_{\tilde{x}^i}^2;$$

for the other operators, we will give explicit expressions in §2.3.

Associated with any 0-differential operator $A \in \text{Diff}_0^m(M)$ is its *indicial family* (see also [48, §2]) $I(A, \lambda) \in \mathcal{C}^\infty(\partial M)$, $\lambda \in \mathbb{C}$, which is defined by

$$A(\tau^\lambda u) = \tau^\lambda I(A, \lambda) u + \mathcal{O}(\tau^{\lambda+1}), \quad u \in \mathcal{C}^\infty(M),$$

for any defining function τ ; this is independent of the choice of defining function. Concretely,

$$A = \sum_{i+|\alpha| \leq m} a_{i\alpha}(\tau, x) (\tau \partial_\tau)^i (\tau \partial_x)^\alpha \Rightarrow I(A, \lambda) = \sum_{i \leq m} a_{i0}(0, x) \lambda^i. \quad (2.5)$$

Thus, $I(A, \lambda)$ is a polynomial of degree m in λ , depending smoothly on $x \in \partial M$. We call the roots of the polynomial $\lambda \mapsto I(A, \lambda)(x)$ the *indicial roots* of A ; if they are independent of x , we say that A has *constant indicial roots*.

If $A \in \text{Diff}_0^k(M; E, F)$ acts between sections of vector bundles $E, F \rightarrow M$, we define $I(A, \lambda) \in \mathcal{C}^\infty(\partial M; \text{Hom}(E, F)|_{\partial M})$ similarly; the indicial roots of A are then those λ (depending on $x \in \partial M$) for which $I(A, \lambda)$ fails to be invertible.

Lower order terms of A as in (2.5) can be defined upon fixing a collar neighborhood $[0, \epsilon)_\tau \times \partial M$ of ∂M : writing

⁵One has $\text{tr}_g \circ \mathbb{G}_g = -\text{tr}_g$ only for $n + 1 = 4$.

$$A \equiv \sum_{k=0}^m \tau^k \sum_{\substack{i+|\alpha| \leq m \\ |\alpha|=k}} a_{i\alpha}^{(k)}(x) (\tau \partial_\tau)^i \partial_x^\alpha.$$

modulo terms of the form $\tau^{k+1} a(\tau, x) (\tau \partial_\tau)^i \partial_x^\alpha$, $a \in \mathcal{C}^\infty$,⁶ we define

$$I(A[\tau^k], \lambda) := \sum_{\substack{i+|\alpha| \leq m \\ |\alpha|=k}} a_{i\alpha}(x) \lambda^i \partial_x^\alpha \in \text{Diff}^k(\partial M). \quad (2.6)$$

Thus, $I(A[1], \lambda) = I(A, \lambda)$. If A acts between sections of vector bundles $E, F \rightarrow M$, the same works, with $a_{i\alpha}^{(k)} \in \mathcal{C}^\infty(\partial M; \text{Hom}(E, F)|_{\partial M})$, upon choosing an identification of E, F in the collar neighborhood with pullbacks of $E|_{\partial M}, F|_{\partial M}$ along the projection $[0, \epsilon) \times \partial M \rightarrow \partial M$.

We record some standard calculations involving the indicial family. If $A \in \text{Diff}_0^m(M; E, F)$ and $\lambda \in \mathbb{C}$ are such that $\ker I(A, \lambda)$ is a \mathcal{C}^∞ vector subbundle of $E|_{\partial M} \rightarrow \partial M$, then $A(\tau^\lambda u) = \mathcal{O}(\tau^{\lambda+1})$ for all $u \in \mathcal{C}^\infty(M)$ with $u|_{\partial M} \in \mathcal{C}^\infty(\partial M; \ker I(A, \lambda))$. For such u , we moreover have

$$A(\tau^\lambda (\log \tau) u) = \tau^\lambda \partial_\lambda I(A, \lambda) u + \mathcal{O}(\tau^{\lambda+1} \log \tau); \quad (2.7)$$

this can be seen by differentiating the relationship $A(\tau^\lambda v) = \tau^\lambda I(A, \lambda) v + \tau^{\lambda+1} \tilde{v}$ (with $\tilde{v} \in \mathcal{C}^\infty(M)$ depending smoothly on λ) in λ and plugging in $v = u$. We also record that in a collar neighborhood of ∂M , we have, for such u , $A(\tau^\lambda u) = \tau^{\lambda+1} I(A[\tau], \lambda) u + \mathcal{O}(\tau^{\lambda+2})$.

The L^2 -based function spaces corresponding to 0-analysis are *weighted 0-Sobolev spaces*

$$\tau^m H_{0, \text{loc}}^k(M) = \{u \in H_{0, \text{loc}}^k(M) : u \in L^2(M)\}.$$

For $k=0$, we define $H_{0, \text{loc}}^0(M) = L_{\text{loc}}^2(M)$ to be the space of locally⁷ square integrable functions on M relative for a smooth 0-density, i.e. a smooth positive section of the 0-density bundle $|\Lambda^{n+1} {}^0 T^* M|$; in local coordinates, such a density takes the form $a(\tau, x) |\frac{d\tau}{\tau^{n+1}} \frac{dx}{x}|$ with $0 < a \in \mathcal{C}^\infty$, a typical example being the volume density $|\text{dg}|$ of a Lorentzian 0-metric g . For $k \in \mathbb{N}$, we define $H_{0, \text{loc}}^k(M)$ to consist of all $u \in L^2(M)$ so that $Pu \in L^2(M)$ for all $P \in \text{Diff}_0^k(M)$. If M is compact, the space $\tau^m H_0^k(M) = \tau^m H_{0, \text{loc}}^k(M)$ carries the structure of a Hilbert space. More generally, if M is noncompact and $\Omega \subset M$ is open with compact closure, then

$$\tau^m \bar{H}_0^k(\Omega) := \{u|_\Omega : u \in \tau^m H_0^k(M)\} \quad (2.8)$$

is a Hilbert space.

For compact M , we can characterize the space $H_0^m(M)$ using a covering of M by ‘uniformly degenerating cubes’ as follows: if a distribution u is supported in a coordinate patch $[0, 2)_\tau \times \mathbb{R}_x^n$, and in fact in $\tau \leq 1$, $|x| \leq 1$, then⁸

⁶The reader familiar with b-analysis [53] will recognize this as the Taylor expansion of A into dilation-invariant (with respect to τ) b-differential operators on $[0, \epsilon)_\tau \times \partial M$.

⁷On M , thus this does encode uniformity down to compact subsets of ∂M .

⁸We write $A \sim B$ to mean the existence of a constant $C > 1$, independent of u , so that $C^{-1}B \leq A \leq CB$.

$$\|u\|_{H_0^m(M)}^2 \sim \sum_{k=0}^{\infty} \sum_{\substack{\alpha \in \mathbb{Z}^n \\ |\alpha| \leq 2^k}} \|u_{k,\alpha}\|_{H^m([-1/2, 1/2]^{n+1})}^2,$$

$$u_{k,\alpha}(T, X) := u(2^{-k}(1+T), 2^{-k}(\alpha+X)), \quad (T, X) \in \mathbb{R} \times \mathbb{R}^n, \quad |T|, |X| \leq \frac{1}{2}.$$

Note that $u_{k,\alpha}$ sees u on a cube of size 2^{-k} centered at a point at a distance 2^{-k} from the boundary, and ∂_T, ∂_X are of the same size as $\tau\partial_\tau, \tau\partial_\alpha$. We leave the notational changes required to drop the support condition to the reader; see also [48, Proof of Corollary (3.23)]. An important consequence of this characterization is that algebra properties of Sobolev spaces on \mathbb{R}^n immediately carry over to 0-Sobolev spaces; in particular:

Lemma 2.2. *On an $(n+1)$ -dimensional compact manifold M with boundary, and for $k > (n+1)/2$, the space $H_0^k(M)$ is an algebra. More generally, we have*

$$u_j \in \tau^{m_j} H_0^k(M), \quad j = 1, 2 \Rightarrow u_1 u_2 \in \tau^{m_1+m_2} H_0^k(M).$$

Solutions of uniformly degenerate equations often have better regularity and are *conformal*, for instance as shown for solutions of the wave equation on de Sitter type spaces in [33]. For $\alpha \in \mathbb{R}$, we define the space of *conormal functions* relative to $\tau^\alpha L^\infty(M)$ by

$$\mathcal{A}^\alpha(M) := \{u \in \mathcal{C}^\infty(M^\circ) : P(\tau^{-\alpha} u) \in L^\infty(M) \quad \forall P \in \text{Diff}_b(M)\},$$

where $\tau \in \mathcal{C}^\infty(M)$ is a boundary defining function, and $\text{Diff}_b(M)$ is the space of all *b-differential operators* on M : locally, these are finite products of the vector fields $\tau\partial_\tau$ and ∂_α with $\mathcal{C}^\infty(M)$ coefficients. The space

$$\mathcal{A}^\alpha \text{Diff}_0^m(M) \tag{2.9}$$

of 0-differential operators with conormal coefficients consists of all locally finite linear combinations of differential operators of the form aP , $a \in \mathcal{A}^\alpha(M)$, $P \in \text{Diff}_0^m(M)$.

2.3. Einstein vacuum equation and its linearization

We make some general observations about the following nonlinear operator for 0-metrics:

Definition 2.3. For a Lorentzian metric g on a manifold M , define

$$P_0(g) := 2(\text{Ric}(g) - \Lambda g). \tag{2.10}$$

For definiteness, we now work in $3+1$ dimensions on the spacetime manifold

$$M = [0, 1]_\tau \times X, \quad \dim X = 3,$$

where X is a 3-dimensional manifold without boundary such as \mathbb{R}^3 or \mathbb{S}^3 ; the boundary $\partial M = \tau^{-1}(0)$ will play the role of the future conformal boundary. This product structure allows us to identify differential operators (in particular: vector fields) on X with ‘spatial’ differential operators on M . In particular, we can pull back TX (along the projection $M \rightarrow X$) to a bundle over M , still denoted TX , which allows us to split 0TM as

$${}^0TM = \mathbb{R}e_0 \oplus \tau TX, \quad e_0 = \tau \partial_\tau;$$

that is, we identify a 0-vector field $ue_0 + \tau V$, $u \in \mathcal{C}^\infty(M)$, $V \in \mathcal{C}^\infty(M; TX) \subset \mathcal{V}(M)$, with the pair (u, V) . This induces splittings

$$\begin{aligned} {}^0T^*M &= \mathbb{R}e^0 \oplus \tau^{-1}T^*X, \quad e^0 = \frac{d\tau}{\tau}, \\ {}^0T^*M &= \mathbb{R}(e^0)^2 \oplus (2e^0 \otimes_s \tau^{-1}T^*X) \oplus \tau^{-2}S^2T^*X; \end{aligned} \quad (2.11)$$

that is, we identify a section $u(e^0)^2 + 2e^0 \otimes \tau^{-1}\omega + \tau^{-2}k$ of $S^2{}^0T^*M$ with the triple (u, ω, k) , where $u \in \mathcal{C}^\infty(M)$, $\omega \in \mathcal{C}^\infty(M; T^*X)$, $k \in \mathcal{C}^\infty(M; S^2T^*X)$. Given a Riemannian metric h on X , we can split S^2T^*X into pure trace ($\mathbb{R}h$) and trace-free parts ($\ker \text{tr}_h$), thereby refining (2.11) to

$$S^2{}^0T^*M = \mathbb{R}(e^0)^2 \oplus (2e^0 \otimes_s \tau^{-1}T^*X) \oplus \mathbb{R}\tau^{-2}h \oplus \tau^{-2}\ker \text{tr}_h. \quad (2.12)$$

We shall denote the components of $\dot{g} \in S^2{}^0T^*M$ in the four summands in (2.12) by $\dot{g}_{NN} \in \mathbb{R}$ (normal-normal), $\dot{g}_{NT} \in T^*X$ (normal-tangential), $\dot{g}_{TT} \in \mathbb{R}$ (tangential-tangential, pure trace), $\dot{g}_{TT0} \in \ker \text{tr}_h$ (tangential-tangential, trace-free).

We shall first study geometric operators associated with a product metric

$$g = (3/\Lambda) \frac{-d\tau^2 + h(x, dx)}{\tau^2}. \quad (2.13)$$

We denote the exterior derivative on X , pulled back to a spatial operator on M , by dx .

Lemma 2.4. *In the splittings (2.11), we have⁹*

$$\delta_g^* = \begin{pmatrix} e_0 & 0 \\ \frac{1}{2}\tau dx & \frac{1}{2}(1+e_0) \\ h & \tau\delta_h^* \end{pmatrix}, \quad 3\Lambda^{-1}\delta_g = \begin{pmatrix} e_0 - 3 & \tau\delta_h & -\text{tr}_h \\ 0 & e_0 - 4 & \tau\delta_h \end{pmatrix},$$

and, as operators on symmetric 2-tensors,

$$\mathbf{G}_g = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2}\text{tr}_h \\ 0 & 1 & 0 \\ \frac{1}{2}h & 0 & \mathbf{G}_h \end{pmatrix}, \quad 3\Lambda^{-1}\square_g = e_0^2 - 3e_0 + \tau^2\Delta_h + \begin{pmatrix} -6 & 4\tau\delta_h & -2\text{tr}_h \\ -2\tau dx & -6 & 2\tau\delta_h \\ -2h & -4\tau\delta_h^* & -2 \end{pmatrix}.$$

Finally, if $R_{\kappa\lambda\mu\nu}$ and $\text{Ric}_{\mu\nu}$ denote the Riemann curvature tensor and Ricci tensor of g , then the operator $\mathcal{R}_g(u)_{\kappa\mu} = R^\nu_{\kappa\mu\rho}u_{\nu\rho} + \frac{1}{2}(\text{Ric}_\kappa^\nu u_{\nu\mu} + \text{Ric}_\mu^\nu u_{\kappa\nu})$ is equal to

$$3\Lambda^{-1}\mathcal{R}_g = \begin{pmatrix} 3 & 0 & \text{tr}_h \\ 0 & 4 & 0 \\ h & 0 & 4 - h\text{tr}_h \end{pmatrix} + \tau^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}\text{Ric}(h) & 0 \\ 0 & 0 & \mathcal{R}_h \end{pmatrix}.$$

Proof. In local coordinates x^1, x^2, x^3 on X , and setting $e_i = \tau\partial_{x^i}$, $e^i = \tau^{-1}dx^i$, we compute $\nabla_{e_0}e^\mu = 0$ for all μ , $\nabla_{e_i}e^0 = h_{ik}e^k$, and $\nabla_{e_i}e^k = \delta_i^k e_0 - \tau\Gamma(h)_{ij}^k e^j$, where δ_i^k is the

⁹Here, h in the (3, 1) component of δ_g^* simply multiplies the scalar λ it acts on by h , producing the tangential-tangential 2-tensor λh ; similarly for other occurrences of h .

Kronecker delta. Moreover, we have $R^{\lambda}_{\kappa\mu}{}^{\nu} = 0$ except for $3\Lambda^{-1}R^0{}_{ki}{}^0 = h_{ik}$, $3\Lambda^{-1}R^0{}_{i0}{}^j = \delta_i^j$, $3\Lambda^{-1}R^l{}_{0i}{}^0 = \delta_i^l$, $3\Lambda^{-1}R^l{}_{00}{}^j = h^{lj}$, $3\Lambda^{-1}R^l{}_{ki}{}^j = \delta_i^l\delta_k^j - h_{ik}h^{jl} + \tau^2 R(h)_{ki}{}^j$, where $h^{jl} = h^{-1}(dx^j, dx^k)$ denotes the coefficients of the dual metric of h ; this gives $\text{Ric}_{00} = -\Lambda$, $\text{Ric}_{i0} = 0$, and $\text{Ric}_{ij} = \Lambda h_{ij} + (\Lambda/3)\tau^2 \text{Ric}(h)_{ij}$. The expressions in the lemma follow from this by direct computation. \square

The calculations in the proof imply that $\text{Ric}(g) - \Lambda g \in \tau^2 \mathcal{C}^\infty(M; S^{2-0}T^*M)$, i.e. any g of the form (2.13) satisfies the Einstein equation modulo $\mathcal{O}(\tau^2)$ errors.

In [37, 49], the linearization of P_0 is computed as

$$L_{0,g}\dot{g} := D_g P_0(\dot{g}) = \square_g - 2\delta_g^* \delta_g \mathbf{G}_g + 2\mathcal{R}_g - 2\Lambda. \quad (2.14)$$

Using Lemma 2.4, we compute the leading and subleading order behavior of $L_{0,g}$:

Corollary 2.5. *For g as in (2.10), and in the bundle splitting (2.12), we have*

$$3\Lambda^{-1}I(L_{0,g}, \lambda) = \begin{pmatrix} 3\lambda - 6 & 0 & -\lambda(3\lambda - 6) & 0 \\ 0 & 0 & 0 & 0 \\ 6 - \lambda & 0 & -\lambda(6 - \lambda) & 0 \\ 0 & 0 & 0 & \lambda(\lambda - 3) \end{pmatrix},$$

$$3\Lambda^{-1}I(L_{0,g}[\tau], \lambda) = \begin{pmatrix} 0 & 2(1 - \lambda)\delta_h & 0 & 0 \\ 2d_X & 0 & -2\lambda d_X & -\lambda\delta_h \\ 0 & \frac{2}{3}(\lambda - 5)\delta_h & 0 & 0 \\ 0 & (4 - 2\lambda)\delta_{h,0}^* & 0 & 0 \end{pmatrix}.$$

Proof. In the calculation of $\delta_g^* \delta_g \mathbf{G}_g$, one needs to use $e_0\tau = \tau(e_0 + 1)$ as well as $\delta_h h = -d_X$ and $\delta_h^* = \delta_{h,0}^* - \frac{1}{3}h\delta_h$ to obtain the stated expression for $I(L_{0,g}[\tau], \lambda)$. \square

Typically, metrics do have τ -dependence. The following two lemmas describe the (linearized) Einstein operator for lower order perturbations of (2.13).

Lemma 2.6. *If $\alpha > 0$, $\tilde{g} \in \mathcal{A}^\alpha(M; S^{2-0}T^*M)$, then $L_{0,g+\tilde{g}} - L_{0,g} \in \mathcal{A}^\alpha \text{Diff}_0^2(M; S^{2-0}T^*M)$, see (2.9). If $\tilde{g} \in \tau^m \mathcal{C}^\infty$ for some $m \in \mathbb{N}$, then $L_{0,g+\tilde{g}} - L_{0,g} \in \tau^m \text{Diff}_0^2$.*

Proof. Using that the space \mathcal{A}^α is a $\mathcal{C}^\infty(M)$ -module which is closed under multiplication, we have $(g + \tilde{g})^{-1} - g^{-1} \in \mathcal{A}^\alpha(M; S^{2-0}T^*M)$. Hence $\mathbf{G}_{g+\tilde{g}} - \mathbf{G}_g \in \mathcal{A}^\alpha(M; \text{End}(S^{2-0}T^*M))$. Similarly, indicating the metric in the notation for the Levi-Civita connection by a superscript,

$$\nabla^{g+\tilde{g}} - \nabla^g \in \mathcal{A}^\alpha \text{Diff}_0^1(M; {}^0T^*M; {}^0T^*M \otimes {}^0T^*M).$$

This implies that $\delta_{g+\tilde{g}}^* - \delta_g^* \in \mathcal{A}^\alpha \text{Diff}_0^1(M; {}^0T^*M; S^{2-0}T^*M)$, similarly for the other operators appearing in (2.14). For the proof of the second part of the lemma, replace \mathcal{A}^α by $\tau^m \mathcal{C}^\infty$. \square

In particular, for $m \geq 2$, the indicial families $I(L_{0,g+\tilde{g}}[\tau^j], \lambda)$, $j=0, 1$, are independent of \tilde{g} ; likewise (suitably interpreted) for $\tilde{g} \in \mathcal{A}^\alpha$, $\alpha > 1$.

Lemma 2.7. *With P_0 defined in (2.10), suppose $g \in C^\infty(M; S^{2-0}T^*M)$. If $\alpha > 0$, $\tilde{g} \in \mathcal{A}^\alpha(M; S^{2-0}T^*M)$, then $P_0(g + \tilde{g}) - P_0(g) - L_{0,g}\tilde{g} \in \mathcal{A}^{2\alpha}(M; S^{2-0}T^*M)$. Similarly, if $\tilde{g} \in \tau^m \mathbb{C}$, $m \in \mathbb{N}$, then $P_0(g + \tilde{g}) - P_0(g) - L_{0,g}\tilde{g} \in \tau^{2m}\mathcal{C}^\infty$.*

Proof. This follows similarly to the proof of Lemma 2.6. Since $P_0(g)$ and $L_{0,g}\tilde{g}$ capture all terms of $P_0(g + \tilde{g})$ which are at most linear in \tilde{g} , the difference $P_0(g + \tilde{g}) - P_0(g) - L_{0,g}\tilde{g}$ only contains terms which are at least quadratic in \tilde{g} , hence its coefficients, as a 0-differential operator, have the stated decay. \square

3. Multi-Schwarzschild-de Sitter spacetimes

In this section, we show how to glue several Schwarzschild-de Sitter metrics into global de Sitter space; we shall work near the future conformal boundary, hence on

$$M = [0, 1)_\tau \times \mathbb{S}^3, \quad \tau = \cos s. \quad (3.1)$$

The de Sitter metric is of the form discussed in Lemmas 2.6–2.7. Indeed, we have

$$g_{\text{dS}} \in 3\Lambda^{-1}\tau^{-2}(-d\tau^2 + h) + \tau^2 \mathcal{C}^\infty, \quad h = 3\Lambda^{-1}g_{\mathbb{S}^3}, \quad (3.2)$$

We recall the Schwarzschild-de Sitter (SdS) metric with mass $m \in \mathbb{R}$:

$$g_m = -\left(\frac{\Lambda r^2}{3} - 1 + \frac{2m}{r}\right)^{-1} dr^2 + \left(\frac{\Lambda r^2}{3} - 1 + \frac{2m}{r}\right) dt^2 + r^2 g_{\mathbb{S}^2}. \quad (3.3)$$

We consider the metric (3.3) for $r > r_+$, where r_+ is the largest positive real root of $\Lambda r^2/3 - 1 + 2m/r$ if one exists; otherwise, fix an arbitrary $r_+ > 0$. As in (2.3d), we put $\tau_s = r^{-1}$, and thus g_m is a smooth 0-metric on

$$M_{m,s} := \left[0, r_+^{-1}\right]_{\tau_s} \times \mathbb{R}_t \times \mathbb{S}^2. \quad (3.4)$$

Comparison with the de Sitter metric, expressed in the same coordinates and on the manifold M_s (see (2.3c)) by (2.3b) and (2.3d) (thus $g_{\text{dS}} = g_m|_{m=0}$), shows that

$$g_m - g_{\text{dS}} \in \tau_s^3 \mathcal{C}^\infty(M_s \cap M_{m,s}; S^{2-0}T^*M_s) \quad (3.5)$$

in their common domain of definition. Note that at $\tau_s = 0$, we have, in the upper half space coordinates (2.3a), $\tilde{\tau} = 0$ and $\tilde{R} = e^{-t\sqrt{3/\Lambda}}$. In particular, $t \rightarrow \infty$ corresponds to $\tilde{R} \rightarrow 0$; let $p_0 = (1, 0, 0, 0) \in \mathbb{S}^3 \subset \mathbb{R}^4$ denote the point $\tilde{R} = 0$ inside $\tilde{\tau} = 0$. Moreover, $t \rightarrow -\infty$ corresponds to $\tilde{R} \rightarrow \infty$, which on global de Sitter space corresponds to the antipodal point $-p_0 \in \mathbb{S}^3$ inside ∂M by inspection of (2.2a).

In summary, by relating the coordinates in (3.4) to the semi-global de Sitter manifold (3.1), g_m can be regarded as gluing an SdS black hole into de Sitter space at the point p_0 at the future conformal boundary $\tau = 0$. Given a point $p \in \mathbb{S}^3$, choose a rotation $R \in \text{SO}(4)$ with $Rp = p_0$; this induces a map $(\tau, \psi) \mapsto (\tau, R(\psi))$ on M . Pulling back g_m along this map, we obtain the metric

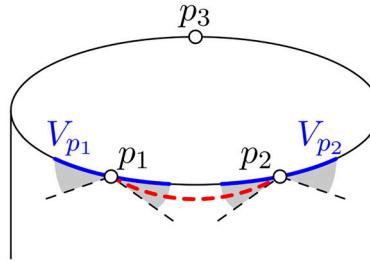


Figure 4. Illustration of Theorem 3.2, focusing on a neighborhood of p_1, p_2 ; the shaded regions indicate where we glue in the SdS metrics g_{p_i, m_i} , $i = 1, 2$. The blue segments indicate the sets V_{p_i} . The red dashed line indicates a piece of the past boundary of the domain U on which we construct the metric g . The difference to Figure 2 is that here we do not require the masses to be subextremal, hence we content ourselves with gluing the far end of the cosmological region of several SdS black holes into de Sitter space.

$$g_{p, m}, \quad p \in \mathbb{S}^3, \quad m \in \mathbb{R}, \quad (3.6)$$

with $g_{p, m}$ defined in a neighborhood of $U_p = \mathbb{S}^3 \setminus \{p, -p\}$. See Figure 3.

Definition 3.1. Let $N \in \mathbb{N}$. We say that $\{(p_i, m_i) : i = 1, \dots, N\} \subset \mathbb{S}^3 \times \mathbb{R}$ is *balanced* if the p_i are pairwise distinct and if, regarding \mathbb{S}^3 as the unit sphere inside \mathbb{R}^4 , the following relation holds:

$$\sum_{i=1}^N m_i p_i = 0 \in \mathbb{R}^4.$$

We can now state our main theorem:

Theorem 3.2. Let $N \in \mathbb{N}$, and suppose $\{(p_1, m_1), \dots, (p_N, m_N)\} \subset \mathbb{S}^3 \times \mathbb{R}$ is balanced. Suppose $V_{p_i} \subset U_{p_i}$ is a ball around p_i with the point p_i removed, and suppose $\overline{V_{p_i}} \cap \overline{V_{p_j}} = \emptyset$ for $i \neq j$. Then there exist a neighborhood U of $\partial M \setminus \{p_1, \dots, p_N\}$ and a Lorentzian 0-metric $g \in C^\infty(U; S^2 \wedge T_U^* M)$ with the following properties:

- (1) g satisfies the Einstein vacuum equation $\text{Ric}(g) - \Lambda g = 0$;
- (2) near V_{p_i} , we have $g = g_{p_i, m_i}$;
- (3) g is $\mathcal{O}(\tau^3)$ -close to the de Sitter metric: $g - g_{\text{dS}} \in \tau^3 C^\infty(U; S^2 \wedge T_U^* M)$.

See Figure 4. In the case of small subextremal masses, we can say more about the domain of existence of g ; we discuss this at the end of §3.3.

Remark 3.3. We only explicitly describe here how to glue a piece of the cosmological region of an SdS black hole into de Sitter space. As is well-known (see e.g. [38, §3.1]), the metric g_m in (3.3) merely has a coordinate singularity at the *cosmological horizon* $r = r_+$ if the mass is subextremal, meaning $0 < 9\Lambda m^2 < 1$. After a suitable (singular) coordinate change, g_m is analytic. There is another coordinate singularity at the event horizon, located at the second largest root of $\Lambda r^2/3 - 1 + 2m/r$, beyond which the metric can again be extended analytically. Thus, one can paste these extended subextremal SdS metrics into neighborhoods of p_i and thus, via Theorem 3.2, glue subextremal SdS metrics, extended as far as one wishes, into de Sitter space. This is depicted in Figure 2.

We show the necessity of the balance assumption under certain decay assumptions on g :

Theorem 3.4. Let $(p_1, \mathbf{m}_1), \dots, (p_N, \mathbf{m}_N) \in \mathbb{S}^3 \times \mathbb{R}$, with the p_i pairwise distinct. Suppose g satisfies the conclusions (1)–(2) of [Theorem 3.2](#). If, for some $\epsilon > 0$, we have $g - g_{\text{ds}} \in \tau^3(\log \tau)\mathcal{C}^\infty + \tau^3\mathcal{C}^\infty + \mathcal{A}^{3+\epsilon}(M; \mathbb{S}^{2-0}T^*M)$, then $\{(p_1, \mathbf{m}_1), \dots, (p_N, \mathbf{m}_N)\}$ is balanced.

In particular, this applies for metrics g satisfying part (3) of [Theorem 3.2](#). The uniqueness theorem is not sharp; the inclusion of a $\tau^3 \log \tau$ term merely serves as a demonstration that the inclusion of a logarithmic term does not help in constructing a (formal) solution of $P_0(g) = 0$ when the balance condition is violated. The determination of *sharp* conditions under which the balance condition of [Definition 3.1](#) is necessary for the existence of a metric g satisfying (1)–(2) is left as an open problem. We remark that the analysis of the Einstein vacuum equation for metrics g with $g - g_{\text{ds}} \in \tau^2\mathcal{C}^\infty$ or \mathcal{A}^2 (or even less decay) is necessarily *nonlinear* on the level of $\mathcal{O}(\tau^4)$ contributions to $\text{Ric}(g) - \Lambda g$.

Let $\chi_i \in \mathcal{C}^\infty(\partial M)$ denote cutoffs, identically 1 near $\overline{V_{p_i}}$, and with mutually disjoint supports; put $\chi_0 := 1 - \sum_{i=1}^N \chi_i$. The starting point of the proof of [Theorem 3.2](#) is the naively glued metric

$$g_0 := \chi_0 g_{\text{ds}} + \sum_{i=1}^N \chi_i g_{p_i, \mathbf{m}_i}. \quad (3.7)$$

Away from the points p_i , we have $g_0 - g_{\text{ds}} \in \tau^3\mathcal{C}^\infty$. We shall show in [§3.1](#) that the failure $P_0(g_0) = 2(\text{Ric}(g_0) - \Lambda g_0)$ of g_0 to solve the Einstein vacuum equation lies in $\tau^4\mathcal{C}^\infty$ and is supported away from the p_i , but it is always nonzero except in the trivial case that $\mathbf{m}_i = 0$ for all i . The goal is to find a correction $\tilde{g} \in \tau^3\mathcal{C}^\infty$, with support disjoint from $\overline{V_{p_i}}$, such that $P_0(g_0 + \tilde{g}) = 0$. To accomplish this, we proceed in several steps:

- (1) We improve the error to $P_0(g_0 + \tilde{g}_0) \in \tau^5\mathcal{C}^\infty$ by solving an underdetermined divergence equation for \tilde{g}_0 ; the balance condition ensures the solvability, while the underdetermined nature of the equation enables us to choose \tilde{g}_0 to vanish identically near the p_i . See [§3.1](#).
- (2) We find g in the wave map gauge with background metric $g^0 := g_0 + \tilde{g}_0$ by solving a suitable gauge-fixed Einstein equation $P(g) = 0$. This is done in two steps.
 - (a) One can construct $g_1 = g^0 + \tilde{g}_1$, $\tilde{g}_1 \in \tau^5\mathcal{C}^\infty$, with $P(g_1) \in \mathcal{C}^\infty(M)$ by inverting the indicial family of $D_{g_{\text{ds}}} P$ and using a Borel summation argument. See [§3.2](#).
 - (b) In order to solve away the final error, we solve the quasilinear wave equation $P(g_1 + \tilde{g}_2) = 0$ backwards from ∂M , with solution $\tilde{g}_2 \in \mathcal{C}^\infty(M)$. See [§3.3](#).
- (3) Also in [§3.3](#), we show that g solves the Einstein vacuum equation by means of the usual argument involving the second Bianchi identity and a unique continuation argument at ∂M .

With P_0 as in [Definition 2.3](#), we shall write from now on

$$L_0 := L_{g_{\text{ds}}, 0} = D_{g_{\text{ds}}} P_0. \quad (3.8)$$

3.1. Obstructed problem for the leading order correction

We will prove:

Proposition 3.5. *Under the assumptions of Theorem 3.2, and with g_0 defined by (3.7), there exists $\tilde{g}_0 \in \tau^3 \mathcal{C}^\infty(M; S^{2-0} T^* M)$, vanishing near $\cup_i \overline{V_{p_i}}$, so that $P_0(g_0 + \tilde{g}_0) \in \tau^5 \mathcal{C}^\infty$.*

We begin by computing the error produced by naively gluing a single SdS black hole into a neighborhood of $p_0 \in \partial M$:

Lemma 3.6. *In the coordinates (3.3), let $\chi \in \mathcal{C}^\infty(\mathbb{R}_t)$ be identically 1 for large t , and put $g_0 = \chi(t)g_m + (1 - \chi(t))g_{\text{dS}}$. With $P_0 = 2(\text{Ric} - \Lambda)$ as in (2.10), we then have*

$$P_0(g_0) \equiv \tau_s^4 \text{Err}_{s,0} \bmod \tau_s^5 \mathcal{C}^\infty, \quad \text{Err}_{s,0} = 2 \frac{d\tau_s}{\tau_s} \otimes_s \frac{12m}{\Lambda} \frac{d\chi}{\tau_s}.$$

Proof. Since g_m and g_{dS} solve the Einstein equation, $P_0(g_0)$ is supported on $\{\chi \neq 0, 1\}$. By Lemma 2.7 and in view of (3.5), we have

$$P_0(g_0) = P_0(g_{\text{dS}} + \chi(g_m - g_{\text{dS}})) \equiv L_0(\chi(g_m - g_{\text{dS}})) \bmod \tau_s^6 \mathcal{C}^\infty; \quad (3.9)$$

but for $\chi \equiv 1$, the left hand side vanishes, hence

$$L_0(g_m - g_{\text{dS}}) \in \tau_s^6 \mathcal{C}^\infty. \quad (3.10)$$

Now, note that $\gamma := g_m - g_{\text{dS}} = \tau_s^3 \gamma_3 + \tau_s^4 \gamma_4 + \mathcal{O}(\tau_s^5)$, with γ_3 and γ_4 independent of τ_s when expressed in terms of the bundle splitting (2.12), with $d\tau_s/\tau_s = -dr/r$, τ_s , and $h_s = (\Lambda^2/9)dt^2 + (\Lambda/3)g_{\mathbb{S}^2}$ (see (2.4)) taking the roles of e^0 , τ , and h ; explicitly,

$$\gamma_3 = \frac{18m}{\Lambda^2} \frac{dr^2}{r^2} + 2m(r dt)^2 = \left(\frac{18m}{\Lambda^2}, 0, \frac{6m}{\Lambda^2}, \frac{4m}{3} dt^2 - \frac{2m}{\Lambda} g_{\mathbb{S}^2} \right).$$

In view of (3.10), or by direct calculation using Corollary 2.5, we have $I(L_0, 3)\gamma_3 = 0$ and $I(L_0[\tau_s], 3)\gamma_3 + I(L_0, 4)\gamma_4 = 0$; thus, (3.9) implies that, modulo $\tau_s^5 \mathcal{C}^\infty$,

$$\begin{aligned} P_0(g_0) &\equiv \tau_s^4 \left(I(L_0[\tau_s], 3)(\chi\gamma_3) + I(L_0, 4)(\chi\gamma_4) \right) \\ &= \tau_s^4 [I(L_0[\tau_s], 3), \chi] \gamma_3 \\ &= \tau_s^4 (\Lambda/3) (0, [-3\delta_{h_s}, \chi](\gamma_3)_{TT0}, 0, 0) \\ &= \tau_s^4 \left(0, \frac{12m}{\Lambda} d\chi, 0, 0 \right) \end{aligned} \quad (3.11)$$

since $[\delta_{h_s}, \chi] = -\iota_{\nabla^{h_s}} \chi$, ι denoting contraction, and $\nabla^{h_s} \chi = \chi' \nabla^{h_s} t = 9\Lambda^{-2} \chi'(t) \partial_t$. \square

Thus, $\text{Err}_{s,0} = \mathcal{O}(\tau_s^4)$ is a tangential-normal tensor. In order to proceed, let us pretend we want to glue a single SdS black hole into M . Since $I(L_0, 4)_{NT} = 0$ by Corollary 2.5, we *cannot* solve away $\text{Err}_{s,0}$ with a $\mathcal{O}(\tau_s^4)$ metric correction. Since $\text{Err}_{s,0} = \mathcal{O}(\tau_s^4)$ is due to the $\mathcal{O}(\tau_s^3)$ difference of the metrics g_m and g_{dS} , we shall instead attempt to solve away $\text{Err}_{s,0}$ with a $\mathcal{O}(\tau_s^3)$ correction *with support not containing p_0* . To this end, note first that by Corollary 2.5,

$$\ker I(L_0, 3) = \ker \text{tr}_{h_s} \oplus \mathbb{R} \left(3(e^0)^2 + h_s \right) \oplus (2e^0 \otimes_s \tau_s^{-1} T^* X). \quad (3.12)$$

Written as a block matrix with respect to this splitting and (2.12), we furthermore have

$$3\Lambda^{-1}I(L_0[\tau_s], 3) = \begin{pmatrix} 0 & 0 & -4\delta_{h_s} \\ -3\delta_{h_s} & 0 & 0 \\ 0 & 0 & -\frac{4}{3}\delta_{h_s} \\ 0 & 0 & -2\delta_{h_s, 0}^* \end{pmatrix} : \mathcal{C}^\infty(M; \ker I(L_0, 3)) \rightarrow \mathcal{C}^\infty(M; S^{2-0}T^*M). \quad (3.13)$$

Thus, we need to find $k \in \mathcal{C}^\infty(\partial M; \ker \text{tr}_{h_s})$ which vanishes near p_0 and which solves $-(\Lambda/3)3\delta_{h_s}k = (\text{Err}_{s, 0})_{NT} = 12\mathfrak{m}\Lambda^{-1}d\chi$. A necessary condition for solvability is that the right hand side be $L^2(\partial M; |dh_s|)$ -orthogonal to the space $\ker \delta_{h_s, 0}^* \in \mathcal{C}^\infty(\partial M; T^*\partial M)$ of conformal Killing 1-forms. Identifying 1-forms with vector fields via the metric h_s , this condition reads

$$\int_{\partial M} V(\text{Err}_{s, 0})_{NT} |dh_s| = 0 \quad \text{for all conformal Killing vector fields } V \text{ on } (\partial M, h_s). \quad (3.14)$$

The space of conformal Killing vector fields only depends on the conformal class of the metric.¹⁰ Note then that h_s is conformal to $g_{\mathbb{S}^3}$; indeed, $h_s = \tau_s^2 \tau^{-2} g_{\mathbb{S}^3}$. The conformal Killing vector fields of the standard n -sphere, $n \geq 3$, are well-known (see e.g. [50, §1.4] and use the stereographic projection):

Proposition 3.7. *The space $\text{conf}_n = \ker \delta_{g_{\mathbb{S}^n}, 0}^* \subset \mathcal{V}(\mathbb{S}^n)$ is a direct sum*

$$\text{conf}_n = \text{iso}_n \oplus \text{scal}_n,$$

where $\text{iso}_n = \ker \delta_{g_{\mathbb{S}^n}}^* \cong \mathfrak{so}_{n+1}$ is the space of Killing vector fields (rotations) on \mathbb{S}^n , and

$$\text{scal}_n = \{S_q : q \in \mathbb{R}^n\}, \quad S_q : \mathbb{S}^n \ni p \mapsto q - \langle q, p \rangle p \in T_p \mathbb{S}^n, \quad (3.15)$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on \mathbb{R}^{n+1} .

Passing from τ_s to the global boundary defining function τ , the error Err_0 in $P_0(g_0) \equiv \tau^4 \text{Err}_0 \bmod \tau^5 \mathcal{C}^\infty$ has normal-tangential component

$$(\text{Err}_0)_{NT} = \tau^{-4} P_0(g_0)(\tau \partial_\tau, \tau \cdot)|_{T\mathbb{S}^3} = \tau^{-3} \tau_s^3 (\text{Err}_{s, 0})_{NT} = \tau^{-3} \tau_s^3 \frac{12\mathfrak{m}}{\Lambda} d\chi. \quad (3.16)$$

Remark 3.8. Since $|dg_{\mathbb{S}^3}| = \tau^3 \tau_s^{-3} |dh_s|$, the solvability condition (3.14) is equivalent to

$$\int_{\partial M} V(\text{Err}_0)_{NT} |dg_{\mathbb{S}^3}| = 0 \quad \forall V \in \text{conf}_3.$$

This has the same form as (3.14); thus, the condition (3.14) is conformally invariant.

Now, at $\tau=0$, both τ_s/τ and χ are functions of t only, thus of $\tilde{R} = |\tilde{x}|$ by (2.3a), and thus of the geodesic distance $d_{\mathbb{S}^3}(p_0, -)$ from the point $p_0 \in \mathbb{S}^3$ by (2.2a); thus, we have $\tau^{-3} \tau_s^3 d\chi = d\tilde{\chi}$ for some $\tilde{\chi} = \tilde{\chi}(d_{\mathbb{S}^3}(p_0, -))$. With $(\text{Err}_0)_{NT} = \frac{12\mathfrak{m}}{\Lambda} d\tilde{\chi}$, we can now compute

¹⁰Indeed, if (X, h) is a Riemannian manifold and V is a conformal Killing vector fields, so $\mathcal{L}_V h = fh$ for some $f \in \mathcal{C}^\infty(X)$, then $\mathcal{L}_V(e^{2\varphi} h) = e^{2\varphi}(f + 2V\varphi)h$ for any $\varphi \in \mathcal{C}^\infty(X)$.

$$\ell_{p_0, \mathfrak{m}} \in (\mathfrak{conf}_3)^*, \quad \mathfrak{conf}_3 \ni V \mapsto \int_{\mathbb{S}^3} V(\text{Err}_0)_{NT} |dg_{\mathbb{S}^3}|. \quad (3.17)$$

Namely, for $V \in \mathfrak{iso}_n = \ker \delta_{g_{\mathbb{S}^3}}^* \subset \ker \delta_{g_{\mathbb{S}^3}}$, integration by parts gives $\ell_{p_0, \mathfrak{m}}(V) = 0$. On the other hand, if $q = p_0 \in \mathbb{S}^3 \subset \mathbb{R}^4$, the vector field S_q is the radial vector field pointing toward p_0 , and

$$\ell_{p_0, \mathfrak{m}}(S_{p_0}) = C_0 \cdot 12\mathfrak{m}/\Lambda,$$

where $C_0 \in \mathbb{R}$ is a universal constant. We claim that $C_0 \neq 0$. Indeed, passing back to (3.14) note that S_q is a radial vector field, i.e. a $C^\infty(\mathbb{R}_R)$ -multiple of ∂_R and thus a $C^\infty(\mathbb{R}_t)$ -multiple of ∂_t , and hence must be a constant nonzero multiple of ∂_t , which is the unique conformal Killing vector field of h_s of this form. (In fact, ∂_t is Killing for h_s .) But

$$\int \partial_t \left(\frac{12\mathfrak{m}}{\Lambda} \chi'(t) dt \right) |dh_s| = \frac{12\mathfrak{m}}{\Lambda} \cdot (0-1) \cdot \frac{\Lambda^2}{9} \text{vol}(\mathbb{S}^2) = -\frac{16\pi\Lambda\mathfrak{m}}{3}$$

is nonzero, proving that $C_0 \neq 0$.

Finally, if $q \perp p_0$, then the integrand $S_q(\text{Err}_0)_{NT}$ in (3.17) is odd with respect to the reflection across the axis $\mathbb{R}p_0$, hence $\ell_{p_0, \mathfrak{m}}(S_q) = 0$ in this case. Therefore,

$$\ell_{p_0, \mathfrak{m}}(S_q) = C_0 \frac{12\mathfrak{m}}{\Lambda} \langle p_0, q \rangle, \quad q \in \mathbb{R}^4. \quad (3.18)$$

In particular, there is a nontrivial obstruction to gluing a single nontrivial ($\mathfrak{m} \neq 0$) SdS black hole into M . We summarize our findings in the following lemma:

Lemma 3.9. *Given $p_i \in \mathbb{S}^3$, $\mathfrak{m}_i \in \mathbb{R}$, and cutoff functions χ_i identically 1 near p_i and vanishing near $-p_i$, for $i = 1, \dots, N$, set $\text{Err}_{NT} := \sum_{i=1}^N (\text{Err}_{p_i, \mathfrak{m}_i})_{NT} \in \mathcal{C}^\infty(\mathbb{S}^3, T^*\mathbb{S}^3)$, where*

$$(\text{Err}_{p_i, \mathfrak{m}_i})_{NT} := \tau^{-4} P_0(\chi_i g_{p_i, \mathfrak{m}_i} + (1 - \chi_i) g_{\text{ds}})(\tau \partial_\tau, \tau W)|_{\tau=0}, \quad W \in T\mathbb{S}^3.$$

Then we have

$$\int_{\mathbb{S}^3} V(\text{Err}_{NT}) |dg_{\mathbb{S}^3}| = 0 \quad \forall V \in \mathfrak{conf}_3$$

if and only if $\{(p_1, \mathfrak{m}_1), \dots, (p_N, \mathfrak{m}_N)\}$ is balanced as in Definition 3.1.

Proof. If every χ_i is a radial cutoff, relative to the point p_i , the claim follows from (3.18) and the fact that $\sum_{i=1}^N \langle \mathfrak{m}_i p_i, q \rangle = 0$ for all $q \in \mathbb{R}^4$ if and only if $\sum_{i=1}^N \mathfrak{m}_i p_i = 0$, which is precisely the balance condition.

It remains to prove the lemma for general cutoffs. Observe that the difference of error terms produced by two different cutoffs $\chi_{i,j}$, $j = 1, 2$, to a neighborhood of the same point p_i lies in the range of δ_h acting on smooth 1-forms supported away from p_i . Indeed, similarly to (3.11), the difference is equal to $(0, -(\Lambda/3)3\delta_h((\chi_{i,1} - \chi_{i,2})(\gamma_3)_{TT0}), 0, 0)$ (in the splitting (2.12)) where $(\gamma_3)_{TT0}$ is the trace-free part of the tangential-tangential component (with respect to (3.1)) of $g_{p_i, \mathfrak{m}_i} - g_{\text{ds}}$; note that $\chi_{i,1} - \chi_{i,2}$ vanishes near p_i . \square

On any Riemannian manifold (X, h) with $\dim X \geq 2$, the divergence δ_h acting on trace-free symmetric 2-tensors has surjective principal symbol and thus closed range; in the present

context then, standard elliptic theory implies that under the balance condition, the error of [Lemma 3.9](#) can be written as $\text{Err}_{NT} = \delta_h k$ for some $k \in \mathcal{C}^\infty(\mathbb{S}^3; S^2 T^* \mathbb{S}^3)$, $\text{tr}_{g_{\mathbb{S}^3}} k = 0$. Crucially, we can do much better, since the overdetermined nature of this equation allows us to find k with strong support restrictions due to the following result due to Delay:

Theorem 3.10. (Delay [31, Theorem 1.3, Proposition 9.7, Corollary 8.4].) *Let (X, h) be a smooth Riemannian manifold, and let $\Omega \subset X$ be open. Suppose $f \in \mathcal{C}^\infty(X; T^* X)$ satisfies $\text{supp } f \Subset \Omega$ and $\int_{\Omega} V(f) |dh| = 0$ for all $V \in \mathcal{V}(\Omega)$ satisfying $\delta_{h,0}^* V^\flat = 0$.¹¹ Then there exists $k \in \mathcal{C}^\infty(X; S^2 T^* X)$ with $\text{tr}_h k = 0$ and $\text{supp } k \subset \bar{\Omega}$ such that $\delta_h k = f$.*

Proof of Proposition 3.5. Define the glued metric g_0 as in (3.7). As in (the proof of) [Lemma 3.6](#), we define Err to be the τ^4 coefficient of

$$P_0(g_0) \equiv \sum_{i=1}^N L_0(g_{\text{ds}} + \chi_i(g_{p_i, \mathfrak{m}_i} - g_{\text{ds}})) \bmod \tau^5 \mathcal{C}^\infty(M; S^{2-0} T^* M).$$

We can thus compute Err using [Lemma 3.6](#), its normal-tangential component is equal to Err_{NT} as defined in [Lemma 3.9](#).

Since the cutoffs χ_i are identically 1 in a neighborhood of $\overline{V_{p_i}}$, there exists an open set $\Omega \subset \mathbb{S}^3$ with $\bar{\Omega} \cap \overline{V_{p_i}} = \emptyset$ for all i , and so that $\text{supp } \text{Err}_{NT} \Subset \Omega$; moreover, we may choose Ω to be connected. Suppose $V \in \mathcal{V}(\Omega)$ is a conformal Killing vector field. We contend that $V = \tilde{V}|_{\Omega}$ for a conformal Killing vector field $\tilde{V} \in \mathcal{V}(\mathbb{S}^3)$. Indeed, on any connected n -dimensional Riemannian manifold, the dimension of the space of conformal Killing vector fields is at most $(n+1)(n+2)/2$, and on \mathbb{S}^n it is equal to this. We conclude that the restriction map $\ker_{\mathcal{V}(\mathbb{S}^3)} \delta_{h,0}^* \rightarrow \ker_{\mathcal{V}(\Omega)} \delta_{h,0}^*$, which is injective (as a consequence of the explicit description in [Proposition 3.7](#)), must be an isomorphism.

By [Lemma 3.9](#), the balance condition implies that the conditions of [Theorem 3.10](#) are satisfied; thus, there exists $k \in \mathcal{C}^\infty(\mathbb{S}^3; S^2 T^* \mathbb{S}^3)$, $\text{tr}_{g_{\mathbb{S}^3}} k = 0$, $\text{supp } k \subset \bar{\Omega}$, with

$$-(\Lambda/3) 3 \delta_h k = -\text{Err}_{NT}. \quad (3.19)$$

In the splitting (2.12), put

$$\tilde{g}_0 = (0, 0, 0, k) \in \mathcal{C}^\infty(M; S^{2-0} T^* M).$$

In view of (3.12), we have $\tilde{g}_0 \in \ker I(L_0, 3)$. Therefore, [Lemma 2.7](#) and [Corollary 2.5](#) imply that, modulo $\tau^5 \mathcal{C}^\infty(M; S^{2-0} T^* M)$,

$$P_0(g_0 + \tilde{g}_0) \equiv \tau^4 ((0, \text{Err}_{NT}, 0, 0) + I(L_0[\tau], 3) \tilde{g}_0) \equiv 0,$$

finishing the proof. \square

Remark 3.11. A direct calculation shows that the error Err_{NT} is, up to a constant rescaling, equal to the divergence (with respect to the induced metric h on ∂M) of the leading order term of the normal-tangential-normal-tangential part of the Weyl tensor of g_0 . Thus, [27, Lemma (3.1)], in particular [27, Equation (3.12)], requires the solution of

¹¹Note that if Ω has several connected components, the space of such V is larger than the space of conformal Killing vector fields on X .

the same divergence [equation \(3.19\)](#). Solving Friedrich's conformal Einstein field equations then produces a solution of the Einstein vacuum equation and proves [Theorem 3.2](#). As motivated in the introduction, we give a different, self-contained proof below.

3.2. Gauge fixing; construction of the formal solution

Following DeTurck [\[49\]](#), we make the following definition:

Definition 3.12. Let g^0 and g denote two Lorentzian metrics on the same manifold.

(1) We define the *gauge 1-form* by

$$\Upsilon(g; g^0) = g(g^0)^{-1} \delta_g \mathbf{G}_g g^0.$$

(2) The *gauge-fixed Einstein operator* is

$$P(g; g^0) := 2 \left(\text{Ric}(g) - \Lambda - \delta_g^* \Upsilon(g; g^0) \right).$$

Its linearization in the first argument is denoted

$$L_{g, g^0}(\dot{g}) = D_1|_g P(\dot{g}; g^0) := \frac{d}{ds} P(g + s\dot{g}; g^0)|_{s=0}.$$

We first discuss general properties of these operators. Following [\[37, §3\]](#), we have, using the Levi-Civita connection of g ,

$$\begin{aligned} D_1|_g \Upsilon(\dot{g}; g^0) &= \frac{d}{ds} \Upsilon(g + s\dot{g}; g^0)|_{s=0} = -\delta_g \mathbf{G}_g \dot{g} + \mathcal{C}(\dot{g}) - \mathcal{D}(\dot{g}), \\ C_{\mu\nu}^\kappa &= \frac{1}{2} \left((g^0)^{-1} \right)^{\kappa\lambda} \left(g_{\mu\lambda;\nu}^0 + g_{\nu\lambda;\mu}^0 - g_{\mu\nu;\lambda}^0 \right), \quad D^\kappa = g^{\mu\nu} C_{\mu\nu}^\kappa, \\ \mathcal{C}(\dot{g})_\kappa &= g_{\kappa\lambda} C_{\mu\nu}^\lambda \dot{g}^{\mu\nu}, \quad \mathcal{D}(\dot{g})_\kappa = D^\lambda \dot{g}_{\kappa\lambda}. \end{aligned}$$

In the special case $g = g^0$, we have $\mathcal{C} \equiv 0$ and $\mathcal{D} \equiv 0$, and therefore by [\(2.14\)](#)

$$L_{g, g} = \square_g + 2\mathcal{R}_g - 2\Lambda.$$

If moreover $g = (3/\Lambda)\tau^{-2}(-dt^2 + h(x, dx))$ is a product metric as in [\(2.13\)](#), then in the splitting [\(2.12\)](#)

$$3\Lambda^{-1} I(L_{g, g}, \lambda) = \lambda^2 - 3\lambda + \begin{pmatrix} -6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.20a)$$

For later use, we note that the indicial roots are, in increasing order,

$$\frac{1}{2}(3 - \sqrt{33}) \in (-2, -1), \quad -1, \quad 4, \quad \frac{1}{2}(3 + \sqrt{33}) \in (4, 5). \quad (3.20b)$$

For more general metrics, arguments similar to those in [Lemmas 2.6–2.7](#) give:

Lemma 3.13. Let $g_{00} \in \mathcal{C}^\infty(M; S^{2-0} T^* M)$, and suppose that $g^0 \in \mathcal{C}^\infty(M; S^{2-0} T^* M)$ is such that $g^0 - g_{00} \in \tau^m \mathcal{C}^\infty$ for some $m \in \mathbb{N}$. Suppose moreover that $\tilde{g} \in \tau^{m'} \mathcal{C}^\infty$ for some

$m' \in \mathbb{N}$, $m' \geq m$, and put $g = g^0 + \tilde{g}$.¹² Then $L_{g,g^0} - L_{g_{00},g_{00}} \in \tau^m \text{Diff}_0^2(M; S^{2-0} T^* M)$. Moreover, if $\dot{g} \in \tau^{m_2} \mathcal{C}^\infty$, then $P(g + \dot{g}; g^0) - P(g; g^0) - L_{g,g^0} \dot{g} \in \tau^{2m_2} \mathcal{C}^\infty$.

Applying this with g_{00} a product metric as in (2.13), we conclude that $I(L_{g,g^0}, \lambda)$ is equal to the right hand side of (3.20a).

Returning to the black hole gluing problem and the notation of Proposition 3.5, we now define the ‘background metric’ g^0 to be

$$g^0 = g_0 + \tilde{g}_0 \in \mathcal{C}^\infty(M; S^{2-0} T^* M). \quad (3.21)$$

Since $\Upsilon(g^0; g^0) = 0$ and $\text{Ric}(g_0) - \Lambda g_0 \in \tau^5 \mathcal{C}^\infty$, we have

$$P(g^0; g^0) \in \tau^5 \mathcal{C}^\infty(M; S^{2-0} T^* M);$$

moreover, by construction, $P(g^0; g^0)$ vanishes near $\cup_i \overline{V_{p_i}}$.

Proposition 3.14. *Under the assumptions of Proposition 3.5, and with g^0 as in (3.21), there exists a metric perturbation $\tilde{g}_1 \in \tau^5 \mathcal{C}^\infty(M; S^{2-0} T^* M)$, vanishing near $\cup_i \overline{V_{p_i}}$, so that $P(g^0 + \tilde{g}_1; g^0) \in \tau^\infty \mathcal{C}^\infty = \cap_{m \in \mathbb{N}} \tau^m \mathcal{C}^\infty$ (i.e. vanishing to infinite order at $\tau = 0$).*

Proof. Suppose we have already found \tilde{g}_1 as in the statement and with $P(g^0 + \tilde{g}_1; g^0) \in \tau^m \mathcal{C}^\infty$ for some $m \geq 5$; note that for $m = 5$, this holds for $\tilde{g}_1 = 0$. Moreover, under these assumptions, $P(g^0 + \tilde{g}; g^0)$ vanishes near $\cup_i \overline{V_{p_i}}$. Then, for $\dot{g} = \tau^m \dot{g}_0 \in \tau^m \mathcal{C}^\infty$, we have, using Lemma 3.13 and noting that $g^0 - g_{\text{ds}} \in \tau^3 \mathcal{C}^\infty$,

$$\begin{aligned} P(g^0 + \tilde{g}_1 + \dot{g}; g^0) &\equiv P(g^0 + \tilde{g}_1; g^0) + L_{g^0 + \tilde{g}_1; g^0} \dot{g} \bmod \tau^{2m} \mathcal{C}^\infty \\ &\equiv P(g^0 + \tilde{g}_1; g^0) + L_{g_{\text{ds}}, g_{\text{ds}}} \dot{g} \bmod \tau^{m+3} \mathcal{C}^\infty \\ &\equiv P(g^0 + \tilde{g}_1; g^0) + \tau^m I(L_{g_{\text{ds}}, g_{\text{ds}}}, m) \dot{g}_0 \bmod \tau^{m+1} \mathcal{C}^\infty. \end{aligned}$$

But for $m \geq 5$, $I(L_{g_{\text{ds}}, g_{\text{ds}}}, m)$ is invertible pointwise on ∂M in view of (3.20b), hence we can find $\dot{g}_0 \in \mathcal{C}^\infty(\partial M; S^{2-0} T_{\partial M}^* M)$, vanishing near $\cup_i \overline{V_{p_i}}$ such that this vanishes (modulo $\tau^{m+1} \mathcal{C}^\infty$). Replacing \tilde{g}_1 by $\tilde{g}_1 + \dot{g}$ improves the order of vanishing of $P(g^0 + \tilde{g}_1; g^0)$ at $\tau = 0$ by one order. A Borel summation argument produces a formal solution $\tilde{g}_1 \in \tau^5 \mathcal{C}^\infty$. \square

3.3. Solving the nonlinear equation: conclusion of the construction

Using indicial operator arguments, one cannot go beyond Proposition 3.14; the remaining (‘trivial’) error can however easily be solved away:

Proposition 3.15. *With $g_1 := g^0 + \tilde{g}_1$ defined using Proposition 3.14, there exists $\tilde{g}_2 \in \tau^\infty \mathcal{C}^\infty(M; S^{2-0} T^* M)$, vanishing near $\cup_i \overline{V_{p_i}}$, so that $g := g_1 + \tilde{g}_2$ satisfies*

$$P(g; g^0) = 0 \text{ near } \tau = 0. \quad (3.22)$$

¹²In particular, $g - g_{00} \in \tau^m \mathcal{C}^\infty$.

Proof. The key point is that forced linear wave equations on de Sitter space, or with any product metric of the form (2.13) or indeed any metric smoothly asymptotic to it, can be solved backwards on function spaces encoding sufficient polynomial decay in τ (i.e. sufficient *exponential* decay in $-\log \tau$), with the solution unique in such spaces; see the proof of [51, Lemma 1] (where N is the order of decay in $|x_1|$, $x_1 := -\tau^2$) for the relevant energy estimate, and the beginning of [33, §3] (where our τ, x are denoted x, y).

Since the error we need to solve away vanishes to all orders at $\tau = 0$, there is no need to choose vector field multipliers and *positive definite* vector bundle inner products on $S^2 \wedge^0 T^* M$ carefully in such energy estimates; rather, fixing *any* smooth positive definite inner product on $S^2 \wedge^0 T^* M$, one obtains an energy estimate using the vector field multiplier $\tau^{-2N} \tau \partial_\tau$ when N is sufficiently large. Indeed, the only contribution to the bulk term in the estimate which comes with a factor N in front arises from differentiating τ^{-2N} and is thus of the form $-2N\tau^{-2N} E(\tau \partial_\tau, \tau \partial_\tau)$, where E is the energy-momentum tensor of the wave u one wishes to estimate; all other bulk terms can be estimated by τ^{-2N} times a *bounded* (independently of N) multiple of $|u|^2 + |\tau \partial_\tau u|^2 + |\tau \partial_x u|^2$. But since $\tau \partial_\tau$ is timelike, choosing N large enough produces a coercive bulk term, and one obtains, for example, an estimate $\|u\|_{\tau^N \bar{H}_0^1(\Omega)} \leq C \|\square_{g_1} u\|_{\tau^N \bar{H}_0^0(\Omega)}$ for sufficiently large N , where $\Omega = \tau^{-1}([0, \frac{1}{2}])$. One can also commute any fixed number of 0-derivatives through the equation and thus (upon increasing N and C) obtain the estimate $\|u\|_{\tau^N \bar{H}_0^{k+1}(\Omega)} \leq C \|\square_{g_1} u\|_{\tau^N \bar{H}_0^k(\Omega)}$.

For the quasilinear wave equation at hand, we work with 0-Sobolev spaces with more than $\frac{1}{2}(\dim M) + 2 = 4$ derivatives; thus, fix $k_0 = 5 > 4$. Then by a simple adaptation of the standard iteration scheme for solving quasilinear wave equations (see e.g. [52, §16]), we obtain, for sufficiently large N_0 , a solution $\tilde{g}_2 \in \tau^{N_0} \bar{H}_0^{k_0}(\Omega_0)$ (unique in this space) of equation (3.22), where $\Omega_0 = \tau^{-1}([0, \epsilon_0])$ for sufficiently small $\epsilon_0 > 0$. Moreover, \tilde{g}_2 vanishes near each $\overline{V_{p_i}}$ since $P(g_1; g^0)$ does; recall that g_1 and g^0 are both equal to the Schwarzschild-de Sitter metric g_{p_i, m_i} near $\overline{V_{p_i}}$.

For any $k \geq k_0$, one can similarly find a solution of (3.22) in the space $\tau^N \bar{H}_0^k(\Omega)$ where Ω is a neighborhood of $\tau = 0$; since solutions of quasilinear wave equations can be continued (backwards, i.e. in the direction of increasing τ) in the same regularity class as long as a *fixed* low regularity norm remains finite, we can in fact take $\Omega = \Omega_0$. We conclude that $\tilde{g}_2 \in \cap_{N,k} \tau^N \bar{H}_0^k(\Omega_0) = \tau^\infty \mathcal{C}^\infty(\Omega_0)$, the final equality following from the fact that $\tau^k \bar{H}_0^{k+3}(\Omega_0) \subset \mathcal{C}^k(\Omega_0)$ by Sobolev embedding and using that $\partial_\tau = \tau^{-1} \cdot \tau \partial_\tau$ and $\partial_x = \tau^{-1} \cdot \tau \partial_x$. \square

By construction, the metric g meets the requirements (2)–(3) of Theorem 3.2. We prove that it also satisfies requirement (1); recall that $g - g^0 \in \tau^5 \mathcal{C}^\infty$ by Propositions 3.5 and 3.15, and $g - g^0$ vanishes near $\cup \overline{V_{p_i}}$.

Lemma 3.16. *Suppose g, g^0 are two Lorentzian metrics defined near $\partial M \setminus \{p_1, \dots, p_N\}$, smooth down to $\tau = 0$ as sections of $S^2 \wedge^0 T^* M$, and equal, modulo $\tau \mathcal{C}^\infty(M; S^2 \wedge^0 T^* M)$, to a metric of product type (2.13) near $\partial M \setminus \{p_1, \dots, p_N\}$. Suppose that near $\cup_i \overline{V_{p_i}}$, we have $g = g^0$ and $\text{Ric}(g) - \Lambda g = 0$. Suppose moreover that $\text{Ric}(g^0) - \Lambda g^0$ and $g - g^0$ lie in $\tau^5 \mathcal{C}^\infty(M; S^2 \wedge^0 T^* M)$. If $P(g; g^0) = 0$, then $\text{Ric}(g) - \Lambda g = 0$ and $\Upsilon(g; g^0) = 0$ near ∂M .*

Proof. The conclusion holds trivially near $\overline{V_{p_i}}$. Now, by the second Bianchi identity, the equation $\delta_g \mathbf{G}_g P(g; g^0) = 0$ implies the wave equation

$$2\delta_g \mathbf{G}_g \delta_g^* \Upsilon(g; g^0) = 0 \quad (3.23)$$

for the gauge 1-form $\Upsilon(g; g^0)$. By assumption, we have $\Upsilon(g; g^0) \in \tau^5 \mathcal{C}^\infty(M; {}^0 T^* M)$. The idea is to view equation (3.23) as a scattering problem ('initial value problem for data at infinity') for $\Upsilon(g; g^0)$. We need to show that the a priori decay of $\Upsilon(g; g^0)$ is a suitable replacement for vanishing Cauchy data in the usual proof of short time existence for the Einstein equation, in that it suffices to conclude $\Upsilon(g; g^0) \equiv 0$.

We first contend that in fact $\Upsilon(g; g^0) \in \tau^\infty \mathcal{C}^\infty$ vanishes to infinite order at ∂M ; this uses an indicial operator argument. Thus, if g_{00} is a product metric on a 4-manifold M as in (2.13), then we have, in the bundle splitting (2.12),

$$3\Lambda^{-1} I\left(2\delta_{g_{00}} \mathbf{G}_{g_{00}} \delta_{g_{00}}^*, \lambda\right) = \begin{pmatrix} \lambda^2 - 3\lambda - 6 & 0 \\ 0 & \lambda^2 - 3\lambda - 4 \end{pmatrix};$$

its indicial roots are given by (3.20b). If $g - g_{00} \in \tau \mathcal{C}^\infty$, then $2\delta_g \mathbf{G}_g \delta_g^* - 2\delta_{g_{00}} \mathbf{G}_{g_{00}} \delta_{g_{00}}^* \in \tau \text{Diff}_0^2$ by arguments similar to Lemmas 2.6 and 2.7.

Now, if we already know $\Upsilon(g; g^0) \in \tau^m \mathcal{C}^\infty$ for some $m \geq 5$ (the case $m = 5$ being our starting point), then, writing $\Upsilon(g; g^0) = \tau^m \Upsilon_0 + \tilde{\Upsilon}$ for $\Upsilon_0 \in \mathcal{C}^\infty(\partial M; {}^0 T_{\partial M}^* M)$ (τ -independent) and $\tilde{\Upsilon} \in \tau^{m+1} \mathcal{C}^\infty$, equation (3.23) implies

$$I\left(2\delta_g \mathbf{G}_g \delta_g^*, m\right) \Upsilon_0 = 0.$$

But the indicial operator appearing here is pointwise invertible, hence $\Upsilon_0 \equiv 0$ and therefore $\Upsilon(g; g^0) = \tilde{\Upsilon} \in \tau^{m+1} \mathcal{C}^\infty$. Since m was arbitrary, this proves our contention.

Finally, the rapid decay of $\Upsilon(g; g^0)$ at $\tau = 0$ (and its vanishing near the p_i where the metric g is singular) implies by a unique continuation argument for the wave equation (3.23), based on an energy estimate with multiplier $\tau^{-2N} \tau \partial_\tau$ for sufficiently large N , that $\Upsilon(g; g^0)$ vanishes identically near $\tau = 0$. See [51, Lemma 1]; a closely related alternative approach is given in [33, Proposition 5.3].

Since $P(g; g^0) = 0$ and $\Upsilon(g; g^0) = 0$, we conclude that $\text{Ric}(g) - \Lambda g = 0$ near $\tau = 0$. \square

The proof of Theorem 3.2 is complete.

Remark 3.17. Given a Riemannian metric h_0 and a transverse traceless tensor h_n on an n -manifold ∂M , the Fefferman–Graham construction [35, §3] produces a formal solution g^0 of $\text{Ric}(g^0) - \Lambda g^0 \in \tau^\infty \mathcal{C}^\infty$ on $M = [0, 1)_\tau \times \partial M$ of the form $g^0 = \tau^{-2}(-d\tau^2 + h(\tau, x, dx))$, where h_0 and h_n are the coefficients of τ^0 and τ^n in the polyhomogeneous expansion of h . (Concretely, h has an expansion into τ^j , $j \in \mathbb{N}_0$, and $\tau^j \log \tau$ for integer $j \geq n$.) Using this formal solution as a background metric for the gauge-fixed Einstein equation, Proposition 3.15 and Lemma 3.16 produce a true solution $g = g^0 + \mathcal{O}(\tau^\infty)$ of $\text{Ric}(g) - \Lambda g = 0$. We stress that this does not require any smallness conditions on the data h_0, h_n . See [42, Theorem 1.3] for a different approach.

We end with a discussion of the domain of existence when all masses are subextremal and small; we show that the cosmological horizons of at least two different black holes intersect nontrivially in the maximally globally hyperbolic development of the glued metric g of [Theorem 3.2](#) and [Remark 3.3](#). Let us work on the partial compactification

$$M = (0, \pi/2]_s \times \mathbb{S}^3, \quad g = (3/\Lambda) \cos^{-2}(s) (-ds^2 + g_{\mathbb{S}^3}),$$

of de Sitter space; the gluing theorem is, so far, local near $s = \pi/2$. Fix $N \geq 2$ distinct points $p_1, \dots, p_N \in \mathbb{S}^3$. If $N = 2$, we assume $p_2 \neq -p_1$. Denoting by d the Riemannian distance on $(\mathbb{S}^3, d_{\mathbb{S}^3})$, we set $d_0 := \min_{i \neq j} d(p_i, p_j) \in (0, \pi)$; without loss of generality, suppose the distance is minimized for p_1, p_2 so that

$$d_0 = d(p_1, p_2).$$

Let moreover $0 < r_0 < \pi/2$ be less than $\frac{1}{2}$ times the smallest radius of any of the balls V_{p_i} in [Theorem 3.2](#). Given subextremal masses $\mathfrak{m}_1, \dots, \mathfrak{m}_N$ so that

$$\mathcal{D} = \{(p_1, \mathfrak{m}_1), \dots, (p_N, \mathfrak{m}_N)\}$$

is balanced, the metric g constructed in [Theorem 3.2](#) is equal to g_{p_i, \mathfrak{m}_i} in the domain of dependence of $B(p_i, 2r_0)$. Fix $0 < \epsilon < r_0/16$, and define $\Sigma_0 \subset M$ as the union of $\mathbb{S}^3 \setminus \cup_{i=1}^N B(p_i, r_0) \subset \partial M$ and the N spacelike surfaces

$$\mathcal{N}_i := \{(s, p) \in M : d(p, p_i) = r_0 - \eta(\pi/2 - s), \quad \pi/2 - s \leq r_0/2 + 4\epsilon\},$$

where $1 < \eta < 2$ is fixed so that $r_0 - \eta(r_0/2 + 4\epsilon) > 0$. Note that \mathcal{N}_i penetrates the cosmological horizon of an observer in de Sitter space tending to p_i , i.e. the backwards light cone from $(\pi/2, p_i) \in \partial M$. We denote by

$$\mathcal{S}_{\eta, i} := \{(s, p) \in M : d(p, p_i) = r_0 - \eta(r_0/2 + \epsilon), \quad \pi/2 - s = r_0/2 + \epsilon\} \subset \mathcal{N}_i$$

a sphere which lies just inside of said cosmological horizon when $\eta - 1 \ll 1$. See [Figure 5](#).

Consider the rescaled gluing data

$$\lambda \mathcal{D} := \{(p_1, \lambda \mathfrak{m}_1), \dots, (p_N, \lambda \mathfrak{m}_N)\}$$

for $\lambda \geq 0$. For $\lambda = 0$, all masses vanish, hence we are gluing pieces of de Sitter space into de Sitter space—the result of course being de Sitter space, with metric defined globally on M . Let $q \in \mathbb{S}^3$ denote the midpoint between p_1 and p_2 (so $d(p_1, q) = d(p_2, q) = d_0/2$), and let

$$\bar{z}_j = (\bar{s}_j, \bar{p}_j) \in \mathcal{S}_{\eta, j}, \quad j = 1, 2,$$

denote the point on $\mathcal{S}_{\eta, j}$ for which \bar{p}_j is closest to q . If we had $\eta = 1$, then for any point $z = (s, q)$ with $\pi/2 - s > d_0/2 + 2\epsilon$, we would have

$$d(q, \bar{p}_j) = \frac{1}{2}(d_0 - r_0) + \epsilon < (\pi/2 - s) - r_0/2 - \epsilon = (\pi/2 - s) - (\pi/2 - \bar{s}_j),$$

so (\bar{s}_1, \bar{p}_1) and (\bar{s}_2, \bar{p}_2) are both contained in the timelike future $I^+(z)$ of z . For $\eta > 1$ sufficiently close to 1, and shrinking $\epsilon > 0$ if necessary, this holds for the point

$$z := (\pi/2 - d_0/2 - 3\epsilon, q) \in M.$$

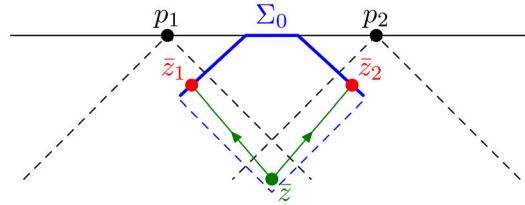


Figure 5. Illustration of the argument giving long-time existence of the metric g in [Theorem 3.2](#) for small subextremal masses. The geometry shown here is that of de Sitter space. The region enclosed by the blue lines is the backwards domain of dependence of the spacelike surface Σ_0 which consists of the three thick blue lines. By Cauchy stability, two observers starting at \bar{z} can reach the points $\bar{z}_1 \in \mathcal{S}_{\eta,1}$ and $\bar{z}_2 \in \mathcal{S}_{\eta,2}$ if we glue sufficiently light black holes into p_1 and p_2 , in which case the black dashed lines become the cosmological horizons of the black holes.

For small $\lambda \geq 0$, one can define the SdS metric $g_{p_i, \lambda m_i}$, extended across the cosmological horizon and defined in $(0, \pi/2]_s$ times a $2r_0$ -neighborhood of p_i inside \mathbb{S}^3 , in such a manner that as $\lambda \rightarrow 0$, the weighted difference $\tau^{-3}(g_{\text{dS}} - g_{p_i, \lambda m_i})$ (cf. [equation \(3.5\)](#)) converges smoothly to 0 as a section of $S^2 T^* M$ away from the line $L_i := \{(s, p_i) : s \in (-\pi/2, \pi/2]\}$.¹³ We claim that for sufficiently small $\lambda > 0$, we can do the SdS gluing with parameters \mathcal{D}_λ in such a way that

- (1) the point z is contained in the maximal globally hyperbolic development of the glued metric g_λ with respect to Σ_0 (and Σ_0 is spacelike for g_λ),
- (2) $\bar{z}_1, \bar{z}_2 \in I^+(z)$ with respect to g_λ ,
- (3) near \bar{z}_j , g_λ is equal to g_{p_j, m_j} for $j = 1, 2$, and
- (4) \bar{z}_j lies *inside* the cosmological horizon of the SdS black hole associated with the point p_j .

To begin, let $\lambda_0 > 0$ be a small fixed constant. Consider then the naively glued metric $g_{\lambda,0} = \chi_0 g_{\text{dS}} + \sum_{i=1}^N \chi_i g_{p_i, \lambda m_i}$ analogously to [\(3.7\)](#); this fails to solve the Einstein vacuum equation by the amount $P_0(g_{\lambda,0}) = \lambda \tau^4 \text{Err}(\lambda)$, where $\text{Err}(\lambda) \in \mathcal{C}^\infty([0, \lambda_0) \times M; S^2 T^* M)$ depends smoothly on $\lambda \geq 0$, and whose leading order term at ∂M can be computed using [Lemma 3.6](#) and [equation \(3.16\)](#); in particular, the leading order (τ^0) coefficient of Err is λ -independent. Here and below, we take $\lambda_0 > 0$ to be a small fixed constant. Thus, we can take the solution of the divergence equation $-(\Lambda/3)3\delta_{\mathbb{S}^3} k_\lambda = -\lambda \text{Err}(\lambda)_{NT}$ (cf. [equation \(3.19\)](#)), to be $k = \lambda k_1$ for some *fixed* $k_1 \in \mathcal{C}^\infty(\mathbb{S}^3; S^2 T^* \mathbb{S}^3)$. For small λ , we then work with the background metric $g_\lambda^0 = g_{\lambda,0} + k_\lambda$.

The remainder of the formal part of the gluing construction does not depend on any further choices; the Borel summation in the proof of [Proposition 3.14](#) can be defined to produce a metric correction $\tilde{g}_{\lambda,1} \in \lambda \tau^5 \mathcal{C}^\infty([0, \lambda_0) \times M; S^2 T^* M)$ with support in a small fixed neighborhood of ∂M . The resulting formal solution $g_{\lambda,1} = g_\lambda^0 + \tilde{g}_{\lambda,1}$ (cf. the

¹³A systematic and more precise way of accomplishing this is to use geometric microlocal techniques [54]. For a single SdS black hole centered at $p \in \mathbb{S}^3$, one starts with the total space $[0, \lambda_0) \times M$ and blows up $[0, \lambda_0) \times \{p\}$ and then $\{0\} \times L$, $L = (0, \pi/2) \times \{p\}$. The first blow-up resolves the singular nature—due to its r -dependence—of the SdS metric near p , and the second blow-up resolves the event horizon, whose r -coordinate goes to 0 roughly linearly with λ . The family of SdS metrics with mass λm can then be defined as a smooth section of the pullback of $S^2 T^* M$ to this resolved space, and, crucially, in such a manner that it equals the de Sitter metric on the lift of $\lambda = 0$.

statement of [Proposition 3.15](#)) satisfies the gauge-fixed Einstein equation with error $P(g_{\lambda,1}; g_{\lambda}^0) \in \lambda \tau^{\infty} \mathcal{C}^{\infty}([0, \lambda_0] \times M; S^{2-0} T^* M)$; the support of this error is disjoint from the \mathcal{N}_i . But then, since $g_{\lambda,1}$ and g_{λ}^0 converge smoothly to the de Sitter metric in the backwards domain of dependence of Σ_0 , Cauchy stability for the solution $\tilde{g}_{\lambda,2} \in \lambda \tau^{\infty} \mathcal{C}^{\infty}$ of the quasilinear equation $P(g_{\lambda,1} + \tilde{g}_{\lambda,2}; g_{\lambda}^0) = 0$ implies that $g_{\lambda} = g_{\lambda,1} + \tilde{g}_{\lambda,2}$ indeed exists (uniquely, by domain of dependence considerations) on a sufficiently large subset of M so that the requirements (1)–(4) are indeed met.

3.4. Necessity of the balance condition

In this section, we prove [Theorem 3.4](#). Thus, suppose that $\text{Ric}(g) - \Lambda g = 0$ for a metric g of the form

$$g = g_{\text{ds}} + \sum_{i=1}^N \chi_i (g_{p_i, \mathfrak{m}_i} - g_{\text{ds}}) + \tau^3 (\log \tau) g_{\ell} + \tau^3 g_3 + \tilde{g}, \quad \tilde{g} \in \mathcal{A}^{3+\epsilon}(M; S^{2-0} T^* M),$$

where $\chi_i \in \mathcal{C}^{\infty}(\partial M)$ is a cutoff localizing to a small neighborhood of p_i ; here $g_{\ell}, g_3 \in \mathcal{C}^{\infty}(\partial M; S^{2-0} T_{\partial M}^* M)$, and g_{ℓ}, g_3, \tilde{g} have supports disjoint from the p_i . Then, with $L_0 = D_{g_{\text{ds}}} P_0$ as in [\(3.8\)](#), [Lemma 2.7](#) gives

$$f := L_0 (\tau^3 (\log \tau) g_{\ell} + \tau^3 g_3 + \tilde{g}) + \tau^4 \text{Err} \in \mathcal{A}^{6-\delta} \quad \forall \delta > 0, \quad (3.24)$$

where the normal-tangential component Err_{NT} of $\text{Err} \in \mathcal{C}^{\infty}(\partial M; S^{2-0} T^* M)$ takes the form given in [Lemma 3.9](#).

Note that, for any g_{ℓ}, g_3, \tilde{g} in the above function spaces, we have $f \in \tau^3 (\log \tau) \mathcal{C}^{\infty} + \tau^3 \mathcal{C}^{\infty} + \mathcal{A}^{3+\epsilon}$. Using [\(3.24\)](#), its $\tau^3 \log \tau$ coefficient is $I(L_0, 3) g_{\ell} = 0$. In view of [\(3.12\)](#) (with h_s, τ_s replaced by h, τ , where h is the boundary metric [\(3.2\)](#)), we thus have, in the bundle splitting [\(2.12\)](#),

$$g_{\ell} = (3u, \eta, u, k),$$

where $u \in \mathcal{C}^{\infty}(\partial M)$, $\eta \in \mathcal{C}^{\infty}(\partial M; T^* \partial M)$, and $k \in \mathcal{C}^{\infty}(\partial M; S^2 T^* \partial M)$ with $\text{tr}_h k = 0$ are supported away from the p_i .

Using [\(2.7\)](#) and [Corollary 2.5](#), we then compute the τ^3 coefficient of f as

$$0 = I(L_0, 3) g_3 + \partial_{\lambda} I(L_0, \lambda)|_{\lambda=3} g_{\ell} = I(L_0, 3) g_3 + (\Lambda/3)(-3u, 0, -3u, 3k).$$

Since the pure trace part of the tangential-tangential component of $I(L_0, 3)$ always vanishes, we must have $k = 0$. But then we can then write

$$g_3 = g_{30} + g_{31}, \quad I(L_0, 3) g_{30} = 0, \quad g_{31} = (u, 0, 0, 0),$$

with g_{31} defined so that it solves $I(L_0, 3) g_{31} + \partial_{\lambda} I(L_0, \lambda)|_{\lambda=3} g_{\ell} = 0$.

Lastly then, projecting to the normal-tangential component, we have $L_0(\tilde{g})_{NT} \in \mathcal{A}^{4+\epsilon}$ by [Corollary 2.5](#), hence the τ^4 component of f is equal to

$$\begin{aligned} 0 &= \text{Err}_{NT} + \partial_{\lambda} I(L_0[\tau], \lambda)|_{\lambda=3} g_{\ell} + I(L_0[\tau], 3) g_{31} + I(L_0[\tau], 3) g_{30} \\ &= \text{Err}_{NT} + (\Lambda/3)(-2d_X)u + (\Lambda/3)2d_Xu + (\Lambda/3)(-3\delta_h(g_{30})_{TT0}) \\ &= \text{Err}_{NT} - \Lambda\delta_h(g_{30})_{TT0}, \end{aligned}$$

where $(g_{30})_{TT0} \in \mathcal{C}^\infty(\partial M; S^2 T^* \partial M)$ is the trace-free part of the tangential-tangential component of g_{30} . Integrating this against a conformal Killing vector field V , an integration by parts implies that the second term does not contribute, and therefore $\int_{\partial M} V(\text{Err}_{NT}) |dh| = 0$ for all $V \in \text{conf}_3$. An application of [Lemma 3.9](#) concludes the proof of [Theorem 3.4](#).

3.5. Gluing with noncompact spatial topology

The balance condition in [Definition 3.1](#) captures the orthogonality of the leading order error term to conformal Killing vector fields on \mathbb{S}^3 . If, however, we allow the trace-free 2-tensor k in [\(3.19\)](#) to be blowing up sufficiently fast at a point $p_\infty \in \mathbb{S}^3$ distinct from the p_i , this obstruction disappears, since elements of the relevant cokernel now need to vanish at sufficiently high order at p_∞ ; since conformal Killing vector fields on the sphere vanish at most quadratically at any given point, this cokernel is empty.

A more natural way to phrase this is to pass to the upper half space picture of de Sitter space, $M_u = [0, \infty)_{\tilde{\tau}} \times \mathbb{R}^3$, see [\(2.2b\)](#); the point p_∞ is the point at infinity within the conformal boundary ∂M_u , and we need to solve [equation \(3.19\)](#), with $h = (3/\Lambda)g_{\mathbb{R}^3}$ now a constant multiple of the Euclidean metric, and the error term having compact support in \tilde{x} . We can *always* solve this, with the solution k having support disjoint from the $\overline{V_{p_i}}$, if we allow k to be nonzero in $|\tilde{x}| \gg 1$ and allow for $\langle \tilde{x} \rangle^{-2+\epsilon}$ decay, $\epsilon > 0$. More precisely, using the function spaces of [\[18, Appendix A\]](#), we can find $k \in H_{\phi, \psi}^1(g_{\mathbb{R}^3})$ with $\phi = \langle \tilde{x} \rangle$ and $\psi = \langle \tilde{x} \rangle^{1/2-2\epsilon}$. (Refining the weights to be exponential at the boundary of the domain in which one wants k to be supported enforces the correct support of k .) Indeed, since $\delta_{g_{\mathbb{R}^3}} : H_{\phi, \psi}^1(g_{\mathbb{R}^3}) \rightarrow L_{\psi\phi}^2(g_{\mathbb{R}^3})$ by [\[18, \(A.4\)\]](#), the relevant cokernel (for $s=1$) consists of conformal Killing 1-forms ω on $(\mathbb{R}^3, g_{\mathbb{R}^3})$ lying in $L_{\psi^{-1}\phi^{-1}}^2(g_{\mathbb{R}^3})$. But since all such ω have size at least 1 near infinity, and since $\int |1|^2 \psi^{-2} \phi^{-2} |d\omega| \sim \int \langle \tilde{x} \rangle^{-3+4\epsilon} r^2 dr$ diverges, the space of such ω is trivial.

The remainder of the gluing construction as before; however, in the final step, the domain of existence of the quasilinear equation might shrink to zero as $|\tilde{x}| \rightarrow \infty$. (Even when not gluing *any* black hole into the upper half space model M_u , we point out that past directed null-geodesics leave M_u in finite affine time; see [\[47, Figure 7\]](#).) In summary:

Theorem 3.18. *Let $N \in \mathbb{N}$, and let $p_1, \dots, p_N \in \mathbb{R}^3$, $m_1, \dots, m_N \in \mathbb{R}$. Suppose $V_{p_i} \subset \mathbb{R}^3$ is a punctured neighborhood of p_i , and suppose $\overline{V_{p_i}} \cap \overline{V_{p_j}} = \emptyset$ for $i \neq j$. Then there exists a neighborhood U of $\partial M_u \setminus \{p_1, \dots, p_N\}$ and a Lorentzian 0-metric $g \in \mathcal{C}^\infty(U; S^2 T_U^* M_u)$ satisfying the properties (1)–(3) of [Theorem 3.2](#).*

Remark 3.19. More generally, we can glue any spacetime into de Sitter space whose metric is defined in an interval (in $\tilde{\tau}$) times an annulus (in \tilde{x}) around a point $p \in \partial M_u$, provided the metric is asymptotic to g_{dS} at a rate τ^3 in this region. This can be further relaxed, but we will not pursue this further.

4. Multi-Kerr-de Sitter spacetimes

The goal is to glue several Kerr-de Sitter (KdS) black holes into a neighborhood of the future conformal boundary of global de Sitter space; we thus continue to work on the manifold $M = [0, 1)_\tau \times \mathbb{S}^3$ as in (3.1).

4.1. Kerr-de Sitter metrics in corotating coordinates; parameterization

We recall the KdS metric with parameters $\mathfrak{m} \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}$ in the form

$$\begin{aligned} g_{\mathfrak{m}, \mathbf{a}} = & -\frac{\Delta_r}{\rho^2} \left(dt_0 - \frac{\mathbf{a} \sin^2 \theta_0}{\Delta_0} d\phi_0 \right)^2 + \frac{\rho^2}{\Delta_r} dr_0^2 + \frac{\rho^2}{\Delta_\theta} d\theta_0^2 \\ & + \sin^2 \theta_0 \frac{\Delta_\theta}{\rho^2} \left(\mathbf{a} dt_0 - \frac{r_0^2 + \mathbf{a}^2}{\Delta_0} d\phi_0 \right)^2, \end{aligned} \quad (4.1)$$

where we define (omitting the dependence on \mathfrak{m}, \mathbf{a} from the notation)

$$\begin{aligned} \Delta_r &= (r_0^2 + \mathbf{a}^2) \left(1 - \frac{\Lambda r_0^2}{3} \right) - 2\mathfrak{m}r_0, & \Delta_\theta &= 1 + \frac{\Lambda}{3} \mathbf{a}^2 \cos^2 \theta_0, \\ \rho^2 &= r_0^2 + \mathbf{a}^2 \cos^2 \theta_0, & \Delta_0 &= 1 + \frac{\Lambda}{3} \mathbf{a}^2. \end{aligned}$$

(This matches the expression in [41, Equations (5.2)–(5.4)] upon adding the subscripts ‘0’, and differs from that in [38, Equation (3.12)] only by a constant rescaling of t by Δ_0 .) Following [41, Appendix B],¹⁴ we recall the coordinate change which displays $g_{\mathfrak{m}, \mathbf{a}}$ as a perturbation of the de Sitter metric (2.3b) up to terms of size r^{-3} (as uniformly degenerate symmetric 2-tensors). Thus, under the change of coordinates¹⁵

$$\begin{aligned} t &= t_0, & \phi &= \phi_0 - \frac{\Lambda}{3} \mathbf{a} t_0, \\ r^2 &= \frac{1}{\Delta_0} (r_0^2 \Delta_\theta + \mathbf{a}^2 \sin^2 \theta_0), & r \cos \theta &= r_0 \cos \theta_0. \end{aligned}$$

the de Sitter metric g_{dS} in (2.3b) takes the form

$$\begin{aligned} g_{\text{dS}} = & \left(\frac{\Lambda}{3} (r_0^2 + \mathbf{a}^2 \sin^2 \theta_0) - 1 \right) dt_0^2 + \frac{\rho^2}{(r_0^2 + \mathbf{a}^2) \left(1 - \frac{\Lambda r_0^2}{3} \right)} dr_0^2 + \frac{\rho^2}{\Delta_\theta^2} d\theta_0^2 \\ & - \frac{2\Lambda}{3} \mathbf{a} \frac{r_0^2 + \mathbf{a}^2}{\Delta_0} \sin^2 \theta_0 dt_0 d\phi_0 + \frac{r_0^2 + \mathbf{a}^2}{\Delta_0} \sin^2 \theta_0 d\phi_0^2. \end{aligned}$$

Therefore

$$g_{\mathfrak{m}, \mathbf{a}} = g_{\text{dS}} + c_{\mathfrak{m}, \mathbf{a}}, \quad c_{\mathfrak{m}, \mathbf{a}} = \frac{2\mathfrak{m}r_0}{\rho^2} \left(dt_0 - \frac{\mathbf{a} \sin^2 \theta_0}{\Delta_0} d\phi_0 \right)^2 + \frac{2\mathfrak{m}r_0 \rho^2}{\Delta_r|_{\mathfrak{m}=0} \Delta_r} dr_0^2. \quad (4.2)$$

We then compute:

¹⁴For consistency with §3, the roles of t_0, r_0, \dots and t, r, \dots are reversed compared to the reference.

¹⁵The definition of r^2 implies that $r_0^2 \cos^2 \theta_0 \leq r^2$, hence θ is well-defined.

Lemma 4.1. Let $\tau_s = r^{-1}$, and denote by \mathcal{C}^∞ the space of functions which are smooth in $(t, \tau_s, \theta, \phi)$, $\tau_s = r^{-1}$, down to $\tau_s = 0$. The symmetric 2-tensor $c_{\mathbf{m}, \mathbf{a}}$ in (4.2) then has components, modulo $\tau_s^4 \mathcal{C}^\infty$, given by

$$\begin{aligned} r^2 (c_{\mathbf{m}, \mathbf{a}})_{rr} &\equiv r^{-3} \left[\left(\frac{\Delta_\theta}{\Delta_0} \right)^{3/2} \frac{18\mathbf{m}}{\Lambda^2} \right], & r^{-2} (c_{\mathbf{m}, \mathbf{a}})_{tt} &\equiv r^{-3} \left[\sqrt{\frac{\Delta_\theta}{\Delta_0}} 2\mathbf{m} \frac{\Delta_\theta^2}{\Delta_0^2} \right], \\ r^{-2} (c_{\mathbf{m}, \mathbf{a}})_{t\phi} &\equiv r^{-3} \left[\sqrt{\frac{\Delta_\theta}{\Delta_0}} \cdot (-2\mathbf{m}) \mathbf{a} \sin^2 \theta_0 \frac{\Delta_\theta}{\Delta_0^2} \right], & r^{-2} (c_{\mathbf{m}, \mathbf{a}})_{\phi\phi} &\equiv r^{-3} \left[\sqrt{\frac{\Delta_\theta}{\Delta_0}} 2\mathbf{m} \frac{\mathbf{a}^2 \sin^4 \theta_0}{\Delta_0^2} \right]. \end{aligned}$$

Furthermore, $(c_{\mathbf{m}, \mathbf{a}})_{rt} = (c_{\mathbf{m}, \mathbf{a}})_{r\phi} = (c_{\mathbf{m}, \mathbf{a}})_{t\theta} = (c_{\mathbf{m}, \mathbf{a}})_{\theta\phi} = 0$, and

$$r^{-2} (c_{\mathbf{m}, \mathbf{a}})_{\theta\theta} \in \tau_s^5 \mathcal{C}^\infty, \quad (c_{\mathbf{m}, \mathbf{a}})_{r\theta} \in \tau_s^3 \mathcal{C}^\infty.$$

Proof. Since $1 \leq \Delta_0, \Delta_\theta \leq 1 + \Lambda \mathbf{a}^2/3$, we record that

$$\frac{r}{r_0} \equiv \sqrt{\frac{\Delta_\theta}{\Delta_0}} \bmod \tau_s^2 \mathcal{C}^\infty, \quad (4.3)$$

and in particular $r/r_0, r_0/r \in \mathcal{C}^\infty$. Now, direct calculations give

$$\begin{aligned} \partial_{t_0} &= \partial_t - \frac{\Lambda}{3} \mathbf{a} \partial_\phi, & \partial_{r_0} &= \frac{\Delta_\theta r_0}{\Delta_0 r} \partial_r - \frac{\mathbf{a}^2 \sin^2 \theta_0 \cos \theta_0}{\Delta_0 r^3 \sin \theta} \\ \partial_\theta, \partial_{\phi_0} &= \partial_\phi, & \partial_{\theta_0} &= \frac{\mathbf{a}^2}{\Delta_0 r} \left(1 - \frac{\Lambda r_0^2}{3} \right) \cos \theta_0 \sin \theta_0 \partial_r \\ && &+ \frac{r_0 \sin \theta_0}{r \sin \theta} \left(1 + \frac{\mathbf{a}^2 \cos^2 \theta_0}{\Delta_0 r^2} \left(1 - \frac{\Lambda r_0^2}{3} \right) \right) \partial_\theta. \end{aligned}$$

The main structure of the right column is captured by

$$\begin{pmatrix} \partial_{r_0} \\ \partial_{\theta_0} \end{pmatrix} = \begin{pmatrix} \frac{\Delta_\theta r_0}{\Delta_0 r} & \tau_s^3 \mathcal{C}^\infty \\ \tau_s^{-1} \mathcal{C}^\infty & \mathcal{C}^\infty \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta \end{pmatrix},$$

with the bottom right entry invertible. Therefore,

$$\partial_t = \partial_{t_0} + \frac{\Lambda}{3} \mathbf{a} \partial_{\phi_0}, \quad \partial_\phi = \partial_{\phi_0}, \quad \begin{pmatrix} \partial_r \\ \partial_\theta \end{pmatrix} = \begin{pmatrix} \frac{\Delta_0 r}{\Delta_\theta r_0} & \tau_s^3 \mathcal{C}^\infty \\ \tau_s^{-1} \mathcal{C}^\infty & \mathcal{C}^\infty \end{pmatrix} \begin{pmatrix} \partial_{r_0} \\ \partial_{\theta_0} \end{pmatrix}.$$

Note also that $\rho^2 \equiv r_0^2 \bmod \mathcal{C}^\infty$, hence $2\mathbf{m}r_0/\rho^2 \equiv 2\mathbf{m}/r_0 \bmod \tau_s \mathcal{C}^\infty$, and moreover $\Delta_r \equiv -\Lambda r_0^4/3 \bmod \tau_s^{-2} \mathcal{C}^\infty$; therefore,

$$c_{\mathbf{m}, \mathbf{a}} \in \left(\frac{2\mathbf{m}}{r_0} + \tau_s^3 \mathcal{C}^\infty \right) \left(dt_0 - \frac{\mathbf{a} \sin^2 \theta_0}{\Delta_0} d\phi_0 \right)^2 + \left(\frac{18\mathbf{m}}{\Lambda^2 r_0^5} + \tau_s^7 \mathcal{C}^\infty \right) dr_0^2.$$

Thus, for instance, we have $r^2 (c_{\mathbf{m}, \mathbf{a}})_{rr} \equiv r^2 \frac{\Delta_0^2 r^2}{\Delta_\theta^2 r_0^2} \frac{18\mathbf{m}}{\Lambda^2 r_0^5}$, which gives the stated result upon using (4.3). The other components are calculated similarly. \square

Note that $r\partial_r = -\tau_s\partial_{\tau_s}$ and $r^{-1}\partial_{\bullet} = \tau_s\partial_{\bullet}$ for $\bullet = t, \theta, \phi$. Let now r_+ be such that $\inf_{\theta \in (0, \pi)} r_0(r_+, \theta)$ is larger than the largest positive real root of Δ_r (as a function of r_0) if one exists, and otherwise fix any $r_+ > 0$. Define the manifold

$$M_{\mathfrak{m}, \mathbf{a}, s} := \left[0, r_+^{-1}\right]_{\tau_s} \times \mathbb{R}_t \times \mathbb{S}_{\theta\phi}^2$$

where $\tau_s = r^{-1}$; then the lemma implies that

$$c_{\mathfrak{m}, \mathbf{a}} \in \tau_s^3 \mathcal{C}^\infty(M_{\mathfrak{m}, \mathbf{a}, s} \cap M_s, S^{2-0} T^* M_s)$$

on the common domain of definition of the KdS metric and the de Sitter metric, cf. (2.3c). (We leave it to the reader to check that $c_{\mathfrak{m}, \mathbf{a}} \in \tau_s^3 \mathcal{C}^\infty$ also at the poles of \mathbb{S}^2 where the polar coordinates break down.) In view of (2.3d), we in particular have $g_{\mathfrak{m}, \mathbf{a}} \in \mathcal{C}^\infty$ on the common domain of definition.

At $\tau_s = 0$, the limits $t \rightarrow \infty$ and $t \rightarrow -\infty$ correspond to $\tilde{R} \rightarrow 0$ and $\tilde{R} \rightarrow \infty$, respectively, in the upper half space coordinates (2.2a). Therefore, as in the SdS case, $g_{\mathfrak{m}, \mathbf{a}}$ is defined in a neighborhood of $U_{p_0} := \mathbb{S}^3 \setminus \{p_0, -p_0\}$ where $p_0 = (1, 0, 0, 0) \in \mathbb{S}^3 \subset \mathbb{R}^4$ is the point defined by $\tilde{R} = 0$ inside $\tilde{\tau} = 0$; it describes a KdS black hole rotating in p_0^\perp around an axis, which we fix to be

$$\hat{\mathbf{a}}_0 = (0, 0, 0, 1),$$

with specific angular momentum \mathbf{a} .

We wish to define KdS metrics located at other points on the future conformal boundary ∂M . To this end, we use a parameterization of the KdS family by triples

$$(p, \mathfrak{m}, \mathbf{a}), \quad p \in \mathbb{S}^3, \quad \mathfrak{m} \in \mathbb{R}, \quad \mathbf{a} \in \mathfrak{so}_4, \quad \mathbf{a}p = 0; \quad (4.4)$$

here, we identify \mathfrak{so}_4 both with $\{A \in \mathbb{R}^{4 \times 4} : A + A^T = 0\}$ and the space \mathfrak{iso}_3 of Killing vector fields on \mathbb{S}^3 where $A \in \mathbb{R}^{4 \times 4}$ corresponds to the vector field $\frac{d}{ds} e^{sA}|_{s=0}$ on the unit sphere $\mathbb{S}^3 \subset \mathbb{R}^4$. Thus, viewing $\mathbf{a} \in \mathfrak{so}_4$ as a matrix, the condition $\mathbf{a}p = 0$ means that $p \in \mathbb{S}^3 \subset \mathbb{R}^4$ lies in its kernel; viewing $\mathbf{a} \in \mathfrak{iso}_3$, it means that the vector field \mathbf{a} vanishes at p .

Definition 4.2. We call a triple $(p, \mathfrak{m}, \mathbf{a}) \in \mathbb{S}^3 \times \mathbb{R} \times \mathfrak{so}_4$ *admissible* if $\mathbf{a}p = 0$.

For $\mathbf{a}, \mathbf{a}_1, \mathbf{a}_2 \in \mathfrak{so}_4$, viewed as 4×4 matrices, we denote

$$\langle \mathbf{a}_1, \mathbf{a}_2 \rangle := \frac{1}{2} \sum_{i,j=1}^4 (\mathbf{a}_1)_{ij} (\mathbf{a}_2)_{ij} = \sum_{i < j} (\mathbf{a}_1)_{ij} (\mathbf{a}_2)_{ij}, \quad |\mathbf{a}|^2 := \langle \mathbf{a}, \mathbf{a} \rangle. \quad (4.5)$$

(Invariantly, $\langle \cdot, \cdot \rangle$ is $(-\frac{1}{2})$ times the Killing form on \mathfrak{so}_4 .) Given an admissible triple $(p, \mathfrak{m}, \mathbf{a})$, we define the metric

$$g_{p, \mathfrak{m}, \mathbf{a}} \quad (4.6)$$

as a smooth Lorentzian 0-metric near $U_p = \partial M \setminus \{p, -p\}$ as follows. First, if $\mathbf{a} = 0$, we let $g_{p, \mathfrak{m}, 0} = g_{p, \mathfrak{m}}$ be equal to the SdS metric with mass \mathfrak{m} centered at p , as defined in the paragraph leading up to (3.6). Otherwise, $\mathbf{a} \in \mathbb{R}^{4 \times 4}$ induces a nontrivial skew-adjoint linear transformation on $p^\perp \subset \mathbb{R}^4$, equivalently a rotation vector field, around an axis

$\hat{\mathbf{a}} \in p^\perp$, $|\hat{\mathbf{a}}| = 1$, with amplitude $\mathbf{a} := |\mathbf{a}|$. Choose then an element $R_1 \in SO(4)$ with $R_1 p = p_0$, and then (noting that $R_1 \hat{\mathbf{a}} \in p_0^\perp$) an element $R_2 \in SO(4)$ with $R_2 p_0 = p_0$ so that $R_2(R_1 \hat{\mathbf{a}}) = \hat{\mathbf{a}}_0$; put $R = R_2 R_1$. Note that the properties $Rp = p_0$ and $R\hat{\mathbf{a}} = \hat{\mathbf{a}}_0$ determine R uniquely up to multiplication from the left by a rotation fixing p_0 and $\hat{\mathbf{a}}_0$, thus a rotation $\phi \mapsto \phi + \phi'$ for some $\phi' \in \mathbb{R}$ —which is an isometry of $g_{\mathbf{m}, \mathbf{a}}$. We then define $g_{p, \mathbf{m}, \mathbf{a}}$ as the pullback of $g_{\mathbf{m}, \mathbf{a}}$ along the map $M \rightarrow M$, $(\tau, \psi) \mapsto (\tau, R(\psi))$. In particular, this parameterizes $g_{\mathbf{m}, \mathbf{a}}$ as

$$g_{\mathbf{m}, \mathbf{a}} = g_{p_0, \mathbf{m}, \mathbf{a}_0}, \quad \mathbf{a}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{a} & 0 \\ 0 & -\mathbf{a} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.7)$$

4.2. Gluing theorem

With the KdS metrics $g_{\mathbf{m}, p, \mathbf{a}}$ defined as in §4.1, we are ready to state the gluing theorem, which holds subject to a balance condition generalizing Definition 3.1; it involves the *effective mass* of an admissible triple $b = (p, \mathbf{m}, \mathbf{a})$, defined as

$$\mathbf{m}_{\text{eff}}(b) := \frac{\mathbf{m}}{(1 + \Lambda|\mathbf{a}|^2/3)^2}.$$

Definition 4.3. Let $N \in \mathbb{N}$. We say that a collection $\{b_1, \dots, b_n\}$ of admissible triples $b_i = (p_i, \mathbf{m}_i, \mathbf{a}_i)$ is *balanced* if the p_i are pairwise distinct and if, regarding \mathbb{S}^3 as the unit sphere inside \mathbb{R}^4 , the following relations hold:

$$\sum_{i=1}^N \mathbf{m}_{\text{eff}}(b_i) p_i = 0 \in \mathbb{R}^4, \quad (4.8a)$$

$$\sum_{i=1}^N \mathbf{m}_{\text{eff}}(b_i) \mathbf{a}_i = 0 \in \mathfrak{so}_4 \subset \mathbb{R}^{4 \times 4}. \quad (4.8b)$$

Theorem 4.4. Let $N \in \mathbb{N}$, and suppose $\{b_1, \dots, b_N\} \subset \mathbb{S}^3 \times \mathbb{R} \times \mathfrak{so}_4$ is balanced, $b_i = (p_i, \mathbf{m}_i, \mathbf{a}_i)$. Suppose $V_{p_i} \subset U_{p_i}$ is a ball around p_i with the point p_i removed, and suppose $\overline{V_{p_i}} \cap \overline{V_{p_j}} = \emptyset$ for $i \neq j$. Then there exist a neighborhood U of $\partial M \setminus \{p_1, \dots, p_N\}$ and a Lorentzian 0-metric $g \in \mathcal{C}^\infty(U; S^2 {}^0 T_U^* M)$ with the following properties:

- (1) g satisfies the Einstein vacuum equation $\text{Ric}(g) - \Lambda g = 0$;
- (2) near V_{p_i} , we have $g = g_{p_i, \mathbf{m}_i, \mathbf{a}_i}$;
- (3) g is $\mathcal{O}(\tau^3)$ -close to the de Sitter metric: $g - g_{\text{dS}} \in \tau^3 \mathcal{C}^\infty(U; S^2 {}^0 T_U^* M)$.

In the special case that $b_i = (p_i, \mathbf{m}_i, 0)$ for all i , this reduces to Theorem 3.2.

Remark 4.5. A remark analogous to Remark 3.3 applies also in the Kerr-de Sitter setting: if the black hole parameters are subextremal, one can extend the glued Kerr-de Sitter metrics across their cosmological and event horizons. See e.g. [38, §3.2]. For small

masses, the domain of existence of g can be shown to include the interaction of several black holes by following the arguments at the end of §3.3.

Remark 4.6. If one passes to the upper half space model M_u , there are no obstructions to gluing anymore, analogously to [Theorem 3.18](#).

The main part of the proof of [Theorem 4.4](#) is the calculation of the obstruction for solving the divergence [equation \(3.19\)](#). First, we compute the failure of the Einstein equation for a naive gluing of a single KdS black hole. Let $P_0 = 2(\text{Ric} - \Lambda)$ and $L_0 = D_{g_{\text{dS}}} P_0$ as in [\(3.8\)](#).

Lemma 4.7. *Let $\chi \in C^\infty(\mathbb{R}_t)$ be identically 1 for large t , and put $g_0 = \chi(t)g_{\mathbb{m},\mathbf{a}} + (1 - \chi(t))g_{\text{dS}}$. With $P_0 = 2(\text{Ric} - \Lambda)$ as in [\(2.10\)](#), we then have $P_0(g_0) = \tau_s^4 \text{Err}_{s,0} \bmod \tau_s^5 \mathcal{C}^\infty$, where $\text{Err}_{s,0} = 2 \frac{d\tau_s}{\tau_s} \otimes_s \frac{(\text{Err}_{s,0})_{\text{NT}}}{\tau_s} + \text{Err}'_{s,0}$ with*

$$(\text{Err}_{s,0})_{\text{NT}} = \frac{18\mathbf{m}}{\Lambda \Delta_0^2} \Delta_\theta \sqrt{\frac{\Delta_\theta}{\Delta_0}} \chi'(t) \left(\left(\Delta_\theta - \frac{1}{3} \Delta_0 \right) dt - \mathbf{a} \sin^2 \theta_0 d\phi \right),$$

and $\text{Err}'_{s,0} = \tau_s^4 I(L_0, 4)\tilde{c}$ for some $\tilde{c} \in C^\infty(\partial M_s; S^{2-0} T_{\partial M_s}^* M_s)$ with $\text{supp } \tilde{c} \subset \text{supp } d\chi$.

Proof. Recall from [\(2.4\)](#) the metric $h_s = (\Lambda^2/9)dt^2 + (\Lambda/3)g_{\mathbb{S}^2}$ induced on the boundary ∂M_s by g_{dS} and the boundary defining function r^{-1} . In the splitting [\(2.11\)](#) with h, τ replaced by h_s, τ_s , the leading order components of $c_{\mathbb{m},\mathbf{a}} = g_{\mathbb{m},\mathbf{a}} - g_{\text{dS}}$ are then, by [Lemma 4.1](#),

$$\begin{aligned} (\gamma_3)_{NN} &= \left(r^3 \cdot r^2 (c_{\mathbb{m},\mathbf{a}})_{rr} \right) \Big|_{\tau_s=0} = \left(\frac{\Delta_\theta}{\Delta_0} \right)^{3/2} \frac{18\mathbf{m}}{\Lambda^2}, \\ (\gamma_3)_{TT} &= r^3 \cdot \left. \left(r^2 \left((c_{\mathbb{m},\mathbf{a}})_{tt} dt^2 + 2(c_{\mathbb{m},\mathbf{a}})_{t\phi} dt d\phi + (c_{\mathbb{m},\mathbf{a}})_{\phi\phi} d\phi^2 \right) \right) \right|_{\tau_s=0}, \end{aligned}$$

and $(\gamma_3)_{\text{NT}}$ is a smooth 1-form on $\mathbb{R}_t \times \mathbb{S}^2$ whose precise form we do not need.

Since $0 = P_0(g_{\mathbb{m},\mathbf{a}}) \equiv L_0(c_{\mathbb{m},\mathbf{a}}) \bmod \tau_s^6 \mathcal{C}^\infty$ as in the proof of [Lemma 3.6](#), and since $c_{\mathbb{m},\mathbf{a}} \equiv r^{-3}\gamma_3 \bmod \tau_s^4 \mathcal{C}^\infty$, we conclude that $I(L_0, 3)\gamma_3 = 0$. In view of [\(3.12\)](#), this implies the relationship $(\gamma_3)_{NN} = \text{tr}_{h_s}(\gamma_3)_{TT}$ (using that $3 = \text{tr}_{h_s}h_s$).¹⁶ Therefore, the trace-free part $(\gamma_3)_{TT0}$ in the refined splitting [\(2.12\)](#) is given by

$$(\gamma_3)_{TT0} = (\gamma_3)_{TT} - \frac{1}{3}(\gamma_3)_{NN}h_s.$$

By following the calculation [\(3.11\)](#), the normal-tangential component of $\text{Err}_{s,0}$ is thus

$$(\text{Err}_{s,0})_{\text{NT}} = -\Lambda \cdot 9\Lambda^{-2} \chi'(t) \cdot (-\iota_{\partial_t}(\gamma_3)_{TT0}),$$

which we can compute by means of [Lemma 4.1](#).

Regarding the remaining components of $\text{Err}_{s,0}$, we note that they lie in the range of the third column of the operator [\(3.13\)](#). But by [Corollary 2.5](#), we have

¹⁶This can also be checked directly. Indeed, the equality of $\text{tr}_{h_s}(\gamma_3)_{TT} = \frac{9}{\Lambda^2}(\gamma_3)_{tt} + \frac{3}{\Lambda} \sin^{-2}\theta (\gamma_3)_{\phi\phi}$ and $(\gamma_3)_{NN}$ is equivalent to $\Delta_\theta^2 + \sin^{-2}\theta \frac{\Lambda}{3} \mathbf{a}^2 \sin^4 \theta_0 = \Delta_\theta \Delta_0$ and thus to $\Delta_\theta = \frac{\sin^2 \theta_0}{\sin^2 \theta}$; this is easily verified by plugging in $\sin^2 \theta = 1 - \frac{r^2}{\Lambda^2} \cos^2 \theta_0 = 1 - \frac{\Delta_0}{\Delta_\theta} \cos^2 \theta_0$, which holds at $\tau_s = 0$.

$$3\Lambda^{-1}I(L_0, 4) = \begin{pmatrix} 6 & 0 & -24 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & -8 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix},$$

whose range is thus spanned by $(1, 0, \frac{1}{3}, 0)$ and $\ker \text{tr}_h$, and hence contains the range of the third column of (3.13). \square

Since the components of $\text{Err}_{s,0}$ other than the normal-tangential component can thus be solved away pointwise on ∂M_s (modulo one order down, i.e. modulo $\tau_s^5 \mathcal{C}^\infty$) with a $\tau_s^4 \mathcal{C}^\infty$ correction, the only obstruction for gluing is again the integral of $(\text{Err}_{s,0})_{NT}$ against conformal Killing vector fields on $(\partial M_s, h_s) = (\mathbb{R}_t \times \mathbb{S}^2, h_s)$ as in (3.14). The volume density in these integrals is

$$|dh_s| = \frac{\Lambda^2}{9} dt \sin \theta d\theta d\phi = \frac{\Lambda^2}{9} \sqrt{\frac{\Delta_0}{\Delta_\theta}} \Delta_\theta^{-1} dt_0 \sin \theta_0 d\theta_0 d\phi_0,$$

since at $\tau_s = 0$ we have, using (4.3),

$$\begin{aligned} \sin \theta d\theta &= -d(\cos \theta) = -d\left(\frac{r_0}{r} \cos \theta_0\right) = -d\left(\sqrt{\frac{\Delta_0}{\Delta_\theta}} \cos \theta_0\right) \\ &= \left(\sin \theta_0 \sqrt{\frac{\Delta_0}{\Delta_\theta}} - \frac{1}{2} \cos \theta_0 \frac{\Delta_0^{1/2}}{\Delta_\theta^{3/2}} \cdot 2 \frac{\Lambda}{3} \mathbf{a}^2 \cos \theta_0 \sin \theta_0 \right) d\theta_0 = \sqrt{\frac{\Delta_0}{\Delta_\theta}} \Delta_\theta^{-1} \sin \theta_0 d\theta_0. \end{aligned}$$

By Lemma 4.7, we therefore have, for $V \in \mathcal{V}(\partial M_s)$,

$$\begin{aligned} \ell_{\mathbf{m}, \mathbf{a}}(V) &:= \int_{\partial M_s} V(\text{Err}_{s,0})_{NT} |dh_s| \\ &= 2\Lambda \mathbf{m}_{\text{eff}} \int_0^\infty \int_0^{2\pi} \int_0^\pi \chi'(t_0) V\left(\left(\Delta_\theta - \frac{1}{3} \Delta_0\right) dt - \mathbf{a} \sin^2 \theta_0 d\phi\right) \sin \theta_0 d\theta_0 d\phi_0 dt_0, \end{aligned} \tag{4.9}$$

where $\mathbf{m}_{\text{eff}} := \mathbf{m}/\Delta_0^2$ is the effective mass of the triple $(p_0, \mathbf{m}, \mathbf{a})$. Particular conformal Killing vector fields V on $(\partial M_s, h_s)$ include ∂_t and ∂_ϕ , and we compute

$$\ell_{\mathbf{m}, \mathbf{a}}(\partial_t) = -\frac{16\pi\Lambda \mathbf{m}_{\text{eff}}}{3}, \quad \ell_{\mathbf{m}, \mathbf{a}}(\partial_\phi) = \frac{16\pi\Lambda \mathbf{m}_{\text{eff}} \mathbf{a}}{3}.$$

If V is a rotation around an axis orthogonal to that corresponding to ∂_ϕ , then the integrand in (4.9) vanishes pointwise, hence $\ell_{\mathbf{m}, \mathbf{a}}(V) = 0$ in this case. Passing to the boundary \mathbb{S}^3 of global de Sitter space, with the KdS black hole sitting at the point $p_0 = (1, 0, 0, 0) \in \mathbb{S}^3$, we have $\partial_t = S_{Cp_0}$ in the notation (3.15) for some constant $C > 0$ (only depending on Λ), while the rotations on the \mathbb{S}^2 -factor of ∂M_s which we considered above span the set $(\mathfrak{so}_4)_{p_0}$ of rotations on \mathbb{S}^3 keeping p_0 fixed.

Consider rotations $V \in \mathfrak{so}_4$ which are orthogonal to $(\mathfrak{so}_4)_{p_0}$ with respect to the inner product $\langle \cdot, \cdot \rangle$ defined in (4.5); the 3-dimensional space of such V is spanned by rotation

vector fields R_j , $j = 2, 3, 4$, which are, say, 90 degree rotations in the planes determined by $p_0 = (1, 0, 0, 0)$ and $\hat{e}_2 = (0, 1, 0, 0)$, $\hat{e}_3 = (0, 0, 1, 0)$, $\hat{e}_4 = (0, 0, 0, 1)$, respectively, and which keep the orthogonal complement of $\text{span}\{p_0, \hat{e}_j\}$ in \mathbb{R}^4 fixed. But then the integrand in (4.9), for each $j = 2, 3, 4$, is odd either with respect to the reflection $\theta_0 \mapsto \pi - \theta_0$ or with respect to the rotation $\phi_0 \mapsto \phi_0 + \pi$, hence $\ell_{\mathbf{m}, \mathbf{a}}(R_j) = 0$. A similar symmetry argument shows that $\ell_{\mathbf{m}, \mathbf{a}}(S_{\hat{e}_j}) = 0$ for $j = 2, 3, 4$.

Note now that ∂_ϕ , written as a rotation matrix (rotating in the plane spanned by \hat{e}_1 and \hat{e}_2 , while leaving the span of p_0 and \hat{e}_3 fixed), is given by

$$\partial_\phi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By comparison with (4.7), we can thus summarize our calculations by

$$\ell_{\mathbf{m}, \mathbf{a}}(S_q) = C_0 \mathbf{m}_{\text{eff}} \langle p_0, q \rangle, \quad q \in \mathbb{R}^4,$$

$$\ell_{\mathbf{m}, \mathbf{a}}(\mathbf{a}) = C_1 \mathbf{m}_{\text{eff}} \langle \mathbf{a}_0, \mathbf{a} \rangle, \quad \mathbf{a} \in \mathfrak{so}_4,$$

where C_0, C_1 are nonzero real constants. We then have the following analogue of Lemma 3.9:

Lemma 4.8. *Given admissible triples b_1, \dots, b_N as in Theorem 4.4, $b_i = (p_i, \mathbf{m}_i, \mathbf{a}_i)$, with the p_i pairwise distinct, suppose $\chi_i \in \mathcal{C}^\infty(\partial M)$ are cutoff functions, which are identically 1 near p_i . Set $\text{Err}_{NT} := \sum_{i=1}^N (\text{Err}_{b_i})_{NT} \in \mathcal{C}^\infty(\mathbb{S}^3; T^* \mathbb{S}^3)$, where*

$$(\text{Err}_{b_i})_{NT}(W) := \tau^{-4} P_0(\chi_i g_{p_i, \mathbf{m}_i, \mathbf{a}_i} + (1 - \chi_i) g_{\text{ds}})(\tau \partial_\tau, \tau W)|_{\tau=0}, \quad W \in T\mathbb{S}^3.$$

Then we have

$$\int_{\mathbb{S}^3} V(\text{Err}_{NT}) |d\mathbf{g}_{\mathbb{S}^3}| = 0 \quad \forall V \in \mathbf{conf}_3$$

if and only if $\{b_1, \dots, b_N\}$ is balanced as in Definition 4.3.

The remainder of the gluing construction is very similar to the SdS gluing:

Proof of Theorem 4.4. The only minor difference compared to the proof of Theorem 3.2 is the analogue of Proposition 3.5. Under the balance condition we can solve away the normal-tangential component of the error term using Delay's result. However, the τ^4 leading order part of the error in general now has other nonvanishing components as well; but as demonstrated in Lemma 4.7, these error terms lie in the range of $I(L_0, 4)$ and can thus be solved away pointwise on ∂M using a $\tau^4 \mathcal{C}^\infty(\partial M; S^2 \wedge T^* M)$ metric correction, with support of this correction contained in $\cup_i \text{supp } d\chi_i$.

The rest of the proof is the same: one constructs a formal solution in a generalized harmonic gauge as in Proposition 3.14, solves away the remaining 'trivial' error as in Proposition 3.15, and thus obtains a solution of the Einstein vacuum equation by appealing to Lemma 3.16. \square

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