Logarithmic structures of Fontaine-Illusie. II. — Logarithmic flat topology.

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Abstract

This is Part II of the author's paper Logarithmic structures of Fontaine-Illusie. We discuss log flat topology and log flat descent. We study the first log flat cohomology $H^1(X_{\log,\mathrm{fl}}, G)$ for various sheaves of groups G, for example, $G = GL_n$, finite flat commutative group schemes, the log multiplicative group M^{gp}, etc.

Contents

Introduction.

This is a continuation of the paper [K$_1$] on the foundation of log geometry in the sense of Fontaine-Illusie. Here we discuss mainly log flat topologies, especially log flat descent theory.

This paper was started around 1991, and was circulated as an incomplete preprint for a long time. Since then, some contents of this paper have been reproduced by several authors with proofs ([Ha], [KS], [Ni], [Na2], [Ol], ...). A. Moriwaki [M] also studied flat descents in the category of log schemes. In the parts which were incomplete in the circulated preprint, we sometimes referred to these papers instead of completing the original proofs. In particular, the author does not claim the results with ∗ (i.e., two theorems 7.1, 7.2 and one proposition 6.5) are his results. Since the paper is already referred to in many published works, in the other parts, we preferred to preserve the original, circulated form. In both parts, we tried to preserve the original numberings of definitions and propositions.

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1 Basic facts.

In this section, we review basic facts about logarithmic schemes in $[\mathrm{K}_1]$, and introduce "fs log schemes" and "log flat morphisms."

1.1. In this paper, a monoid is always assumed to be commutative and has a unit element. A homomorphism of monoids is assumed to respect the unit elements. The semi-group law of a monoid is usually written multiplicatively.

For a monoid P, we denote by $P^\times$ the group of all invertible elements of P, and by P^{gp} the group hull $P^{-1}P = \{a^{-1}b \; ; \; a, b \in P\}$ of P.

1.2. *Pre-logarithmic structures, logarithmic structures, logarithmic schemes.* A pre-log structure on a scheme X is a sheaf of monoids M on the étale site $X_{\mathrm{ét}}$ endowed with a homomorphism of sheaves of monoids

$$\alpha\colon M \to \mathcal{O}_X,$$

where $\mathcal{O}_X$ is regarded as a monoid for the multiplicative law.

A pre-log structure M is called a log structure if

$$\alpha^{-1}(\mathcal{O}_X^\times) \overset{\cong}{\to} \mathcal{O}_X^\times \qquad \text{via } \alpha.$$

A log scheme is a scheme endowed with a log structure. A morphism of log schemes is defined in a natural way. For a log scheme X, the log structure of X is usually denoted by M_X.

1.3. *The log structure associated to a pre-log structure.* For a scheme X and a pre-log structure $M \overset{\alpha}{\to} \mathcal{O}_X$, the log structure $M^\sim$ associated to M is defined to be the pushout of the diagram

$$M \leftarrow \alpha^{-1}(\mathcal{O}_X^\times) \overset{\alpha}{\to} \mathcal{O}_X^\times$$

in the category of sheaves of monoids, which is endowed with the homomorphism $M^\sim \to \mathcal{O}_X$ induced by α and by $\mathcal{O}_X^\times \overset{\subset}{\to} \mathcal{O}_X$.

1.4. *The inverse image of a log structure.* Let $f\colon X \to Y$ be a morphism of schemes, and let M be a log structure on Y. Then the inverse image f^*M of M is defined to be the log structure on X associated to the pre-log structure $f^{-1}(M)$ endowed with $f^{-1}(M) \to f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$. Here $f^{-1}(M)$ denotes the sheaf-theoretic inverse image of M.

Definition 1.5. (1) We say a monoid P is *integral* if the canonical map $P \to P^{\mathrm{gp}}$ is injective (that is, if $ab = ac$ $(a, b, c \in P)$ implies $b = c$).

We regard an integral monoid P as a submonoid of P^{gp}.

(2) We say a monoid P is *saturated* if P is integral and satisfies the following condition: If $a \in P^{\mathrm{gp}}$ and $a^n \in P$ for some $n \geq 1$, then $a \in P$.

(3) We call a finitely generated integral monoid a *fine monoid*. We call a finitely generated saturated monoid an *fs monoid*.

1.6. *Fine log schemes and fs log schemes.* We call a log scheme X a fine (resp. an fs) log scheme if the following condition is satisfied: Étale locally on X, there exists a fine (resp. an fs) monoid P and a homomorphism $\alpha \colon P \to \mathcal{O}_X$ such that M_X is isomorphic to the log structure associated to the constant sheaf P on X regarded as a pre-log structure with respect to α.

An fs log scheme is fine. A fine log scheme X is fs if and only if M_X is a sheaf of saturated monoids.

For a fine (resp. fs) log scheme X, the stalk $(M_X/\mathcal{O}_X^\times)_{\overline{x}}$ is a fine monoid (resp. an fs monoid) for any $x \in X$.

1.7. *Charts.* (1) A chart of a fine (resp. an fs) log scheme X is a pair (P, h), where P is a fine (resp. an fs) monoid and h is a homomorphism $P \to M_X$ satisfying the following condition: Let $P^\sim$ be the log structure associated to $P \to \mathcal{O}_X$ induced by h. Then the induced map $P^\sim \to M_X$ is an isomorphism.

A chart for X exists étale locally on X.

(2) Let $f \colon X \to Y$ be a morphism of fine (resp. fs) log schemes. A chart of f is a diagram $M_Y \overset{\alpha}{\leftarrow} P \to Q \overset{\beta}{\to} M_X$ such that α is a chart for Y, β is a chart for X, and the diagram

$$
\begin{array}{ccc}
P & \longrightarrow & Q \\
\downarrow & & \downarrow \\
f^{-1}(M_Y) & \longrightarrow & M_X
\end{array}
$$

is commutative.

A chart of f exists étale locally on X and Y. This is proved in [K$_1$] in the fine case and the fs case is proved similarly.

For a morphism $f \colon X \to Y$ and a fine (resp. an fs) log structure M on Y, the inverse image f^*M is also a fine (resp. an fs) log structure. A chart of M gives naturally a chart of f^*M.

1.8. *Finite inverse limits.* The category of fine (resp. fs) log schemes has finite inverse limits. For a finite inverse system Σ, its inverse limit is a fine (resp. an fs) log scheme whose underlying scheme is finite and of finite presentation over the inverse limit of the underlying diagram of schemes of Σ. To see this, it is enough to show it for fiber products. The fiber product of $\Sigma \colon X_1 \overset{f_1}{\to} X_0 \overset{f_2}{\leftarrow} X_2$ is constructed étale locally on X_i's as follows. Étale locally, take charts $\alpha_i \colon P_i \to M_{X_i}$ of X_i and charts $M_{X_0} \overset{\alpha_0}{\leftarrow} P_0 \overset{h_j}{\to} P_j \overset{\alpha_j}{\to} M_{X_j}$ of f_j $(j = 1, 2)$ (this is possible by 1.7). Let X be the fiber product of the underlying diagram of schemes of Σ and let P be the pushout of the diagram $P_1 \overset{h_1}{\leftarrow} P_0 \overset{h_0}{\to} P_2$ in the

category of monoids. Then we have a canonical homomorphism $P \to \mathcal{O}_X$. Let P^{int} be the image of $P \to P^{\mathrm{gp}}$ (resp. $P^{\mathrm{sat}} = \{x \in P^{\mathrm{gp}} \; ; \; x^n \in \mathrm{Image}\,(P \to P^{\mathrm{gp}})$ for some $n \geq 1\}$). Then the fiber product of Σ in the category of fine (resp. fs) log schemes is the scheme $T = X \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P^{\mathrm{int}}]$ (resp. $X \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P^{\mathrm{sat}}]$) endowed with the log structure associated to P^{int} (resp. $P^{\mathrm{sat}}) \to \mathcal{O}_T$.

1.9. *Advantages of fs log schemes.* In §2–§10 of this paper, we consider fs log schemes but do not consider fine log schemes, though we considered fine log schemes in $[\mathrm{K}_1]$. The category of fs log schemes has, for example, the following advantages.

1.9.1. The category of fs log schemes has much more group objects than the category of fine log schemes. This point is explained in the preprint $[\mathrm{K}_2]$.

1.9.2. The exactness of the sequence

$$0 \to \Gamma(X, \mathbb{Z}/n(1)) \to \Gamma(X, M_X^{\mathrm{gp}}) \xrightarrow{n} \Gamma(X, M_X^{\mathrm{gp}})$$

$(n \in \mathbb{Z}, n \neq 0)$, which plays important roles in log algebraic geometry (see §4), holds for an fs log scheme X, but does not hold in general for a fine log scheme X.

1.9.3. "Log Galois theory" works well for fs log schemes (see §10) but not for fine log schemes. For example, let A be a discrete valuation ring, $n \geq 2$ an integer which is invertible in A, π a prime element of A, $B = A[\pi^{1/n}]$, and assume that A contains a primitive n-th root of 1. Let $G = \mathrm{Aut}_A(B) \cong \mathbb{Z}/n$. Endow $Y = \mathrm{Spec}(A)$ (resp. $X = \mathrm{Spec}(B)$) with the standard log structure which is associated to $\mathbb{N} \to A; 1 \mapsto \pi$ (resp. $\mathbb{N} \to B; 1 \mapsto \pi^{1/n}$). Then,

$$G \times X \xrightarrow{\cong} X \times_Y X; (g, x) \mapsto (x, gx)$$

(an isomorphism as in the étale Galois theory in classical algebraic geometry) holds in the category of fs log schemes, but does not hold in the category of fine log schemes. (Indeed, the underlying scheme of the fiber product $X \times_Y X$ in the category of fine log schemes is the same as the fiber product of the underlying schemes, and hence is connected (not the disjoint union of $\sharp(G)$-copies of X).)

1.10. *Log flatness, log smoothness, log étaleness.* Let $f: X \to Y$ be a morphism in either the category of fine log schemes or the category of fs log schemes. We say f is log flat (resp. log smooth, resp. log étale) if classical fppf (classical étale, classical étale) locally on X and on Y, there exists a chart $M_Y \leftarrow P \to Q \to M_X$ of f satisfying the following two conditions (i) and (ii).

(i) The induced map $P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ is injective (resp. is injective and the order of the torsion part of its cokernel is invertible on X, resp. is injective and its cokernel is finite with an order invertible on X).

(ii) The induced morphism of schemes $X \to Y \times_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ is flat (resp. smooth, resp. étale) in the classical sense.

A morphism of fs log schemes is log flat (resp. log smooth, resp. log étale) if and only if it is so as a morphism of fine log schemes.

Log flat (resp. log smooth, resp. log étale) morphisms are stable under compositions and base changes. The stability for base changes is seen easily, and the stability of log smoothness and that of log étaleness for compositions are seen from the following infinitesimal characterization of log smoothness and that of log étaleness, respectively.

A morphism $f : X \to Y$ of fine log schemes is log smooth (resp. log étale) if and only if the following condition is satisfied:

($*$) For any diagram of fine log schemes including f

$$
\begin{array}{ccc}
T & \xrightarrow{\ t\ } & X \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\
S & \xrightarrow{\ s\ } & Y
\end{array}
$$

such that $i^*(M_S) \xrightarrow{\cong} M_T$ and such that the underlying scheme of T is (via i) a closed subscheme of the underlying scheme of S defined by a nilpotent quasi-coherent ideal of $\mathcal{O}_S$, then there exists classical étale locally on S a morphism (resp. there exists a unique morphism) $g : S \to X$ such that $g \circ i = t$ and $f \circ g = s$.

See [K_1] (3.5) for the proof.

A morphism $f : X \to Y$ of fs log schemes is log smooth (resp. log étale) if and only if the condition

($*$)' the same as ($*$) except that we assume that T and S are fs log schemes, is satisfied.

This is proved in the same way as the case of fine log schemes.

We prove the stability of log flatness for compositions. This stability is also proved in [Ol] Corollary 4.12 (ii) by another method. See also Ogus' book [Og] Chapter IV, Proposition 4.1.2 (4).

Lemma 1.11. *Let $f : X \to Y$ be a log flat morphism of fine (resp. fs) log schemes and let $\beta : P \to M_Y$ be a chart. Then fppf locally on X and on Y in the classical sense, there exists a chart $(P \xrightarrow{\beta} M_Y, Q \to M_X, P \to Q)$ of f including β satisfying the conditions (i) and (ii) in the definition of log flatness in 1.10.*

Proof. Let $(M_Y \leftarrow P' \to Q' \to M_X)$ be a chart of f satisfying these conditions (i) and (ii) in the definition of log flatness.

Claim 1. We may assume that there is a homomorphism $P \to P'$ such that $\beta : P \to M_Y$ factors as $P \to P' \to M_Y$ and such that the homomorphism $P^{\mathrm{gp}} \to (P')^{\mathrm{gp}}/(P')^{\times}$ is surjective.

Proof of Claim 1. Fix $x \in X$, $y = f(x) \in Y$. By replacing P' with the inverse image P'' of $M_{Y,\overline{y}}$ under

$$
P^{\mathrm{gp}} \times (P')^{\mathrm{gp}} \to M_{Y,\overline{y}}^{\mathrm{gp}} \ ; \ (a, b) \mapsto ab,
$$

and by replacing Q' with the pushout of $P'' \leftarrow P' \to Q'$ in the category of fine (resp. fs) monoids, we obtain the situation stated in Claim 1.

Assume we have $P \to P'$ as in Claim 1.

Claim 2. We may assume that there are a finitely generated abelian group H which contains P^{gp} as a subgroup and a homomorphism $H \to (Q')^{\mathrm{gp}}$ such that the diagram

$$
\begin{array}{ccc}
P^{\mathrm{gp}} & \to & H \\
\downarrow & & \downarrow \\
(P')^{\mathrm{gp}} & \to & (Q')^{\mathrm{gp}}
\end{array}
$$

is commutative and is co-cartesian in the category of abelian groups.

Proof of Claim 2. Let $R = (Q')^{\mathrm{gp}}/(P')^{\mathrm{gp}}$ and let S be the inverse image of $(P')^{\times}$ in P^{gp}. Then $P^{\mathrm{gp}}/S \overset{\cong}{\to} (P')^{\mathrm{gp}}/(P')^{\times}$. Consider the commutative diagram of exact sequences

$$
\begin{array}{ccccccc}
\mathrm{Ext}^1(R, S) & \to & \mathrm{Ext}^1(R, P^{\mathrm{gp}}) & \to & \mathrm{Ext}^1(R, P^{\mathrm{gp}}/S) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Ext}^1(R, (P')^{\times}) & \to & \mathrm{Ext}^1(R, (P')^{\mathrm{gp}}) & \to & \mathrm{Ext}^1(R, (P')^{\mathrm{gp}}/(P')^{\times}) & \to & 0.
\end{array}
$$

(Note that $\mathrm{Ext}^2 = 0$ for $\mathbb{Z}$-modules.) Let $b \in \mathrm{Ext}^1(R, (P')^{\mathrm{gp}})$ be the class of $(Q')^{\mathrm{gp}}$. Since the right vertical arrow is an isomorphism, this diagram shows that there is $a \in \mathrm{Ext}^1(R, P^{\mathrm{gp}})$ such that $a' - b$, where a' denotes the image of a in $\mathrm{Ext}^1(R, (P')^{\mathrm{gp}})$, comes from an element c of $\mathrm{Ext}^1(R, (P')^{\times})$. Since c dies in $\mathrm{Ext}^1(R, I)$ for a divisible abelian group I containing $(P')^{\times}$ as a subgroup, c dies in $\mathrm{Ext}^1(R, G)$ for some finitely generated abelian group G containing $(P')^{\times}$ as a subgroup. Replace P' by the pushout of $P' \leftarrow (P')^{\times} \to G$, Q' by the pushout of $Q' \leftarrow (P')^{\times} \to G$, X by $\mathbb{Z}[G] \otimes_{\mathbb{Z}[(P')^{\times}]} X$, and Y by $\mathbb{Z}[G] \otimes_{\mathbb{Z}[(P')^{\times}]} Y$. Then $a' = b$ in $\mathrm{Ext}^1(R, (P')^{\mathrm{gp}})$. We have the desired abelian group H with an exact sequence $0 \to P^{\mathrm{gp}} \to H \to R \to 0$ corresponding to $a \in \mathrm{Ext}^1(R, P^{\mathrm{gp}})$. This proves Claim 2.

Now let the situation be as in Claim 2, and let Q be the inverse image of Q' in H. Since $Q/S \overset{\cong}{\to} Q'/(P')^{\times}$, $Q \to M_X$ is a chart of X. We have the chart $M_Y \overset{\beta}{\leftarrow} P \to Q \to M_X$ of f having the desired property. $\qquad\square$

Now we complete the proof of the stability of the log flatness for composition. Let $f : X \to Y$ and $g : Y \to Z$ be log flat morphisms of fine (resp. fs) log schemes. Let $M_Z \leftarrow P \to Q \overset{\beta}{\to} M_Y$ be a chart of g which satisfies the conditions (i) and (ii) in the definition of log flatness. By 1.11, we may assume that there is a chart $M_Y \overset{\beta}{\leftarrow} Q \to R \to M_X$ of f including β which satisfies the conditions (i) and (ii) of log flatness. Then the chart $M_Z \leftarrow P \to R \to M_X$ of $g \circ f$ satisfies the conditions (i) and (ii) of log flatness.

A morphism $f \colon X \to Y$ of fine log schemes such that the homomorphism $f^* M_X \to M_Y$ is an isomorphism is log flat (resp. log smooth, resp. log étale) if and only if the underlying morphism of schemes is flat (resp. smooth, resp. étale).

2 The logarithmic flat topology and the logarithmic étale topology.

Definition 2.1. A homomorphism of fs monoids $h \colon P \to Q$ is said to be *of Kummer type* if the following condition is satisfied: It is injective, and for any $a \in Q$, there exists $n \geq 1$ such that $a^n \in h(P)$.

Definition 2.2. We say a morphism $f\colon X \to Y$ of fs log schemes is *of Kummer type* if for any $x \in X$, the homomorphism of fs monoids $M_{Y,\bar{y}}/\mathcal{O}^\times_{Y,\bar{y}} \to M_{X,\bar{x}}/\mathcal{O}^\times_{X,\bar{x}}$ with $y = f(x)$ is of Kummer type in the sense of 2.1.

A standard example is the following. Let Y be an fs log scheme with a chart $P \to M_Y$, and assume that we are given an fs monoid Q and a homomorphism $P \to Q$ of Kummer type. Let X be any scheme over $Y \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ endowed with the log structure associated to $Q \to \mathcal{O}_X$. Then $f\colon X \to Y$ is of Kummer type.

If $X = Y \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ here, f is log flat and surjective, and X is a standard example of a covering of Y for the log flat topology introduced in the following 2.3.

In the log flat case of 1.10, if f is of Kummer type and locally of finite presentation, we can take the chart satisfying the additional conditions that $P \to Q$ is of Kummer type and that $X \to Y \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ is surjective. See [INT] Proposition 1.3.

Now we define the logarithmic flat topology and the logarithmic étale topology on an fs log scheme. (In the sequel we neglect questions of universes, which can be treated as in the non-log case.)

Definition 2.3. For an fs log scheme T, we say a family of morphisms $\{f_i\colon U_i \to T\}_i$ of fs log schemes is a covering of T for the log flat (resp. log étale) topology, if the following conditions (i) (resp. (i)$'$) and (ii) are satisfied.

(i) f_i are log flat and of Kummer type, and the underlying morphisms of schemes of f_i are locally of finite presentation.

(i)$'$ f_i are log étale and of Kummer type.

(ii) $T = \bigcup_i f_i(U_i)$ (set theoretically).

Let X be an fs log scheme and let (fs$/X$) be the category of fs log schemes over X. We define the Grothendieck topology called the log flat (resp. log étale) topology on (fs$/X$) by taking coverings as above. We denote the site (fs$/X$) endowed with this topology by $X^{\log}_{\mathrm{fl}}$ (resp. $X^{\log}_{\text{ét}}$).

To see that these really give definitions of Grothendieck topologies, we need the following lemma whose (2) is a case of [Na1] (2.2.2). (In there, it is proved that the assumption that f is of Kummer type in 2.4 (2) can be replaced by the weaker assumption that f is exact.)

Lemma 2.4. *Let $f\colon X \to Y$ be a morphism of fs log schemes of Kummer type, let Y' be an fs log scheme over Y, and let $f'\colon X' := X \times_Y Y' \to Y'$.*

(1) The morphism f' is of Kummer type.

(2) Assume that f is surjective. Then f' is surjective. More strongly, for any $x \in X$ and $y \in Y'$ having the same image in Y, there exists $z \in X'$ with image x in X and y in Y'.

Here and in the rest of this paper, when we are discussing fs log schemes, the notation of the fiber product stands for the fiber product in the category of fs log schemes unless the contrary is explicitly stated.

Proof. We may assume that X, Y and Y' have the same underlying scheme which is the spectrum of an algebraically closed field k. Take a submonoid P' of $M_{Y'}$ such that

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$M_{Y'} = k^\times \times P'$. Let P be the inverse image of P' in M_Y under the homomorphism $M_Y \to M_{Y'}$. Then $M_Y = k^\times \times P$. We have a chart $M_Y \leftarrow P \to P' \to M_{Y'}$ of $Y' \to Y$.

Take a submonoid Q of M_X such that $M_X = k^\times \times Q$. Then the map $M_Y \to M_X$ is written as $k^\times \times P \to k^\times \times Q$; $(s,t) \mapsto (sh(t), b(t))$ for some homomorphism $h : P \to k^\times$, where b is the composition $P \cong M_Y/k^\times \to M_X/k^\times \cong Q$. Since $k^\times$ is a divisible abelian group and b is injective, h extends to a homomorphism $Q \to k^\times$ which we still denote by h. We have a commutative diagram

$$
\begin{array}{ccc}
P & \overset{b}{\to} & Q \\
\downarrow & & \downarrow \\
M_Y & \to & M_X
\end{array}
$$

where the left vertical arrow is the inclusion map and the right vertical arrow is $y \mapsto h(y)y$ which we denote by γ. We have a chart $M_Y \leftarrow P \overset{b}{\to} Q \overset{\gamma}{\to} M_X$ of $X \to Y$.

Let Q' be the pushout of $P' \leftarrow P \to Q$ in the category of fs monoids. Then the fiber product X' is $\mathrm{Spec}(R)$ where $R = k[Q'] \otimes_{k[P']} k$ with the log structure associated to $Q' \to R$. Let Δ be the torsion subgroup of $(Q')^{\mathrm{gp}}$. Then $\Delta \subset Q'$ and the map $P' \to Q'/\Delta$ is injective. For any $a \in Q' \smallsetminus \Delta$, there is some $n \geq 1$ such that a^n comes from $P' \smallsetminus \{1\}$. Hence there is a nilpotent ideal I of R and a nilpotent ideal J of $k[\Delta]$ such that the map $k[\Delta] \to R$ induces $k[\Delta]/J \overset{\cong}{\to} R/I$. Hence X' is not empty and $Q'/\Delta \overset{\cong}{\to} (M_{X'}/\mathcal{O}_{X'}^\times)_x$ for any $x \in X'$. This proves (1) and (2). □

We prove the following three propositions 2.5, 2.6, 2.7 concerning coverings for the logarithmic flat topology. First, the following 2.5 says that the family $\{f_i(U_i)\}_i$ in 2.3 (ii) is an open covering of T.

Proposition 2.5. *Let $f : X \to Y$ be a morphism of fs log schemes. Assume that f is log flat and of Kummer type, and the underlying morphism of schemes of f is locally of finite presentation. Then f is an open map.*

Proposition 2.6. *Let $f : X \to Y$ be a morphism of fs log schemes, and assume that f is a covering for the logarithmic flat topology. Then, the underlying diagram of topological spaces of*

$$ X \times_Y X \rightrightarrows X \to Y $$

is exact, that is, Y is the coequalizer of the left two arrows in the category of topological spaces. (Here $X \times_Y X$ is the fiber product in the category of fs log schemes.)

Proposition 2.7. *Let $f : X \to Y$ be a morphism of fs log schemes.*
(1) The following two conditions are equivalent.
(i) f is of Kummer type.
(ii) Étale locally on X, there exists a covering $Y' \to Y$ for the log flat topology such that the log structure of $X \times_Y Y'$ is the inverse image of the log structure of Y'.
(2) Assume that f is of Kummer type, X is quasi-compact, and assume that we are given a chart $P \to M_Y$. Then, globally on X, we can take as Y' in (ii) a log scheme over Y of the type $Y \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ endowed with the log structure associated to $Q \to \mathcal{O}_{Y'}$, for some homomorphism of fs monoids $P \to Q$ of Kummer type.

Proof of 2.5. Since classical fppf morphisms are open maps, it suffices to prove 2.5 in the following case: Y has a chart $P \to M_Y$, and there are an fs monoid Q and a homomorphism $P \to Q$ of Kummer type such that $X = Y \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ with the log structure associated to $Q \to \mathcal{O}_X$. Let G be the group scheme $\mathrm{Spec}(\mathbb{Z}[Q^{\mathrm{gp}}/P^{\mathrm{gp}}])$ over $\mathbb{Z}$ with the trivial log structure. Then, G acts on X over Y by

$$\mathcal{O}_X = \mathcal{O}_Y \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q] \to \mathcal{O}_{G \times X} = \mathcal{O}_Y \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q \oplus (Q^{\mathrm{gp}}/P^{\mathrm{gp}})]$$
$$1 \otimes a \mapsto 1 \otimes (a, a \bmod P^{\mathrm{gp}}) \qquad (a \in Q).$$

We have $G \times X \xrightarrow{\cong} X \times_Y X; (g, x) \mapsto (x, gx)$. (Recall that the fiber products are taken in the category of fs log schemes.)

Note that f is a closed map (for the underlying morphisms of schemes of f is finite) and surjective (indeed, since $\mathbb{Z}[P] \to \mathbb{Z}[Q]$ is injective and finite, $\mathrm{Spec}(\mathbb{Z}[Q]) \to \mathrm{Spec}(\mathbb{Z}[P])$ is surjective, and the surjectivity of a morphism of schemes is preserved by base changes). Hence Y has the quotient topology of the topology of X.

Let U be an open set of X. By the above remark, to see that $f(U)$ is open in Y, it is enough to show that $f^{-1}(f(U))$ is open in X. But this last fact follows from 2.8 below and from the fact the action $G \times X \to X$ is fppf (since G is fppf over $\mathbb{Z}$).

Lemma 2.8. *With the notation as above, $f^{-1}(f(U))$ coincides with the image of $G \times U$ under the action $G \times X \to X$.*

Proof. This is reduced to the case where Y is the spectrum of a field. $\qquad\square$

Proof of 2.6. By 2.4 (2), the sequence of the underlying sets is exact. Hence, it is enough to show that the topology of Y is the quotient topology of that of X. Taking local charts, we may assume that there exists a chart $M_Y \leftarrow P \xrightarrow{h} Q \to M_X$ of f with h being of Kummer type such that the induced morphism of schemes $u \colon X \to Y \times_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ is flat, surjective and locally of finite presentation. Since $Y \times_{\mathbb{Z}[P]} \mathbb{Z}[Q] \to Y$ is closed and surjective, the topology of Y is the quotient topology of that of $Y \times_{\mathbb{Z}[P]} \mathbb{Z}[Q]$. Further, since u is open and surjective, the topology of $Y \times_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ is the quotient topology of that of X. Thus we conclude that the topology of Y is the quotient topology of that of X.

Proof of 2.7. Assume that the condition (ii) in (1) is satisfied. Then, $X \times_Y Y' \to X$ is of Kummer type by 2.4 (1) and $X \times_Y Y' \to Y$ is of Kummer type, and hence $X \to Y$ is of Kummer type, that is, the condition (i) is satisfied.

Next, assume that the condition (i) is satisfied. Taking local charts for f, we see that, after localizing X, there is a positive integer n such that the cokernel of $(f^* M_Y)^{\mathrm{gp}} \to M_X^{\mathrm{gp}}$ is killed by n. We show that, under the assumption of the existence of such an n, globally on X, there exists a covering $Y' \to Y$ such that the log structure of $X \times_Y Y'$ is the inverse image of the log structure of Y', which completes the proof of (1). To show this, by étale localizing Y, we may and will assume that there is a chart $P \to M_Y$ of Y. We show that, under this further assumption of the existence of a chart of Y, there exists a covering of the type $Y \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ as in (2). This completes not only the proof of (1) but also the proof of (2) because there is an n which kills the cokernel of $(f^* M_Y)^{\mathrm{gp}} \to M_X^{\mathrm{gp}}$ under the assumption of (2). Let $h \colon P \to Q$ be a homomorphism $n \colon P \to P$ of Kummer type, let $Y' = Y \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$, and let $X' = X \times_Y Y'$. It is enough to show that $f'^* M_{Y'} \to M_{X'}$ is

an isomorphism, where f' is the projection $X' \to Y'$. Since f' is of Kummer type, it is sufficient to show that $(f'^* M_{Y'})^{\mathrm{gp}} \to M_{X'}^{\mathrm{gp}}$ is an isomorphism, which is easily checked at stalks (cf. [Na1] 2.1.1).

3 Descent theory. I.

Concerning the descent theory for logarithmic flat topology, it seems a right philosophy is that *morphisms descent, properties of morphisms also descent, but objects do not descent.* We show in §3 (resp. §7) that morphisms (resp. properties of morphisms) descent for the log flat topology (3.1) (resp. (7.1)) and in §8 that log objects descent for the classical flat topology (8.1).

Theorem 3.1. *Let X be an fs log scheme, and let Y be an fs log scheme over X. Then, the functor*

$$\mathrm{Mor}_X(\ , Y) \colon T \mapsto \mathrm{Mor}_X(T, Y)$$

on (fs/X) is a sheaf for the log flat topology.

Theorem 3.2. *Let $\mathbb{G}_{m,\log}$ be the functor $T \mapsto \Gamma(T, M_T^{\mathrm{gp}})$ on (fs/X). Then $\mathbb{G}_{m,\log}$ is a sheaf for the log flat topology.*

We remark that in the circulated version (cf. Introduction), $\mathbb{G}_{m,\log}$ was denoted by $\mathbb{G}_m^{\mathrm{cpt}}$.

Since a sheaf for the log flat topology is also a sheaf for the log étale topology, the above results show that the functors $\mathrm{Mor}_X(\ , Y)$ in 3.1 and $\mathbb{G}_{m,\log}$ are sheaves for the log étale topology.

3.3. We prove Theorem 3.1. In this 3.3, we show that it is sufficient to prove that the functors

(3.3.1) $$T \mapsto \Gamma(T, \mathcal{O}_T) \qquad \text{and} \qquad T \mapsto \Gamma(T, M_T)$$

are sheaves for the log flat topology.

We may assume that $X = \mathrm{Spec}(\mathbb{Z})$ with the trivial log structure and Y is affine as a scheme. Further, we may assume that Y has a chart $P \to M_Y$ (for this, see [KKN] §5).

Let F, G, H be functors $(\mathrm{fs}/\mathrm{Spec}(\mathbb{Z})) \to (\mathrm{Set})$ defined by

$$F(T) = \{\text{ring homomorphisms } \Gamma(Y, \mathcal{O}_Y) \to \Gamma(T, \mathcal{O}_T)\}$$
$$G(T) = \{\text{homomorphisms } P \to \Gamma(T, M_T)\}$$
$$H(T) = \{\text{homomorphisms } P \to \Gamma(T, \mathcal{O}_T)\}.$$

Then, $\mathrm{Mor}_X(\ , Y)$ is the fiber product of $F \to H \leftarrow G$, where the first arrow is induced by $P \to \Gamma(Y, \mathcal{O}_Y)$ and the second arrow is induced by $\Gamma(T, M_T) \to \Gamma(T, \mathcal{O}_T)$. Hence it is sufficient to prove that F, G, H are sheaves.

Take a presentation

$$\Gamma(Y, \mathcal{O}_Y) = \mathbb{Z}[T_i \; ; \; i \in I]/(f_j \; ; \; j \in J),$$
$$\mathbb{N}^r \rightrightarrows \mathbb{N}^s \to P \qquad \text{(exact)}.$$

Then, $F(T)$ is the kernel of $\Gamma(T, \mathcal{O}_T)^I \to \Gamma(T, \mathcal{O}_T)^J$, and $G(T)$ and $H(T)$ are the equalizers of $\Gamma(T, M_T)^s \rightrightarrows \Gamma(T, M_T)^r$ and $\Gamma(T, \mathcal{O}_T)^s \rightrightarrows \Gamma(T, \mathcal{O}_T)^r$, respectively. Hence it is sufficient to prove that the functors in 3.3.1 are sheaves.

3.4. We prove that $T \mapsto \Gamma(T, \mathcal{O}_T)$ is a sheaf for the log flat topology. It is sufficient to prove

Lemma 3.4.1. *Let T be an fs log scheme which is affine as a scheme and which has a chart $P \to M_T$. Let Q be an fs monoid, let $P \to Q$ be a homomorphism of Kummer type, let $T' = T \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ which we endow with the log structure associated to $Q \to \mathcal{O}_{T'}$, and let $T'' = T' \times_T T'$ be the fiber product in the category of fs log schemes. Then*

$$(*) \qquad \qquad \Gamma(T, \mathcal{O}_T) \to \Gamma(T', \mathcal{O}_{T'}) \rightrightarrows \Gamma(T'', \mathcal{O}_{T''})$$

is exact.

Proof. Let $A = \Gamma(T, \mathcal{O}_T)$. Then, the diagram $(*)$ is isomorphic to

$$(**) \qquad \qquad A \xrightarrow{\alpha} A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q] \overset{\beta_1}{\underset{\beta_2}{\rightrightarrows}} A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q \oplus (Q^{\mathrm{gp}}/P^{\mathrm{gp}})],$$

where β_1 (resp. β_2) is the homomorphism of A-algebras which sends $1 \otimes a$ ($a \in Q$) to $1 \otimes (a, 1)$ (resp. $1 \otimes (a, a \bmod P^{\mathrm{gp}})$). Let $s \colon A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q] \to A$ be the homomorphism of A-modules which sends $1 \otimes a$ ($a \in Q$) to a if $a \in P$, and to 0 if $a \notin P$. Let $\iota \colon A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q \oplus (Q^{\mathrm{gp}}/P^{\mathrm{gp}})] \to A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ be the homomorphism of A-modules which sends $1 \otimes (a, b)$ ($a \in Q, b \in Q^{\mathrm{gp}}/P^{\mathrm{gp}}$) to $1 \otimes a$ if $b \neq 0$, and to 0 if $b = 0$. (Here we use the fact that $P \to Q$ is of Kummer type to show that s is well-defined.) Then, $s \circ \alpha$ is the identity map of A, and $\alpha \circ s + \iota \circ (\beta_2 - \beta_1)$ is the identity map of $A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$. This proves the exactness of $(**)$. $\qquad \square$

3.5. We prove that $T \mapsto \Gamma(T, M_T)$ is a sheaf for the log flat topology.

Let $T' \to T$ be a classical fppf covering (recall that this means that the log structure of T' is the inverse image of that of T and the underlying morphism of schemes of $T' \to T$ is an fppf covering), and let $T'' = T' \times_T T'$. Then

$$\Gamma(T, \mathcal{O}_T^\times) \to \Gamma(T', \mathcal{O}_{T'}^\times) \rightrightarrows \Gamma(T'', \mathcal{O}_{T''}^\times)$$

is exact (a classical result). Further,

$$\Gamma(T, M_T/\mathcal{O}_T^\times) \to \Gamma(T', M_{T'}/\mathcal{O}_{T'}^\times) \rightrightarrows \Gamma(T'', M_{T''}/\mathcal{O}_{T''}^\times)$$

is exact. To see it, taking local charts, we may assume that $M_T/\mathcal{O}_T^\times$ is the inverse image of a sheaf on the Zariski site, and use the fact that $M_{T'}/\mathcal{O}_{T'}^\times$ and $M_{T''}/\mathcal{O}_{T''}^\times$ are the inverse images of the sheaf $M_T/\mathcal{O}_T^\times$ and the fact that the diagram of topological spaces

$$T'' \rightrightarrows T' \to T$$

is exact. We can deduce from these two exactness that $T \mapsto \Gamma(T, M_T)$ is a sheaf for the classical flat topology.

Since $T \mapsto \Gamma(T, M_T)$ commutes with the limit with respect to the inverse system of the étale neighborhoods of a geometric point of the underlying scheme of T, it is sufficient to prove the following.

Lemma 3.5.1. *Let $T, T', T'', P \to M_T$ and $P \to Q$ be as in the hypothesis of Lemma 3.4.1, and assume $T = \mathrm{Spec}(A)$ for a local ring A, $P \xrightarrow{\cong} (M_T/\mathcal{O}_T^\times)_{\bar{t}}$, where t is the closed point of T, and Q has no torsion. Then,*

$$\Gamma(T, M_T) \to \Gamma(T', M_{T'}) \rightrightarrows \Gamma(T'', M_{T''})$$

is exact.

Proof. Let $A' = \Gamma(T', \mathcal{O}_{T'}) = A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$, $A'' = \Gamma(T'', \mathcal{O}_{T''}) = A \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q \oplus (Q^{\mathrm{gp}}/P^{\mathrm{gp}})]$. By 3.4.1, $A \to A' \rightrightarrows A''$ is exact. Let I (resp. I', resp. I'') be the ideal of A (resp. A', resp. A'') generated by the image of $P \setminus \{1\}$ (resp. $Q \setminus \{1\}$, resp. $Q \setminus \{1\}$), and let V (resp. V', resp. V'') be the subgroup of $A^\times$ (resp. $(A')^\times$, resp. $(A'')^\times$) consisting of elements which are congruent to 1 modulo I (resp. I', resp. I''). Since $A/I \xrightarrow{\cong} A'/I'$, we see that $V \to V' \rightrightarrows V''$ is exact. It remains to show that

$(*) \qquad\qquad \Gamma(T, M_T)/V \to \Gamma(T', M_{T'})/V' \rightrightarrows \Gamma(T'', M_{T''})/V''$

is exact. Consider the exact diagram

$(**) \qquad\qquad P \oplus (A/I)^\times \to Q \oplus (A/I)^\times \overset{\beta_1}{\underset{\beta_2}{\rightrightarrows}} Q \oplus \{(A/I)[Q^{\mathrm{gp}}/P^{\mathrm{gp}}]\}^\times,$

where β_1 (resp. β_2) is the map $(a, u) \mapsto (a, u)$ (resp. (a, au)). The exactness of this follows from the exactness of $P \to Q \rightrightarrows Q \oplus (Q^{\mathrm{gp}}/P^{\mathrm{gp}})$. Now there is a natural homomorphism of diagrams from $(**)$ to $(*)$, which induces an isomorphism on the first terms, an isomorphism on the second terms, and an injection on the last terms. Since $(**)$ is exact, $(*)$ is also exact. $\qquad\square$

3.6. We prove that $\mathbb{G}_{m,\log}$ is a sheaf for the log flat topology. Let $T' \to T$ be a log flat morphism of Kummer type which is surjective and locally of finite presentation as a morphism of schemes. Assume that T has a chart $P \to M_T$ and is quasi-compact. Let $T'' = T' \times_T T'$ (the fiber product in the category of fs log schemes). Then, $\Gamma(T, M_T^{\mathrm{gp}})$ (resp. $\Gamma(T', M_{T'}^{\mathrm{gp}})$, resp. $\Gamma(T'', M_{T''}^{\mathrm{gp}})$) is isomorphic to $\varinjlim_a \Gamma(T, a^{-1}M_T)$ (resp. $\varinjlim_a \Gamma(T', a^{-1}M_{T'})$, resp. $\varinjlim_a \Gamma(T'', a^{-1}M_{T''})$), where a ranges over all elements of P. (This follows from the fact that T' and T'' are of Kummer type over T.) Hence the exactness of $\Gamma(T, M_T^{\mathrm{gp}}) \to \Gamma(T', M_{T'}^{\mathrm{gp}}) \rightrightarrows \Gamma(T'', M_{T''}^{\mathrm{gp}})$ is reduced to the exactness of $\Gamma(T, M_T) \to \Gamma(T', M_{T'}) \rightrightarrows \Gamma(T'', M_{T''})$. This implies that $\mathbb{G}_{m,\log}$ is a sheaf for the log flat topology.

4 Cohomology.

For an fs log scheme X, $X_{\mathrm{fl}}^{\mathrm{cl}}$ denotes the category (fs/X) endowed with the classical fppf topology. That is, $(U_i \to T)_i$ is a covering in $X_{\mathrm{fl}}^{\mathrm{cl}}$ means that T is an object of (fs/X), the log structure of U_i is the inverse image of M_T for any i, the underlying scheme of U_i is flat and locally of finite presentation over that of T, and the images of U_i in T cover T set theoretically.

The aim of this section is to prove

Theorem 4.1. *Let X be an fs log scheme and assume that X is locally Noetherian as a scheme. Let $\varepsilon\colon X_{\mathrm{fl}}^{\log} \to X_{\mathrm{fl}}^{\mathrm{cl}}$ be the canonical morphism of sites. Let G be a commutative group scheme over the underlying scheme of X satisfying either one of the following two conditions.*

(i) G is finite flat over the underlying scheme of X.

(ii) G is smooth and affine over the underlying scheme of X.

We endow G with the inverse image of the log structure of X. Then we have a canonical isomorphism

$$R^1\varepsilon_*G \cong \varinjlim_{n\neq 0} \mathcal{H}om(\mathbb{Z}/n(1), G) \otimes_{\mathbb{Z}} (\mathbb{G}_{m,\log}/\mathbb{G}_m),$$

where n ranges over all non-zero integers and the inductive limit is taken with respect to the canonical projections $\mathbb{Z}/mn(1) \to \mathbb{Z}/n(1)$.

Here $\mathbb{G}_m$ is the functor $T \to \Gamma(T, \mathcal{O}_T^\times)$ on (fs/X) and $\mathbb{Z}/n(1) = \mathrm{Ker}\,(n\colon \mathbb{G}_m \to \mathbb{G}_m)$ $(n \neq 0)$. The quotient $\mathbb{G}_{m,\log}/\mathbb{G}_m$ here is taken in the categories of sheaves on $X_{\mathrm{fl}}^{\mathrm{cl}}$.

The following Kummer sequence for $\mathbb{G}_{m,\log}$ on $X_{\mathrm{fl}}^{\log}$ will be a starting point for the proof of 4.1.

Proposition 4.2. *Let X be an fs log scheme. Then the sequences of sheaves on $X_{\mathrm{fl}}^{\log}$ (resp. $X_{\mathrm{\acute{e}t}}^{\log}$)*

$$0 \to \mathbb{Z}/n(1) \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 0$$

$$0 \to \mathbb{Z}/n(1) \to \mathbb{G}_{m,\log} \xrightarrow{n} \mathbb{G}_{m,\log} \to 0$$

are exact for any non-zero integer n (resp. for any integer n which is invertible on X).

Proof. Similarly as in [KN] 2.3. $\qquad\qquad\square$

4.3. Let X be an fs log scheme and let G be a sheaf of abelian groups on $X_{\mathrm{fl}}^{\log}$. We define a canonical homomorphism of sheaves on $X_{\mathrm{fl}}^{\log}$

(4.3.1) $\qquad\qquad \varinjlim \mathcal{H}om(\mathbb{Z}/n(1), G) \otimes (\mathbb{G}_{m,\log}/\mathbb{G}_m) \to R^1\varepsilon_*G$

as follows. Let h be a local section of $\mathcal{H}om(\mathbb{Z}/n(1), G)$. By the Kummer sequence

$$0 \to \mathbb{Z}/n(1) \to \mathbb{G}_{m,\log} \xrightarrow{n} \mathbb{G}_{m,\log} \to 0$$

on $X_{\mathrm{fl}}^{\log}$, we have

$$\mathbb{G}_{m,\log} = \varepsilon_*\mathbb{G}_{m,\log} \xrightarrow{\delta} R^1\varepsilon_*(\mathbb{Z}/n(1)) \xrightarrow{h} R^1\varepsilon_*G,$$

where δ is the connecting homomorphism. The map δ kills $\mathbb{G}_m$, for $n\colon \mathbb{G}_m \to \mathbb{G}_m$ is surjective on $X_{\mathrm{fl}}^{\mathrm{cl}}$. Thus we obtain the map 4.3.1.

4.4. We show that the case (i) of Theorem 4.1 follows from the case (ii) of Theorem 4.1. Let G be a finite flat commutative group over the underlying scheme of X. Let G^* be

the Cartier dual of G, and let $L = \mathcal{M}or(G^*, \mathbb{G}_m)$, where $\mathcal{M}or$ means the sheaf of the morphisms of sheaves of sets. Then, we have an exact sequence

$$0 \to G \to L \to L' \to 0, \qquad L' = L/G$$

and L, L' are affine and smooth over the underlying scheme of X. By endowing L, L' with the inverse images of the M_X, we have

$$0 \to R^1\varepsilon_* G \to R^1\varepsilon_* L \to R^1\varepsilon_* L' \qquad \text{exact}$$

(for $L = \varepsilon_* L \to L' = \varepsilon_* L'$ is surjective on $X_{\mathrm{fl}}^{\mathrm{cl}}$). Hence the bijectivity of 4.3.1 for G is reduced to the bijectivities of 4.3.1 for L and L'.

4.5. Assume that we are given a chart $P \to M_X$ such that P is fs and torsion free (such chart exists classical étale locally on X). We study $R^1\varepsilon_*$ by using Čech cohomology.

For $n \geq 1$, let

$$X_n = X \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P^{1/n}]$$

with the log structure associated to $\mathbb{Z}[P^{1/n}] \to \mathcal{O}_{X_n}$, and let $X_{n,i}$ ($i \geq 0$) be the fiber product of $i+1$ copies of X_n over X in the category of fs log schemes. Note that $X_n \to X$ is a covering in $X_{\mathrm{fl}}^{\log}$. For a sheaf G of abelian groups on $X_{\mathrm{fl}}^{\log}$, we have a Čech complex $C_{G,n}$

$$C_{G,n} : \Gamma(X_{n,0}, G) \to \Gamma(X_{n,1}, G) \to \Gamma(X_{n,2}, G) \to \cdots .$$

Lemma 4.6. *Assume $X = \mathrm{Spec}(A)$ as a scheme with A a strict local ring (i.e. a Henselian local ring with separably closed residue field), and assume that G is represented by a smooth commutative group scheme over the underlying scheme of X endowed with the inverse image of the log structure of X. Then*

$$\varinjlim_n H^1(C_{G,n}) \overset{\cong}{\to} H^1(X_{\mathrm{fl}}^{\log}, G).$$

Proof. By the general theory of Čech cohomology, $\varinjlim_n H^1(C_{G,n}) \to H^1(X_{\mathrm{fl}}^{\log}, G)$ is injective and the cokernel is embedded in $\varinjlim_n H^1((X_n)_{\mathrm{fl}}^{\log}, G)$. We show $\varinjlim_n H^1((X_n)_{\mathrm{fl}}^{\log}, G) = 0$. Let a be an element of $H^1((X_n)_{\mathrm{fl}}^{\log}, G)$. Then there exists a quasi-compact fs log scheme T and a surjective log flat morphism of Kummer type $T \to X_n$ such that a vanishes in $H^1(T_{\mathrm{fl}}^{\log}, G)$. By 2.7 (2), for some $m \neq 0$, $T \times_{X_n} X_{mn} \to X_{mn}$ is with the inverse image log structure and flat in the classical sense. Hence a vanishes classical flat locally on X_{mn}, so the image of a in $H^1((X_{mn})_{\mathrm{fl}}^{\log}, G)$ comes from $H^1((X_{mn})_{\mathrm{fl}}^{\mathrm{cl}}, G)$. Since G is smooth, we have $H^1((X_{mn})_{\mathrm{fl}}^{\mathrm{cl}}, G) = H^1((X_{mn})_{\mathrm{ét}}^{\mathrm{cl}}, G)$ ([G] 11.7). Since X_{mn} is the disjoint union of a finite number of Spec of strict local rings, we have $H^1((X_{mn})_{\mathrm{ét}}^{\mathrm{cl}}, G) = 0$. $\qquad \square$

4.7. For $n \geq 1$, let H_n be the classical commutative group scheme $\mathrm{Spec}(\mathbb{Z}[P^{\mathrm{gp}}/(P^{\mathrm{gp}})^n]) = \mathcal{H}om(P^{\mathrm{gp}}, \mathbb{Z}/n(1))$ over $\mathrm{Spec}(\mathbb{Z})$. Then H_n acts on X_n over X, and $H_n \times X_n \overset{\cong}{\to} X_n \times_X X_n$ (the fiber product is taken in the category of fs log schemes). Hence we have

$$X_{n,i} \cong \underbrace{H_n \times \cdots \times H_n}_{i \text{ times}} \times X_n.$$

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4.8. Let G be a sheaf of abelian groups on $X^{\log}_{\mathrm{fl}}$, and let F be the sheaf of abelian groups on $X^{\log}_{\mathrm{fl}}$ defined by $F(T) = \Gamma(T \times_X X_n, G)$. Then H_n acts on F. By 4.7, $C_{G,n}$ is isomorphic to the complex

$$(*) \qquad \mathrm{Mor}(\{1\}, F) \xrightarrow{\delta_0} \mathrm{Mor}(H_n, F) \xrightarrow{\delta_1} \mathrm{Mor}(H_n \times H_n, F) \to \cdots ,$$

where Mor means the set of morphisms as sheaves of sets on $X^{\log}_{\mathrm{fl}}$,

$$\delta_0(x) = (\sigma \mapsto \sigma x - x), \qquad \delta_1(x) = ((\sigma, \tau) \mapsto x(\sigma\tau) - x(\sigma) - \sigma(x(\tau))), \dots.$$

4.9. Let G and F be as in 4.8. Assume that X is strictly local as a scheme and $P \xrightarrow{\cong} (M_X/\mathcal{O}_X^\times)_x$, where x is the closed point of X.
Consider the trivial action of H_n on G. Then, the natural homomorphism $G \to F$ preserves the action of H_n and hence we get a homomorphism $H^1(H_n, G) \to H^1(H_n, F)$. Note

$$H^1(H_n, G) = \mathrm{Hom}(H_n, G) = \mathrm{Hom}(\mathbb{Z}/n(1), G) \otimes_{\mathbb{Z}} P^{\mathrm{gp}}.$$

The composite map

$$\mathrm{Hom}(\mathbb{Z}/n(1), G) \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \cong H^1(H_n, G) \to H^1(H_n, F) \cong H^1(C_{G,n}) \to H^1(X^{\log}_{\mathrm{fl}}, G)$$

sends $h \otimes a$ to the image of a under

$$P^{\mathrm{gp}} \to H^0(X^{\log}_{\mathrm{fl}}, \mathbb{G}_{m,\log}) \to H^1(X^{\log}_{\mathrm{fl}}, \mathbb{Z}/n(1)) \xrightarrow{h} H^1(X^{\log}_{\mathrm{fl}}, G),$$

where the second arrow is the connecting homomorphism of

$$0 \to \mathbb{Z}/n(1) \to \mathbb{G}_{m,\log} \xrightarrow{n} \mathbb{G}_{m,\log} \to 0 \qquad (\text{cf. } (4.2)).$$

Together with 4.6, the next proposition 4.10 implies 4.1 under the assumption on X in 4.10.

Proposition 4.10. *Assume that as a scheme, $X = \mathrm{Spec}(A)$ for a Noetherian complete local ring A with separably closed residue field. Let G be a smooth commutative group scheme over the underlying scheme of X and endow G with the inverse image of the log structure of X. Assume $P \xrightarrow{\cong} (M_X/\mathcal{O}_X^\times)_x$, where x is the closed point of X. Then,*

$$\mathrm{Hom}(\mathbb{Z}/n(1), G) \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \xrightarrow{\cong} H^1(C_{G,n}) \qquad \text{for any } n \geq 1.$$

4.11. We prove 4.10 in the case where A is an Artinian local ring. Let I (resp. J) be the ideal of A (resp. $\mathcal{O}_{X_n}$) generated by the image of $P \setminus \{1\}$ (resp. $P^{1/n} \setminus \{1\}$). Then I (resp. J) is a nilpotent ideal. We define a descending filtrations $\{\mathrm{fil}^i G\}_{i \geq 0}$ on the H_n-module G and $\{\mathrm{fil}^i F\}_{i \geq 0}$ on the H_n-module F by

$$(\mathrm{fil}^i G)(T) = \mathrm{Ker}\left(G(T) \to G(T \times_X \mathrm{Spec}(\mathcal{O}_X/I^i))\right)$$
$$(\mathrm{fil}^i F)(T) = \mathrm{Ker}\left(F(T) \to G(T \times_X \mathrm{Spec}(\mathcal{O}_{X_n}/J^i))\right).$$

By the nilpotence of I (resp. J), we have $\mathrm{fil}^i = 0$ for a sufficiently large i. Since G is smooth, we have

$$(4.11.1) \qquad \mathrm{gr}^i(G)(T) \cong \mathrm{Lie}(G) \otimes_A \Gamma(T, I^i \mathcal{O}_T / I^{i+1} \mathcal{O}_T),$$

$$(4.11.2) \qquad \mathrm{gr}^i(F)(T) \cong \mathrm{Lie}(G) \otimes_A \Gamma(T, J^i \mathcal{O}_T / J^{i+1} \mathcal{O}_T)$$

$(\mathrm{gr}^i = \mathrm{fil}^i / \mathrm{fil}^{i+1}, i \geq 1)$. We have also

$$(4.11.3) \qquad \mathrm{gr}^0(G) \overset{\cong}{\to} \mathrm{gr}^0(F).$$

By 4.11.3 and by the following 4.12, we have

$$\mathrm{Hom}(\mathbb{Z}/n(1), G) \otimes_{\mathbb{Z}} P^{\mathrm{gp}} = H^1(H_n, G)$$
$$\overset{\cong}{\to} H^1(H_n, \mathrm{gr}^0(G)) \overset{\cong}{\to} H^1(H_n, \mathrm{gr}^0(F)) \overset{\cong}{\leftarrow} H^1(H_n, F) = H^1(C_{G,n}).$$

This proves 4.10 in the case A is Artinian.

Lemma 4.12. *For any $i \geq 1$ and any $m \geq 1$, $H^m(H_n, \mathrm{gr}^i(G))$ and $H^m(H_n, \mathrm{gr}^i(F))$ are zero.*

Proof. Since $\mathrm{gr}^i(G)$ and $\mathrm{gr}^i(F)$ are coherent, this follows from [SGA3] I, 5.3.3. $\square$

4.13. We prove Proposition 4.10. We denote the maximal ideal of A by m_A. For $i \geq 0$, define the abelian groups D_i and E_i by the exact sequences

$$0 \to G(A/m_A^i) \to G(X_{n,0} \otimes_A A/m_A^i) \to D_i \to 0$$
$$0 \to E_i \to G(X_{n,1} \otimes_A A/m_A^i) \to G(X_{n,2} \otimes_A A/m_A^i).$$

Then $D_i \subset E_i$ and E_i/D_i is H^1 of the complex $C_{G,n}$ for $\mathrm{Spec}(A/m_A^i)$ which is endowed with the inverse image of the log structure of X. Let $E = \varprojlim_i E_i$ and $D = \varprojlim_i D_i$. Since $G(A) \overset{\cong}{\to} \varprojlim G(A/m_A^i)$, $G(X_{n,k}) = \varprojlim G(X_{n,k} \otimes_A A/m_A^i)$, and since $G(A/m_A^{i+1}) \to G(A/m_A^i)$ and $G(X_{n,0} \otimes_A A/m_A^{i+1}) \to G(X_{n,0} \otimes_A A/m_A^i)$ are surjective by the smoothness of G, we have exact sequences

$$0 \to G(A) \to G(X_{n,0}) \to D \to 0$$
$$0 \to E \to G(X_{n,1}) \to G(X_{n,2})$$

and an isomorphism $E/D \cong \varprojlim E_i/D_i$. Hence $E/D \cong H^1(C_{G,n})$.

On the other hand, by [SGA3] XV 1.6, $\mathrm{Hom}(\mathbb{Z}/n(1), G)$ does not change when X is replaced by $\mathrm{Spec}(A/m_A^i)$. This and 4.11 show

$$\mathrm{Hom}(\mathbb{Z}/n(1), G) \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \overset{\cong}{\to} E_i/D_i \qquad \text{for any } i.$$

Hence $\mathrm{Hom}(\mathbb{Z}/n(1), G) \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \overset{\cong}{\to} E/D \cong H^1(C_{G,n})$.

4.14. We prove 4.1 in general. We may assume $X = \mathrm{Spec}(A)$ as a scheme for a Noetherian strict local ring and let G be a smooth affine commutative group scheme over A endowed with the inverse image of M_X.

Let $\hat{X} = \mathrm{Spec}(\hat{A})$, where $\hat{A}$ is the completion of A, and endow X with the inverse image of the log structure of X. Since the problem is already solved for $\hat{X}$, and since $\mathrm{Hom}(\mathbb{Z}/n(1), G)$ does not change when we replace X by $\hat{X}$, it is sufficient to prove that $H^1(X_{\mathrm{fl}}^{\log}, G) \to H^1(\hat{X}_{\mathrm{fl}}^{\log}, G)$ is injective.

Let α be an element of $H^1(X_{\mathrm{fl}}^{\log}, G)$ which vanishes in $H^1(\hat{X}_{\mathrm{fl}}^{\log}, G)$. Since α vanishes classical fpqc locally, by the fpqc descent (here we use the fact that G is affine), α is the class of a representable smooth affine G-torsor Y over the underlying scheme of X which is endowed with the inverse image of the log structure of X. Since X is strictly local, Y has an X-rational point and hence Y is a trivial G-torsor. Hence $\alpha = 0$.

5   Hilbert 90.

Theorem 5.1. *Let X be an fs log scheme whose underlying scheme is locally Noetherian. Then the canonical map from $H^1(X_{\mathrm{\acute{e}t}}^{\mathrm{cl}}, \mathbb{G}_{m,\log})$ to $H^1(X_{\mathrm{\acute{e}t}}^{\log}, \mathbb{G}_{m,\log})$ (resp. to $H^1(X_{\mathrm{fl}}^{\mathrm{cl}}, \mathbb{G}_{m,\log})$, resp. $H^1(X_{\mathrm{fl}}^{\log}, \mathbb{G}_{m,\log})$) is bijective.*

Since $\mathbb{G}_{m,\log}$ on the classical étale site is the direct image of $\mathbb{G}_{m,\log}$ on the other sites, this theorem is equivalent to its local form

Corollary 5.2. *Let X be an fs log scheme whose underlying scheme is Spec of a Noetherian strict local ring. Then $H^1(X_{\mathrm{fl}}^{\mathrm{cl}}, \mathbb{G}_{m,\log})$, $H^1(X_{\mathrm{\acute{e}t}}^{\log}, \mathbb{G}_{m,\log})$, $H^1(X_{\mathrm{fl}}^{\log}, \mathbb{G}_{m,\log})$ are zero.*

We prove 5.2. Since the natural projections from $X_{\mathrm{fl}}^{\log}$ to $X_{\mathrm{fl}}^{\mathrm{cl}}$ and to $X_{\mathrm{\acute{e}t}}^{\log}$ send $\mathbb{G}_{m,\log}$ to $\mathbb{G}_{m,\log}$, it is enough to show $H^1(X_{\mathrm{fl}}^{\log}, \mathbb{G}_{m,\log}) = 0$.

Consider the exact sequence $0 \to \mathbb{G}_m \to \mathbb{G}_{m,\log} \to \mathbb{G}_{m,\log}/\mathbb{G}_m \to 0$ on $X_{\mathrm{fl}}^{\log}$. First, we show that the connecting map $H^0(X_{\mathrm{fl}}^{\log}, \mathbb{G}_{m,\log}/\mathbb{G}_m) \to H^1(X_{\mathrm{fl}}^{\log}, \mathbb{G}_m)$ is surjective by using the result 4.1 as follows. Since $H^i(X_{\mathrm{fl}}^{\mathrm{cl}}, \mathbb{G}_m) = 0$ for $i > 0$ by [G] 11.7, $H^1(X_{\mathrm{fl}}^{\log}, \mathbb{G}_m) = H^0(X_{\mathrm{fl}}^{\mathrm{cl}}, R^1\varepsilon_*\mathbb{G}_m)$ and the latter is isomorphic to $H^0(X_{\mathrm{fl}}^{\mathrm{cl}}, (\mathbb{Q}/\mathbb{Z}) \otimes (\mathbb{G}_{m,\log}/\mathbb{G}_m))$ by 4.1. On the other hand, $H^0(X_{\mathrm{fl}}^{\log}, \mathbb{G}_{m,\log}/\mathbb{G}_m) = H^0(X_{\mathrm{fl}}^{\mathrm{cl}}, \varepsilon_*(\mathbb{G}_{m,\log}/\mathbb{G}_m)) = H^0(X_{\mathrm{fl}}^{\mathrm{cl}}, \mathbb{Q} \otimes (\mathbb{G}_{m,\log}/\mathbb{G}_m))$. Hence, the cokernel of the above connecting map injects into $H^1(X_{\mathrm{fl}}^{\mathrm{cl}}, \mathbb{G}_{m,\log}/\mathbb{G}_m)$, which is isomorphic to $H^1(X_{\mathrm{\acute{e}t}}^{\mathrm{cl}}, \mathbb{G}_{m,\log}/\mathbb{G}_m) = 0$ by [G] 11.9.

Hence it is sufficient to prove $H^1(X_{\mathrm{fl}}^{\log}, \mathbb{G}_{m,\log}/\mathbb{G}_m) = 0$. To show this, we apply the same argument in the proof of 4.6. Take a chart by $P := (M_X/\mathcal{O}_X^\times)_x$. In the notation there, since $H^1((X_{mn})_{\mathrm{fl}}^{\mathrm{cl}}, \varepsilon_*(\mathbb{G}_{m,\log}/\mathbb{G}_m)) = H^1((X_{mn})_{\mathrm{fl}}^{\mathrm{cl}}, \mathbb{Q} \otimes (\mathbb{G}_{m,\log}/\mathbb{G}_m)) = H^1((X_{mn})_{\mathrm{\acute{e}t}}^{\mathrm{cl}}, \mathbb{Q} \otimes (\mathbb{G}_{m,\log}/\mathbb{G}_m)) = 0$ by [G] 11.9 again, it is enough to show $H^1(C_{\mathbb{G}_{m,\log}/\mathbb{G}_m, n}) = 0$ for each fixed $n \geq 1$. But the complex $C_{\mathbb{G}_{m,\log}/\mathbb{G}_m, n}$ is easy to be described. In fact, let n' be the greatest divisor of n which is invertible on X. Let $R := \Gamma(X_n, \mathbb{G}_{m,\log}/\mathbb{G}_m) = \mathbb{Q} \otimes (P^{1/n})^{\mathrm{gp}}$ and $S := \{1, \ldots, n'\}$. Then, $X_{n,i}$ is a disjoint union of $(n')^i$ strict local fs log schemes, and the complex $C_{\mathbb{G}_{m,\log}/\mathbb{G}_m, n}$ is isomorphic to $R \xrightarrow{\delta_0} \mathrm{Map}(S, R) \xrightarrow{\delta_1} \mathrm{Map}(S^2, R) \to \cdots$, where $(\delta_0(a))(i) = a$ for any $a \in R, i \in S$, and $(\delta_1(b))(i, j) = b(i) - b(j)$ for any $b \in \mathrm{Map}(S, R), i, j \in S$. Hence $H^1(C_{\mathbb{G}_{m,\log}/\mathbb{G}_m, n}) = 0$ as desired.

6 Locally free modules.

6.1. In this section, for an fs log scheme X, we denote the sheaf $T \mapsto \Gamma(T, \mathcal{O}_T)$ on $X_{\mathrm{fl}}^{\log}$ by $\mathcal{O}_X$.

Consider the map

$$\text{(6.1.1)} \qquad \prod^n H^1(X_{\mathrm{fl}}^{\log}, \mathbb{G}_m) \to H^1(X_{\mathrm{fl}}^{\log}, GL_n(\mathcal{O}_X))$$

($\prod^n$ denotes the product of n copies) induced by the embedding $\prod^n \mathbb{G}_m \overset{\subseteq}{\to} GL_n(\mathcal{O}_X)$ as diagonal matrices. If $X = \mathrm{Spec}(A)$ as a scheme for a Noetherian strict local ring, $H^1(X_{\mathrm{fl}}^{\log}, \mathbb{G}_m)$ is isomorphic to $(M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x \otimes (\mathbb{Q}/\mathbb{Z})$, where x denotes the closed point of X (4.1).

Theorem 6.2. *Let X be an fs log scheme and assume $X = \mathrm{Spec}(A)$ as a scheme for a Noetherian strict local ring A. Then the map 6.1.1 induces an isomorphism of pointed sets*

$$H^1(X_{\mathrm{fl}}^{\log}, GL_n(\mathcal{O}_X)) \cong \mathfrak{S}_n \backslash (\prod^n ((M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x \otimes (\mathbb{Q}/\mathbb{Z}))).$$

Here $\mathcal{O}_X$ denotes the sheaf $T \to \Gamma(T, \mathcal{O}_T)$ on $X_{\mathrm{fl}}^{\log}$, and $\mathfrak{S}_n \backslash$ means the quotient by the natural action of the symmetric group of degree n on the product of n-copies.

Remark 6.3. For an fs log scheme X, $H^1(X_{\mathrm{fl}}^{\log}, GL_n(\mathcal{O}_X))$ is identified with the set of isomorphism classes of $\mathcal{O}_X$-modules on $X_{\mathrm{fl}}^{\log}$ which are locally free of finite rank.

A way to obtain such modules is the following. Let X be an fs log scheme having a chart $P \to M_X$, and take an fs monoid Q and an integral homomorphism $P \to Q$ of Kummer type. Let Y be the scheme $X \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ endowed with the log structure associated to $Q \to \mathcal{O}_Y$. Then if f denotes the canonical morphism $Y_{\mathrm{fl}}^{\log} \to X_{\mathrm{fl}}^{\log}$, the $\mathcal{O}_X$-module $f_*\mathcal{O}_Y$ on $X_{\mathrm{fl}}^{\log}$ is locally free of finite rank. We have the following direct decomposition of $f_*\mathcal{O}_Y$ into invertible modules. Let H be the group scheme $\mathrm{Spec}(\mathbb{Z}[Q^{\mathrm{gp}}/P^{\mathrm{gp}}])$ over $\mathbb{Z}$. Then H acts on Y over X, and hence on $f_*\mathcal{O}_Y$. For $a \in Q^{\mathrm{gp}}/P^{\mathrm{gp}}$, let L_a be the part of $f_*\mathcal{O}_Y$ on which H acts via the character $H \to \mathbb{G}_m$ corresponding to a. Then

$$f_*\mathcal{O}_Y = \bigoplus_a L_a,$$

where a ranges over all elements of $Q^{\mathrm{gp}}/P^{\mathrm{gp}}$ (this can be checked log flat locally). Let $a \in Q^{\mathrm{gp}}$ and let m be a non-zero integer such that $a^m \in P^{\mathrm{gp}}$. Then the element of $H^1(X_{\mathrm{fl}}^{\log}, \mathbb{G}_m)$ corresponding to the invertible module $L_{a \bmod P^{\mathrm{gp}}}$ coincides with the image of a^m under $H^0(X_{\mathrm{fl}}^{\log}, \mathbb{G}_{m,\log}) \to H^1(X_{\mathrm{fl}}^{\log}, \mathbb{Z}/m(1)) \to H^1(X_{\mathrm{fl}}^{\log}, \mathbb{G}_m)$, where the first arrow is the connecting map of the Kummer sequence $0 \to \mathbb{Z}/m(1) \to \mathbb{G}_{m,\log} \overset{m}{\to} \mathbb{G}_{m,\log} \to 0$. If X is as in 6.2 and x is its closed point, this element of $H^1(X_{\mathrm{fl}}^{\log}, \mathbb{G}_m) = H^1(X_{\mathrm{fl}}^{\log}, GL_1(\mathcal{O}_X))$ corresponds to $a^m \otimes m^{-1}$ of $(M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x \otimes (\mathbb{Q}/\mathbb{Z})$ in the isomorphism of 6.2.

Corollary 6.4. *Let X be as in 6.2, and let F be an $\mathcal{O}_X$-module on $X_{\mathrm{fl}}^{\log}$ which is locally free of finite rank. Then F is a direct sum of invertible $\mathcal{O}_X$-modules on $X_{\mathrm{fl}}^{\log}$.*

The method of proving of 6.2 is as follows. Let x be the closed point of X. By the similar method to the proof of Theorem 4.1, we find

$$H^1(X_{\mathrm{fl}}^{\log}, GL_n(\mathcal{O}_X)) \cong \varinjlim_{m} \mathrm{Hom}(H_m, GL_n)/\sim,$$

where H_m is the group scheme $\mathrm{Hom}((M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x, \mathbb{Z}/m(1))$ and $/\sim$ means the quotient set by the inner conjugation by elements of $GL_n(A)$. Theorem 6.2 can be deduced from this.

This Theorem 6.2 is quoted in [Ni] and a proof with details is given there. See [Ni] Theorem 3.22.

We prove some propositions concerning $\mathcal{O}_X$-modules on log flat sites. Some results concerning GL_n-torsors and vector bundles on log flat sites are given in §9.

Proposition* 6.5. *Let X be an fs log scheme whose underlying scheme is affine, and let F be an $\mathcal{O}_X$-module satisfying the following condition: There is a covering $Y \to X$ in $X_{\mathrm{fl}}^{\log}$ such that the pullback of F on $Y_{\mathrm{fl}}^{\log}$ is isomorphic to the module theoretic inverse image on $Y_{\mathrm{fl}}^{\log}$ of some quasi-coherent module on the small Zariski site of Y. Then,*

$$H^m(X_{\mathrm{fl}}^{\log}, F) = 0 \qquad for~any~m \geq 1.$$

The proof of 6.5 is by the computation of Čech cohomology.

This Proposition 6.5 is quoted in [Ni] and a proof with details is given there. See [Ni] Proposition 3.27.

Proposition 6.6. *Let X be an fs log scheme and let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of $\mathcal{O}_X$-modules on $X_{\mathrm{fl}}^{\log}$ which are locally free and of finite rank.*

(1) *If the underlying scheme of X is affine, this exact sequence splits.*

(2) *F is classical if and only if F' and F'' are classical.*

Here we say an $\mathcal{O}_X$-module F on $X_{\mathrm{fl}}^{\log}$ which is locally free of finite rank is classical if the restriction $\overline{F}$ of F to the small Zariski site of X is locally free and F is the module theoretic inverse image of $\overline{F}$ on $X_{\mathrm{fl}}^{\log}$.

Proof of (1). Consider the exact sequence of cohomology groups associated to the exact sequence

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(F'', F') \to \mathcal{H}om_{\mathcal{O}_X}(F'', F) \to \mathcal{H}om_{\mathcal{O}_X}(F'', F'') \to 0$$

of sheaves on $X_{\mathrm{fl}}^{\log}$. By 6.5, $H^1(X_{\mathrm{fl}}^{\log}, \mathcal{H}om_{\mathcal{O}_X}(F'', F')) = 0$. Hence the identity map of F'' comes from $\mathrm{Hom}_{\mathcal{O}_X}(F'', F)$, and hence we obtain the splitting.

Proof of (2). We may assume that the underlying scheme of X is affine. Then, we are reduced to (1).

7 Descent theory. II. Logarithmic flat descent.

In the circulated version (cf. Introduction), we have announced the following results (under some finiteness conditions). The original proofs are so long and complicated, but for the first three cases of 7.1 and for 7.2, Illusie–Nakayama–Tsuji ([INT]) gave considerable short proofs based on a result of Olsson in [Ol]. So we do not include here the original proofs. The last two cases of 7.1 were also proved in Tani's [T].

Theorem* 7.1. *Let $f\colon X \to Y$ be a morphism of fs log schemes, and let $g\colon Y' \to Y$ be a covering in $Y_{\mathrm{fl}}^{\log}$. Let $X' = X \times_Y Y'$ (the fiber product in the category of fs log schemes) and let $f'\colon X' \to Y'$ be a morphism induced by f.*

Then f is log étale (resp. log smooth, resp. log flat, resp. of Kummer type, resp. with finite underlying morphism of schemes) if and only if so is f'.

Theorem* 7.2. *Let $X' \xrightarrow{g} X \xrightarrow{f} Y$ be morphisms of fs log schemes, and assume that g is surjective and of Kummer type.*

If g and $f \circ g$ are log étale (resp. log smooth, resp. log flat), then f is log étale (resp. log smooth, resp. log flat).

8 Descent theory. III.

In the theory of fs log schemes, the descent theory for objects works for the classical fppf topology.

Theorem 8.1. *Let X be an fs log scheme and let F be a sheaf on $X_{\mathrm{fl}}^{\mathrm{cl}}$. Assume that there is a covering $Y \to X$ in $X_{\mathrm{fl}}^{\mathrm{cl}}$ such that the inverse image of F on (fs/Y) is represented by an fs log scheme T over Y which is of Kummer type over Y and whose underlying scheme is affine over that of Y.*

Then, F is represented by an fs log scheme S over X which is of Kummer type over X and whose underlying scheme is affine over that of X.

We remark that this theorem also follows from [Ol] Appendix Corollary A.5.

Proof. Let X, F, Y, T be as in the hypothesis of the theorem. By the classical fppf descent, there exists a scheme S over the scheme X which is affine over the scheme X and which is endowed with an isomorphism $T \cong S \times_X Y$ of schemes over the scheme Y. Our task is to descend the log structure of T to S. Let T' be the fiber product $T \times_Y (Y \times_X Y)$ in the category of fs log schemes, which represents the pullback of F on $Y \times_X Y$. (As a scheme, T' is the fiber product $T \times_S T$ in the category of schemes.) Let $f\colon T \to S$ and $g\colon T' \to S$ be the canonical morphisms of schemes. Note f is faithfully flat and locally of finite presentation. Define the sheaf M_S on $S_{\text{ét}}$ to be the equalizer of $f_* M_T \rightrightarrows g_* M_{T'}$ where the two arrows are induced by the first and the second projections $T' \to T$. Then $M_S \to \mathcal{O}_S$ is induced from the exactness of $\mathcal{O}_S \to \mathcal{O}_T \rightrightarrows \mathcal{O}_{T'}$, and it is easy to see that M_S is a log structure. Now 8.1 is reduced to

Lemma 8.2. *Let the notation be as above. Then, M_S is an fs log structure and M_T is the inverse image of M_S.*

We deduce 8.2 from

Lemma 8.3. *With the notation as above, $f^{-1}(M_S/\mathcal{O}_S^\times) \xrightarrow{\cong} M_T/\mathcal{O}_T^\times$.*

Proof of 8.2 assuming 8.3. Let $t \in T$ and $s = f(t) \in S$, and let $P = (M_S/\mathcal{O}_S^\times)_{\bar{s}}$. Since P is isomorphic to $(M_T/\mathcal{O}_T^\times)_{\bar{t}}$, it is an fs monoid (1.5) (3). Take a homomorphism $h\colon P \to M_{S,\bar{s}}$ such that the composite $P \xrightarrow{h} M_{S,\bar{s}} \to P$ is the identity. Since P is of finite presentation as a monoid, we can extend h to a homomorphism $P \to M_S|_U$ for an étale neighbourhood

U of $\bar{s}$. Since $P \overset{\cong}{\to} (M_T/\mathcal{O}_T^\times)_{\bar{t}}$, $P \to M_T|_V$ is a chart for an étale neighbourhood V of $\bar{t}$ in $f^{-1}(U)$. Let W be the image of V in U. Since f is flat and locally of finite presentation, it is an open map ([EGAIV] Théorème 2.4.6), and hence W is an open set of U. Since the inverse image of $M_W/\mathcal{O}_W^\times$ on V coincides with $M_V/\mathcal{O}_V^\times$, we have that M_W coincides with the log structure associated to $P \to \mathcal{O}_S$.

We prove 8.3 by dividing it into the two parts (1) (2) of

Lemma 8.4. *Define the sheaf N on $S_{\text{ét}}$ to be the equalizer of $f_*(M_T/\mathcal{O}_T^\times) \rightrightarrows g_*(M_{T'}/\mathcal{O}_{T'}^\times)$. Then:*
(1) $f^{-1}(N) \overset{\cong}{\to} M_T/\mathcal{O}_T^\times$.
(2) $M_S/\mathcal{O}_S^\times \overset{\cong}{\to} N$.

Proof of (1). First note that all sheaves are for the classical étale topology here. Let $t \in T$, $a \in (M_T/\mathcal{O}_T^\times)_{\bar{t}}$ and let s (resp. x) be the image of t in S (resp. X). We prove a comes from $N_{\bar{s}}$. Since T is of Kummer type over X, there exists $n \geq 1$ such that a^n comes from $b \in (M_X/\mathcal{O}_X^\times)_{\bar{x}}$. Working classical étale locally on S and on X, we may assume that a comes from an element $\tilde{a}$ of $\Gamma(U, M_T/\mathcal{O}_T^\times)$ for some étale neighbourhood $U \to T$ of $\bar{t} \to T$ such that $U \to S$ is surjective, b comes from an element $\tilde{b}$ of $\Gamma(X, M_X/\mathcal{O}_X^\times)$, and $\tilde{a}^n$ comes from $\tilde{b}$. (Here we used the fact f is an open map.)

Lemma 8.5. *For any $y \in T$, the image of $\tilde{b}$ in $(M_T/\mathcal{O}_T^\times)_{\bar{y}}$ is the n-th power of some element.*

Proof. Take $y' \in T$ which is in the image of U in T such that $f(y) = f(y')$, and take $z \in T'$ such that $p_1(z) = y$ and $p_2(z) = y'$ (here p_i is the i-th projection $T' \to T$). We have isomorphisms

$$(M_T/\mathcal{O}_T^\times)_{\bar{y}} \overset{\cong}{\to} (M_{T'}/\mathcal{O}_{T'}^\times)_{\bar{z}} \overset{\cong}{\leftarrow} (M_T/\mathcal{O}_T^\times)_{\overline{y'}} \quad \text{over } (M_X/\mathcal{O}_X^\times)_{\bar{x}}.$$

Now 8.5 follows from the fact that the image of $\tilde{b}$ in $(M_T/\mathcal{O}_T^\times)_{\overline{y'}}$ is an n-th power of the image of $\tilde{a}$. $\qquad\square$

By 8.5, the image of $\tilde{b}$ in $M_T/\mathcal{O}_T^\times$ is locally an n-th power. Since $M_T/\mathcal{O}_T^\times$ is torsion free, the n-th root of $\tilde{b}$ in $M_T/\mathcal{O}_T^\times$ exists globally on T, and it should coincide on U with $\tilde{a}$. This n-th root of $\tilde{b}$ on T is a global section of N on S by the uniqueness of n-th root in $M_{T'}/\mathcal{O}_{T'}^\times$ of the image of $\tilde{b}$. This shows that a comes from $N_{\bar{s}}$. This completes the proof of (1) of 8.4.

Proof of 8.4 (2). The problem is the surjectivity of $M_S \to N$. Let $h \in \Gamma(S, N)$. It is sufficient to prove that h comes from M_S étale locally on S. The inverse image of h in M_T under $M_T \to M_T/\mathcal{O}_T^\times$ is an $\mathbb{G}_m$-torsor on T which is endowed with descent data on T'. By the descent theory of line bundles, this $\mathbb{G}_m$-torsor descends to a $\mathbb{G}_m$-torsor L on S. Étale locally on S, L has a section which is regarded as a section of M_S with image h in N. Hence $M_S \to N$ is surjective. $\qquad\square$

9   Torsors.

The aim of this section is to prove

Theorem 9.1. *Let X be an fs log scheme which is locally Noetherian as a scheme. Let G be a finite flat commutative group scheme over the underlying scheme of X, which we endow with the inverse image of M_X, and let F be a G-principal homogeneous space in the category of sheaves on $X_{\mathrm{fl}}^{\log}$. Then, F is representable by an fs log scheme over X which is log flat of Kummer type and whose underlying scheme is finite over that of X.*

9.2. Let X and G be as in the hypothesis of 9.1. We denote by $H_{\mathrm{r}}^1(X_{\mathrm{fl}}^{\log}, G)$ the subset of $H^1(X_{\mathrm{fl}}^{\log}, G)$ consisting of elements whose corresponding G-principal homogeneous space on $X_{\mathrm{fl}}^{\log}$ is represented by an fs log scheme over X, which is log flat and of Kummer type and whose underlying scheme is finite over that of X. Theorem 9.1 states that $H_{\mathrm{r}}^1(X_{\mathrm{fl}}^{\log}, G)$ coincides with $H^1(X_{\mathrm{fl}}^{\log}, G)$. Note that

$$H^1(X_{\mathrm{fl}}^{\mathrm{cl}}, G) \subset H_{\mathrm{r}}^1(X_{\mathrm{fl}}^{\log}, G) \qquad \text{in } H^1(X_{\mathrm{fl}}^{\log}, G)$$

by the classical fppf descent theory.

Lemma 9.3. *Let X and G be as in the hypothesis of 9.1.*
  (1) If $\chi_1 \in H_{\mathrm{r}}^1(X_{\mathrm{fl}}^{\log}, G)$ and $\chi_2 \in H^1(X_{\mathrm{fl}}^{\mathrm{cl}}, G)$, then $\chi_1 + \chi_2 \in H_{\mathrm{r}}^1(X_{\mathrm{fl}}^{\log}, G)$.
  (2) Let G' be a finite flat commutative group scheme over the underlying scheme of X, which we endow with the inverse image of the log structure of X, and let $h\colon G' \to G$ be an injective homomorphism. Then h sends $H_{\mathrm{r}}^1(X_{\mathrm{fl}}^{\log}, G')$ into $H_{\mathrm{r}}^1(X_{\mathrm{fl}}^{\log}, G)$.

Proof of (1). By §8, we may work classical fppf locally on X. Since χ_2 vanishes classical fppf locally, we may assume $\chi_2 = 0$.
Proof of (2). Let F' be the G'-principal homogeneous space on $X_{\mathrm{fl}}^{\log}$ whose class belongs to $H_{\mathrm{r}}^1(X_{\mathrm{fl}}^{\log}, G')$. Then the image of the class of F' in $H^1(X_{\mathrm{fl}}^{\log}, G)$ is represented by the induced G-principal homogeneous space $F = G' \backslash (G \times F')$, where G' acts on $G \times F'$ by (σ^{-1}, σ) $(\sigma \in G')$. We have a cartesian diagram

$$
\begin{array}{ccc}
G \times F' & \longrightarrow & F \\
{\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow \\
G & \longrightarrow & G' \backslash G.
\end{array}
$$

Since $G \to G' \backslash G$ is fppf, the diagram implies that F over $G' \backslash G$ is represented classical fppf locally on $G' \backslash G$, by a log flat fs log scheme of Kummer type whose underlying scheme is finite over the base. Hence by §8, $F \to G' \backslash G$ is represented by an fs log scheme over $G' \backslash G$ which is log flat and of Kummer type whose underlying scheme is finite over $G' \backslash G$. $\qquad\square$

Lemma 9.4. *Let X be an fs log scheme and let $n \geq 1$. Then the image of the connecting map $H^0(X, M_X^{\mathrm{gp}}) \to H^1(X_{\mathrm{fl}}^{\log}, \mathbb{Z}/n(1))$ of the Kummer sequence is contained in $H_{\mathrm{r}}^1(X_{\mathrm{fl}}^{\log}, \mathbb{Z}/n(1))$.*

Proof. Let $a \in H^0(X, M_X^{\mathrm{gp}})$, and let F_a be the $\mathbb{Z}/n(1)$-principal homogeneous space corresponding to the image of a in $H^1(X_{\mathrm{fl}}^{\log}, \mathbb{Z}/n(1))$. That is, $F_a(T) = \{b \in \Gamma(T, M_T^{\mathrm{gp}}) \; ; \; b^n = a\}$ for any fs log scheme T over X. By working classical étale locally on X, we may assume that there are a chart $P \to M_X$ and an element $\tilde{a}$ of P whose image in $H^0(X, M_X^{\mathrm{gp}})$ is a. Let L be the abelian group generated by P^{gp} and a letter b which is subject to the relation $b^n = \tilde{a}$. Let $Q = \{x \in L \; ; \; x^n \in P\}$ and let $Y = X \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ which is endowed with the log structure associated to $Q \to \mathcal{O}_Y$. Then, $b \in F(Y)$ defines an isomorphism $\mathrm{Mor}_X(\ , Y) \overset{\cong}{\to} F_a$. Furthermore by the construction, Y is log flat of Kummer type over X and the underlying scheme of Y is finite over that of X. $\qquad\square$

9.5. Now we prove 9.1. By §8 and by a limit argument, we may assume that as a scheme, $X = \mathrm{Spec}(A)$ for some Noetherian strict local ring A. Take a chart $P \to M_X$ which induces $P \overset{\cong}{\to} M_{X,x}/\mathcal{O}_{X,x}^\times$ where x is the closed point of x. Let

$$(9.5.1) \qquad \varinjlim_n \mathrm{Hom}(\mathbb{Z}/n(1), G) \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \to H^1(X_{\mathrm{fl}}^{\log}, G)$$

be the homomorphism which sends $h \otimes a$ ($h \in \mathrm{Hom}(\mathbb{Z}/n(1), G)$, $a \in P^{\mathrm{gp}}$) to the image of a under $P^{\mathrm{gp}} \to H^0(X, M_X^{\mathrm{gp}}) \to H^1(X_{\mathrm{fl}}^{\log}, \mathbb{Z}/n(1)) \overset{h}{\to} H^1(X_{\mathrm{fl}}^{\log}, G)$. By §4, we have an isomorphism

$$H^1(X_{\mathrm{fl}}^{\mathrm{cl}}, G) \oplus (\varinjlim_n \mathrm{Hom}(\mathbb{Z}/n(1), G) \otimes_{\mathbb{Z}} P^{\mathrm{gp}}) \overset{\cong}{\to} H^1(X_{\mathrm{fl}}^{\log}, G).$$

By 9.3 (1), it is enough to show that the image of $\varinjlim_n \mathrm{Hom}(\mathbb{Z}/n(1), G) \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \to H^1(X_{\mathrm{fl}}^{\log}, G)$ is contained in $H_{\mathrm{r}}^1(X_{\mathrm{fl}}^{\log}, G)$. Let G^{mult} be the multiplicative part of G. Then, any homomorphism $\mathbb{Z}/n(1) \to G$ factors through G^{mult}. By 9.3 (2), we may assume G is multiplicative. Then, $G \cong \oplus_i (\mathbb{Z}/m_i\mathbb{Z})(1)$ for some finite family of non-zero integers $(m_i)_i$. Since a G-principal homogeneous space is the product of $\mathbb{Z}/m_i(1)$-principal homogeneous spaces, we may assume $G = \mathbb{Z}/m(1)$ for some $m \neq 0$. Then,

$$\varinjlim_n \mathrm{Hom}(\mathbb{Z}/n(1), G) \otimes_{\mathbb{Z}} P^{\mathrm{gp}} \cong P^{\mathrm{gp}}/(P^{\mathrm{gp}})^m,$$

and the map 9.5.1 is identified with the map $P^{\mathrm{gp}}/(P^{\mathrm{gp}})^m \to H^1(X_{\mathrm{fl}}^{\log}, \mathbb{Z}/m(1))$ of the Kummer sequence. Hence we are reduced to 9.4.

9.6. We add some remarks on GL_n-torsors and vector bundles on $X_{\mathrm{fl}}^{\log}$. We have seen in §6 that $\mathcal{O}_X$-modules on $X_{\mathrm{fl}}^{\log}$ which are locally free of finite rank need not be classical ones. The following 9.7 (resp. 9.8) says that representable GL_n-torsors (resp. vector bundles) on $X_{\mathrm{fl}}^{\log}$ *need not be* (resp. *need be*) classical ones.

   (Recall that the sheaf $\mathcal{O}_X$ on $X_{\mathrm{fl}}^{\log}$ is defined by $T \to \Gamma(T, \mathcal{O}_T)$.)

Proposition 9.7. *Let F be a $GL_n(\mathcal{O}_X)$-torsor in the category of sheaves on $X_{\mathrm{fl}}^{\log}$. Then F is representable.*

Proposition 9.8. *Let F be an $\mathcal{O}_X$-module on $X_{\mathrm{fl}}^{\log}$ which is locally free of finite rank. Then F is representable if and only if F is a classical one.*

Proof of 9.7. By a limit argument we may assume that $X = \mathrm{Spec}(A)$ for a Noetherian strict local ring A. Let $\alpha \in H^1(X_{\mathrm{fl}}^{\log}, GL_n(\mathcal{O}_X))$ and let F be the $GL_n(\mathcal{O}_X)$-torsor on $X_{\mathrm{fl}}^{\log}$ corresponding to α. We show that F is representable. Let $P \to M_X$ be a chart such that $P \overset{\cong}{\to} (M_X/\mathcal{O}_X^\times)_x$ where x is the closed point of X. By §6, α comes from an element $(\alpha_1, \ldots, \alpha_n)$ of $(P^{\mathrm{gp}} \otimes (\mathbb{Q}/\mathbb{Z}))^n$. Take $m_i \geq 1$ such that $m_i \alpha_i = 0$, and write $\alpha_i = a_i \otimes m_i^{-1}$, $a_i \in P^{\mathrm{gp}}$ and let F_i be the $\mathbb{Z}/m_i(1)$-torsor corresponding to the image in $H^1(X_{\mathrm{fl}}^{\log}, \mathbb{Z}/m_i(1))$ of a_i under the connecting map of $0 \to \mathbb{Z}/m_i(1) \to \mathbb{G}_{m,\log} \overset{m_i}{\to} \mathbb{G}_{m,\log} \to 0$. Let $H = \prod_{1 \leq i \leq n} \mathbb{Z}/m_i(1)$. Then the $GL_n(\mathcal{O}_X)$-torsor F is induced from the H-torsor $\prod_{1 \leq i \leq n} F_i$ by the diagonal embedding $H \to GL_n(\mathcal{O}_X)$. Since the quotient $H \backslash GL_n(\mathcal{O}_X)$ is representable and $GL_n(\mathcal{O}_X) \to H \backslash GL_n(\mathcal{O}_X)$ is fppf covering, the cartesian diagram

$$
\begin{array}{ccc}
GL_n(\mathcal{O}_X) \times \prod_{1 \leq i \leq n} F_i & \longrightarrow & F \\
\downarrow & & \downarrow \\
GL_n(\mathcal{O}_X) & \longrightarrow & H \backslash GL_n(\mathcal{O}_X)
\end{array}
$$

shows (§8) that F is representable.

Proof of 9.8. The "if" part is clear and so we consider the "only if" part. We may assume $X = \mathrm{Spec}\, A$ as a scheme for a Noetherian strict local ring. Assume F is representable but not classical. Then F comes from an element $(a_i \otimes m_i^{-1})_{1 \leq i \leq n}$ of $((M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x \otimes (\mathbb{Q}/\mathbb{Z}))^n$ ($n = \mathrm{rank}\,(F)$, x is the closed point, $a_i \in (M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x$, $m_i \in \mathbb{Z}, m_i \geq 1$) such that a_1 is not an m_1-th power in $(M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x$.

Let Y be the $GL_n(\mathcal{O}_Y)$-torsor defined to be the subsheaf of $\overset{n}{\prod} F$ consisting of bases of F. Then Y is representable (9.7). We denote by the same letter F (resp. Y) the fs log scheme over X which represents F (resp. Y). We prove the following $(*)$ and $(**)$.

$(*)$ There is an open set U of $\overset{n}{\prod} F$ which contains the image of the zero section $X \to \overset{n}{\prod} F$ such that M_U coincides with the inverse image of M_X.

$(**)$ For any $y \in Y$ lying over x, $(M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_{\overline{x}} \to (M_Y^{\mathrm{gp}}/\mathcal{O}_Y^\times)_{\overline{y}}$ is not bijective.

These $(*)$ and $(**)$ imply $(U \cap Y) \times_X \{x\} = \varnothing$. But if $X' \to X$ is a covering in $X_{\mathrm{fl}}^{\log}$ such that the pullback of F on X' is classical and if x' is a point of X' lying over x, $(U \cap Y) \times_X \{x'\} \neq \varnothing$ because a classical vector bundle over a field is irreducible (so any two non-empty open sets intersect).

Proof of $(*)$. $\overset{n}{\prod} F$ is of Kummer type over X since log flat locally on X, F becomes a classical vector bundle with the inverse image of M_X. Let $y \in \overset{n}{\prod} F$ be a point in the image of the zero section $X \to \overset{n}{\prod} F$, and let s be the image of y in X. Then, the zero section defines $g \colon (M_Y^{\mathrm{gp}}/\mathcal{O}_Y^\times)_{\overline{y}} \to (M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_{\overline{s}}$ and the composite $g \circ h$ with the canonical map $h \colon (M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_{\overline{s}} \to (M_Y^{\mathrm{gp}}/\mathcal{O}_Y^\times)_{\overline{y}}$ is the identity map of $(M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_{\overline{s}}$. Since h is of Kummer type, this means that h is bijective. This proves $(*)$.

Proof of $(**)$. Since the pullback of F on Y is isomorphic to $\overset{n}{\prod} \mathcal{O}_Y$, the image of α_i in $(M_Y^{\mathrm{gp}}/\mathcal{O}_Y^\times)_{\overline{y}} \otimes \mathbb{Q}/\mathbb{Z}$ vanishes for any $y \in Y$. Hence a_1 is an m_1-th power in $(M_Y^{\mathrm{gp}}/\mathcal{O}_Y^\times)_{\overline{y}}$ for any $y \in Y$.

10 Logarithmic fundamental groups.

10.1. As is said in §3, the log flat descent for objects does not work well. An exception is the descent for log étale finite objects of Kummer type.

For an fs log scheme X, let $\text{Ét}^{\log}(X)$ be the category of fs log schemes over X which are log étale and of Kummer type over X and whose underlying scheme is finite over that of X. Let $\text{lcf}(X_{\text{ét}}^{\log})$ (resp. $\text{lcf}(X_{\text{fl}}^{\log})$) be the category of sheaves on $X_{\text{ét}}^{\log}$ (resp. $X_{\text{fl}}^{\log}$) which are locally constant and finite.

Theorem 10.2. *Let X be an fs log scheme whose underlying scheme is locally Noetherian.*

(1) For an object Y of $\text{Ét}^{\log}(X)$, the sheaf $\text{Mor}_X(\ ,Y)$ on $X_{\text{ét}}^{\log}$ (resp. $X_{\text{fl}}^{\log}$) belongs to $\text{lcf}(X_{\text{ét}}^{\log})$ (resp. $\text{lcf}(X_{\text{fl}}^{\log})$).

(2) We have equivalences of categories

$$\text{Ét}^{\log}(X) \xrightarrow{\simeq} \text{lcf}(X_{\text{ét}}^{\log}) \xrightarrow{\simeq} \text{lcf}(X_{\text{fl}}^{\log}).$$

(3) If X is connected, there exists a profinite group $\pi_1^{\log}(X)$, which is unique up to isomorphism, such that the equivalent categories in (2) are equivalent to the category of finite $\pi_1^{\log}(X)$-sets.

(4) If $X = \text{Spec}(A)$ as a scheme for a strict local ring A, we have an isomorphism

$$\pi_1^{\log}(X) \cong \varprojlim_n \text{Hom}((M_X^{\text{gp}}/\mathcal{O}_X^\times)_x, \mathbb{Z}/n(1))$$

where n ranges over all integers which are invertible in A, x is the closed point, and $\mathbb{Z}/n(1)$ denotes $\Gamma(\text{Spec}(A), \mathbb{Z}/n(1))$.

Here in (3), for a profinite group G, a G-set means a discrete set endowed with a continuous action of G.

The log fundamental group is first treated in [FK]. A large part of 10.2 is taken from [FK]. See also [I], [S], [V], [Ho].

Remark 10.3. Let X be a regular locally Noetherian scheme and let D be a divisor on X with normal crossings. Endow X with the log structure defined by D (that is, $M_X = \{f \in \mathcal{O}_X \ ; \ f$ is invertible outside $D\}$. Then $\pi_1^{\log}(X)$ coincides with the tame fundamental group of Grothendieck-Murre $\pi_1(X, D)$ which controls finite étale schemes over $X \setminus D$ which are at worst tamely ramified at generic points of D.

In the rest of this section, we prove 10.2.

10.4. We prove 10.2 (1). Note that $\text{Mor}_X(\ ,Y)$ is indeed a sheaf by 3.1. Since a finite kummer log étale fs log scheme over an fs log scheme is kummer log étale locally a finite étale scheme in the classical sense endowed with the pullback log structure from the base, (1) is reduced to the nonlog corresponding result.

Proposition 10.5. *Let X and x be as in the assumption of 10.2 (4) and let G be a finite group. For an integer $n \geq 1$, let H_n be the group scheme $\mathcal{H}om((M_X^{\text{gp}}/\mathcal{O}_X^\times)_x, \mathbb{Z}/n(1))$.*

(1) *Take a chart $P \to M_X$ such that $P \overset{\cong}{\to} (M_X/\mathcal{O}_X^\times)_x$. For each integer $n \geq 1$, let $X_n \to X$ be as in 4.5. Let T be a G-torsor on $X_{\mathrm{fl}}^{\log}$. Then there is an integer $n \geq 1$ which is invertible on X such that the pullback of T on $(X_n)_{\mathrm{fl}}^{\log}$ is a trivial G-torsor.*

(2) *For Let $* = $ ét or fl, $H^1(X_*^{\log}, G)$ is canonically isomorphic to $\varinjlim_n \mathrm{Hom}(H_n(k), G)/ \sim$, where n ranges over all integers ≥ 1 which are invertible on X, k is the residue field of x, and $\sim$ is the G-conjugacy.*

Proof. By 2.7 (2), if T is a G-torsor on $X_{\mathrm{fl}}^{\log}$, there is an integer $n \geq 1$ such that the pullback of T on $(X_n)_{\mathrm{fl}}^{\log}$ is a G-torsor over X_n in the classical sense with the inverse image of the log structure of X_n. Since this G-torsor over X_n is finite étale over X_n and X_n is strict local, it is a trivial G-torsor. Thus we have proved a weaker version of (1) without the condition n is invertible on X.

Hence $H^1(X_{\mathrm{fl}}^{\log}, G)$ is isomorphic to the inductive limit of the H^1- Čech cohomology $\{g \in G(X_n \times_X X_n) \mid p_{13}^*(g) = p_{12}^*(g)p_{23}^*(g)\}/ \sim$ of $(X_n/X, G)$, where p_{ij}^* is the pullback by the (i,j)-th projection $p_{ij} : X_n \times_X X_n \times_X X_n \to X_n \times_X X_n$ and $\sim$ is the following equivalence relation. $g' \sim g$ if and only if $g' = p_1^*(u)gp_2^*(u)^{-1}$ for some $u \in G(X_n)$ where $p_i : X_n \times_X X_n \to X_n$ is the i-th projection.

For any Noetherian scheme S, $G(S)$ is identified with the set of all maps $\pi_0(S) \to G$ where $\pi_0(S)$ denotes the set of connected components of S. If $n' \geq 1$ denotes the largest divisor of n which is invertible on X, we have $\pi_0(X_n \times_X X_n) \cong \pi_0(H_n \times X_n) = H_{n'}(k)$ and $\pi_0(X_n \times_X X_n \times_X X_n) \cong \pi_0(H_n \times H_n \times X_n) = H_{n'}(k) \times H_{n'}(k)$, and hence the H^1- Čech cohomology of $(X_n/X, G)$ is identified with $\mathrm{Hom}(H_{n'}(k), G)/ \sim$ where $\sim$ is the conjugacy by G. Hence we obtain the isomorphism in (2) for $H^1(X_{\mathrm{fl}}^{\log}, G)$. This isomorphism is canonical (independent of the choice of the above chart $P \to M_X$) because the element of $H^1(X_{\mathrm{fl}}^{\log}, H_{n'}(k))$ corresponding to the identity map of $H_{n'}(k)$ comes from the canonical isomorphism in 4.1.

This argument shows that the H^1-Čech cohomology of $(X_n/X, G)$ is the same as that of $(X_{n'}/X, G)$. This completes the proof of (1).

Since $H^1(X_{\mathrm{ét}}^{\log}, G) \subset H^1(X_{\mathrm{fl}}^{\log}, G)$ and X_n is log étale over X if n is invertible on X, we have $H^1(X_{\mathrm{ét}}^{\log}, G) = H^1(X_{\mathrm{fl}}^{\log}, G)$. This completes the proof of (2). $\qquad\square$

10.6. Let X be as in the assumption of 10.2 (3). Let $* = $ ét or fl and let $\mathcal{C} = \mathrm{lcf}(X_*^{\log})$. We prove that there exists a profinite group $\pi_{1,*}^{\log}(X)$ such that $\mathcal{C}$ is equivalent to the category of finite $\pi_{1,*}^{\log}(X)$-sets.

Let k be an algebraically closed field with divisible integral log, let $n \geq 1$, and let $a : \mathrm{Spec}(k) \to X$ be a morphism (we call this morphism a base point). Let $\mathcal{C}$ be the category of sheaves on $X_*^{\log}$ which are locally constant and finite. We prove that with the pullback functor by a, $\mathcal{C}$ is a Galois category in the sense of [SGA1] Section V 5, which implies the above.

It is clear that $\mathcal{C}$ satisfies the condition (G1)–(G5) of Galois category in Section V 5 of [SGA1]. We prove that $\mathcal{C}$ satisfies the remaining condition (G6) of the Galois category, that is, a morphism $S \to T$ of $\mathcal{C}$ is an isomorphism if the induced map $S_a \to T_a$ is bijective. Let I be the image of $S \to T$. Let C be the complement of I in T, so T is the disjoint union of I and C. Let D be the complement of the diagonal image of S in $S \times_I S$. Since $S_a \to T_a$ is bijective, C_a and D_a are empty. It is sufficient to prove that these C and D are

empty. We have the vector bundles $\mathcal{O}_C$ and $\mathcal{O}_D$ on the site $X_*^{\log}$. Since X is connected, $\mathcal{O}_C$ and $\mathcal{O}_D$ are of constant rank. Since they are 0 at a, they are 0. Hence C and D are empty.

10.7. We prove the equivalence $\mathrm{lcf}(X_{\text{ét}}^{\log}) \overset{\cong}{\to} \mathrm{lcf}(X_{\text{fl}}^{\log})$ in 10.2 (2). We may assume that as a scheme, $X = \mathrm{Spec}(A)$ for a strict local ring A (and hence is connected). Then by 10.6, the first (resp. second) category is equivalent to the category of finite $\pi_{1,\text{ét}}^{\log}(X)$ (resp. $\pi_{1,\text{fl}}^{\log}(X)$)-sets. By 10.5, we have

(1) $\pi_{1,\text{ét}}^{\log}(X) = \pi_{1,\text{fl}}^{\log}(X) \cong \varprojlim_n \mathrm{Hom}((M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x, \mathbb{Z}/n(1))$

where n ranges over all integers which are invertible on X and x denotes the closed point of X. This proves the part $\mathrm{lcf}(X_{\text{ét}}^{\log}) \overset{\cong}{\to} \mathrm{lcf}(X_{\text{fl}}^{\log})$ of 10.2 (2).

Hence for any connected X, $\pi_{1,\text{ét}}^{\log}(X) = \pi_{1,\text{fl}}^{\log}(X)$ and we denote this group by $\pi_1^{\log}(X)$.

We prove $\mathrm{\acute{E}t}^{\log}(X) \overset{\cong}{\to} \mathrm{lcf}(X_{\text{ét}}^{\log})$ in 10.2 (2). We may assume that as a scheme, $X = \mathrm{Spec}(A)$ for a strict local ring A. By the above (1), it is sufficient to prove that for any commutative finite group G, any G-torsor over X is represented by an object of $\mathrm{\acute{E}t}^{\log}(X)$. But this follows from Theorem 9.1.

This completes the proof of 10.2.

Correction to [K₁]. There are two corrections to [K₁]. The first is about Section 4 of [K₁], which was pointed by A. Ogus. The definition of a morphism of weakly purely inseparable in [K₁] (4.9) is strange and [K₁] PROPOSITION (4.10) (2) and [K₁] LEMMA (4.11) are false as they stand. PROPOSITION (4.10) (1) is correct. We modify the paper as follows. We delete [K₁] (4.9), (4.10)(2), (4.11). In [K₁] THEOREM (4.12), we add the assumption that f is of Cartier type, and $(X'', M'') = (X', M')$. Note that in the rest of [K₁], PROPOSITION (4.12) is not used. In most applications, only morphisms of Cartier type are important.

The next is about PROPOSITION (3.14) (4) of [K₁], which was pointed by Y. Nakkajima. For this statement, it is necessary to assume that f is integral. Further, $I\mathcal{O}_{\tilde{X}}$ there (which is not defined) should be replaced by $I \otimes_{\mathcal{O}_Y} \mathcal{O}_X$.

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