

PAPER

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11D supergravity on $AdS_4 \times S^7$ versus $AdS_7 \times S^4$

Ergin Sezgin 

George and Cynthia Woods Mitchell Institute for Fundamental Physics and Astronomy, Texas A & M University, College Station, TX 77843, United States of America

E-mail: sezgin@tamu.edu

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Abstract

The maximally supersymmetric Freund–Rubin vacua for 11 dimensional supergravity, namely $AdS_4 \times S^7$ and $AdS_7 \times S^4$, admit an analytic continuation to $S^4 \times S^7$. From the full harmonic expansions on $S^4 \times S^7$, it is shown that by analytical continuation to either AdS_4 , or to AdS_7 , the detailed structure of the Kaluza–Klein spectrum can be obtained for both vacua in a unified manner. The results are shown to be related by a simple rule which interchanges the spacetime and internal space representations. We also obtain the linearized field equations for the singletons and doubletons but they can be gauged away by fixing certain Stueckelberg shift symmetries inherited from the Kaluza–Klein reduction.

Keywords: M-theory, flux compactifications, singletons and doubletons

1. Introduction

This paper is dedicated to the memory of Freund. I did not have the opportunity to work with him but I was greatly influenced by his work on Kaluza–Klein supergravity [1], like many others. In fact, as put very well in [2], in the beginning of the 80s, Peter’s paper on Freund–Rubin compactifications of the eleven dimensional supergravity, and another two papers, one by Witten [3] and another by Salam and Strathdee [4], started a renaissance in Kaluza–Klein theories. Nowadays we take for granted the idea spontaneous compactification, but in the early days of Kaluza–Klein supergravity, moving from dimensional reduction to spontaneous compactification by addressing the underlying dynamics, as highlighted in the title of the Freund–Rubin paper as ‘the dynamics of dimensional reduction’, and the fact that it works so naturally to give four dimensional spacetime, was a very impactful development. The Freund–Rubin paper not only emphasized this point but it also elevated greatly the stakes for the 11 dimensional (11D) supergravity, which has proven to be so important for what has become M-theory since the mid 90s.

In this note, it is fitting to revisit the maximally symmetric Freund–Rubin compactifications of 11D supergravity, namely $AdS_{4/7} \times S^{7/4}$ with 4-form flux turned on. Firstly, we would like to find out if the resulting Kaluza–Klein spectrum of states can be described in a unified manner. Second, we aim at probing the question of whether singletons and doubleton field equations can be identified in the bulk. We will see that a unified treatment of the KK spectra is indeed possible, by exploiting the fact that both of the maximally supersymmetric Freund–Rubin vacua admit analytic continuation to $S^4 \times S^7$. As a result, we will see that the detailed supermultiplet structure of the spectrum, as well as the 11D origin of the fluctuations emerges from a simple rule. So far, these spectra have been obtained by separate computations [5–10].

It has been observed for the S^7 compactification in [6, 7], and S^4 compactification in [10], that the group theoretical structure of the KK spectrum suggests the presence of singletons and doubletons. In [8], it was argued that these states vanish identically, while in [6, 7] they appeared as nonpropagating modes, as the saturated propagator has vanishing residue for the associated poles. Examining the issue of whether they can arise as boundary states, we find the linearized field equations for the singletons in AdS_4 and doubletons in AdS_7 , but we also find that there are certain Stueckelberg shift symmetries inherited from the Kaluza–Klein reduction which can be used to gauge them away. Section 5 is devoted to these issues, which are further discussed in the conclusions.

2. Preliminaries

The Freund–Rubin compactifying solutions of 11D supergravity on $AdS_{4/7} \times S^{7/4}$ can be given in a unified fashion such that the only nonvanishing fields are

$$\begin{aligned} \bar{R}_{\mu\nu\rho\sigma} &= -4\epsilon m^2 (\bar{g}_{\mu\rho} \bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma} \bar{g}_{\nu\rho}), \\ 2\bar{R}_{\alpha\beta\gamma\delta} &= \epsilon m^2 (\bar{g}_{\alpha\gamma} \bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta} \bar{g}_{\beta\gamma}), \\ 2\bar{F}_{\mu\nu\rho\sigma} &= 3m \epsilon_{\mu\nu\rho\sigma}, \end{aligned} \tag{1}$$

where m is an arbitrary constant, the Levi-Civita tensor $\epsilon_{\mu\nu\rho\sigma}$ is evaluated in the background, and the following index notation is used

$$\begin{aligned} \epsilon = +1 : AdS_4 \times S^7, \quad \mu = 0, 1, 2, 3, \quad \alpha = 1, 2, \dots, 7, \\ \epsilon = -1 : AdS_7 \times S^4, \quad \alpha = 0, 1, \dots, 6, \quad \mu = 1, 2, 3, 4, \end{aligned} \tag{2}$$

and accordingly, $\epsilon_{\mu\nu\rho\sigma} \epsilon^{\mu\nu\rho\sigma} = -4!\epsilon$. We parametrize the linearized fluctuations around a background as

$$g_{MN} = \bar{g}_{MN} + h_{MN}, \quad A_{MNP} = \bar{A}_{MNP} + a_{MNP}. \tag{3}$$

Introducing the notation $\Phi = (h_{AB}, a_{CAB})$, where A is the 11D tangent space index, and coupling to the source $\mathcal{J} = (T_{AB}, J_{CAB})$, the action quadratic in fluctuations can be written as

$$I^{(2)} = \int d^{11}x \left(-\frac{1}{2} \Phi \mathcal{O} \Phi + \mathcal{J} \Phi \right), \tag{4}$$

where \mathcal{O} is the wave operator. Since the action is invariant under the background gauge transformations

$$\delta h_{MN} = \bar{\nabla}_M \xi_N + \bar{\nabla}_N \xi_M, \quad \delta a_{MNP} = \xi^L \bar{F}_{LMNP} + 3\bar{\nabla}_{[M} \Lambda_{NP]}. \tag{5}$$

it follows that the sources must satisfy the constraints (using the normalizations chosen in [7])

$$\bar{\nabla}^M T_{MN} + \frac{2}{3} \bar{F}_{NPQR} J^{PQR} = 0, \quad \bar{\nabla}^M J_{MNP} = 0. \quad (6)$$

The following gauge was chosen in [7] (slightly different from the gauge chosen in [6])

$$\bar{\nabla}^M \left(h_{MN} - \frac{1}{2} \bar{g}_{MN} \bar{g}^{RS} h_{RS} \right) = 0, \quad \bar{\nabla}^P a_{PMN} = 0. \quad (7)$$

Using the gauge condition, the wave operator \mathcal{O} can be inverted. Substituting the result into $I^{(2)}$, and importantly using the source constraints, we obtain the *saturated propagator*

$$I^{(2)} = \frac{1}{2} \int d^{11}x \mathcal{J} \mathcal{O}^{-1} \mathcal{J}. \quad (8)$$

This procedure, in the case of $\text{Minkowski}_4 \times S^2$ compactification of 6D Maxwell–Einstein theory was employed in [11]. In that case, the harmonic expansion on the Minkowski spacetime is the usual Fourier transform, while here, where we are dealing with AdS spacetimes, harmonic expansions reduce to those on spheres. After harmonic expansions in the total Euclideanized spacetime, the physical states are determined from the analysis of the poles in the principle lowest weight, that is the lowest AdS energy E_0 plane, in the expression for the saturated propagator. The nonvanishing and positive residues describe the physical states. The manner in which the representation function on spheres and AdS space are related under the analytic continuation was examined in detail in [6]. It was also shown that the eigenvalues of the second order Casimir operators for the isometry group of the sphere and AdS space are related by the simple rule where one identifies the leading lowest label with an opposite sign. This simple rule facilitates the physical interpretation of the poles in the lowest weight plane [6, 7].

Turning to the Freund–Rubin compactifications of 11D supergravity, both of the maximally supersymmetric vacua can be treated simultaneously by analytically continuing the equations to $S^4 \times S^7$. In this ‘democratic’ approach, starting from the universal result on the product of the spheres, one can analytically continue either S^4 to AdS_4 , or with equal ease S^7 to AdS_7 , thereby obtaining not only the group theoretical content of the full spectrum of physical states but also the full information about how they are formed out of the 11D supergravity fields.

3. Analytic continuation and harmonic expansions on $S_4 \times S^7$

Analytic continuation from $AdS_4 \times S^7$ to $S^4 \times S^7$ was described in [6, 7]. Here we shall formulate it in a way that enables us to analytically continue also $AdS_7 \times S^4$ to $S^4 \times S^7$ such that all the harmonic analysis performed on $S^4 \times S^7$ in [6, 7] are exactly the same as before. With this strategy in mind, we consider the metric for AdS_{d+2}

$$ds^2 = m^2(-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_d), \quad (9)$$

where $d\Omega_d$ is the metric on unit radius d -sphere. The Euclideanization rule appropriate for our purposes here is to send

$$\rho \mapsto i\rho, \quad (10)$$

Table 1. The highest weights of the G representations occurring in the harmonic expansions of the fields listed in the first columns which have the H -representation content listed in the second column. In representations $(m_1)(\ell\ell_1\ell_2\ell_3)$ it is understood that $n \geq n_1$ and $\ell \geq \ell_1 \geq \ell_2 \geq |\ell_3|$. The last entries in each row, and the second to the last entries for $K_{\mu\alpha}$ and $Y_{\mu\alpha}$ turn out to be unphysical gauge modes. There are six $(n)(\ell)$ entries but only four of them are independent. We have chosen to eliminate those in $H_{\mu\nu}$ and $L_{\alpha\beta}$.

Fields	$SO(4) \times SO(7)$ content	$SO(5) \times SO(8)$ content	Restrictions
$H_{\mu\nu}$	(20)(000)	$(n2)(\ell), (n)(\ell), (n1)(\ell)$	$n \geq 2, \ell \geq 0$
M	(00)(000)	$(n)(\ell)$	$n \geq 0, \ell \geq 0$
$K_{\mu\alpha}$	(10)(100)	$(n1)(\ell1), (n)(\ell), (n1)(\ell), (n)(\ell1)$	$n \geq 1, \ell \geq 1$
$L_{\alpha\beta}$	(00)(200)	$(n)(\ell2), (n)(\ell), (n)(\ell1)$	$n \geq 0, \ell \geq 2$
N	(00)(000)	$(n)(\ell)$	$n \geq 0, \ell \geq 0$
W_μ	(10)(000)	$(n)(\ell), (n1)(\ell)$	
$X_{\mu\nu\alpha}^\pm$	$(1, \pm 1)(100)$	$(n1)(\ell1), (n1)(\ell)$	$n \geq 1, \ell \geq 0$
$Y_{\mu\alpha\beta}$	(10)(110)	$(n1)(\ell11), (n1)(\ell1), (n)(\ell1), (n)(\ell11)$	$n \geq 1, \ell \geq 1$
$Z_{\alpha\beta\gamma}$	(00)(111)	$(n)(\ell11, \pm 1), (n)(\ell11)$	$n \geq 0, \ell \geq 1$
$\eta_{r\alpha}$	$(\frac{3}{2}, \frac{1}{2}) (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$(n, \frac{3}{2}) (\ell, \frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}), (n, \frac{1}{2}) (\ell, \frac{1}{2}, \frac{1}{2} \pm \frac{1}{2})$	$n \geq \frac{3}{2}, \ell \geq \frac{1}{2}$
$\eta_{r\dot{\alpha}}$	$(\frac{3}{2}, -\frac{1}{2}) (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$(n, \frac{3}{2}) (\ell, \frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}), (n, \frac{1}{2}) (\ell, \frac{1}{2}, \frac{1}{2} \pm \frac{1}{2})$	$n \geq \frac{3}{2}, \ell \geq \frac{1}{2}$
$\chi_{i\alpha}$	$(\frac{1}{2}, \frac{1}{2}) (\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$	$(n, \frac{1}{2}) (\ell, \frac{3}{2}, \frac{1}{2}, \pm \frac{1}{2}), (n, \frac{1}{2}) (\ell, \frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2})$	$n \geq \frac{1}{2}, \ell \geq \frac{3}{2}$
$\chi_{i\dot{\alpha}}$	$(\frac{1}{2}, -\frac{1}{2}) (\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$	$(n, \frac{1}{2}) (\ell, \frac{3}{2}, \frac{1}{2}, \pm \frac{1}{2}), (n, \frac{1}{2}) (\ell, \frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2})$	$n \geq \frac{1}{2}, \ell \geq \frac{3}{2}$
λ_α	$(\frac{1}{2}, \frac{1}{2}) (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$(n, \frac{1}{2}) (\ell, \frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2})$	$n \geq \frac{1}{2}, \ell \geq \frac{1}{2}$
$\lambda_{\dot{\alpha}}$	$(\frac{1}{2}, -\frac{1}{2}) (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$(n, \frac{1}{2}) (\ell, \frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2})$	$n \geq \frac{1}{2}, \ell \geq \frac{1}{2}$

which gives

$$ds^2 \mapsto -m^2(\cos^2 \rho dt^2 + d\rho^2 + \sin^2 \rho d\Omega_d) = -ds_{(E)}^2, \tag{11}$$

which is locally the metric of the $(d + 2)$ -sphere with negative-definite signature. Therefore, in evaluating the linearized field equations around the Euclideanized vacuum solution, we need to send $\bar{g}^{AdS} \rightarrow -\bar{g}^{\text{sphere}}$ and $\bar{\text{Riem}}^{AdS} \rightarrow -\bar{\text{Riem}}^{\text{sphere}}$. In this way, the analytical continuation that enables us to treat both cases simultaneously takes the form

$$\bar{g}_{\mu\nu} \rightarrow -\epsilon \bar{g}_{\mu\nu}^{S^4}, \quad \bar{g}_{\alpha\beta} \rightarrow \epsilon \bar{g}_{\alpha\beta}^{S^7}. \tag{12}$$

In the linearized field equations, there will be ϵ dependence coming from the Freund–Rubin solution. However, with the analytic continuation prescription described above, the ϵ factors work out in such a way that the linearized field equations take the same form as those which have already been analysed for the $AdS_4 \times S^7$ vacuum, with the harmonic expansions on $S^4 \times S^7$ fully performed. The treatment of the Levi-Civita symbols requires some care but the key point is that one can take over the results of [6, 7] and either continue them back to $AdS_4 \times S^7$ as was done in [7], or continue back to $AdS_7 \times S^4$ readily.

Next, we proceed with the harmonic expansions on $S^4 \times S^7 = G/H$ with $G = SO(5) \times SO(8)$ and $H = SO(4) \times SO(7)$. According to the framework described in [4], and applied to the case at hand in [7], one expands the fluctuations with a given H -content in terms of all G -representation functions that contain the H -representation. The highest weight labelling of the representations is more convenient than the Dynkin labels for this purpose. Using the notation of [7], let the highest weight of an H -representation be

$$H : (a_1 a_2)(b_1 b_2 b_3); \quad a_1 \geq a_2, \quad b_1 \geq b_2 \geq b_3, \tag{13}$$

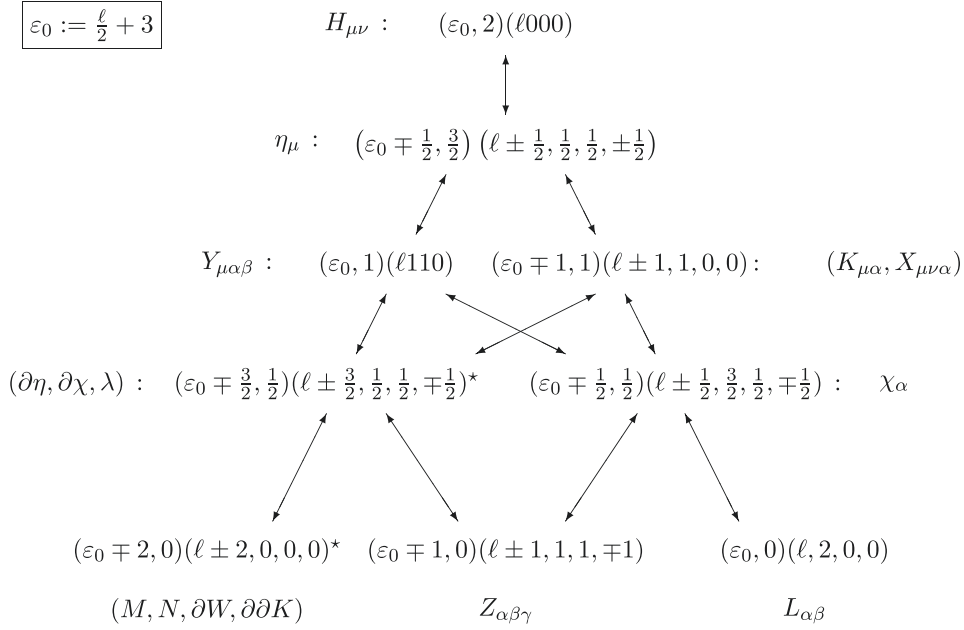


Figure 1. The spectrum of 11D supergravity on $AdS_4 \times S^7$. The 11D supergravity fluctuation fields from which the states come from are defined in (20), and shown in the figure. Here $\mu = 0, 1, \dots, 3$ labels the AdS_4 spacetime, and $\alpha = 1, \dots, 7$ labels the S^7 coordinates. The representations are labelled by the highest weights $(E_0, s)(\ell, \ell_1, \ell_2, \ell_3)$, where $E_0 \geq s$ and $\ell \geq \ell_1 \geq \ell_2 \geq |\ell_3|$. The corresponding Dynkin labels for $SO(8)$ are $(\ell - \ell_1, \ell_1 - \ell_2, \ell_2 - \ell_3, \ell_2 + \ell_3)$. Each value of ℓ gives a $OSp(8|4)$ multiplet. $\ell = 0$ gives the massless 4D maximal supergravity multiplets for which $E_0 = s + 1$, while $\ell = -1$ gives the singleton multiplet, contained in the towers marked by \star in the figure. States for $\ell \geq 1$ are massive multiplets with $d_\ell \times (128_B + 128_F)$ degrees of freedom, where d_ℓ is the dimension of the ℓ th rank totally symmetric and traceless $SO(8)$ tensor. One can define m_B^2 and m_F^2 for bosons and fermions, respectively, such that they actually vanish for the AdS_4 massless states with $s = 0, 1/2, 1$, as follows: $m_B^2 = 4E_0(E_0 - 3) + 8$ and $m_F^2 = (2E_0 - 3)^2$.

All G -representations that contain this H -representations have the with highest weight

$$G : (n n_1)(\ell \ell_1 \ell_2 \ell_3); \quad n \geq n_1, \quad \ell \geq \ell_1 \geq \ell_2 \geq |\ell_3|, \tag{14}$$

subject to the conditions

$$n \geq a_1 \geq n_1 \geq |a_2|, \quad \ell \geq b_1 \geq \ell_1 \geq b_2 \geq \ell_2 \geq b_3 \geq |\ell_3|. \tag{15}$$

Using these embedding conditions a field $\phi(x, y)$ with a fixed $SO(4) \times SO(7)$ representation $(a_1, a_2)(b_1, b_2, b_3)$ can be expanded in terms of the representation functions of $SO(5) \times SO(8)$ as follows:

$$\begin{aligned} \phi_{(a_1 a_2)(b_1 b_2 b_3)}(x, y) &= \text{vol.}(S^4 \times S^7) \sum \sqrt{\frac{d_{n n_1} d_{\ell \ell_1 \ell_2 \ell_3}}{d_{a_1 a_2} d_{b_1 b_2 b_3}}} D_{(a_1 a_2), p}^{(n n_1)}(L_x^{-1}) \\ &\times D_{(b_1 b_2 b_3), q}^{(\ell \ell_1 \ell_2 \ell_3)}(L_y^{-1}) \phi_{pq}^{(n n_1)(\ell \ell_1 \ell_2 \ell_3)}, \end{aligned} \tag{16}$$

where the summation ranges are as given in (13) and (14), the d 's are the dimensions of the relevant representations, L_x and L_y are the coset representative elements, $D_{(a_1 a_2), p}^{(n m_1)}(L_x^{-1})$ is the $(n m_1)$ representation of L_x^{-1} , with rows labelled by $(a_1 a_2)$ and columns by $p = 1, 2, \dots, d_{n m_1}$. The representation matrices (functions) of L_y^{-1} are to be interpreted similarly, and $\phi_{pq}^{(m_1)(\ell_1 \ell_2 \ell_3)}$ are x and y independent expansion coefficients. In the computation of saturated propagator, the orthogonality relations for the harmonics are needed. For example, on S^4 they take the form

$$\int_{S^4} d^4 x (\det \bar{g}_{\mu\nu})^{1/2} D_{(a), p}^{(n)}(L_x^{-1}) D_{(a), p'}^{(n')}(L_x^{-1}) = \text{Vol.}(S^4) \frac{d_{(a)}}{d_{(n)}} \delta_{pp'} \delta^{(n)(n')}, \quad (17)$$

where (a) denotes the row label for the H -representations, e.g. $\mu, (\mu\nu)$, etc, and (n) is shorthand for $(n m_1)$. Summation over (a) is understood. Similar formula holds on S^7 .

In these computations repeated use of the following relations are made

$$\bar{\square} L_x^{-1} = -4 m^2 (C_2[SO(5)] - C_2[SO(4)]) L_x^{-1}, \quad (18)$$

$$2\bar{\nabla}_\mu D_{(a_1 a_2), p}^{(n m_2)}(L_x^{-1}) = -2 m \langle p | Q_\mu | a_1 a_2 \rangle D_{(a_1 a_2), p}^{(n m_2)}(L_x^{-1}), \quad (19)$$

where $Q_\mu = M_{\mu 5}, \mu = 1, \dots, 4$ are the $SO(5)/SO(4)$ coset generators. Similar formula hold for the $SO(8)/SO(7)$ coset. The matrix elements $\langle p | Q_\mu | a_1 a_2 \rangle$, indeed all matrix elements of $SO(N)$ for any N , can be found in [12, 13], where they are given in Gelfand–Zeitlin (GZ) basis. Thus, one needs to find the relation between these basis elements and the tensorial one. Many of these relations can be found in [7].

If one is only interested in determining the KK spectrum of the theory, it is worth noting that there are shortcuts for doing so. In that context, the kind of data provided in table 1 is very powerful. Indeed, following the approach of [14], in the $AdS_4 \times S^7$ compactification here, one can start with $(n2)(\ell 000)$ state that describes the graviton tower, and simply compute the tensor product with the supercharge representation $(\frac{1}{2}, \frac{1}{2})(\frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2})$, repeatedly. Comparing with the available representations listed in table 1, one can deduce the content of figure 1. The representations that are left over from table 1 can then be interpreted as being non-propagating. This method works well especially if there is high degree of supersymmetry, and one ‘pyramid’ of states. For less amount of supersymmetry, one would have to determine the top member of more than one pyramid of states [14, 15], and repeat the procedure until all supermultiplets are accounted for. However, the details of exactly how the 11D fluctuations organize themselves to produce the physical states, and the analysis of possible boundary states may not be available in this approach.

4. The spectrum on $AdS_4 \times S^7$ and $AdS_7 \times S^4$

With full harmonic expansions on $S^4 \times S^7$, the problem of finding the saturated propagator reduces to an algebraic one. The following notation is introduced for the fluctuations [6, 7]

$$\begin{aligned} h_{\mu\nu} &= H_{\mu\nu} + \bar{g}_{\mu\nu} M, & \bar{g}^{\mu\nu} H_{\mu\nu} &= 0, & h_{\mu\alpha} &= K_{\mu\alpha}, \\ 2h_{\alpha\beta} &= L_{\alpha\beta} + \bar{g}_{\alpha\beta} N, & \bar{g}^{\alpha\beta} L_{\alpha\beta} &= 0, \\ 2a_{\mu\nu\rho} &= \varepsilon_{\mu\nu\rho\sigma} W^\sigma, & a_{\mu\nu\alpha}^\pm &= \frac{1}{2}(a_{\mu\nu\alpha} \pm \frac{i}{2} \varepsilon_{\mu\nu}{}^{\rho\sigma} a_{\rho\sigma\alpha}) \equiv X_{\mu\nu\alpha}^\pm, \\ 2a_{\mu\alpha\beta} &= Y_{\mu\alpha\beta}, & a_{\alpha\beta\gamma} &= Z_{\alpha\beta\gamma}. \end{aligned} \quad (20)$$

The full harmonic expansions on $S^4 \times S^7$ for the bosonic sector give the result [7]¹

$$\begin{aligned}
 I^{(2)} = \sum_{n,\ell} & \left\{ \frac{|T_H^{(n2)(\ell000)}|^2 + |T_L^{(n0)(\ell200)}|^2 + 4|J_Y^{(n1)(\ell110)}|^2}{8(2n + \ell + 6)(2n - \ell)} \right. \\
 & + \frac{2|J_Z^{(n0)(\ell111)}|^2}{(2n + \ell + 9)(2n - \ell - 3)} + \frac{2|J_Z^{(n0)(\ell11-1)}|^2}{(2n + \ell + 3)(2n - \ell + 3)} \\
 & + \frac{1}{2(2n + \ell + 3)(2n - \ell - 3)} \left| \frac{\sqrt{n+1} T_K^{(n1)(\ell100)} + 4\sqrt{n+2} J_X^{(n1)(\ell100)}}{\sqrt{2n+3}} \right|^2 \\
 & + \frac{8}{(2n + \ell + 9)(2n - \ell + 3)} \left| \frac{\sqrt{n+2} T_K^{(n1)(\ell100)} - 4\sqrt{n+1} J_X^{(n1)(\ell100)}}{\sqrt{2n+3}} \right|^2 \\
 & \left. + I_{\text{scalars}}^{(2)} + I_{\text{nonpropagating}}^{(2)} \right\}. \tag{21}
 \end{aligned}$$

The last two terms will be discussed further below. Using this formula, we can continue to either $AdS_4 \times S^7$ or $AdS_7 \times S^4$. In the first case, we set $n = -E_0$, and examine the pole in the E_0 -plane. In the second case, we set $\ell = -E_0$, and look for the poles in the E_0 -plane, with E_0 now denoting the lowest energy in AdS_7 .

In the first three terms above, one of the poles in the n -plane gives $E_0 = 3 + \frac{\ell}{2}$. They describe towers of physical states with the following $SO(3, 2) \times SO(8)$ representation content

$$AdS_4 \times S^7 : \begin{cases} H_{\mu\nu} : (E_0, 2)(\ell000), & E_0 = \frac{\ell}{2} + 3, & \ell \geq 0 \\ Y_{\mu\alpha\beta} : (E_0, 1)(\ell110), & E_0 = \frac{\ell}{2} + 3, & \ell \geq 1 \\ L_{\alpha\beta} : (E_0, 0)(\ell200), & E_0 = \frac{\ell}{2} + 3, & \ell \geq 2. \end{cases} \tag{22}$$

The second poles are related to the pole discussed above by the replacement $E_0 \rightarrow (3 - E_0)$, which describe the conjugate representations. We see that $H_{\mu\nu}$ contains the irrep $(3, 2)(0000)$ which is the massless graviton, as its energy $E_0 = 3$ saturates the unitarity bound $E_0 = s + 1$ for $s = 2$.

From the same saturated propagator (21), it also easy to read of the spectrum in $AdS_7 \times S^4$. To do so, we simply look for the poles in the ℓ -plane. Those are at $\ell = 2n$ and $\ell = -2n - 6$, again related to each other by the rule $\ell \rightarrow -6 - \ell$. Next, we identify $\ell = -E_0$, where E_0 now represents the lowest energy in AdS_7 . Thus, the tower of physical states in this sector are given by

$$AdS_7 \times S^4 : \begin{cases} L_{\alpha\beta} : (E_0, 2, 0, 0)(n0), & E_0 = 2n + 6, & n \geq 0 \\ Y_{\mu\alpha\beta} : (E_0, 1, 1, 0)(n1), & E_0 = 2n + 6, & n \geq 1 \\ H_{\mu\nu} : (E_0, 0, 0, 0)(n2), & E_0 = 2n + 6, & n \geq 2. \end{cases} \tag{23}$$

Now it is the field $L_{\alpha\beta}$ at the bottom floor of the tower with $n = 0$ that describes the massless graviton in AdS_7 as it has the lowest energy $E_0 = 6$ that saturates the unitarity bound for the

¹ We are being cavalier about the overall signs in the individual terms here, with the understand that the sign of the residues at the poles, whether in the n -plane, or the ℓ -plane are always positive, upon properly taking into accounts the rules of the analytic continuations involved.

unitary discrete representation of $SO(6, 2)$, as $E_0 = \ell_1 + 2$ with $\ell_1 = 2$, while $H_{\mu\nu}$ describes a tower of massive scalars.

Turning to the $(n)(\ell, 1, 1, \pm 1)$ and $(n0)(\ell 100)$ representations, in order to clarify how the poles in the n -plane fit into $OSp(8|4)$ multiplets, we relabel $\ell \rightarrow \ell - 1$ for the first terms, and $\ell \rightarrow \ell + 1$ in the second terms in the saturated propagator in this sector. Thus one finds the following towers of physical states

$$AdS_4 \times S^7 : \begin{cases} Z_{\alpha\beta\gamma} : & (E_0^\pm, 0)(\ell \pm 1, 0, 0, 0), & E_0^\pm = \frac{\ell}{2} + 3 \mp 1, \\ (K_{\mu\alpha}, X_{\mu\nu\alpha}^\pm) : & (E_0^\pm, 1)(\ell \pm 1, 1, 0, 0), & E_0^\pm = \frac{\ell}{2} + 3 \mp 1, \end{cases} \quad (24)$$

where $\ell \geq 0$ for the upper sign tower, with $\ell = 0$ states being the massless scalars in the 35_v -plet and massless vectors in the 28-plet of $SO(8)$.

In a similar fashion, analytically continuing to AdS_7 instead, this time letting $n \rightarrow n + 1$ for the first terms, and $n \rightarrow n - 1$ in the second terms discussed above, we easily obtain the following spectrum of states

$$AdS_7 \times S^4 : \begin{cases} Z_{\alpha\beta\gamma} : & (E_0^\pm, 1, 1, \pm 1)(n \pm 1, 0), & E_0^\pm = 2n + 6 \mp 1, \\ (K_{\mu\alpha}, X_{\mu\nu\alpha}^\pm) : & (E_0^\pm, 1, 0, 0)(n \pm 1, 1), & E_0^\pm = 2n + 6 \mp 1, \end{cases} \quad (25)$$

where $n \geq 0$ for the upper sign tower, with $n = 0$ states being the massless 3-form fields in the 5-plet, and massless vectors in the 10-plet of $SO(5)$, while $n \geq 2$ for the lower sign towers consisting of massive 3-form fields and vectors only. In the first tower, the case of $n = -1$ is special. It will be analysed in more detail in the next section, where we will see that it describes a doubleton.

The case of massless 3-form fields also deserves a further comment. In this case, the harmonically expanded field equation becomes $(\ell + 5)(\ell - 5)Z^{\ell, 1, 1, -1(00)} = 0$. As shown in [16, 17], this means that the field equation for this mode factorizes as

$$\left(\delta_{\alpha\beta\gamma}^{\alpha'\beta'\gamma'} + \frac{1}{12} \varepsilon_{\alpha\beta\gamma} \alpha'\beta'\gamma'\delta' \nabla_{\delta'} \right) \left(\delta_{\alpha'\beta'\gamma'}^{\alpha''\beta''\gamma''} - \frac{1}{48} \varepsilon_{\alpha'\beta'\gamma'} \alpha''\beta''\gamma''\delta'' \nabla_{\delta''} \right) Z_{\alpha''\beta''\gamma''}^I(y) = 0, \quad (26)$$

where $I = 1, \dots, 5$ is the $SO(5)$ vector index. This can be checked by expanding $Z_{\alpha\beta\gamma}^I(y) = \sum_p Z_p^{I, (\ell 11-1)} D_{\alpha\beta\gamma, p} (L_y^{-1})$ (we ignore the normalization factors here), and using the relation

$$\nabla_{[\delta} D_{\alpha\beta\gamma], p}^{(\ell 11-1)} = -\frac{1}{24} (\ell + 3) \varepsilon_{\alpha\beta\gamma\delta} \alpha'\beta'\gamma' D_{\alpha'\beta'\gamma', p}^{(\ell 11-1)}. \quad (27)$$

Recalling the analytical continuation by which $E_0 = -\ell$, we see that the first factor in (26) gives the lowest energy $E_0 = 5$ appropriate for the massless 3-form field, as $E_0 = \ell_1 + 4$ with $\ell_1 = 1$.

Next we turn to the scalar fields arising in the sector where the fields carry the $(n0)(\ell 000)$ representation. This is the most complicated sector as the linearized equation mix the fields $(M, N, \partial W, \partial \partial K, \partial \partial H, \partial \partial L)$. The last two can be eliminated in terms of the remaining ones by

means of the gauge conditions. The resulting four coupled linearized equations were analysed in [7] where it is found that the residues at two of the resulting poles in the n -plane in the saturated propagator vanish, while the other two poles give physical states. To see how these states fit into the supermultiplets, in this case the shifts $\ell \rightarrow \ell \pm 2$ are appropriate, and we find the towers

$$AdS_4 \times S^7 : (M, N, \partial W, \partial \partial K) : (E_0^\pm, 0)(\ell \pm 2, 0, 0, 0), \quad E_0^\pm = \frac{\ell}{2} + 3 \mp 2. \quad (28)$$

For $\ell = 0$ the first tower gives the massless scalars in the 35-plet of $SO(8)$. At $\ell = -1$, scalars in 8_v of $SO(8)$ reside and have been shown to be gauge modes in [7]. The question of whether they can be part of a supersingleton boundary supermultiplet will be discussed in the next section.

Analytical continuation from S^7 to AdS_7 instead, we find that two of the poles give vanishing residue, and the remaining two, upon letting $n \rightarrow n + 2$ for one of the poles, and $n \rightarrow n - 2$ for the other, again for the supermultiplet interpretation, give the towers

$$AdS_7 \times S^4 : (M, N, \partial W, \partial \partial K) : (E_0^\pm, 0, 0, 0)(n \pm 2, 0), \quad E_0^\pm = 2n + 6 \mp 2, \quad (29)$$

with the upper sign tower starting at $n = 0$, and the lower one at $n = 2$. The first tower at $n = 0$ contains massless scalars 14-plet of $SO(5)$. At $n = -1$, there are scalars in 5-plet of $SO(5)$ which turn out to describe part of the superdoubleton, as we shall see in the next section.

Finally, we turn to the term $I_{\text{unphysical}}^{(2)}$ in (21). This term refers to the remaining sectors:

$$\text{Nonpropagating} : (n0)(\ell 100), \quad (n0)(\ell 110), \quad (n1)(\ell 000). \quad (30)$$

In the case of analytic continuation to $AdS_4 \times AS^7$, it was shown in [7] that their contribution to the saturated propagator in which the source squared term is divided by a quadratic expression in ℓ , without any n dependence. Thus, not having a pole in the n -plane, these are interpreted as being nonpropagating. In the case of analytic continuation to $AdS_4 \times AS^7$, we can easily show that their contribution to the saturated propagator this time has the form of source squared term divided by a quadratic expression in n , without any ℓ dependence. Thus, not having a pole in the ℓ -plane, we again see that these states are nonpropagating.

So far we have discussed the bosonic sector of the 11D supergravity. In the fermionic sector, the analytic continuation from $AdS_4 \times S^7$ to $S^4 \times S^7$ was presented in [7] where the complete spectrum, bosonic and fermionic, was worked out. The analytic continuation from $AdS_7 \times S^4$ along the lines described above can be extended to fermionic sector as well, just as in the case of $AdS_4 \times S^7$ described in detail in [7].

In the fermionic sector the local supersymmetry transformations of the fluctuations is given by

$$\delta\psi^A = \bar{\nabla}_A \epsilon - \frac{1}{144} \left(\Gamma_A^{B_1 \dots B_4} - 8\Gamma^{B_1 \dots B_3} \delta_A^{B_4} \right) \bar{F}_{B_1 \dots B_4} \epsilon. \quad (31)$$

Introducing the source term with a suitable normalization, one finds that local supersymmetry imposes the constraint [7]

$$\bar{\nabla}_A J^A - \frac{1}{144} \left(\Gamma_A^{B_1 \dots B_4} + 8\Gamma^{B_1 \dots B_3} \delta_A^{B_4} \right) \bar{F}_{B_1 \dots B_4} \epsilon = 0. \quad (32)$$

In [7] the following gauge is chosen

$$\Gamma^A \psi_A = 0. \tag{33}$$

Writing $\Gamma^r = \gamma^r \times 1$, and $\Gamma^i = \gamma_5 \times \gamma^i$, where the 11D tangent space index is split as $A = (r, i)$, with $r = 0, 1, 2, 3$ and $i = 4, \dots, 10$, and defining the fluctuations fields

$$\psi_r = \left(\frac{1 + i\gamma_5}{\sqrt{2}} \right) (\eta_r + \gamma_r \lambda), \quad \psi_i = \left(\frac{1 + i\gamma_5}{\sqrt{2}} \right) (\chi_i + \gamma_i \theta), \tag{34}$$

where η_r and χ_i are γ -traceless, and the γ_5 dependent prefactors are introduced for convenience. We shall skip the details of the analytic continuation in this sector, as it is similar to the one described in [7]. The procedure is exactly as the one explained for the bosonic sector, and the saturated propagator for this sector provided in [7] yields the result

$$AdS_4 \times S^7 : \begin{cases} \eta_r : & \left(E_0^\pm, \frac{3}{2} \right) \left(\ell \pm \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2} \right), & E_0^\pm = \frac{\ell}{2} + 3 \mp \frac{1}{2}, \\ \chi_i : & \left(E_0^\pm, \frac{1}{2} \right) \left(\ell \pm \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \mp \frac{1}{2} \right), & E_0^\pm = \frac{\ell}{2} + 3 \mp \frac{1}{2}, \\ (\partial\eta, \partial\chi, \lambda) : & \left(E_0^\pm, \frac{1}{2} \right) \left(\ell \pm \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \mp \frac{1}{2} \right), & E_0^\pm = \frac{\ell}{2} + 3 \mp \frac{3}{2}. \end{cases} \tag{35}$$

In the first equation, the $\ell = 0$ states are the massless gravitini in the 8_s of $SO(8)$. In the last result above, three coupled linearize equations were analysed in [7], and additional poles in the saturated propagator were shown to give vanishing residue; hence only the towers displayed above arise as physical. Note also the shifts of ℓ by $\pm \frac{1}{2}$ or $\pm \frac{3}{2}$, again for the purposes of supermultiplet interpretation; see figure 1. Furthermore, $\ell = -1$ in the last tower gives fermions which were shown to be gauge modes in [7]. Whether they can survive as the fermionic partner of a supersingleton will be examined in the next section.

Using the saturated propagator given in [7] this time to continue analytically to $AdS_7 \times S^4$ instead, we easily obtain the result

$$AdS_7 \times S^4 : \begin{cases} \chi_i : & \left(E_0 \pm \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \mp \frac{1}{2} \right) \left(n \pm \frac{1}{2}, \frac{1}{2} \right), & E_0 = 2n + 6 \mp \frac{1}{2}, \\ \eta_r : & \left(E_0^\pm, \frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2} \right) \left(n \pm \frac{1}{2}, \frac{3}{2} \right), & E_0^\pm = 2n + 6 \mp \frac{1}{2}, \\ (\partial\eta, \partial\chi, \lambda) : & \left(E_0^\pm, \frac{1}{2}, \frac{1}{2}, \mp \frac{1}{2} \right) \left(n \pm \frac{3}{2}, \frac{1}{2} \right), & E_0^\pm = 2n + 6 \mp \frac{3}{2}. \end{cases} \tag{36}$$

In the first equation, the upper sign tower starts at $n = 0$, which is the *massless* gravitino. In the last equation $n = -1$ is an acceptable representation, and it will be shown in the next section to have the appropriate field equation for a fermionic partner of a superdoubleton.

In summary, the results for the full spectrum in the $AdS_4 \times S^7$ and $AdS_7 \times S^4$ are given in figures 1 and 2. In [7], it was observed that the following relation holds for a particular combination of the second order Casimir operator eigenvalues at each level:

$$2C_2[SO(3, 2)] + C_2[SO(8)] = \frac{3}{2} (\ell + 2)(\ell + 4). \tag{37}$$

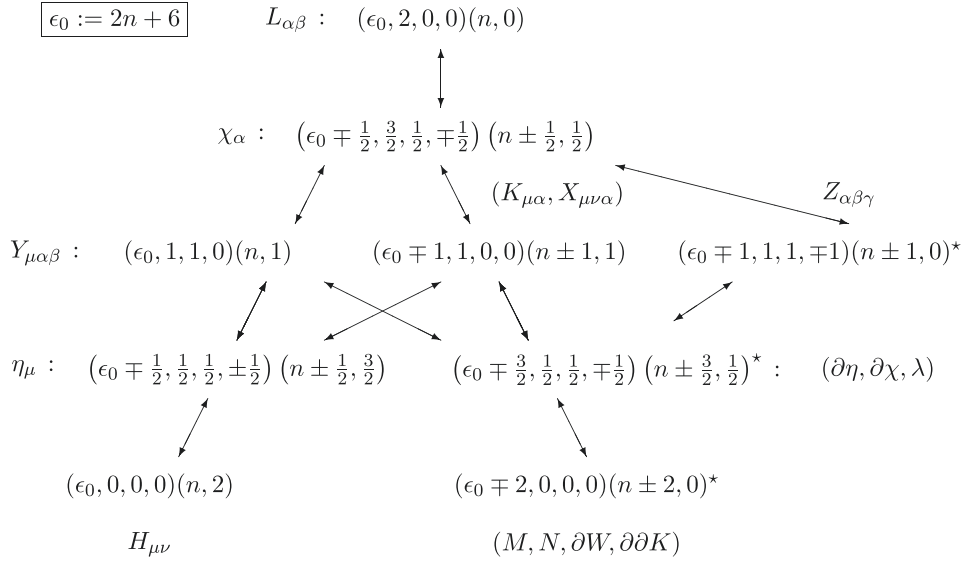


Figure 2. The spectrum of 11D supergravity on $AdS_7 \times S^4$. The 11D supergravity fluctuation fields from which the states come from are defined in (20), and shown in the figure. Here $\alpha = 0, 1, \dots, 6$ labels the AdS_7 spacetime, and $\mu = 1, \dots, 4$ labels the S^4 coordinates. The representations are labelled by the highest weights of $SO(6, 2) \times SO(5)$ as $(E_0, \ell_1, \ell_2, \ell_3)(n, n_1)$, where $E_0 \geq \ell_1 \geq \ell_2 \geq |\ell_3|$ and $n \geq n_1$. The Dynkin labels for $SO(5)$ are $(n - n_1, 2n_1)$ and the corresponding Dynkin labels for $USp(4)$ are $(2n_1, n - n_1)$. Each value of ℓ gives an $OSp(6, 2|4)$ multiplet. $n = 0$ gives the 7D maximal supergravity multiplet, for which $E_0 = \ell_1 + 4$, saturating the unitarity bound. $n = -1$ gives the doubleton multiplet. These sit in the towers marked by $*$ in the Figure. States for $n \geq 1$ are massive multiplets with $d_n \times (128_B + 128_F)$ degrees of freedom, where d_n is the dimension of n th rank totally symmetric and traceless $SO(5)$ tensor.

In a similar fashion, here we find that the following relation holds in the case of $AdS_7 \times S^4$ Kaluza–Klein spectrum organized into levels labelled by n :

$$2C_2[SO(5)] + C_2[SO(6, 2)] = 6(n + 1)(n + 2). \tag{38}$$

Interestingly, this vanishes for the doubleton multiplet for which $n = -1$.

5. Search for singletons and doubletons

Figure 1 shows the full spectrum of 11D sugra compactified on $AdS_4 \times S^7$ compactification, in such a way that each value of $\ell = 0, 1, 2, \dots$, represents an $OSp(8|4)$ multiplet. For $\ell = 0$ one has the massless maximal 4D supergravity multiplet, and the rest are massive supermultiplets. In [6, 7], while it was shown that the representations for $\ell = -1$ are gauge modes, it was observed that these form the singleton supermultiplet of $OSp(8|4)^2$. The question arises as to whether the gauge fixing procedure allows their existence as boundary states. We begin by noting that for $\ell = -1$ (corresponding to $\ell = 1$ before the shift $\ell \rightarrow \ell + 2$ that defines the universal KK level ℓ), the scalar $\partial\partial K$ can be eliminated by using the gauge condition, and this

²The conjecture for its existence in the spectrum, appeared in the first reference in [6].

leads to field equations for the scalars $(M, N, \partial W)$, which we recall are the fields $h_\mu^\mu, h_\alpha^\alpha$ and $\varepsilon^{\mu\nu\rho\sigma}(F_{\mu\nu\rho\sigma})^{\text{lin}}$. Using the results given in [7], one then finds a linear combination of their equations of motion that takes the form³ (henceforth all covariant derivatives are understood to be evaluated in the background):

$$\ell = -1 : (\square_{AdS_4} + 5m^2) (6M - 12N - \nabla_\mu W^\mu)^I = 0, \quad (39)$$

where we have re-introduced the parameter $m = 1/(2L_{AdS_4})$, and $I = 1, \dots, 8$ labels the $(1, 0, 0, 0)$ representation of $SO(8)$. This is the appropriate field equation for a singleton with lowest energy $E_0 = \frac{1}{2}$. After examining the fermionic sector, we shall come back to the question of whether the boundary states described by this equations survive the fixing of local symmetries.

In the fermionic sector the candidate singleton carries the representation $(1, \frac{1}{2})(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$. To better understand the role of supersymmetry gauge fixing, let us not impose the gauge condition (33) to begin with. Thus, we examine the γ -trace of the $4 + 7$ split of the linearized gravitino equation prior to any gauge fixing. Making use of the fact that for $\ell = \frac{1}{2}$ we have $\nabla_i \chi^i = 0$ [7], these equations determine $\nabla_r \eta^r$ in terms of $(\lambda, i\theta)$, and furthermore give

$$\left(\nabla_4 - \frac{12}{7}i\nabla_7 + 2\right) i\theta(x, y) - \frac{4}{7} \left(i\nabla_7 + \frac{7}{2}\right) \lambda(x, y) = 0, \quad (40)$$

and the linearized supersymmetry transformations take the form

$$\delta\lambda(x, y) = \frac{1}{4}(\nabla_4 - 4)\epsilon(x, y), \quad \delta i\theta(x, y) = \frac{1}{7} \left(i\nabla_7 + \frac{7}{2}\right) \epsilon(x, y). \quad (41)$$

Denoting any of the spinors occurring above generically by $\psi(x, y)$, it is understood that its harmonic expansion on S^7 is of the form $\psi(x, y) = \psi_+(x)D_+(L_y^{-1}) + \psi_-(x)D_-(L_y^{-1})$, where $D_\pm(L_y^{-1})$ denote the $SO(8)$ representation functions in $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2})$ obeying $i\nabla_7 D_\pm(L_y^{-1}) = \mp\frac{7}{2}D_\pm(L_y^{-1})$. Thus equations (40) and (41) give

$$(\nabla_4 - 4)i\theta_-(x) - 4\lambda_-(x) = 0, \quad (42)$$

invariant under⁴

$$\delta i\theta_-(x) = \epsilon_-(x), \quad \delta\lambda_-(x) = \frac{1}{4}(\nabla_4 - 4)\epsilon_-(x), \quad (43)$$

and $(\nabla_4 + 8)\theta_+(x) = 0$ with $\delta\theta_+(x) = 0$. Harmonic expansion of $\theta_+(x)$ on AdS_4 gives the lowest energy $E_0 = \frac{11}{2}$, thus describing the physical state at level $\ell = 2$ shown in figure 1. As for $\theta_-(x)$, it can be gauged away by using the parameter $\epsilon_-(x)$, in which case $\lambda_-(x)$ vanishes by its field equation. However, suppose we fix the following gauge instead

$$4\lambda + i\beta\theta_-(x) = 0, \quad (44)$$

where β is a constant parameter. Then, the field equation becomes

$$(\nabla_4 - 4 + \beta)\theta_-(x) = 0. \quad (45)$$

³ For a detailed analysis of the singleton field equations, see [18].

⁴ Correcting the sign of the last term in the variation of the gravitino given in equation (3) of [7].

This has a solution with AdS lowest energy $E_0 = \frac{1}{2}(\beta - 1)$, with the unitarity bound imposing the condition $\beta \geq 3$. Interestingly, the choice $\beta = 3$ which saturates the unitarity bound gives the singleton field equation⁵. On the other hand, maintaining the gauge condition imposes the condition involving the same wave operator, namely, $(\nabla_4 - 4 + \beta)\epsilon_-(x) = 0$. Since the field equation satisfied by the residual symmetry parameter $\epsilon_-(x)$ coincides with that of the fermionic field $\theta_-(x)$ for any value of β , it follows that the latter can be removed entirely by using this residual symmetry, again for any value of β . Therefore, even though $\beta = 3$ gives the singleton field equation for $\theta_-(x)$ this field can nonetheless be removed entirely by fixing the Stuckelberg symmetry. By supersymmetry, we expect that similar phenomenon must be present for the bosonic singleton equation (39) as well, namely the KK reduction of the 11D general coordinate and tensor gauge transformations must provide the required residual Stuckelberg shift symmetries to remove them. The nature of these symmetries is similar to those described in detail in [15] for 6D supergravity on $AdS_3 \times S^3$.

Let us now examine the linearized field equations for $n = -1$ (corresponding to $n = \frac{1}{2}$ after the relabelling $n \rightarrow n + \frac{3}{2}$ to define a universal KK level number n) in the $AdS_7 \times S^4$ compactification. Again one can eliminate $\partial\partial K$, and using the results of [7] one finds that a particular linear combination of the field equations for $(M, N, \partial W)$ in AdS_7 takes the form

$$n = -1 : (\bar{\square}_{AdS_7} + 8m^2) (5M + 28N - 6\nabla_\mu W^\mu)^I = 0, \tag{46}$$

where $m = 1/L_{AdS_7}$ and $I = 1, \dots, 5$ labels the vector representation of $SO(5)$. This equation admits a solution with lowest energy $E_0 = 2$ which is appropriate for a doubleton scalar. The only other bosonic state at $n = -1$ in figure 2, carries the $SO(6, 2) \times SO(5)$ representation $(3, 1, 1, -1)(0, 0)$. Its field equation is that of $Z_{\alpha\beta\gamma}$ expanded on S^4 with the $n = -1$ mode kept. The result is

$$n = -1 : (\bar{\square}_{AdS_7} + 12m^2) Z_{\alpha\beta\gamma} + \varepsilon_{\alpha\beta\gamma}{}^{\alpha'\beta'\gamma'\delta'} \bar{\nabla}_{\alpha'} Z_{\beta'\gamma'\delta'} = 0, \tag{47}$$

where $Z_{\alpha\beta\gamma}$ depends only on the $7D$ coordinates, and it is a singlet of $SO(5)$. This equation factorizes as

$$n = -1 : \left(\delta_{\alpha\beta\gamma}{}^{\alpha'\beta'\gamma'} + \frac{1}{36} \varepsilon_{\alpha\beta\gamma}{}^{\alpha'\beta'\gamma'\delta'} \bar{\nabla}_{\delta'} \right) \left(\varepsilon_{\alpha'\beta'\gamma'}{}^{\alpha''\beta''\gamma''\delta''} \bar{\nabla}_{\delta''} \right) Z_{\alpha''\beta''\gamma''} = 0, \tag{48}$$

The general solution is a linear combination of those annihilated by the first or second first order wave operator, the second one giving the lowest energy $E_0 = 3$ state which is appropriate for the doubleton representation.

Turning to fermions, at $n = -1$, making use of the fact that $\nabla_r \eta^r = 0$ in this sector, the γ -trace of the linearized gravitino equation of motion on $AdS_7 \times S^4$, prior to any gauge fixing, now determines $\nabla_i \chi^i$ in terms of $(\theta, i\lambda)$, and furthermore gives

$$\left(\nabla_7 + \frac{3}{2} i \nabla_4 + \frac{5}{2} \right) i\lambda(y, x) - \frac{7}{4} (i \nabla_4 - 4) \theta(y, x) = 0, \tag{49}$$

⁵The gauge condition (33) instead gives $(\nabla_4 + 3)\theta = 0$, yielding the lowest energy $E_0 = 3$ [7]. This field equation arises in the $N = 1$ supersymmetric Wess–Zumino model in AdS_4 [19], where the states $(\frac{3}{2}, 0) + (3, \frac{1}{2}) + (\frac{7}{2}, 0)$ form a massive scalar multiplet. The free action for this case is given in equation (B1) of [19] with $\mu = \frac{3}{2}$. In [20], the value $\mu = \frac{3}{2}$ was mentioned as accommodating a boundary $N = 1$ supersingleton. We correct that statement here by noting that it should have read $\mu = \frac{1}{2}$. I thank Tanii for pointing this out, and also for noting that the supersingletons for $\mu = \frac{1}{2}$ is related to the one for $\mu = -\frac{1}{2}$ by a field redefinition.

invariant under

$$\delta\theta(y, x) = \frac{1}{7} \left(\nabla_7 - \frac{7}{2} \right) \epsilon(y, x), \quad \delta i\lambda(y, x) = \frac{1}{4} (i\nabla_4 - 4)\epsilon(y, x). \quad (50)$$

Denoting any of the spinors occurring above generically by $\psi(y, x)$, it is understood that its harmonic expansion on S^4 is of the form $\psi(y, x) = \psi_+(y)D_+(L_x^{-1}) + \psi_-(y)D_-(L_x^{-1})$, where $D_\pm(L_x^{-1})$ are the $SO(5)$ representation of the $SO(5)/SO(4)$ coset representative elements L_x^{-1} with the row labelled by $(\frac{1}{2}, \pm\frac{1}{2})$ representation of $SO(4) \subset SO(5)$, and the column by the $(\frac{1}{2}, \frac{1}{2})$ representation of $SO(5)$. They obey $i\nabla_4 D_\pm = \mp 4D_\mp$. Thus equations (49) and (50) give

$$\left(\nabla_7 - \frac{7}{2} \right) i\lambda_-(y) + 14\theta_-(y) = 0, \quad (51)$$

invariant under

$$\delta i\lambda_-(y) = -2\epsilon_-(y), \quad \delta\theta_-(y) = \frac{1}{7} \left(\nabla_7 - \frac{7}{2} \right) \epsilon_-(y), \quad (52)$$

and $(\nabla_7 + \frac{7}{2})\lambda_+(y) = 0$ with $\delta\lambda_+(y) = 0$. In the latter equation, harmonic expansion of $\lambda_+(y)$ on AdS_7 gives the lowest energy $E_0 = \frac{23}{2}$ which is the physical state at level $n = 2$ shown in figure 2. As for $\lambda_-(y)$, it can be gauged away by using the parameter $\epsilon_-(x)$, in which case $\theta_-(y)$ vanishes by its field equation. However, if we choose the following gauge condition

$$7i\theta_-(y) + \tilde{\beta}\lambda_-(y) = 0, \quad (53)$$

the fermionic field equation becomes

$$\left(\nabla_7 - \frac{7}{2} + 2\tilde{\beta} \right) \lambda_-(y) = 0, \quad (54)$$

which gives the lowest energy $E_0 = \frac{1}{2}(13 - 4\tilde{\beta})$ with the unitarity bound requiring $\tilde{\beta} \geq 2$. Saturating this bound by taking $\tilde{\beta} = 2$ gives the fermionic doubleton field equation yielding the AdS_7 lowest energy $E_0 = \frac{5}{2}$ solution. Maintaining the gauge condition imposes the constraint $\left(\nabla_7 - \frac{7}{2} + 2\tilde{\beta} \right) \epsilon_-(y) = 0$. Thus, the picture which emerges here is similar to the one we encountered for the singletons in AdS_4 , and we see that the residual symmetries can be used to remove entirely the field $\lambda_-(y)$ for any value of $\tilde{\beta}$, even though it satisfies the singleton field equation for $\tilde{\beta} = 2$. By supersymmetry, we deduce that the fields which we found to obey the doubleton field equations above can also be removed by residual Stueckelberg shift symmetries coming from the KK reduction of the 11D general coordinate and tensor gauge transformations.

6. Conclusions

We have found a simple rule that relates the spectrum of physicals states in the Freund–Rubin compactifications of 11D supergravity on $AdS_4 \times S^7$ and $AdS_7 \times S^4$. Thus, from the $SO(3, 2) \times SO(8)$ lowest weights of the spectrum in the $AdS_4 \times S^7$ compactification given by

$$AdS_4 \times S^7 : (\varepsilon_0 \mp a, n_1)(\ell \pm a, \ell_1, \ell_2, \ell_3), \quad \varepsilon_0 := \frac{\ell}{2} + 3, \quad a = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \tag{55}$$

we deduce the $SO(6, 2) \times SO(5)$ lowest weights in the $AdS_7 \times S^4$ compactification by a remarkably simple rule that gives

$$AdS_7 \times S^4 : (\varepsilon_0 \mp a, \ell_1, \ell_2, \ell_3)(n \pm a, n_1), \quad \varepsilon_0 := 2n + 6, \quad a = 0, \frac{1}{2}, 1, \frac{3}{2}, 2. \tag{56}$$

The rule is to interchange the spacetime and internal symmetry labels such that (ε_0, ℓ, a) goes to $(n, \varepsilon_0, -a)$. In figures 1 and 2 each value of the integer $\ell \geq 0$ in figure 1, and $n \geq 0$ in figure 2, describe the states which form supermultiplets of $OSp(8|4)$ and $OSp(6, 2|4)$, respectively (in the latter case see [23, 25] for the AdS lowest energies). The 11D origin of the states, and how they transform to each other is also displayed in these figures.

In obtaining the results summarized above, we have used the fact that $M_1 = AdS_4 \times S^7$ and $M_2 = AdS_7 \times S^4$ admit analytic continuation to $M = S^4 \times S^7$. We have performed harmonic expansions on M , and shown that analytic continuations of either S^4 to AdS_4 or S^7 to AdS_7 give the spectrum of physical states on M_1 and M_2 , respectively. The discrete unitary representations of the relevant AdS groups with well known boundary conditions arise in these computations. The spectra obtained in this approach are in perfect agreement with those obtained by other methods. However, it should be noted that the analytic continuation of AdS_n to S^n may have limitations in the computation of the effective action [26–30]. In particular, it has been shown that in computing the effective potential, while the direct AdS_n mode sum gives physically reasonable result and properly incorporates the supersymmetric boundary conditions, the analytic continuation from S^n back to AdS_n fails to produce the correct supersymmetric effective action [29]. Similar results were also obtained in [30]. It is interesting to note, however, that in computing the beta functions in a framework where AdS_n is analytically continued to S^n , the results agree with those obtained from continuation to Euclidean AdS_n [31].

Turning to the spectral relation we have found between those of $AdS_4 \times S^7$ and $AdS_7 \times S^4$ compactifications, its further uses remain to be seen. For example, the vanishing of Casimir energies that has been shown for $AdS_4 \times S^7$ in [21, 22] and for $AdS_7 \times S^4$ in [23], may possibly be understood from a different angle afforded by these relations. One may also investigate whether the spectral relationship of the kind presented here exists for other compactifications as well [24]. It would also be interesting to explore possible uses of our approach in the computation of interactions involving the KK states, which are of considerable interest in the context of consistent KK truncation schemes and holography [25]. It would also be interesting to apply our analytic continuation approach to the Freund–Rubin compactifications of 11D supergravities with signatures (9, 2) and (6, 5) signatures [32], and study of branes propagating in these backgrounds [33, 34].

In this paper, we have also found the linearized field equations for the singletons and doubletons in the bulk. However, these turns out to be gauge dependent results, and we have shown that residual Stueckelberg shift symmetries inherited from the Kaluza–Klein reduction can be used to remove them. We have displayed these symmetries explicitly for the fermions but they are expected to arise in a similar fashion in the bosonic sector as well, as described in detail in [15] in the context of 6D supergravity on $AdS_3 \times S^3$. These gauge symmetries are not to be confused with the AdS symmetry that operates on the solution space. In the latter case, as

explained in detail in [18, 35], there is a sense in which the singletons can be treated in the framework of a gauge theory in which the solution space, after modding out by gauge transformations that fall off rapidly in the direction of spatial infinity, does support the singletons as boundary states. What we have seen in the Freund–Rubin compactification of 11D supergravity is that there is an additional local Stueckelberg symmetry coming from 11D, other than the AdS symmetry of the background, which removes these states.

It would be useful to study the fate of the singletons and the attendant Stueckelberg symmetries in the $N = 1$ supersymmetric compactification of 11D supergravity on $AdS_4 \times LS^7$ [36], where LS^7 is the left-squashed 7-sphere. It has been noted that the spectrum around this vacuum can be related to that of round S^7 by a Higgs mechanism [37] if the singletons are included in the spectrum, and it has been suggested that a field theoretic mechanism may exist by which the singleton eats an ordinary field of the same spin to become an ordinary bulk field [38]. Shedding light on this puzzle would require an analysis of how turning on the squashing parameter, with the attendant phenomenon of switching vacuum expectation value to the 35 scalar fields of $N = 8$ supergravity [37], effects the singleton field equations and importantly the Stueckelberg symmetries.

The general expectation that singletons have a role to play in AdS/CFT holography [39, 40], motivates a further study of singletons in the context of KK supergravity. The arguments that have been given in support of their presence tend to involve BF type bosonic topological field theories in the bulk. In particular, a detailed study of the AdS_5 , and a general discussion of the BF type theories in this context exists (see [40], and references therein). However, the way in which suitable topological field theories may arise from the flux compactifications and how coupling to supergravity may occur apparently has not been investigated so far. The fact that the BF type theories considered involve p-forms of supergravities, and their behaviour on the boundaries plays an essential role, suggests that the 3-form potential arising in the singleton and doubleton field equations we have found may involve some global considerations that make them survive on the boundary, despite the presence of the Stueckelberg shift symmetries. Whether this is the case remains to be investigated.

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ORCID iDs

Ergin Sezgin  <https://orcid.org/0000-0002-1415-8023>

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