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A CANONICAL PARAMETERIZATION OF PATHS IN $\mathbb{R}^n$

L. C. HOEHN, L. G. OVERSTEEGEN, AND E. D. TYMCHATYN

Abstract. For sufficiently tame paths in $\mathbb{R}^n$, Euclidean length provides a canonical parametrization of a path by length. In this paper we provide such a parametrization for all continuous paths. This parametrization is based on an alternative notion of path length, which we call len. Like Euclidean path length, len is invariant under isometries of $\mathbb{R}^n$, is monotone with respect to sub-paths, and for any two points in $\mathbb{R}^n$ the straight line segment between them has minimal len length.

Unlike Euclidean path length, the len length of *any* path is defined (i.e., finite) and len is continuous relative to the uniform distance between paths. We use this notion to obtain characterizations of those families of paths which can be reparameterized to be equicontinuous or compact. Finally, we use this parametrization to obtain a canonical homeomorphism between certain families of arcs.

1. Introduction

A *path* in $\mathbb{R}^n$ is a continuous function γ from a closed interval $[a,b] \subset \mathbb{R}$ to $\mathbb{R}^n$. Given $z_1, z_2 \in \mathbb{R}^n$, denote by $\overline{z_1 z_2}$ the straight line segment path $t \mapsto (1-t)z_1 + t z_2$, $t \in [0,1]$.

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Given a path $\gamma : [a,b] \to \mathbb{R}^n$, the *Euclidean path length* of γ, denoted $L_E(\gamma)$, is defined by the formula

$$L_E(\gamma) = \sup \left\{ \sum_{i=1}^{n} |\gamma(x_{i-1}) - \gamma(x_i))| : a = x_0 < x_1 < \cdots < x_n = b,\ n \in \mathbb{Z}^+ \right\} \in [0,\infty],$$

where $|z_1 - z_2|$ denotes the Euclidean distance between points $z_1, z_2 \in \mathbb{R}^n$.

If a sequence of smooth paths γ_i converges to γ_∞ in C^1 (in the sense that the paths γ_i and their derivatives γ_i' converge uniformly to γ_∞ and γ_∞', respectively), then $L_E(\gamma_i) \to L_E(\gamma_\infty)$ and, hence, Euclidean path length provides a canonical parameterization of this entire family. One of the main goals in this paper is to extend such results to the topological category. For this reason we introduce a new notion of path length which is defined for all paths and behaves well with respect to uniform convergence of paths.

The Euclidean path length function $L = L_E$ satisfies the following basic properties for the path $\gamma : [a,b] \to \mathbb{R}^n$:

(**L1**) If $A \subset [a,b]$ is a closed subinterval, then $L(\gamma\restriction_A) \leq L(\gamma)$;
(**L2**) If $c \in (a,b)$, then $L(\gamma) = L(\gamma\restriction_{[a,c]}) + L(\gamma\restriction_{[c,b]})$;
(**L3**) If $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry, then $L(\Phi \circ \gamma) = L(\gamma)$;
(**L4**) $L(\gamma) =$ the supremum, taken over all partitions $a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$, of the values $L(P)$ where P is the polygonal path with vertices $\gamma(x_0), \ldots, \gamma(x_n)$;

and moreover, we have

(**L5**) $L(\overline{0e}) = 1$, where $e = (1,0,\ldots,0) \in \mathbb{R}^n$.

Conversely, any function L defined on the set of all paths which satisfies the properties (**L1**) through (**L5**) must be equal to L_E (and any function L which satisfies the properties (**L1**) through (**L4**) must be a scalar multiple $c \cdot L_E$ of L_E, where $c = L(\overline{0e})$). Indeed, one can use properties (**L1**), (**L2**), (**L3**), and (**L5**) to show that $L(\overline{ab}) = b - a$ for any $a, b \in \mathbb{R} \subset \mathbb{R}^n$ with $a < b$. Then by properties (**L2**) and (**L3**) it follows that the length of any polygonal path is equal to the sum of the (Euclidean) distances between consecutive vertices. We then conclude by (**L4**) that $L(\gamma) = L_E(\gamma)$ for all paths γ.

There are a number of results in metric geometry pertaining to when a given metric on a Euclidean space is equal to the Euclidean metric; [1], [3], and [8] each survey a variety of such results. Much of this work is related to Hilbert's fourth problem. The length function introduced in this paper contributes to the corresponding program for path length functions by illustrating that there are other length functions which have many properties in common with the Euclidean length.

In light of the above discussion, to provide a genuinely different path length function from the Euclidean length, one must give up at least one of the properties (**L1**) through (**L4**). In Section 2, we define a path length function, called "len", such that $L = \mathsf{len}$ satisfies properties (**L1**), (**L3**), and (**L4**) (see Propositions 3.1(iii), 3.2(i), and 3.4 below), as well as the following weaker form of (**L2**) (see Proposition 3.2(ii)):

(**L2′**) If $c \in (a, b)$, then $L(\gamma) \leq L(\gamma{\restriction}_{[a,c]}) + L(\gamma{\restriction}_{[c,b]})$;

Furthermore, this length function has the following additional properties not enjoyed by the Euclidean length L_E:

- $\mathsf{len}(\gamma) < 1$ for any path γ;
- len is continuous as a function from the space of all maps $[0, 1] \to \mathbb{R}^n$ (with the uniform metric) to $\mathbb{R}$.
- len is defined for any continuous function γ from a locally connected continuum X to $\mathbb{R}^n$;

Moreover, this length function can differentiate between paths whose Euclidean lengths are infinite. For instance, if $\gamma : [a, b] \to \mathbb{R}^n$ is a path and $[c, d]$ is a subinterval of $[a, b]$ such that γ is non-constant on at least one component of $[a, b] \setminus [c, d]$, then $\mathsf{len}(\gamma) > \mathsf{len}(\gamma{\restriction}_{[c,d]})$, even if both of these paths have infinite Euclidean length.

A very similar function is developed by Cannon et. al. in [4], called the *total oscillation* of a path. The most notable difference is that the total oscillation is not invariant under isometries of $\mathbb{R}^n$.

Another similar function is given by Morse in [9], called the *μ-length*, which is defined for paths into any metric space[1].

After establishing the above properties in Section 3, we use len in Section 4 to obtain a standard parameterization of all paths in $\mathbb{R}^n$. This yields characterizations of those families of paths which may be reparameterized so as to be equicontinuous or compact, related to similar characterizations obtained by Silverman in [12]. These results extend classical results on families of paths having finite Euclidean length. In Section 5, we develop a second canonical parameterization of all paths in $\mathbb{R}^n$ with all of the above properties, and which also

[1]We are indebted to J. Keesling for pointing out this paper of Morse to us.

commutes with a reversal of orientation of a path. This second parameterization yields a canonical extension of a bijection between the endpoint sets of two arcs to a homeomorphism between the two arcs. We use this notion to construct homeomorphisms between certain families of pairwise disjoint arcs in Theorem 5.2.

The results of this paper are already seeing use in two other papers. In [11] it was shown that any isotopy of a planar continuum can be extended to an isotopy of the entire plane. Using Theorem 5.1, this result is extended in [6] to a more general class of planar compacta. It was shown in [2] that any two points in a closed topological disk D in the plane can be connected by a unique arc A in D which has the property that any subarc of A which connects two points, neither of which is an endpoint of A, has minimal (finite) Euclidean length among all such arcs. In [7] this result is generalized to shortest paths (in the sense of len length, and in the above Euclidean sense for proper subpaths) in the closure of any homotopy class in an open connected subset of the plane with arbitrary boundary.

2. Definition of the function len

A *generalized path* is a continuous function $\gamma : X \to \mathbb{R}^n$, where X is a locally connected metric continuum.

Given $n \geq 2$, there is a length function len_n defined for generalized paths $X \to \mathbb{R}^n$. In this section, to simplify the definition and arguments below, we restrict our attention to the case $n = 2$, and give a definition of $\mathsf{len} = \mathsf{len}_2$. The case $n > 2$ proceeds similarly; the primary differences being that we cut by $(n-1)$-dimensional hyperplanes instead of lines (see below), and the parameter t below varies through the upper half of the $(n-1)$-dimensional sphere instead of $[0, 1]$ (which we use to parameterize the semi-circle $\{e^{t\pi i} : t \in [0, 1]\}$). We offer some further details for the case $n > 2$ in section 2.1 below.

For notational convenience, we will identify $\mathbb{R}^2$ with $\mathbb{C}$. The reader may find it easier to work through this construction for an ordinary path $\gamma : [a, b] \to \mathbb{C}$ instead of a generalized path, on a first reading.

For $j \in \mathbb{Z}$, let S_j denote the closed horizontal strip $\{a+ib : a \in \mathbb{R}, b \in [j, j+1]\}$ in the plane $\mathbb{C}$. Given $x, t \in [0, 1]$, $\mu \in (0, 1]$, and $j \in \mathbb{Z}$, let

$$S_j^{x,t,\mu} = \mu e^{t\pi i}(S_j + ix).$$

If $A \subset \mathbb{C}$, define $\|A\|_t = \operatorname{diam}(\operatorname{proj}_t^{\perp}(A))$, where $\operatorname{proj}_t^{\perp}$ denotes the orthogonal projection of $\mathbb{C}$ onto the line $\{re^{(t+\frac{1}{2})\pi i} : r \in \mathbb{R}\}$ and diam denotes the diameter in the Euclidean metric. Note that the map which assigns to each $t \in [0, 1]$ the line $\{re^{(t+\frac{1}{2})\pi i} \mid r \in \mathbb{R}\}$ is bijective except on the set $\{0,1\}$ while this set has zero measure in $[0,1]$. Hence the integration used in the definition of $\mathsf{len}(\gamma)$ below is not affected by this ambiguity.

Fix a generalized path $\gamma : X \to \mathbb{C}$. The following lemma will be used in the definition of the function len below.

Lemma 2.1. *For any $(x,t,\mu) \in [0,1]\times[0,1]\times(0,1]$ and any $\varepsilon > 0$, there are only finitely many components C of the sets $\gamma^{-1}(S_j^{x,t,\mu})$ ($j \in \mathbb{Z}$) with $\|\gamma(C)\|_t \geq \varepsilon$.*

Proof. We may assume that $\varepsilon \leq \frac{\mu}{2}$. Suppose for a contradiction that there are infinitely many distinct components $\{C_n\}_{n=0}^{\infty}$ of the sets $\gamma^{-1}(S_j^{x,t,\mu})$ ($j \in \mathbb{Z}$) with $\|\gamma(C)\|_t \geq \varepsilon$.

For each n let $j(n) \in \mathbb{Z}$ be the integer for which $\gamma(C_n) \subset S_{j(n)}^{x,t,\mu}$, and let $p_n \in C_n$ be such that $d(\gamma(p_n), \partial S_{j(n)}^{x,t,\mu}) = \varepsilon$, where d denotes the Euclidean metric in $\mathbb{R}^n$. Observe that by local connectivity of X, for each n we have $\gamma(\partial C_n) \subset \partial S_{j(n)}^{x,t,\mu}$.

Let $p \in X$ be an accumulation point of the set $\{p_n\}_{n=0}^{\infty}$, and let U be an open neighborhood of p which is small enough so that $\operatorname{diam}(\gamma(U)) < \varepsilon$. Then for any n such that $p_n \in U$, we have $\gamma(U) \cap \partial S_{j(n)}^{x,t,\mu} = \emptyset$, hence $U \cap \partial C_n = \emptyset$, and so $U \cap C_n$ is closed and open in U. It follows that U cannot be connected, which is a contradiction since X is locally connected. $\square$

Given $x,t \in [0,1]$ and $\mu \in (0,1]$, let $\langle C_n^{x,t,\mu}\rangle_{n=0}^{\infty}$ enumerate the collection of all components of the sets $\gamma^{-1}(S_j^{x,t,\mu})$ ($j \in \mathbb{Z}$) which have non-degenerate image under the map $\operatorname{proj}_t^{\perp}$, ordered so that $\|\gamma(C_n^{x,t,\mu})\|_t \geq \|\gamma(C_{n+1}^{x,t,\mu})\|_t$ for all n (this is possible by Lemma 2.1).

Define

$$L^{x,t,\mu}(\gamma) = \sum_{n=0}^{\infty} \frac{\|\gamma(C_n^{x,t,\mu})\|_t}{2^n}$$

and define the *length of* γ by

$$\mathsf{len}(\gamma) = \int_0^1 \int_0^1 \int_0^1 L^{x,t,\mu}(\gamma)\, dx\, dt\, d\mu.$$

Observe that if σ is any injective function of the non-negative integers to themselves, then

$$(*) \qquad \sum_{n=0}^{\infty} \frac{\|\gamma(C_{\sigma(n)}^{x,t,\mu})\|_t}{2^n} \leq L^{x,t,\mu}(\gamma).$$

It remains to show that the function $L^{x,t,\mu}(\gamma)$ is in fact integrable, so that the above definition of the function len makes sense. This is accomplished in Lemma 2.3 below.

Lemma 2.2. *Let C be a component of $\gamma^{-1}(S_j^{x,t,\mu})$ for some x,t,μ,j which has non-degenerate image under the map $\mathrm{proj}_t^\perp$, and let $\varepsilon > 0$. Then there exists a subcontinuum $D \subset C$ such that $\gamma(D) \subset \mathrm{int}(S_j^{x,t,\mu})$ and $\|\gamma(D)\|_t \geq \|\gamma(C)\|_t - \varepsilon$.*

PROOF. For the purposes of this argument, let us naturally identify $\mathbb{R}$ with the rotated line $\{re^{(t+\frac{1}{2})\pi i} : r \in \mathbb{R}\}$ which is the range of the map $\mathrm{proj}_t^\perp$.

Let $s_1, s_2 \in \mathbb{R}$ be such that $s_1 < s_2$ and $\mathrm{proj}_t^\perp(\gamma(C)) = [s_1, s_2]$ (and hence $\|\gamma(C)\|_t = s_2 - s_1$). We may assume that $\varepsilon < \frac{s_2 - s_1}{2}$. Let S' denote the narrower (closed) strip $(\mathrm{proj}_t^\perp)^{-1}([s_1 + \frac{\varepsilon}{2}, s_2 - \frac{\varepsilon}{2}]) \subset \mathrm{int}(S_j^{x,t,\mu})$. Then $C \cap \gamma^{-1}(S')$ must have a component D such that $\mathrm{proj}_t^\perp(\gamma(D)) = [s_1 + \frac{\varepsilon}{2}, s_2 - \frac{\varepsilon}{2}]$ (see e.g. Theorem 5.2 of [10]). This D is as desired. $\square$

A real-valued function f is *lower semicontinuous* if $f^{-1}((\alpha, \infty))$ is open for every $\alpha \in \mathbb{R}$. Note that a lower semicontinuous function is Borel, hence (Lebesgue) integrable.

Lemma 2.3. *For a fixed generalized path $\gamma : X \to \mathbb{C}$, put $L(x,t,\mu) = L^{x,t,\mu}(\gamma)$. Then the function $L(x,t,\mu)$ from $[0,1]\times[0,1]\times(0,1]$ to $\mathbb{R}$ is lower semicontinuous, hence integrable.*

PROOF. Fix a number $\alpha \in \mathbb{R}$, and suppose $L^{x,t,\mu}(\gamma) > \alpha$. Choose N large enough so that $\sum_{n=0}^{N} \frac{\|\gamma(C_n^{x,t,\mu})\|_t}{2^n} > \alpha$.

For each $n \in \{0, 1, \dots, N\}$ let $j(n)$ be such that $C_n^{x,t,\mu}$ is a component of $\gamma^{-1}(S_{j(n)}^{x,t,\mu})$. Then, by Lemma 2.2, for each n we can find a proper subcontinuum $D_n \subset C_n^{x,t,\mu}$ such that $\gamma(D_n)$ is contained in the interior of $S_{j(n)}^{x,t,\mu}$, and so that

$$\sum_{n=0}^{N} \frac{\|\gamma(D_n)\|_t}{2^n} > \alpha.$$

Let $\varepsilon_1 > 0$ be small enough so that if $|x' - x|, |t' - t|, |\mu' - \mu| < \varepsilon_1$, then $\gamma(D_n) \subset S_{j(n)}^{x',t',\mu'}$ for each $n \in \{0, 1, \dots, N\}$, and moreover

$$\sum_{n=0}^{N} \frac{\|\gamma(D_n)\|_{t'}}{2^n} > \alpha. \tag{1}$$

For each pair of numbers $n_1 < n_2$ in $\{0, 1, \dots, N\}$ with $j(n_1) = j(n_2)$, find an open set $A_{n_1,n_2} \subset X$ such that $C_{n_1}^{x,t,\mu} \subset A_{n_1,n_2} \subset \overline{A_{n_1,n_2}} \subset X \smallsetminus C_{n_2}^{x,t,\mu}$ and $\partial A_{n_1,n_2} \cap \gamma^{-1}(S_{j(n_1)}^{x,t,\mu}) = \emptyset$; that is, $\gamma(\partial A_{n_1,n_2}) \cap S_{j(n_1)}^{x,t,\mu} = \emptyset$.

Let $0 < \varepsilon_2 < \varepsilon_1$ be small enough so that if $|x' - x|, |t' - t|, |\mu' - \mu| < \varepsilon_2$, then $\gamma(\partial A_{n_1,n_2}) \cap S^{x',t',\mu'}_{j(n_1)} = \emptyset$ for every pair of numbers $n_1 < n_2$ in $\{0, 1, \dots, N\}$ with $j(n_1) = j(n_2)$. Since $\partial A_{n_1,n_2}$ separates D_{n_1} from D_{n_2} in X, it follows that D_{n_1} and D_{n_2} are contained in distinct components of $\gamma^{-1}(S^{x',t',\mu'}_{j(n_1)})$. Therefore, for such x', t', μ', by $(*)$ and (1) we have

$$L^{x',t',\mu'}(\gamma) \geq \sum_{n=0}^{N} \frac{\|\gamma(D_n)\|_{t'}}{2^n} > \alpha.$$

Thus, the set $\{(x, t, \mu) : L^{x,t,\mu}(\gamma) > \alpha\}$ is open in $[0, 1] \times [0, 1] \times (0, 1]$, and so $L(x, t, \mu)$ is a lower semicontinuous function. □

Thus the function len is well-defined. Observe that the set $\gamma(C^{x,t,\mu}_n)$ is contained in some strip $S^{x,t,\mu}_j$ having width μ, hence $\|\gamma(C^{x,t,\mu}_n)\|_t \leq \mu$. It follows that $L^{x,t,\mu}(\gamma) < 2\mu$, and therefore $\mathsf{len}(\gamma) < 1$.

It can easily be seen that $\mathsf{len}(\overline{0x}) \to 1$ as $x \to \infty$, $x \in \mathbb{R}$. It follows from Propositions 3.1(iii) and 3.3 below that if $\gamma_m : X_m \to \mathbb{C}$, $m \in \mathbb{N}$ is a sequence of generalized paths such that $\operatorname{diam}(\gamma_m(X_m)) \to \infty$ as $m \to \infty$, then $\mathsf{len}(\gamma_m) \to 1$ as $m \to \infty$.

On the other hand, if we define $\gamma_m : [0, 1] \to \mathbb{C}$ by $\gamma_m(t) = e^{2\pi imt}$, then $\mathsf{len}(\gamma_m) \to 1$ as $m \to \infty$, even though $\operatorname{diam}(\gamma_m([0, 1])) = 2$ for all m.

2.1. **Definition of len in $\mathbb{R}^n$ for $n \geq 2$.** Let $n \geq 2$ be fixed. In this section we give an overview of the general definition of $\mathsf{len} = \mathsf{len}_n$.

Let $\mathbb{S}^{n-1}_+$ denote the upper hemi-sphere of $\mathbb{S}^{n-1}$; that is

$$\mathbb{S}^{n-1}_+ = \{\langle t_1, \dots, t_n\rangle \in \mathbb{R}^n : t_1^2 + \cdots + t_n^2 = 1 \text{ and } t_n \geq 0\},$$

equipped with the usual spherical measure $\mathfrak{m}$, normalized so that $\mathfrak{m}(\mathbb{S}^{n-1}_+) = 1$.

Let $\mathbf{t} \in \mathbb{S}^{n-1}_+$. If $A \subset \mathbb{R}^n$, define $\|A\|_{\mathbf{t}} = \operatorname{diam}(\operatorname{proj}^\perp_{\mathbf{t}}(A))$, where $\operatorname{proj}^\perp_{\mathbf{t}}$ denotes the orthogonal projection of $\mathbb{R}^n$ onto the line containing the unit vector $\mathbf{t}$.

Consider the hyperplane $H_{\mathbf{t}} = \{\mathbf{z} \in \mathbb{R}^n : \mathbf{z} \cdot \mathbf{t} = 0\}$. Thus we use $\mathbb{S}^{n-1}_+$ to parameterize the collection of all $(n-1)$-dimensional hyperplanes through the origin, via the mapping $\mathbf{t} \mapsto H_{\mathbf{t}}$. We remark that this mapping is one-to-one except on the set $\{\langle t_1, \dots, t_n\rangle \in \mathbb{S}^{n-1}_+ : t_n = 0\}$, and this set has measure zero in $\mathbb{S}^{n-1}_+$ with respect to $\mathfrak{m}$.

Given $j \in \mathbb{Z}$, let $S^{\mathbf{t}}_j$ denote the region between the two translates $H_{\mathbf{t}} + j\mathbf{t}$ and $H_{\mathbf{t}} + (j+1)\mathbf{t}$ of the hyperplane $H_{\mathbf{t}}$. In other words,

$$S^{\mathbf{t}}_j = \{\mathbf{z} + r\mathbf{t} : \mathbf{z} \bullet \mathbf{t} = 0 \text{ and } j \leq r \leq j+1\}.$$

Given $x \in [0,1]$, $\mathbf{t} \in \mathbb{S}^{n-1}_+$, $\mu \in (0,1]$, and $j \in \mathbb{Z}$, let

$$S_j^{x,\mathbf{t},\mu} = \mu(S_j^{\mathbf{t}} + x\mathbf{t}).$$

Fix a generalized path $\gamma : X \to \mathbb{R}^n$. Given $x \in [0,1]$, $\mathbf{t} \in \mathbb{S}^{n-1}_+$, and $\mu \in (0,1]$, let $\langle C_\ell^{x,\mathbf{t},\mu} \rangle_{\ell=0}^\infty$ enumerate the collection of all components of the sets $\gamma^{-1}(S_j^{x,\mathbf{t},\mu})$ ($j \in \mathbb{Z}$) which have non-degenerate image under the map $\mathrm{proj}_{\mathbf{t}}^\perp$, ordered so that $\|\gamma(C_\ell^{x,\mathbf{t},\mu})\|_{\mathbf{t}} \geq \|\gamma(C_{\ell+1}^{x,\mathbf{t},\mu})\|_{\mathbf{t}}$ for all ℓ.

Define

$$L^{x,\mathbf{t},\mu}(\gamma) = \sum_{\ell=0}^{\infty} \frac{\|\gamma(C_\ell^{x,\mathbf{t},\mu})\|_{\mathbf{t}}}{2^\ell}$$

and define the *length of* γ by

$$\mathsf{len}(\gamma) = \mathsf{len}_n(\gamma) = \int_0^1 \int_{\mathbb{S}^{n-1}_+} \int_0^1 L^{x,\mathbf{t},\mu}(\gamma)\, dx\, d\mathfrak{m}\, d\mu.$$

3. Properties of the function len

Let $n \geq 2$ be fixed. All results in this section will be stated for $\mathsf{len} = \mathsf{len}_n$, but for simplicity all proofs will be given only for the case $n = 2$.

The following basic properties follow immediately from the definition of the function len.

Proposition 3.1. *Let $\gamma : X \to \mathbb{R}^n$ be a generalized path.*

(i) $\mathsf{len}(\gamma) = 0$ *if and only if γ is a constant function.*
(ii) If $h : X \to Y$ is a homeomorphism, then $\mathsf{len}(\gamma \circ h) = \mathsf{len}(\gamma)$.
(iii) If $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry, then $\mathsf{len}(\Phi \circ \gamma) = \mathsf{len}(\gamma)$.

For the next properties, we need to consider a more restricted class of locally connected continua, namely dendrites. A *dendrite* is a locally connected continuum which contains no simple closed curve. A characteristic feature of dendrites is that they are *hereditarily unicoherent*; that is, given any two intersecting subcontinua A and B of a dendrite X, the intersection $A \cap B$ is connected. See Section 6 for examples to illustrate how these properties can fail when the domain of a generalized path is not a dendrite.

Proposition 3.2. *Let X be a dendrite, and let $\gamma : X \to \mathbb{R}^n$ be a generalized path.*

(i) If A is a subcontinuum of X, then $\mathsf{len}(\gamma{\restriction}_A) \leq \mathsf{len}(\gamma)$. *Moreover,* $\mathsf{len}(\gamma{\restriction}_A) = \mathsf{len}(\gamma)$ *if and only if γ is constant on each component of $X \setminus A$.*
(ii) If A, B are subcontinua of X with $A \cup B = X$, then

$$\mathsf{len}(\gamma) \leq \mathsf{len}(\gamma{\restriction}_A) + \mathsf{len}(\gamma{\restriction}_B).$$

PROOF. We treat the case $n = 2$.

Fix x, t, μ, and for convenience denote $S_j^{x,t,\mu}$ and $C_n^{x,t,\mu}$ (defined as in Section 2) simply by S_j and C_n, respectively.

Let $A \subseteq X$ be a subcontinuum. Given $j \in \mathbb{Z}$ and a component C of $(\gamma{\restriction_A})^{-1}(S_j)$, there exists some n such that $C \subseteq C_n$. Since $C_n \cap A$ is connected (by hereditary unicoherence), it follows that $C = C_n \cap A$.

Therefore there exists an injective function σ from the non-negative integers to themselves such that $\langle C_{\sigma(n)} \cap A\rangle_{n=0}^{\infty}$ enumerates the collection of all components of the sets $(\gamma{\restriction_A})^{-1}(S_j^{x,t,\mu})$ ($j \in \mathbb{Z}$) which have non-degenerate image under the map $\mathrm{proj}_t^{\perp}$, so that $\|\gamma(C_{\sigma(n)} \cap A)\|_t \geq \|\gamma(C_{\sigma(n+1)} \cap A)\|_t$ for all n. Then

$$\begin{aligned} L^{x,t,\mu}(\gamma{\restriction_A}) &= \sum_{n=0}^{\infty} \frac{\|\gamma(C_{\sigma(n)} \cap A)\|_t}{2^n} \\ &\leq \sum_{n=0}^{\infty} \frac{\|\gamma(C_{\sigma(n)})\|_t}{2^n} \\ &\leq L^{x,t,\mu}(\gamma) \qquad \text{(by the observation } (*)). \end{aligned}$$

Since this holds for all x, t, μ, we have established the first statement of (i).

For the second statement of (i), suppose γ is non-constant on some component K of $X \setminus A$. The intersection $\overline{K} \cap A$ consists of a single point (see e.g. 10.9 and 10.24 of [10]). Let $\{p\} = \overline{K} \cap A$, and let $q \in K$ be such that $\gamma(p) \neq \gamma(q)$. There is a positive measure set of parameters x, t, μ and an integer $j \in \mathbb{Z}$ for which $\gamma(q) \in \mathrm{int}(S_j^{x,t,\mu})$ and $\gamma(p) \notin S_j^{x,t,\mu}$. For such x, t, μ, j, there is a component of $\gamma^{-1}(S_j^{x,t,\mu})$ contained in K, which contributes positively to the sum $L^{x,t,\mu}(\gamma)$, thereby making it larger than $L^{x,t,\mu}(\gamma{\restriction_A})$. It follows that $\mathsf{len}(\gamma) > \mathsf{len}(\gamma{\restriction_A})$. The converse implication is immediate.

Now suppose $A, B \subseteq X$ are subcontinua with $A \cup B = X$. As above, for any $j \in \mathbb{Z}$, each component of $(\gamma{\restriction_A})^{-1}(S_j)$ (respectively $(\gamma{\restriction_B})^{-1}(S_j)$) has the form $C_n \cap A$ (respectively $C_n \cap B$) for some n.

Let $\langle n(\alpha)\rangle_{\alpha=0}^{\infty}$ and $\langle m(\beta)\rangle_{\beta=0}^{\infty}$ be the strictly increasing sequences of non-negative integers such that $\langle C_{n(\alpha)} \cap A\rangle_{\alpha=0}^{\infty}$ enumerates the collection of all components of the sets $(\gamma{\restriction_A})^{-1}(S_j^{x,t,\mu})$ ($j \in \mathbb{Z}$) which have non-degenerate image under the map $\mathrm{proj}_t^{\perp} \circ \gamma$, and $\langle C_{m(\beta)} \cap B\rangle_{\beta=0}^{\infty}$ enumerates the collection of all components of the sets $(\gamma{\restriction_B})^{-1}(S_j^{x,t,\mu})$ ($j \in \mathbb{Z}$) which have non-degenerate image under the map $\mathrm{proj}_t^{\perp} \circ \gamma$. Note that these enumerations are not necessarily ordered according to the sizes of the images under $\mathrm{proj}_t^{\perp} \circ \gamma$.

For any n, we clearly have $\|\gamma(C_n)\|_t \leq \|\gamma(C_n \cap A)\|_t + \|\gamma(C_n \cap B)\|_t$. Therefore

$$\begin{aligned} L^{x,t,\mu}(\gamma) &= \sum_{n=0}^{\infty} \frac{\|\gamma(C_n)\|_t}{2^n} \\ &\leq \sum_{n=0}^{\infty} \frac{\|\gamma(C_n \cap A)\|_t}{2^n} + \sum_{n=0}^{\infty} \frac{\|\gamma(C_n \cap B)\|_t}{2^n} \\ &= \sum_{\alpha=0}^{\infty} \frac{\|\gamma(C_{n(\alpha)} \cap A)\|_t}{2^{n(\alpha)}} + \sum_{\beta=0}^{\infty} \frac{\|\gamma(C_{m(\beta)} \cap B)\|_t}{2^{m(\beta)}} \\ &\leq \sum_{\alpha=0}^{\infty} \frac{\|\gamma(C_{n(\alpha)} \cap A)\|_t}{2^{\alpha}} + \sum_{\beta=0}^{\infty} \frac{\|\gamma(C_{m(\beta)} \cap B)\|_t}{2^{\beta}} \qquad \text{(since } \alpha \leq n(\alpha),\ \beta \leq m(\beta)) \\ &\leq L^{x,t,\mu}(\gamma{\restriction}_A) + L^{x,t,\mu}(\gamma{\restriction}_B) \qquad \text{(by the observation } (*)). \end{aligned}$$

Since this holds for all x, t, μ, we have established (ii). □

Proposition 3.3. *Let $z_1, z_2 \in \mathbb{R}^n$. If $\gamma : X \to \mathbb{R}^n$ is any generalized path such that $z_1, z_2 \in \gamma(X)$, then $\mathsf{len}(\overline{z_1 z_2}) \leq \mathsf{len}(\gamma)$. Moreover, if $\gamma(X)$ is not the straight line segment joining z_1 and z_2, or if $\gamma^{-1}(w)$ is disconnected for some w on the straight line segment between z_1 and z_2, then $\mathsf{len}(\overline{z_1 z_2}) < \mathsf{len}(\gamma)$.*

Proposition 3.3 can be proved directly from the definition of the function len, and we leave this to the reader. Note that it also follows that if in a path $\gamma : [0,1] \to \mathbb{R}^n$ we replace the subpath $\gamma{\restriction}_{[a,b]}$ with the straight line segment $\overline{\gamma(a)\gamma(b)}$ and if we denote the resulting path by γ^*, then $\mathsf{len}(\gamma^*) \leq \mathsf{len}(\gamma)$ with strict inequality if $\gamma{\restriction}_{[a,b]}$ is not a monotone parametrization of the straight line segment $\overline{\gamma(a)\gamma(b)}$.

Next we consider $C(X) = C(X, \mathbb{R}^n)$, the set of all generalized paths $X \to \mathbb{R}^n$. This is a metric space with the usual metric $d_{\sup}(\gamma_1, \gamma_2) = \sup_{p \in X} |\gamma_1(p) - \gamma_2(p)|$.

Proposition 3.4. *The function $\mathsf{len} : C(X) \to \mathbb{R}$ is continuous.*

Proof. We treat the case $n = 2$. Let γ_0 be in $C(X)$.

Suppose $\alpha < \mathsf{len}(\gamma_0) < \beta$. We will prove that for small enough $\xi > 0$, if $\gamma \in C(X)$ with $d_{\sup}(\gamma, \gamma_0) < \xi$, then $\alpha < \mathsf{len}(\gamma_0) < \beta$.

A simple modification of the proof of Lemma 2.3 shows that for $\xi > 0$ small enough, if $d_{\sup}(\gamma, \gamma_0) < \xi$ then $\mathsf{len}(\gamma) > \alpha$. Thus it remains to show $\mathsf{len}(\gamma) < \beta$ for sufficiently small $\xi > 0$.

Fix a countable dense set $\{q_k\}_{k=1}^{\infty} \subset X$. Given $k \neq l$ and $j \in \mathbb{Z}$, let

$$\begin{aligned} B^j_{kl} = \{&(x,t,\mu) \in [0,1] \times [0,1] \times (0,1] : \\ &\text{there is a continuum } C \subseteq \gamma_0^{-1}(S_j^{x,t,\mu}) \text{ with } q_k, q_l \in C \text{ and} \\ &\text{for every such } C \text{ we have } \gamma_0(C) \cap \partial S_j^{x,t,\mu} \neq \emptyset\} \end{aligned}$$

and let $B = \bigcup_{\substack{k\neq l\\ j\in\mathbb{Z}}} B^j_{kl}$. It is easy to see that $([0,1] \times [0,1] \times (0,1]) \smallsetminus B^j_{kl}$ is open, and so B is F_σ, hence measurable.

Claim 3.4.1. B has measure zero.

Proof of Claim 3.4.1. Fix $k \neq l$ and $j \in \mathbb{Z}$. It will be convenient to change variables from (x,t,μ) to (z,t,μ) so that for any fixed rotation angle t and translation parameter z, as the strip width μ shrinks, the j-th strip itself shrinks inwards, nesting down on a line.

Given $(x,t,\mu) \in [0,1] \times [0,1] \times (0,1]$, let $z = \mu(x + \frac{1}{2} + j) \in (-\infty,\infty)$, and define $\Phi(x,t,\mu) = (z,t,\mu)$.

Observe that for (z,t,μ) in the image of Φ, $\Phi^{-1}(z,t,\mu) = (\frac{z}{\mu} - \frac{1}{2} - j, t, \mu)$. Thus $\Phi(B^j_{kl})$ is contained in the set

$$\begin{aligned} B' = \{&(z,t,\mu) : \text{there is a continuum } C \subseteq \gamma_0^{-1}(T^{z,t,\mu}) \text{ with } q_k, q_l \in C \text{ and} \\ &\text{for every such } C \text{ we have } \gamma_0(C) \cap \partial T^{z,t,\mu} \neq \emptyset\} \end{aligned}$$

where $T^{z,t,\mu} = \mu e^{t\pi i}(S_j + i(\frac{z}{\mu} - \frac{1}{2} - j)) = e^{t\pi i}(\mu(S_j - \frac{1}{2}i - ji) + iz)$. Observe that the strip $T^{z,t,\mu}$ is centered about the line $e^{t\pi i}(\mathbb{R} + iz)$, and if $\mu' < \mu$, then $T^{z,t,\mu'}$ is contained in the interior of $T^{z,t,\mu}$. Thus for any fixed z, t, there can be at most one μ for which $(z,t,\mu) \in B'$. By Fubini's theorem, this implies B' has measure zero. Since $\Phi(B^j_{kl}) \subseteq B'$, we have that $\Phi(B^j_{kl})$ has measure zero as well.

A straightforward calculation shows that Φ is a C^1-diffeomorphism on $[0,1] \times [0,1] \times (0,1]$ with Jacobian equal to μ. Thus by the change of variables theorem [5, Theorem 2.47], the measure of $\Phi(B^j_{kl})$ is equal to

$$\iiint_{B^j_{kl}} \mu \, dx \, dt \, d\mu.$$

Since $\mu > 0$ and $\Phi(B^j_{kl})$ has measure zero, it follows that B^j_{kl} has measure zero as well. Since $B = \bigcup_{\substack{k\neq l\\ j\in\mathbb{Z}}} B^j_{kl}$, the Claim follows. $\square$(Claim 3.4.1)

Claim 3.4.2. Given $(x_0, t_0, \mu_0) \in ([0,1] \times [0,1] \times (0,1)) \setminus B$ and $\varepsilon > 0$, there exists $\delta > 0$ and $\xi_0 > 0$ such that if $|x - x_0| < \delta$, $|t - t_0| < \delta$, $|\mu - \mu_0| < \delta$, and $d_{\sup}(\gamma, \gamma_0) < \xi_0$, then $L^{x,t,\mu}(\gamma) < L^{x_0,t_0,\mu_0}(\gamma_0) + \varepsilon$.

Proof of Claim 3.4.2. For $j \in \mathbb{Z}$, let S'_j denote the narrower (closed) strip obtained from $S_j^{x_0,t_0,\mu_0}$ by moving the boundary lines in towards the middle a distance of $\frac{\varepsilon}{20}$ each.

Let $\langle C_n \rangle_{n=0}^{\infty}$ enumerate the collection of all components of the sets $\gamma_0^{-1}(S_j^{x_0,t_0,\mu_0})$ ($j \in \mathbb{Z}$) which have non-degenerate image under the map $\mathrm{proj}_t^{\perp}$, ordered so that $\|\gamma_0(C_n)\|_{t_0} \geq \|\gamma_0(C_{n+1})\|_{t_0}$ for all n. For each n, let $j(n)$ be the integer such that $\gamma_0(C_n) \subset S_{j(n)}^{x_0,t_0,\mu_0}$. By Lemma 2.1, there are only finitely many components $C_0, \ldots, C_N$ such that $\gamma_0(C_n)$ meets the narrower strip $S'_{j(n)}$, for $0 \leq n \leq N$.

Fix some n with $0 \leq n \leq N$. Let $U_1, \ldots, U_r$ be a finite cover of $C_n \cap \gamma_0^{-1}(S'_{j(n)})$ by connected open subsets of X whose closures are mapped by γ_0 into the interior of $S_{j(n)}^{x_0,t_0,\mu_0}$. Let k be such that $q_k \in U_1$, and for each $2 \leq i \leq r$ let $l(i)$ be such that $q_{l(i)} \in U_i$. Then for each $2 \leq i \leq r$, since $(x_0, t_0, \mu_0) \notin B_{k\,l(i)}^{j(n)}$ and C_n is a continuum in $\gamma_0^{-1}(S_{j(n)}^{x_0,t_0,\mu_0})$ containing q_k and $q_{l(i)}$, there exists a continuum K_i containing q_k and $q_{l(i)}$ which is mapped by γ_0 into the interior of the strip $S_{j(n)}^{x_0,t_0,\mu_0}$. Let $C'_n = \overline{U_1} \cup \bigcup_{2 \leq i \leq r} (\overline{U_i} \cup K_i)$. Then C'_n is a continuum which is mapped by γ_0 into the interior of the strip $S_{j(n)}^{x_0,t_0,\mu_0}$ and such that $C_n \cap \gamma_0^{-1}(S'_j) \subseteq C'_n \subset C_n$.

Having done this for each $0 \leq n \leq N$, let $\delta > 0$ be small enough and let $\xi_0 > 0$ be small enough so that if $|x - x_0| < \delta$, $|t - t_0| < \delta$, $|\mu - \mu_0| < \delta$, and $d_{\sup}(\gamma, \gamma_0) < \xi_0$, then for each $0 \leq n \leq N$ we have:

(i) $\gamma(C'_n)$ is contained in the interior of the strip $S_{j(n)}^{x,t,\mu}$,
(ii) $\|\gamma(C'_n)\|_t < \|\gamma_0(C'_n)\|_{t_0} + \frac{\varepsilon}{4}$, and
(iii) if $A \subset X$ with $\gamma_0(A)$ contained in between two consecutive narrowed strips S'_j and S'_{j+1}, then $\|\gamma(A)\|_t < \frac{\varepsilon}{8}$.

Note that if $0 \leq n \leq N$ and if C is the component of $\gamma^{-1}(S_{j(n)}^{x,t,\mu})$ containing C'_n, then C consists of C'_n plus some part which γ_0 maps in between $S'_{j(n)}$ and $S'_{j(n)-1}$, and some part which γ_0 maps in between $S'_{j(n)}$ and $S'_{j(n)+1}$. Therefore, by (ii) and (iii) we have

$$\|\gamma(C)\|_t < \|\gamma_0(C'_n)\|_{t_0} + \tfrac{\varepsilon}{4} + 2 \cdot \tfrac{\varepsilon}{8} = \|\gamma_0(C'_n)\|_{t_0} + \tfrac{\varepsilon}{2}.$$

Every other component $\tilde{C}$ of $\gamma^{-1}(S_j^{x,t,\mu})$ satisfies $\|\gamma(\tilde{C})\|_t < \frac{\varepsilon}{8}$ by (iii). It follows that

$$
\begin{aligned}
L^{x,t,\mu}(\gamma) &< \sum_{n=0}^{N} \frac{\|\gamma_0(C'_n)\|_{t_0} + \frac{\varepsilon}{2}}{2^n} + \sum_{n=N+1}^{\infty} \frac{\varepsilon/8}{2^n} \\
&< \sum_{n=0}^{N} \frac{\|\gamma_0(C'_n)\|_{t_0}}{2^n} + \varepsilon \\
&\leq L^{x_0,t_0,\mu_0}(\gamma_0) + \varepsilon.
\end{aligned}
$$

$\square$(Claim 3.4.2)

We are now ready to show that $\mathsf{len}(\gamma) < \beta$ for γ sufficiently close to γ_0.

Recalling that $L^{x,t,\mu}(\gamma_0) < 2\mu \leq 2$, choose a step function $\psi = 2 - \sum_{i=0}^{k} c_i\, \chi_{A_i}$, where the $A_i \subset [0,1] \times [0,1] \times (0,1]$ are pairwise disjoint compact sets and χ_{A_i} is the characteristic function of the set A_i, with

$$L^{x,t,\mu}(\gamma_0) \leq \psi(x,t,\mu) \quad \text{for all } x,t,\mu$$

and

$$\int_0^1 \int_0^1 \int_0^1 \psi(x,t,\mu)\, dx\, dt\, d\mu < \beta.$$

Let $\eta = \beta - \int_0^1 \int_0^1 \int_0^1 \psi\, dx\, dt\, d\mu > 0$. By Claim 3.4.1, we can find a compact set $\Omega \subset ([0,1] \times [0,1] \times (0,1]) \smallsetminus B$ of measure $\geq 1 - \frac{\eta}{4}$.

Using Claim 3.4.2 and compactness of the sets $A_i \cap \Omega$, we can find ξ_i small enough so that if $d_{\sup}(\gamma, \gamma_0) < \xi_i$, then $L^{x,t,\mu}(\gamma) < \psi(x,t,\mu) + \frac{\eta}{4}$ for all $(x,t,\mu) \in A_i \cap \Omega$. Letting $\xi = \min_i \xi_i$, it follows that if $d_{\sup}(\gamma, \gamma_0) < \xi$, then

$$
\begin{aligned}
\mathsf{len}(\gamma) &= \int_0^1 \int_0^1 \int_0^1 L^{x,t,\mu}(\gamma)\, dx\, dt\, d\mu \\
&\leq \iiint_\Omega L^{x,t,\mu}(\gamma)\, dx\, dt\, d\mu + 2 \cdot \tfrac{\eta}{4} \\
&\leq \iiint_\Omega \left(\psi(x,t,\mu) + \tfrac{\eta}{4}\right)\, dx\, dt\, d\mu + 2 \cdot \tfrac{\eta}{4} \\
&\leq (\beta - \eta) + \tfrac{\eta}{4} + 2 \cdot \tfrac{\eta}{4} \\
&< \beta.
\end{aligned}
$$

$\square$

It follows from Proposition 3.4 that for any path $\gamma_0 : [0,1] \to \mathbb{R}^n$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that if $d_{\sup}(\gamma, \gamma_0) < \delta$, then $|\mathsf{len}(\gamma{\restriction}_{[0,t]}) - \mathsf{len}(\gamma_0{\restriction}_{[0,t]})| < \varepsilon$ for all $t \in [0,1]$.

To see this, note that by Proposition 3.4, for any $t_0 \in [0,1]$, there is a small open interval J_0 around t_0 and $\delta_0 > 0$ small enough such that if $d_{\sup}(\gamma, \gamma_0) < \delta_0$, then $|\mathsf{len}(\gamma{\restriction}_{[0,t]}) - \mathsf{len}(\gamma_0{\restriction}_{[0,t_0]})| < \frac{\varepsilon}{2}$ for any $t \in J_0$. Take a finite cover of $[0,1]$ by such intervals J_0 and take δ to be the minimum of the corresponding numbers δ_0. Suppose $d_{\sup}(\gamma, \gamma_0) < \delta$. Given any $t \in [0,1]$, take one of the intervals J_0 from the cover such that $t \in J_0$. Then we have

$$\begin{aligned}|\mathsf{len}(\gamma{\restriction}_{[0,t]}) - \mathsf{len}(\gamma_0{\restriction}_{[0,t]})| &\leq |\mathsf{len}(\gamma{\restriction}_{[0,t]}) - \mathsf{len}(\gamma_0{\restriction}_{[0,t_0]})| + |\mathsf{len}(\gamma_0{\restriction}_{[0,t]}) - \mathsf{len}(\gamma_0{\restriction}_{[0,t_0]})| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

A consequence of Proposition 3.4 is that for a path $\gamma : [a,b] \to \mathbb{R}^n$, $\mathsf{len}(\gamma)$ is small if and only if $\mathrm{diam}(\gamma([a,b]))$ is small. This will suffice for our purposes, but in fact one can argue from the definition of $\mathsf{len} = \mathsf{len}_n$ that there are constants $c_1(n), c_2(n) > 0$ such that:

(∗∗): If $\gamma : [a,b] \to \mathbb{R}^n$ is a path with $\mathrm{diam}(\gamma([a,b])) \leq \frac{1}{2}$, then

$$c_1(n) \cdot \mathrm{diam}(\gamma([a,b])) \leq \mathsf{len}_n(\gamma) \leq c_2(n) \cdot \mathrm{diam}(\gamma([a,b])).$$

4. Parameterization by len

Let $n \geq 2$ be fixed. As before, all results in this section will be stated for $\mathsf{len} = \mathsf{len}_n$, and proofs will be given for the case $n = 2$.

In this section, we work with $\mathcal{C}[0,1] = \mathcal{C}([0,1], \mathbb{R}^n)$, the set of all paths $\gamma : [0,1] \to \mathbb{R}^n$. This is a metric space with the usual metric $d_{\sup}(\gamma_1, \gamma_2) = \sup_{t \in [0,1]} |\gamma_1(t) - \gamma_2(t)|$.

Definition 1. Given two paths $\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^n$, we say that γ_2 is a *reparameterization* of γ_1 if there are non-decreasing onto maps $m_1, m_2 : [0,1] \to [0,1]$ such that γ_i is constant on each fiber $m_i^{-1}(s)$, $s \in [0,1]$, for both $i = 1, 2$, and $\gamma_1 \circ m_1^{-1} = \gamma_2 \circ m_2^{-1}$. In this case, we write $\gamma_1 \approx \gamma_2$.

Thus $\gamma_1 \approx \gamma_2$ if they both parameterize the same path, with the same orientation, where we disregard any constant sections. Note that if $\gamma_1 \approx \gamma_2$, then $\mathsf{len}(\gamma_1) = \mathsf{len}(\gamma_2)$. It is easy to see that $\approx$ is an equivalence relation on $\mathcal{C}[0,1]$. Denote by $[\gamma]$ the equivalence class of γ with respect to $\approx$.

Let Π denote the collection of all equivalence classes $[\gamma]$. We define a metric ρ on Π as follows:

$$\rho([\gamma_1],[\gamma_2]) = \inf\{\sup_{t\in[0,1]} |\lambda_1(t)-\lambda_2(t)| : \lambda_1 \in [\gamma_1], \lambda_2 \in [\gamma_2]\}.$$

In fact, by reparameterizing, this can be expressed as $\rho([\gamma_1],[\gamma_2]) = \inf\{\sup_{t\in[0,1]} |\lambda_1(t)-\gamma_2(t)| : \lambda_1 \in [\gamma_1]\}$. It is easy to show that ρ is a metric, and that the resultant metric topology on Π coincides with the quotient topology induced from $\mathcal{C}[0,1]$.

One can deduce from Propositions 3.2(i) and 3.4 that given a path $\gamma : [0,1] \to \mathbb{R}^n$, the function $[0,1] \to [0,1)$ defined by $t \mapsto \mathsf{len}(\gamma\restriction_{[0,t]})$ is continuous and non-decreasing. As a result, we can make the following definition:

Definition 2. The *standard parameterization* $\widetilde{\gamma} : [0,1] \to \mathbb{R}^n$ of γ, also called the parameterization of γ by len, is defined as follows. If γ is constant, then $\widetilde{\gamma} = \gamma$. Otherwise, given $s \in [0,1]$, $\widetilde{\gamma}(s) = \gamma(t)$, where $t \in [0,1]$ is such that $\mathsf{len}(\gamma\restriction_{[0,t]}) = s \cdot \mathsf{len}(\gamma)$.

Note that this value t may not be unique, but by Proposition 3.2(i), the point $\gamma(t)$ is uniquely determined by s. One can easily check that $\widetilde{\gamma}$ is a path (i.e. is a continuous function), $\widetilde{\gamma} \approx \gamma$, and $\mathsf{len}(\widetilde{\gamma}\restriction_{[0,s]}) = s\cdot\mathsf{len}(\gamma)$ for any $s \in [0,1]$. However, note that in general $\mathsf{len}(\widetilde{\gamma}\restriction_{[s_1,s_2]}) \neq (s_2-s_1)\mathsf{len}(\gamma)$ when $0 < s_1 < s_2 \leq 1$.

For the Euclidean path length, such a parameterization is only available for rectifiable paths, i.e. those paths with finite Euclidean length.

Observe that the standard parameterization is unique within each equivalence class of paths, in the sense that if $\gamma_1 \approx \gamma_2$, then $\widetilde{\gamma_1} = \widetilde{\gamma_2}$.

Consider the standard parameterization as a function $\Pi \to \mathcal{C}[0,1]$ which maps each class $[\gamma]$ to the unique standard parameterization $\widetilde{\gamma} \in [\gamma]$. Denote by $\widetilde{\Pi}$ the range of this function; that is, $\widetilde{\Pi}$ is the set of all standard parameterizations of paths $[0,1] \to \mathbb{R}^n$.

Theorem 4.1. *$\widetilde{\Pi}$ is a closed subset of $\mathcal{C}[0,1]$, and the function $[\gamma] \mapsto \widetilde{\gamma}$ is a homeomorphism from Π to $\widetilde{\Pi}$.*

Proof. Suppose $\gamma \in \mathcal{C}[0,1] \setminus \widetilde{\Pi}$, which means that $\mathsf{len}(\gamma\restriction_{[0,s]}) \neq s \cdot \mathsf{len}(\gamma)$ for some $s \in [0,1]$. Then for all $\lambda \in \mathcal{C}[0,1]$ which are uniformly close to γ, we have that $\lambda\restriction_{[0,s]}$ is uniformly close to $\gamma\restriction_{[0,s]}$ as well, hence by Proposition 3.4 we have that $\mathsf{len}(\lambda)$ and $\mathsf{len}(\lambda\restriction_{[0,s]})$ are close to $\mathsf{len}(\gamma)$ and $\mathsf{len}(\gamma\restriction_{[0,s]})$, respectively. It follows that $\mathsf{len}(\lambda\restriction_{[0,s]}) \neq s \cdot \mathsf{len}(\lambda)$ if λ is sufficiently close to γ, hence $\lambda \notin \widetilde{\Pi}$. Thus $\mathcal{C}[0,1] \setminus \widetilde{\Pi}$ is open, and so $\widetilde{\Pi}$ is closed.

It is clear that $[\gamma] \mapsto \widetilde{\gamma}$ is one-to-one, and that the inverse of this map is continuous, by definition of the metric ρ on Π (indeed the map $\widetilde{\gamma} \mapsto [\widetilde{\gamma}]$ is Lipschitz continuous with constant 1).

To see that $[\gamma] \mapsto \widetilde{\gamma}$ is continuous, suppose $[\gamma_i]$ is a sequence in Π converging to $[\gamma_\infty] \in \Pi$ (in the metric ρ on Π). By changing representatives if necessary, we may assume that $\gamma_i \to \gamma_\infty$ uniformly. By Proposition 3.4 (and the statements immediately after), it follows that for every $\varepsilon > 0$ there exists n_0 such that for all $i \geq n_0$ and all $t \in [0,1]$, $|\mathsf{len}(\gamma_i{\restriction}_{[0,t]}) - \mathsf{len}(\gamma_\infty{\restriction}_{[0,t]})| < \varepsilon$.

Fix $\varepsilon > 0$. Let $\delta > 0$ be small enough so that for all $i \geq 1$ and all $t_1, t_2 \in [0,1]$, if $|\mathsf{len}(\gamma_i{\restriction}_{[0,t_1]}) - \mathsf{len}(\gamma_i{\restriction}_{[0,t_2]})| < \delta$ then $\mathrm{diam}(\gamma_i([t_1,t_2])) < \frac{\varepsilon}{2}$. Let n_0 be large enough so that for all $i \geq n_0$ and $t \in [0,t]$, $|\mathsf{len}(\gamma_i{\restriction}_{[0,t]}) - \mathsf{len}(\gamma_\infty{\restriction}_{[0,t]})| < \frac{\delta}{2}$ and $|\gamma_i(t) - \gamma_\infty(t)| < \frac{\varepsilon}{2}$.

Given $s \in [0,1]$ and $i \geq n_0$, let $t_i, t_\infty \in [0,1]$ be such that $\mathsf{len}(\gamma_i{\restriction}_{[0,t_i]}) = s \cdot \mathsf{len}(\gamma_i)$ and $\mathsf{len}(\gamma_\infty{\restriction}_{[0,t_\infty]}) = s \cdot \mathsf{len}(\gamma_\infty)$, so that $\widetilde{\gamma}_i(s) = \gamma_i(t_i)$ and $\widetilde{\gamma}_\infty(s) = \gamma_\infty(t_\infty)$. We have

$$\begin{aligned}
|\mathsf{len}(\gamma_i{\restriction}_{[0,t_i]}) - \mathsf{len}(\gamma_i{\restriction}_{[0,t_\infty]})| &\leq |\mathsf{len}(\gamma_i{\restriction}_{[0,t_i]}) - \mathsf{len}(\gamma_\infty{\restriction}_{[0,t_\infty]})| + |\mathsf{len}(\gamma_\infty{\restriction}_{[0,t_\infty]}) - \mathsf{len}(\gamma_i{\restriction}_{[0,t_\infty]})| \\
&= |s \cdot \mathsf{len}(\gamma_i) - s \cdot \mathsf{len}(\gamma_\infty)| + |\mathsf{len}(\gamma_\infty{\restriction}_{[0,t_\infty]}) - \mathsf{len}(\gamma_i{\restriction}_{[0,t_\infty]})| \\
&\leq s \cdot \frac{\delta}{2} + \frac{\delta}{2} \\
&\leq \delta.
\end{aligned}$$

By the definition of δ, it follows that $\mathrm{diam}(\gamma_i([t_i, t_\infty])) < \frac{\varepsilon}{2}$. This implies

$$\begin{aligned}
|\widetilde{\gamma}_i(s) - \widetilde{\gamma}_\infty(s)| &= |\gamma_i(t_i) - \gamma_\infty(t_\infty)| \\
&\leq |\gamma_i(t_i) - \gamma_i(t_\infty)| + |\gamma_i(t_\infty) - \gamma_\infty(t_\infty)| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus $\widetilde{\gamma}_i \to \widetilde{\gamma}_\infty$ uniformly. Therefore, $[\gamma] \mapsto \widetilde{\gamma}$ is continuous. $\square$

Given a family $\mathcal{F} \subseteq \Pi$, define $\widetilde{\mathcal{F}} = \{\widetilde{\gamma} : [\gamma] \in \mathcal{F}\}$.

Corollary 4.2. *A set $\mathcal{F} \subseteq \Pi$ is closed (respectively, compact) if and only if $\widetilde{\mathcal{F}}$ is a closed (respectively, compact) subset of $\mathcal{C}[0,1]$.*

A classical result from metric geometry (see e.g. [3]) is that if $L > 0$ and $\langle \gamma_m \rangle_{m=1}^{\infty}$ is a sequence of paths in a bounded set, with Euclidean path lengths $\leq L$, and if $\widetilde{\gamma}_m : [0,1] \to \mathbb{R}^n$ is the parameterization of γ_m by Euclidean path length (with domain linearly rescaled to $[0,1]$), then the sequence $\langle \widetilde{\gamma}_m \rangle_{m=1}^{\infty}$ has a subsequence which converges uniformly to a path of finite Euclidean length.

This reparameterization is necessary, as standard examples show (consider e.g. $\gamma_m : [0,1] \to [0,1]$ defined by $\gamma_m(s) = s^m$).

We will now prove a version of this result for the function len, where the uniform bound on length assumption is replaced by a weaker restriction on the number of long sections of the paths. Moreover, we prove that this condition is in fact a characterization of those families of paths which can be parameterized so as to be equicontinuous. A similar result is proved in [12] using Morse's length function.

Theorem 4.3. *Let $\mathcal{F} \subseteq \Pi$. Suppose that*

> (†): *for each $\varepsilon > 0$, there is a positive integer N such that for every $[\gamma] \in \mathcal{F}$, there is no collection of more than N pairwise disjoint subintervals of $[0,1]$ whose images under γ have diameters $\geq \varepsilon$.*

Then the family $\widetilde{\mathcal{F}} = \{\widetilde{\gamma} : [\gamma] \in \mathcal{F}\}$ is equicontinuous.

Conversely, if an equicontinuous family can be formed by choosing parameterizations of all the paths in $\mathcal{F}$, then $\mathcal{F}$ satisfies the property (†).

Proof. We treat the case $n = 2$. As usual, we identify $\mathbb{R}^2$ with $\mathbb{C}$.

Fix $\varepsilon > 0$. Let $N \geq 1$ be such that for every γ with $[\gamma] \in \mathcal{F}$, there is no collection of more than N pairwise disjoint subintervals of $[0,1]$ whose images under γ have diameters $\geq \frac{\varepsilon}{16}$. Let $\delta = \frac{\varepsilon^2}{2^{N+7} \cdot N}$.

Suppose for a contradiction that for some $[\gamma] \in \mathcal{F}$ there exist $0 \leq s_1 < s_2 \leq 1$ with $s_2 - s_1 < \delta$ and $\rho(\widetilde{\gamma}(s_1), \widetilde{\gamma}(s_2)) \geq \varepsilon$. Note that

$$\begin{aligned} \mathsf{len}(\widetilde{\gamma}\restriction_{[0,s_2]}) &= s_2 \cdot \mathsf{len}(\gamma) \\ &< s_1 \cdot \mathsf{len}(\gamma) + \delta \cdot \mathsf{len}(\gamma) \\ &< \mathsf{len}(\widetilde{\gamma}\restriction_{[0,s_1]}) + \delta. \end{aligned}$$

Let $t_0 \in [0,1]$ be such that the line $\{re^{t_0 \pi i} : r \in \mathbb{R}\}$ is orthogonal to the segment $\overline{\widetilde{\gamma}(s_1)\widetilde{\gamma}(s_2)}$. Define $W \subset [0,1] \times [0,1] \times (0,1]$ by

$$W = [0,1] \times [t_0 - \tfrac{1}{4}, t_0 + \tfrac{1}{4}] \times [\tfrac{\varepsilon}{8}, \tfrac{\varepsilon}{4}],$$

where the interval $[t_0 - \frac{1}{4}, t_0 + \frac{1}{4}]$ should be considered reduced mod 1 (i.e. it represents the set of all $t \in [0,1]$ such that one of $|t - t_0|$, $|t - (t_0 - 1)|$, or $|t - (t_0 + 1)|$ is $\leq \frac{1}{4}$). Note that for any $(x, t, \mu) \in W$, any strip $S_j^{x,t,\mu}$ ($j \in \mathbb{Z}$) covers less than half of the line segment $\overline{\widetilde{\gamma}(s_1)\widetilde{\gamma}(s_2)}$.

Consider a fixed $x, t, \mu \in W$. Let $C_0, \ldots, C_N$ and $D_0, \ldots, D_N$ be the first $N+1$ components of $(\widetilde{\gamma}\restriction_{[0,s_1]})^{-1}(S_j^{x,t,\mu})$ and $(\widetilde{\gamma}\restriction_{[0,s_2]})^{-1}(S_j^{x,t,\mu})$ ($j \in \mathbb{Z}$), respectively, ordered so that $\|\widetilde{\gamma}(C_i)\|_t \geq \|\widetilde{\gamma}(C_{i+1})\|_t$ and $\|\widetilde{\gamma}(D_i)\|_t \geq \|\widetilde{\gamma}(D_{i+1})\|_t$ for each

$i = 0, 1, \ldots, N-1$. So $\sum_{i=0}^{N} \frac{\|\widetilde{\gamma}(C_i)\|_t}{2^i}$ and $\sum_{i=0}^{N} \frac{\|\widetilde{\gamma}(D_i)\|_t}{2^i}$ are the first $N+1$ terms of the sums $L^{x,t,\mu}(\widetilde{\gamma}\restriction_{[0,s_1]})$ and $L^{x,t,\mu}(\widetilde{\gamma}\restriction_{[0,s_2]})$, respectively.

Note that

$$\|\widetilde{\gamma}(D_i)\|_t \geq \|\widetilde{\gamma}(C_i)\|_t \text{ for each } i = 0, 1, \ldots, N. \tag{1}$$

Moreover, there is some $j \in \{0, 1, \ldots, N-1\}$ such that $D_j \subset (s_1, s_2)$ and $\|\widetilde{\gamma}(D_j)\|_t = \mu$. Since such a component is absent in the list $C_0, \ldots, C_N$, we have $\|\widetilde{\gamma}(D_{i+1})\|_t \geq \|\widetilde{\gamma}(C_i)\|_t$ for each $i = j, \ldots, N-1$.

Now $\|\widetilde{\gamma}(D_j)\|_t = \mu \geq \frac{\varepsilon}{8}$, and $\|\widetilde{\gamma}(D_N)\|_t < \frac{\varepsilon}{16}$ by choice of N, so there must be some i between j and $N-1$ such that $\|\widetilde{\gamma}(D_i)\|_t > \|\widetilde{\gamma}(D_{i+1})\|_t + \frac{\varepsilon}{8N}$. Hence

$$\|\widetilde{\gamma}(D_i)\|_t > \|\widetilde{\gamma}(C_i)\|_t + \frac{\varepsilon}{8N}. \tag{2}$$

It follows from (1) and (2) that

$$\begin{aligned} L^{x,t,\mu}(\widetilde{\gamma}\restriction_{[0,s_2]}) &> L^{x,t,\mu}(\widetilde{\gamma}\restriction_{[0,s_1]}) + \frac{\varepsilon/8N}{2^i} \\ &> L^{x,t,\mu}(\widetilde{\gamma}\restriction_{[0,s_1]}) + \frac{\varepsilon/8N}{2^N} \\ &= L^{x,t,\mu}(\widetilde{\gamma}\restriction_{[0,s_1]}) + \frac{\varepsilon}{2^{N+3} \cdot N} \end{aligned}$$

Noting that the measure of W is $1 \cdot \frac{1}{2} \cdot (\frac{\varepsilon}{4} - \frac{\varepsilon}{8}) = \frac{\varepsilon}{16}$, it follows that

$$\begin{aligned} \mathsf{len}(\widetilde{\gamma}\restriction_{[0,s_2]}) &\geq \mathsf{len}(\widetilde{\gamma}\restriction_{[0,s_1]}) + \frac{\varepsilon}{2^{N+3} \cdot N} \cdot \frac{\varepsilon}{16} \\ &= \mathsf{len}(\widetilde{\gamma}\restriction_{[0,s_1]}) + \delta. \end{aligned}$$

But this contradicts the assumption that $s_2 - s_1 < \delta$.

Thus for every $[\gamma] \in \mathcal{F}$, if $0 \leq s_1 < s_2 \leq 1$ with $s_2 - s_1 < \delta$, then $\rho(\widetilde{\gamma}(s_1), \widetilde{\gamma}(s_2)) < \varepsilon$.

For the converse, suppose there is some $\varepsilon > 0$ such that for any positive integer N, there exists a path γ_N with $[\gamma_N] \in \mathcal{F}$ and a collection of N disjoint subintervals of $[0,1]$ whose images under γ_N have diameters $\geq \varepsilon$. Note that at least one of these subintervals must have width $\leq \frac{1}{N}$; denote it by J_N.

Let $s \in [0,1]$ be an accumulation point of the centers of the intervals J_N, $N = 1, 2, 3, \ldots$. Then for any $\delta > 0$, there is some N such that $J_N \subset (s - \delta, s + \delta)$, and hence $\gamma_N((s - \delta, s + \delta))$ has diameter $\geq \varepsilon$. Thus we cannot choose parameterizations of the paths in $\mathcal{F}$ to obtain an equicontinuous family. $\square$

Theorem 4.3 implies in particular that if it is possible to parameterize the paths of a family $\mathcal{F}$ to obtain an equicontinuous family, then the standard parameterization will accomplish this.

Theorem 4.4. *Let $\mathcal{F} \subseteq \Pi$. Then $\overline{\mathcal{F}}$ is compact if and only if the following two properties are satisfied:*

(1) the set $\{\gamma(0) : [\gamma] \in \mathcal{F}\}$ is bounded; and
(2) $\mathcal{F}$ satisfies the property (†) (from Theorem 4.3).

In particular, if $\mathcal{F}$ satisfies properties (1) and (2), then the closure of $\widetilde{\mathcal{F}}$ in $\mathcal{C}[0,1]$ is compact.

Proof. By the Arzelà-Ascoli theorem [5, Theorem 4.43], the closure of $\widetilde{\mathcal{F}}$ is compact if and only if $\widetilde{\mathcal{F}}$ is equicontinuous and pointwise bounded, i.e. for every $t \in [0,1]$ the set $\{\widetilde{\gamma}(t) : \widetilde{\gamma} \in \widetilde{\mathcal{F}}\}$ is bounded. By Theorem 4.3, equicontinuity of $\widetilde{\mathcal{F}}$ is equivalent to $\mathcal{F}$ satisfying the property (†). Moreover, in the presence of (†), the condition (1) is clearly equivalent to $\widetilde{\mathcal{F}}$ being pointwise bounded.

Finally, by Theorem 4.1, $\overline{\mathcal{F}}$ is compact if and only if the closure of $\widetilde{\mathcal{F}}$ is compact. □

5. Midpoint parameterization

One drawback to the standard parameterization of a path, introduced in the previous section, is that it does not commute with reversal of orientation of a path. That is, if we define $r : [0,1] \to [0,1]$ by $r(t) = 1-t$, then for a non-constant path γ, the standard parameterization of $\gamma \circ r$ is not the same as $\widetilde{\gamma} \circ r$.

In this section we introduce a second parameterization of a path and consider some applications, including a way to canonically define a homeomorphism between two arcs (or even between two families of arcs) once the endpoints have been assigned.

Definition 3. Given a path γ, the *midpoint parameterization* $\gamma^* : [0,1] \to \mathbb{C}$ is defined as follows. Let $m \in (0,1)$ be such that $\mathsf{len}(\gamma\restriction_{[0,m]}) = \mathsf{len}(\gamma\restriction_{[m,1]}) = L$. Define $\gamma_1, \gamma_2 : [0,1] \to \mathbb{C}$ by $\gamma(t) = \gamma_1(m - mt)$ and $\gamma_2(t) = \gamma(m + (1-m)t)$, and consider their standard parameterizations $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2 : [0,1] \to \mathbb{C}$. Then

$$\gamma^*(t) = \begin{cases} \widetilde{\gamma}_1(1-2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \widetilde{\gamma}_2(2t-1) & \text{if } \frac{1}{2} < t \leq 1 \end{cases}$$

Observe that $\gamma^* \circ r = (\gamma \circ r)^*$.

As with the standard parameterization, the midpoint parameterization is unique within each equivalence class of paths, in the sense that if $\gamma_1 \approx \gamma_2$, then $\gamma_1^* = \gamma_2^*$.

We leave it to the reader to see that the following analogue of Theorem 4.1 also holds (where the standard parameterization is replaced by the midpoint paramterization). We denote by Π^* the set of all midpoint parameterizations of paths $[0,1] \to \mathbb{R}^n$. Recall that the topology on Π is given by the metric ρ and that Π^* is a subspace of $\mathcal{C}[0,1] = \mathcal{C}([0,1], \mathbb{R}^n)$ with the metric $d_{\sup}(\gamma_1, \gamma_2) = \sup_{t \in [0,1]} |\gamma_1(t) - \gamma_2(t)|$.

Theorem 5.1. *Π^* is a closed subset of $\mathcal{C}[0,1]$, and the function $[\gamma] \mapsto \gamma^*$ is a homeomorphism from Π to Π^*.*

An *arc* is a space A which is homeomorphic to the interval $[0,1]$. By a *parametrization of an arc* A we mean a homeomorphism $\gamma : [0,1] \to A$.

Given two arcs A_1 and A_2, and a bijection f between their endpoint sets, we can extend f to a canonical homeomorphism $F : A_1 \to A_2$ as follows. Choose a parameterization γ_1 of A_1, and a parameterization γ_2 of A_2 such that $\gamma_2(0) = f(\gamma_1(0))$ and $\gamma_2(1) = f(\gamma_1(1))$. Then given $x \in A_1$, define $F(x) = \gamma_2^*((\gamma_1^*)^{-1}(x))$. Observe that F is independent of the choice of orientation of γ_1, since $\gamma_1^* \circ r = (\gamma_1 \circ r)^*$ and $\gamma_2^* \circ r = (\gamma_2 \circ r)^*$ (hence the word "canonical").

In the next result, we show that this canonical homeomorphism has some useful convergence properties.

The following definition is inspired by the concept of a *lamination*, which appears in dynamics (see e.g. [13]) and plane topology (see e.g. [11]). In those contexts one defines a space X by starting with a smaller "base" space, which we call $B(X)$ here, then adding a family of arcs with endpoints attached to $B(X)$, with disjoint interiors and which converge nicely to each other.

Definition 4. A *laminated space* is a space $X \subset \mathbb{R}^n$ together with a closed (in X) subspace $B(X)$ and a collection $\mathcal{A}(X)$ of arcs in X, such that:

(1) $X = B(X) \cup \bigcup \mathcal{A}(X)$;
(2) each arc in $\mathcal{A}(X)$ has endpoints in $B(X)$, but is otherwise disjoint from $B(X)$;
(3) any two distinct arcs in $\mathcal{A}(X)$ meet at most in one common endpoint;
(4) given a sequence $\langle A_i \rangle_{i=1}^{\infty}$ of arcs in $\mathcal{A}(X)$:
    (a) if $\mathrm{diam} A_i \to 0$, then the set of accumulation points in $\mathbb{R}^n$ of $\bigcup_i A_i$ is disjoint from $X \setminus B(X)$;

(b) otherwise, there is an arc $A_\infty \in \mathcal{A}(X)$, a subsequence $\langle A_{i_j}\rangle_{j=1}^\infty$ and homeomorphisms $h_j : A_\infty \to A_{i_j}$ such that $d_{\sup}(h_j, \mathrm{id}_{A_\infty}) \to 0$ as $j \to \infty$.

Note that the conclusion in condition (4)(b) is equivalent to the statement that $A_\infty \cup \bigcup_j A_{i_j}$ is homeomorphic to the product of $[0,1]$ and the convergent sequence $\{0\} \cup \{\frac{1}{n} : n = 1, 2, \ldots\}$.

Theorem 5.2. *Let X and Y be laminated spaces, and let $f : B(X) \to B(Y)$ be a continuous function. Suppose f maps the endpoints of any arc in $\mathcal{A}(X)$ onto the set of endpoints of some arc in $\mathcal{A}(Y)$. Then there exists a continuous extension $F : X \to Y$ of f which is one-to-one on each arc in $\mathcal{A}(X)$.*

Moreover, if additionally f is a homeomorphism and f^{-1} maps the endpoints of any arc in $\mathcal{A}(Y)$ to the endpoints of some arc in $\mathcal{A}(X)$, then F is a homeomorphism.

Proof. Let $\mathcal{P} = \{[\gamma] : \gamma$ parameterizes some arc in $\mathcal{A}(X)\} \subset \Pi$ and $\mathcal{Q} = \{[\lambda] : \lambda$ parameterizes some arc in $\mathcal{A}(Y)\} \subset \Pi$. Define the function $\mathbf{g} : \mathcal{P} \to \mathcal{Q}$ by $\mathbf{g}([\gamma]) = [\lambda]$ if $f(\gamma(0)) = \lambda(0)$ and $f(\gamma(1)) = \lambda(1)$.

Claim 5.2.1. $\mathbf{g}$ is continuous.

Proof of Claim 5.2.1. Suppose that $\langle [\gamma_i]\rangle_{i=1}^\infty$ is a sequence of elements of $\mathcal{P}$ converging to $[\gamma_\infty] \in \mathcal{P}$. By continuity of f, $\lim_{i\to\infty} f(\gamma_i(0)) = f(\gamma_\infty(0))$ and $\lim_{i\to\infty} f(\gamma_i(1)) = f(\gamma_\infty(1))$, and it follows from property (4) (for Y) that $\lim_{i\to\infty} \mathbf{g}([\gamma_i]) = \mathbf{g}([\gamma_\infty])$. □(Claim 5.2.1)

Define $F : X \to Y$ as follows. Given $A \in \mathcal{A}(X)$, choose γ parameterizing A. Let γ^* be the midpoint parameterization of γ, and let λ^* be the midpoint parameterization of $\mathbf{g}([\gamma])$. Now for each $x \in A$, define $F(x) = \lambda^*((\gamma^*)^{-1}(x))$.

Observe that the definition of F on an arc $A \in \mathcal{A}(X)$ does not depend on the choice of orientation of the parameterization γ of A, because $\gamma^* \circ r = (\gamma \circ r)^*$, $\lambda^* \circ r = (\lambda \circ r)^*$, and $\mathbf{g}([\gamma \circ r]) = [\lambda \circ r]$. Thus F is well-defined on each arc A. Moreover, if $x \in \mathcal{E}(\mathcal{A}(X))$, then $F(x) = f(x)$. Thus F is well-defined on X (since two arcs in $\mathcal{A}(X)$ meet at most in an endpoint), and extends f. It is also clear that F is one-to-one on any arc in $\mathcal{A}(X)$ since γ^* and λ^* are homeomorphisms.

Claim 5.2.2. F is continuous.

Proof of Claim 5.2.2. Suppose $\langle x_i\rangle_{i=1}^\infty$ is a sequence in X converging to $x_\infty \in X$. We will show that $f(x_i) \to f(x_\infty)$. By continuity of f and closedness of $B(X)$

in X, we may assume that the points x_i belong to $X \setminus B(X)$. For each i, let $A_i \in \mathcal{A}(X)$ be the arc containing x_i, and let $Q_i = F(A_i) \in \mathcal{A}(Y)$.

If $\mathrm{diam}(A_i) \to 0$, then by property (4) x_∞ is a point in $B(X)$. For each i, let x_i' be an endpoint of A_i. Then $x_i' \to x_\infty$, so by continuity of f, $f(x_i') \to f(x_\infty)$. Thus both endpoints of the arcs Q_i converge to $f(x_\infty)$, so again by (4) (for Y) we have $\mathrm{diam}(Q_i) \to 0$, and so $f(x_i) \to f(x_\infty)$.

Otherwise, by property (4) we may assume (by taking a subsequence of x_i), that there exists an arc $A_\infty \in \mathcal{A}(X)$ and homeomorphisms $h_i : A_\infty \to A_i$ such that $d_{\sup}(h_i, \mathrm{id}_{A_\infty}) \to 0$. Let $Q_\infty = F(A_\infty) \in \mathcal{A}(Y)$. For each $n \in \mathbb{N} \cup \{\infty\}$, let γ_n parameterize A_n, and let $\mathbf{g}([\gamma_n]) = [\lambda_n]$, so that λ_n parameterizes Q_n. We may assume (by choosing appropriate orientations) that $\gamma_i(0) \to \gamma_\infty(0)$ and $\gamma_i(1) \to \gamma_\infty(1)$. Then $[\gamma_i] \to [\gamma_\infty]$, and by continuity of $\mathbf{g}$, $[\lambda_i] \to [\lambda_\infty]$.

By Theorem 5.1, we have $\gamma_i^* \to \gamma_\infty^*$ and $\lambda_i^* \to \lambda_\infty^*$ uniformly. For each $n \in \mathbb{N} \cup \{\infty\}$, let $t_n \in [0,1]$ be such that $\gamma_n(t_n) = x_n$. Then by uniform convergence and continuity, $t_i \to t_\infty$, and $\lambda_i^*(t_i) \to \lambda_\infty^*(t_\infty)$. Thus

$$F(x_i) = \lambda_i^*((\gamma_i^*)^{-1}(x_i)) = \lambda_i^*(t_i) \to \lambda_\infty^*(t_\infty) = F(x_\infty)$$

as needed. □(Claim 5.2.2)

If, in addition, f is a homeomorphism and f^{-1} also maps the endpoints of any arc in $\mathcal{A}(Y)$ to the endpoints of some arc in $\mathcal{A}(X)$, then f^{-1} extends in the same way to a continuous function $Y \to X$ which is the inverse of F. Thus F is a homeomorphism. □

For a fixed laminated space $X \subset \mathbb{R}^n$, let

$$\begin{aligned} M = \{(Y,f) : \; & Y \subset \mathbb{R}^n \text{ is a bounded laminated space, and} \\ & f : B(X) \to B(Y) \text{ satisfies the hypotheses of Theorem 5.2}\}. \end{aligned}$$

Theorem 5.2 affords an operator Θ from M to the set $\mathcal{C}_b(X, \mathbb{R}^n)$ of bounded continuous functions from X into $\mathbb{R}^n$, where if $F = \Theta(Y,f)$ then $F(X) \subset Y$ and $F{\restriction}_{B(X)} = f$.

We next prove that this operator is continuous, in the sense that if Y_1 and Y_2 are nearby laminated spaces, and if $f_1 : B(X) \to B(Y_1)$ and $f_2 : B(X) \to B(Y_2)$ are close functions as in Theorem 5.2, then the extensions $F_1 : X \to Y_1$ and $F_2 : X \to Y_2$ are close as well. To make this precise, we define a metric $\mathbf{d}$ on M below. Let $(Y_1, f_1), (Y_2, f_2) \in M$.

First, given $A \in \mathcal{A}(X)$, let γ be a parameterization of A, and for $i = 1, 2$, let λ_i parameterize the corresponding arc in $\mathcal{A}(Y_i)$ with $\lambda_i(0) = f_i(\gamma(0))$ and

$\lambda_i(1) = f_i(\gamma(1))$. Define $s_A(f_1, f_2) = \rho([\lambda_1], [\lambda_2])$. Clearly $s_A(f_1, f_2)$ does not depend on the choice of parameterizations γ, λ_1, and λ_2.

Now define

$$\mathbf{d}\left((Y_1, f_1), (Y_2, f_2)\right) = d_{\sup}(f_1, f_2) + \sup\{s_A(f_1, f_2) : A \in \mathcal{A}(X)\}.$$

It is straightforward to verify that $\mathbf{d}$ is a metric on M. On $\mathcal{C}_b(X, \mathbb{R}^n)$, we use the metric $d_{\sup}$.

Theorem 5.3. *Let $X \subset \mathbb{R}^n$ be a laminated space, and let M be as defined above. The operator $\Theta : M \to \mathcal{C}_b(X, \mathbb{R}^n)$ given by Theorem 5.2 is continuous. Moreover, $\Theta(X, \mathrm{id}_{B(X)}) = \mathrm{id}_X$.*

Proof. Let $(Y_0, f_0) \in M$, and let $F_0 = \Theta(Y_0, f_0) \in \mathcal{C}_b(X, \mathbb{R}^n)$. Let $\varepsilon > 0$.

Since Y_0 is bounded, by the definition of a laminated space, the set $\{[\lambda] : \lambda$ parameterizes an arc $Q_0 \in \mathcal{A}(Y_0)$ with $\mathrm{diam}(Q_0) \geq \frac{\varepsilon}{2}\}$ is compact in Π. Therefore, by Theorem 5.1, there exists $\delta > 0$ such that if λ_0 parameterizes an arc in $\mathcal{A}(Y_0)$ of diameter $\geq \frac{\varepsilon}{2}$ and if λ is any path with $\rho([\lambda_0], [\lambda]) < \delta$, then $d_{\sup}(\lambda_0^*, \lambda^*) < \varepsilon$. We may assume that $\delta \leq \frac{\varepsilon}{2}$.

Observe that if $\mathrm{diam}(\lambda([0,1])) < \frac{\varepsilon}{2}$ and $\rho([\lambda_0], [\lambda]) < \frac{\varepsilon}{2}$, then every point in the range of λ_0 is within ε of each point in the range of λ, hence $d_{\sup}(\lambda_0^*, \lambda^*) < \varepsilon$ as well. Thus in fact for any arc $Q_0 \in \mathcal{A}(Y_0)$, if λ_0 parameterizes Q_0 and if λ is any path with $\rho([\lambda_0], [\lambda]) < \delta$, then $d_{\sup}(\lambda_0^*, \lambda^*) < \varepsilon$.

Let $(Y, f) \in M$ with $\mathbf{d}((Y_0, f_0), (Y, f)) < \delta$, and let $F = \Theta(Y, f)$.

Clearly, by definition of $\mathbf{d}$, for any point $x \in B(X)$ we have $|f_1(x) - f_2(x)| < \varepsilon$. Thus to confirm $d_{\sup}(F_1, F_2) < \varepsilon$, we need only consider points in $\bigcup \mathcal{A}(X)$.

Let $A \in \mathcal{A}(X)$, let γ parameterize A, and let λ_0 and λ parameterize the corresponding arcs $Q_0 \in \mathcal{A}(Y_0)$ and $Q \in \mathcal{A}(Y)$ with $\lambda_0(0) = f_0(\gamma(0))$, $\lambda_0(1) = f_0(\gamma(1))$, $\lambda(0) = f(\gamma(0))$, and $\lambda(1) = f(\gamma(1))$. By definition of $\mathbf{d}$, we have $\rho([\lambda_0], [\lambda]) < \delta$, hence by the choice of δ, $d_{\sup}(\lambda_0^*, \lambda^*) < \varepsilon$. Moreover, by the definition of F and F_0 from Theorem 5.2, $d_{\sup}(F_0{\restriction}_A, F{\restriction}_A) = d_{\sup}(\lambda_0^*, \lambda^*)$. Thus since A was arbitrary, we have $d_{\sup}(F_0, F) < \varepsilon$.

The second part of this Theorem is clear from the definition of Θ. $\square$

6. Generalized paths

To see that the assumption that X is a dendrite in Proposition 3.2 is necessary, consider the identity function $\mathrm{id}_{\mathbb{D}}$ on the unit disk $\mathbb{D} \subset \mathbb{C}$ with boundary circle $\mathbb{S}^1$. It is not difficult to see that $\mathsf{len}(\mathrm{id}_{\mathbb{D}}) < \mathsf{len}(\mathrm{id}_{\mathbb{S}^1})$.

Moreover, consider the embedding O of the circle $\mathbb{S}^1$ depicted in Figure 1. Let $\gamma : [0,1] \to O$ be a path which goes exactly once around the circle O, starting

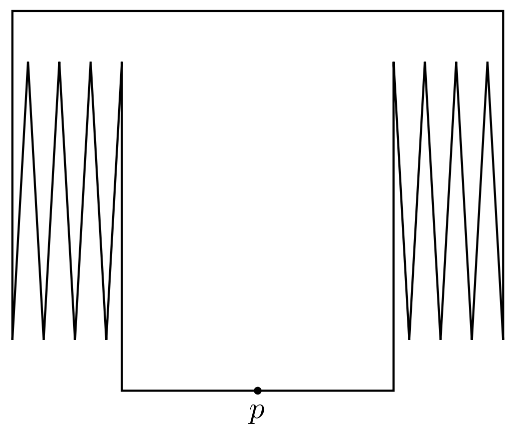

FIGURE 1. A particular embedding of the circle in the plane.

and ending at the indicated point p, and otherwise one-to-one. We claim that $\mathsf{len}(\gamma) < \mathsf{len}(\mathrm{id}_O)$, which can be argued as follows:

Given a strip $S_j^{x,t,\mu}$ containing the point p, the component C of $O \cap S_j^{x,t,\mu}$ containing p corresponds to two components $[0,c]$ and $[d,1]$ of $\gamma^{-1}(S_j^{x,t,\mu})$. For nearly horizontal strips (i.e. for t values close to 0 or 1) the sets $\mathrm{proj}_t^{\perp}(\gamma([0,c]))$ and $\mathrm{proj}_t^{\perp}(\gamma([d,1]))$ may overlap; however, because of the oscillation up and down on the left and right sides of the circle O, for such parameters x,t,μ there are many other components of $[0,1] \cap \gamma^{-1}(S_j^{x,t,\mu})$ and of $O \cap S_j^{x,t,\mu}$ ($j \in \mathbb{Z}$) with large projections, hence the weighted sums $L^{x,t,\mu}(\mathrm{id}_O)$ and $L^{x,t,\mu}(\gamma)$ will differ only very slightly. For all other values of x,t,μ, the sets $\mathrm{proj}_t^{\perp}(\gamma([0,c]))$ and $\mathrm{proj}_t^{\perp}(\gamma([d,1]))$ share only the point $\mathrm{proj}_t^{\perp}(p)$, and one of them will be added with a smaller weight in the sum $L^{x,t,\mu}(\gamma)$ than that of C in $L^{x,t,\mu}(\mathrm{id}_O)$. In particular, this is so for values of x,t,μ for which the strips $S_j^{x,t,\mu}$ are wide and nearly vertical, and for these values resulting difference between $L^{x,t,\mu}(\mathrm{id}_O)$ and $L^{x,t,\mu}(\gamma)$ will be more pronounced due to the small number of terms in these sums. Thus, with an appropriate amount of oscillation, we obtain that $\mathsf{len}(\gamma) < \mathsf{len}(\mathrm{id}_O)$.

Now if we let A be a very small arc in O containing the point p and such that $\mathsf{len}(\mathrm{id}_A) < \mathsf{len}(\mathrm{id}_O) - \mathsf{len}(\gamma)$, and let $A' = \overline{O \setminus A}$, then it follows that $\mathsf{len}(\mathrm{id}_O) > \mathsf{len}(\mathrm{id}_A) + \mathsf{len}(\mathrm{id}_{A'})$.

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(L. C. Hoehn) Nipissing University, Department of Computer Science & Mathematics, 100 College Drive, Box 5002, North Bay, Ontario, Canada, P1B 8L7

Email address: `loganh@nipissingu.ca`

(L. G. Oversteegen) University of Alabama at Birmingham, Department of Mathematics, Birmingham, AL 35294, USA

Email address: `overstee@uab.edu`

(E. D. Tymchatyn) University of Saskatchewan, Department of Mathematics and Statistics, 106 Wiggins road, Saskatoon, Canada, S7N 5E6

Email address: `tymchat@math.usask.ca`