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4D $\mathcal{N} = 1$ Kaluza-Klein superspace

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Abstract: Motivated by recent efforts to encode 11D supergravity in 4D $\mathcal{N} = 1$ superfields, we introduce a general covariant framework relevant for describing any higher dimensional supergravity theory in external 4D $\mathcal{N} = 1$ superspace with n additional internal coordinates. The superspace geometry admits both external and internal diffeomorphisms and provides the superfields necessary to encode the components of the higher dimensional vielbein, except for the purely internal sector, in a universal way that depends only on the internal dimension n. In contrast, the $\mathcal{N} = 1$ superfield content of the internal sector of the metric is expected to be highly case dependent and involve covariant matter superfields, with additional hidden higher dimensional Lorentz and supersymmetry transformations realized in a non-linear manner.

Contents

1 Introduction and motivation

A major difficulty in studying higher-dimensional supergravity theories is the absence of a (finite) off-shell formulation. This leads to a number of complications, a major one being the difficulty in writing down generic higher-derivative supersymmetric actions. This is in sharp contrast to the situations in lower dimensions (and fewer supersymmetries) where off-shell superspaces are available.

Standard techniques to address this always involve trade-offs. One can introduce infinite auxiliary fields, using harmonic superspace [1, 2] (which is related to projective superspace [3–8]) or pure spinor superspace [9–12], but the former does not seem applicable beyond six dimensions and the latter leads to a very complicated Batalin-Vilkovisky form whose on-shell component structure proves difficult to extract (see discussions in [13, 14]). One could take the opposite extreme – eliminating auxiliary fields altogether – by working in light cone superspace [15–18], but this breaks manifest Lorentz symmetry and leads to other complications – for example, having to work only with gauge-fixed physical degrees of freedom.

A plausible middle ground is to keep manifest some number of auxiliary fields and some amount of supersymmetry by working in some convenient low dimensional, low $\mathcal{N}$ superspace. 4D $\mathcal{N} = 1$ superspace is the obvious choice, given its relative simplicity and presence of certain features (e.g. holomorphic superpotentials) absent in even simpler superspaces. Already in 1983, Marcus, Sagnotti, and Siegel took this approach with the prototypical globally supersymmetric case by showing how to recast 10D super Yang-Mills in 4D $\mathcal{N} = 1$ language [19]. This breaks the 10D Lorentz group to $\mathrm{SO}(3,1) \times \mathrm{SU}(3) \times \mathrm{U}(1)$, but keeps off-shell 1/4 of the supersymmetry.[1]

The natural next step, discussed already in the conclusion of [19], would be to repeat the exercise for 11D supergravity, but as Marcus et al. noted even $\mathcal{N} = 2$ supergravity had not yet been fully written in $\mathcal{N} = 1$ superfields at that time. In the intervening 35 years, a number of papers have examined how to rewrite higher dimensional supergravity theories in $\mathcal{N} = 1$ superspace. These have included 4D $\mathcal{N} = 2$ supergravity [21–26], 5D $\mathcal{N} = 1$ supergravity [27–30], and 6D $\mathcal{N} = (1,0)$ supergravity [31, 32], but the 11D case has remained open.

[1] This was extended to lower dimensions in [20], motivated in part by brane-world scenarios.

In the last few years, that remaining case has been explored step-by-step. The initial papers [33, 34] identified the structure of the $\mathcal{N} = 1$ tensor hierarchy that descends from the M-theory 3-form and constructed the unique cubic $\mathcal{N} = 1$ Chern-Simons action. The superfields in this tensor hierarchy turned out to encode all the spin ≤ 1 fields. In particular, one of them contained a gauge-invariant 3-form field with which one can endow a Riemannian 7-manifold with a G_2 structure [35]. Remarkably, the $\mathcal{N} = 1$ Chern-Simons action, combined with a natural choice of Kähler potential, led to a 4D scalar potential that reproduces the internal sector of the 11D action. Most of the kinetic terms were also correctly reproduced, except for those terms involving fields in the **7** of G_2. The explanation offered in [35] was that the gravitino superfield, which encodes the additional seven gravitini, should include auxiliary vector and tensor fields that when integrated out modify the kinetic terms in the **7**. This was demonstrated indeed to be the case in [36] where the entire linearized action was written down in $\mathcal{N} = 1$ superspace. Then it was shown in [37] that the full action for fields of spin ≤ 1 could be linearly coupled to the gravitino and graviton supermultiplets consistently using its supercurrent. What remained was to include the gravitino and graviton couplings to all orders.

The main stumbling block to this task turns out to be $\mathcal{N} = 1$ superspace itself. Unlike in globally supersymmetric cases, the superspace covariant derivatives carry geometric data in their connections and these must be made dependent on the internal coordinates. Put another way, one must introduce new "internal" derivatives in $\mathcal{N} = 1$ superspace, and these must have non-trivial commutators with the "external" superspace derivatives. This introduces anew the old problem of solving superspace Bianchi identities, but with the added wrinkle of an additional set of coordinates and a slew of new superfields describing the mixed curvatures.

It turns out this can be done in a rather universal way, which seems as applicable to minimal 5D supergravity as to 11D supergravity, although the details of intermediate cases have not yet been worked out. In this paper, we provide such a generic reformulation of 4D $\mathcal{N} = 1$ superspace with n additional internal coordinates. (Our interest is $n = 7$, of course, but the formulae are agnostic to the specific choice.) Our construction will be motivated by the requirement that it consistently covariantize the 11D supergravity results. It will also make contact with existing 5D [30] and 6D results [32], where the linearized version of this supergeometry was built explicitly out of prepotential superfields.

This paper is organized as follows. In section 2, we review some details of how 11D supergravity is recast in 4D $\mathcal{N} = 1$ language to motivate a number of choices we will make for the Kaluza-Klein supergeometry. Section 3 is devoted to a general discussion of bosonic Kaluza-Klein geometry that readily generalizes to superspace. In sections 4 through 6, we discuss how to solve the superspace Bianchi identities. Section 4 provides a general discussion in terms of abstract curvature superfields and shows how, with a certain minimal set of constraints, the Bianchi identities can be rewritten in terms of simpler abstract curvature operators. This leads to a set of six abstract operator equations that must be satisfied. In section 5, we discuss the linearized solution to these identities, and then in section 6, we address the full solution. The existence of full superspace and chiral superspace actions is

established in section 7. In the conclusion, we sketch the remaining steps needed to rewrite 11D supergravity in $\mathcal{N}=1$ language, which will be the subject of a subsequent publication.

2 Elements of 11D supergravity and an $\mathcal{N}=1$ wishlist

In order to lay the groundwork for the Kaluza-Klein superspace we will construct, it will be helpful to sketch what is currently known about the rewriting of 11D supergravity in $\mathcal{N}=1$ language [33–37]. (See the introduction of [37] for more details.)

2.1 A sketch of 11D supergravity

Locally, we decompose 11D spacetime into four external coordinates x^m and seven internal coordinates $y^{\underline{m}}$. Four local supersymmetries are made manifest by introducing local Grassmann coordinates $(\theta^\mu, \bar\theta_{\dot\mu})$, which are combined with x^m to give an external 4D $\mathcal{N}=1$ superspace. The 11D spectrum comprises a metric, 3-form, and a 32-component gravitino, each of which must be decomposed into 4D $\mathcal{N}=1$ multiplets. The structure of the 3-form is easiest to understand as its abelian gauge structure has a unique encoding in $\mathcal{N}=1$ superspace. The component form decomposes directly into a tensor hierarchy of forms

$$C_3 \quad\to\quad C_{\underline{mnp}}\ ,\quad C_{m\,\underline{np}}\ ,\quad C_{mn\,\underline{p}}\ ,\quad C_{mnp} \tag{2.1}$$

extending from a 0-form to a 3-form in external spacetime. The $\mathcal{N}=1$ superspace encoding of such a tensor hierarchy is known. In terms of $\mathcal{N}=1$ superfields, it comprises a chiral superfield $\Phi_{\underline{mnp}}$, a real vector superfield $V_{\underline{mn}}$, a chiral spinor superfield $\Sigma_{\underline{m}\,\alpha}$, and a real superfield X. They transform under abelian gauge transformations as (in form notation)

$$\delta\Phi = \partial\Lambda\ , \tag{2.2a}$$
$$\delta V = \tfrac{1}{2i}(\Lambda-\bar\Lambda) - \partial U\ , \tag{2.2b}$$
$$\delta\Sigma_\alpha = -\tfrac{1}{4}\bar{\mathcal{D}}^2\mathcal{D}_\alpha U + \partial\Upsilon_\alpha + \mathcal{W}_\alpha \lrcorner \Lambda\ , \tag{2.2c}$$
$$\delta X = \tfrac{1}{2i}(\mathcal{D}^\alpha\Upsilon_\alpha - \bar{\mathcal{D}}_{\dot\alpha}\bar\Upsilon^{\dot\alpha}) - \omega_{\mathsf{h}}(\mathcal{W}_\alpha, U) \tag{2.2d}$$

with $\lrcorner$ and ∂ denoting the interior product and de Rham differential on the internal space, and where we have used the shorthand

$$\omega_{\mathsf{h}}(\chi_\alpha, v) := \chi^\alpha \lrcorner \mathcal{D}_\alpha v + \bar\chi_{\dot\alpha}\lrcorner\bar{\mathcal{D}}^{\dot\alpha}v + \frac{1}{2}\left(\mathcal{D}^\alpha\chi_\alpha\lrcorner v + \bar{\mathcal{D}}_{\dot\alpha}\bar\chi^{\dot\alpha}\lrcorner v\right)\ . \tag{2.3}$$

The gauge parameters are a chiral superfield $\Lambda_{\underline{mn}}$, a real superfield $U_{\underline{m}}$, and a chiral spinor superfield Υ_α. The 4D $\mathcal{N}=1$ derivatives involve a Kaluza-Klein connection, $\mathcal{D} := \mathrm{d} - \mathscr{L}_{\mathcal{A}}$, which acts via the internal Lie derivative, i.e. $\mathscr{L}_{\mathcal{A}}U := \mathcal{A}\lrcorner\partial U + \partial(\mathcal{A}\lrcorner U)$. The prepotential $\mathcal{V}^{\underline{m}}$ for the connection $\mathcal{A}$ describes the $\mathcal{N}=1$ vector multiplet that includes the Kaluza-Klein vector component of the higher dimensional vielbein. In this covariant formulation, the Kaluza-Klein prepotential does not appear explicitly, but rather only via the covariant derivative and its chiral field strength $\mathcal{W}_\alpha{}^{\underline{m}}$ obeying the usual Bianchi identities,

$$\bar{\mathcal{D}}_{\dot\alpha}\mathcal{W}_\alpha{}^{\underline{m}} = 0\ , \qquad \mathcal{D}^\alpha\mathcal{W}_\alpha{}^{\underline{m}} = \bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha\underline{m}}\ . \tag{2.4}$$

The field strength superfields invariant under the gauge transformations (2.2) are given in form notation as

$$E = \partial\Phi \ , \tag{2.5a}$$
$$F = \tfrac{1}{2i}(\Phi - \bar{\Phi}) - \partial V \ , \tag{2.5b}$$
$$W_\alpha = -\tfrac{1}{4}\bar{\mathcal{D}}^2\mathcal{D}_\alpha V + \partial\Sigma_\alpha + \mathcal{W}_\alpha \lrcorner \Phi \ , \tag{2.5c}$$
$$H = \tfrac{1}{2i}(\mathcal{D}^\alpha\Sigma_\alpha - \bar{\mathcal{D}}_{\dot\alpha}\bar{\Sigma}^{\dot\alpha}) - \partial X - \omega_{\mathsf{h}}(\mathcal{W}_\alpha, V) \ , \tag{2.5d}$$
$$G = -\tfrac{1}{4}\bar{\mathcal{D}}^2 X + \mathcal{W}^\alpha \lrcorner \Sigma_\alpha \tag{2.5e}$$

and satisfy Bianchi identities

$$0 = \partial E \ , \tag{2.6a}$$
$$0 = \tfrac{1}{2i}(E - \bar{E}) - \partial F \ , \tag{2.6b}$$
$$0 = -\tfrac{1}{4}\bar{\mathcal{D}}^2\mathcal{D}_\alpha F + \partial W_\alpha + \mathcal{W}_\alpha \lrcorner E \ , \tag{2.6c}$$
$$0 = \tfrac{1}{2i}(\mathcal{D}^\alpha W_\alpha - \bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\alpha}) - \partial H - \omega_{\mathsf{h}}(\mathcal{W}_\alpha, F) \ , \tag{2.6d}$$
$$0 = -\tfrac{1}{4}\bar{\mathcal{D}}^2 H + \partial G + \mathcal{W}^\alpha \lrcorner W_\alpha \ . \tag{2.6e}$$

From these superfields, one can construct the $\mathcal{N} = 1$ supersymmetrization of the 11D Chern-Simons term:

$$\begin{aligned} -12\kappa^2\, S_{CS} = \int \mathrm{d}^{11}x\, \mathrm{d}^2\theta \Big\{ & i\Phi \wedge \Big(EG + \frac{1}{2} W^\alpha \wedge W_\alpha - \frac{i}{4}\bar{\mathcal{D}}^2(F \wedge H) \Big) \\ & + i\Sigma^\alpha \wedge \Big(E \wedge W_\alpha - \frac{i}{4}\bar{\mathcal{D}}^2(F \wedge \mathcal{D}_\alpha F) \Big) \Big\} \\ + \int \mathrm{d}^{11}x\, \mathrm{d}^4\theta \Big\{ & V \wedge \Big(E \wedge H + F \wedge \mathcal{D}^\alpha W_\alpha + 2\mathcal{D}^\alpha F \wedge (W_\alpha - i\mathcal{W}_\alpha \lrcorner F) \Big) \\ & - XE \wedge F \Big\} \\ + \text{h.c.} & \end{aligned} \tag{2.7}$$

It turns out that the on-shell field content of the $\mathcal{N} = 1$ superfields above involves more than just the 3-form fields. They also encode all of the spin-1/2 components of the 11D gravitino and all components of the 11D metric except for the purely external part. It also turns out that the above superspace Chern-Simons action encodes the kinetic terms for the 4D vector fields. Together with just one other Kähler-type term (whose precise form does not concern us here), nearly the entire action of 11D supergravity for the spin ≤ 1 fields can be encoded in $\mathcal{N} = 1$ superspace [35].

The above description turns out to miss a few critical elements. The external graviton and spin-3/2 part of the gravitino must belong to additional higher superspin multiplets. Naturally, the $\mathcal{N} = 1$ gravitino combines with the external graviton into a single supermultiplet, described by a prepotential superfield $H_{\alpha\dot\alpha} = (\sigma^a)_{\alpha\dot\alpha} H_a$, subject to a linearized gauge transformation

$$\delta H_{\alpha\dot\alpha} = D_\alpha \bar{L}_{\dot\alpha} - \bar{D}_{\dot\alpha} L_\alpha \ . \tag{2.8}$$

This is the linearized prepotential of $\mathcal{N}=1$ conformal supergravity. The remaining seven spin-3/2 components of the gravitino live in a superfield $\Psi_{\underline{m}\alpha}$, subject to the linearized transformations

$$\delta\Psi_{\underline{m}\alpha} = \Xi_{\underline{m}\alpha} + D_\alpha \Omega_{\underline{m}} + 2i\,\partial_{\underline{m}} L_\alpha \ , \tag{2.9}$$

where $\Xi_{\underline{m}\alpha}$ is a chiral spinor and $\Omega_{\underline{m}}$ is an unconstrained real superfield. This describes the so-called $\mathcal{N}=1$ conformal graviton multiplet.

The parameter L_α encodes local $\mathcal{N}=1$ superconformal transformations, while Ξ and Ω are usually interpreted as encoding extended supersymmetry. The matter fields of the tensor hierarchy necessarily also vary under these transformations, but the precise form will not concern us here, except for the following observation. As discussed in [37], the Ξ transformations of Ψ and the other matter fields take a very simple form and do not strongly constrain the action. Therefore, we are going to take the point of view that Ξ *is not really an extended supersymmetry transformation*, but rather a symmetry naturally associated with the prepotential structure of the $\mathcal{N}=1$ superspace we want to construct. The transformation involving Ω will then be interpreted as an honest extended supersymmetry transformation. Since any such transformation necessarily breaks manifest $\mathcal{N}=1$ supersymmetry, it will not play any further role in our discussion.

If we ignore the Ω parameter in the linearized transformation (2.9), it is possible to combine the linearized $H_{\alpha\dot\alpha}$ and $\Psi_{\underline{m}\alpha}$ into an abelian tensor hierarchy, just like the 3-form fields, where the Kaluza-Klein gauge field appears encoded in the covariant derivatives. It turns out that a further chiral spinor superfield $\Phi_{\underline{mn}\alpha}$ is needed. The linearized gauge transformations read

$$\delta H_{\alpha\dot\alpha} = \mathcal{D}_\alpha \bar L_{\dot\alpha} - \bar{\mathcal{D}}_{\dot\alpha} L_\alpha \ , \tag{2.10a}$$

$$\delta\Psi_{1\alpha} = \Xi_{1\alpha} + 2i\,\partial L_\alpha \ , \tag{2.10b}$$

$$\delta\Phi_{2\alpha} = -\partial\,\Xi_{1\alpha} \tag{2.10c}$$

where we have written the internal degrees of the forms explicitly. The derivatives $\mathcal{D}$ include the Kaluza-Klein connection as with the matter fields. The corresponding curvatures are

$$W_{\gamma\beta\alpha} = \frac{1}{16}\bar{\mathcal{D}}^2\Big[i\mathcal{D}_\gamma{}^{\dot\alpha}\mathcal{D}_\beta H_{\alpha\dot\alpha} + \mathcal{D}_\gamma \mathcal{W}_{\beta\lrcorner}\Psi_{1\alpha} - 2\mathcal{W}_{\gamma\lrcorner}\mathcal{D}_\beta\Psi_{1\alpha}\Big]_{(\gamma\beta\alpha)} + \frac{1}{2i}\mathcal{W}_{(\gamma\lrcorner}\mathcal{W}_{\beta\lrcorner}\Phi_{2\alpha)} \ , \tag{2.11a}$$

$$X_{1\alpha\dot\alpha} = \frac{1}{2i}(\bar{\mathcal{D}}_{\dot\alpha}\Psi_{1\alpha} + \mathcal{D}_\alpha\bar\Psi_{1\dot\alpha}) + \partial H_{\alpha\dot\alpha} \ , \tag{2.11b}$$

$$\Psi_{2\alpha} = \Phi_{2\alpha} + \partial\Psi_{1\alpha} \ , \tag{2.11c}$$

$$\Phi_{3\alpha} = \partial\Phi_{2\alpha} \ , \tag{2.11d}$$

and they satisfy the Bianchi identities[2]

$$\partial W_{\gamma\beta\alpha} = \bar{\mathcal{D}}^2\Big[\frac{i}{16}\mathcal{D}_\gamma{}^{\dot\alpha}\mathcal{D}_\beta X_{1\alpha\dot\alpha} - \frac{1}{16}\mathcal{D}_\gamma\mathcal{W}_{\beta}\lrcorner\Psi_{2\alpha} + \frac{1}{8}\mathcal{W}_\gamma\lrcorner\mathcal{D}_\beta\Psi_{2\alpha}\Big]_{(\gamma\beta\alpha)} + \frac{1}{2i}\mathcal{W}_{(\gamma}\lrcorner\mathcal{W}_{\beta}\lrcorner\Phi_{3\alpha)} \ , \tag{2.12a}$$

$$\partial X_{1\alpha\dot\alpha} = \frac{1}{2i}(\mathcal{D}_\alpha\bar\Psi_{2\dot\alpha} + \bar{\mathcal{D}}_{\dot\alpha}\Psi_{2\alpha}) \ , \tag{2.12b}$$

$$\partial\Psi_{2\alpha} = \Phi_{3\alpha} \ , \tag{2.12c}$$

$$\partial\Phi_{3\alpha} = 0 \ . \tag{2.12d}$$

The chiral spinor $\Phi_{2\,\alpha}$ is necessary to ensure gauge invariance of the curvatures under Ξ transformations. The Ξ transformations are important because they preserve the "gauge-for-gauge" symmetry of L_α, whereby a shift of L_α by a chiral spinor superfield can always be balanced by some compensating gauge transformation elsewhere. For the case of $\Psi_{\underline{m}\alpha}$, this requires that $\Xi_{\underline{m}\alpha}$ shift by the internal derivative of that chiral spinor. This is another reason to consider Ξ transformations as part of the purely $N = 1$ sector and not an honest extended supersymmetry transformation.

It is puzzling that $\Phi_{2\alpha}$ was not encountered in our prior linearized analysis [36] or in the supercurrent analysis [37]. The only sensible explanation is that when the prepotentials above are coupled to the tensor hierarchy fields of 11D supergravity, it becomes possible to absorb $\Phi_{2\alpha}$ by a field redefinition. We will argue in the conclusion that this is indeed so.

2.2 In search of a covariant completion

The complete action involving the spin ≤ 1 fields was presented in [37], but this was only to linear order in $H_{\alpha\dot\alpha}$ and $\Psi_{\underline{m}\alpha}$. The main obstruction was that unlike the tensor hierarchy fields, $H_{\alpha\dot\alpha}$ and $\Psi_{\underline{m}\alpha}$ are expected to appear intrinsically non-polynomially in the action, just like the Kaluza-Klein prepotential $\mathcal{V}^{\underline{m}}$.[3] The solution to this should be, just as with $\mathcal{V}^{\underline{m}}$, to introduce new covariant derivatives in which these prepotentials are encoded, so that they appear only via minimal substitution and their associated field strengths. When $H_{\alpha\dot\alpha}$ is y-independent and decouples from the supergravity hierarchy, this can be done using $\mathcal{N} = 1$ conformal superspace [38], where new covariant derivatives ∇_A are introduced so that the only field strength is the $\mathcal{N} = 1$ super-Weyl tensor $W_{\alpha\beta\gamma}$. Then the L_α *pregauge* transformations of the prepotential $H_{\alpha\dot\alpha}$ are absorbed into superdiffeomorphisms and all terms are manifestly supercovariant. However, if the supergraviton multiplet depends on $y^{\underline{m}}$, the corresponding supergeometry must be rebuilt from scratch to accommodate this. This means that we must look for an $\mathcal{N} = 1$ superspace involving not only the usual covariant external derivatives $\nabla_A = (\nabla_a, \nabla_\alpha, \bar\nabla^{\dot\alpha})$ but also curved internal derivatives. We can motivate the constraints

[2]There is an additional Bianchi identity of the form $\mathcal{D}_{\dot\alpha}{}^\beta\mathcal{D}^\gamma W_{\gamma\beta\alpha} + \mathcal{D}_\alpha{}^{\dot\beta}\bar{\mathcal{D}}^{\dot\gamma}\bar W_{\dot\gamma\dot\beta\dot\alpha} = \cdots$, but this occurs at rather high dimension, so we won't have much use for it.

[3]One could adopt a Wess-Zumino gauge and work to a certain order in these prepotentials, but that is cumbersome in practice.

and structure of this new superspace by requiring it to consistently covariantize the matter and actions we already have.

For example, one might expect that the internal derivatives should be valued in the internal tangent space group, e.g. $\nabla_{\underline{a}}$ with $\underline{a}$ a vector of SO(n), but this would lead to an immediate complication: we would need to introduce an internal vielbein $e_{\underline{m}}{}^{\underline{a}}$ absent in [37]. Now, this is not an independent field, but is equivalent (up to an SO(n) transformation) to the internal metric $g_{\underline{mn}}$; however, the internal metric is, in our approach, a composite field related to the tensor hierarchy field strength $F_{\underline{mnp}}$ via the G_2 structure relation

$$\sqrt{g}\, g_{\underline{mn}} = -\frac{1}{144}\varepsilon^{\underline{p_1\cdots p_7}}\, F_{\underline{m\, p_1 p_2}} F_{\underline{p_3 p_4 p_5}} F_{\underline{p_6 p_7\, n}} \ . \tag{2.13}$$

This equation is complicated enough without trying to take its square root to define $e_{\underline{m}}{}^{\underline{a}}$. So we should avoid introducing any SO(n) symmetry and take our internal derivative to carry a GL(n) index,

$$\nabla_{\underline{m}} = \partial_{\underline{m}} + \cdots \ , \tag{2.14}$$

where new connection terms (which may include external derivatives) must be added. Presumably such terms would be absent in flat space.

Let's impose a number of additional conditions for the superspace we seek:

- In the absence of an internal vielbein, all the matter fields should be p-forms on the internal manifold. This includes chiral superfields $\Phi_{\underline{mnp}}$ and $\Sigma_{\underline{m}\,\alpha}$. The $\mathcal{N}=1$ superspace geometry must support the existence of such chiral superfields. This implies certain constraints on the torsion and curvatures of the $\mathcal{N}=1$ derivatives ∇_A. In addition, the structure of the Chern-Simons action requires the existence of chiral superspace in addition to full superspace, leading to additional restrictions.

- In flat space, the Chern-Simons action as well as the entire structure of the 3-form tensor hierarchy is written in differential form notation, and the internal derivative appears only via the de Rham differential. We assume that the same should be true of its curved space version. This means that we should only expect to build covariant quantities using suitable internal covariant de Rham differentials.

- A related point follows: we should avoid introducing a metric-compatible affine connection into $\nabla_{\underline{m}}$, since we're going to use only covariant de Rham differentials. This means the composite $g_{\underline{mn}}$ should play no role in the superspace derivatives. In fact, we're going to assume that *none* of the superfields of the tensor hierarchy play a role in the construction of this superspace. They should appear only as consistent "matter", that is covariant superfields consistent with but not required by the supergeometry. This means that the underlying supergeometry should be built only out of four fundamental prepotentials, $H_{\alpha\dot\alpha}$, $\Psi_{\underline{m}\alpha}$, $\Phi_{\underline{mn}\alpha}$, and $\mathcal{V}^{\underline{m}}$. The curvatures of the superspace geometry should be built only out of their five corresponding field strengths, $W_{\alpha\beta\gamma}$, $X_{\underline{m}\alpha\dot\alpha}$, $\Psi_{\underline{mn}\alpha}$, $\Phi_{\underline{mnp}\alpha}$, and $\mathcal{W}_\alpha{}^{\underline{m}}$.

- The tensor hierarchy transformation of $\Phi_{\underline{mnp}}$ should be something like $\delta\Phi_3 = \nabla_1\Lambda_2$ where $\Lambda_{\underline{mn}}$ is also chiral. This means that $\nabla_{\underline{m}}$ should preserve chirality. But if $\nabla_{\underline{m}}$ were to commute with both $\bar{\nabla}_{\dot{\alpha}}$ and ∇_α, then it would commute with ∇_a as well, and the supergeometry would trivialize to be y-independent. The only way to make sense of this is to assume $\delta\Phi_3 = \nabla_1^+\Lambda_2$ where $\nabla^+_{\underline{m}}$ is a modified *complex* version of $\nabla_{\underline{m}}$ that preserves chirality. Similarly, there will be a $\nabla^-_{\underline{m}}$ for anti-chiral superfields, with $\nabla^-_{\underline{m}} = (\nabla^+_{\underline{m}})^*$. These should also look roughly like (2.14), meaning that they differ from each other and from $\nabla_{\underline{m}}$ only by connection terms. In fact, such derivatives have already been built at the linearized level to describe 5D [30] and 6D supergravity [32] in $N = 1$ superspace.

3 Kaluza-Klein (super)geometry

The first step towards constructing a Kaluza-Klein supergeometry is to understand its differential geometry. It will be useful to first review how Kaluza-Klein decompositions work in more familiar bosonic spaces. Extending these results to superspace amounts to just extending commuting world and tangent space coordinates to include anticommuting ones, i.e. replacing $m \to M$ and $a \to A$ below. This makes no difference in formulae but complicates notation, so we will restrict to a bosonic space here for clarity.

In addition, this will allow us to overload notation in this section and use M and A as indices for the coordinates and tangent space of the higher-dimensional theory that we are decomposing. Since these higher-dimensional coordinates will only appear here, we hope there will be no confusion later on when we restore super-indices.

3.1 Decomposition of the vielbein

Suppose we begin with some D-dimensional bosonic space with local coordinates $\hat{x}^M$. We are interested in locally decomposing this space into a d-dimensional "external space" with local coordinates x^m and an n-dimensional "internal space" with local coordinates $y^{\underline{m}}$. If the $D = (d+n)$-dimensional space is equipped with a vielbein $\hat{e}_M{}^A$, a natural choice for its decomposition is

$$\hat{e}_M{}^A = \begin{pmatrix} e_m{}^a + A_m{}^{\underline{m}}\chi_{\underline{m}}{}^a & A_m{}^{\underline{m}}e_{\underline{m}}{}^{\underline{a}} \\ \chi_{\underline{m}}{}^a & e_{\underline{m}}{}^{\underline{a}} \end{pmatrix} , \tag{3.1}$$

where we have split the tangent space index as $A = (a, \underline{a})$, with a and $\underline{a}$ indices associated to the external and internal tangent spaces. We assume that the external and internal vielbeins $e_m{}^a$ and $e_{\underline{m}}{}^{\underline{a}}$ are invertible, with inverses $e_a{}^m$ and $e_{\underline{a}}{}^{\underline{m}}$. Then the inverse of (3.1) is

$$\hat{e}_A{}^M = \begin{pmatrix} e_a{}^m & -e_a{}^m A_m{}^{\underline{m}} \\ -e_{\underline{a}}{}^{\underline{n}}\chi_{\underline{n}}{}^b e_b{}^m & e_{\underline{a}}{}^{\underline{m}} + e_{\underline{a}}{}^{\underline{n}}\chi_{\underline{n}}{}^b e_b{}^n A_n{}^{\underline{m}} \end{pmatrix} . \tag{3.2}$$

In conventional Kaluza-Klein scenarios, the higher dimensional tangent space is $\mathrm{SO}(D-1,1)$ and allows one to choose an upper triangular gauge where $\chi = 0$. But this choice will not

be available to us for two reasons. The first is that we are actually interested in the situation where x^m above is extended to include the θ variables of 4D $\mathcal{N}=1$ superspace – that is, x^m will be extended to $(x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}})$, with a extended similarly to $(a, \alpha, \dot{\alpha})$; in this case, the local symmetries are insufficient to fix all of χ to zero. The second reason is that manifest $\mathcal{N}=1$ supersymmetry actually seems to *require* even the bosonic part of χ to be non-zero, at least prior to a Wess-Zumino gauge fixing. We will elaborate on this momentarily.

The precise choice of decomposition above is motivated by how the fields transform under external and internal diffeomorphisms. We denote these transformations by ξ^m and $\Lambda^{\underline{m}}$ and embed them into higher dimensional diffeomorphisms $\hat{\xi}^M$ via $\hat{\xi}^m = \xi^m$ and $\hat{\xi}^{\underline{m}} = \Lambda^{\underline{m}} - \xi^m A_m{}^{\underline{m}}$. Under internal diffeomorphisms,

$$\delta_\Lambda e_m{}^a = \Lambda^{\underline{n}} \partial_{\underline{n}} e_m{}^a \ , \tag{3.3a}$$

$$\delta_\Lambda \chi_{\underline{m}}{}^a = \partial_{\underline{m}} \Lambda^{\underline{n}} \chi_{\underline{n}}{}^a + \Lambda^{\underline{n}} \partial_{\underline{n}} \chi_{\underline{m}}{}^a \ , \tag{3.3b}$$

$$\delta_\Lambda A_m{}^{\underline{m}} = \partial_m \Lambda^{\underline{m}} - A_m{}^{\underline{n}} \partial_{\underline{n}} \Lambda^{\underline{m}} + \Lambda^{\underline{n}} \partial_{\underline{n}} A_m{}^{\underline{m}} \equiv \hat{D}_m \Lambda^{\underline{m}} \ , \tag{3.3c}$$

$$\delta_\Lambda e_{\underline{m}}{}^{\underline{a}} = \partial_{\underline{m}} \Lambda^{\underline{n}} e_{\underline{n}}{}^{\underline{a}} + \Lambda^{\underline{n}} \partial_{\underline{n}} e_{\underline{m}}{}^{\underline{a}} \ . \tag{3.3d}$$

That is $e_m{}^a$ transforms as an internal scalar, while $\chi_{\underline{m}}{}^a$ and $e_{\underline{m}}{}^{\underline{a}}$ transform as internal 1-forms, and $A_m{}^{\underline{m}}$ transforms as a connection. Under external diffeomorphisms,

$$\delta_\xi e_m{}^a = \hat{D}_m \xi^n e_n{}^a + \xi^n \hat{D}_n e_m{}^a \ , \tag{3.4a}$$

$$\delta_\xi \chi_{\underline{m}}{}^a = \xi^n \hat{D}_n \chi_{\underline{m}}{}^a + \partial_{\underline{m}} \xi^n e_n{}^a \ , \tag{3.4b}$$

$$\delta_\xi A_m{}^{\underline{m}} = \xi^n F_{nm}{}^{\underline{m}} \ , \tag{3.4c}$$

$$\delta_\xi e_{\underline{m}}{}^{\underline{a}} = \xi^n \hat{D}_n e_{\underline{m}}{}^{\underline{a}} \ , \tag{3.4d}$$

where we have defined $\hat{D}_m := \partial_m - \delta_\Lambda(A_m)$ as the covariant external derivative (with ξ^m understood to be an internal scalar). The fields $e_m{}^a$ and $e_{\underline{m}}{}^{\underline{a}}$ transform as an external 1-form and a scalar respectively. $\chi_{\underline{m}}{}^a$ transforms as a scalar with an anomalous piece involving $\partial_{\underline{m}} \xi^n$, and $A_m{}^{\underline{m}}$ transforms as a connection with field strength $F_{nm}{}^{\underline{p}}$ given by

$$F_{nm}{}^{\underline{p}} := 2\, \partial_{[n} A_{m]}{}^{\underline{p}} - 2 A_{[n}{}^{\underline{n}} \partial_{\underline{n}} A_{m]}{}^{\underline{p}} \ . \tag{3.5}$$

The field strength automatically obeys the Bianchi identity $\hat{D}_{[p} F_{nm]}{}^{\underline{q}} = 0$ and transforms as an internal vector and external 2-form under internal and external diffeomorphisms.

It is conspicuous in the transformation laws above that not all of the components of the higher dimensional vielbein transform into each other. In particular, the external vielbein $e_m{}^a$, the Kaluza-Klein gauge field $A_m{}^{\underline{m}}$, and the additional field $\chi_{\underline{m}}{}^a$ can be separated from the internal vielbein $e_{\underline{m}}{}^{\underline{a}}$. This is fortuitous, as this is exactly the sort of situation we require. The two prepotentials $H_{\alpha\dot{\alpha}}$ and $\mathcal{V}^{\underline{m}}$ already encode $e_m{}^a$ and $A_m{}^{\underline{m}}$. It is natural to suppose $\Psi_{\underline{m}\alpha}$ encodes $\chi_{\underline{m}}{}^a$, and this is not hard to see. At the linearized level, $\delta \chi_{\underline{m}}{}^a = \partial_{\underline{m}} \xi^a$ arises from $\delta \Psi_{\underline{m}\alpha} = 2i\, \partial_{\underline{m}} L_\alpha$ provided we identify χ as

$$\chi_{\underline{m}}{}^a = \frac{1}{4} (\bar{\sigma}^a)^{\dot{\alpha}\alpha} (\bar{D}_{\dot{\alpha}} \Psi_{\underline{m}\alpha} - D_\alpha \bar{\Psi}_{\underline{m}\dot{\alpha}})|_{\theta=0} \ . \tag{3.6}$$

Observe that the opposite combination $\bar{D}_{\dot{\alpha}}\Psi_{\underline{m}\alpha} + D_\alpha \bar{\Psi}_{\underline{m}\dot{\alpha}}$ is what contributes to the field strength $X_{\underline{m}}{}^a$ (2.11b).

In 11D supergravity, it is possible to set the component field $\chi_{\underline{m}}{}^a$ (as well as the bottom component of $X_{\underline{m}}{}^a$) to zero by an Ω transformation. That is, the usual Lorentz gauge-fixing of Kaluza-Klein theory corresponds here to a choice of Wess-Zumino gauge, and precisely this choice was made in the linearized analysis of [36]. But Wess-Zumino gauge fixing is awkward at the superfield level. While we can set the bottom component of χ to zero, higher θ components will survive, and so we cannot discard it completely. It is simpler to just keep the Ω gauge unfixed.

As we have already mentioned, the internal vielbein $e_{\underline{m}}{}^{\underline{a}}$ should not play any role. This is because, at least in the $\mathcal{N} = 1$ case, the internal metric is not its own independent superfield but is encoded in the bottom component of the 3-form field strength $F_{\underline{mnp}}$. Moreover, in the $\mathcal{N} = 1$ spectrum we have already constructed, there is no internal Lorentz group. All superfields are in 4D representations or representations of the internal diffeomorphism group GL(7). This means that as we build covariant external and internal derivatives, we must forbid the use of $e_{\underline{m}}{}^{\underline{a}}$ at any point. It is an important fact that this will be possible.

3.2 Covariant internal p-forms and a covariant de Rham differential

The transformation rules (3.3) and (3.4) motivate a uniform notion for how external and internal covariant forms transform under external and internal diffeomorphisms. We remark first that a field ϕ is a *covariant scalar field* if it transforms as

$$\delta\phi = \xi^n \hat{D}_n \phi + \Lambda^{\underline{n}} \partial_{\underline{n}} \phi \tag{3.7}$$

under external and internal diffeomorphisms. This definition naturally arises by taking ϕ to be a scalar on the full space, i.e. $\delta\phi = \hat{\xi}^N \hat{\partial}_N \phi$, and then decomposing $\hat{\xi}^N$. The extension to external or internal 1-forms is obvious. A field $\omega_{\underline{m}}$ is a *covariant internal 1-form* if it transforms as

$$\delta\omega_{\underline{m}} = \xi^n \hat{D}_n \omega_{\underline{m}} + \Lambda^{\underline{n}} \partial_{\underline{n}} \omega_{\underline{m}} + \partial_{\underline{m}} \Lambda^{\underline{n}} \omega_{\underline{n}} \ . \tag{3.8}$$

Similarly, a field ω_m is a *covariant external 1-form* if it transforms as

$$\delta\omega_m = \xi^n \hat{D}_n \omega_m + \hat{D}_m \xi^n \, \omega_n + \Lambda^{\underline{n}} \partial_{\underline{n}} \omega_m \ . \tag{3.9}$$

Both of these definitions arise by requiring $\omega_{\underline{a}}$ and ω_a to transform as scalars under external and internal diffeomorphisms, and then defining $\omega_m := e_m{}^a \omega_a$ and $\omega_{\underline{m}} := e_{\underline{m}}{}^{\underline{a}} \omega_{\underline{a}}$. This is quite natural if we have a 1-form $\hat{\omega}_M$ on the full spacetime. Then the usual way to define its external and internal components is to flatten the indices with the higher-dimensional vielbein and then to unflatten with the external or internal vielbein, i.e. $\omega_m := e_m{}^a \hat{e}_a{}^M \hat{\omega}_M$ and $\omega_{\underline{m}} := e_{\underline{m}}{}^{\underline{a}} \hat{e}_{\underline{a}}{}^M \hat{\omega}_M$. This can be generalized to higher degree forms or mixed internal/external forms. However, for the remainder of this section, we will mainly be interested in internal

p-forms, since the $\mathcal{N} = 1$ superfields we encounter for 11D supergravity will be in such representations.

We now want to introduce a notion of internal and external covariant derivatives – that is, generalizations of $\partial_{\underline{m}}$ and ∂_m that preserve covariance. Let's start with a covariant scalar field ϕ transforming as (3.7). It is obvious that $\hat{e}_a{}^M \partial_M \phi$ and $\hat{e}_{\underline{a}}{}^M \partial_M \phi$ transform covariantly, as these are just the external and internal components of $D_A \phi$. This suggests the definitions

$$D_m \equiv e_m{}^a D_a := e_m{}^a \hat{e}_a{}^M \partial_M = \partial_m - A_m{}^{\underline{m}} \partial_{\underline{m}} \ , \tag{3.10a}$$

$$D_{\underline{m}} := e_{\underline{m}}{}^{\underline{a}} \hat{e}_{\underline{a}}{}^M \partial_M = \partial_{\underline{m}} - \chi_{\underline{m}}{}^a e_a{}^m (\partial_m - A_m{}^{\underline{n}} \partial_{\underline{n}}) \ . \tag{3.10b}$$

Then $D_m \phi$ and $D_{\underline{m}} \phi$ indeed transform covariantly. Note that the former coincides with $\hat{D}_m$ for a scalar field, but $D_{\underline{m}}$ is *not* identical to $\partial_{\underline{m}}$. In neither case does $e_{\underline{m}}{}^{\underline{a}}$ appear in the fundamental definition of the derivative.

For an internal 1-form $\omega_{\underline{m}}$, the situation is a bit more subtle. The derivative $\hat{D}_m$ acts as

$$\hat{D}_m \omega_{\underline{m}} = D_m \omega_{\underline{m}} - \partial_{\underline{m}} A_m{}^{\underline{n}} \omega_{\underline{n}}$$

This turns out to be covariant under internal diffeomorphisms, but it fails to be covariant under external ones. One finds that

$$\delta_\xi (\hat{D}_n \omega_{\underline{m}}) = \hat{D}_n \xi^p \hat{D}_p \omega_{\underline{m}} + \xi^p \hat{D}_p \hat{D}_n \omega_{\underline{m}} - \partial_{\underline{m}} \xi^p F_{pn}{}^{\underline{p}} \omega_{\underline{p}} \ , \tag{3.11}$$

The extra third term can be cancelled if we use instead the combination $\hat{D}_n \omega_{\underline{m}} + \chi_{\underline{m}}{}^m F_{mn}{}^{\underline{p}} \omega_{\underline{p}}$. This suggests that we introduce an external $\mathrm{GL}(n)$ connection acting on the internal indices, $\mathring{\Gamma}_{m\underline{n}}{}^{\underline{p}}$, so that

$$\mathring{\nabla}_n \omega_{\underline{m}} := D_n \omega_{\underline{m}} - \mathring{\Gamma}_{n\underline{m}}{}^{\underline{p}} \omega_{\underline{p}} \ , \qquad \mathring{\Gamma}_{n\underline{m}}{}^{\underline{p}} := \partial_{\underline{m}} A_n{}^{\underline{p}} - \chi_{\underline{m}}{}^a F_{an}{}^{\underline{p}} \ . \tag{3.12}$$

It is convenient here to consider the term $\partial_{\underline{m}} A_n{}^{\underline{p}}$, originally part of the internal Lie derivative, as part of the $\mathrm{GL}(n)$ connection. Or to put it another way, we define $\mathring{\nabla}_m$ in terms of D_m instead of $\hat{D}_m$. We have denoted this specific choice of $\mathrm{GL}(n)$ connection with a circle accent $\mathring{}$ to emphasize that it is the simplest choice to make. Any other connection Γ will differ from $\mathring{\Gamma}$ by some tensor field. A different choice might seem artificial, but when we choose natural $\mathcal{N} = 1$ superspace constraints, they will turn out to lead to such a modified $\mathrm{GL}(n)$ connection.

Now consider the internal derivative of $\omega_{\underline{m}}$. From flat space experience, we expect that we should make do by covariantizing the de Rham differential, that is, $\partial_{[\underline{m}} \omega_{\underline{n}]}$. From the scalar field case, we expect to use $\hat{D}_{\underline{m}} := \partial_{\underline{m}} - \chi_{\underline{m}}{}^a \hat{D}_a$ plus some additional piece. Under internal diffeomorphisms, we find that

$$\delta_\Lambda (\hat{D}_{\underline{n}} \omega_{\underline{m}}) = \partial_{\underline{n}} \Lambda^{\underline{p}} \hat{D}_{\underline{p}} \omega_{\underline{m}} + \partial_{\underline{m}} \Lambda^{\underline{p}} \hat{D}_{\underline{n}} \omega_{\underline{p}} + \Lambda^{\underline{p}} \partial_{\underline{p}} \hat{D}_{\underline{n}} \omega_{\underline{m}} + \partial_{\underline{n}} \partial_{\underline{m}} \Lambda^{\underline{p}} \omega_{\underline{p}} \ . \tag{3.13}$$

This indeed becomes covariant if we antisymmetrize $\underline{n}$ and $\underline{m}$. However, under external diffeomorphisms,

$$\delta_\xi (\hat{D}_{\underline{n}} \omega_{\underline{m}}) = \xi^p D_p \hat{D}_{\underline{n}} \omega_{\underline{m}} + \xi^p \partial_{\underline{n}} \partial_{\underline{m}} A_p{}^{\underline{p}} \omega_{\underline{p}} - \chi_{\underline{n}}{}^a \partial_{\underline{m}} \xi^m F_{am}{}^{\underline{p}} \omega_{\underline{p}} \ . \tag{3.14}$$

While the second term drops out upon antisymmetrizing, the third term remains. To cancel it, we now introduce an internal leg to the GL(n) connection so that the antisymmetric part of $\mathring{\nabla}_{\underline{n}}\omega_{\underline{m}}$ is covariant. We write this contribution as

$$\mathring{\nabla}_{\underline{n}}\omega_{\underline{m}} := \partial_{\underline{n}}\omega_{\underline{m}} - \chi_{\underline{n}}{}^{a}\nabla_{a}\omega_{\underline{m}} - \mathring{\Gamma}_{\underline{nm}}{}^{\underline{p}}\omega_{\underline{p}} \ , \qquad \mathring{\Gamma}_{\underline{nm}}{}^{\underline{p}} := \frac{1}{2}\chi_{\underline{n}}{}^{c}\chi_{\underline{m}}{}^{b}F_{cb}{}^{\underline{p}} \ .$$

Note that there is a contribution to the GL(n) connection coming from the second term, so that $\chi_{\underline{n}}{}^{a}e_{a}{}^{n}\mathring{\Gamma}_{n\underline{m}}{}^{\underline{p}}$ is being added to the explicit $\mathring{\Gamma}_{\underline{nm}}{}^{\underline{p}}$.

The generalization to internal p-forms is obvious, but with the caveat that only the totally antisymmetric part of the internal covariant derivative is actually covariant. *In other words, we covariantize only the internal de Rham differential, not the internal derivative in general.* It is remarkable that χ and A alone are needed to build an internal covariant de Rham differential.

Naturally, the next objects one might consider are external p-forms, with an eye to generalize to mixed external/internal forms, but the situation grows more complicated. For example, if ω_m is an external 1-form, we find that $2\hat{D}_{[n}\omega_{m]} + F_{nm}{}^{\underline{p}}\chi_{\underline{p}}{}^{c}e_{c}{}^{p}\omega_{p}$ transforms as an external 2-form. This might suggest introducing a connection for the external coordinate indices, but we should avoid doing this. Eventually, we want to reproduce as much as possible the structure of existing 4D $\mathcal{N}=1$ superspace, and no affine connection plays any role there. Instead, one deals solely with the Lorentz and other tangent space connections. In addition, experience with 11D supergravity suggests we will deal only with covariant $\mathcal{N}=1$ superfields without any external coordinate indices, but only internal GL(n) indices (and possibly Lorentz spinor or vector indices), and so such objects won't be directly encountered.[4]

3.3 Including tangent space connections

We want to include additional connections for gauge symmetries that act on $e_{m}{}^{a}$ and other tensor fields. The prototypical example is Lorentz symmetry but we will be rather general since later on we will be considering the $\mathcal{N}=1$ superconformal group. Suppose we have a group $\mathcal{H}$ that acts on $e_{m}{}^{a}$ and $\chi_{\underline{m}}{}^{a}$ as

$$\delta_{\mathcal{H}}e_{m}{}^{a} = e_{m}{}^{b}\lambda^{x}f_{xb}{}^{a} \ , \qquad \delta_{\mathcal{H}}\chi_{\underline{m}}{}^{a} = \chi_{\underline{m}}{}^{b}\lambda^{x}f_{xb}{}^{a} \ , \tag{3.15}$$

where λ^{x} is a local gauge parameter and we use $x, y, \cdots$ to label the generators g_x of $\mathcal{H}$. We assume $A_{m}{}^{\underline{m}}$ is invariant. We suppose further that we are furnished an $\mathcal{H}$ connection with external and internal components, $h_{m}{}^{x}$ and $h_{\underline{m}}{}^{x}$, transforming under $\mathcal{H}$ transformations as

$$\begin{aligned} \delta_{\mathcal{H}}h_{m}{}^{x} &= \hat{D}_{m}g^{x} + e_{m}{}^{a}g^{y}f_{ya}{}^{x} + h_{m}{}^{y}g^{z}f_{zy}{}^{x} \ , \\ \delta_{\mathcal{H}}h_{\underline{m}}{}^{x} &= \partial_{\underline{m}}g^{x} + \chi_{\underline{m}}{}^{a}g^{y}f_{ya}{}^{x} + h_{\underline{m}}{}^{y}g^{z}f_{zy}{}^{x} \ . \end{aligned} \tag{3.16}$$

[4]The superfields of the $\mathcal{N}=1$ tensor hierarchy discussed in section 2.1 will turn out to be components of mixed superforms in superspace.

The constants f should obey the Jacobi identity associated with a Lie algebra that extends $\mathcal{H}$ by a generator P_a, with commutation relations[5]

$$[g_x, g_y] = -f_{xy}{}^z g_z \ , \qquad [g_x, P_a] = -f_{xa}{}^b P_b - f_{xa}{}^y g_y \ , \qquad [P_a, P_b] = 0 \ . \tag{3.17}$$

One can check that the commutator of $\delta_{\mathcal{H}}$ transformations reproduces the $[g, g]$ algebra.

Now we augment the covariant derivatives defined in the previous section with the $\mathcal{H}$-connections. At the same time, we will allow the GL(n) connection Γ to differ from the simplest choice $\mathring{\Gamma}$. Explicitly, we have

$$\begin{aligned} \nabla_a &= e_a{}^m(\partial_m - A_m{}^{\underline{n}}\partial_{\underline{n}} - \Gamma_{m\underline{n}}{}^{\underline{p}} g_{\underline{p}}{}^{\underline{n}} - h_m{}^x g_x) \ , \\ \nabla_{\underline{m}} &= \partial_{\underline{m}} - \chi_{\underline{m}}{}^a \nabla_a - \Gamma_{\underline{mn}}{}^{\underline{p}} g_{\underline{p}}{}^{\underline{n}} - h_{\underline{m}}{}^x g_x \ . \end{aligned} \tag{3.18}$$

The operator $g_{\underline{m}}{}^{\underline{n}}$ generates GL(n) transformations, i.e. $g_{\underline{m}}{}^{\underline{n}}\omega_{\underline{p}} = \delta_{\underline{p}}{}^{\underline{n}}\omega_{\underline{m}}$. In order for the above covariant derivatives to remain covariant with respect to external and internal diffeomorphisms, we must take the $\mathcal{H}$ connections to transform as[6]

$$\begin{aligned} \delta h_m{}^x &= \hat{D}_m \xi^n h_n{}^x + \xi^n \hat{D}_n h_m{}^x + \Lambda^{\underline{n}}\partial_{\underline{n}} h_m{}^x \ , \\ \delta h_{\underline{m}}{}^x &= \partial_{\underline{m}}\Lambda^{\underline{n}} h_{\underline{n}}{}^x + \partial_{\underline{m}}\xi^n h_n{}^x + \xi^n \hat{D}_n h_{\underline{m}}{}^x + \Lambda^{\underline{n}}\partial_{\underline{n}} h_{\underline{m}}{}^x \ . \end{aligned} \tag{3.19}$$

They are also $\mathcal{H}$-covariant in the sense that if Φ is some field transforming as $\delta_{\mathcal{H}}\Phi = \lambda^x g_x \Phi$, then

$$\begin{aligned} \delta_{\mathcal{H}}\nabla_a\Phi &\equiv \lambda^x g_x \nabla_a \Phi = \lambda^x(\nabla_a g_x \Phi - f_{xa}{}^b \nabla_b \Phi) \ , \\ \delta_{\mathcal{H}}\nabla_{\underline{m}}\Phi &\equiv \lambda^x g_x \nabla_{\underline{m}}\Phi = \lambda^x \nabla_{\underline{m}} g_x \Phi \ , \end{aligned} \tag{3.20}$$

which amounts to the formal operator algebra

$$[g_x, \nabla_a] = -f_{xa}{}^b \nabla_b - f_{xa}{}^y g_y \ , \qquad [g_x, \nabla_{\underline{m}}] = 0 \ . \tag{3.21}$$

Above, ∇_a is playing the role of P_a in the flat algebra (3.17). The vanishing commutators of P_a and $\partial_{\underline{m}}$ with each other are replaced with field-dependent curvature tensors

$$\begin{aligned} [\nabla_a, \nabla_b] &= -T_{ab}{}^c \nabla_c - \mathcal{L}_{F_{ab}} - \mathcal{R}_{ab\underline{m}}{}^{\underline{n}} g_{\underline{n}}{}^{\underline{m}} - R_{ab}{}^x g_x \ , \\ [\nabla_a, \nabla_{\underline{m}}] &= -T_{a\underline{m}}{}^b \nabla_b - \mathcal{L}_{F_{a\underline{n}}} - \mathcal{R}_{a\underline{mn}}{}^{\underline{p}} g_{\underline{p}}{}^{\underline{n}} - R_{a\underline{m}}{}^x g_x \ , \\ [\nabla_{\underline{m}}, \nabla_{\underline{n}}] &= -T_{\underline{mn}}{}^a \nabla_a - \mathcal{L}_{F_{\underline{mn}}} - \mathcal{R}_{\underline{mnp}}{}^{\underline{q}} g_{\underline{q}}{}^{\underline{p}} - R_{\underline{mn}}{}^x g_x \ , \end{aligned} \tag{3.22}$$

where T^a is the external torsion tensor, $F^{\underline{m}}$ is the internal Kaluza-Klein curvature, $\mathcal{R}_{\underline{m}}{}^{\underline{n}}$ is the GL(n) curvature, and R^x is the $\mathcal{H}$-curvature. $\mathcal{L}$ denotes the internal covariant Lie derivative, defined so that any lower internal form indices of $F^{\underline{m}}$ are spectators, e.g.

$$\mathcal{L}_{F_{\underline{mn}}}\omega_{\underline{p}} := 2\, F_{\underline{mn}}{}^{\underline{q}}\nabla_{[\underline{q}}\omega_{\underline{p}]} + \nabla_{\underline{p}}(F_{\underline{mn}}{}^{\underline{q}}\omega_{\underline{q}}) = F_{\underline{mn}}{}^{\underline{q}}\nabla_{\underline{q}}\omega_{\underline{p}} + \nabla_{\underline{p}}F_{\underline{mn}}{}^{\underline{q}}\omega_{\underline{q}} \ . \tag{3.23}$$

[5] We treat g_x as an operator acting from the left that takes a covariant field to a covariant field, so that $\delta_1\delta_2\Phi = \lambda_1^x\lambda_2^y g_x g_y \Phi$.

[6] Note the anomalous term in the transformation of $h_{\underline{m}}{}^x$ that rotates it into $h_m{}^x$: it is similar in structure to the anomalous term in the transformation of $\chi_{\underline{m}}{}^a$ that rotates it into $e_m{}^a$.

We have chosen to package the internal curvature term in (3.22) as a covariant Lie derivative (rather than a covariant derivative) because this ensures covariance of the curvature terms separately when the commutator acts on an internal p-form. This amounts to a redefinition of the GL(n) curvature $\mathcal{R}_{\underline{m}}{}^{\underline{n}}$.

The external torsion tensors in (3.22) are given by

$$\begin{aligned} T_{nm}{}^{a} &= 2D_{[n}e_{m]}{}^{a} + 2\,e_{[n}{}^{b}h_{m]}{}^{x}f_{xb}{}^{a} + F_{nm}{}^{\underline{p}}\chi_{\underline{p}}{}^{a}\ , \\ T_{n\underline{m}}{}^{a} + \chi_{\underline{m}}{}^{b}T_{nb}{}^{a} &= \hat{D}_{n}\chi_{\underline{m}}{}^{a} - \partial_{\underline{m}}e_{n}{}^{a} + e_{n}{}^{b}\,h_{\underline{m}}{}^{x}f_{xb}{}^{a} - \chi_{\underline{m}}{}^{b}h_{n}{}^{x}f_{xb}{}^{a}\ , \\ T_{\underline{mn}}{}^{a} + 2\chi_{[\underline{m}}{}^{c}T_{c\underline{n}]}{}^{a} + \chi_{\underline{m}}{}^{c}\chi_{\underline{n}}{}^{b}T_{cb}{}^{a} &= 2\partial_{[\underline{n}}\chi_{\underline{m}]}{}^{a} + 2\,\chi_{[\underline{n}}{}^{b}\,h_{\underline{m}]}{}^{x}f_{xb}{}^{a}\ . \end{aligned} \tag{3.24}$$

Here one must plug the first equation into the second and both into the third to solve for $T_{n\underline{m}}{}^{a}$ and $T_{\underline{mn}}{}^{a}$, and then flatten external world indices with $e_{a}{}^{m}$. Similarly, the $\mathcal{H}$ curvatures are given by

$$\begin{aligned} R_{mn}{}^{x} &= 2\hat{D}_{[m}h_{n]}{}^{x} + 2h_{[m}{}^{y}e_{n]}{}^{b}f_{by}{}^{x} + h_{m}{}^{y}h_{n}{}^{z}f_{zy}{}^{x} + F_{mn}{}^{\underline{p}}h_{\underline{p}}{}^{x}\ , \\ R_{m\underline{n}}{}^{x} + \chi_{\underline{n}}{}^{b}R_{mb}{}^{x} &= \hat{D}_{m}h_{\underline{n}}{}^{x} - \partial_{\underline{n}}h_{m}{}^{x} + h_{m}{}^{y}\chi_{\underline{n}}{}^{b}f_{by}{}^{x} + e_{m}{}^{b}h_{\underline{n}}{}^{y}f_{yb}{}^{x} \\ &\quad + h_{m}{}^{y}h_{\underline{n}}{}^{z}f_{zy}{}^{x}\ , \\ R_{\underline{mn}}{}^{x} + 2\chi_{[\underline{m}}{}^{c}R_{c\underline{n}]}{}^{x} + \chi_{\underline{m}}{}^{c}\chi_{\underline{n}}{}^{b}R_{cb}{}^{x} &= 2\partial_{[\underline{m}}h_{\underline{n}]} + 2h_{[\underline{m}}{}^{x}\chi_{\underline{n}]}{}^{a}f_{ay}{}^{x} + h_{\underline{m}}{}^{y}h_{\underline{n}}{}^{z}f_{zy}{}^{x}\ . \end{aligned} \tag{3.25}$$

We do not give explicit expressions the GL(n) curvatures, although they can be worked out straightforwardly. $F_{ab}{}^{\underline{m}}$ is given by flattening the form indices of $F_{mn}{}^{\underline{m}}$ in (3.5) with the external vielbein. The expressions for the mixed $F_{a\underline{n}}{}^{\underline{m}}$ and internal $F_{\underline{pn}}{}^{\underline{m}}$ tensors can be worked out explicitly. However, it is more helpful to observe that when Γ is chosen to be $\mathring{\Gamma}$, one finds that $\mathring{F}_{a\underline{m}}{}^{\underline{n}} = \mathring{F}_{\underline{mn}}{}^{\underline{p}} = 0$. Then deforming the GL($n$) connection by a purely covariant pieces $\Delta\Gamma$, defined so that

$$\nabla_{a} := \mathring{\nabla}_{a} - \Delta\Gamma_{a\underline{n}}{}^{\underline{p}}g_{\underline{p}}{}^{\underline{n}}\ , \qquad \nabla_{\underline{m}} := \mathring{\nabla}_{\underline{m}} - \Delta\Gamma_{\underline{mn}}{}^{\underline{p}}g_{\underline{p}}{}^{\underline{n}}\ , \tag{3.26}$$

one can show that

$$F_{a\underline{m}}{}^{\underline{n}} = \Delta\Gamma_{a\underline{m}}{}^{\underline{n}}\ , \qquad F_{\underline{mn}}{}^{\underline{p}} = 2\,\Delta\Gamma_{[\underline{mn}]}{}^{\underline{p}}\ . \tag{3.27}$$

By construction the curvatures above must obey Bianchi identities,

$$\sum_{[abc]}[\nabla_{a},[\nabla_{b},\nabla_{c}]] = 0\ , \qquad \sum_{[ab]}[\nabla_{a},[\nabla_{b},\nabla_{\underline{m}}]] = -[\nabla_{\underline{m}},[\nabla_{a},\nabla_{b}]]\ , \qquad \text{etc.}$$

Particularly useful are the Kaluza-Klein field strength Bianchi identities, which read

$$
\begin{aligned}
0 &= \sum_{[abc]} \Big(\nabla_c F_{ab}{}^{\underline{m}} + T_{ab}{}^e F_{ec}{}^{\underline{m}} + F_{ab}{}^{\underline{n}} F_{\underline{n}c}{}^{\underline{m}} \Big) \ , \\
0 &= \sum_{[ab]} \Big(- \mathcal{R}_{ab\underline{m}}{}^{\underline{n}} + 2\nabla_a F_{b\underline{m}}{}^{\underline{n}} + T_{ab}{}^e F_{e\underline{m}}{}^{\underline{n}} + 2T_{\underline{m}a}{}^e F_{eb}{}^{\underline{n}} + F_{ab}{}^{\underline{p}} F_{\underline{p}\underline{m}}{}^{\underline{n}} + 2F_{\underline{m}a}{}^{\underline{p}} F_{\underline{p}b}{}^{\underline{n}} \Big) \ , \\
0 &= \sum_{[\underline{mn}]} \Big(- 2\mathcal{R}_{a\underline{mn}}{}^{\underline{p}} + \nabla_a F_{\underline{mn}}{}^{\underline{p}} + T_{\underline{mn}}{}^c F_{ca}{}^{\underline{p}} + 2T_{a\underline{m}}{}^e F_{e\underline{n}}{}^{\underline{p}} + F_{\underline{mn}}{}^{\underline{q}} F_{\underline{q}a}{}^{\underline{p}} + 2F_{a\underline{m}}{}^{\underline{q}} F_{\underline{q}\underline{n}}{}^{\underline{p}} \Big) \ , \\
0 &= \sum_{[\underline{mnp}]} \Big(- \mathcal{R}_{\underline{mnp}}{}^{\underline{q}} + T_{\underline{mn}}{}^e F_{e\underline{p}}{}^{\underline{q}} + F_{\underline{mn}}{}^{\underline{r}} F_{\underline{rp}}{}^{\underline{q}} \Big) \ .
\end{aligned}
\tag{3.28}
$$

The first equation ensures that $F_{ab}{}^{\underline{m}}$ is covariantly closed. The other three determine the parts of the GL(n) curvature $\mathcal{R}$ that are antisymmetric in lower internal indices. Because we will only be constructing internal covariant de Rham differentials, only the (internal) antisymmetric parts of $\mathcal{R}$ will ever appear, and these are completely determined in terms of the other quantities.

Finally, for reference we give the covariantized external diffeomorphisms of the various connections, which arise by combining an external diffeomorphism with $\xi^m = \xi^a e_a{}^m$ and an $\mathcal{H}$ transformation with $\lambda^x = -\xi^a e_a{}^m h_m{}^x$:

$$
\begin{aligned}
\delta^{\rm cov}_\xi e_m{}^a &= D_m \xi^a + h_m{}^x \xi^c f_{cx}{}^a + e_m{}^b \xi^c (T_{cb}{}^a - F_{cb}{}^{\underline{p}} \chi_{\underline{p}}{}^a) \ , \\
\delta^{\rm cov}_\xi \chi_{\underline{m}}{}^a &= \partial_{\underline{m}} \xi^a + h_{\underline{m}}{}^x \xi^c f_{cx}{}^a + \xi^c (T_{c\underline{m}}{}^a + \chi_{\underline{m}}{}^b T_{cb}{}^a) \ , \\
\delta^{\rm cov}_\xi h_m{}^x &= e_m{}^b \xi^c (R_{cb}{}^x - F_{cb}{}^{\underline{p}} h_{\underline{p}}{}^x) + h_m{}^y \xi^b f_{by}{}^x \ , \\
\delta^{\rm cov}_\xi h_{\underline{m}}{}^x &= \xi^c (R_{c\underline{m}}{}^x + \chi_{\underline{m}}{}^b R_{cb}{}^x) + h_{\underline{m}}{}^y \xi^b f_{by}{}^x \ , \\
\delta^{\rm cov}_\xi A_m{}^{\underline{m}} &= e_m{}^b \xi^c F_{cb}{}^{\underline{m}} \ .
\end{aligned}
\tag{3.29}
$$

These transformations are relevant when the relations discussed above are promoted to superspace; then the fermionic component of ξ^A is identified with the local supersymmetry parameter. Then it is crucial that the above transformations involve covariant tensors; this ensures that the SUSY transformations are sensibly defined.

4 The supergeometry of 4D $\mathcal{N} = 1$ Kaluza-Klein superspace

Now we are in a position to start building the general supergeometry of 4D $\mathcal{N} = 1$ Kaluza-Klein superspace. The first step is to extend the discussion of section 3 by allowing the external space considered there to be a superspace. This is just a cosmetic change, promoting the coordinates x^m to supercoordinates $z^M = (x^m, \theta^\mu, \bar\theta_{\dot\mu})$ and the tangent indices a to $A = (a, \alpha, \dot\alpha)$. This requires promoting the external vielbein $e_m{}^a$, the Kaluza-Klein gauge field $A_m{}^{\underline{m}}$, and the additional field $\chi_m{}^m$ to superfields in the obvious manner, i.e.

$$
e_m{}^a \to E_M{}^A \ , \qquad A_m{}^{\underline{m}} \to A_M{}^{\underline{m}} \ , \qquad \chi_{\underline{m}}{}^a \to \chi_{\underline{m}}{}^A \ .
\tag{4.1}
$$

However, we do not modify the internal space – it remains a bosonic manifold with a GL(n) index $\underline{m}$. Let us not reproduce every formula, but only give a few that are directly relevant. The superspace external covariant derivatives ∇_A and internal $\nabla_{\underline{m}}$ are given by

$$\begin{aligned}\nabla_A &= E_A{}^M(\partial_M - A_M{}^{\underline{n}}\partial_{\underline{n}} - \Gamma_{M\underline{n}}{}^{\underline{p}}g_{\underline{p}}{}^{\underline{n}}) - H_A{}^x g_x \ , \\ \nabla_{\underline{m}} &= \partial_{\underline{m}} - \chi_{\underline{m}}{}^A\nabla_A - \Gamma_{\underline{mn}}{}^{\underline{p}}g_{\underline{p}}{}^{\underline{n}} - H_{\underline{m}}{}^x g_x \ ,\end{aligned} \tag{4.2}$$

with the GL(n) connections defined as

$$\begin{aligned}\Gamma_{A\underline{m}}{}^{\underline{p}} &:= E_A{}^N\partial_{\underline{m}}A_N{}^{\underline{p}} - \chi_{\underline{m}}{}^B F_{BA}{}^{\underline{p}} + \Delta\Gamma_{A\underline{m}}{}^{\underline{n}} \ , \\ \Gamma_{\underline{nm}}{}^{\underline{p}} &:= -\frac{1}{2}\chi_{\underline{n}}{}^B\chi_{\underline{m}}{}^A F_{AB}{}^{\underline{p}} + \Delta\Gamma_{\underline{nm}}{}^{\underline{p}} + \chi_{\underline{n}}{}^A\Delta\Gamma_{A\underline{m}}{}^{\underline{n}} \ ,\end{aligned} \tag{4.3}$$

where the $\Delta\Gamma$ terms transform covariantly. The covariant derivative algebra reads

$$[\nabla_A, \nabla_B] = -T_{AB}{}^C\nabla_C - \mathcal{L}_{F_{AB}} - \mathcal{R}_{AB\underline{m}}{}^{\underline{n}}g_{\underline{n}}{}^{\underline{m}} - R_{AB}{}^x g_x \ , \tag{4.4a}$$

$$[\nabla_A, \nabla_{\underline{m}}] = -T_{A\underline{m}}{}^B\nabla_B - \mathcal{L}_{F_{A\underline{n}}} - \mathcal{R}_{A\underline{mn}}{}^{\underline{p}}g_{\underline{p}}{}^{\underline{n}} - R_{A\underline{m}}{}^x g_x \ , \tag{4.4b}$$

$$[\nabla_{\underline{m}}, \nabla_{\underline{n}}] = -T_{\underline{mn}}{}^A\nabla_A - \mathcal{L}_{F_{\underline{mn}}} - \mathcal{R}_{\underline{mnp}}{}^{\underline{q}}g_{\underline{q}}{}^{\underline{p}} - R_{\underline{mn}}{}^x g_x \ . \tag{4.4c}$$

In superspace, one is not generally interested in the precise expressions for the various torsions and curvatures in terms of the potentials. Rather, one imposes some constraints on the torsions/curvatures and solves the Bianchi identities in terms of some fundamental curvature superfields (which obey Bianchi identities themselves).[7] These quantities, e.g. $W_{\alpha\beta\gamma}$ in $\mathcal{N} = 1$ conformal superspace, or $W_{\alpha\beta\gamma}$, R, and $G_{\alpha\dot\alpha}$ in the conventional $\mathcal{N} = 1$ Wess-Zumino superspace (see e.g. [39–41]), are, along with the covariant derivatives, supermeasures, and any covariant matter superfields, sufficient to construct covariant Lagrangians. In our case, we expect these curvature superfields to be built out of the basic curvatures $W_{\alpha\beta\gamma}$, $X_{\underline{m}\alpha\dot\alpha}$, $\Psi_{\underline{mn}\alpha}$, $\Phi_{\underline{mnp}\alpha}$, and $\mathcal{W}_\alpha{}^{\underline{m}}$.

4.1 Abstract solution of the Bianchi identities

A great deal of progress can be made working almost entirely abstractly if a very strong set of constraints is imposed from the beginning:[8]

$$\{\nabla_\alpha, \nabla_\beta\} = \{\bar\nabla_{\dot\alpha}, \bar\nabla_{\dot\beta}\} = 0 \ , \qquad \{\nabla_\alpha, \bar\nabla_{\dot\beta}\} = -2i\nabla_{\alpha\dot\beta} \ . \tag{4.5}$$

These coincide with the constraints of $\mathcal{N} = 1$ super Yang-Mills and were shown in $\mathcal{N} = 1$ conformal superspace to be the appropriate constraints to describe $\mathcal{N} = 1$ conformal supergravity [38]. These imply the existence of a coordinate system and a gauge where $\bar\nabla^{\dot\alpha} = \partial/\partial\bar\theta_{\dot\alpha}$. In

[7] In principle, the fundamental curvature superfields as well as all the potentials can in turn be solved in terms of prepotential superfields explicitly. Usually this is highly non-polynomial and not immediately useful. Typically only the linearized solution around a given background (e.g. flat space) is necessary.

[8] The last constraint is mainly a *conventional constraint* – that is, a definition of the connections in ∇_a.

such a gauge, covariantly chiral superfields are simply independent of $\bar{\theta}$. Such a set of constraints is not actually necessary for supergravity (conventional Wess-Zumino superspace does not satisfy these constraints, for example), but we will find them to be the right constraints in our case.

An immediate consequence of these constraints is the simplification of the external spinor/vector commutator to only a spin-1/2 part:

$$[\nabla_\alpha, \nabla_{\beta\dot{\beta}}] = 2\,\epsilon_{\alpha\beta}\bar{\mathcal{W}}_{\dot{\beta}} \ , \qquad [\bar{\nabla}_{\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] = 2\,\epsilon_{\dot{\alpha}\dot{\beta}}\mathcal{W}_\beta \ , \tag{4.6}$$

where $\mathcal{W}_\alpha$ is a fermionic operator, that is, it has an expansion $\mathcal{W}_\alpha = \mathcal{W}_\alpha{}^B\nabla_B + \mathcal{L}_{\mathcal{W}_\alpha{}^{\underline{m}}} + \mathcal{W}_{\alpha\underline{m}}{}^{\underline{n}}g_{\underline{n}}{}^{\underline{m}} + \mathcal{W}_\alpha{}^x g_x$. It must satisfy

$$\{\bar{\nabla}_{\dot{\alpha}}, \mathcal{W}_\alpha\} = 0 \ , \qquad \{\nabla^\alpha, \mathcal{W}_\alpha\} = \{\bar{\nabla}_{\dot{\alpha}}, \bar{\mathcal{W}}^{\dot{\alpha}}\} \ .$$

The first relation implies that $\mathcal{W}_\alpha$ is a chiral operator – it takes chiral superfields to chiral superfields. The second relation implies a reality condition reducing by half the number of independent pieces in the θ expansion of $\mathcal{W}$. These two identities together guarantee that the $[\nabla_{[A}, [\nabla_B, \nabla_{C]}]] = 0$ Bianchi identity holds. The final external commutator is vector/vector and is determined by the Bianchi identities to be

$$[\nabla_a, \nabla_b] = -\frac{i}{2}(\sigma_{ab})^{\alpha\beta}\{\nabla_\alpha, \mathcal{W}_\beta\} + \frac{i}{2}(\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}}\{\bar{\nabla}_{\dot{\alpha}}, \bar{\mathcal{W}}_{\dot{\beta}}\} \ . \tag{4.7}$$

The upshot is that the external curvatures are completely determined by $\mathcal{W}_\alpha$. Because the constraints (4.5) are the same as imposed in $\mathcal{N} = 1$ super Yang-Mills, the solution looks formally identical to that case.

Now we turn to the mixed curvature. Identifying the mixed curvature operator $R_{\underline{m}A}$,

$$[\nabla_{\underline{m}}, \nabla_A] = -R_{\underline{m}A} \ , \tag{4.8}$$

we can abstractly solve the $[\nabla_A, [\nabla_B, \nabla_{\underline{m}}]] + \cdots = 0$ Bianchi identity. The lowest dimension identities involving spinor derivatives imply that

$$R_{\underline{m}\,\alpha\dot{\alpha}} := (\sigma^a)_{\alpha\dot{\alpha}}R_{\underline{m}a} = \frac{i}{2}\{\nabla_\alpha, R_{\underline{m}\dot{\alpha}}\} + \frac{i}{2}\{\bar{\nabla}_{\dot{\alpha}}, R_{\underline{m}\alpha}\} \ , \qquad \nabla_{(\alpha}R_{\underline{m}\beta)} = 0 \ . \tag{4.9}$$

Because of the constraints (4.5), the second identity suggests to identify $R_{\underline{m}\alpha}$ as the spinor derivative of some other operator. By redefining $\nabla_{\underline{m}}$, one can always choose that operator to be imaginary, so that

$$R_{\underline{m}\alpha} = i[\nabla_\alpha, X_{\underline{m}}] \tag{4.10}$$

for some real operator $X_{\underline{m}}$. This operator is thus responsible for generating all of the mixed curvatures.

The existence of such a real operator lets us satisfy another of the entries on our wish list in section 2.2 – the existence of a modified internal derivative $\nabla^+_{\underline{m}}$ that preserves chirality:

$$\nabla^\pm_{\underline{m}} := \nabla_{\underline{m}} \pm iX_{\underline{m}} \ , \qquad [\nabla_\alpha, \nabla^-_{\underline{m}}] = [\bar{\nabla}_{\dot{\alpha}}, \nabla^+_{\underline{m}}] = 0 \ . \tag{4.11}$$

Provided $X_{\underline{m}}$ preserves covariance (and we will ensure it does), $\nabla^+_{\underline{m}}$ provides a chirality-preserving internal de Rham differential. The remainder of the $[\nabla_A, [\nabla_B, \nabla_{\underline{m}}]] + \cdots = 0$ Bianchi is then solved provided

$$[\nabla^+_{\underline{m}}, \mathcal{W}_\alpha] = -\frac{1}{4}[\bar\nabla_{\dot\beta}, \{\bar\nabla^{\dot\beta}, [\nabla_\alpha, X_{\underline{m}}]\}]$$

This intertwines the external curvatures with the mixed curvatures, implying they cannot be fixed separately.

For later use, we define $R^+_{\underline{m}\alpha}$ and $R^-_{\underline{m}\dot\alpha}$ as the mixed spinor curvatures arising from $\nabla^+_{\underline{m}}$ and $\nabla^-_{\underline{m}}$, respectively. They turn out to be twice the original curvatures $R_{\underline{m}\alpha}$ and $R_{\underline{m}\dot\alpha}$,

$$[\nabla^+_{\underline{m}}, \nabla_\alpha] \equiv -R^+_{\underline{m}\alpha} = -2R_{\underline{m}\alpha} \ , \qquad [\nabla^-_{\underline{m}}, \bar\nabla_{\dot\alpha}] \equiv -R^+_{\underline{m}\dot\alpha} = -2R_{\underline{m}\dot\alpha} \ . \tag{4.12}$$

It is helpful to give $R^+_{\underline{m}\alpha}$ a name distinct from $R_{\underline{m}\alpha}$ because when we expand them out in terms of derivatives and generators, we will write $R^+_{\underline{m}\alpha}$ in terms of $\nabla^+_{\underline{m}}$ while $R_{\underline{m}\alpha}$ will be written in terms of $\nabla_{\underline{m}}$. For example,

$$R^+_{\underline{m}A} = T^+_{\underline{m}A}{}^C\nabla_C + \mathcal{L}^+_{\mathcal{F}^+_{\underline{m}A}} + \mathcal{R}^+_{\underline{m}A\underline{n}}{}^{\underline{p}}g_{\underline{p}}{}^{\underline{n}} + R^+_{\underline{m}A}{}^x g_x \ , \tag{4.13}$$

where $\mathcal{L}^+$ denotes the covariant Lie derivative built from $\nabla^+_{\underline{m}}$.

Now let us address the internal curvature. The existence of chiral internal derivative $\nabla^+_{\underline{m}}$ suggests we should examine their curvatures, defined as

$$[\nabla^+_{\underline{m}}, \nabla^+_{\underline{n}}] = -R^+_{\underline{mn}} \tag{4.14}$$

and similarly for $R^-_{\underline{mn}}$. The content of the $[\nabla_A, [\nabla_{\underline{m}}, \nabla_{\underline{n}}]]$ Bianchi identity is now succinctly encoded in two conditions. The first is that $R^+_{\underline{mn}}$ is a chiral operator, $[\bar\nabla^{\dot\alpha}, R^+_{\underline{mn}}] = 0$. The second condition is that $R^+_{\underline{mn}}$ is related to the real $R_{\underline{mn}}$ via

$$R^+_{\underline{mn}} = R_{\underline{mn}} - 2i\nabla_{[\underline{m}}X_{\underline{n}]} + [X_{\underline{m}}, X_{\underline{n}}] \ . \tag{4.15}$$

The real part of this expression defines $R_{\underline{mn}}$, while its imaginary part links the mixed curvatures to the internal curvatures.

The final Bianchi identity, $[[\nabla_{[\underline{m}}, \nabla_{\underline{n}}], \nabla_{\underline{p}]}] = 0$, can equivalently be formulated in terms of $\nabla^+_{\underline{m}}$. It leads immediately to $[\nabla^+_{[\underline{p}}, R^+_{\underline{mn}]}] = 0$.

In summation, we have uncovered three basic operators: a complex spinor $\mathcal{W}_\alpha$, a real 1-form $X_{\underline{m}}$, and a complex 2-form $R^+_{\underline{mn}}$ that must obey six abstract Bianchi identities:

$$\{\bar\nabla_{\dot\alpha}, \mathcal{W}_\alpha\} = 0 \tag{BI.1}$$

$$\{\nabla^\alpha, \mathcal{W}_\alpha\} = \{\bar\nabla_{\dot\alpha}, \bar{\mathcal{W}}^{\dot\alpha}\} \tag{BI.2}$$

$$[\nabla^+_{\underline{m}}, \mathcal{W}_\alpha] = -\frac{1}{4}[\bar\nabla_{\dot\beta}, \{\bar\nabla^{\dot\beta}, [\nabla_\alpha, X_{\underline{m}}]\}] \tag{BI.3}$$

$$[\bar\nabla^{\dot\alpha}, R^+_{\underline{mn}}] = 0 \tag{BI.4}$$

$$\frac{i}{4}R^+_{\underline{mn}} - \frac{i}{4}R^-_{\underline{mn}} = [\nabla_{[\underline{m}}, X_{\underline{n}]}] \tag{BI.5}$$

$$[\nabla^+_{[\underline{p}}, R^+_{\underline{mn}]}] = 0 \tag{BI.6}$$

In terms of these, the external curvatures $R_{AB} = -[\nabla_A, \nabla_B]$ are given by

$$\begin{aligned} R_{\alpha\beta} = 0 \ , \qquad R_{\dot\alpha\dot\beta} = 0 \ , \qquad R_{\alpha\dot\beta} = 2i\,\nabla_{\alpha\dot\beta} \ , \\ R_{\alpha b} = -(\sigma_b)_{\alpha\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha} \ , \qquad R^{\dot\alpha}{}_b = (\bar\sigma_b)^{\dot\alpha\alpha}\mathcal{W}_\alpha \ , \\ R_{ab} = \frac{i}{2}(\sigma_{ab})^{\alpha\beta}\{\nabla_\alpha, \mathcal{W}_\beta\} - \frac{i}{2}(\bar\sigma_{ab})^{\dot\alpha\dot\beta}\{\bar\nabla_{\dot\alpha}, \bar{\mathcal{W}}_{\dot\beta}\} \ , \end{aligned} \tag{4.16}$$

the internal curvatures $R_{\underline{mn}} = -[\nabla_{\underline{m}}, \nabla_{\underline{n}}]$ are given by

$$R_{\underline{mn}} = \frac{1}{2}(R^+_{\underline{mn}} + R^-_{\underline{mn}}) - [X_{\underline{m}}, X_{\underline{n}}] \ , \tag{4.17}$$

and the mixed curvatures $R_{\underline{m}A} = -[\nabla_{\underline{m}}, \nabla_A]$ are given by

$$\begin{aligned} R_{\underline{m}\alpha} = i[\nabla_\alpha, X_{\underline{m}}] \ , \qquad R_{\underline{m}\dot\alpha} = -i[\bar\nabla_{\dot\alpha}, X_{\underline{m}}] \ , \\ R_{\underline{m}a} = -\frac{1}{4}(\bar\sigma_a)^{\dot\alpha\alpha}\{\nabla_\alpha, [\bar\nabla_{\dot\alpha}, X_{\underline{m}}]\} + \frac{1}{4}(\bar\sigma_a)^{\dot\alpha\alpha}\{\bar\nabla_{\dot\alpha}, [\nabla_\alpha, X_{\underline{m}}]\} \end{aligned} \tag{4.18}$$

For reference, it is also useful to give the mixed curvatures when written in terms of $\nabla^+_{\underline{m}}$:

$$\begin{aligned} R^+_{\underline{m}\alpha} = 2i[\nabla_\alpha, X_{\underline{m}}] \ , \qquad R^+_{\underline{m}\dot\alpha} = 0 \ , \\ R^+_{\underline{m}a} = \frac{1}{2}(\bar\sigma_a)^{\dot\alpha\alpha}\{\bar\nabla_{\dot\alpha}, [\nabla_\alpha, X_{\underline{m}}]\} \ . \end{aligned} \tag{4.19}$$

Our goal in subsequent sections will be to impose further constraints on the operators appearing above and to identify the fundamental curvature superfields that comprise them. Before doing that, we need to elaborate a bit more on the structure group $\mathcal{H}$ we will be using.

4.2 The superconformal structure group

The conformal superspace approach to $\mathcal{N} = 1$ conformal supergravity introduced in [38] involves choosing the generators g_x to be the set of Lorentz transformations (M_{ab}), dilatations and $\mathrm{U}(1)_R$ transformations ($\mathbb{D}$ and $\mathbb{A}$), S-supersymmetry transformations (S_α and $\bar S^{\dot\alpha}$), and finally special conformal boosts (K_a). Together with the covariant derivatives $\nabla_A = (\nabla_a, \nabla_\alpha, \bar\nabla^{\dot\alpha})$, they furnish a representation of the $\mathcal{N} = 1$ superconformal algebra with (anti)commutators

$$\begin{aligned} [M_{ab}, \nabla_c] &= 2\,\eta_{c[a}\nabla_{b]} \ , & [M_{ab}, K_c] &= 2\,\eta_{c[a}K_{b]} \ , \\ [\mathbb{D}, \nabla_a] &= \nabla_a \ , & [\mathbb{D}, K_a] &= -K_a \ , \\ [K_a, \nabla_b] &= 2\,\eta_{ab}\mathbb{D} + 2M_{ab} \ , & [M_{ab}, M_{cd}] &= 2\,\eta_{c[a}M_{b]d} - 2\,\eta_{d[a}M_{b]c} \ , \end{aligned}$$

$$\begin{aligned} [M_{ab}, \nabla_\gamma] &= -(\sigma_{ab})_\gamma{}^\beta\nabla_\beta \ , & [M_{ab}, S_\gamma] &= -(\sigma_{ab})_\gamma{}^\beta S_\beta \ , \\ [M_{ab}, \bar\nabla^{\dot\gamma}] &= -(\sigma_{ab})^{\dot\gamma}{}_{\dot\beta}\bar\nabla^{\dot\beta} \ , & [M_{ab}, \bar S^{\dot\gamma}] &= -(\sigma_{ab})^{\dot\gamma}{}_{\dot\beta}\bar S^{\dot\beta} \ , \\ [\mathbb{D}, \nabla_\alpha] &= \tfrac{1}{2}\nabla_\alpha, & [\mathbb{D}, S_\alpha] &= -\tfrac{1}{2}S_\alpha, \\ [\mathbb{D}, \bar\nabla^{\dot\alpha}] &= \tfrac{1}{2}\bar\nabla^{\dot\alpha} \ , & [\mathbb{D}, \bar S^{\dot\alpha}] &= -\tfrac{1}{2}\bar S^{\dot\alpha} \ , \end{aligned}$$

$$
\begin{aligned}
&[\mathbb{A}, \nabla_\alpha] = -i\nabla_\alpha, && [\mathbb{A}, S_\alpha] = +iS_\alpha, \\
&[\mathbb{A}, \bar\nabla^{\dot\alpha}] = +i\bar\nabla^{\dot\alpha}\ , && [\mathbb{A}, \bar S^{\dot\alpha}] = -i\bar S^{\dot\alpha}\ , \\
&[K_a, \nabla_\alpha] = i(\sigma_a)_{\alpha\dot\beta}\bar S^{\dot\beta}, && [K_a, \bar\nabla^{\dot\alpha}] = i(\bar\sigma_a)^{\dot\alpha\beta}S_\beta\ , \\
&[S_\alpha, \nabla_a] = i(\sigma_a)_{\alpha\dot\beta}\bar\nabla^{\dot\beta}, && [\bar S^{\dot\alpha}, \nabla_a] = i(\bar\sigma_a)^{\dot\alpha\beta}\nabla_\beta\ , \\
&\{S_\alpha, \nabla_\beta\} = \epsilon_{\alpha\beta}(2\,\mathbb{D} - 3i\mathbb{A}) - 4M_{\alpha\beta}\ , && \{\bar S^{\dot\alpha}, \bar\nabla^{\dot\beta}\} = \epsilon^{\dot\alpha\dot\beta}(2\,\mathbb{D} + 3i\mathbb{A}) - 4M^{\dot\alpha\dot\beta}\ , \\
&\{S_\alpha, \bar S_{\dot\alpha}\} = 2i(\sigma^a)_{\alpha\dot\alpha}K_a\ . &&
\end{aligned}
\tag{4.20}
$$

The operators $g_x = \{M_{ab}, \mathbb{D}, \mathbb{A}, S_\alpha, \bar S^{\dot\alpha}, K_a\}$ are taken to commute with $\nabla_{\underline m}$. Here we use $M_{\alpha\beta} = -\frac{1}{2}(\sigma^{ab})_{\alpha\beta}M_{ab}$ for the anti-self-dual part of M_{ab} and similarly for $M^{\dot\alpha\dot\beta}$.

If the ∇_A obeyed the flat $\mathcal{N} = 1$ superspace algebra, the algebra of the operators g_x and ∇_A would just be the $\mathcal{N} = 1$ superconformal algebra. Because the ∇_A curvatures instead involve the curvature operator $\mathcal{W}_\alpha$, the flat superconformal algebra becomes deformed. This is the sense in which the $\mathcal{N} = 1$ superconformal algebra has been gauged.

Consistency of the above relations with the algebra of covariant derivatives implies that the basic curvature operators $\mathcal{W}_\alpha$, $X_{\underline m}$, and $R^+_{\underline{mn}}$ are conformal primaries. That is, their (anti)commutators with S_α, $\bar S^{\dot\alpha}$, and K_a all vanish. These imply a number of conditions on the various pieces of these operators, which were useful in our analysis as checks, but we will not comment on them explicitly. $\mathcal{W}_\alpha$ additionally carries dilatation and $\mathrm{U}(1)_R$ weights $3/2$ and $+1$, whereas the other operators are inert.

5 The linearized solution to the Bianchi identities

In this section, we are going to sketch a solution to the Bianchi identities (BI.1) – (BI.6) at the linearized level, where it is possible to be very explicit about how the prepotentials appear. This will allow us also to make more transparent contact with the 5D [30] and 6D cases [32], which worked to linear order in the gravitino superfield $\Psi_{\underline m\alpha}$.

We treat the supergeometry as linearized around a nearly flat background, whose only non-vanishing curvature is the Kaluza-Klein curvature. The background covariant derivatives are $\boldsymbol{\nabla}_M = \mathcal{D}_M = \partial_M - \mathcal{L}_{A_M}$ and $\boldsymbol{\nabla}_{\underline m} = \partial_{\underline m}$ with curvature operators

$$
\boldsymbol{\mathcal{W}}_\alpha = \mathcal{W}_\alpha{}^{\underline m}\partial_{\underline m} + \partial_{\underline m}\mathcal{W}_\alpha{}^{\underline n}g_{\underline n}{}^{\underline m}\ , \qquad \boldsymbol{X}_{\underline m} = 0\ , \qquad \boldsymbol{R}^+_{\underline{mn}} = 0\ . \tag{5.1}
$$

The linearized fluctuations around this background are denoted

$$
\nabla_M = \boldsymbol{\nabla}_M + \delta\nabla_M\ , \qquad \nabla_{\underline m} = \partial_{\underline m} + \delta\nabla_{\underline m} \tag{5.2}
$$

with linearized curvatures

$$
\mathcal{W}_\alpha = \boldsymbol{\mathcal{W}}_\alpha + \delta\mathcal{W}_\alpha\ , \qquad X_{\underline m} = \delta X_{\underline m}\ , \qquad R^+_{\underline{mn}} = \delta R^+_{\underline{mn}}\ . \tag{5.3}
$$

The basic constraints (4.5) are solved (up to a gauge transformation) by choosing

$$\delta\nabla_\alpha = -i[\nabla_\alpha, \mathcal{V}] \ , \qquad \delta\bar{\nabla}_{\dot\alpha} = +i[\bar{\nabla}_{\dot\alpha}, \mathcal{V}] \ , \tag{5.4}$$

with the linearized external curvature $\delta\mathcal{W}_\alpha$ being given by

$$\delta\mathcal{W}_\alpha = -\frac{1}{4}[\bar{\mathcal{D}}_{\dot\beta}, \{\bar{\mathcal{D}}^{\dot\beta}, [\mathcal{D}_\alpha, \mathcal{V}]\}] + i[\boldsymbol{\mathcal{W}}_\alpha, \mathcal{V}] \ . \tag{5.5}$$

If we were discussing an abelian gauge theory, $\mathcal{V}$ would be the vector multiplet prepotential and $\delta\mathcal{W}_\alpha$ would be its linearized field strength. Here both become operators, whose form we will discuss shortly. Preserving the chirality constraint $[\nabla^+_{\underline{m}}, \bar{\nabla}_{\dot\alpha}] = 0$ then tells us that

$$\delta\nabla^+_{\underline{m}} = i[\partial_{\underline{m}}, \mathcal{V}] - \Lambda_{\underline{m}} \ , \tag{5.6}$$

where $\Lambda_{\underline{m}}$ is a chiral operator, $[\bar{\mathcal{D}}_{\dot\alpha}, \Lambda_{\underline{m}}] = 0$. It follows that $\delta\nabla_{\underline{m}} = -\frac{1}{2}(\Lambda_{\underline{m}} + \bar{\Lambda}_{\underline{m}})$ and

$$X_{\underline{m}} = [\partial_{\underline{m}}, \mathcal{V}] - \frac{1}{2i}(\Lambda_{\underline{m}} - \bar{\Lambda}_{\underline{m}}) \ , \qquad R^+_{\underline{mn}} = 2\,[\partial_{[\underline{m}}, \Lambda_{\underline{n}]}] \ , \tag{5.7}$$

Specifying the linearized geometry amounts to specifying the operators $\mathcal{V}$ and $\Lambda_{\underline{m}}$. There is some redundancy to this choice, as they can be taken to transform under pregauge transformations

$$\delta\mathcal{V} = \frac{1}{2i}(\Lambda - \bar{\Lambda}) \ , \qquad \delta\Lambda_{\underline{m}} = [\partial_{\underline{m}}, \Lambda] \ , \tag{5.8}$$

where Λ is a chiral operator.

5.1 Structure of the prepotentials

The operator $\mathcal{V}$ is real but as yet unconstrained, with an expansion

$$\begin{aligned}\mathcal{V} = {} & H^A\mathcal{D}_A + \mathcal{V}^{\underline{m}}\partial_{\underline{m}} + (\partial_{\underline{m}}\mathcal{V}^{\underline{n}} + \mathcal{V}_{\underline{m}}{}^{\underline{n}})g_{\underline{n}}{}^{\underline{m}} + \frac{1}{2}\mathcal{V}(M)^{ab}M_{ab} \\ & + \mathcal{V}(D)\,\mathbb{D} + \mathcal{V}(A)\,\mathbb{A} + \mathcal{V}(S)^\alpha S_\alpha + \mathcal{V}(S)_{\dot\alpha}\bar{S}^{\dot\alpha} + \mathcal{V}(K)^a K_a \ .\end{aligned} \tag{5.9}$$

We have denoted $\mathcal{V}^A$ by H^A, which is common in superspace literature. The superfield H^a is the $\mathcal{N} = 1$ gravitational prepotential. The superfield $\mathcal{V}^{\underline{m}}$ describes fluctuations of the Kaluza-Klein prepotential about the background. All the other prepotentials must be constrained in some way or turn out to be gauge artifacts as a consequence of the pregauge freedom Λ. The proper way to uncover the constraints is to take certain curvature tensors to vanish and to derive conditions on the prepotentials from these. We assume that $\mathcal{V}$ is a primary operator, but, aside from H^a, $\mathcal{V}^{\underline{m}}$, and $\mathcal{V}_{\underline{m}}{}^{\underline{n}}$, the individual prepotentials in its expansion are not primary.

The chiral operator $\Lambda_{\underline{m}}$ has a similar expansion

$$\begin{aligned}\Lambda_{\underline{m}} = {} & \Lambda_{\underline{m}}{}^A\mathcal{D}_A + \Lambda_{\underline{m}}{}^{\underline{n}}\partial_{\underline{n}} + (\partial_{\underline{n}}\Lambda_{\underline{m}}{}^{\underline{p}} + \Lambda_{\underline{m}}{}^{\underline{p}})g_{\underline{p}}{}^{\underline{n}} + \frac{1}{2}\Lambda_{\underline{m}}(M)^{ab}M_{ab} \\ & + \Lambda_{\underline{m}}(D)\,\mathbb{D} + \Lambda_{\underline{m}}(A)\,\mathbb{A} + \Lambda_{\underline{m}}(S)^\alpha S_\alpha + \Lambda_{\underline{m}}(S)_{\dot\alpha}\bar{S}^{\dot\alpha} + \Lambda_{\underline{m}}(K)^b K_b \ .\end{aligned} \tag{5.10}$$

The chirality constraint implies that the components of $\Lambda_{\underline{m}}{}^{A}$ are given by

$$\Lambda_{\underline{m}}{}^{\dot\alpha\alpha} = -\bar{\mathcal{D}}^{\dot\alpha}\Psi_{\underline{m}}{}^{\alpha} \ , \quad \Lambda_{\underline{m}}{}^{\alpha} = \frac{i}{8}\bar{\mathcal{D}}^2\Psi_{\underline{m}}{}^{\alpha} \ , \quad \Lambda_{\underline{m}\dot\alpha} \ \text{ unconstrained} \tag{5.11}$$

where $\Psi_{\underline{m}}{}^{\alpha}$ will play the role of the gravitino superfield. The internal diffeomorphism and GL(n) parameters are given as

$$\Lambda_{\underline{m},}{}^{\underline{n}} = \varphi_{\underline{m},}{}^{\underline{n}} - \mathcal{W}^{\beta\underline{n}}\Psi_{\underline{m}\beta} \ , \qquad \Lambda_{\underline{m},\underline{n}}{}^{\underline{p}} = \varphi_{\underline{m},\underline{n}}{}^{\underline{p}} + \partial_{\underline{n}}\Psi_{\underline{m}}{}^{\beta}\mathcal{W}_{\beta}{}^{\underline{p}} \tag{5.12}$$

where $\varphi_{\underline{m},}{}^{\underline{n}}$ and $\varphi_{\underline{m},\underline{n}}{}^{\underline{p}}$ are chiral superfields and $\mathcal{W}_{\alpha}{}^{\underline{m}}$ is the background Kaluza-Klein field strength. The other parameters are found to be

$$\begin{gathered}
\Lambda_{\underline{m}}(D) = \varphi_{\underline{m}} - \frac{1}{2}\bar{\mathcal{D}}^{\dot\gamma}\Lambda_{\underline{m}\dot\gamma} \ , \qquad \Lambda_{\underline{m}}(A) = \frac{i}{2}\varphi_{\underline{m}} - \frac{3i}{4}\bar{\mathcal{D}}^{\dot\gamma}\Lambda_{\underline{m}\dot\gamma} \ , \\
\Lambda_{\underline{m}}(M)_{\dot\alpha\dot\beta} = \bar{\mathcal{D}}_{(\dot\alpha}\Lambda_{\underline{m}\dot\beta)} \ , \qquad \Lambda_{\underline{m}}(M)_{\alpha\beta} = \varphi_{\underline{m}\alpha\beta} \ , \\
\Lambda_{\underline{m}}(S)_{\alpha} = \sigma_{\underline{m}\alpha} \ , \qquad \Lambda_{\underline{m}}(S)^{\dot\alpha} = \frac{1}{8}\bar{\mathcal{D}}^2\Lambda_{\underline{m}}{}^{\dot\alpha} \ , \\
\Lambda_{\underline{m}}(K)^{\dot\alpha\alpha} = -i\bar{\mathcal{D}}^{\dot\alpha}\sigma_{\underline{m}}{}^{\alpha} \ ,
\end{gathered} \tag{5.13}$$

where $\varphi_{\underline{m}}$ and $\varphi_{\underline{m}\alpha\beta}$ are chiral and $\sigma_{\underline{m}\alpha}$ is complex linear. If $\Lambda_{\underline{m}}$ is required to be primary, then the extra superfields $\Lambda_{\underline{m}\dot\alpha}$, $\varphi_{\underline{m}}$, $\varphi_{\underline{m}\alpha\beta}$ and $\sigma_{\underline{m}\alpha}$ are not primary and should be written in terms of other superfields that are.

The operator Λ describing pregauge transformations is identical to $\Lambda_{\underline{m}}$, but with the $\underline{m}$ index deleted. We relabel some of its components as

$$\begin{gathered}
\Psi_{\underline{m}}{}^{\alpha} \to 2i\,L^{\alpha} \ , \qquad \varphi_{\underline{m},}{}^{\underline{n}} \to \ell^{\underline{n}} \ , \qquad \varphi_{\underline{m},\underline{n}}{}^{\underline{p}} \to \ell_{\underline{n}}{}^{\underline{p}} \ , \\
\varphi_{\underline{m}} \to \ell \ , \qquad \varphi_{\underline{m}\alpha\beta} \to \ell_{\alpha\beta} \ , \qquad \sigma_{\underline{m}\alpha} \to \sigma_{\alpha} \ .
\end{gathered} \tag{5.14}$$

We emphasize that the ℓ's above are chiral while σ_α is complex linear. A few prepotentials can already be eliminated by a gauge choice using the pregauge Λ transformations. $\Lambda_{\dot\alpha}$ and $\bar\Lambda^{\alpha}$ are unconstrained superfields and can be used to fix $H_{\dot\alpha}$ and H^{α}. In order to keep $\mathcal{V}$ as a primary operator, one actually should choose

$$H^{\alpha} \overset{*}{=} -\frac{i}{8}\bar{\mathcal{D}}_{\dot\alpha}H^{\dot\alpha\alpha} \ , \qquad H_{\dot\alpha} \overset{*}{=} -\frac{i}{8}\mathcal{D}^{\alpha}H_{\alpha\dot\alpha} \ . \tag{5.15}$$

We denote this equality with a $*$ to emphasize that this is a choice. Similarly, the chiral superfield $\ell_{\underline{m}}{}^{\underline{n}}$ can be used to eliminate $\varphi_{\underline{m},}{}^{\underline{n}}$,

$$\varphi_{\underline{m},}{}^{\underline{n}} \overset{*}{=} 0 \ . \tag{5.16}$$

The other extra parameters in $\mathcal{V}$ and $\Lambda_{\underline{m}}$ must be eliminated by imposing curvature constraints so that only $H_{\alpha\dot\alpha}$, $\mathcal{V}^{\underline{m}}$, and $\Psi_{\underline{m}\alpha}$ (and possibly some chiral superfield $\Phi_{\underline{m}\underline{n}\alpha}$) remain.

5.2 Choosing curvature constraints on $\mathcal{W}_\alpha$ and $X_{\underline{m}}$

From the definition of $X_{\underline{m}}$, one can show that

$$X_{\underline{m}}{}^{\dot\alpha\alpha} = \partial_{\underline{m}} H^{\dot\alpha\alpha} - \frac{i}{2}(\bar{\mathcal{D}}^{\dot\alpha}\Psi_{\underline{m}}{}^{\alpha} + \mathcal{D}^{\alpha}\bar\Psi_{\underline{m}}{}^{\dot\alpha}) \ , \tag{5.17a}$$

$$X_{\underline{m}}{}^{\alpha} = \partial_{\underline{m}} H^{\alpha} - \frac{1}{16}\bar{\mathcal{D}}^2\Psi_{\underline{m}}{}^{\alpha} - \frac{i}{2}\bar\Lambda_{\underline{m}}{}^{\alpha} \ , \tag{5.17b}$$

$$X_{\underline{m}\dot\alpha} = \partial_{\underline{m}} H_{\dot\alpha} - \frac{1}{16}\mathcal{D}^2\bar\Psi_{\underline{m}\dot\alpha} + \frac{i}{2}\Lambda_{\underline{m}\dot\alpha} \ . \tag{5.17c}$$

The presence of the unconstrained $\bar\Lambda_{\underline{m}}{}^{\alpha}$ and $\Lambda_{\underline{m}\dot\alpha}$ mean $X_{\underline{m}}{}^{\alpha}$ and $X_{\underline{m}\dot\alpha}$ can be set however we wish, in analogy to H^α and $H_{\dot\alpha}$. $X_{\underline{m}}{}^{\dot\alpha\alpha}$ matches the linearized curvature (2.11b). The equations for $X_{\underline{m}}{}^{\alpha}$ and $X_{\underline{m}\dot\alpha}$ can be interpreted as *definitions* of $\bar\Lambda_{\underline{m}}{}^{\alpha}$ and $\Lambda_{\underline{m}\dot\alpha}$ in terms of these arbitrary curvatures. If we want $X_{\underline{m}}$ to be a primary operator, the natural choice is

$$X_{\underline{m}}{}^{\alpha} \overset{*}{=} -\frac{i}{8}\bar{\mathcal{D}}_{\dot\alpha}X_{\underline{m}}{}^{\dot\alpha\alpha} \ , \qquad X_{\underline{m}\dot\alpha} \overset{*}{=} -\frac{i}{8}\mathcal{D}^{\alpha}X_{\underline{m}\alpha\dot\alpha} \ . \tag{5.18}$$

It follows that

$$\Lambda_{\underline{m}\dot\alpha} \overset{*}{=} -\frac{i}{8}\mathcal{D}_\beta\bar{\mathcal{D}}_{\dot\alpha}\Psi_{\underline{m}}{}^{\beta} \ , \qquad \bar\Lambda_{\underline{m}}{}^{\alpha} \overset{*}{=} \frac{i}{8}\bar{\mathcal{D}}^{\dot\beta}\mathcal{D}^{\alpha}\bar\Psi_{\underline{m}\dot\beta} \ . \tag{5.19}$$

Next, let's impose a constraint on $\delta\mathcal{W}_\alpha$. The simplest constraint we can impose is that $\mathcal{W}_\alpha{}^b = \delta\mathcal{W}_\alpha{}^b = 0$. Using (5.5) and being careful to account for the variation of the covariant derivatives in the operator $\delta\mathcal{W}_\alpha$, one finds

$$\begin{aligned}\mathcal{V}(M)_\alpha{}^{\beta} + \frac{1}{2}\delta_\alpha{}^{\beta}(\mathcal{V}(D) - 2i\mathcal{V}(A)) = \mathcal{D}_\alpha H^\beta - \delta_\alpha{}^{\beta}\bar{\mathcal{D}}^{\dot\alpha}H_{\dot\alpha} - \frac{i}{4}\bar{\mathcal{D}}_{\dot\gamma}\mathcal{D}_\alpha H^{\dot\gamma\beta} \\ - \frac{i}{2}\mathcal{W}_\alpha{}^{\underline{m}}\Psi_{\underline{m}}{}^{\beta} + \text{chiral superfield} \ .\end{aligned} \tag{5.20}$$

The entire expression appears under $\bar{\mathcal{D}}^{\dot\beta}$ and so there is an undetermined chiral superfield on the right-hand side. Assuming $\mathcal{V}$ is primary, this chiral superfield is also primary. In fact, it can be eliminated using the chiral superfields ℓ and $\ell_{\alpha\beta}$ in the Λ pregauge freedom:

$$\mathcal{V}(M)_\alpha{}^{\beta} + \frac{1}{2}\delta_\alpha{}^{\beta}(\mathcal{V}(D) - 2i\mathcal{V}(A)) \overset{*}{=} \mathcal{D}_\alpha H^\beta - \delta_\alpha{}^{\beta}\bar{\mathcal{D}}^{\dot\alpha}H_{\dot\alpha} - \frac{i}{4}\bar{\mathcal{D}}_{\dot\gamma}\mathcal{D}_\alpha H^{\dot\gamma\beta} - \frac{i}{2}\mathcal{W}_\alpha{}^{\underline{m}}\Psi_{\underline{m}}{}^{\beta} \ . \tag{5.21}$$

From this and its complex conjugate, one can determine $\mathcal{V}(M)_{ab}$, $\mathcal{V}(D)$, and $\mathcal{V}(A)$.

Taking the same combination of $X_{\underline{m}}$'s, one can show that

$$\begin{aligned}X_{\underline{m}}(M)_\alpha{}^{\beta} + \frac{1}{2}\delta_\alpha{}^{\beta}(X_{\underline{m}}(D) - 2iX_{\underline{m}}(A)) = \mathcal{D}_\alpha X_{\underline{m}}{}^{\beta} - \delta_\alpha{}^{\beta}\bar{\mathcal{D}}^{\dot\alpha}X_{\underline{m}\dot\alpha} - \frac{i}{4}\bar{\mathcal{D}}_{\dot\gamma}\mathcal{D}_\alpha X_{\underline{m}}{}^{\dot\gamma\gamma} \\ + \frac{i}{2}\mathcal{W}_\alpha{}^{\underline{n}}\Psi_{\underline{nm}}{}^{\beta} - \frac{i}{2}\mathcal{W}_\alpha{}^{\underline{n}}\Phi_{\underline{nm}}{}^{\beta} \\ - \frac{1}{16}\bar{\mathcal{D}}^2\mathcal{D}_\alpha\Psi_{\underline{m}}{}^{\beta} + \frac{i}{2}\varphi_{\underline{m}}\delta_\alpha{}^{\beta} + \frac{i}{2}\varphi_{\underline{m}\alpha}{}^{\beta} \ .\end{aligned} \tag{5.22}$$

In computing the above, we have introduced a new field $\Phi_{\underline{mn}}{}^{\alpha}$ and required

$$\Psi_{\underline{nm}}{}^{\beta} = 2\,\partial_{[\underline{n}}\Psi_{\underline{m}]}{}^{\beta} + \Phi_{\underline{nm}}{}^{\beta} \tag{5.23}$$

to be invariant under Ξ transformations. At this point, the introduction of $\Phi_{\underline{mn}}{}^{\alpha}$ was *ad hoc* in order to ensure a Ξ invariance we are imposing by hand. The last four terms of (5.22) are chiral potentials, while the rest are curvatures. Making the choice

$$\frac{i}{2}\varphi_{\underline{m}}\delta_{\alpha}{}^{\beta} + \frac{i}{2}\varphi_{\underline{m}\alpha}{}^{\beta} \overset{*}{=} \frac{1}{16}\bar{\mathcal{D}}^2\mathcal{D}_{\alpha}\Psi_{\underline{m}}{}^{\beta} + \frac{i}{2}\mathcal{W}_{\alpha}{}^{\underline{n}}\Phi_{\underline{nm}}{}^{\beta}\ , \tag{5.24}$$

which also determines the chiral prepotentials $\varphi_{\underline{m}}$ and $\varphi_{\underline{m}\alpha\beta}$ separately, one finds

$$\begin{aligned} X_{\underline{m}}(M)_{\alpha}{}^{\beta} + \frac{1}{2}\delta_{\alpha}{}^{\beta}(X_{\underline{m}}(D) - 2iX_{\underline{m}}(A)) \overset{*}{=} \mathcal{D}_{\alpha}X_{\underline{m}}{}^{\beta} - \delta_{\alpha}{}^{\beta}\bar{\mathcal{D}}^{\dot{\alpha}}X_{\underline{m}\dot{\alpha}} \\ - \frac{i}{4}\bar{\mathcal{D}}_{\dot{\gamma}}\mathcal{D}_{\alpha}X_{\underline{m}}{}^{\dot{\gamma}\beta} + \frac{i}{2}\mathcal{W}_{\alpha}{}^{\underline{n}}\Psi_{\underline{nm}}{}^{\beta}\ . \end{aligned} \tag{5.25}$$

From this expression, one can determine $X_{\underline{m}}(M)_{ab}$, $X_{\underline{m}}(D)$, and $X_{\underline{m}}(A)$.

Now let's compute another curvature in $\mathcal{W}_{\alpha}$. It turns out that $\mathcal{W}_{\alpha}{}^{\beta}$ vanishes as a consequence of $\mathcal{W}_{\alpha}{}^{b}$ vanishing. The next curvature is $\mathcal{W}_{\alpha\dot{\beta}}$. Without going into great detail, one can show that

$$\mathcal{W}_{\alpha\dot{\alpha}} = -2i\mathcal{W}_{\alpha}{}^{\underline{m}}X_{\underline{m}\dot{\alpha}} - \frac{1}{2}\mathcal{W}^{\beta\underline{m}}\mathcal{D}_{\beta}X_{\underline{m}\alpha\dot{\alpha}} - \frac{1}{4}\mathcal{D}^{\beta}\mathcal{W}_{\beta}{}^{\underline{m}}X_{\underline{m}\alpha\dot{\alpha}} + iY_a(\sigma^a)_{\alpha\dot{\alpha}} \tag{5.26}$$

where Y_a is a real quantity given by

$$\begin{aligned} Y_{\alpha\dot{\alpha}} &= 2\mathcal{V}(K)_{\alpha\dot{\alpha}} + 2i\mathcal{D}_{\alpha}\mathcal{V}(S)_{\dot{\alpha}} + 2i\bar{\mathcal{D}}_{\dot{\alpha}}\mathcal{V}(S)_{\alpha} - \frac{i}{4}\mathcal{D}_{\alpha}\bar{\mathcal{D}}^2H_{\dot{\alpha}} - \frac{i}{4}\bar{\mathcal{D}}_{\dot{\alpha}}\mathcal{D}^2H_{\alpha} \\ &\quad - \frac{1}{32}(\mathcal{D}^{\beta}\bar{\mathcal{D}}^2\bar{\mathcal{D}}_{\dot{\beta}} + \bar{\mathcal{D}}_{\dot{\beta}}\mathcal{D}^2\bar{\mathcal{D}}^{\dot{\beta}})H_{\alpha\dot{\alpha}} + \frac{1}{8}\mathcal{D}^{\beta}\mathcal{W}_{\beta}{}^{\underline{m}}(\mathcal{D}_{\alpha}\bar{\Psi}_{\underline{m}\dot{\alpha}} - \bar{\mathcal{D}}_{\dot{\alpha}}\Psi_{\underline{m}\alpha}) \\ &\quad - \frac{i}{4}\mathcal{W}^{\beta\underline{m}}\mathcal{D}_{\beta}(\partial_{\underline{m}}H_{\alpha\dot{\alpha}} - i\bar{\mathcal{D}}_{\dot{\alpha}}\Psi_{\underline{m}\alpha}) + \frac{i}{4}\bar{\mathcal{W}}_{\dot{\beta}}{}^{\underline{m}}\bar{\mathcal{D}}^{\dot{\beta}}(\partial_{\underline{m}}H_{\alpha\dot{\alpha}} - i\mathcal{D}_{\alpha}\bar{\Psi}_{\underline{m}\dot{\alpha}})\ . \end{aligned} \tag{5.27}$$

The choice of $\mathcal{V}(K)_a$ amounts to a choice of Y_a. Two natural choices are to fix $Y_a \overset{*}{=} 0$ or to choose Y_a so that $\mathcal{W}_{\alpha\dot{\alpha}} \overset{*}{=} -\bar{\mathcal{W}}_{\dot{\alpha}\alpha}$, but the specific choice does not affect the following analysis.

Identifying $W_{\alpha\beta\gamma}$ as (proportional to) the totally symmetric part of $\mathcal{W}_{\alpha}(M)_{\beta\gamma}$, we find

$$\mathcal{W}_{(\alpha}(M)_{\beta\gamma)} = 2i\,W_{\alpha\beta\gamma} = \mathcal{W}_{\alpha}{}^{\underline{n}}\varphi_{\underline{n}\beta\alpha} - \frac{1}{4}\bar{\mathcal{D}}^2\mathcal{D}_{(\alpha}\mathcal{V}(M)_{\beta\gamma)}\ . \tag{5.28}$$

From the above expressions, one can show that

$$W_{\alpha\beta\gamma} \overset{*}{=} \frac{1}{16}\bar{\mathcal{D}}^2\Big[i\mathcal{D}_{\alpha}{}^{\dot{\gamma}}\mathcal{D}_{\beta}H_{\gamma\dot{\gamma}} + \mathcal{D}_{\alpha}(\mathcal{W}_{\beta}\lrcorner\Psi_{1\gamma}) - \mathcal{W}_{\alpha}\lrcorner\mathcal{D}_{\beta}\Psi_{1\gamma}\Big]_{(\alpha\beta\gamma)} - \frac{i}{2}\mathcal{W}_{(\alpha}\lrcorner\mathcal{W}_{\beta}\lrcorner\Phi_{2\gamma)}\ . \tag{5.29}$$

This is exactly the expression for $W_{\alpha\beta\gamma}$ that we have been seeking. The remaining trace part is also a chiral superfield Z_{α}. Writing

$$\mathcal{W}_{\alpha}(M)_{\beta\gamma} = -\epsilon_{\alpha(\beta}Z_{\gamma)} + 2i\,W_{\alpha\beta\gamma} \tag{5.30}$$

we find that

$$Z_\alpha = -\frac{1}{4}\bar{\mathcal{D}}^2\Big[4\,\mathcal{V}(S)_\alpha + \mathcal{D}_\alpha\mathcal{V}(D) + \frac{2i}{3}\mathcal{D}_\alpha\mathcal{V}(A) + \frac{i}{6}\mathcal{W}_\alpha\lrcorner\mathcal{D}_\beta\Psi_1{}^\beta - \frac{2}{3}\bar{\mathcal{W}}^{\dot\alpha}\lrcorner X_{1\alpha\dot\alpha}\Big] + \frac{1}{3}\mathcal{W}_\alpha\lrcorner\mathcal{W}_\beta\lrcorner\Phi_2{}^\beta \ . \tag{5.31}$$

The remaining undetermined prepotential $\mathcal{V}(S)_\alpha$ lets one choose Z_α however one wishes, at least in principle. A natural choice is

$$Z_\alpha \stackrel{*}{=} 0 \quad \Longrightarrow \quad \mathcal{V}(S)_\alpha \text{ determined} \ . \tag{5.32}$$

This determines $\mathcal{V}(S)_\alpha$ up to a complex linear superfield, which corresponds to the pregauge freedom $\ell(S)_\alpha$ within the chiral Λ operator. A curious feature of this choice is that it seems to require a non-covariant expression for $\mathcal{V}(S)_\alpha$, as one must introduce a prepotential for the background $\mathcal{W}_\alpha$ or for the field $\Phi_{2\alpha}$ in order to extract a $\bar{\mathcal{D}}^2$ from the last term in (5.31).

We have nearly exhausted all of the freedom to choose the components of the operator $\mathcal{V}$. The last element is $\mathcal{V}_{\underline{m}}{}^{\underline{n}}$. This can be fixed by observing that

$$X_{\underline{m}}{}^{\underline{n}} = -\mathcal{V}_{\underline{m}}{}^{\underline{n}} + \partial_{\underline{m}}\mathcal{V}^{\underline{n}} + \frac{i}{2}(\Lambda_{\underline{m},}{}^{\underline{n}} - \bar\Lambda_{\underline{m},}{}^{\underline{n}}) \ . \tag{5.33}$$

We will then make the simplifying choice

$$X_{\underline{m}}{}^{\underline{n}} \stackrel{*}{=} 0 \quad \Longrightarrow \quad \mathcal{V}_{\underline{m}}{}^{\underline{n}} \stackrel{*}{=} \partial_{\underline{m}}\mathcal{V}^{\underline{n}} + \frac{i}{2}(\Lambda_{\underline{m},}{}^{\underline{n}} - \bar\Lambda_{\underline{m},}{}^{\underline{n}}) \ . \tag{5.34}$$

Now that all components of the operator $\mathcal{V}$ have been fixed, all components of $\mathcal{W}_\alpha$ must now be determined, up to terms coming from undetermined pieces in the chiral $\Lambda_{\underline{m}}$ operator. Indeed, we find for the other dimension-1 components of $\mathcal{W}_\alpha$ that

$$\begin{aligned}
\mathcal{W}_\alpha(M)_{\dot\alpha\dot\beta} &= -\bar{\mathcal{D}}_{(\dot\alpha}\mathcal{W}_{\alpha\dot\beta)} \ , \\
\mathcal{W}_\alpha(D) &= \frac{1}{2}\bar{\mathcal{D}}^{\dot\gamma}\mathcal{W}_{\alpha\dot\gamma} - \frac{1}{4}\bar{\mathcal{D}}^2(X_{\underline{m}\alpha\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha\underline{m}}) + \frac{3}{2}Z_\alpha \ , \\
\mathcal{W}_\alpha(A) &= \frac{3i}{4}\bar{\mathcal{D}}^{\dot\gamma}\mathcal{W}_{\alpha\dot\gamma} - \frac{i}{8}\bar{\mathcal{D}}^2(X_{\underline{m}\alpha\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha\underline{m}}) + \frac{3i}{4}Z_\alpha \ , \\
\mathcal{W}_{\alpha\underline{m}}{}^{\underline{n}} &= \frac{1}{4}\bar{\mathcal{D}}^2(X_{\underline{m}\alpha\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha\underline{n}}) + \mathcal{W}_\alpha{}^{\underline{p}}\Big(2\,\varphi_{[\underline{p},\underline{m}]}{}^{\underline{n}} + \Phi_{[\underline{pm}]}{}^\beta\mathcal{W}_\beta{}^{\underline{n}}\Big) \\
&\stackrel{*}{=} \frac{1}{4}\bar{\mathcal{D}}^2(X_{\underline{m}\alpha\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha\underline{n}}) \ .
\end{aligned} \tag{5.35}$$

In the last equality, we have chosen $\varphi_{\underline{m},\underline{n}}{}^{\underline{p}}$ to simplify the expression and build a curvature. This does not determine the symmetric part of $\varphi_{\underline{m},\underline{n}}{}^{\underline{p}}$, but this will drop out of explicit expressions because lower form indices generally end up antisymmetrized.

The only remaining piece of $\Lambda_{\underline{m}}$ that is undetermined is the complex linear component $\sigma_{\underline{m}\alpha}$. This contributes to $X_{\underline{m}}(S)_\alpha$,

$$\begin{aligned}
X_{\underline{m}}(S)_\alpha &= \partial_{\underline{m}}\mathcal{V}(S)_\alpha - \frac{1}{2i}(\Lambda_{\underline{m}}(S)_\alpha - \bar\Lambda_{\underline{m}}(S)_\alpha) \\
&= \partial_{\underline{m}}\mathcal{V}(S)_\alpha - \frac{1}{2i}(\sigma_{\underline{m}\alpha} - \tfrac{1}{8}\mathcal{D}^2\Lambda_{\underline{m}\alpha})
\end{aligned} \tag{5.36}$$

This curvature then obeys the Bianchi identity

$$\frac{1}{4}\bar{\mathcal{D}}^2\Big[4\,X_{\underline{m}}(S)_\alpha + \mathcal{D}_\alpha X_{\underline{m}}(D) + \frac{2i}{3}\mathcal{D}_\alpha X_{\underline{m}}(A) - \frac{i}{6}\mathcal{W}_\alpha{}^{\underline{n}}\mathcal{D}_\beta\Psi_{\underline{nm}}{}^\beta - \frac{2}{3}\partial_{\underline{m}}(\bar{\mathcal{W}}^{\dot\alpha\underline{n}}X_{\underline{n}\alpha\dot\alpha})\Big]$$
$$= -\partial_{\underline{m}}Z_\alpha + \frac{1}{3}\mathcal{W}_\alpha{}^{\underline{n}}\mathcal{W}_\beta{}^{\underline{p}}\Phi_{\underline{pnm}}{}^\beta\ . \tag{5.37}$$

5.3 Some lower dimension results for $R^+_{\underline{mn}}$

The remaining curvature operator we have not directly addressed is $R^+_{\underline{mn}}$, which is built by taking the curl of the chiral operator $\Lambda_{\underline{m}}$. We find for $T^+_{\underline{mn}}{}^A$ and $F^+_{\underline{mn}}{}^{\underline{p}}$ the results

$$T^+_{\underline{mn}}{}^{\dot\alpha\alpha} = -\bar{\mathcal{D}}^{\dot\alpha}\Psi_{\underline{mn}}{}^\alpha\ , \tag{5.38a}$$
$$T^+_{\underline{mn}}{}^\alpha = \frac{i}{8}\bar{\mathcal{D}}^2\Psi_{\underline{mn}}{}^\alpha\ , \tag{5.38b}$$
$$T^+_{\underline{mn}\dot\alpha} = 2\,\partial_{[\underline{m}}\Lambda_{\underline{n}]\dot\alpha} \overset{*}{=} -\frac{i}{8}\mathcal{D}_\beta\bar{\mathcal{D}}_{\dot\alpha}\Psi_{\underline{mn}}{}^\beta\ , \tag{5.38c}$$
$$F^+_{\underline{mn}}{}^{\underline{p}} = 2\Lambda_{[\underline{m},\underline{n}]}{}^{\underline{p}} \overset{*}{=} -\Psi_{\underline{mn}}{}^\alpha\mathcal{W}_\alpha{}^{\underline{p}}\ . \tag{5.38d}$$

The other linearized curvatures can be computed directly in a similar way, but their forms will not be terribly enlightening.

5.4 Comparison to 5D and 6D results and summary

We have now accounted for all of the prepotentials and field strengths and uncovered the appropriate curvature constraints to remove all but one unconstrained spinor prepotential $\mathcal{V}(S)_\alpha$ and a complex linear prepotential $\sigma_{\underline{m}\alpha}$. These unfixed prepotentials can be eliminated by breaking manifest background covariance, but we will find it simpler to just leave them unfixed in the remainder, keeping in mind that they appear mainly in two curvatures – a chiral spinor superfield Z_α and the curvature superfield $X_{\underline{m}}(S)_\alpha$, which are related in terms of a complicated Bianchi identity (5.37) involving the field strength $\Phi_{\underline{mnp}\alpha}$.

At this stage, we can make a few brief comments connecting with the existing 5D and 6D work involving linearized supergravity. In the explicit linearized 5D construction of Sakamura [30], the covariant derivative $\hat\partial_y$ can be identified with the linearized $\nabla^+_{\underline{m}}$ when acting on chiral superfields, with the rescaling of $\Psi^{5\text{D}}_\alpha \to \frac{i}{2}\Psi_{\underline{m}\alpha}$.[9] Similarly, the gravitational superfield U^μ is identified with $-H_a$ here (keeping in mind $\sigma^\mu \to -\sigma^a$) and U^4 there is identified with $-\mathcal{V}^{\underline{m}}$ here. Similar comments pertain to the 6D results of Abe, Aoki, and Sakamura [32].

The major difference between our linearized results and previous results is that those papers fully describe 5D and 6D supergravity, whereas we aim only to describe the minimal extension of $\mathcal{N}=1$ conformal supergravity necessary to encode y-dependent superfields. We make no effort to identify the internal sector of the metric, with the understanding that from an $\mathcal{N}=1$ perspective that sector must correspond to "matter", i.e. some appropriately defined covariant superfields. Thus, in our formulation, there is no analogue to their gauge

[9] The precise identification requires going to a (complex) chiral gauge where $\bar\nabla_{\dot\alpha} = \bar D_{\dot\alpha}$ and $\nabla^+_{\underline{m}} = \partial_{\underline{m}} - \Lambda_{\underline{m}}$.

parameter N; that parameter is the 5D or 6D analogue of the complex 11D parameter $\Omega^{\underline{m}}$, and it encodes details of the higher dimensional sector beyond what is purely required for a covariant $\mathcal{N}=1$ supergeometry.

At this stage, we have developed enough intuition to address the full non-linear geometry. That will be our next task.

6 Exploring the non-linear Bianchi identities

Solving the Bianchi identities (BI.1) – (BI.6) in terms of curvature superfields $W_{\alpha\beta\gamma}$, $X_{\underline{m}\alpha\dot{\alpha}}$, $\Psi_{\underline{mn}\alpha}$, $\Phi_{\underline{mnp}\alpha}$, and $\mathcal{W}_\alpha{}^{\underline{m}}$ is a rather involved task, as most of the Bianchi identities just serve as consistency checks on lower dimension ones. Typically in superspace, one can invoke some version of Dragon's theorem [42], which states that the curvature superfields are completely determined by the torsion superfields, so solving the torsion tensor Bianchi identities is the only necessary step. In its original formulation, Dragon's theorem is limited to dimensions higher than three and for a tangent space group consisting of the Lorentz and R-symmetry groups. For this reason, it does not directly apply to either conformal superspace (where S and K curvatures are present) or to its extension here with internal torsion and GL(n) curvatures. Moreover, in boiling the Bianchi identities down to (BI.1) – (BI.6), we have *already* solved a number of them! It is possible that a modification of Dragon's theorem is possible, but we found it more direct to analyze the identities (BI.1) – (BI.6) exhaustively, taking guidance from the linearized case. In this section, we provide a summary of their solution, with some guideposts for the enterprising reader to reproduce. The reader interested only in the result may consult Appendix A where we summarize the supergeometry.

6.1 The chiral Bianchi identities (BI.1) and (BI.4)

The easiest Bianchi identities to solve are the ones imposing chirality on $\mathcal{W}_\alpha$ and $R^+_{\underline{mn}}$, eqs. (BI.1) and (BI.4). In both cases, to make the chirality analysis simpler, it is convenient to choose a chiral basis of derivatives – that is, we will choose to use $\nabla^+_{\underline{m}}$ instead of $\nabla_{\underline{m}}$. In general, this means $\mathcal{W}_\alpha$ must possess an expansion of the form

$$\begin{aligned}\mathcal{W}_\alpha = {}& \mathcal{W}_\alpha{}^b\nabla_b + \mathcal{W}_\alpha{}^\beta\nabla_\beta + \mathcal{W}_{\alpha\dot\beta}\bar\nabla^{\dot\beta} + \mathcal{W}_\alpha{}^{\underline{m}}\nabla^+_{\underline{m}} + \Big(\nabla^+_{\underline{n}}\mathcal{W}_\alpha{}^{\underline{m}} + \mathcal{W}_{\alpha\underline{n}}{}^{\underline{m}}\Big)g_{\underline{m}}{}^{\underline{n}} \\ & + \mathcal{W}_\alpha(D)\mathbb{D} + \mathcal{W}_\alpha(A)\mathbb{A} + \mathcal{W}_\alpha(M)^{\beta\gamma}M_{\beta\gamma} + \mathcal{W}_\alpha(M)_{\dot\beta\dot\gamma}M^{\dot\beta\dot\gamma} \\ & + \mathcal{W}_\alpha(S)^\beta S_\beta + \mathcal{W}_\alpha(S)_{\dot\beta}\bar S^{\dot\beta} + \mathcal{W}_\alpha(K)^b K_b\ .\end{aligned} \tag{6.1}$$

We have chosen to include an explicit $\nabla^+_{\underline{n}}\mathcal{W}_\alpha{}^{\underline{m}}$ term in the GL(n) piece so that it combines with $\mathcal{W}_\alpha{}^{\underline{m}}\nabla^+_{\underline{m}}$ to give the covariant internal Lie derivative $\mathcal{L}^+$ built from $\nabla^+_{\underline{m}}$.

Now we impose the constraint

$$\mathcal{W}_\alpha{}^b = 0\ . \tag{6.2}$$

In the linearized theory, recall this has the effect of fixing the underlying prepotentials $\mathcal{V}(M)$, $\mathcal{V}(D)$, and $\mathcal{V}(A)$. The Bianchi identity implies several simple conditions:

$$\mathcal{W}_\alpha{}^\beta = 0 \ , \qquad \bar\nabla^{\dot\alpha}\mathcal{W}_\alpha{}^{\underline{m}} = 0 \ , \qquad \bar\nabla^{\dot\alpha}\mathcal{W}_{\alpha\underline{m}}{}^{\underline{n}} = 0 \ , \qquad \bar\nabla^{\dot\alpha}\mathcal{W}_\alpha(M)_{\beta\gamma} = 0 \ . \tag{6.3}$$

No condition is imposed yet on $\mathcal{W}_{\alpha\dot\alpha}$, but higher curvatures are determined in terms of it:

$$\begin{aligned} \mathcal{W}_\alpha(D) &= \frac{1}{2}\bar\nabla^{\dot\gamma}\mathcal{W}_{\alpha\dot\gamma} + \phi_\alpha \ , \qquad & \mathcal{W}_\alpha(A) &= \frac{3i}{4}\bar\nabla^{\dot\gamma}\mathcal{W}_{\alpha\dot\gamma} + \frac{i}{2}\phi_\alpha \ , \\ \mathcal{W}_\alpha(M)_{\dot\alpha\dot\beta} &= -\bar\nabla_{(\dot\alpha}\mathcal{W}_{\alpha\dot\beta)} \ , & \mathcal{W}_\alpha(S)_{\dot\alpha} &= \frac{1}{8}\bar\nabla^2\mathcal{W}_{\alpha\dot\alpha} \ , \end{aligned} \tag{6.4}$$

where ϕ_α is an undetermined chiral superfield. The remaining chirality conditions amount to

$$\bar\nabla^2\mathcal{W}_\alpha(S)^\beta = 0 \ , \qquad \mathcal{W}_\alpha(K)^{\dot\beta\beta} = i\bar\nabla^{\dot\beta}\mathcal{W}_\alpha(S)^\beta \ . \tag{6.5}$$

The superfield $\mathcal{W}_\alpha{}^{\underline{m}}$ corresponds in the flat limit to the Kaluza-Klein field strength, and we have recovered its chirality condition. As in the linearized case, we expect the totally symmetric part of $\mathcal{W}_\alpha(M)_{\beta\gamma}$ to be the superfield $W_{\alpha\beta\gamma}$, and this is what happens if we drop the internal derivatives to recover $\mathcal{N} = 1$ conformal superspace. The other superfields will turn out to be composite, or correspond to curvatures that can be turned off by redefining certain connections.

The chirality condition on $R^+_{\underline{mn}}$ is also simple to analyze. Taking a similar decomposition

$$\begin{aligned} R^+_{\underline{mn}} = {} & T^+_{\underline{mn}}{}^B\nabla_B + F^+_{\underline{mn}}{}^{\underline{p}}\nabla^+_{\underline{p}} + \Big(\nabla^+_{\underline{p}}F^+_{\underline{mn}}{}^{\underline{q}} + \mathcal{R}_{\underline{mn}\underline{p}}{}^{\underline{q}}\Big)g_{\underline{q}}{}^{\underline{p}} \\ & + R^+_{\underline{mn}}(D)\mathbb{D} + R^+_{\underline{mn}}(A)\mathbb{A} + R^+_{\underline{mn}}(M)^{\beta\gamma}M_{\beta\gamma} + R^+_{\underline{mn}}(M)_{\dot\beta\dot\gamma}M^{\dot\beta\dot\gamma} \\ & + R^+_{\underline{mn}}(S)^\beta S_\beta + R^+_{\underline{mn}}(S)_{\dot\beta}S^{\dot\beta} + R^+_{\underline{mn}}(K)^c K_c \end{aligned} \tag{6.6}$$

one immediately finds that the chirality condition implies

$$T^+_{\underline{mn}}{}^{\dot\beta\beta} = -\bar\nabla^{\dot\beta}\Psi_{\underline{mn}}{}^\beta \ , \qquad T^+_{\underline{mn}}{}^\beta = \frac{i}{8}\bar\nabla^2\Psi_{\underline{mn}}{}^\beta \ , \tag{6.7}$$

for some 2-form spinor superfield $\Psi_{\underline{mn}\alpha}$. The remaining components of the Bianchi identify

impose no condition on $T^+_{\underline{mn}\dot\beta}$. The other components are

$$F^+_{\underline{mn}}{}^{\underline{p}} = -\Psi_{\underline{mn}}{}^{\alpha}\mathcal{W}_\alpha{}^{\underline{p}} + \Phi_{\underline{mn}}{}^{\underline{p}} \ , \tag{6.8a}$$

$$R^+_{\underline{mn}}(M)_{\alpha\beta} = -\Psi_{\underline{mn}}{}^{\gamma}\mathcal{W}_\gamma(M)_{\alpha\beta} + \Phi_{\underline{mn}\alpha\beta} \ , \tag{6.8b}$$

$$R^+_{\underline{mn}}(M)_{\dot\alpha\dot\beta} = \bar\nabla_{(\dot\alpha}\Psi_{\underline{mn}}{}^{\gamma}W_{\gamma\dot\beta)} + \bar\nabla_{(\dot\alpha}T^+_{\underline{mn}\dot\beta)} \ , \tag{6.8c}$$

$$R^+_{\underline{mn}}(D) = -\Psi_{\underline{mn}}{}^{\alpha}\phi_\alpha + \Phi_{\underline{mn}} - \frac{1}{2}\bar\nabla^{\dot\gamma}T^+_{\underline{mn}\dot\gamma} - \frac{1}{2}\bar\nabla^{\dot\gamma}\Psi_{\underline{mn}}{}^{\gamma}W_{\gamma\dot\gamma} \ , \tag{6.8d}$$

$$R^+_{\underline{mn}}(A) = -\frac{i}{2}\Psi_{\underline{mn}}{}^{\alpha}\phi_\alpha + \frac{i}{2}\Phi_{\underline{mn}} - \frac{3i}{4}\bar\nabla^{\dot\gamma}T^+_{\underline{mn}\dot\gamma} - \frac{3i}{4}\bar\nabla^{\dot\gamma}\Psi_{\underline{mn}}{}^{\gamma}W_{\gamma\dot\gamma} \ , \tag{6.8e}$$

$$R^+_{\underline{mn}}(S)_\alpha = -\Psi_{\underline{mn}}{}^{\gamma}\mathcal{W}_\gamma(S)_\alpha + \Sigma_{\underline{mn}\alpha} \ , \tag{6.8f}$$

$$R^+_{\underline{mn}}(S)_{\dot\alpha} = \frac{1}{8}\bar\nabla^2 T^+_{\underline{mn}\dot\alpha} + \frac{1}{8}\bar\nabla^2\Psi_{\underline{mn}}{}^{\gamma}W_{\gamma\dot\gamma} - \frac{1}{4}\bar\nabla_{\dot\phi}\Psi_{\underline{mn}}{}^{\gamma}\bar\nabla^{\dot\phi}W_{\gamma\dot\gamma} \ , \tag{6.8g}$$

$$R^+_{\underline{mn}}(K)_{\alpha\dot\alpha} = -i\bar\nabla_{\dot\alpha}R^+_{\underline{mn}}(S)_\alpha - i\bar\nabla^{\dot\alpha}\Psi_{\underline{mn}}{}^{\beta}\mathcal{W}_\beta(S)_\alpha \ , \tag{6.8h}$$

$$\mathcal{R}^+_{\underline{mnp}}{}^{\underline{q}} = -\Psi_{\underline{mn}}{}^{\alpha}\mathcal{W}_{\alpha\underline{p}}{}^{\underline{q}} + \nabla^+_{\underline{p}}\Psi_{\underline{mn}}{}^{\alpha}\mathcal{W}_\alpha{}^{\underline{q}} + \Phi_{\underline{mnp}}{}^{\underline{q}} \tag{6.8i}$$

In deriving these results, we used the explicit forms of some of the $\mathcal{W}_\alpha$ superfields. But there remain certain undetermined pieces. These are the chiral superfields $\Phi_{\underline{mn}}{}^{\underline{p}}$, $\Phi_{\underline{mn}\alpha\beta}$, $\Phi_{\underline{mn}}$, and $\Phi_{\underline{mnp}}{}^{\underline{q}}$, as well as the complex linear superfield $\Sigma_{\underline{mn}\alpha}$, which obeys $\bar\nabla^2\Sigma_{\underline{mn}\alpha} = 0$. From the linearized analysis, we know that $\Phi_{\underline{mn}}{}^{\underline{p}}$ can be eliminated by redefining a connection, so we choose it to vanish,

$$\Phi_{\underline{mn}}{}^{\underline{p}} = 0 \ .$$

6.2 Interlude: The $X_{\underline{m}}$ operator and variant covariant derivatives

Let us pause to make a few comments that will be useful very soon. We take the operator $X_{\underline{m}}$ that translates between the chiral internal derivative and the antichiral one to have an expansion as

$$X_{\underline{m}} = X_{\underline{m}}{}^{A}\nabla_A + X_{\underline{m}}{}^{x}g_x + X_{\underline{mn}}{}^{\underline{p}}g_{\underline{p}}{}^{\underline{n}} \ . \tag{6.9}$$

That is, we explicitly turn off any $X_{\underline{m}}{}^{\underline{n}}\nabla_{\underline{n}}$ term. This is sensible because $X_{\underline{m}}{}^{\underline{n}}$ has dimension zero and no such superfield seems possible to construct given our constituents. It is also justified from the linearized analysis. Recall that $X_{\underline{m}}{}^{a}$ coincides at the linearized level to (2.11b). Other $X_{\underline{m}}$ fields are of higher dimension and will correspond either to composite quantities or fields that can be removed by redefinitions.

We observe that the primary condition, $[S_\alpha, X_{\underline{m}}] = 0$, implies for the lowest three $X_{\underline{m}}$ fields that

$$S_\beta X_{\underline{m}}{}^{a} = 0 \ , \qquad S_\beta X_{\underline{m}}{}^{\alpha} = 0 \ , \qquad S_\beta X_{\underline{m}\dot\alpha} = -iX_{\underline{m}\beta\dot\alpha} \ , \tag{6.10}$$

and similarly for their complex conjugates. So $X_{\underline{m}}{}^{a}$ is primary, as expected for a fundamental curvature. The conditions on $X_{\underline{m}}{}^{\alpha}$ suggest that it be written as

$$X_{\underline{m}}{}^{\alpha} = -\frac{i}{8}\bar\nabla_{\dot\alpha}X_{\underline{m}}{}^{\dot\alpha\alpha} + \text{primary superfield} \tag{6.11}$$

and it is tempting to set the primary superfield above to zero, just as at the linearized level. However, it is going to be more useful to keep the non-primary superfield $X_{\underline{m}}{}^{\alpha}$ unfixed and work with it directly.

Assuming the above structure for the X operator, we can already compute some parts of the mixed curvature $R_{\underline{m}\alpha}$. We are interested in the Kaluza-Klein curvature piece,

$$F_{\underline{m}\alpha}{}^{\underline{n}} = i\, X_{\underline{m}\alpha\dot{\alpha}}\bar{\mathcal{W}}^{\dot{\alpha}\underline{n}} \,. \tag{6.12}$$

As anticipated, this is non-vanishing, which means the ∇_A we are using do not coincide with the $\mathring{\nabla}$ introduced in section 3.3 with the simplest $\mathrm{GL}(n)$ connections. Rather, we find that

$$\mathring{\nabla}_\alpha = \nabla_\alpha - F_{\underline{m}\alpha}{}^{\underline{n}}\, g_{\underline{n}}{}^{\underline{m}} \,. \tag{6.13}$$

These derivatives do not satisfy the first constraint of (4.5), whereas they do lead to vanishing mixed Kaluza-Klein curvatures, $\mathring{F}_{\underline{m}A}{}^{\underline{n}} = 0$. The advantage of using ∇_α is that it anticommutes with itself and the natural superfields we will be using are chiral or antichiral with respect to it.

Actually, we are going to discover that, at least when working with $\nabla^+_{\underline{m}}$, there is yet another spinor derivative that makes an appearance. It is defined by

$$\hat{\nabla}_\alpha := \nabla_\alpha - 2F_{\underline{m}\alpha}{}^{\underline{n}}\, g_{\underline{n}}{}^{\underline{m}} \,. \tag{6.14}$$

It is not hard to see that $[\hat{\nabla}_\alpha, \nabla^+_{\underline{m}}]$ has no Kaluza-Klein curvature. For this reason, $\{\hat{\nabla}_\alpha, \bar{\nabla}_{\dot{\alpha}}, \nabla^+_{\underline{m}}\}$ turn out to be a convenient set of derivatives to use when dealing with chiral objects as we have shoved all of the $\mathrm{GL}(n)$ connection into $\hat{\nabla}_\alpha$. We will see this derivative begin to make appearances very soon. Similarly, $\{\nabla_\alpha, \hat{\bar{\nabla}}_{\dot{\alpha}}, \nabla^-_{\underline{m}}\}$ turn out to be convenient to use with antichiral objects, where $\hat{\bar{\nabla}}^{\dot{\alpha}} := \bar{\nabla}^{\dot{\alpha}} - 2F_{\underline{m}}{}^{\dot{\alpha}\underline{n}}\, g_{\underline{n}}{}^{\underline{m}}$.

However, we emphasize that when we discuss the curvature tensors $R^+_{\underline{mn}}$ and $R^+_{\underline{m}\alpha}$, they are always here to be understood to be built using ∇_α, rather than $\hat{\nabla}_\alpha$, so as to avoid confusion.

6.3 The $\mathcal{W}_\alpha$ reality Bianchi identity (BI.2)

We introduce the abstract operator

$$\mathcal{Y} := -\frac{1}{4}\{\nabla^\alpha, \mathcal{W}_\alpha\} \tag{6.15}$$

The content of the Bianchi identity (BI.2) is that this is a real operator. Let's take its lowest engineering dimension components, $\mathcal{Y}^{\underline{m}}$ and $\mathcal{Y}^a$. The first leads to

$$\mathcal{Y}^{\underline{m}} = -\frac{1}{4}\hat{\nabla}^\alpha \mathcal{W}_\alpha{}^{\underline{m}} = -\frac{1}{4}\hat{\bar{\nabla}}_{\dot{\alpha}}\bar{\mathcal{W}}^{\dot{\alpha}\underline{m}} \,, \tag{6.16}$$

which reduces in flat space to the Bianchi identity for the Kaluza-Klein field strength. Note that it is $\hat{\nabla}_\alpha$ above, rather than ∇_α. The other lowest engineering dimension component is

$$\begin{aligned} i\mathcal{Y}_{\alpha\dot{\alpha}} &= \mathcal{W}_{\alpha\dot{\alpha}} - \frac{i}{2}\mathcal{W}^{\beta\underline{m}} T_{\underline{m}\beta\,\alpha\dot{\alpha}} + \frac{1}{4}\nabla\mathcal{W}^{\underline{m}} X_{\underline{m}\alpha\dot{\alpha}} \\ &= \mathcal{W}_{\alpha\dot{\alpha}} - \frac{i}{4}\mathcal{W}^{\beta\underline{m}} T^+_{\underline{m}\beta\,\alpha\dot{\alpha}} + \frac{1}{4}\hat{\nabla}\mathcal{W}^{\underline{m}} X_{\underline{m}\alpha\dot{\alpha}} \end{aligned} \tag{6.17}$$

where the mixed torsion tensor is given in Appendix A. Because $\mathcal{Y}^a$ is real, this constrains the real part of $\mathcal{W}_{\alpha\dot\alpha}$ to be

$$\mathcal{W}_{\alpha\dot\alpha} - \bar{\mathcal{W}}_{\dot\alpha\alpha} = -2i\,(\mathcal{W}_\alpha{}^{\underline{m}} X_{\underline{m}\dot\alpha} + \bar{\mathcal{W}}_{\dot\alpha}{}^{\underline{m}} X_{\underline{m}\alpha}) - \frac{1}{2}\mathcal{W}^{\beta\underline{m}}\hat\nabla_\beta X_{\underline{m}\alpha\dot\alpha} - \frac{1}{2}\bar\nabla_{\dot\beta}(\bar{\mathcal{W}}^{\dot\beta\underline{m}} X_{\underline{m}\alpha\dot\alpha}) \ . \quad (6.18)$$

From the linearized analysis, we know that $\mathcal{Y}^a$ can be fixed by a connection redefinition. One convenient choice is

$$\mathcal{Y}_a \stackrel{*}{=} 0 \qquad \Longrightarrow \qquad \mathcal{W}_{\alpha\dot\alpha} \stackrel{*}{=} \frac{i}{4}\mathcal{W}^{\beta\underline{m}} T^+_{\underline{m}\beta\,\alpha\dot\alpha} - \frac{1}{4}\hat\nabla\mathcal{W}^{\underline{m}} X_{\underline{m}\alpha\dot\alpha} \ . \quad (6.19)$$

Another choice is to take

$$\mathcal{W}_{\alpha\dot\alpha} \stackrel{*}{=} -\bar{\mathcal{W}}_{\dot\alpha\alpha} \stackrel{*}{=} -i\,(\mathcal{W}_\alpha{}^{\underline{m}} X_{\underline{m}\dot\alpha} + \bar{\mathcal{W}}_{\dot\alpha}{}^{\underline{m}} X_{\underline{m}\alpha}) - \frac{1}{4}\mathcal{W}^{\beta\underline{m}}\hat\nabla_\beta X_{\underline{m}\alpha\dot\alpha} - \frac{1}{4}\bar\nabla_{\dot\beta}(\bar{\mathcal{W}}^{\dot\beta\underline{m}} X_{\underline{m}\alpha\dot\alpha}) \ . \quad (6.20)$$

The Bianchi identity involving $\mathcal{Y}^\alpha$ is a bit more intricate. It allows one to determine the non-linear version of the combination (5.25),

$$\begin{aligned} X_{\underline{m}}(M)_\alpha{}^\beta + \frac{1}{2}\delta_\alpha{}^\beta(X_{\underline{m}}(D) - 2iX_{\underline{m}}(A)) = \hat\nabla_\alpha X_{\underline{m}}{}^\beta - \delta_\alpha{}^\beta\bar\nabla^{\dot\gamma} X_{\underline{m}\dot\gamma} + \frac{i}{4}\bar\nabla^{\dot\gamma}\hat\nabla_\alpha X_{\underline{m}}{}^\beta{}_{\dot\gamma} \\ + X_{\underline{m}\alpha\dot\beta}\bar{\mathcal{W}}^{\dot\beta\beta} + \frac{i}{2}\mathcal{W}_\alpha{}^{\underline{n}}\Psi_{\underline{n}\underline{m}}{}^\beta \end{aligned} \quad (6.21)$$

where $\hat\nabla_\alpha = \nabla_\alpha - 2F_{\underline{m}\alpha}{}^{\underline{n}} g_{\underline{n}}{}^{\underline{m}}$. In principle, there is an undetermined chiral superfield on the right-hand side, but it can be set to zero by a connection redefinition as in the linearized analysis. Separating $\mathcal{W}_\gamma(M)_{\beta\alpha}$ into spin-1/2 and spin-3/2 pieces as in the linearized analysis,

$$\mathcal{W}_\alpha(M)_{\beta\gamma} = -\epsilon_{\alpha(\beta} Z_{\gamma)} + 2i\,W_{\alpha\beta\gamma} \quad (6.22)$$

the $\mathcal{Y}^\alpha$ Bianchi then relates Z_α to ϕ_α in (6.4) as

$$\frac{2}{3}\phi_\alpha = Z_\alpha - \frac{1}{6}\bar\nabla^2(\mathcal{W}^{\dot\alpha\underline{m}} X_{\underline{m}\alpha\dot\alpha}) \ . \quad (6.23)$$

The remaining Bianchi identities in (BI.2) are more complicated. The ones at dimension two allow us to determine $\mathcal{W}_\alpha(S)^\beta$. Employing the shorthand,

$$\begin{aligned} \mathscr{Z}^x &:= \nabla^\alpha\mathcal{W}_\alpha{}^x + i\nabla^\alpha\mathcal{W}_\alpha{}^{\underline{m}} X_{\underline{m}}{}^x + 2\mathcal{W}^{\alpha\underline{m}} R_{\underline{m}\alpha}{}^x \\ &= \nabla^\alpha\mathcal{W}_\alpha{}^x + i\hat\nabla^\alpha\mathcal{W}_\alpha{}^{\underline{m}} X_{\underline{m}}{}^x + \mathcal{W}^{\alpha\underline{m}} R^+_{\underline{m}\alpha}{}^x \end{aligned} \quad (6.24)$$

the remaining Bianchi identities provide a definition for $\mathcal{W}_\alpha(S)^\beta$ as

$$\mathcal{W}_\alpha(S)^\beta := \frac{1}{8}\delta_\alpha{}^\beta(\mathscr{Z}(D) - \bar{\mathscr{Z}}(D)) + \frac{i}{12}\delta_\alpha{}^\beta(\mathscr{Z}(A) - \bar{\mathscr{Z}}(A)) - \frac{1}{4}(\mathscr{Z}(M)_\alpha{}^\beta - \bar{\mathscr{Z}}(M)_\alpha{}^\beta) \ . \quad (6.25)$$

In addition, one finds a consistency condition

$$\bar{\nabla}_{\dot{\alpha}}\mathcal{W}_\alpha(S)^\alpha = \mathscr{Z}(S)_{\dot{\alpha}} - \bar{\mathscr{Z}}(S)_{\dot{\alpha}} \tag{6.26}$$

and a Bianchi identity

$$\mathscr{Z}(K)^a = \bar{\mathscr{Z}}(K)^a \ . \tag{6.27}$$

The last corresponds to a complicated modification of the dimension-3 Bianchi identity that relates derivatives of $W_{\gamma\beta\alpha}$ to its complex conjugate. This is one of the fundamental Bianchi identities of the geometry, mentioned in footnote 2, but it lies at such high dimension one does not usually need its explicit form. As we have not worked out a useful compact way of writing it, we do not give it explicitly here.

These relations are compact, but not necessarily useful. For example, it is not immediately clear that the expression for $\mathcal{W}_\alpha(S)^\beta$ satisfies the complex linearity condition (6.5). This can be made more apparent by expanding it out:

$$\begin{aligned}\mathcal{W}_\alpha(S)^\beta = &-\frac{1}{4}\Big(\nabla^\gamma\mathcal{W}_\gamma(M)_\alpha{}^\beta + 2F_{\underline{m}}{}^{\gamma\underline{m}}\mathcal{W}_\gamma(M)_\alpha{}^\beta + \mathcal{W}^{\gamma\underline{m}}R^+_{\underline{m}\gamma}(M)_\alpha{}^\beta\Big)\\ &+\frac{1}{12}\delta_\alpha{}^\beta(\nabla^\gamma\phi_\gamma + 2F_{\underline{m}}{}^{\gamma\underline{m}}\phi_\gamma) + \frac{1}{8}\delta_\alpha{}^\beta\mathcal{W}^{\gamma\underline{m}}(R^+_{\underline{m}\gamma}(D) + \frac{2i}{3}R^+_{\underline{m}\gamma}(A))\\ &+\bar{\nabla}_{\dot{\gamma}}\Big[\frac{1}{4}\bar{\mathcal{W}}^{\dot{\gamma}}(M)_\alpha{}^\beta - \frac{i}{2}\bar{\mathcal{W}}^{\dot{\gamma}\underline{m}}X_{\underline{m}}(M)_\alpha{}^\beta\Big]\\ &+\delta_\alpha{}^\beta\,\bar{\nabla}_{\dot{\gamma}}\Big[-\frac{1}{8}\bar{\mathcal{W}}^{\dot{\gamma}}(D) + \frac{i}{4}\bar{\mathcal{W}}^{\dot{\gamma}\underline{m}}X_{\underline{m}}(D) - \frac{i}{12}\bar{\mathcal{W}}^{\dot{\gamma}}(A) - \frac{1}{6}\bar{\mathcal{W}}^{\dot{\gamma}\underline{m}}X_{\underline{m}}(A)\Big]\ .\end{aligned} \tag{6.28}$$

The last two lines are manifestly complex linear. The first two lines are complex linear by virtue of the Bianchi identities involving $\nabla^+_{\underline{m}}\mathcal{W}_\gamma(M)_\alpha{}^\beta$ and $\nabla^+_{\underline{m}}\phi_\alpha$, which we will encounter below. The expression could be evaluated further but we will postpone that for now.

6.4 The $\boldsymbol{\nabla^+_{\underline{m}}\mathcal{W}_\alpha}$ Bianchi identity (BI.3)

The Bianchi identity that directly links $\mathcal{W}_\alpha$ to $X_{\underline{m}}$ is (BI.3), which can be rewritten as

$$[\nabla^+_{\underline{m}}, \mathcal{W}_\alpha] = \frac{i}{8}[\bar{\nabla}_{\dot{\alpha}}, \{\bar{\nabla}^{\dot{\alpha}}, R^+_{\underline{m}\alpha}\}] \ . \tag{6.29}$$

We expand $R^+_{\underline{m}\alpha}$ in terms of $\nabla^+_{\underline{m}}$, leading to

$$R^+_{\underline{m}\alpha} = F^+_{\underline{m}\alpha}{}^{\underline{n}}\nabla^+_{\underline{n}} + T^+_{\underline{m}\alpha}{}^B\nabla_B + R^+_{\underline{m}\alpha}{}^x X_x + \nabla^+_{\underline{n}}F^+_{\underline{m}\alpha}{}^{\underline{p}}g_{\underline{p}}{}^{\underline{n}} + \mathcal{R}^+_{\underline{m}\alpha\underline{n}}{}^{\underline{p}}g_{\underline{p}}{}^{\underline{n}} \ . \tag{6.30}$$

Expanding out both sides of (6.29) leads to a number of identities. Simplifications occur upon using

$$\frac{i}{8}\Big(8iT^+_{\underline{m}\alpha}{}^\beta + 2\bar{\nabla}_{\dot{\beta}}T^+_{\underline{m}\alpha}{}^{\dot{\beta}\beta}\Big) = -\mathcal{W}_\alpha{}^{\underline{n}}\Psi_{\underline{n}\underline{m}}{}^\beta \ , \tag{6.31}$$

which holds on account of the explicit expression (6.21). The terms in (6.29) involving covariant derivatives become

$$0 = -\mathcal{W}_{\alpha\underline{m}}{}^{\underline{n}} + \mathcal{W}_{\alpha}{}^{\underline{p}}\Big(F^{+}_{\underline{pm}}{}^{\underline{n}} + \Psi_{\underline{pm}}{}^{\beta}\mathcal{W}_{\beta}{}^{\underline{n}}\Big) - \frac{i}{8}\bar{\nabla}^2 F^{+}_{\underline{m}\alpha}{}^{\underline{n}} \,, \tag{6.32a}$$

$$0 = \mathcal{W}_{\alpha}{}^{\underline{n}}T^{+}_{\underline{nm}}{}^{a} - \frac{i}{8}\bar{\nabla}^2 T^{+}_{\underline{m}\alpha}{}^{a} + \frac{1}{2}\bar{\nabla}^{\dot\beta}T^{+}_{\underline{m}\alpha}{}^{\beta}(\sigma^a)_{\beta\dot\beta} \,, \tag{6.32b}$$

$$0 = \mathcal{W}_{\alpha}{}^{\underline{n}}T^{+}_{\underline{nm}}{}^{\beta} - \frac{i}{8}\bar{\nabla}^2 T^{+}_{\underline{m}\alpha}{}^{\beta} \,, \tag{6.32c}$$

$$\begin{aligned} 0 = {}& \nabla^{+}_{\underline{m}}\mathcal{W}_{\alpha\dot\alpha} + \mathcal{W}_{\alpha}{}^{\underline{n}}\Big(T^{+}_{\underline{nm}\dot\alpha} + \Psi_{\underline{nm}}{}^{\gamma}\mathcal{W}_{\gamma\dot\alpha}\Big) \\ & - \frac{i}{8}\Big(\bar{\nabla}^2 T^{+}_{\underline{m}\alpha\dot\alpha} + \bar{\nabla}_{\dot\alpha}R^{+}_{\underline{m}\alpha}(D) + 2i\bar{\nabla}_{\dot\alpha}R^{+}_{\underline{m}\alpha}(A) + 2\bar{\nabla}_{\dot\beta}R^{+}_{\underline{m}\alpha}(M)^{\dot\beta}{}_{\dot\alpha} + 8R^{+}_{\underline{m}\alpha}(S)_{\dot\alpha}\Big) \,. \end{aligned} \tag{6.32d}$$

The first identity is solved by

$$\mathcal{W}_{\alpha\underline{m}}{}^{\underline{n}} = -\frac{i}{4}\bar{\nabla}^2 F_{\underline{m}\alpha}{}^{\underline{n}} \,. \tag{6.33}$$

The second and third identities hold automatically. The fourth identity leads to a definition of $X_{\underline{m}}(K)_a$ in terms of lower dimension quantities:

$$\begin{aligned} iX_{\underline{m}}(K)_{\alpha\dot\alpha} = {}& \frac{1}{4}\nabla^{+}_{\underline{m}}\mathcal{W}_{\alpha\dot\alpha} + \frac{1}{4}\mathcal{W}_{\alpha}{}^{\underline{n}}(T^{+}_{\underline{nm}\dot\alpha} + \Psi_{\underline{nm}}{}^{\gamma}\mathcal{W}_{\gamma\dot\alpha}) \\ & + \bar{\nabla}_{\dot\alpha}X_{\underline{m}}(S)_{\alpha} - F_{\underline{m}\dot\alpha}{}^{\underline{n}}X_{\underline{n}}(S)_{\alpha} + \frac{1}{16}\bar{\nabla}^2\Big[\nabla_{\alpha}X_{\underline{m}\dot\alpha} - 2F_{\underline{m}\alpha}{}^{\underline{n}}X_{\underline{n}\dot\alpha}\Big] \\ & + \frac{1}{8}\bar{\nabla}_{\dot\beta}\Big(\delta_{\underline{m}}{}^{\underline{n}}\nabla_{\alpha} - 2F_{\underline{m}\alpha}{}^{\underline{n}}\Big)\Big(X_{\underline{n}}(M)^{\dot\beta}{}_{\dot\alpha} + \frac{1}{2}X_{\underline{n}}(D)\,\delta^{\dot\beta}{}_{\dot\alpha} + iX_{\underline{n}}(A)\,\delta^{\dot\beta}{}_{\dot\alpha}\Big) \\ & - \text{h.c.} \end{aligned} \tag{6.34}$$

At dimension two, we find the Bianchi identities

$$\begin{aligned} 0 = {}& \nabla^{+}_{\underline{m}}\mathcal{W}_{\alpha}(D) + \mathcal{W}_{\alpha}{}^{\underline{n}}\Big(R^{+}_{\underline{nm}}(D) + \Psi_{\underline{nm}}{}^{\gamma}\mathcal{W}_{\gamma}(D)\Big) - \frac{i}{8}\Big[\bar{\nabla}^2 R^{+}_{\underline{m}\alpha}(D) + 4\epsilon^{\dot\beta\dot\alpha}\bar{\nabla}_{\dot\alpha}R^{+}_{\underline{m}\alpha}(S)_{\dot\beta}\Big] \,, \\ 0 = {}& \nabla^{+}_{\underline{m}}\mathcal{W}_{\alpha}(A) + \mathcal{W}_{\alpha}{}^{\underline{n}}\Big(R^{+}_{\underline{nm}}(A) + \Psi_{\underline{nm}}{}^{\gamma}\mathcal{W}_{\gamma}(A)\Big) - \frac{i}{8}\Big[\bar{\nabla}^2 R^{+}_{\underline{m}\alpha}(A) + 6i\epsilon^{\dot\beta\dot\alpha}\bar{\nabla}_{\dot\alpha}R^{+}_{\underline{m}\alpha}(S)_{\dot\beta}\Big] \,, \\ 0 = {}& \nabla^{+}_{\underline{m}}\mathcal{W}_{\alpha}(M)_{\dot\beta\dot\gamma} + \mathcal{W}_{\alpha}{}^{\underline{n}}\Big(R^{+}_{\underline{nm}}(M)_{\dot\beta\dot\gamma} + \Psi_{\underline{nm}}{}^{\gamma}\mathcal{W}_{\gamma}(M)_{\dot\beta\dot\gamma}\Big) - \frac{i}{8}\Big[\bar{\nabla}^2 R^{+}_{\underline{m}\alpha}(M)_{\dot\beta\dot\gamma} - 8\bar{\nabla}_{(\dot\beta}R^{+}_{\underline{m}\alpha}(S)_{\dot\gamma)}\Big] \,, \\ 0 = {}& \nabla^{+}_{\underline{m}}\mathcal{W}_{\alpha}(M)^{\beta\gamma} + \mathcal{W}_{\alpha}{}^{\underline{n}}\Big(R^{+}_{\underline{nm}}(M)^{\beta\gamma} + \Psi_{\underline{nm}}{}^{\delta}\mathcal{W}_{\delta}(M)^{\beta\gamma}\Big) - \frac{i}{8}\Big[\bar{\nabla}^2 R^{+}_{\underline{m}\alpha}(M)^{\beta\gamma}\Big] \,. \end{aligned} \tag{6.35}$$

The first three Bianchi identities hold on account of (6.32d) provided that

$$X_{\underline{m}}(S)_{\alpha} = -\frac{1}{4}\Big[\hat{\nabla}_{\alpha}X_{\underline{m}}(D) + X_{\underline{m}\alpha\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha}(D)\Big] - \frac{i}{6}\Big[\hat{\nabla}_{\alpha}X_{\underline{m}}(A) + X_{\underline{m}\alpha\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha}(A)\Big] + \Sigma^{(1)}_{\underline{m}\alpha} \,, \tag{6.36}$$

where $\Sigma^{(1)}_{\underline{m}\alpha}$ is a non-primary superfield obeying

$$-\frac{3}{2}\bar{\nabla}^2\Sigma^{(1)}_{\underline{m}\alpha} = \nabla^{+}_{\underline{m}}\phi_{\alpha} + \mathcal{W}_{\alpha}{}^{\underline{n}}\Phi_{\underline{nm}} \,. \tag{6.37}$$

From the trace part of $\nabla^+_{\underline{m}}\mathcal{W}_\alpha(M)^{\beta\gamma}$, we find a similar relation

$$X_{\underline{m}}(S)_\alpha = \frac{1}{6}\hat{\nabla}^\beta X_{\underline{m}}(M)_{\beta\alpha} - \frac{1}{6}X_{\underline{m}\beta\dot{\beta}}\bar{\mathcal{W}}^{\dot{\beta}}(M)^\beta{}_\alpha + \frac{i}{6}\nabla^+_{\underline{m}}F_{\underline{p}\alpha}{}^{\underline{p}} + \Sigma^{(2)}_{\underline{m}\alpha} \tag{6.38}$$

where

$$-\frac{3}{2}\bar{\nabla}^2\Sigma^{(2)}_{\underline{m}\alpha} = \nabla^+_{\underline{m}}\phi_\alpha - \mathcal{W}^{\beta\underline{n}}\Phi_{\underline{nm}\beta\alpha}\ . \tag{6.39}$$

Equating the two competing expressions for $X_{\underline{m}}(S)_\alpha$, one can compute the difference between $\Sigma^{(1)}_{\underline{m}\alpha}$ and $\Sigma^{(2)}_{\underline{m}\alpha}$. This leads to

$$\mathcal{W}^{\beta\underline{n}}\Big(\epsilon_{\alpha\beta}\Phi_{\underline{nm}} + \Phi_{\underline{nm}\alpha\beta}\Big) = \mathcal{W}^{\beta\underline{n}}\,\bar{\nabla}^2\Big[-\frac{i}{8}\hat{\nabla}_\beta\Psi_{\underline{nm}\alpha} - \frac{i}{4}\Psi_{\underline{nm}\beta}F_{\underline{p}\alpha}{}^{\underline{p}} + \frac{1}{16}T^+_{\underline{n}(\beta,\gamma)\dot{\gamma}}T^+_{\underline{m}(\alpha,}{}^{\gamma)\dot{\gamma}}\Big]\ , \tag{6.40}$$

which implies

$$\begin{aligned}\Phi_{\underline{nm}\,\alpha\beta} &= \bar{\nabla}^2\Big[-\frac{i}{8}\hat{\nabla}_{(\beta}\Psi_{\underline{nm}\alpha)} - \frac{i}{4}\Psi_{\underline{nm}(\beta}F_{\underline{p}\alpha)}{}^{\underline{p}} - \frac{1}{4}\hat{\nabla}_{(\beta}X_{[\underline{n}\gamma)\dot{\gamma}}\hat{\nabla}_{(\alpha}X_{\underline{m}]}{}^{\gamma)\dot{\gamma}}\Big] - \mathcal{W}_{(\alpha}{}^{\underline{p}}\Phi_{\underline{pnm}\,\beta)}\ ,\\ \Phi_{\underline{nm}} &= \bar{\nabla}^2\Big[\frac{i}{16}\hat{\nabla}^\gamma\Psi_{\underline{nm}\gamma} + \frac{i}{8}\Psi_{\underline{nm}}{}^\gamma F_{\underline{p}\gamma}{}^{\underline{p}}\Big] + \frac{1}{2}\mathcal{W}^{\beta\underline{p}}\Phi_{\underline{pnm}\,\beta}\ ,\end{aligned} \tag{6.41}$$

for some chiral primary 3-form superfield $\Phi_{\underline{pnm}\alpha}$. From the linearized analysis, we know this should indeed be the curvature $\Phi_{3\alpha}$ whose linearized form is $\partial\Phi_{2\alpha}$. Then one may define a primary superfield $\Sigma_{\underline{m}\alpha}$ by the relation

$$\begin{aligned}X_{\underline{m}}(S)_\alpha = &-\frac{1}{4}\Big[\hat{\nabla}_\alpha X_{\underline{m}}(D) + X_{\underline{m}\alpha\dot{\alpha}}\bar{\mathcal{W}}^{\dot{\alpha}}(D)\Big] - \frac{i}{6}\Big[\hat{\nabla}_\alpha X_{\underline{m}}(A) + X_{\underline{m}\alpha\dot{\alpha}}\bar{\mathcal{W}}^{\dot{\alpha}}(A)\Big] - \frac{1}{4}\bar{\nabla}^{\dot{\alpha}}\hat{\nabla}_\alpha X_{\underline{m}\dot{\alpha}}\\ &- \frac{2}{3}\mathcal{W}_\alpha{}^{\underline{n}}\Big(\frac{i}{16}\hat{\nabla}^\beta\Psi_{\underline{nm}\beta} + \frac{i}{8}\Psi_{\underline{nm}}{}^\beta F_{\underline{p}\beta}{}^{\underline{p}}\Big) - \frac{i}{6}\nabla^+_{\underline{m}}F_{\underline{p}\alpha}{}^{\underline{p}} + \Sigma_{\underline{m}\alpha}\ ,\end{aligned} \tag{6.42}$$

where $\Sigma_{\underline{m}\alpha}$ obeys

$$\bar{\nabla}^2\Sigma_{\underline{m}\alpha} = -\nabla^+_{\underline{m}}Z_\alpha - \frac{1}{3}\mathcal{W}_\alpha{}^{\underline{n}}\mathcal{W}^{\beta\underline{p}}\Phi_{\underline{pnm}\beta}\ . \tag{6.43}$$

This is a natural generalization of (6.36), where we have aimed to make $\Sigma_{\underline{m}\alpha}$ primary and to express it in terms of Z_α rather than ϕ_α. One could instead have aimed for a generalization of (6.38) (or some combination of (6.36) and (6.38)). This would involve shifting $\Sigma_{\underline{m}\alpha}$ by some primary complex linear superfield.

From the totally symmetric part of the $\nabla^+_{\underline{m}}\mathcal{W}_\alpha(M)^{\beta\gamma}$ Bianchi identity, we find

$$\begin{aligned}\nabla^+_{\underline{m}}W_{\alpha\beta\gamma} = &\frac{1}{16}\bar{\nabla}^2\Big[\hat{\nabla}_\alpha{}^{\dot{\gamma}}\hat{\nabla}_\beta X_{\underline{m}\gamma\dot{\gamma}} - \hat{\nabla}_\alpha(\mathcal{W}_\beta{}^{\underline{n}}\Psi_{\underline{nm}\gamma}) + 4\hat{\nabla}_\alpha X_{\underline{m}\beta\dot{\gamma}}\bar{\mathcal{W}}^{\dot{\gamma}\underline{n}}X_{\underline{n}\gamma} + 2i\,\hat{\nabla}_\alpha X_{\underline{m}\beta\dot{\gamma}}\bar{\mathcal{W}}^{\dot{\gamma}}{}_\gamma\Big]_{(\alpha\beta\gamma)}\\ &+ \frac{i}{2}\mathcal{W}_{(\alpha}{}^{\underline{n}}\Phi_{\underline{nm}\,\beta\gamma)}\ .\end{aligned} \tag{6.44}$$

This is the non-linear generalization of the fundamental Bianchi identity (2.12a).

The three highest dimension Bianchi identities are

$$
\begin{aligned}
0 &= \nabla^+_{\underline{m}}\mathcal{W}_\alpha(S)^\beta + \mathcal{W}_\alpha{}^{\underline{n}}\Big(R^+_{\underline{nm}}(S)^\beta + \Psi_{\underline{nm}}{}^\gamma\mathcal{W}_\gamma(S)^\beta\Big) - \frac{i}{8}\Big[\bar\nabla^2 R^+_{\underline{m}\alpha}(S)^\beta + 2i\bar\nabla_{\dot\beta}R^+_{\underline{m}\alpha}(K)^{\dot\beta\beta}\Big] \,, \\
0 &= \nabla^+_{\underline{m}}\mathcal{W}_\alpha(S)_{\dot\beta} + \mathcal{W}_\alpha{}^{\underline{n}}\Big(R^+_{\underline{nm}}(S)_{\dot\beta} + \Psi_{\underline{nm}}{}^\gamma\mathcal{W}_\gamma(S)_{\dot\beta}\Big) - \frac{i}{8}\Big[\bar\nabla^2 R^+_{\underline{m}\alpha}(S)_{\dot\beta}\Big] \,, \\
0 &= \nabla^+_{\underline{m}}\mathcal{W}_\alpha(K)^b + \mathcal{W}_\alpha{}^{\underline{n}}\Big(R^+_{\underline{nm}}(K)^b + \Psi_{\underline{nm}}{}^\gamma\mathcal{W}_\gamma(K)^b\Big) - \frac{i}{8}\Big[\bar\nabla^2 R^+_{\underline{m}\alpha}(K)^b\Big] \,.
\end{aligned}
\tag{6.45a}
$$

The first should be a consequence of the explicit form of $\mathcal{W}_\alpha(S)^\beta$ that we have derived in (6.28). It is not hard to show that the second and third are consequences of lower dimension identities.

6.5 The $\nabla^+_{[\underline{m}}R^+_{\underline{np}]} = 0$ Bianchi identity (BI.6)

Next, we analyze (BI.6). The lower dimension ones are

$$
\begin{aligned}
0 &= -\mathcal{R}^+_{[\underline{mnp}]}{}^{\underline{q}} + T^+_{[\underline{mn}}{}^B F^+_{B\underline{p}]}{}^{\underline{q}} + F^+_{[\underline{mn}}{}^{\underline{r}} F^+_{\underline{rp}]}{}^{\underline{q}} \,, && (6.46a)\\
0 &= \nabla^+_{[\underline{p}}T^+_{\underline{mn}]}{}^B + T^+_{[\underline{mn}}{}^C T^+_{C\underline{p}]}{}^B + F^+_{[\underline{mn}|}{}^{\underline{q}} T^+_{\underline{q}|\underline{p}]}{}^B \,, && (6.46b)\\
0 &= \nabla^+_{[\underline{p}}R^+_{\underline{mn}]}(D) + T^+_{[\underline{mn}}{}^C R^+_{C\underline{p}]}(D) + F^+_{[\underline{mn}|}{}^{\underline{q}} R^+_{\underline{q}|\underline{p}]}(D) \,, && (6.46c)\\
0 &= \nabla^+_{[\underline{p}}R^+_{\underline{mn}]}(A) + T^+_{[\underline{mn}}{}^C R^+_{C\underline{p}]}(A) + F^+_{[\underline{mn}|}{}^{\underline{q}} R^+_{\underline{q}|\underline{p}]}(A) \,, && (6.46d)\\
0 &= \nabla^+_{[\underline{p}}R^+_{\underline{mn}]}(M)^{\beta\gamma} + T^+_{[\underline{mn}}{}^C R^+_{C\underline{p}]}(M)^{\beta\gamma} + F^+_{[\underline{mn}|}{}^{\underline{q}} R^+_{\underline{q}|\underline{p}]}(M)^{\beta\gamma} \,, && (6.46e)\\
0 &= \nabla^+_{[\underline{p}}R^+_{\underline{mn}]}(M)_{\dot\beta\dot\gamma} + T^+_{[\underline{mn}}{}^C R^+_{C\underline{p}]}(M)_{\dot\beta\dot\gamma} + F^+_{[\underline{mn}|}{}^{\underline{q}} R^+_{\underline{q}|\underline{p}]}(M)_{\dot\beta\dot\gamma} \,, && (6.46f)
\end{aligned}
$$

and the higher dimension ones involving S_α, $\bar S^{\dot\alpha}$ and K_a follow the same pattern.

In analyzing the Bianchi identity on $T^+_{\underline{mn}}{}^a$, one discovers that

$$
\nabla^+_{[\underline{p}}\Psi_{\underline{mn}]\,\alpha} = -\frac{i}{4}\bar\nabla^{\dot\beta}\Psi_{[\underline{mn}}{}^\beta T^+_{\underline{p}]\beta\,\alpha\dot\beta} + \Psi_{[\underline{mn}|}{}^\beta\mathcal{W}_\beta{}^{\underline{q}}\Psi_{\underline{q}|\underline{p}]\,\alpha} + \frac{1}{3}\Phi_{\underline{pmn}\,\alpha} \,,
\tag{6.47}
$$

generalizing the linearized result (2.12c). This identity is found under an antichiral derivative, so the chiral superfield $\Phi_{3\alpha}$ is undetermined. From our linearized analysis, we know it involves $\partial\Phi_{2\alpha}$.

The Bianchi identity involving $T^+_{\underline{mn}\dot\beta}$ is not immediately useful because we do not yet have an independent expression for it. The remainder of the Bianchi identities lead to

$$
\begin{aligned}
\nabla^+_{\underline{m}}\phi_\alpha &= -W_\alpha{}^{\underline{n}}\Phi_{\underline{nm}} + \frac{i}{8}\bar\nabla^2\Big[\tfrac{3}{2}R^+_{\underline{p}\alpha}(D) + iR^+_{\underline{p}\alpha}(A)\Big] \,, \\
\nabla^+_{[\underline{p}}\Phi_{\underline{mn}]} &= \frac{1}{3}\Phi_{\underline{pmn}}{}^\alpha\phi_\alpha + \frac{i}{8}\bar\nabla^2\Big[\Psi_{[\underline{mn}}{}^\alpha\Big(\tfrac{3}{2}R^+_{\underline{p}]\alpha}(D) + iR^+_{\underline{p}]\alpha}(A)\Big)\Big] \,, \\
\nabla^+_{[\underline{p}}\Phi_{\underline{mn}]\,\alpha\beta} &= \frac{1}{3}\Phi_{\underline{pmn}}{}^\gamma W_\gamma(M)_{\alpha\beta} + \frac{i}{8}\bar\nabla^2(\Psi_{[\underline{mn}}{}^\gamma R^+_{\underline{p}]\gamma}(M)_{\alpha\beta}) \,, \\
\nabla^+_{[\underline{p}}\Sigma_{\underline{mn}]\alpha} &= \frac{1}{3}\Phi_{\underline{pmn}}{}^\beta W_\beta(S)_\alpha + \frac{i}{8}\bar\nabla^2(\Psi_{[\underline{mn}}{}^\beta R^+_{\underline{p}]\beta}(S)_\alpha) - \frac{1}{4}\bar\nabla^{\dot\beta}(\Psi_{[\underline{mn}}{}^\beta R^+_{\underline{p}]\beta}(K)_{\alpha\dot\beta}) \,, \\
\Phi_{[\underline{mnp}]}{}^{\underline{q}} &= -\frac{1}{3}\Phi_{\underline{mnp}}{}^\alpha W_\alpha{}^{\underline{q}} - \frac{i}{8}\bar\nabla^2(\Psi_{[\underline{mn}}{}^\beta F^+_{\underline{p}]\beta}{}^{\underline{q}}) \,.
\end{aligned}
\tag{6.48}
$$

The first corresponds to an identity we have seen already. The remaining ones should hold on account of the definitions of these various quantities, although we do not here give explicit forms for $\Sigma_{\underline{mn}\alpha}$ and $\Phi_{\underline{mnp}}{}^{\underline{q}}$.

As an integrability condition, one can now check $\nabla_1^+\nabla_1^+\Psi_{2\alpha}$. This leads to

$$0 = \nabla^+\Phi_{3\alpha} + \frac{1}{8}\bar{\nabla}^2\Big[i\Psi_2{}^\beta\hat{\nabla}_\beta\Psi_{2\alpha} + i\Psi_2{}^\beta\Psi_{2\beta}\,F_{\underline{m}\alpha}{}^{\underline{m}} - X_{1\alpha\gamma\dot\gamma}X_1{}^{\beta\dot\gamma\gamma}\Psi_{2\beta}\Big] \tag{6.49}$$

where $X_{1\,\alpha\beta\dot\beta} := \hat{\nabla}_{(\alpha}X_{1\beta)\dot\beta}$. This is the non-linear generalization of (2.12d). It confirms that one *cannot* set the field strength $\Phi_{3\alpha}$ consistently to zero.

6.6 The $R^+ - R^-$ Bianchi identity (BI.5)

The last batch of Bianchi identities to discuss are those arising from (BI.5),

$$\nabla_{[\underline{m}}X_{\underline{n}]} = \frac{i}{4}R^+_{\underline{mn}} - \frac{i}{4}R^-_{\underline{mn}}\ .$$

In expanding this expression, we must write both sides in terms of $\nabla_{\underline{m}}$ rather than $\nabla^+_{\underline{m}}$ or $\nabla^-_{\underline{m}}$. The lowest dimension pieces read

$$X_{[\underline{mn}]}{}^{\underline{p}} + X_{[\underline{m}}{}^B F_{\underline{n}]B}{}^{\underline{p}} = \frac{i}{4}F^+_{\underline{mn}}{}^{\underline{p}} - \frac{i}{4}F^-_{\underline{mn}}{}^{\underline{p}}\ , \tag{6.50a}$$

$$\nabla_{[\underline{m}}X_{\underline{n}]}{}^a + X_{[\underline{m}}{}^B T_{\underline{n}]B}{}^a = \frac{i}{4}(T^+_{\underline{mn}}{}^a - T^-_{\underline{mn}}{}^a) - \frac{1}{4}(T^+_{\underline{mn}}{}^{\underline{p}} + T^-_{\underline{mn}}{}^{\underline{p}})X_{\underline{p}}{}^a\ , \tag{6.50b}$$

$$\nabla_{[\underline{m}}X_{\underline{n}]}{}^\alpha + X_{[\underline{m}}{}^B T_{\underline{n}]B}{}^\alpha = \frac{i}{4}(T^+_{\underline{mn}}{}^\alpha - T^-_{\underline{mn}}{}^\alpha) - \frac{1}{4}(T^+_{\underline{mn}}{}^{\underline{p}} + T^-_{\underline{mn}}{}^{\underline{p}})X_{\underline{p}}{}^\alpha\ , \tag{6.50c}$$

$$\nabla_{[\underline{m}}X_{\underline{n}]\dot\alpha} + X_{[\underline{m}}{}^B T_{\underline{n}]B\dot\alpha} = \frac{i}{4}(T^+_{\underline{mn}\dot\alpha} - T^-_{\underline{mn}\dot\alpha}) - \frac{1}{4}(T^+_{\underline{mn}}{}^{\underline{p}} + T^-_{\underline{mn}}{}^{\underline{p}})X_{\underline{p}\dot\alpha}\ . \tag{6.50d}$$

The first equation defines $X_{\underline{mn}}{}^{\underline{p}}$ up to the symmetric part. There is no constraint on the symmetric part because lower GL(n) indices will always be antisymmetrized in our approach. Writing it as a vector-valued 2-form, we have several equivalent expressions:

$$\begin{aligned} X_2{}^1 &= -\frac{i}{4}\Psi_2{}^\alpha\mathcal{W}_\alpha{}^1 - \frac{1}{8}X_{1\beta\dot\alpha}X_1{}^{\dot\alpha\alpha}\hat{\nabla}^\beta\mathcal{W}_\alpha{}^1 - \frac{1}{4}X_{1\gamma\dot\gamma}\hat{\nabla}^\gamma X_1{}^{\dot\gamma\alpha}\mathcal{W}_\alpha{}^1 + \frac{i}{2}X_1{}^{\dot\alpha\alpha}F_{1\alpha}{}^1\lrcorner F_{1\dot\alpha}{}^1 + \text{h.c.} \\ &= -\frac{i}{4}\Psi_2{}^\alpha\mathcal{W}_\alpha{}^1 - \frac{1}{8}X_{1\beta\dot\alpha}X_1{}^{\dot\alpha\alpha}\nabla^\beta\mathcal{W}_\alpha{}^1 - \frac{1}{4}X_{1\gamma\dot\gamma}\nabla^\gamma X_1{}^{\dot\gamma\alpha}\mathcal{W}_\alpha{}^1 \\ &\quad - \frac{i}{4}X_{1\gamma\dot\gamma}X_1{}^{\gamma\dot\phi}\bar{\mathcal{W}}_{\dot\phi}{}^1\lrcorner X_1{}^{\dot\gamma\alpha}\mathcal{W}_\alpha{}^1 + \text{h.c.} \end{aligned} \tag{6.51}$$

A useful chiral form of this expression is

$$X_2{}^1 = \frac{i}{4}(F_2^{+1} - F_2^{-1}) + \frac{i}{4}X_1{}^{\dot\beta\beta}\bar{\nabla}_{\dot\beta}F_{1\beta}{}^1 + \frac{1}{4}X_1{}^{\dot\beta\beta}\hat{\nabla}_\beta X_{1\gamma\dot\beta}\mathcal{W}^{\gamma 1} + \frac{i}{2}X_1{}^a X_1{}^b F_{ba}{}^1\ . \tag{6.52}$$

The second equation (6.50b) gives

$$\nabla_{[\underline{m}}X_{\underline{n}]}{}^a + X_{[\underline{m}}{}^B T_{\underline{n}]B}{}^a = \frac{i}{4}(T^+_{\underline{mn}}{}^a - T^-_{\underline{mn}}{}^a) - \frac{1}{4}(F^+_{\underline{mn}}{}^{\underline{p}} + F^-_{\underline{mn}}{}^{\underline{p}})X_{\underline{p}}{}^a\ . \tag{6.53}$$

This gives the generalization of the linearized Bianchi identity (2.12b) relating $X_{\underline{m}a}$ to $\Psi_{\underline{mn}\alpha}$.

The remaining two equations (6.50c) and its complex conjugate (6.50d) give

$$\begin{aligned} T^+_{\underline{mn}\dot\alpha} &= T^-_{\underline{mn}\dot\alpha} - i(F^+_{\underline{mn}}{}^{\underline{p}} + F^-_{\underline{mn}}{}^{\underline{p}})X_{\underline{p}\dot\alpha} - 4i\Big(\nabla_{[\underline{m}}X_{\underline{n}]\dot\alpha} + X_{\underline{m}}{}^{B}T_{\underline{n}B\dot\alpha}\Big) \\ &= -\frac{i}{8}\nabla^2\Psi_{\underline{mn}\dot\alpha} - i(F^+_{\underline{mn}}{}^{\underline{p}} + F^-_{\underline{mn}}{}^{\underline{p}})X_{\underline{p}\dot\alpha} - 4i\Big(\nabla_{[\underline{m}}X_{\underline{n}]\dot\alpha} + X_{\underline{m}}{}^{B}T_{\underline{n}B\dot\alpha}\Big) \end{aligned} \tag{6.54}$$

as well as it complex conjugate. This defines the expression $T^+_{\underline{mn}\dot\alpha}$, which previously had not been determined.

The remaining identities, which we have not explicitly written out, lead to, among other consistency relations, explicit but complicated expressions for $\Phi_{\underline{mn}}$, $\Phi_{\underline{mn}\alpha\beta}$ and $\Sigma_{\underline{mn}}$. For example,

$$\begin{aligned} \frac{i}{2}\Phi_{\underline{mn}} &= \frac{i}{2}\Psi_{\underline{mn}}{}^{\alpha}\phi_\alpha + \frac{i}{2}\bar\nabla^{\dot\gamma}T^+_{\underline{mn}\dot\gamma} + \frac{i}{4}\nabla_\gamma T^-_{mn}{}^{\gamma} + \frac{i}{2}\bar\nabla^{\dot\gamma}\Psi_{\underline{mn}}{}^{\gamma}W_{\gamma\dot\gamma} + \frac{i}{4}\nabla^\gamma\bar\Psi_{\underline{mn}}{}^{\dot\gamma}\bar W_{\dot\gamma\gamma} \\ &\quad + \nabla_{[\underline{m}}(X_{\underline{n}]}(D) - 2iX_{\underline{n}]}(A)) + X_{[\underline{m}}{}^{B}(R_{\underline{n}]B}(D) - 2iR_{\underline{n}]B}(A)) \\ &\quad + \frac{1}{4}(F^+_{\underline{mn}}{}^{\underline{p}} + F^-_{\underline{mn}}{}^{\underline{p}})(X_{\underline{p}}(D) - 2iX_{\underline{p}}(A)) \ , \end{aligned} \tag{6.55}$$

$$\begin{aligned} \frac{i}{4}\Phi_{\underline{mn}\alpha\beta} &= \nabla_{[\underline{m}}X_{\underline{n}]}(M)_{\alpha\beta} + X_{[\underline{m}}{}^{D}R_{\underline{n}]D}(M)_{\alpha\beta} + \frac{1}{4}(F^+_{\underline{mn}}{}^{\underline{p}} + F^-_{\underline{mn}}{}^{\underline{p}})X_{\underline{p}}(M)_{\alpha\beta} \\ &\quad + \frac{i}{4}\nabla_{(\alpha}\Psi_{\underline{mn}\dot\gamma}\bar W^{\dot\gamma}{}_{\beta)} + \frac{i}{4}\nabla_{(\alpha}T^-_{\underline{mn}\beta)} + \frac{i}{4}\Psi_{\underline{mn}}{}^{\gamma}\mathcal{W}_\gamma(M)_{\alpha\beta} \ . \end{aligned} \tag{6.56}$$

It is a complicated exercise to check that the explicit solutions (6.41) that we found somewhat indirectly are consistent with these relations. We have confirmed this to leading order in curvatures.

7 Action principles

Having established the superspace geometry, we now turn to establishing the existence of superspace actions and the various technical rules for manipulating these actions, both in superspace and in components. The results in this section will not come as a surprise to the superspace expert. In short order, we establish:

- the consistency of both full and chiral superspace integration, provided one is given a suitable Lagrangian,
- the formula for converting a full superspace to a chiral superspace integral,
- the rules for integrating by parts in full and chiral superspace, and
- the expression for a component action arising from a chiral superspace integral.

Because the details are rather technical and only the results are important, we mainly sketch the computations required.

What we will not be concerned with here is describing how to build the Lagrangians required. As mentioned elsewhere, this will be the concern of a subsequent paper. The reader may keep in mind the 11D Chern-Simons action (2.7) as a prototype. It will turn out (with some minor modifications) to take the same form in this superspace.

7.1 Consistency of full and chiral superspace integration

A full superspace integral can be written

$$S = \int \mathrm{d}^4x\, \mathrm{d}^n y\, \mathrm{d}^4\theta\, E\, \mathscr{L} = \frac{1}{n!} \int \mathrm{d}^4x\, \mathrm{d}^n y\, \mathrm{d}^4\theta\, E\, \epsilon^{\underline{m}_1 \cdots \underline{m}_n} \omega_{\underline{m}_1 \cdots \underline{m}_n} \tag{7.1}$$

where $\omega_{\underline{m}_1 \cdots \underline{m}_n}$ is a real covariant n-form on the internal space and $E = \mathrm{sdet}(E_M{}^A)$ is the full superspace measure, defined as the superdeterminant (or Berezinian) of the supervielbein. Above we are denoting $\mathscr{L} \equiv \frac{1}{n!}\, \epsilon^{\underline{m}_1 \cdots \underline{m}_n}\, \omega_{\underline{m}_1 \cdots \underline{m}_n}$ where the antisymmetric tensor density $\epsilon^{\underline{m}_1 \cdots \underline{m}_n}$ has constant entries of ± 1. Thus $\omega_{\underline{m}_1 \cdots \underline{m}_n}$ is a top-form on the internal manifold and $\mathscr{L}$ is its scalar density.

In order for the action to be gauge invariant, $\omega_{\underline{m}_1 \cdots \underline{m}_n}$ (equivalently, $\mathscr{L}$) must be a conformal primary (annihilated by S-supersymmetry) of Weyl weight two.

The vielbein transforms under external diffeomorphisms (with parameter ξ^M), $\mathcal{H}$-gauge transformations (with parameter g^x), and internal diffeomorphisms (with parameter $\Lambda^{\underline{m}}$) as

$$\delta E_M{}^A = \hat{D}_M \xi^N E_N{}^A + \xi^N \hat{D}_N E_M{}^A + E_M{}^B g^x f_{xB}{}^A + \Lambda^{\underline{n}} \partial_{\underline{n}} E_M{}^A \ . \tag{7.2}$$

This means that the full superspace measure transforms as

$$\delta E = \hat{D}_M(\xi^M E) + \Lambda^{\underline{m}} \partial_{\underline{m}} E + g^x f_{xA}{}^A (-)^A\, E \ . \tag{7.3}$$

We require $\mathscr{L}$ to transform as

$$\delta \mathscr{L} = \hat{D}_M \mathscr{L} + \partial_{\underline{m}}(\Lambda^{\underline{m}} \mathscr{L}) - g^x f_{xA}{}^A\, (-)^A\, \mathscr{L} \ . \tag{7.4}$$

This is consistent with requiring $\omega_{\underline{m}_1 \cdots \underline{m}_m}$ to transform as an n-form under internal diffeomorphisms, a scalar field under external diffeomorphisms, and as a tensor with weight $-f_{xA}{}^A (-)^A$ under $\mathcal{H}$-gauge transformations. The action (7.1) is manifestly invariant under all but external diffeomorphisms. For these, we find (using internal form notation)

$$\delta S = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \hat{D}_M \Big(\xi^M E\, \omega \Big) = \int \mathrm{d}^4x\, \mathrm{d}^4\theta \left\{ \partial_M \Big(\xi^M E\, \omega \Big) - \partial \Big(\imath_{\xi^M A_M} \omega E \Big) \right\} = 0 \tag{7.5}$$

where we have used the property that ω is a top form on the internal space.

Showing consistency of chiral superspace integration is more involved. The basic integral looks like

$$S_c = \int \mathrm{d}^4x\, \mathrm{d}^n y\, \mathrm{d}^2\theta\, \mathcal{E}\, \mathscr{L}_c = \frac{1}{n!} \int \mathrm{d}^4x\, \mathrm{d}^n y\, \mathrm{d}^2\theta\, \mathcal{E}\, \epsilon^{\underline{m}_1 \cdots \underline{m}_n} \omega^c_{\underline{m}_1 \cdots \underline{m}_n} \tag{7.6}$$

where $\omega^c_{\underline{m}_1\cdots\underline{m}_n}$ is a covariant chiral n-form. The meaning of chirality here is that $\bar\nabla^{\dot\alpha}\omega^c_{\underline{m}_1\cdots\underline{m}_n} = 0$. The measure $\mathcal{E}$ must be defined. We are going to take the approach used in Appendix A of [43]. Write the full supervielbein and its inverse as

$$E_M{}^A = \begin{pmatrix} \mathcal{E}_\mathcal{M}{}^\mathcal{A} & E_{\mathcal{M}\dot\alpha} \\ E^{\dot\mu\mathcal{A}} & E^{\dot\mu}{}_{\dot\alpha} \end{pmatrix} , \qquad E_A{}^M = \begin{pmatrix} E_\mathcal{A}{}^\mathcal{M} & E_{\mathcal{A}\dot\mu} \\ E_\mathcal{A}{}^\mathcal{M} & \bar{\mathcal{E}}^{\dot\alpha}{}_{\dot\mu} \end{pmatrix} \tag{7.7}$$

with $\mathcal{M} = (m, \mu)$ and $\mathcal{A} = (a, \alpha)$ describing the coordinates and tangent space of chiral superspace. We have given special names to the blocks $\mathcal{E}_\mathcal{M}{}^\mathcal{A}$ and $\bar{\mathcal{E}}^{\dot\alpha}{}_{\dot\mu}$ and assume both of these are invertible with inverses $\mathcal{E}_\mathcal{A}{}^\mathcal{M}$ and $\bar{\mathcal{E}}^{\dot\mu}{}_{\dot\alpha}$. The chiral measure is $\mathcal{E} = \operatorname{sdet} \mathcal{E}_\mathcal{M}{}^\mathcal{A}$.

Since $\mathcal{E}_\mathcal{M}{}^\mathcal{A} = E_\mathcal{M}{}^\mathcal{A}$, we can use (7.2) for its transformations. Invariance of S_c under internal diffeomorphisms proceeds as before because $\mathcal{E}$ is a scalar and $\mathscr{L}_c$ is a scalar density. Invariance under $\mathcal{H}$-gauge transformations requires $f_x{}^{\dot\beta\,\mathcal{A}} = 0$. This can be understood as an integrability condition for the existence of $\mathcal{H}$-invariant chiral superfields. It also means that the chiral part of the vielbein only transforms into itself under $\mathcal{H}$ transformations, leading to

$$\delta_\mathcal{H}\mathcal{E} = \mathcal{E}\, g^x f_{x\mathcal{A}}{}^\mathcal{A}(-)^\mathcal{A} \qquad \Longrightarrow \qquad \delta_\mathcal{H}\mathscr{L}_c = -\mathscr{L}_c\, g^x f_{x\mathcal{A}}{}^\mathcal{A}(-)^\mathcal{A} \ . \tag{7.8}$$

To show invariance under external diffeomorphisms, it helps to consider *covariant* external diffeomorphisms: these are a special combination of external diffeomorphisms and $\mathcal{H}$-gauge transformations with $g^x = \xi^M H_M{}^x$. For the full supervielbein, these become

$$\begin{aligned} \delta_{cov}(\xi) E_M{}^A &= \hat D_M \xi^N E_N{}^A + \xi^N \hat D_N E_M{}^A + E_M{}^B \xi^N H_N{}^x f_{xB}{}^A \\ &= \nabla_M \xi^A + E_M{}^B \xi^C T_{CB}{}^A - E_M{}^B \xi^C F_{CB}{}^{\underline{m}} \chi_{\underline{m}}{}^A \end{aligned} \tag{7.9}$$

where we have rewritten the last line in terms of $\xi^A = \xi^M E_M{}^A$. We remind the reader that the field $\chi_{\underline{m}}{}^A$, discussed in detail in the bosonic case in section 3.1, can be understood as a component of a *super-sehrvielbein* on a larger superspace.

We now consider separately chiral external diffeomorphisms with $\xi^M = (\xi^\mathcal{M}, 0)$ and anti-chiral covariant external diffeomorphisms with $\xi^A = (0, \xi_{\dot\alpha})$.[10] Chiral external diffeomorphisms lead to an invariant action just as before. Under anti-chiral covariant external diffeomorphisms, one finds

$$\begin{gathered} \delta\mathcal{E}_\mathcal{M}{}^\mathcal{A} = E_\mathcal{M}{}^B \xi_{\dot\gamma} \Big(T^{\dot\gamma}{}_B{}^\mathcal{A} - F^{\dot\gamma}{}_B{}^{\underline{m}} \chi_{\underline{m}}{}^\mathcal{A} \Big) \qquad \Longrightarrow \\ \mathcal{E}^{-1}\delta\mathcal{E} = \xi_{\dot\gamma} \Big(T^{\dot\gamma}{}_\mathcal{A}{}^\mathcal{A} - F^{\dot\gamma}{}_\mathcal{A}{}^{\underline{m}} \chi_{\underline{m}}{}^\mathcal{A} \Big) + \mathcal{E}_\mathcal{A}{}^\mathcal{M} E_{\mathcal{M}\dot\beta} \xi_{\dot\gamma} \Big(T^{\dot\gamma\dot\beta\mathcal{A}} - F^{\dot\gamma\dot\beta\underline{m}} \chi_{\underline{m}}{}^\mathcal{A} \Big) \end{gathered} \tag{7.10}$$

Provided we satisfy the conditions

$$T^{\dot\gamma\dot\beta\mathcal{A}} = 0 \ , \qquad F^{\dot\gamma\dot\beta\underline{m}} = 0 \ , \tag{7.11}$$

the second batch of terms vanishes. The first batch of terms does not. In our case, it leads to

$$\mathcal{E}^{-1}\delta\mathcal{E} = \xi_{\dot\gamma} W_\gamma{}^{\underline{m}} (\bar\sigma_b)^{\dot\gamma\gamma} \Big(i X_{\underline{m}}{}^b - \chi_{\underline{m}}{}^b \Big) \tag{7.12}$$

[10] These span the entire space of external diffeomorphisms only when $\bar{\mathcal{E}}^{\dot\alpha}{}_{\dot\mu}$ is invertible.

In order for invariance to be maintained $\mathscr{L}_c$ must obey

$$\hat{\mathcal{D}}^{\dot\alpha}\mathscr{L}_c = -\mathscr{L}_c\,(\bar\sigma_b)^{\dot\alpha\beta}W_\beta{}^{\underline{m}}\Big(iX_{\underline{m}}{}^b - \chi_{\underline{m}}{}^b\Big)\ . \tag{7.13}$$

This derivative $\hat{\mathcal{D}}$ is the original $\hat{D}$ derivative augmented with the $\mathcal{H}$ connection. It does not possess the GL(n) connection. Recall the GL(n) connection involves

$$\Gamma_{M\underline{n}}{}^{\underline{p}} \sim -\chi_{\underline{n}}{}^N F_{NM}{}^{\underline{p}} + \Delta\Gamma_{M\underline{n}}{}^{\underline{p}} \quad\Longrightarrow\quad \Gamma^{\dot\alpha}{}_{\underline{n}}{}^{\underline{p}} \sim -(\bar\sigma_b)^{\dot\alpha\alpha}W_\alpha{}^{\underline{p}}\Big(iX_{\underline{n}}{}^b - \chi_{\underline{n}}{}^b\Big) \tag{7.14}$$

where we have used (3.27) for the shifted part of the GL(n) connection $\Delta\Gamma$. (The piece involving $\partial_{\underline{n}}A_M{}^{\underline{p}}$ is already contained in $\hat{D}$.) It follows that

$$\bar\nabla^{\dot\alpha}\mathscr{L}_c = D^{\dot\alpha}\mathscr{L}_c - \Gamma^{\dot\alpha}{}_{\underline{p}}{}^{\underline{p}}\mathscr{L}_c = 0 \tag{7.15}$$

as the condition for chiral integration to be well-defined.

We emphasize that the redefinition of the GL(n) connection was key to finding this simple chirality condition. With the original connection, we would have found $\mathring{\bar\nabla}^{\dot\alpha}\mathscr{L}_c \neq 0$, which is less convenient to work with.

7.2 Converting full superspace to chiral superspace

Now that we know that full superspace and chiral superspace separately exist, we should establish how to move from one to the other. We claim that (generalizing the flat superspace result)

$$\int \mathrm{d}^4x\,\mathrm{d}^ny\,\mathrm{d}^4\theta\,E\,\mathscr{L} = -\frac{1}{4}\int \mathrm{d}^4x\,\mathrm{d}^ny\,\mathrm{d}^2\theta\,\mathcal{E}\,\bar\nabla^2\mathscr{L}\ . \tag{7.16}$$

The proof goes as follows. Because of the basic condition $\{\bar\nabla^{\dot\alpha},\bar\nabla^{\dot\beta}\}=0$, we can adopt a chiral gauge where

$$\bar\nabla^{\dot\alpha} = \frac{\partial}{\partial\theta_{\dot\alpha}} - \Gamma^{\dot\alpha}{}_{\underline{m}}{}^{\underline{n}}g_{\underline{n}}{}^{\underline{m}}\ . \tag{7.17}$$

The GL(n) connection is

$$\Gamma^{\dot\alpha}{}_{\underline{m}}{}^{\underline{n}} = -\chi_{\underline{m}}{}^b F_b{}^{\dot\alpha\underline{n}} + F^{\dot\alpha}{}_{\underline{m}}{}^{\underline{n}} = (\chi_{\underline{m}}{}^{\dot\alpha\alpha} - iX_{\underline{m}}{}^{\dot\alpha\alpha})W_\alpha{}^{\underline{n}}\ . \tag{7.18}$$

The full superspace and chiral superspace measures are equal, $E=\mathcal{E}$, and furthermore,

$$\partial^{\dot\alpha}\mathcal{E} = \mathcal{E}\Big(T^{\dot\alpha}{}_b{}^b - F^{\dot\alpha}{}_b{}^{\underline{p}}\chi_{\underline{p}}{}^b\Big) = \mathcal{E}W_\alpha{}^{\underline{m}}(\bar\sigma_b)^{\dot\alpha\alpha}\Big(iX_{\underline{m}}{}^b - \chi_{\underline{m}}{}^b\Big) = -\Gamma^{\dot\alpha}{}_{\underline{m}}{}^{\underline{m}} \tag{7.19}$$

It follows that

$$\int \mathrm{d}^4x\,\mathrm{d}^ny\,\mathrm{d}^4\theta\,E\,\mathscr{L} = -\frac{1}{4}\int \mathrm{d}^4x\,\mathrm{d}^ny\,\mathrm{d}^2\theta\,\mathcal{E}\,(\bar\partial_{\dot\alpha} - \Gamma_{\dot\alpha\underline{n}}{}^{\underline{n}})(\bar\partial^{\dot\alpha} - \Gamma^{\dot\alpha}{}_{\underline{m}}{}^{\underline{m}})\mathscr{L} \tag{7.20}$$

The operators appearing in parentheses are just $\bar\nabla_{\dot\alpha}$ in chiral gauge, so it follows that the two sides of (7.16) are equal to each other in chiral gauge. But because they are both gauge invariant expressions, they must be equal in all gauges.

7.3 Rules for integrations by parts

There turn out to be three useful expressions for integrating by parts in superspace. These are most simply formulated in terms of the vanishing (or near vanishing) of certain total covariant derivatives.

The first expression is relevant for integrating by parts with external covariant derivatives in full superspace. Suppose $V^A = (V^a, V^\alpha, V_{\dot\alpha})$ is some covariant expression, with not necessarily all of these entries nonzero. (Of course, V^A must be a scalar density under internal diffeomorphisms.) Then one can show that

$$\begin{aligned}\int \mathrm{d}^4x\,\mathrm{d}^n y\,\mathrm{d}^4\theta\,E\,\nabla_A V^A\,(-)^A &= -\int \mathrm{d}^4x\,\mathrm{d}^n y\,\mathrm{d}^4\theta\,\Big(E\,V^B(T_{BA}{}^A(-)^A + F_{B\underline{m}}{}^{\underline{m}}) + H_M{}^x g_x(EV^M)\Big)\\ &= -\int \mathrm{d}^4x\,\mathrm{d}^n y\,\mathrm{d}^4\theta\,H_M{}^x g_x(EV^M)\ .\end{aligned} \tag{7.21}$$

The first equality follows rather generally, while the second follows for the particular constraints on our superspace torsion and curvature tensors we have chosen. The residual term arises if $E\,V^M = E\,V^A E_A{}^M$ is not a gauge singlet; in practice, this involves only the S and K connections and such terms cancel out if, after a series of integrations by parts, the initial and final forms are both primary.

The other expressions involve integrating by parts with internal covariant derivatives. In full superspace, one can use either $\nabla_{\underline{m}}$ or $\nabla^\pm_{\underline{m}}$ and the results are structurally similar:

$$\int \mathrm{d}^4x\,\mathrm{d}^n y\,\mathrm{d}^4\theta\,E\,\nabla_{\underline{m}} V^{\underline{m}} = -\int \mathrm{d}^4x\,\mathrm{d}^n y\,\mathrm{d}^4\theta\,E\,V^{\underline{m}}\Big(T^\pm_{\underline{m}A}{}^A(-)^A + F^\pm_{\underline{mn}}{}^{\underline{n}}\Big) = 0 \tag{7.22}$$

$$\int \mathrm{d}^4x\,\mathrm{d}^n y\,\mathrm{d}^4\theta\,E\,\nabla^\pm_{\underline{m}} V^{\underline{m}} = -\int \mathrm{d}^4x\,\mathrm{d}^n y\,\mathrm{d}^4\theta\,E\,V^{\underline{m}}\Big(T^\pm_{\underline{m}A}{}^A(-)^A + F^\pm_{\underline{mn}}{}^{\underline{n}}\Big) = 0\ . \tag{7.23}$$

Here we assume $EV^{\underline{m}}$ is $\mathcal{H}$-invariant for simplicity (as well as an internal vector density) so that $\mathcal{H}$ connections do not appear. This will always be the case when we need to integrate internal covariant derivatives by parts. As before, the expressions involving the traces of the torsion and curvature tensors cancel out for our superspace geometry. In chiral superspace, we will only need to integrate $\nabla^\pm_{\underline{m}}$ by parts. Its rule is similar:

$$\int \mathrm{d}^4x\,\mathrm{d}^n y\,\mathrm{d}^2\theta\,\mathcal{E}\,\nabla^+_{\underline{m}}\mathcal{V}^{\underline{m}} = -\int \mathrm{d}^4x\,\mathrm{d}^n y\,\mathrm{d}^2\theta\,\mathcal{E}\,\mathcal{V}^{\underline{m}}\Big(T^+_{\underline{m}\mathcal{A}}{}^{\mathcal{A}}(-)^{\mathcal{A}} + F^+_{\underline{mn}}{}^{\underline{n}}\Big) = 0\ . \tag{7.24}$$

To be well-defined, $\mathcal{V}^{\underline{m}}$ must be chiral, a vector density, and transform so that $\mathcal{E}\mathcal{V}^{\underline{m}}$ is $\mathcal{H}$-invariant.

The proof of (7.21) is completely standard. The proof of (7.22) is only a bit more involved. We give a few steps to guide the reader. Discarding total derivatives in equalities

and suppressing gradings,

$$\begin{aligned} E\nabla_{\underline{m}} V^{\underline{m}} &= -\nabla_{\underline{m}} E V^{\underline{m}} - \chi_{\underline{m}}{}^{N} \nabla_N (E V_{\underline{m}}) - 2\,\Gamma_{[\underline{mn}]}{}^{\underline{n}} (E V^{\underline{m}}) \\ &= -\nabla_N \Big(E V^{\underline{m}} \chi_{\underline{m}}{}^{N} \Big) - E V^{\underline{m}} (T_{\underline{m}A}{}^{A} + F_{N\underline{m}}{}^{\underline{n}} \chi_{\underline{n}}{}^{N}) - 2\,\Gamma_{[\underline{mn}]}{}^{\underline{n}} (E V^{\underline{m}}) \\ &= (F_{N\underline{n}}{}^{\underline{n}} - \chi_{\underline{n}}{}^{M} F_{MN}{}^{\underline{n}}) \Big(E V^{\underline{m}} \chi_{\underline{m}}{}^{N} \Big) - E V^{\underline{m}} (T_{\underline{m}A}{}^{A} + F_{N\underline{m}}{}^{\underline{n}} \chi_{\underline{n}}{}^{N}) - 2\,\Gamma_{[\underline{mn}]}{}^{\underline{n}} (E V^{\underline{m}}) \\ &= -E V^{\underline{m}} (T_{\underline{m}A}{}^{A} + F_{\underline{mn}}{}^{\underline{n}}) \ . \end{aligned} \tag{7.25}$$

The corresponding expression for (7.23) follows just by affixing $\pm$ superscripts to the internal connections and curvatures, defining them with respect to $\nabla^{\pm}_{\underline{m}}$. The rule for (7.24) is a bit more involved but the fact that it vanishes follows from (7.23) by converting to chiral superspace and identifying $\mathcal{V}^{\underline{m}} = -\frac{1}{4}\bar{\nabla}^2 V^{\underline{m}}$.

7.4 Chiral superspace to components

The final result we should discuss is how to convert a superspace integral to components. Since any full superspace integral may be converted to chiral superspace using (7.16), it suffices to show how to evaluate the chiral θ integrations. The result we want to establish is

$$S_c = \int \mathrm{d}^4x\, \mathrm{d}^n y\, \mathrm{d}^2\theta\, \mathcal{E}\, \mathscr{L}_c = \int \mathrm{d}^4x\, \mathrm{d}^n y\, e\, \mathcal{L}_c \tag{7.26}$$

where

$$\begin{aligned} \mathcal{L}_c = &-\frac{1}{4}\nabla^2 \mathscr{L}_c + \frac{i}{2}(\bar{\psi}_m \bar{\sigma}^m)^\alpha \nabla_\alpha \mathscr{L}_c - \bar{\psi}_m \bar{\sigma}^{mn} \bar{\psi}_n\, \mathscr{L}_c \\ &- 2i\, \bar{\mathcal{W}}_{\dot{\alpha}}{}^{\underline{m}} (\chi_{\underline{m}}{}^{\dot{\alpha}} + i X_{\underline{m}}{}^{\dot{\alpha}})\, \mathscr{L}_c + i\, e_a{}^m \bar{\psi}_{m\dot{\alpha}}\, \mathcal{W}^{\dot{\alpha}\underline{m}} (\chi_{\underline{m}}{}^{a} + i X_{\underline{m}}{}^{a})\, \mathscr{L}_c \ . \end{aligned} \tag{7.27}$$

Some of the above result may be guessed without much work. The first term is the flat superspace result, and the rest of the first line is its generalization to conformal supergravity. Additional terms essentially can only involve the terms found in the second line, and some of the relative coefficients can be determined by S-invariance.

A standard way of deriving the above result is to exploit the ectoplasmic approach [44, 45]. In conventional $N = 1$ superspace, this amounts to treating the component Lagrangian as a 4-form in superspace, writing

$$S_c = \int_M \frac{1}{4!} E^A E^B E^C E^D\, J_{DCBA} \tag{7.28}$$

where the integral is restricted to the bosonic spacetime M lying at $\theta = 0$. The condition that the action is supersymmetric amounts to J being a closed superform. Choosing the components of J appropriately then leads to the desired result. While the ectoplasmic approach does lead to (7.27), it is a bit subtle because in our case the full superspace is actually extended by the n internal coordinates and so J is actually a $(4+n)$-form. Care must be taken to account for this.

A technically simpler approach is to use a brute force normal coordinate method. Starting from chiral superspace lying at $\bar\theta = 0$, use the residual θ-dependent coordinate and gauge transformations to set $\nabla_\alpha = \partial_\alpha - \Gamma_{\alpha\underline{m}}{}^{\underline{n}} g_{\underline{n}}{}^{\underline{m}}$ where $\Gamma_{\alpha\underline{m}}{}^{\underline{n}} = -(\chi_{\underline{m}}{}^a + iX_{\underline{m}}{}^a)(\sigma_a)_{\alpha\dot\alpha}\mathcal{W}^{\dot\alpha\underline{n}}$. In this gauge, $\mathcal{E} = \det(e_m{}^a) = e$, so evaluating the θ integrals gives

$$e\,\mathcal{L}_c = -\frac{1}{4}\partial^\alpha\partial_\alpha(e\mathscr{L}_c)\ . \tag{7.29}$$

To evaluate these terms, the following results are useful (in this gauge):

$$\begin{aligned}
\partial_\alpha e_m{}^a &= T_{\alpha m}{}^a - F_{\alpha m}{}^{\underline{p}}\chi_{\underline{p}}{}^a = i(\sigma^a\bar\psi_m)_\alpha + (\sigma_m)_{\alpha\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha\underline{p}}(\chi_{\underline{p}}{}^a + iX_{\underline{p}}{}^a)\ ,\\
e^{-1}\partial_\alpha e &= i(\sigma^m\bar\psi_m)_\alpha - \Gamma_{\alpha\underline{m}}{}^{\underline{m}}\ ,\\
\frac{1}{2}\partial_\alpha\psi_{m\dot\beta} &= T_{\alpha m\dot\beta} - F_{\alpha m}{}^{\underline{p}}\chi_{\underline{p}\dot\beta} = (\sigma_m)_{\alpha\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha\underline{p}}(\chi_{\underline{p}\dot\beta} + iX_{\underline{p}\dot\beta})\ ,\\
\partial_\alpha(e\mathscr{L}_c) &= e\nabla_\alpha\mathscr{L}_c + i(\sigma^m\bar\psi_m)_\alpha e\mathscr{L}_c\ .
\end{aligned} \tag{7.30}$$

Putting these results together leads to (7.27).

8 Conclusion and outlook

The goal of this paper has been to construct a general framework in 4D $\mathcal{N} = 1$ superspace that is suitable for describing a higher-dimensional supergravity theory in $4+n$ dimensions. While this is motivated by previous work on 11D supergravity [33–37] and 5D minimal supergravity, it is expected to be applicable to other cases.

Let us say a few words on that point. One potential argument *against* the wider applicability of this framework is that both 11D and 5D minimal supergravities correspond to very particular cases where the number of internal dimensions and the number of hidden supersymmetries coincide (respectively, 7 and 1). This is important because the superfield $\Psi_{\underline{m}\alpha}$, which here plays only the role of the prepotential of the lower left block of the higher-dimensional vielbein, should pull double duty as a prepotential for the additional spin-3/2 gravitino multiplets. The simplest way this can work is when the number of additional gravitini matches the internal dimension. Nevertheless, we can learn a lesson from the 6D situation [32]. There, one indeed has two fields $\Psi_{\underline{m}\alpha}$, but in constructing minimal 6D supergravity, one encounters a constraint that permits one of these fields to be eliminated (see section 5 of [32]) – this is important as there is only one additional gravitino (not two) in this framework. This may well persist for other cases where the number of extra gravitini is smaller than the number of extra dimensions. For the reverse situation, where the number of extra gravitini is larger than the internal dimension, we may point to IIA supergravity, which can be constructed by dimensionally reducing 11D supergravity in this framework. In that case, one of the gravitini, say $\Psi_{7\alpha}$, is "ungeometrized" and becomes a matter superfield, albeit a high superspin one. It would be interesting to understand both of these cases better.

There are several topics that we did not directly address in this paper. One outstanding issue is the application to 11D supergravity itself. This paper only provides the geometric

superspace framework necessary to describe that case. We still must analyze how the flat results reviewed in section 2.1 are generalized. This involves constructing the abelian tensor hierarchy descending from C_3 in the curved supergeometry we have introduced. In principle, this should be fully fixed by the supergeometry itself so that the intricate structure described in flat space in section 2.1 is maintained. As we have stressed throughout, this will be the subject of a future publication.

A technical issue that we have sidestepped is how to address the differences in the supergeometry we have encountered relative to the linearized results [36, 37]. The point of mismatch is the three additional prepotentials – the chiral superfield $\Phi_{\underline{mn}\alpha}$, the complex linear superfield $\sigma_{\underline{m}\alpha}$, and the unconstrained superfield $\mathcal{V}(S)_\alpha$. These appear in the curvatures $\Phi_{\underline{mnp}\alpha}$, $\Sigma_{\underline{m}\alpha}$ (which is a part of $X_{\underline{m}}(S)_\alpha$), and Z_α. These are related by (6.43) and it is tempting to declare them all to vanish. However, we have shown this is not possible due to the integrability condition (6.49). While Z_α may consistently be turned off, $\Phi_{\underline{mnp}\alpha}$ and thus $\Sigma_{\underline{m}\alpha}$ appear inescapable.

One potential solution to this is that the additional prepotentials can be eliminated by field redefinitions even in the presence of additional matter fields of the tensor hierarchy. This would be similar to the way in which the conventional Wess-Zumino superspace (see e.g. [39]), which manifestly describes old minimal Poincaré supergravity, may actually be understood to describe conformal supergravity, by introducing a super-Weyl transformation that acts as a Weyl rescaling of the metric. Provided one only couples to matter in a super-Weyl invariant way, only the conformal part of the gravity multiplet survives. It is plausible that the same sort of mechanism occurs here.

Indeed, we have already seen in the linearized case that Z_α and $\Sigma_{\underline{m}\alpha}$ can be shifted around by redefinitions of underlying prepotentials $\mathcal{V}(S)_\alpha$ and $\sigma_{\underline{m}\alpha}$. The same can be done by analyzing linearized fluctuations about a generic curved background. Showing that the same is true for $\Phi_{\underline{mn}\alpha}$ is a bit more involved, as we have introduced that prepotential by hand in defining the linearized $\Psi_{\underline{mn}\alpha}$. Seeing this at the non-linear level is a bit involved. The key idea is to introduce a shift $\delta_\rho \Psi_{\underline{mn}\alpha} = \rho_{\underline{mn}\alpha}$, where $\rho_{\underline{mn}\alpha}$ is a chiral superfield 2-form. This corresponds just to a shift $\delta\Phi_{\underline{mn}\,\alpha} = \rho_{\underline{mn}\alpha}$ in the underlying extraneous prepotential. One must then demonstrate that ρ-transformations can be consistently imposed at the non-linear level on curvatures and covariant derivatives. One finds, for example, that $W_{\alpha\beta\gamma}$ shifts under ρ exactly as one would expect from its linearized expression (5.29). Provided ρ transformations can be extended to the p-form superfields of 11D supergravity, one can guarantee that the extraneous prepotential can always be removed. We will describe this in greater detail elsewhere.

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A Superspace curvatures

Our superspace and spinor conventions follow [39]. The curvatures of GL(n) Kaluza-Klein superspace are abstractly given in terms of three operators $\mathcal{W}_\alpha$, $X_{\underline{m}}$, and $R^+_{\underline{mn}}$ in the following manner. Letting $R_{\hat{A}\hat{B}} := -[\nabla_{\hat{A}}, \nabla_{\hat{B}}]$ where $\hat{A} = (a, \alpha, \dot{\alpha}, \underline{m})$, we find that

$$\begin{aligned}
R_{\alpha\beta} = 0 \ , \qquad R_{\dot{\alpha}\dot{\beta}} = 0 \ , \qquad R_{\alpha\dot{\beta}} = 2i\,\nabla_{\alpha\dot{\beta}} \ , \\
R_{\alpha b} = -(\sigma_b)_{\alpha\dot{\alpha}}\bar{\mathcal{W}}^{\dot{\alpha}} \ , \qquad R^{\dot{\alpha}}{}_b = (\bar{\sigma}_b)^{\dot{\alpha}\alpha}\mathcal{W}_\alpha \ , \\
R_{ab} = \frac{i}{2}(\sigma_{ab})^{\alpha\beta}\{\nabla_\alpha, \mathcal{W}_\beta\} - \frac{i}{2}(\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}}\{\bar{\nabla}_{\dot{\alpha}}, \bar{\mathcal{W}}_{\dot{\beta}}\} \ , \\
R_{\underline{m}\alpha} = i[\nabla_\alpha, X_{\underline{m}}] \ , \qquad R_{\underline{m}\dot{\alpha}} = -i[\bar{\nabla}_{\dot{\alpha}}, X_{\underline{m}}] \ , \\
R_{\underline{m}a} = -\frac{1}{4}(\bar{\sigma}_a)^{\dot{\alpha}\alpha}\{\nabla_\alpha, [\bar{\nabla}_{\dot{\alpha}}, X_{\underline{m}}]\} + \frac{1}{4}(\bar{\sigma}_a)^{\dot{\alpha}\alpha}\{\bar{\nabla}_{\dot{\alpha}}, [\nabla_\alpha, X_{\underline{m}}]\} \ , \\
R_{\underline{mn}} = \frac{1}{2}(R^+_{\underline{mn}} + R^-_{\underline{mn}}) - [X_{\underline{m}}, X_{\underline{n}}] \ .
\end{aligned} \tag{A.1}$$

When written in terms of $\nabla^+_{\underline{m}} = \nabla_{\underline{m}} + iX_{\underline{m}}$, the mixed curvatures become

$$\begin{aligned}
R^+_{\underline{m}\alpha} = 2i[\nabla_\alpha, X_{\underline{m}}] \ , \qquad R^+_{\underline{m}\dot{\alpha}} = 0 \ , \\
R^+_{\underline{m}a} = \frac{1}{2}(\bar{\sigma}_a)^{\dot{\alpha}\alpha}\{\bar{\nabla}_{\dot{\alpha}}, [\nabla_\alpha, X_{\underline{m}}]\} \ .
\end{aligned} \tag{A.2}$$

A.1 Expressions for $\mathcal{W}_\alpha$

The chiral operator $\mathcal{W}_\alpha$ is defined as:

$$\begin{aligned}
\mathcal{W}_\alpha = \mathcal{W}_{\alpha\dot{\beta}}\bar{\nabla}^{\dot{\beta}} + \mathcal{W}_\alpha{}^{\underline{m}}\nabla^+_{\underline{m}} + \Big(\nabla^+_{\underline{n}}\mathcal{W}_\alpha{}^{\underline{m}} + \mathcal{W}_{\alpha\underline{n}}{}^{\underline{m}}\Big)g_{\underline{m}}{}^{\underline{n}} \\
+ \mathcal{W}_\alpha(D)\mathbb{D} + \mathcal{W}_\alpha(A)\mathbb{A} + \mathcal{W}_\alpha(M)^{\beta\gamma}M_{\beta\gamma} + \mathcal{W}_\alpha(M)_{\dot{\beta}\dot{\gamma}}M^{\dot{\beta}\dot{\gamma}} \\
+ \mathcal{W}_\alpha(S)^\beta S_\beta + \mathcal{W}_\alpha(S)_{\dot{\beta}}\bar{S}^{\dot{\beta}} + \mathcal{W}_\alpha(K)^b K_b \ .
\end{aligned} \tag{A.3}$$

The Kaluza-Klein superfield $\mathcal{W}_\alpha{}^{\underline{m}}$ is chiral and obeys a reduced chirality condition,

$$\hat{\nabla}^\alpha\mathcal{W}_\alpha{}^{\underline{m}} = \hat{\bar{\nabla}}_{\dot{\alpha}}\bar{\mathcal{W}}^{\dot{\alpha}\underline{m}} \ , \qquad \hat{\nabla}_\alpha := \nabla_\alpha - 2F_{\underline{m}\alpha}{}^{\underline{n}}g_{\underline{n}}{}^{\underline{m}} \ , \qquad F_{\underline{m}\alpha}{}^{\underline{n}} := iX_{\underline{m}\alpha\dot{\alpha}}\bar{\mathcal{W}}^{\dot{\alpha}\underline{n}} \ . \tag{A.4}$$

The superfield $\mathcal{W}_{\alpha\dot{\alpha}}$ is related to its conjugate by

$$\mathcal{W}_{\alpha\dot{\alpha}} - \bar{\mathcal{W}}_{\dot{\alpha}\alpha} = -2i\,(\mathcal{W}_\alpha{}^{\underline{m}}X_{\underline{m}\dot{\alpha}} + \bar{\mathcal{W}}_{\dot{\alpha}}{}^{\underline{m}}X_{\underline{m}\alpha}) - \frac{1}{2}\mathcal{W}^{\beta\underline{m}}\hat{\nabla}_\beta X_{\underline{m}\alpha\dot{\alpha}} - \frac{1}{2}\bar{\nabla}_{\dot{\beta}}(\bar{\mathcal{W}}^{\dot{\beta}\underline{m}}X_{\underline{m}\alpha\dot{\alpha}}) \ . \tag{A.5}$$

The other linear combination $\mathcal{W}_{\alpha\dot\alpha} + \bar{\mathcal{W}}_{\dot\alpha\alpha}$ can be fixed to whatever we wish by a redefinition of the superspace connections. Other terms are related to those above:

$$\begin{aligned}
\mathcal{W}_\alpha(D) &= \frac{1}{2}\bar\nabla^{\dot\gamma}\mathcal{W}_{\alpha\dot\gamma} + \phi_\alpha \ , & \mathcal{W}_\alpha(A) &= \frac{3i}{4}\bar\nabla^{\dot\gamma}\mathcal{W}_{\alpha\dot\gamma} + \frac{i}{2}\phi_\alpha \ , \\
\mathcal{W}_\alpha(M)_{\beta\gamma} &= -\epsilon_{\alpha(\beta}Z_{\gamma)} + 2i\, W_{\alpha\beta\gamma} \ , & \mathcal{W}_\alpha(M)_{\dot\alpha\dot\beta} &= -\bar\nabla_{(\dot\alpha}\mathcal{W}_{\alpha\dot\beta)} \ , \\
\mathcal{W}_{\alpha\underline{m}}{}^{\underline{n}} &= -\frac{i}{4}\bar\nabla^2 F_{\underline{m}\alpha}{}^{\underline{n}} \ , & \mathcal{W}_\alpha(S)_{\dot\alpha} &= \frac{1}{8}\bar\nabla^2\mathcal{W}_{\alpha\dot\alpha} \ , \\
\mathcal{W}_\alpha(K)^{\dot\beta\beta} &= i\bar\nabla^{\dot\beta}\mathcal{W}_\alpha(S)^\beta \ . & & \qquad (A.6)
\end{aligned}$$

The chiral superfields ϕ_α and Z_α are related by

$$\phi_\alpha = \frac{3}{2}Z_\alpha - \mathcal{W}_{\alpha\underline{m}}{}^{\underline{m}} \qquad (A.7)$$

Z_α can be fixed to whatever we wish by a connection redefinition. The superfield $\mathcal{W}_\alpha(S)^\beta$ is complex linear and can be written

$$\begin{aligned}
\mathcal{W}_\alpha(S)^\beta = &-\frac{1}{4}\Big(\nabla^\gamma\mathcal{W}_\gamma(M)_\alpha{}^\beta + 2F_{\underline{m}}{}^{\gamma\underline{m}}\mathcal{W}_\gamma(M)_\alpha{}^\beta + \mathcal{W}^{\gamma\underline{m}}R^+_{\underline{m}\gamma}(M)_\alpha{}^\beta\Big) \\
&+ \frac{1}{12}\delta_\alpha{}^\beta(\nabla^\gamma\phi_\gamma + 2F_{\underline{m}}{}^{\gamma\underline{m}}\phi_\gamma) + \frac{1}{8}\delta_\alpha{}^\beta\mathcal{W}^{\gamma\underline{m}}(R^+_{\underline{m}\gamma}(D) + \frac{2i}{3}R^+_{\underline{m}\gamma}(A)) \\
&+ \bar\nabla_{\dot\gamma}\Big[\frac{1}{4}\bar{\mathcal{W}}^{\dot\gamma}(M)_\alpha{}^\beta - \frac{i}{2}\bar{\mathcal{W}}^{\dot\gamma\underline{m}}X_{\underline{m}}(M)_\alpha{}^\beta\Big] \\
&+ \delta_\alpha{}^\beta\,\bar\nabla_{\dot\gamma}\Big[-\frac{1}{8}\bar{\mathcal{W}}^{\dot\gamma}(D) + \frac{i}{4}\bar{\mathcal{W}}^{\dot\gamma\underline{m}}X_{\underline{m}}(D) - \frac{i}{12}\bar{\mathcal{W}}^{\dot\gamma}(A) - \frac{1}{6}\bar{\mathcal{W}}^{\dot\gamma\underline{m}}X_{\underline{m}}(A)\Big] \ . \qquad (A.8)
\end{aligned}$$

A.2 Expressions for $X_{\underline{m}}$

The operator $X_{\underline{m}} = -\frac{i}{2}(\nabla^+_{\underline{m}} - \nabla^-_{\underline{m}})$ is given by

$$\begin{aligned}
X_{\underline{m}} = &\, X_{\underline{m}}{}^A\nabla_A + X_{\underline{m}}(D)\,\mathbb{D} + X_{\underline{m}}(A)\,\mathbb{A} + X_{\underline{m}}(M)^{\alpha\beta}M_{\alpha\beta} + X_{\underline{m}}(M)_{\dot\alpha\dot\beta}M^{\dot\alpha\dot\beta} \\
&+ X_{\underline{m}\underline{n}}{}^{\underline{p}}g_{\underline{p}}{}^{\underline{n}} + X_{\underline{m}}(S)^\alpha S_\alpha + X_{\underline{m}}(S)_{\dot\alpha}\bar S^{\dot\alpha} + X_{\underline{m}}(K)^a K_a \ . \qquad (A.9)
\end{aligned}$$

We leave $X_{\underline{m}\alpha}$ and $X_{\underline{m}}{}^{\dot\alpha}$ unfixed and give other quantities in terms of these. $X_{\underline{m}}(M)$, $X_{\underline{m}}(D)$, and $X_{\underline{m}}(A)$ are determined by

$$\begin{aligned}
X_{\underline{m}}(M)_\alpha{}^\beta + \frac{1}{2}\delta_\alpha{}^\beta(X_{\underline{m}}(D) - 2iX_{\underline{m}}(A)) = &\,\hat\nabla_\alpha X_{\underline{m}}{}^\beta - \delta_\alpha{}^\beta\bar\nabla^{\dot\gamma}X_{\underline{m}\dot\gamma} + \frac{i}{4}\bar\nabla^{\dot\gamma}\hat\nabla_\alpha X_{\underline{m}}{}^\beta{}_{\dot\gamma} \\
&+ X_{\underline{m}\alpha\dot\beta}\bar{\mathcal{W}}^{\dot\beta\beta} + \frac{i}{2}\mathcal{W}_\alpha{}^{\underline{n}}\Psi_{\underline{n}\underline{m}}{}^\beta \ . \qquad (A.10)
\end{aligned}$$

The antisymmetric part of the GL(n) component $X_{\underline{m}\underline{n}}{}^{\underline{p}}$ is

$$\begin{aligned}
X_{[\underline{m}\underline{n}]}{}^{\underline{p}} = &-\frac{i}{4}\Psi_{\underline{m}\underline{n}}{}^\alpha\mathcal{W}_\alpha{}^{\underline{p}} - \frac{1}{4}X_{[\underline{m}\beta\dot\alpha}X_{\underline{n}]}{}^{\dot\alpha\alpha}\hat\nabla^\beta\mathcal{W}_\alpha{}^{\underline{p}} - \frac{1}{2}X_{[\underline{m}\gamma\dot\gamma}\hat\nabla^\gamma X_{\underline{n}]}{}^{\dot\gamma\alpha}\mathcal{W}_\alpha{}^{\underline{p}} \\
&+ iX_{[\underline{m}}{}^{\dot\alpha\alpha}F_{\underline{n}]\alpha}{}^{\underline{q}}F_{\underline{q}\dot\alpha}{}^{\underline{p}} + \text{h.c.} \qquad (A.11)
\end{aligned}$$

The S-supersymmetry piece is

$$X_{\underline{m}}(S)_\alpha = -\frac{1}{4}\Big[\hat{\nabla}_\alpha X_{\underline{m}}(D) + X_{\underline{m}\alpha\dot{\alpha}}\bar{\mathcal{W}}^{\dot{\alpha}}(D)\Big] - \frac{i}{6}\Big[\hat{\nabla}_\alpha X_{\underline{m}}(A) + X_{\underline{m}\alpha\dot{\alpha}}\bar{\mathcal{W}}^{\dot{\alpha}}(A)\Big] - \frac{1}{4}\bar{\nabla}^{\dot{\alpha}}\hat{\nabla}_\alpha X_{\underline{m}\dot{\alpha}}$$
$$- \frac{i}{24}\mathcal{W}_\alpha{}^{\underline{n}}\Big(\hat{\nabla}^\beta\Psi_{\underline{nm}\beta} + 2\Psi_{\underline{nm}}{}^\beta F_{\underline{p}\beta}{}^{\underline{p}}\Big) - \frac{i}{6}\nabla^+_{\underline{m}}F_{\underline{p}\alpha}{}^{\underline{p}} + \Sigma_{\underline{m}\alpha}\ , \tag{A.12}$$

where $\Sigma_{\underline{m}\alpha}$ obeys an inhomogeneous complex linearity condition,

$$\bar{\nabla}^2\Sigma_{\underline{m}\alpha} = -\nabla^+_{\underline{m}}Z_\alpha - \frac{1}{3}\mathcal{W}_\alpha{}^{\underline{n}}\mathcal{W}^{\beta\underline{p}}\Phi_{\underline{pnm}\beta}\ . \tag{A.13}$$

Because $\Phi_{\underline{pnm}\,\alpha}$ cannot be set to zero, we cannot eliminate $\Sigma_{\underline{m}\alpha}$. However, a connection redefinition shifts it by an arbitrary complex linear superfield.

Finally, $X_{\underline{m}}(K)^a$ is given by

$$iX_{\underline{m}}(K)_{\alpha\dot{\alpha}} = \frac{1}{4}\nabla^+_{\underline{m}}\mathcal{W}_{\alpha\dot{\alpha}} + \frac{1}{4}\mathcal{W}_\alpha{}^{\underline{n}}(T^+_{\underline{nm}\dot{\alpha}} + \Psi_{\underline{nm}}{}^\gamma\mathcal{W}_{\gamma\dot{\alpha}})$$
$$+ \bar{\nabla}_{\dot{\alpha}}X_{\underline{m}}(S)_\alpha - F_{\underline{m}\dot{\alpha}}{}^{\underline{n}}X_{\underline{n}}(S)_\alpha + \frac{1}{16}\bar{\nabla}^2\Big[\nabla_\alpha X_{\underline{m}\dot{\alpha}} - 2F_{\underline{m}\alpha}{}^{\underline{n}}X_{\underline{n}\dot{\alpha}}\Big]$$
$$+ \frac{1}{8}\bar{\nabla}_{\dot{\beta}}\Big(\delta_{\underline{m}}{}^{\underline{n}}\nabla_\alpha - 2F_{\underline{m}\alpha}{}^{\underline{n}}\Big)\Big(X_{\underline{n}}(M)^{\dot{\beta}}{}_{\dot{\alpha}} + \frac{1}{2}X_{\underline{n}}(D)\,\delta^{\dot{\beta}}{}_{\dot{\alpha}} + iX_{\underline{n}}(A)\,\delta^{\dot{\beta}}{}_{\dot{\alpha}}\Big)$$
$$- \text{h.c.} \tag{A.14}$$

A.3 Expressions for $R^+_{\underline{mn}}$

The operator $R^+_{\underline{mn}}$ is given abstractly by

$$R^+_{\underline{mn}} = T^+_{\underline{mn}}{}^B\nabla_B + F^+_{\underline{mn}}{}^{\underline{p}}\nabla^+_{\underline{p}} + \Big(\nabla^+_{\underline{p}}F^+_{\underline{mn}}{}^{\underline{q}} + \mathcal{R}_{\underline{mnp}}{}^{\underline{q}}\Big)g_{\underline{q}}{}^{\underline{p}}$$
$$+ R^+_{\underline{mn}}(D)\mathbb{D} + R^+_{\underline{mn}}(A)\mathbb{A} + R^+_{\underline{mn}}(M)^{\beta\gamma}M_{\beta\gamma} + R^+_{\underline{mn}}(M)_{\dot{\beta}\dot{\gamma}}M^{\dot{\beta}\dot{\gamma}}$$
$$+ R^+_{\underline{mn}}(S)^\beta S_\beta + R^+_{\underline{mn}}(S)_{\dot{\beta}}S^{\dot{\beta}} + R^+_{\underline{mn}}(K)^cK_c\ . \tag{A.15}$$

The torsion and KK curvature parts are

$$T^+_{\underline{mn}}{}^{\dot{\beta}\beta} = -\bar{\nabla}^{\dot{\beta}}\Psi_{\underline{mn}}{}^\beta\ , \qquad T^+_{\underline{mn}}{}^\beta = \frac{i}{8}\bar{\nabla}^2\Psi_{\underline{mn}}{}^\beta\ ,$$
$$T^+_{\underline{mn}\dot{\alpha}} = -\frac{i}{8}\nabla^2\Psi_{\underline{mn}\dot{\alpha}} - i(F^+_{\underline{mn}}{}^{\underline{p}} + F^-_{\underline{mn}}{}^{\underline{p}})X_{\underline{p}\dot{\alpha}} - 4i\Big(\nabla_{[\underline{m}}X_{\underline{n}]\dot{\alpha}} + X_{\underline{m}}{}^BT_{\underline{n}B\dot{\alpha}}\Big)\ ,$$
$$F^+_{\underline{mn}}{}^{\underline{p}} = -\Psi_{\underline{mn}}{}^\alpha\mathcal{W}_\alpha{}^{\underline{p}}\ . \tag{A.16}$$

The other components are

$$
\begin{aligned}
R^+_{\underline{mn}}(M)_{\alpha\beta} &= -\Psi_{\underline{mn}}{}^{\gamma}\mathcal{W}_\gamma(M)_{\alpha\beta} + \Phi_{\underline{mn}\alpha\beta}\ , \\
R^+_{\underline{mn}}(M)_{\dot\alpha\dot\beta} &= \bar\nabla_{(\dot\alpha}\Psi_{\underline{mn}}{}^{\gamma}W_{\gamma\dot\beta)} + \bar\nabla_{(\dot\alpha}T^+_{\underline{mn}\dot\beta)}\ , \\
R^+_{\underline{mn}}(D) &= -\Psi_{\underline{mn}}{}^{\alpha}\phi_\alpha + \Phi_{\underline{mn}} - \frac{1}{2}\bar\nabla^{\dot\gamma}T^+_{\underline{mn}\dot\gamma} - \frac{1}{2}\bar\nabla^{\dot\gamma}\Psi_{\underline{mn}}{}^{\gamma}\mathcal{W}_{\gamma\dot\gamma}\ , \\
R^+_{\underline{mn}}(A) &= -\frac{i}{2}\Psi_{\underline{mn}}{}^{\alpha}\phi_\alpha + \frac{i}{2}\Phi_{\underline{mn}} - \frac{3i}{4}\bar\nabla^{\dot\gamma}T^+_{\underline{mn}\dot\gamma} - \frac{3i}{4}\bar\nabla^{\dot\gamma}\Psi_{\underline{mn}}{}^{\gamma}\mathcal{W}_{\gamma\dot\gamma}\ , \\
R^+_{\underline{mn}}(S)_\alpha &= -\Psi_{\underline{mn}}{}^{\gamma}\mathcal{W}_\gamma(S)_\alpha + \Sigma_{\underline{mn}\alpha}\ , \\
R^+_{\underline{mn}}(S)_{\dot\alpha} &= \frac{1}{8}\bar\nabla^2 T^+_{\underline{mn}\dot\alpha} + \frac{1}{8}\bar\nabla^2\Psi_{\underline{mn}}{}^{\gamma}W_{\gamma\dot\gamma} - \frac{1}{4}\bar\nabla_{\dot\phi}\Psi_{\underline{mn}}{}^{\gamma}\bar\nabla^{\dot\phi}\mathcal{W}_{\gamma\dot\gamma}\ , \\
R^+_{\underline{mn}}(K)_{\alpha\dot\alpha} &= -i\bar\nabla_{\dot\alpha}R^+_{\underline{mn}}(S)_\alpha - i\bar\nabla^{\dot\alpha}\Psi_{\underline{mn}}{}^{\beta}\mathcal{W}_\beta(S)_\alpha\ , \\
\mathcal{R}^+_{\underline{mnp}}{}^{\underline{q}} &= -\Psi_{\underline{mn}}{}^{\alpha}\mathcal{W}_{\alpha\underline{p}}{}^{\underline{q}} + \nabla^+_{\underline{p}}\Psi_{\underline{mn}}{}^{\alpha}\mathcal{W}_\alpha{}^{\underline{q}} + \Phi_{\underline{mnp}}{}^{\underline{q}}\ ,
\end{aligned}
\tag{A.17}
$$

where

$$
\begin{aligned}
\Phi_{\underline{nm}\,\alpha\beta} &= -\frac{i}{8}\bar\nabla^2\Big[\hat\nabla_{(\beta}\Psi_{\underline{nm}\alpha)} + 2\Psi_{\underline{nm}(\beta}F_{\underline{p}\alpha)}{}^{\underline{p}} - 2i\,\hat\nabla_{(\beta}X_{[\underline{n}\gamma)\dot\gamma}\hat\nabla_{(\alpha}X_{\underline{m}]}{}^{\gamma)\dot\gamma}\Big] - \mathcal{W}_{(\alpha}{}^{\underline{p}}\Phi_{\underline{pnm}\,\beta)}\ , \\
\Phi_{\underline{nm}} &= \frac{i}{16}\bar\nabla^2\Big[\hat\nabla^\gamma\Psi_{\underline{nm}\gamma} + 2\Psi_{\underline{nm}}{}^{\gamma}F_{\underline{p}\gamma}{}^{\underline{p}}\Big] + \frac{1}{2}\mathcal{W}^{\beta\underline{p}}\Phi_{\underline{pnm}\,\beta}\ , \\
\Phi_{[\underline{mnp}]}{}^{\underline{q}} &= -\frac{1}{3}\Phi_{\underline{mnp}}{}^{\alpha}\mathcal{W}_\alpha{}^{\underline{q}} - \frac{i}{4}\bar\nabla^2(\Psi_{[\underline{mn}}{}^{\alpha}F_{\underline{p}]\alpha}{}^{\underline{q}})\ .
\end{aligned}
\tag{A.18}
$$

In practice, only the antisymmetric part of $\Phi_{\underline{mnp}}{}^{\underline{q}}$ given in the last line is relevant. We do not give an explicit expression for $\Sigma_{\underline{mn}\alpha}$ but it can be worked out.

A.4 Fundamental Bianchi identities

Below we list the fundamental Bianchi identities that generalize (2.12):

$$
\begin{aligned}
\nabla^+_{\underline{m}}W_{\alpha\beta\gamma} &= \frac{1}{16}\bar\nabla^2\Big[\hat\nabla_\alpha{}^{\dot\gamma}\hat\nabla_\beta X_{\underline{m}\gamma\dot\gamma} - \hat\nabla_\alpha(\mathcal{W}_\beta{}^{\underline{n}}\Psi_{\underline{nm}\gamma}) + 4\hat\nabla_\alpha X_{\underline{m}\beta\dot\gamma}\bar{\mathcal{W}}^{\dot\gamma\underline{n}}X_{\underline{n}\gamma} \\
&\qquad + 2i\,\hat\nabla_\alpha X_{\underline{m}\beta\dot\gamma}\bar{\mathcal{W}}^{\dot\gamma}{}_\gamma\Big]_{(\alpha\beta\gamma)} + \frac{i}{2}\mathcal{W}_{(\alpha}{}^{\underline{n}}\Phi_{\underline{nm}\,\beta\gamma)}\ ,
\end{aligned}
\tag{A.19a}
$$

$$
\nabla_{[\underline{m}}X_{\underline{n}]}{}^a = -X_{[\underline{m}}{}^B T_{\underline{n}]B}{}^a + \frac{i}{4}(T^+_{\underline{mn}}{}^a - T^-_{\underline{mn}}{}^a) - \frac{1}{4}(F^+_{\underline{mn}}{}^{\underline{p}} + F^-_{\underline{mn}}{}^{\underline{p}})X_{\underline{p}}{}^a\ ,
\tag{A.19b}
$$

$$
\nabla^+_{[\underline{p}}\Psi_{\underline{mn}]\,\alpha} = -\frac{i}{4}\bar\nabla^{\dot\beta}\Psi_{[\underline{mn}}{}^{\beta}T^+_{\underline{p}]\beta\,\alpha\dot\beta} + \Psi_{[\underline{mn}|}{}^{\beta}\mathcal{W}_\beta{}^{\underline{q}}\Psi_{\underline{q}|\underline{p}]\,\alpha} + \frac{1}{3}\Phi_{\underline{pmn}\,\alpha}\ ,
\tag{A.19c}
$$

$$
\nabla^+_1\Phi_{3\alpha} = -\frac{1}{8}\bar\nabla^2\Big[i\Psi_2{}^\beta\hat\nabla_\beta\Psi_{2\alpha} + i\Psi_2{}^\beta\Psi_{2\beta}\,F_{\underline{m}\alpha}{}^{\underline{m}} - X_{1\alpha\gamma\dot\gamma}X_1{}^{\beta\dot\gamma\gamma}\Psi_{2\beta}\Big]
\tag{A.19d}
$$

The last we have written in form notation. In addition to these, one must also specify the inhomogeneous complex linearity condition of $\Sigma_{\underline{m}\alpha}$, see (A.13).

A.5 Some explicit expressions for torsions and KK curvatures

For reference, we give some explicit expressions for torsion and Kaluza-Klein curvatures. Some of the mixed torsion tensors are particularly simple in the + basis:

$$T^+_{\underline{m}\alpha,\beta\dot\beta} = 2i\hat\nabla_\alpha X_{\underline{m}\beta\dot\beta} + 8\,\epsilon_{\alpha\beta}X_{\underline{m}\dot\beta}\ , \tag{A.20a}$$

$$T^+_{\underline{m}\alpha}{}^{\beta} = \frac{i}{4}\bar\nabla_{\dot\phi}T^{+\ \dot\phi\beta}_{\underline{m}\alpha,} + \mathcal{W}_\alpha{}^{\underline{n}}\Psi_{\underline{nm}}{}^{\beta}\ , \tag{A.20b}$$

$$T^+_{\underline{m}\alpha\dot\beta} = 2i\hat\nabla_\alpha X_{\underline{m}\dot\beta}\ , \tag{A.20c}$$

$$T^+_{\underline{m}\,\alpha\dot\alpha,\,\beta\dot\beta} = \frac{i}{2}\bar\nabla_{\dot\beta}T^+_{\underline{m}\alpha,\,\beta\dot\alpha} + 2\,\epsilon_{\dot\alpha\dot\beta}\mathcal{W}_\alpha{}^{\underline{n}}\Psi_{\underline{nm}\beta}\ , \tag{A.20d}$$

$$T^+_{\underline{m}\,\alpha\dot\alpha}{}^{\beta} = \frac{1}{16}\bar\nabla^2 T^{+\ \beta}_{\underline{m}\alpha,\dot\alpha} - \frac{i}{2}\mathcal{W}_\alpha{}^{\underline{n}}\bar\nabla_{\dot\alpha}\Psi_{\underline{nm}}{}^{\beta} \tag{A.20e}$$

The mixed Kaluza-Klein curvatures are

$$F^+_{\underline{m}\alpha}{}^{\underline{n}} = 2i\,X_{\underline{m}\alpha\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha\underline{n}}\ , \qquad F^+_{\underline{m}\,\alpha\dot\alpha}{}^{\underline{n}} = \frac{i}{2}\bar\nabla_{\dot\alpha}F^+_{\underline{m}\alpha}{}^{\underline{n}} - \frac{i}{2}T^+_{\underline{m}\alpha,\gamma\dot\alpha}\mathcal{W}^{\gamma\underline{n}}\ . \tag{A.21a}$$

In a real basis written in terms of $\nabla_{\underline{m}}$, one has

$$F_{\underline{m}\alpha}{}^{\underline{n}} = \frac{1}{2}F^+_{\underline{m}\alpha}{}^{\underline{n}}\ , \qquad T_{\underline{m}\alpha}{}^{B} = \frac{1}{2}(T^+_{\underline{m}\alpha}{}^{B} + iF^+_{\underline{m}\alpha}{}^{\underline{n}}X_{\underline{p}}{}^{B})\ , \tag{A.22a}$$

$$F_{\underline{m}a}{}^{\underline{n}} = \mathrm{Re}\,F^+_{\underline{m}a}{}^{\underline{n}}\ , \qquad T_{\underline{m}a}{}^{B} = \mathrm{Re}\left(T^+_{\underline{m}a}{}^{B} + iF^+_{\underline{m}a}{}^{\underline{n}}X_{\underline{n}}{}^{B}\right)\ . \tag{A.22b}$$

The external torsion and Kaluza-Klein curvatures found in the $[\nabla_\alpha, \nabla_b]$ commutator are

$$F_{b\alpha}{}^{\underline{n}} = (\sigma_b)_{\alpha\dot\beta}\bar{\mathcal{W}}^{\dot\beta\underline{n}}\ , \qquad F_b{}^{\dot\alpha\underline{n}} = -(\sigma_b)^{\dot\alpha\beta}\mathcal{W}_\beta{}^{\underline{n}}\ , \tag{A.23a}$$

$$T_{b\alpha}{}^{c} = -i(\sigma_b)_{\alpha\dot\beta}\bar{\mathcal{W}}^{\dot\beta\underline{n}}X_{\underline{n}}{}^{c} \qquad T_b{}^{\dot\alpha c} = -i(\sigma_b)^{\dot\alpha\beta}\mathcal{W}_\beta{}^{\underline{n}}X_{\underline{n}}{}^{c} \tag{A.23b}$$

$$T_{b\alpha}{}^{\gamma} = (\sigma_b)_{\alpha\dot\beta}(\bar{\mathcal{W}}^{\dot\beta\gamma} - i\bar{\mathcal{W}}^{\dot\beta\underline{n}}X_{\underline{n}}{}^{\gamma}) \qquad T_b{}^{\dot\alpha}{}_{\dot\gamma} = -(\sigma_b)^{\dot\alpha\beta}(\mathcal{W}_{\beta\dot\gamma} + i\mathcal{W}_\beta{}^{\underline{n}}X_{\underline{n}\dot\gamma}) \tag{A.23c}$$

$$T_{b\alpha\dot\gamma} = -i(\sigma_b)_{\alpha\dot\beta}\bar{\mathcal{W}}^{\dot\beta\underline{n}}X_{\underline{n}\dot\gamma} \qquad T_b{}^{\dot\alpha\gamma} = -i(\sigma_b)^{\dot\alpha\beta}\mathcal{W}_\beta{}^{\underline{n}}X_{\underline{n}}{}^{\gamma}\ . \tag{A.23d}$$

Those found in the vector-vector commutator are most easily written by decomposing the curvature operator into self-dual and anti-self-dual pieces, $R_{ab} = -(\sigma_{ab})^{\alpha\beta}R_{\underbrace{\alpha\beta}} - (\bar\sigma_{ab})^{\dot\alpha\dot\beta}R_{\underbrace{\dot\alpha\dot\beta}}$,

$$F_{\underbrace{\alpha\beta}}{}^{\underline{m}} = -\frac{i}{2}\hat\nabla_{(\alpha}\mathcal{W}_{\beta)}{}^{\underline{m}}\ , \tag{A.24a}$$

$$T_{\underbrace{\alpha\beta}}{}^{c} = iF_{\underbrace{\alpha\beta}}{}^{\underline{n}}X_{\underline{n}}{}^{c} + \frac{i}{2}\mathcal{W}_{(\alpha}{}^{\underline{n}}T^{+}_{\underline{n}\beta)}{}^{c} - W_{(\alpha\dot\gamma}(\sigma^c)_{\beta)}{}^{\dot\gamma}\ , \tag{A.24b}$$

$$T_{\underbrace{\alpha\beta}}{}^{\gamma} = iF_{\underbrace{\alpha\beta}}{}^{\underline{n}}X_{\underline{n}}{}^{\gamma} + \frac{i}{2}\mathcal{W}_{(\alpha}{}^{\underline{n}}T^{+}_{\underline{n}\beta)}{}^{\gamma} - \frac{i}{2}\delta_{(\alpha}{}^{\gamma}\left(\bar\nabla^{\dot\gamma}\mathcal{W}_{\beta)\dot\gamma} + 2Z_{\beta)} - \mathcal{W}_{\beta)\underline{n}}{}^{\underline{n}}\right) + W_{\alpha\beta}{}^{\gamma}\ , \tag{A.24c}$$

$$T_{\underbrace{\alpha\beta}\dot\gamma} = iF_{\underbrace{\alpha\beta}}{}^{\underline{n}}X_{\underline{n}\dot\gamma} + \frac{i}{2}\mathcal{W}_{(\alpha}{}^{\underline{n}}T^{+}_{\underline{n}\beta)\dot\gamma} - \frac{i}{2}\nabla_{(\alpha}\mathcal{W}_{\beta)\dot\gamma}\ . \tag{A.24d}$$

From the second equation, one can see that $T_{ab}{}^{c}$ does not vanish.

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