The de Rham cohomology of the Suzuki curves

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Abstract. For a natural number m, let $\mathcal{S}_m/\mathbb{F}_2$ be the mth Suzuki curve. We study the mod 2 Dieudonné module of $\mathcal{S}_m$, which gives the equivalent information as the Ekedahl-Oort type or the structure of the 2-torsion group scheme of its Jacobian. We accomplish this by studying the de Rham cohomology of $\mathcal{S}_m$. For all m, we determine the structure of the de Rham cohomology as a 2-modular representation of the mth Suzuki group and the structure of a submodule of the mod 2 Dieudonné module. For $m = 1$ and 2, we determine the complete structure of the mod 2 Dieudonné module.

1. Introduction

The structure of the de Rham cohomology of the Hermitian curves as a representation of $\mathrm{PGU}(3, q)$ was studied in [**3, 4, 12**]. The mod p Dieudonné module and the Ekedahl-Oort type of the Hermitian curves were determined in [**22**]. In this paper, we study the analogous structures for the Suzuki curves.

For $m \in \mathbb{N}$, let $q_0 = 2^m$, and let $q = 2^{2m+1}$. The Suzuki curve $\mathcal{S}_m$ is the smooth projective connected curve over $\mathbb{F}_2$ given by the affine equation:

$$z^q + z = y^{q_0}(y^q + y).$$

It has genus $g_m = q_0(q - 1)$.

The number of points of $\mathcal{S}_m$ over $\mathbb{F}_q$ is $\#\mathcal{S}_m(\mathbb{F}_q) = q^2 + 1$; which is *optimal* in that it reaches Serre's improvement to the Hasse-Weil bound [**14**, Proposition 2.1]. In fact, $\mathcal{S}_m$ is the unique $\mathbb{F}_q$-optimal curve of genus g_m [**8**]. Because of the large number of rational points relative to their genus, the Suzuki curves provide good examples of Goppa codes [**9**],[**10**], [**14**].

The automorphism group of $\mathcal{S}_m$ is the Suzuki group $\mathrm{Sz}(q)$. The order of $\mathrm{Sz}(q)$ is $q^2(q-1)(q^2+1)$ which is very large compared with g_m. In fact, $\mathcal{S}_m$ is the Deligne-Lusztig curve associated with the group $\mathrm{Sz}(q) = {}^2B_2(q)$ [**13**, Proposition 4.3].

The L-polynomial of $\mathcal{S}_m/\mathbb{F}_q$ is $(1 + \sqrt{2q}t + qt^2)^{g_m}$ and so $\mathcal{S}_m$ is supersingular for each $m \in \mathbb{N}$ [**13**, Proposition 4.3]. This implies that the Jacobian $\mathrm{Jac}(\mathcal{S}_m)$

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is isogenous over $\bar{\mathbb{F}}_2$ to a product of supersingular elliptic curves. In particular, $\mathrm{Jac}(\mathcal{S}_m)$ has 2-rank 0; it has no points of order 2 over $\bar{\mathbb{F}}_2$.

The 2-torsion group scheme $\mathrm{Jac}(\mathcal{S}_m)[2]$ is a BT_1-group scheme of rank 2^{2g_m}. By [**7**], the a-number of $\mathrm{Jac}(\mathcal{S}_m)[2]$ is $a_m = q_0(q_0+1)(2q_0+1)/6$; in particular, $\lim_{m\to\infty} a_m/g_m = 1/6$. However, the Ekedahl-Oort type of $\mathrm{Jac}(\mathcal{S}_m)[2]$ is not known. Understanding the Ekedahl-Oort type is equivalent to understanding the structure of the de Rham cohomology or the mod 2 reduction of the Dieudonné module as a module under the actions of the operators Frobenius F and Verschiebung V.

In this paper, we study the de Rham cohomology group $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ of the Suzuki curves. The 2-modular representations of the Suzuki group are understood from [**18, 2, 23, 16**]. Using results about the cohomology of Deligne-Lusztig varieties from [**17**] and [**11**], we determine the multiplicity of each irreducible 2-modular representation of $\mathrm{Sz}(q)$ in $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ in Corollary 2.2.

Let D_m denote the mod 2 reduction of the Dieudonné module of (the Jacobian of) $\mathcal{S}_m$. It is an $\mathbb{E}$-module where $\mathbb{E}$ is the non-commutative ring generated over $\bar{\mathbb{F}}_2$ by F and V with the relations $FV = VF = 0$. As explained in Section 3.1, there is an $\mathbb{E}$-module decomposition $D_m = D_{m,0} \oplus D_{m,\neq 0}$, where the $\mathbb{E}$-submodule $D_{m,0}$ is the trivial eigenspace for the action of an automorphism τ of order $q-1$.

In Proposition 3.1, we determine the structure of $D_{m,0}$ completely by finding that its Ekedahl-Oort type is $[0, 1, 1, 2, 2, \ldots, q_0 - 1, q_0]$. This yields the following corollary.

Corollary 1.1. *(Corollary 3.10) If $2^m \equiv 2^e \bmod 2^{e+1}+1$, then the $\mathbb{E}$-module $\mathbb{E}/\mathbb{E}(V^{e+1} + F^{e+1})$ occurs as an $\mathbb{E}$-submodule of the mod 2 Dieudonné module D_m of $\mathcal{S}_m$. In particular,*

1. *$\mathbb{E}/\mathbb{E}(V^{m+1} + F^{m+1})$ occurs as an $\mathbb{E}$-submodule of D_m for all m;*
2. *$\mathbb{E}/\mathbb{E}(V + F)$ occurs as an $\mathbb{E}$-submodule of D_m if m is even; and*
3. *$\mathbb{E}/\mathbb{E}(V^2 + F^2)$ occurs as an $\mathbb{E}$-submodule of D_m if $m \equiv 1 \bmod 4$.*

We have less information about $D_{m,\neq 0}$, the sum of the non-trivial eigenspaces for τ. In Section 3.3, we explain a connection between the Ekedahl-Oort type and irreducible subrepresentations of $H^1_{\mathrm{dR}}(\mathcal{S}_m)$. This motivates Conjecture 3.2, in which we conjecture that the $\mathbb{E}$-module $\mathbb{E}/\mathbb{E}(V^{2m+1} + F^{2m+1})$ occurs with multiplicity 4^m in D_m.

We determine the complete structure of the mod 2 Dieudonné module D_m for $m = 1$ and $m = 2$ in Propositions 3.3-3.4. To do this, we explicitly compute a basis for $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ for all $m \in \mathbb{N}$ in Section 4 and, for $m = 1, 2$, we compute the actions of F and V on this basis.

There is a similar result in [**5**] for the first Ree curve, which is defined over $\mathbb{F}_3$, namely the authors determine its mod 3 Dieudonné module.

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1.1. Notation. We begin by establishing some notation regarding p-torsion group schemes, mod p Dieudonné modules, and Ekedahl-Oort types, taken directly from [**22**, Section 2].

Let k be an algebraically closed field of characteristic $p > 0$. Suppose A is a principally polarized abelian variety of dimension g defined over k. Consider the multiplication-by-p morphism $[p] : A \to A$ which is a finite flat morphism of degree

p^{2g}. It factors as $[p] = V \circ F$. Here $F : A \to A^{(p)}$ is the relative Frobenius morphism coming from the p-power map on the structure sheaf; it is purely inseparable of degree p^g. The Verschiebung morphism $V : A^{(p)} \to A$ is the dual of $F_{A^{\mathrm{dual}}}$.

The *p-torsion group scheme* of A, denoted $A[p]$, is the kernel of $[p]$. It is a finite commutative group scheme annihilated by p, again having morphisms F and V, with $\mathrm{Ker}(F) = \mathrm{Im}(V)$ and $\mathrm{Ker}(V) = \mathrm{Im}(F)$. The principal polarization of A induces a symmetry on $A[p]$ as defined in [**20**, 5.1]; when $p = 2$, there are complications with the polarization which are resolved in [**20**, 9.2, 9.5, 12.2].

There are two important invariants of (the p-torsion of) A: the p-rank and a-number. The *p-rank* of A is $f = \dim_{\mathbb{F}_p} \mathrm{Hom}(\mu_p, A[p])$ where μ_p is the kernel of Frobenius on $\mathbb{G}_m$. Then p^f is the cardinality of $A[p](k)$. The *a-number* of A is $a = \dim_k \mathrm{Hom}(\alpha_p, A[p])$ where α_p is the kernel of Frobenius on $\mathbb{G}_a$.

One can describe the group scheme $A[p]$ using the *mod p Dieudonné module*, i.e., the modulo p reduction of the covariant Dieudonné module, see e.g., [**20**, 15.3]. More precisely, there is an equivalence of categories between finite commutative group schemes over k annihilated by p and left $\mathbb{E}$-modules of finite dimension. Here $\mathbb{E} = k[F, V]$ denotes the non-commutative ring generated by semi-linear operators F and V with the relations $FV = VF = 0$ and $F\lambda = \lambda^p F$ and $\lambda V = V\lambda^p$ for all $\lambda \in k$. Let $\mathbb{E}(A_1, \ldots)$ denote the left ideal of $\mathbb{E}$ generated by $A_1, \ldots$.

Furthermore, there is a bijection between isomorphism classes of $2g$ dimensional left $\mathbb{E}$-modules and *Ekedahl-Oort types*. To find the Ekedahl-Oort type, let N be the mod p Dieudonné module of $A[p]$. The canonical filtration of N is the smallest filtration of N stabilized by the action of F^{-1} and V; denote it by

$$0 = N_0 \subset N_1 \subset \cdots N_z = N.$$

The canonical filtration can be extended to a final filtration; the Ekedahl-Oort type is the tuple $[\nu_1, \ldots, \nu_g]$, where the ν_i are the dimensions of the images of V on the subspaces in the final filtration.

For example, let $I_{t,1}$ denote the p-torsion group scheme of rank p^{2t} having p-rank 0 and a-number 1. Then $I_{t,1}$ has Dieudonné module $\mathbb{E}/\mathbb{E}(F^t + V^t)$ and Ekedahl-Oort type $[0, 1, \ldots, t-1]$ [**21**, Lemma 3.1].

For a smooth projective curve X, by [**19**, Section 5], there is an isomorphism of $\mathbb{E}$-modules between the contravariant mod p Dieudonné module of the p-torsion group scheme $\mathrm{Jac}(X)[p]$ and the de Rham cohomology $H^1_{\mathrm{dR}}(X)$.[1]

In the rest of the paper, $p = 2$.

2. The de Rham cohomology as a representation for the Suzuki group

In this section, we analyze the de Rham cohomology $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ of the Suzuki curve as a 2-modular representation of the Suzuki group.

2.1. Some ordinary representations. Suzuki determined the irreducible ordinary characters and representations of $\mathrm{Sz}(q)$ [**24**]. Consider the following four unipotent representations of $\mathrm{Sz}(q)$. Let W_S denote the Steinberg representation of dimension q^2. Let W_0 be the trivial representation of dimension 1. Let W_+ and W_- be the two unipotent cuspidal representations of $\mathrm{Sz}(q)$, associated to the two ordinary characters of $\mathrm{Sz}(q)$ of degree $q_0(q-1)$ [**24**]. Then W_+ and W_- each have dimension $q_0(q-1)$.

[1]Differences between the covariant and contravariant theory do not cause a problem in this paper since all objects we consider are symmetric.

In [**17**, Theorem 6.1], Lusztig studied the compactly supported ℓ-adic cohomology of the affine Deligne-Lusztig curves. For the Suzuki curves, he proved that the ordinary representations W_S, W_+, W_-, W_0 are the eigenspaces under Frobenius and that each appears with multiplicity 1.

2.2. Modular representations of the Suzuki group. The absolutely irreducible 2-modular representations of $\mathrm{Sz}(q)$ are well-understood [**18, 2, 23, 16**].

Let $q = 2^{2m+1}$. We recall some results about the 2-modular representations of the Suzuki group $\mathrm{Sz}(q)$ from [**18**]. Fix a generator ζ of $\mathbb{F}_q^*$. Let $\theta \in \mathrm{Aut}(\mathbb{F}_q)$ be such that $\theta^2(\alpha) = \alpha^2$ for all $\alpha \in \mathbb{F}_q$, i.e., θ is the square root of Frobenius.

The Suzuki group acts on $\mathcal{S}_m$. Let $\tau \in \mathrm{Sz}(q)$ be an element of order $q-1$; without loss of generality, we suppose that τ acts on $\mathcal{S}_m$ by

$$\tau : y \mapsto \zeta y, \ z \mapsto \zeta^{2^m+1} z.$$

Then $\mathrm{Sz}(q)$ has an irreducible 4-dimensional 2-modular representation V_0 in which $\tau \mapsto M$, where $M \in \mathrm{GL}_4(\mathbb{F}_q)$ is the matrix

$$M = \begin{pmatrix} \zeta^{\theta+1} & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 \\ 0 & 0 & \zeta^{-1} & 0 \\ 0 & 0 & 0 & \zeta^{-(\theta+1)} \end{pmatrix}.$$

For $0 \leq i \leq 2m$, consider the automorphism α_i of $\mathrm{Sz}(q)$ induced by the automorphism $x \mapsto x^{2^i}$ of $\mathbb{F}_q$. Let V_i be the 4-dimensional $\mathbb{F}_q\mathrm{Sz}(q)$-module where $g \in \mathrm{Sz}(q)$ acts as g^{α_i} on V_0.

Let I be a subset of $N = \mathbb{Z}/(2m+1)\mathbb{Z}$. Define $V_I = \otimes_{j\in I} V_j$, with $V_\emptyset$ being the trivial module. Then V_I is an absolutely irreducible 2-modular representation of $\mathrm{Sz}(q)$. By [**18**, Lemma 1], if $I \neq J$ then V_I and V_J are geometrically non-isomorphic and $\{V_I \mid I \subset N\}$ is the complete set of simple $\overline{\mathbb{F}}_2\mathrm{Sz}(q)$-modules. Note that V_I has dimension $4^{|I|}$ and that V_N is the Steinberg module.

By [**23**, Theorem, page 1], for $I, J \subset N$, there are no non-trivial extensions of V_I by V_J, namely $\mathrm{Ext}^1_{\mathbb{F}_2\mathrm{Sz}(q)}(V_I, V_J) = 0$.

The Frobenius $x \mapsto x^2$ on $\mathbb{F}_q$ acts on $\{V_i\}$ taking $V_i \mapsto V_{i+1 \bmod 2m+1}$. Note that $\oplus_{I\in\mathcal{I}} V_I$ is an $\mathbb{F}_2\mathrm{Sz}(q)$-module if and only if $\mathcal{I}$ is invariant under Frobenius or, equivalently, if and only if $\{I \mid I \in \mathcal{I}\}$ is invariant under the translation $i \mapsto i+1 \bmod 2m+1$.

For $i \in N$, let ϕ_i denote the Brauer character associated to the 4-dimensional module V_i. For $I \subseteq N$, let $\phi_I = \prod_{i\in I} \phi_i$, so ϕ_I is the character associated to the module V_I. Then $\{\phi_I : I \subseteq N\}$ is a complete set of Brauer characters for $\mathrm{Sz}(q)$.

By [**2**, Theorem 3.4], $\phi_i^2 = 4 + 2\phi_{i+m+1} + \phi_{i+1}$. Using this relation, Liu constructs a graph with vertex set N and edge set $\{(i, i+1), (i, i+1+m) : i \in N\}$. Edges of the form $(i, i+1)$ are called short edges and edges of the form $(i, i+1+m)$ are called long edges. Two vertices i, j are called adjacent if they are connected by a long edge, i.e., if $i - j \equiv \pm m \bmod 2m+1$. A set $I' \subseteq N$ is called circular if no vertices of $I = N \setminus I'$ are adjacent. A set $I \subseteq S$ is called good if $I' = N \setminus I$ is circular.

The decompositions of W_+ and W_- into irreducible 2-modular representations are known.

Theorem 2.1. *Liu* [**16**, Theorem 3.4] *The irreducible 2-modular representation V_I occurs in $W_\pm$ if and only if I is good, i.e., if and only if there do not exist $i, j \in I$*

such that $j - i \equiv \pm m \bmod 2m+1$. In this case, the multiplicity of V_I in $W_\pm$ is $2^{m-|I|}$.

2.3. Modular representation of the de Rham cohomology. The de Rham cohomology $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ is an $\mathbb{F}_2[\mathrm{Sz}(q)]$-module of dimension $2g_m = 2q_0(q-1)$. We consider the decomposition of $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ into irreducible 2-modular representations of the Suzuki group $\mathrm{Sz}(q)$.

COROLLARY 2.2. *The irreducible 2-modular representation V_I occurs in $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ if and only if there do not exist $i, j \in I$ such that $j - i \equiv \pm m \bmod 2m+1$. If V_I occurs in $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ then its multiplicity is $2^{m+1-|I|}$. Thus the 2-modular $\mathrm{Sz}(q)$-representation of $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ is:*

$$H^1_{\mathrm{dR}}(\mathcal{S}_m) \simeq \bigoplus_{I \text{ good}} V_I^{2^{m+1-|I|}}. \tag{2.1}$$

PROOF. In [**11**, page 2535], Gross uses [**17**, Theorem 6.1] to prove that, as a $\mathrm{Sz}(q)$-representation, the ℓ-adic cohomology of the smooth projective curve $\mathcal{S}_m$ is:

$$H^1(\mathcal{S}_{m,\bar{\mathbb{F}}_2}, \bar{\mathbb{Q}}_\ell) \simeq W_+ \oplus W_-.$$

By [**15**, Theorem 2], the characters of $H^1(\mathcal{S}_{m,\bar{\mathbb{F}}_2}, \bar{\mathbb{Q}}_\ell)$ and $H^1_{\mathrm{crys}}(\mathcal{S}_m, \mathrm{Frac}(W(\bar{\mathbb{F}}_2)))$ as representations of $\mathrm{Sz}(q)$ are the same, and thus the representations are isomorphic. The de Rham cohomology is the reduction modulo 2 of the crystalline cohomology. Thus the result follows from Theorem 2.1. □

EXAMPLE 2.3. When $m = 1$, then $H^1_{\mathrm{dR}}(\mathcal{S}_m) \simeq (V_0 \oplus V_1 \oplus V_2)^2 \oplus V_\emptyset^4$.

EXAMPLE 2.4. When $m = 2$, then

$$H^1_{\mathrm{dR}}(\mathcal{S}_m) \simeq \left(V_{\{0,1\}} \oplus V_{\{1,2\}} \oplus V_{\{2,3\}} \oplus V_{\{3,4\}} \oplus V_{\{4,0\}}\right)^2 \oplus (V_0 \oplus V_1 \oplus V_2 \oplus V_3 \oplus V_4)^4 \oplus V_\emptyset^8.$$

REMARK 2.5. For $m \leq 10$, we verified Corollary 2.2 using the multiplicity of the eigenvalues for τ on $H^1_{\mathrm{dR}}(\mathcal{S}_m)$.

3. The Dieudonné module and de Rham cohomology

In this section, we study the structure of the mod 2 Dieudonné module D_m of the Suzuki curve $\mathcal{S}_m$ or, equivalently, the structure of $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ as an $\mathbb{E}$-module.

3.1. Results and conjectures. The chosen element $\tau \in \mathrm{Sz}(q)$ of order $q-1$ acts on the mod 2 Dieudonné module D_m. Let $D_{m,0}$ denote the trivial eigenspace and $D_{m,\neq 0}$ denote the direct sum of the non-trivial eigenspaces. Since F and V commute with τ, they stabilize $D_{m,0}$ and $D_{m,\neq 0}$; thus there is an $\mathbb{E}$-module decomposition $D_m = D_{m,0} \oplus D_{m,\neq 0}$.

In Section 3.2, we prove the next proposition; it determines the $\mathbb{E}$-module structure of $D_{m,0}$.

PROPOSITION 3.1. *Let $m \in \mathbb{N}$ and let $q_0 = 2^m$. The trivial eigenspace $D_{m,0}$ of the mod 2 Dieudonné module of $\mathcal{S}_m$ has Ekedahl-Oort type $[0, 1, 1, 2, 2, \ldots, q_0 - 1, q_0]$; in particular, it has rank $2q_0$, 2-rank 0, and a-number 2^{m-1}.*

We have less information about the $\mathbb{E}$-module structure of $D_{m,\neq 0}$. In Section 3.3, we explain how the non-trivial representations V_I in $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ lead to $\mathbb{E}$-submodules D_I of the mod 2 Dieudonné module of $H^1_{\mathrm{dR}}(\mathcal{S}_m)$. We would like to understand how to determine the $\mathbb{E}$-module structure of D_I from the representation

V_I for the subset $I \subset N = \mathbb{Z}/(2m+1)\mathbb{Z}$. In Section 3.3, we consider a particular representation W_m, and make the following conjecture.

CONJECTURE 3.2. *The multiplicity of $\mathbb{E}/\mathbb{E}(F^{2m+1} + V^{2m+1})$ in the mod 2 Dieudonné module D_m of $\mathcal{S}_m$ is 4^m.*

We verify Conjecture 3.2 for $m = 1$ and $m = 2$ in Propositions 3.3 and 3.4. In fact, for $m = 1$ and $m = 2$, we determine the mod 2 Dieudonné module D_m completely. To do this, we find a basis for $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ for all m in Section 4. For $m = 1$, we explicitly compute the action of F and V on this basis, proving that:

PROPOSITION 3.3. *When $m = 1$, then the mod 2 Dieudonné module of $\mathcal{S}_1$ is*

$$D_1 = (\mathbb{E}/\mathbb{E}(F^3 + V^3))^4 \oplus \mathbb{E}/\mathbb{E}(F^2 + V^2).$$

For $m = 2$, we determine the action of F and V on $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ using Magma [**1**]. Consider the $\mathbb{E}$-module $\mathbb{E}(Z)$ generated by X_1, X_2, X_3 with the following relations: $V^3X_1 - F^3X_2 = 0$; $V^4X_2 - F^3X_3 = 0$; and $V^3X_3 - F^4X_1 = 0$. Then $\mathbb{E}(Z)$ is symmetric and has rank 20, p-rank 0, and a-number 3.

PROPOSITION 3.4. *When $m = 2$, then the mod 2 Dieudonné module of $\mathcal{S}_2$ is*

$$D_2 = \left(\mathbb{E}/\mathbb{E}(F^5 + V^5)\right)^{16} \oplus (\mathbb{E}(Z))^4 \oplus (\mathbb{E}/\mathbb{E}(F^3 + V^3) \oplus \mathbb{E}/\mathbb{E}(F + V)).$$

3.2. The trivial eigenspace. The eigenspace $D_{m,0}$ is the subspace of $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ of elements fixed by τ. Since τ acts fixed point freely on the 4-dimensional module V_i for each i [**18**, proof of Lemma 3], the generators of $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ which are fixed by τ are exactly those in V_I for $I = \emptyset$. In other words, the representation for $D_{m,0}$ consists of the $2^{m+1} = 2q_0$ copies of the trivial representation in (2.1).

PROOF. (Proof of Proposition 3.1) Let $\mathcal{C}_{m,0}$ be the quotient curve of $\mathcal{S}_m$ by the subgroup $\langle\tau\rangle$. Then $\mathcal{C}_{m,0}$ is a hyperelliptic curve of genus q_0 [**10**, Theorem 6.9].

The de Rham cohomology $H^1_{\mathrm{dR}}(\mathcal{C}_{m,0})$ of $\mathcal{C}_{m,0}$ is isomorphic as an $\mathbb{E}$-module to $D_{m,0}$. Thus the trivial eigenspace $D_{m,0}$ for the mod 2 Dieudonné module of $\mathcal{S}_m$ is isomorphic to the mod 2 Dieudonné module of $C_{m,0}$; in particular, it has rank $2q_0$.

Since $\mathcal{S}_m$ has 2-rank 0, so does $\mathcal{C}_{m,0}$. Thus $C_{m,0}$ is a hyperelliptic curve of 2-rank 0. By [**6**, Corollary 5.3], the Ekedahl-Oort type of $C_{m,0}$ is $[0, 1, 1, 2, 2, \ldots, q_0 - 1, q_0]$; this implies that the a-number is 2^{m-1}. $\square$

We determine the $\mathbb{E}$-module structure of $D_{m,0}$ by applying results from [**6**, Section 5].

PROPOSITION 3.5. [**6**, Proposition 5.8] *The mod 2 Dieudonné module $D_{m,0}$ is the $\mathbb{E}$-module generated as a k-vector space by $\{X_1, \ldots, X_{q_0}, Y_1, \ldots, Y_{q_0}\}$ with the actions of F and V given by:*

(1) $F(Y_j) = 0$.

(2) $V(Y_j) = \begin{cases} Y_{2j} & \text{if } j \leq q_0/2, \\ 0 & \text{if } j > q_0/2. \end{cases}$

(3) $F(X_i) = \begin{cases} X_{j/2} & \text{if } j \text{ is even}, \\ Y_{q_0-(j-1)/2} & \text{if } j \text{ is odd}. \end{cases}$

(4) $V(X_j) = \begin{cases} 0 & \text{if } j \leq (q_0-1)/2, \\ -Y_{2q_0-2j+1} & \text{if } j > (q_0-1)/2. \end{cases}$

We have an explicit description of the generators and relations of $D_{m,0}$ as follows.

NOTATION 3.6. [**6**, Notation 5.9] Fix $c = q_0 \in \mathbb{N}$. Consider the set

$$I = \{j \in \mathbb{N} \mid \lceil (c+1)/2 \rceil \leq j \leq c\},$$

which has cardinality $\lfloor (c+1)/2 \rfloor$. For $j \in I$, let $\ell(j)$ be the odd part of j and let $e(j) \in \mathbb{Z}^{\geq 0}$ be such that $j = 2^{e(j)}\ell(j)$. Let $s(j) = c - (\ell(j) - 1)/2$. Then $\{s(j) \mid j \in I\} = I$. Also, let $m(j) = 2c - 2j + 1$ and let $\epsilon(j) \in \mathbb{Z}^{\geq 0}$ be such that $t(j) := 2^{\epsilon(j)} m(j) \in I$. Then $\{t(j) \mid j \in I\} = I$. Thus, there is a unique bijection $\iota : I \to I$ such that $t(\iota(j)) = s(j)$ for each $j \in I$.

PROPOSITION 3.7. [**6**, Proposition 5.10] *The set $\{X_j \mid j \in I\}$ generates the mod 2 Dieudonné module $D_{m,0}$ as an $\mathbb{E}$-module subject to the following relations, for $j \in I$: $F^{e(j)+1}(X_j) + V^{\epsilon(\iota(j))+1}(X_{\iota(j)})$.*

EXAMPLE 3.8. (1) When $m = 1$ and the Ekedahl-Oort type is $[0, 1]$, then $D_{1,0} \simeq \mathbb{E}/\mathbb{E}(F^2 + V^2)$ (group scheme $I_{2,1}$).
(2) When $m = 2$ and the Ekedahl-Oort type is $[0, 1, 1, 2]$, then one checks that $D_{2,0} \simeq \mathbb{E}/\mathbb{E}(F + V) \oplus \mathbb{E}/\mathbb{E}(F^3 + V^3)$ (group scheme $I_{1,1} \oplus I_{3,1}$).

In the next result, we determine some $\mathbb{E}$-submodules of $D_{m,0}$ for general m.

PROPOSITION 3.9. *The $\mathbb{E}$-module $\mathbb{E}/\mathbb{E}(V^{e+1} + F^{e+1})$ occurs as an $\mathbb{E}$-submodule of $D_{m,0}$ if and only if $2^m \equiv 2^e \bmod 2^{e+1} + 1$. In particular:*
(1) *$\mathbb{E}/\mathbb{E}(V^{m+1} + F^{m+1})$ occurs for all m;*
(2) *$\mathbb{E}/\mathbb{E}(V + F)$ occurs if and only if m is even; and*
(3) *$\mathbb{E}/\mathbb{E}(V^2 + F^2) = 0$ occurs if and only if $m \equiv 1 \bmod 4$.*

PROOF. Let $e \in \mathbb{Z}^{\geq 0}$. By Proposition 3.7, the relation $(V^{e+1} + F^{e+1})X_j = 0$ is only possible if $j = 2^e \ell$ where ℓ is odd. Write $s(j) = c - (\ell - 1)/2$. Then $F^{e+1}(X_j) = F(X_\ell) = Y_{s(j)}$. Now $V(X_j) = -Y_{m(j)}$ where $m(j) = 2c - 2j + 1$. Also $V^{e+1}(X_j) = 2^e m(j)$. Thus we need $s(j) = 2^e m(j)$. This is equivalent to $2^{e+1}c - (j - 2^e) = 2^{2e+1}(2c - 2j + 1)$, which is equivalent to

$$j = \frac{c2^{e+1}(2^{e+1} - 1) + 2^e(2^{2e+1} - 1)}{2^{2e+2} - 1} = \frac{c2^{e+1} + 2^e}{2^{e+1} + 1}.$$

This value of j is integral if and only if $c \equiv 2^e \bmod 2^{e+1} + 1$. Thus, the relation $(V^{e+1} + F^{e+1})X_j = 0$ occurs if and only if $2^m \equiv 2^e \bmod 2^{e+1} + 1$ and also $j = (2^{e+1}q_0 + 2^e)/(2^{e+1} + 1)$. In particular, one checks that:
(1) $(V^{m+1} + F^{m+1})X_{2^m} = 0$;
(2) the relation $(V + F)X_j = 0$ occurs if and only if m is even and $j = (2 \cdot 2^m + 1)/3$;
(3) the relation $(V^2 + F^2)X_j = 0$ occurs if and only if $m \equiv 1 \bmod 4$ and $j = (4 \cdot 2^m + 2)/5$.

□

As a corollary, we determine cases when the $\mathbb{E}$-module $\mathbb{E}/\mathbb{E}(V^{e+1}+F^{e+1})$ occurs in D_m.

COROLLARY 3.10. *If $2^m \equiv 2^e \bmod 2^{e+1} + 1$, then $\mathbb{E}/\mathbb{E}(V^{e+1} + F^{e+1})$ occurs as an $\mathbb{E}$-submodule of the mod 2 Dieudonné module D_m of $\mathcal{S}_m$. In particular,*

(1) $\mathbb{E}/\mathbb{E}(V^{m+1}+F^{m+1})$ *occurs as an* $\mathbb{E}$*-submodule of* D_m *for all* m*;*
(2) $\mathbb{E}/\mathbb{E}(V+F)$ *occurs as an* $\mathbb{E}$*-submodule of* D_m *if* m *is even; and*
(3) $\mathbb{E}/\mathbb{E}(V^2+F^2)$ *occurs as an* $\mathbb{E}$*-submodule of* D_m *if* $m \equiv 1 \bmod 4$*.*

Proof. By Proposition 3.9, $\mathbb{E}/\mathbb{E}(V^{e+1}+F^{e+1})$ occurs as an $\mathbb{E}$-submodule of the mod 2 Dieudonné module $D_{m,0}$. The result follows since $D_{m,0}$ is an $\mathbb{E}$-submodule of D_m. $\square$

3.3. The nontrivial eigenspaces. Recall that $D_{m,\neq 0}$ is the direct sum of the non-trivial eigenspaces for τ. Consider the canonical filtration of $D_{m,\neq 0}$, which is the smallest filtration stabilized under the action of F^{-1} and V; denote it by

$$0 = N_0 \subset N_1 \subset \cdots N_t = N.$$

By [**20**, Chapter 2] (see also [**5**, Section 2.2]), the blocks $B_i = N_{i+1}/N_i$ in the canonical filtration are representations for $H^1_{\mathrm{dR}}(\mathcal{S}_m)$. On each block B_i, either (i) $V|_{B_i} = 0$ in which case $B_i \subset \mathrm{Im}(F)$ and $F^{-1} : B_i^{(p)} \to B_j$ is an isomorphism to another block with index $j > i$; or (ii) $V : B_i^{(p)} \to B_j$ is an isomorphism to another block with index $j < i$. This action of V and F^{-1} yields a permutation π of the set of blocks B_i. Cycles in the permutation are in bijection with orbits $\mathcal{O}$ of the blocks under the action of V and F^{-1}.

Fix an orbit $\mathcal{O}$ of a block B_i under the action of F^{-1} and V. As in [**22**, Section 5.2], this yields a word w in F^{-1} and V. From this, we produce a symmetric $\mathbb{E}$-module $\mathbb{E}(w)$ whose dimension over k is the length of w. Then $\mathbb{E}(w)$ is an isotypic component of $D_{m,0}$. The multiplicity of $\mathbb{E}(w)$ in $D_{m,\neq 0}$ is the dimension of the block B_i in $\mathcal{O}$.

By Corollary 2.2, the representations occurring in $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ are the representations in $W_\pm$, namely the representations V_I for I a good subset of $N = \mathbb{Z}/(2m+1)\mathbb{Z}$.

We now explain the motivation for Conjecture 3.2. Let $I_m = \{0, \ldots, m-1\}$. The smallest power of F that stabilizes I_m is $2m+1$. Consider the 2-modular representation of Sz(q) given by $W_m = \oplus_{i=0}^{2m} F^i(V_{I_m})$. For example, when $m = 1$ then $W_1 = V_0 \oplus V_1 \oplus V_2$ and when $m = 2$ then

$$W_2 = (V_0 \otimes V_1) \oplus (V_1 \otimes V_2) \oplus (V_2 \otimes V_3) \oplus (V_3 \otimes V_4) \oplus (V_4 \otimes V_0).$$

By definition, W_m is an $\mathbb{F}_2\mathrm{Sz}(q)$-module of dimension $(2m+1)4^m$. By Corollary 2.2, the 2-modular representation W_m appears with multiplicity 2 in $H^1_{\mathrm{dR}}(\mathcal{S}_m)$. Consider the $\mathbb{E}$-module $\mathbb{E}/\mathbb{E}(F^{2m+1}+V^{2m+1})$; it has dimension $2(2m+1)$ over k.

The idea behind Conjecture 3.2 is that the subrepresentation W_m^2 of $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ should correspond to a submodule of D_m with structure $(\mathbb{E}/\mathbb{E}(F^{2m+1}+V^{2m+1}))^{4^m}$. More precisely, Conjecture 3.2 would follow from the claims that there is a unique i such that V_{I_m} is a subrepresentation of B_i, that B_i is irreducible and thus equal to V_{I_m}, and that the word w on the orbit of B_i is $(F^{-1})^{2m+1}V^{2m+1}$.

4. An Explicit Basis for the de Rham cohomology

In this section, we compute an explicit basis for $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ for all m. This material is needed to determine the mod 2 Dieudonné module of $\mathcal{S}_m$ when $m = 1$ and $m = 2$ in Propositions 3.3 and 3.4. We determine the action of F and V on the basis elements explicitly here when $m = 1$ and using Magma [**1**] when $m = 2$.

4.1. Preliminaries. Consider the affine equation $z^q + z = y^{q_0}(y^q + y)$ for $\mathcal{S}_m$. Let P_∞ be the point at infinity on $\mathcal{S}_m$. Let $P_{(y,z)}$ denote the point (y, z) on $\mathcal{S}_m$.

Define the functions $h_1, h_2 \in \mathbb{F}_2(\mathcal{S}_m)$ by:

$$h_1 := z^{2q_0} + y^{2q_0+1}, \;\; h_2 := z^{2q_0} y + h_1^{2q_0}.$$

LEMMA 4.1. (1) *The function y has divisor*

$$\operatorname{div}(y) = \sum_{z \in \mathbb{F}_q} P_{(0,z)} - qP_\infty.$$

(2) *The function z has divisor*

$$\operatorname{div}(z) = \sum_{y \in \mathbb{F}_q^\times} P_{(y,0)} + (q_0+1)P_{(0,0)} - (q+q_0)P_\infty.$$

(3) *Let $S = \left\{(y,z) \in \mathbb{F}_q^2 : y^{2q_0+1} = z^{2q_0}, (y,z) \neq (0,0)\right\}$. The function h_1 has divisor*

$$\operatorname{div}(h_1) = \sum_{(y,z) \in S} P_{(y,z)} + (2q_0+1)P_{(0,0)} - (q+2q_0)P_\infty.$$

(4) *The function h_2 has divisor*

$$\operatorname{div}(h_2) = (q + 2q_0 + 1)(P_{(0,0)} - P_\infty).$$

PROOF. The pole orders of these functions are determined in [**14**, Proposition 1.3]. The orders of the zeros can be determined using the equation for the curve and the definitions of h_1 and h_2. □

Let $\mathcal{E}_m$ be the set of $(a, b, c, d) \subset \mathbb{Z}^4$ satisfying

$$\begin{gathered} 0 \le a, \quad 0 \le b \le 1, \quad 0 \le c \le q_0 - 1, \quad 0 \le d \le q_0 - 1, \\ aq + b(q+q_0) + c(q+2q_0) + d(q+2q_0+1) \le 2g-2. \end{gathered}$$

LEMMA 4.2. *The following set is a basis of $H^0(\mathcal{S}_m, \Omega^1)$:*

$$\mathcal{B}_m := \left\{ g_{a,b,c,d} := y^a z^b h_1^c h_2^d \, dy \mid (a,b,c,d) \in \mathcal{E}_m \right\}.$$

PROOF. See [**7**, Proposition 3.7]. □

A basis for $H^1(\mathcal{S}_m, \mathcal{O})$ can be built similarly. Define the map

$$\pi : \mathcal{S}_m \to \mathbb{P}^1_y, \;\; (y,z) \mapsto y, \quad P_\infty \mapsto \infty_y.$$

Let 0_y be the point on $\mathbb{P}^1_y$ with $y = 0$. Then $\pi^{-1}(0_y) = \{(0, z) : z \in \mathbb{F}_q\}$ has cardinality q.

LEMMA 4.3. *The following set represents a basis of $H^1(\mathcal{S}_m, \mathcal{O})$:*

$$\mathcal{A}_m := \left\{ f_{a,b,c,d} := \frac{1}{y^a z^b h_1^c h_2^d} \frac{z h_1^{q_0-1} h_2^{q_0-1}}{y} \mid (a,b,c,d) \in \mathcal{E}_m \right\}.$$

PROOF. Let $U_\infty = \mathcal{S}_m \setminus \pi^{-1}(\infty_y) = \mathcal{S}_m \setminus P_\infty$ and $U_0 = \mathcal{S}_m \setminus \pi^{-1}(0_y)$. The elements of $H^1(\mathcal{S}_m, \mathcal{O})$ can be represented by classes of functions that are regular on $U_\infty \cap U_0$, but are not regular on U_∞ or regular on U_0. In other words, these functions have a pole at P_∞ and at some point in $\pi^{-1}(0_y)$.

Let $f = f_{a,b,c,d}$ for some $(a,b,c,d) \in \mathcal{E}_m$. Then f has poles only in $\{P_\infty, \pi^{-1}(0_y)\}$ by Lemma 4.1. Let $Q = (0, \alpha)$ for some $\alpha \in \mathbb{F}_q^\times$. Then $v_Q(f) = -(a+1) \le -1$. Also, let $t = q + 2q_0 + 1$, then

$$\begin{aligned} v_{P_\infty}(f) &= (a+1)(q) - (1-b)(q+q_0) - (q_0 - 1 - c)(q + 2q_0) - (q_0 - 1 - d)t \\ &= aq + b(q+q_0) + c(q+2q_0) + d(q + 2q_0 + 1) + (2q_0 - 2q_0 q + 1) \\ &\le 2g_m - 2 + (2q_0 - 2q_0 q + 1) \\ &= (2q_0 q - 2q_0 - 2) + (2q_0 - 2q_0 q + 1) \\ &= -1. \end{aligned}$$

So f is regular on $U_\infty \cap U_0$ but not on U_∞ or U_0. By a calculation similar to [**7**, Proposition 3.7], the elements of $\mathcal{A}_m$ are independent because each element has a different pole order at P_∞. The cardinality of $\mathcal{A}_m$ is $g_m = \dim(H^1(\mathcal{S}_m, \mathcal{O}))$. Thus $\mathcal{A}$ is a basis for $H^1(\mathcal{S}_m, \mathcal{O})$.

□

4.2. Constructing the de Rham cohomology. Let $\mathcal{U}$ be the open cover of $\mathcal{S}_m$ given by U_∞ and U_0 from the proof of Lemma 4.3. For a sheaf $\mathcal{F}$ on $\mathcal{S}_m$, let

$$\begin{aligned} C^0(\mathcal{U}, \mathcal{F}) &:= \{g = (g_\infty, g_0) \mid g_i \in \Gamma(U_i, \mathcal{F})\}, \\ C^1(\mathcal{U}, \mathcal{F}) &:= \{\phi \in \Gamma(U_\infty \cap U_0, \mathcal{F})\}. \end{aligned}$$

Define the coboundary operator $\delta : \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^1(\mathcal{U}, \mathcal{F})$ by $\delta g = g_\infty - g_0$. The closed de Rham cocycles are the set

$$Z^1_{\mathrm{dR}}(\mathcal{U}) := \{(f, g) \in \mathcal{C}^1(\mathcal{U}, \mathcal{O}) \times \mathcal{C}^0(\mathcal{U}, \Omega^1) : df = \delta g\}.$$

The de Rham coboundaries are the set

$$B^1_{\mathrm{dR}}(\mathcal{S}_m) := \{(\delta\kappa, d\kappa) \in Z^1_{\mathrm{dR}}(\mathcal{U}) : \kappa \in \mathcal{C}^0(\mathcal{U}, \mathcal{O})\},$$

where $d\kappa = (d(\kappa_0), d(\kappa_\infty))$. The de Rham cohomology $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ is

$$H^1_{\mathrm{dR}}(\mathcal{S}_m) \cong H^1_{\mathrm{dR}}(\mathcal{S}_m)(\mathcal{U}) := Z^1_{\mathrm{dR}}(\mathcal{U}) / B^1_{\mathrm{dR}}(\mathcal{U}).$$

There is an injective homomorphism $\lambda : H^0(\mathcal{S}_m, \Omega^1) \to H^1_{\mathrm{dR}}(\mathcal{S}_m)$ denoted informally by $g \mapsto (0, \mathbf{g})$, where the second coordinate is a tuple $\mathbf{g} = (g_\infty, g_0)$ defined by $g_i = g|_{U_i}$. Define another homomorphism $\gamma : H^1_{\mathrm{dR}}(\mathcal{S}_m) \to H^1(\mathcal{S}_m, \mathcal{O})$ with $(f, \mathbf{g}) \mapsto f$. These create a short exact sequence

$$0 \longrightarrow H^0(\mathcal{S}_m, \Omega^1) \xrightarrow{\lambda} H^1_{\mathrm{dR}}(\mathcal{S}_m) \xrightarrow{\gamma} H^1(\mathcal{S}_m, \mathcal{O}) \longrightarrow 0. \tag{4.1}$$

Let A be a basis for $H^1(\mathcal{S}_m, \mathcal{O})$ and B a basis for $H^0(\mathcal{S}_m, \Omega^1)$. A basis for $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ is then given by $\psi(A) \cup \lambda(B)$, where ψ is defined as follows. Given $f \in H^1(\mathcal{S}_m, \mathcal{O})$, one can write $df = df_\infty + df_0$, where $df_i \in \Gamma(U_i, \Omega^1)$ for $i \in \{0, \infty\}$. For convenience, define $\mathbf{df} = (df_\infty, df_0)$. Define a section of (4.1) by:

$$\psi : H^1(\mathcal{S}_m, \mathcal{O}) \to H^1_{\mathrm{dR}}(\mathcal{S}_m),\ \psi(f) = (f, \mathbf{df}).$$

The image of ψ is a complement in $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ to $\lambda(H^0(\mathcal{S}_m, \Omega^1))$.

4.2.1. *The Frobenius and Verschiebung operators.* The Frobenius F and Verschiebung V act on $H^1_{\mathrm{dR}}(\mathcal{S}_m)$ by

$$F(f, \mathbf{g}) := (f^p, (0,0)) \text{ and } V(f, \mathbf{g}) := (0, \mathscr{C}(\mathbf{g}))$$

where $\mathscr{C}$ is the Cartier operator, which acts componentwise on $\mathbf{g}$. The Cartier operator is defined by the properties that it annihilates exact differentials, preserves logarithmic differentials, and is p^{-1}-linear. It follows from the definitions that

$$\ker(F) = \lambda(H^0(\mathcal{S}_m, \Omega^1)) = \mathrm{im}(V).$$

4.3. The case $m = 1$**.** When $m = 1$, then $q_0 = 2$, $q = 8$, and $g = 14$. The Suzuki curve $\mathcal{S}_1$ has affine equation

$$z^8 + z = y^2(y^8 + y).$$

The set $\mathcal{E}_1$ consists of the 14 tuples

$$\begin{aligned} \mathcal{E}_1 \quad = \quad & \{(0,0,0,0),(0,0,0,1),(0,0,1,0),(0,0,1,1),(0,1,0,0),(0,1,0,1),(0,1,1,0), \\ & (1,0,0,0),(1,0,0,1),(1,0,1,0),(1,1,0,0),(2,0,0,0),(2,1,0,0),(3,0,0,0)\}. \end{aligned}$$

By Lemmas 4.2 and 4.3, $\mathcal{B}_1$ is a basis for $H^0(\mathcal{S}_1, \Omega^1)$ and $\mathcal{A}_1$ is a basis for $H^1(\mathcal{S}_1, \mathcal{O})$. Based on the action of Frobenius and Verschiebung, the following sets make more convenient bases:

Lemma 4.4. (1) *A basis for* $H^1(\mathcal{S}_1, \mathcal{O})$ *is given by the set*

$$\begin{aligned} A \quad = \quad & \{f_{(0,0,0,0)}, f_{(2,0,0,0)}, f_{(0,1,0,0)} + f_{(3,0,0,0)}, f_{(2,1,0,0)} + f_{(0,0,1,0)}, \\ & f_{(0,0,0,1)} + f_{(1,0,1,0)}, f_{(1,0,0,0)}, f_{(2,1,0,0)}, f_{(1,0,0,1)}, f_{(0,0,1,1)}, \\ & f_{(1,0,1,0)}, f_{(3,0,0,0)}, f_{(1,1,0,0)}, f_{(0,1,1,0)}, f_{(0,1,0,1)}\}. \end{aligned}$$

(2) *A basis for* $H^0(\mathcal{S}_1, \Omega^1)$ *is given by the set*

$$\begin{aligned} B \quad = \quad & \{g_{(0,0,0,0)}, g_{(2,0,0,0)}, g_{(0,1,0,0)} + g_{(3,0,0,0)}, g_{(2,1,0,0)} + g_{(0,0,1,0)}, \\ & g_{(0,0,0,1)} + g_{(1,0,1,0)}, g_{(1,0,0,0)}, g_{(2,1,0,0)}, g_{(1,0,0,1)}, g_{(0,0,1,1)}, g_{(1,0,1,0)}, \\ & g_{(3,0,0,0)}, g_{(1,1,0,0)}, g_{(0,1,1,0)}, g_{(0,1,0,1)}\}. \end{aligned}$$

Proof. By Lemma 4.2 (resp. 4.3), these 1-forms (resp. functions) have distinct pole orders at P_∞, are therefore linearly independent, and thus form a basis of $H^1(\mathcal{S}_1, \mathcal{O})$ (resp. $H^0(\mathcal{S}_1, \Omega^1)$). □

It is now possible to calculate the action of F and V on $\psi(A) \cup \lambda(B)$, a basis for $H^1_{\mathrm{dR}}(\mathcal{S}_1)$.

4.3.1. *The action of Frobenius when* $m = 1$. The action of F is summarized in the right column of Table 2. Note that $F(g) = 0$ for $g \in B$ since $\ker(F) = \mathrm{im}(V) \cong H^0(\mathcal{S}_1, \Omega^1)$. For the action of F on $\psi(f)$ for $f \in A$, note that $F(\psi(f)) = (f^2, (0,0))$. Then

$$\begin{aligned} f^2 = (f_{(a,b,c,d)})^2 \quad &= \quad (y^{-1-a} z^{1-b} h_1^{1-c} h_2^{1-d})^2 \\ &= \quad (y^{-2})^{1+a}(yh_1 + h_2)^{1-b}(z + y^3)^{1-c}(h_1 + zy^2)^{1-d}. \end{aligned}$$

To do these calculations, we simplify f^2 and write it as a sum of quotients of monomials in $\{y, z, h_1, h_2\}$. These monomials can then be classified as belonging to $\Gamma(U_0)$ or $\Gamma(U_\infty)$, or can otherwise be rewritten in terms of the basis for $H^1(\mathcal{S}_1, \mathcal{O})$. It is then possible to use coboundaries to write $(f^2, (0,0))$ in terms of the given basis for $H^1_{\mathrm{dR}}(\mathcal{S}_1)$.

EXAMPLE 4.5. To compute that $F(\psi(f_{(0,1,0,1)})) = \lambda(g_{(0,0,0,0,0)})$, note first that

$$\left(f_{(0,1,0,1)}\right)^2 = y^{-2}(z+y^3) = \frac{z}{y^2} + y.$$

Also,

$$d\left(\frac{z}{y^2}\right) = \frac{1}{y^2}dz - 2\frac{z}{y^3}dy = dy \text{ and } d(y) = dy.$$

Since $y \in \Gamma(U_\infty, \mathcal{O})$ and $\frac{z}{y^2} \in \Gamma(U_0, \mathcal{O})$, the pair $\left(\frac{z}{y^2}, y\right)$ is in $C^0(\mathcal{U}, \mathcal{O})$ and one sees that $(\frac{z}{y^2} + y, (dy, dy))$ is a coboundary. Thus

$$\begin{aligned} F\left(\psi\left(f_{(0,1,0,1)}\right)\right) &= \left(\frac{z}{y^2} + y, (0,0)\right) + \left(\frac{z}{y^2} + y, (dy, dy)\right) \\ &= (0, (dy, dy)) = \lambda(dy) = (0, \mathbf{g}_{(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})}). \end{aligned}$$

EXAMPLE 4.6. We compute that $F\left(\psi(f_{(0,0,1,1)})\right) = \psi\left(f_{(0,1,0,1)}\right)$. This is true because

$$\left(f_{(0,0,1,1)}\right)^2 = y^{-2}(yh_1 + h_2) = \frac{h_1}{y} + \frac{h_2}{y^2}.$$

Note that $\frac{h_2}{y^2} \in \Gamma(U_0, \mathcal{O})$, so $(\frac{h_2}{y^2}, 0) \in C^0(\mathcal{U}, \mathcal{O})$, and $d\left(\frac{h_2}{y^2}\right) = \frac{z^4}{y^2}dy$. So one sees that $\left(\frac{h_2}{y^2}, (\frac{z^4}{y^2}dy, 0)\right)$ is a coboundary. Also, $d\left(\frac{h_1}{y}\right) = \frac{z^4}{y^2}dy$. Thus

$$\begin{aligned} F\left(\psi\left(f_{(0,0,1,1)}\right)\right) &= \left(\frac{h_1}{y} + \frac{h_2}{y^2}, (0,0)\right) + \left(\frac{h_2}{y^2}, \left(\frac{z^4}{y^2}dy, 0\right)\right) \\ &= \left(\frac{h_1}{y}, \left(\frac{z^4}{y^2}dy, 0\right)\right) = \psi\left(f_{(0,1,0,1)}\right). \end{aligned}$$

4.3.2. *The action of Verschiebung when $m = 1$.* The action of V is summarized in the middle column of Table 2. In [**7**], the authors calculate the action of the Cartier operator $\mathscr{C}$ (see Table 1). This determines the action of V on $\lambda(g)$ for $g \in B$. It also helps determine the action of V on $\psi(f)$ for $f \in A$.

EXAMPLE 4.7. We compute that $V\left(\psi(f_{(0,1,0,1)})\right) = (0, \mathbf{0})$. Writing

$$f = f_{(0,1,0,1)} = \frac{h_1}{y} = \frac{z^4}{y} + y^4,$$

then

$$df = \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \left(-\frac{z^4}{y^2} + 4y^3\right)dy + 4\frac{z^3}{y}dz = \frac{z^4}{y^2}dy.$$

Considering the pole orders of y, z, and dy, define $df = df_0 \in \Omega_0$ and $df_\infty = 0$, so $\mathbf{df} = (0, df)$. Thus $\mathscr{C}(df) = \frac{z^2}{y}\mathscr{C}(dy) = 0$. Thus $\mathscr{C}(\mathbf{df}) = (0,0) = \mathbf{0}$ and $V(\psi(f_{(0,1,0,1)})) = (0, \mathbf{0})$.

EXAMPLE 4.8. We compute that $V\left(\psi(f_{(2,1,0,0)})\right) = (0, \mathbf{g}_{(0,1,0,0)})$. This is because

$$f_{(2,1,0,0)} = \frac{h_1 h_2}{y^3},$$

so

$$df = y^{-3}d(h_1h_2) + y^{-4}h_1h_2dy = y^{-3}h_1d(h_2) + y^{-3}h_2d(h_1) + y^{-4}h_1h_2dy.$$

Then

$$d(h_1) = d(z^4 + y^5) = y^4dy \text{ and } d(h_2) = d(z^4y + h_1^4) = z^4dy,$$

so

$$
\begin{aligned}
df &= y^{-3}z^4 h_1 dy + y h_2 dy + y^{-4} h_1 h_2 \\
&= y^{-3} h_1 (h_1 + y^5) dy + y h_2 dy + y^{-4} h_1 h_2 \\
&= \frac{h_1^2}{y^3} dy + \frac{h_1 h_2}{y^4} dy + y^2 h_1 dy + y h_2 dy,
\end{aligned}
$$

using the fact that $z^4 = h_1 + y^5$. Considering the orders of the poles, define $df_0 = \frac{h_1^2}{y^3} dy + \frac{h_1 h_2}{y^4} dy \in \Omega_0$ and $df_\infty = y^2 h_1 dy + y h_2 dy \in \Omega_\infty$. Using Table 1 and the fact that $h_1^2 = z + y^3$, then

$$
\begin{aligned}
\mathscr{C}(df_\infty) &= y\mathscr{C}(h_1 dy) + \mathscr{C}(y h_2 dy) \\
&= y^3 dy + h_1^2 dy = (y^3 + z + y^3) dy = z dy.
\end{aligned}
$$

Thus $V\left(\psi(f_{(2,1,0,0)})\right) = (0, \mathbf{g}_{(0,1,0,0)})$.

The actions of F and V are summarized in Table 2.

TABLE 1. Cartier Operator on $H^0(\mathcal{S}_1, \Omega^1)$

f	$\mathscr{C}(f dy)$
1	0
y	dy
z	$y^{q_0/2}\, dy$
h_1	$y^{q_0}\, dy$
h_2	$\left((yh_1)^{q_0/2} + h_2\right)\, dy$
yz	$h_1^{q_0/2}\, dy$
yh_1	$\left((yh_1)^{q_0/2} + h_2\right)\, dy$
zh_1	$(yh_2)^{q_0/2}\, dy$
zh_2	$(h_1 h_2)^{q_0/2}\, dy$
$h_1 h_2$	$\left(h_1 + z y^{q_0}\right)\, dy$
yzh_1	$\left(y^{q_0/2} z + (h_1 h_2)^{q_0/2}\right)\, dy$
yzh_2	$\left(z h_1^{q_0/2} + y^{q_0/2+1} h_2^{q_0/2}\right)\, dy$
$zh_1 h_2$	$\left(z y^{q_0/2} h_2^{q_0/2} + h_1^{q_0/2+1}\right)\, dy$
$yh_1 h_2$	$\left((yh_1)^{q_0/2} z + h_2^{q_0/2} z\right)\, dy$
$yzh_1 h_2$	$\left(y^{q_0/2} h_2 + z h_1^{q_0/2} h_2^{q_0 2}\right)\, dy$

To conclude, we use the tables to give an explicit proof of Proposition 3.3.

PROPOSITION 4.9. *When $m = 1$, then the mod 2 Dieudonné module of $\mathcal{S}_1$ is*

$$
D_1 \simeq \mathbb{E}/\mathbb{E}(F^2 + V^2) \oplus (\mathbb{E}/\mathbb{E}(F^3 + V^3))^4.
$$

PROOF. As an $\mathbb{E}$-module, D_1 is isomorphic to $H^1_{\mathrm{dR}}(\mathcal{S}_1)$. From Table 2, $H^1_{\mathrm{dR}}(\mathcal{S}_1)$ has a summand of rank 4 generated by $X_1 = \psi(f_{(1,0,1,0)})$ with the relation given by $(F^2 + V^2)X_1 = 0$. There are 4 summands of rank 6 generated by $X_2 = \psi(f_{(2,1,0,0)})$, $X_3 = \psi(f_{(2,0,0,0)})$, $X_4 = \psi(f_{(3,0,0,0)})$, and $X_5 = \psi(f_{(0,0,0,0)})$ with the relations given

TABLE 2. Action of Verschiebung and Frobenius on $H^1_{\mathrm{dR}}(\mathcal{S}_1)$

$(f,\mathbf{g})$	$V(f,\mathbf{g})$	$F(f,\mathbf{g})$
$(0,\mathbf{g}_{(0,0,0,0)})$	$(0,\mathbf{0})$	$(0,\mathbf{0})$
$(0,\mathbf{g}_{(2,0,0,0)})$	$(0,\mathbf{0})$	$(0,\mathbf{0})$
$(0,\mathbf{g}_{(0,1,0,0)}+\mathbf{g}_{(3,0,0,0)})$	$(0,\mathbf{0})$	$(0,\mathbf{0})$
$(0,\mathbf{g}_{(2,1,0,0)}+\mathbf{g}_{(0,0,1,0)})$	$(0,\mathbf{0})$	$(0,\mathbf{0})$
$(0,\mathbf{g}_{(0,0,0,1)}+\mathbf{g}_{(1,0,1,0)})$	$(0,\mathbf{0})$	$(0,\mathbf{0})$
$(0,\mathbf{g}_{(1,0,0,0)})$	$(0,\mathbf{g}_{(0,0,0,0)})$	$(0,\mathbf{0})$
$(0,\mathbf{g}_{(0,0,1,0)})$	$(0,\mathbf{g}_{(2,0,0,0)})$	$(0,\mathbf{0})$
$(0,\mathbf{g}_{(1,0,0,1)})$	$(0,\mathbf{g}_{(0,1,0,0)}+\mathbf{g}_{(3,0,0,0)})$	$(0,\mathbf{0})$
$(0,\mathbf{g}_{(0,0,1,1)})$	$(0,\mathbf{g}_{(2,1,0,0)}+\mathbf{g}_{(0,0,1,0)})$	$(0,\mathbf{0})$
$(0,\mathbf{g}_{(1,0,1,0)})$	$(0,\mathbf{g}_{(0,0,0,1)}+\mathbf{g}_{(1,0,1,0)})$	$(0,\mathbf{0})$
$(0,\mathbf{g}_{(0,1,0,0)})$	$(0,\mathbf{g}_{(1,0,0,0)})$	$(0,\mathbf{0})$
$(0,\mathbf{g}_{(1,1,0,0)})$	$(0,\mathbf{g}_{(0,0,1,0)})$	$(0,\mathbf{0})$
$(0,\mathbf{g}_{(0,1,1,0)})$	$(0,\mathbf{g}_{(1,0,0,1)})$	$(0,\mathbf{0})$
$(0,\mathbf{g}_{(0,1,0,1)})$	$(0,\mathbf{g}_{(0,0,1,1)})$	$(0,\mathbf{0})$
$\psi(f_{(0,1,0,1)})$	$(0,\mathbf{0})$	$(0,\mathbf{g}_{(0,0,0,0)})$
$\psi(f_{(0,1,1,0)})$	$(0,\mathbf{0})$	$(0,\mathbf{g}_{(2,0,0,0)})$
$\psi(f_{(1,1,0,0)})$	$(0,\mathbf{0})$	$(0,\mathbf{g}_{(0,1,0,0)}+\mathbf{g}_{(3,0,0,0)})$
$\psi(f_{(0,1,0,0)}+f_{(3,0,0,0)})$	$(0,\mathbf{0})$	$(0,\mathbf{g}_{(2,1,0,0)}+\mathbf{g}_{(0,0,1,0)})$
$\psi(f_{(0,0,0,1)}+f_{(1,0,1,0)})$	$(0,\mathbf{0})$	$(0,\mathbf{g}_{(0,0,0,1)}+\mathbf{g}_{(1,0,1,0)})$
$\psi(f_{(0,0,1,1)})$	$(0,\mathbf{0})$	$\psi(f_{(0,1,0,1)})$
$\psi(f_{(1,0,0,1)})$	$(0,\mathbf{0})$	$\psi(f_{(0,1,1,0)})$
$\psi(f_{(2,1,0,0)}+f_{(0,0,1,0)})$	$(0,\mathbf{0})$	$\psi(f_{(1,1,0,0)})$
$\psi(f_{(1,0,0,0)})$	$(0,\mathbf{0})$	$\psi(f_{(0,1,1,0)})$
$\psi(f_{(1,0,1,0)})$	$(0,\mathbf{g}_{(1,0,1,0)})$	$\psi(f_{(0,0,0,1)}+f_{(1,0,1,0)})$
$\psi(f_{(2,1,0,0)})$	$(0,\mathbf{g}_{(0,1,0,0)})$	$\psi(f_{(0,0,1,1)})$
$\psi(f_{(3,0,0,0)})$	$(0,\mathbf{g}_{(1,1,0,0)})$	$\psi(f_{(1,0,0,1)})$
$\psi(f_{(2,0,0,0)})$	$(0,\mathbf{g}_{(0,1,1,0)})$	$\psi(f_{(2,1,0,0)}+f_{(0,0,1,0)})$
$\psi(f_{(0,0,0,0)})$	$(0,\mathbf{g}_{(0,1,0,1)})$	$\psi(f_{(1,0,0,0)})$

by $(F^3+V^3)X_i=0$. This yields the $\mathbb{E}$-module $\mathbb{E}/\mathbb{E}(F^2+V^2)\oplus\big(\mathbb{E}/\mathbb{E}(F^3+V^3)\big)^4$. □

Note that the trivial eigenspace $D_{1,0}$ appears as the summand $\mathbb{E}/(F^2+V^2)$. It is spanned by

$$\{\psi(f_{(1,0,1,0)}),\psi(f_{(0,0,0,1)}+f_{(1,0,1,0)}),(0,\mathbf{g}_{(1,0,1,0)}),(0,\mathbf{g}_{(0,0,0,1)}+\mathbf{g}_{(1,0,1,0)})\}.$$

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