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Mean field verification theorem*

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It is well known in deterministic and stochastic control that an optimal control can be obtained through a theory of sufficient conditions, so-called Bellman or Dynamic Programming approach. In Bellman's approach, one constructs a control under sufficient conditions and proves that this control is optimal by an argument called verification theorem. This presentation aims at describing the basic ideas of the verification theorem for mean field type control theory.

Keywords and phrases: Verification theorem, mean field type control, sufficient condition.

1. Introduction

Mean Field Type Control Theory is an extension of stochastic control. As well known in stochastic control (as well as in deterministic control), two types of theory exist for obtaining an optimal control, a theory of necessary conditions (Pontryagin approach) and a theory of sufficient conditions (Bellman or Dynamic Programming approach). In the Pontryagin approach, an optimal control if it exists needs to satisfy a necessary condition of optimality. In Bellman's approach, under some conditions (sufficient conditions) one constructs a control, and, by an argument, called "verification theorem", one proves that this control is optimal. These basic ideas can be extended to mean field type control theory. The objective of this brief presentation is to describe the basic ideas of the verification theorem for mean field type control theory. We do not present all the technical proofs. We also do it in

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a relatively simple framework, where the mean field type control aspect is limited to the presence of the expected value of the final state in the payoff functional. For more developments on Mean Field Theory, see [1, 2, 3, 4].

2. Mean field type control

The problem we want to solve is described as follows. Let $(\Omega, \mathcal{A}, P)$ be a probability space, on which is constructed an n-dimensional standard Wiener process, denoted by $w(t)$. We define next functions $g(x,v)$ and $\sigma(x,v)$ from $R^n \times R^m$ to (respectively) R^n and $\mathcal{L}(R^n;R^n)$, which are sufficiently smooth. They will be the drift and diffusion term of a diffusion process in the state space R^n, depending on a control v. A control will be defined by a feedback on the state $v(x,s)$. This feedback is the unknown of the control problem, we describe now. First, the state of the dynamic system denoted by $x(t)$ is the solution of the SDE (stochastic differential equation)

$$dx = g(x, v(x,s))ds + \sigma(x, v(x,s))dw(s), \quad x(0) = x_0. \tag{1}$$

We are assuming here that, in the class of admissible controls $v(x,s)$ we are considering, we can solve this SDE and obtain a unique solution $x^{v(.)}(s)$, which is a continuous process adapted to the filtration generated by the Wiener process. To simplify notation, we write $x(s) = x^{v(.)}(s)$ and $v(s) = v(x^{v(.)}(s), s)$.

The payoff to maximize is given by

$$J(v(.)) = \int_0^T e^{-rs}\, Ef(x(s), v(s))ds + e^{-rT}\, (Eh(x(T)) + F(Ex(T))). \tag{2}$$

Because of the last term in the pay-off, this problem is not a classical stochastic control problem. The objective is to extend Bellman equation (Dynamic Programming) of stochastic control to this situation and to obtain a verification theorem, for a specific feedback to be optimal.

3. Sufficient condition of optimality

3.1. Notation

Define $a(x,v) = \frac{1}{2}\sigma(x,v)\sigma^*(x,v)$. We introduce the Lagrangian

$$L(x,q,M,v) = f(x,v) + q.g(x,v) + \mathrm{tr}(a(x,v)M), \tag{3}$$

where $q \in R^n,\ M \in \mathcal{L}(R^n;R^n)$. We suppose that there exists a measurable function $\hat{v}(x,q,M)$, which attains the maximum in v of the Lagrangian. We then introduce the functions

$$\begin{aligned} H(x,q,M) &= L(x,q,M,\hat{v}(x,q,M)),\\ G(x,q,M) &= g(x,\hat{v}(x,q,M)),\\ P(x,q,M) &= a(x,\hat{v}(x,q,M)). \end{aligned}$$

3.2. System of optimality

Let $\rho\in R^n$. We solve, for ρ fixed, the PDE (partial differential equation)

$$\begin{aligned} -\frac{\partial}{\partial t}u_\rho + ru_\rho &= H(x,Du_\rho,D^2u_\rho),\\ u_\rho(x,T) &= h(x)+F(\rho)+DF(\rho).(x-\rho), \end{aligned} \tag{4}$$

and we assume that we can solve this PDE for any fixed ρ. We supposethat the solution, denoted u_ρ, uniquely defined, is C^1, with second order derivative existing a.e., and growth conditions in x same as the highest growth conditions for $f(x,v)$ and $h(x)$. For each component x_i of x, we then consider the linear second order PDE, whose solution is denoted by $\Psi_{\rho,i}(x,t)$

$$\begin{aligned} &-\frac{\partial}{\partial t}\Psi_{\rho,i} = D\Psi_{\rho,i}.G(x,Du_\rho,D^2u_\rho) + \operatorname{tr}\left(P(x,Du_\rho,D^2u_\rho)D^2\Psi_{\rho,i}\right),\\ &\Psi_{\rho,i}(x,T) = x_i. \end{aligned} \tag{5}$$

We denote by $\Psi_\rho(x,t)$ the vector in R^n, whose components are $\Psi_{\rho,i}(x,t)$. We then consider the fixed point equation

$$\rho = \Psi_\rho(x_0,0), \tag{6}$$

and we suppose that this fixed point equation has a unique solution. We still denote it by ρ to save notation the unique solution of the fixed point equation (6). We next define

$$\hat{v}_\rho(x,t) = \hat{v}(x,Du_\rho,D^2u_\rho), \tag{7}$$

which is the candidate for optimal feedback. We state the

Theorem 3.1. *We assume all the steps described above, leading to the definition of $\hat{v}_\rho(x,t)$. Then the feedback $\hat{v}_\rho(x,t)$ is optimal and the optimal cost is*

$$J(\hat{v}_\rho(.)) = u_\rho(x_0,0) = \sup_{v(.)} J(v(.)). \tag{8}$$

The result (8) will be a consequence of a more general result below, Theorem 5.2 (verification theorem).

4. More general theory

4.1. General comments

The result (8) will be a particular case of a more general theory. We first notice that, when $F = 0$, the problem becomes a standard stochastic control problem. Then $u_\rho(x,t) = u(x,t)$, solution of Bellman equation of stochastic control

$$(9) \qquad \begin{aligned} -\frac{\partial}{\partial t}u + ru &= H(x, Du, D^2u), \\ u(x,T) &= h(x), \end{aligned}$$

and $\hat{v}_\rho(x,t) = \hat{v}(x,t)$, the classical optimal feedback of stochastic control. Moreover

$$(10) \qquad u(x_0, 0) = \sup_{v(.)} J(v(.)).$$

So, the system (5), (6) and the fixed point problem (7) are a generalization of the standard Dynamic Programming argument.

4.2. Invariant embedding in stochastic control

A key element of Dynamic Programming is invariant embedding. One embeds the original control problem, in a family of control problems, indexed by the initial conditions (x,t) instead of $(x_0, 0)$. We then have

$$(11) \qquad u(x,t) = \sup_{v(.)} J_{xt}(v(.)),$$

where

$$(12) \qquad J_{xt}(v(.)) = \int_t^T e^{-r(s-t)}\, Ef(x(s), v(s))ds + e^{-r(T-t)}\, Eh(x(T)),$$

and the dynamic system $x(s)$ evolves as follows

$$(13) \qquad \begin{aligned} dx &= g(x, v(x,s))ds + \sigma(x, v(x,s))dw(s),\ s > t, \\ x(t) &= x. \end{aligned}$$

In standard stochastic control, the function $u(x,t)$ is called the value function. It is the solution of Bellman equation (9).

4.3. Invariant embedding in mean field type control

For mean field type control, the situation is more complicated. We do not get invariant embedding by replacing $(x_0, 0)$ by (x,t). We need to consider a pair (m,t), in which m is a probability measure on R^n. By writing $(x_0,0) = (\delta_{x_0}, 0)$, (m,t) is a generalization of $(x_0, 0)$. We need to take the initial value of the dynamic system (13) at time t to be random. We state the problem as follows

$$\begin{aligned} dx &= g(x, v(x,s))ds + \sigma(x, v(x,s))dw(s), \ s > t, \\ x(t) &= \xi, \end{aligned} \tag{14}$$

where ξ is a random variable in R^n, independent of the σ-algebra $\mathcal{W}_t = \sigma(w(s) - w(t), \forall s > t)$, whose probability distribution is m. The pay-off is

$$\begin{aligned} J_{m,t}(v(.)) = & \int_t^T e^{-r(s-t)}\, Ef(x(s), v(s))ds \\ & + e^{-r(T-t)}\, (Eh(x(T)) + F(Ex(T))). \end{aligned} \tag{15}$$

The value function is then

$$V(m,t) = \sup_{v(.)} J_{m,t}(v(.)). \tag{16}$$

4.4. Reformulation of the mean field type control problem

We have written $J_{m,t}(v(.))$ instead of $J_{\xi,t}(v(.))$ because the right hand side of (15) depends on the initial condition ξ only through its probability distribution m. This is a very important property of diffusion processes. To simplify a little, but it is not at all necessary, we assume that the probability distribution m on R^n has a density with respect to the Lebesgue measure on R^n, which we call $m(x)$ to save notation. Then the probability distribution of the random variable $x(s)$ solution of the SDE (14) has also a density with respect to the Lebesgue measure on R^n, which we denote by $m(x,s)$, solution of the Fokker-Planck equation

$$\begin{aligned} & \frac{\partial m}{\partial s} - \sum_{ij=1}^n \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}(x, v(x,s))m) + \text{div}\,(g(x, v(x,s))m) = 0, \ s > t, \\ & m(x,t) = m(x). \end{aligned} \tag{17}$$

To emphasize the dependence with respect to the feedback control $v(.)$ and the initial conditions m, t, we shall write, when useful $m_{mt}^{v(.)}(x,s)$, or $m^{v(.)}(x,s)$. The right hand side of (15) becomes

$$J_{m,t}(v(.)) = \int_t^T e^{-r(s-t)} \int_{R^n} f(x,v(x,s)) m_{mt}^{v(.)}(x,s)dxds$$
$$+ e^{-r(T-t)} \left(\int_{R^n} h(x) m_{mt}^{v(.)}(x,T)dx + F(\int_{R^n} x m_{mt}^{v(.)}(x,T)dx) \right). \tag{18}$$

The interesting aspect of the writing (17), (18) is that the stochastic control problem (14), (15) has been transformed into a deterministic problem for a dynamic system, whose state at time s is the density $m(x,s)$, whose evolution is governed by a PDE, the Fokker-Planck equation (17). It is then clear that the pay-off functional (18) depends on the random variable ξ only through the initial density m. The value function is still defined by (16) with this new interpretation.

5. Bellman equation for the mean field type control problem

5.1. Prelimininaries

We introduce for each component x_i of x the function $\Psi_i^{v(.)}(x,s)$ solution of

$$-\frac{\partial \Psi_i^{v(.)}}{\partial s} = D\Psi_i^{v(.)}.g(x,v(x,s)) + \mathrm{tr}(a^{v(.)}(x,s)D^2\Psi_i^{v(.)}), \tag{19}$$
$$\Psi_i^{v(.)}(x,T) = x_i.$$

We call $\Psi^{v(.)}(x,s)$ the vector of components $\Psi_i^{v(.)}(x,s)$. We next define

$$\rho_{mt}^{v(.)} = \int_{R^n} \Psi^{v(.)}(x,t)m(x)dx. \tag{20}$$

We state the

Lemma 5.1. *We have the formula*

$$\int_{R^n} x m^{v(.)}(x,T)dx = \rho_{mt}^{v(.)}. \tag{21}$$

Proof. The statement (21) follows from the property

$$\frac{d}{ds}\int_{R^n} m^{v(.)}(x,s)\Psi^{v(.)}(x,s)dx = 0,$$

which is a consequence of the two PDEs (17) and (19). □

5.2. Formula for $J_{m,t}(v(.))$

We next introduce the linear PDE, whose solution is denoted by $u_{mt}^{v(.)}(x,s)$

$$\begin{aligned} -\frac{\partial u_{mt}^{v(.)}}{\partial s} + ru_{mt}^{v(.)} - Du_{mt}^{v(.)}.g(x,v(x,s)) - \mathrm{tr}(a(x,v(x,s))D^2u_{mt}^{v(.)}) \\ = f(x,v(x,s)), \\ u_{mt}^{v(.)}(x,T) = h(x) + F(\rho_{mt}^{v(.)}) + DF(\rho_{mt}^{v(.)}).(x-\rho_{mt}^{v(.)}). \end{aligned} \tag{22}$$

We state

Proposition 5.1. *We have the formula*

$$J_{m,t}(v(.)) = \int_{R^n} u_{mt}^{v(.)}(x,t)m(x)dx. \tag{23}$$

Proof. We compute

$$\frac{d}{ds}[e^{-r(s-t)}\int_{R^n} u_{mt}^{v(.)}(x,s)m_{mt}^{v(.)}(x.s)dx],$$

and comparing the PDEs (17) and (22) we obtain, like in Lemma 5.1

$$\begin{aligned} &\frac{d}{ds}[e^{-r(s-t)}\int_{R^n} u_{mt}^{v(.)}(x,s)m_{mt}^{v(.)}(x.s)dx] \\ &= -e^{-r(s-t)}\int_{R^n} m_{mt}^{v(.)}(x.s)f(x,v(x,s))dx. \end{aligned} \tag{24}$$

Also, from (21) we state, thanks to Lemma 5.1

$$\begin{aligned} \int_{R^n} u_{mt}^{v(.)}(x,T)m_{mt}^{v(.)}(x.T)dx &= \int_{R^n} h(x)m_{mt}^{v(.)}(x.T)dx + F(\rho_{mt}^{v(.)}) \\ &= \int_{R^n} h(x)m_{mt}^{v(.)}(x.T)dx + F(\int_{R^n} xm_{mt}^{v(.)}(x,T))dx. \end{aligned} \tag{25}$$

Integrating (24) for s between t and T, and comparing with (18) we obtain immediately (23). This completes the proof of the Proposition. □

5.3. System

We consider for ρ given, like in (4), (5), the PDEs

$$-\frac{\partial}{\partial s}u_\rho + ru_\rho = H(x, Du_\rho, D^2u_\rho), \quad u_\rho(x,T) = h(x) + F(\rho) + DF(\rho).(x-\rho), \tag{26}$$

and

$$-\frac{\partial}{\partial s}\Psi_{\rho,i} = D\Psi_{\rho,i}.G(x, Du_\rho, D^2u_\rho) + \mathrm{tr}\left(P(x, Du_\rho, D^2u_\rho)D^2\Psi_{\rho,i}\right), \quad \Psi_{\rho,i}(x,T) = x_i. \tag{27}$$

We define ρ_{mt} by the fixed point equation

$$\rho_{mt} = \int_{R^n} \Psi_{\rho_{mt}}(x,t)m(x)dx, \tag{28}$$

and we assume that we can solve (26), (27) and the fixed point equation (28). We set

$$u_{mt}(x,s) = u_{\rho_{mt}}(x,s), \; \Psi_{mt}(x,s) = \Psi_{\rho_{mt}}(x,s). \tag{29}$$

We next define the feedback control

$$\hat{v}_{mt}(x,s) = \hat{v}(x, Du_{mt}(x,s), D^2u_{mt}(x,s)). \tag{30}$$

From (19), (20), (22) it is clear that

$$u_{mt}(x,s) = u_{mt}^{\hat{v}_{mt}(.)}(x,s), \tag{31}$$

and thus, from (23), we can assert that

$$J_{m,t}(\hat{v}_{mt}(.)) = \int_{R^n} u_{mt}(x,t)m(x)dx. \tag{32}$$

We define

$$V(m,t) = \int_{R^n} u_{mt}(x,t)m(x)dx. \tag{33}$$

We have used intentionally the notation $V(m,t)$ for the right hand side, which has been used in (16) to define the value function. We want to prove that the right hand side is indeed the value function and that $\hat{v}_{mt}(.)$ is the optimal feedback. However, for the time being, $V(m,t)$ denotes only the right hand side of (33).

5.4. First order condition

We want to prove the following first-order condition.

Proposition 5.2. *For any feedback $v(x,s)$ such that $\hat{v}_{mt}(x,s)+\epsilon v(x,s)$ is admissible, we have*

$$\frac{J_{m,t}(\hat{v}_{mt}(.)+\epsilon v(.))-J_{m,t}(\hat{v}_{mt}(.))}{\epsilon}\to 0,\ \text{as}\ \epsilon\to 0. \tag{34}$$

Proof. From formula (23), it is sufficient to check that

$$\lim_{\epsilon\to 0}\frac{\int_{R^n}u_{mt}^{\hat{v}_{mt}(.)+\epsilon v(.)}(x,t)m(x)dx-\int_{R^n}u_{mt}^{\hat{v}_{mt}(.)}(x,t)m(x)dx}{\epsilon}=0. \tag{35}$$

We denote

$$\begin{aligned}
\widetilde{u}_\epsilon(x,s)&=\frac{u_{mt}^{\hat{v}_{mt}(.)+\epsilon v(.)}(x,s)-u_{mt}^{\hat{v}_{mt}(.)}(x,s)}{\epsilon},\\
\widetilde{\Psi}_\epsilon(x,s)&=\frac{\Psi_{mt}^{\hat{v}_{mt}(.)+\epsilon v(.)}(x,s)-\Psi_{mt}^{\hat{v}_{mt}(.)}(x,s)}{\epsilon},\\
\widetilde{\rho}_\epsilon&=\frac{\rho_{mt}^{\hat{v}_{mt}(.)+\epsilon v(.)}-\rho_{mt}^{\hat{v}_{mt}(.)}}{\epsilon}.
\end{aligned}$$

After some technical steps, we can see that $\widetilde{u}_\epsilon(x,s)\to\widetilde{u}(x,s)$, $\widetilde{\Psi}_\epsilon(x,s)\to\widetilde{\Psi}(x,s)$, $\widetilde{\rho}_\epsilon\to\widetilde{\rho}$ with the relations (writing $\hat{v}$ for $\hat{v}_{mt}(x,s)$)

$$\begin{aligned}
-\frac{\partial\widetilde{u}}{\partial s}+r\widetilde{u}-D\widetilde{u}.g(x,\hat{v})-\operatorname{tr}(a(x,\hat{v})D^2\widetilde{u})&\\
=L_v(x,Du_{mt},D^2u_{mt},\hat{v})v(x.s)&,\\
\widetilde{u}(x,T)=D^2F(\rho_{mt})\widetilde{\rho}(x-\rho_{mt})&,
\end{aligned} \tag{36}$$

where $L_v(x,q,M,v)$ denotes the gradient in the argument v of the Lagrangian $L(x,q,M,v)$. Also

$$-\frac{\partial\widetilde{\Psi}}{\partial s}-D\widetilde{\Psi}.g(x,\hat{v})-\operatorname{tr}(a(x,\hat{v})D^2\widetilde{\Psi})$$
$$=\left(D\Psi_{mt}.g_v(x,\hat{v})+\operatorname{tr}(a_v(x,\hat{v})D^2\Psi_{mt})\right)v(x,s),$$
$$\widetilde{\Psi}(x,T)=0,$$
$$\widetilde{\rho}=\int_{R^n}\widetilde{\Psi}(x,t)m(x)dx.$$

From the definition of $\hat{v}_{mt}(x,s)$ the right hand side of (36) vanishes. Therefore

$$\frac{d}{ds}e^{-r(s-t)}\int_{R^n}\widetilde{u}(x,s)m^{\hat{v}(.)}(x.s)dx=0.$$

Moreover, from (21),

$$\int_{R^n}\widetilde{u}(x,T)m^{\hat{v}(.)}(x.T)dx=D^2F(\rho_{mt})\widetilde{\rho}(\int_{R^n}xm^{\hat{v}(.)}(x.T)dx-\rho_{mt})=0$$

therefore $\int_{R^n}\widetilde{u}(x,t)m(x)dx=0$, which is the assertion (35) and thus also (34). □

5.5. Derivatives of $V(m,t)$

We want to get a PDE for $V(m,t)$. This requires to define the derivative $\frac{\partial V}{\partial m}$, where the argument m is a probability measure. This is more complex than in the case of the usual gradient in R^n because the space of probability measures is infinite dimensional. Several concepts are possible. To keep things as simple as possible, we shall take the case of densities $m\equiv m(x)$ which are in the space $L^2(R^n)$. If $\Phi(m)$ is a functional on R^n, its Gâteaux derivative is also an element of $L^2(R^n)$, denoted $\frac{d\Phi(m)}{dm}(x)$, such that

$$\frac{\Phi(m+\epsilon m')-\Phi(m)}{\epsilon}\to\int_{R^n}\frac{d\Phi(m)}{dm}(x)m'(x)dx \quad \text{as } \epsilon\to 0,\ \forall m'\in L^2(R^n). \tag{37}$$

We start with the

Proposition 5.3. *V(m,t) has a Gâteaux derivative in m given by*

$$\frac{\partial V(m,t)}{\partial m}(x)=u_{mt}(x,t). \tag{38}$$

Proof. We recall that

$$V(m,t)=\int_{R^n}u_{mt}(\xi,t)m(\xi)d\xi \tag{39}$$

is a composed functional of m. To prove (38) we have to prove that

$$\int_{R^n}\frac{\partial}{\partial m}(u_{mt}(\xi,t))(x)m(\xi)d\xi=0. \tag{40}$$

We set

$$U_{mt}(\xi,x,t)=\frac{\partial}{\partial m}(u_{mt}(\xi,t))(x). \tag{41}$$

We recall that $u_{mt}(\xi,t)$ satisfies, see (26)

$$\begin{aligned}-\frac{\partial}{\partial s}u_{mt}+ru_{mt}&=H(\xi,Du_{mt},D^2u_{mt}),\\ u_{mt}(\xi,T)&=h(\xi)+F(\rho_{mt})+DF(\rho_{mt}).(\xi-\rho_{mt}).\end{aligned} \tag{42}$$

Differentiating the equation (42) in m and using the envelope theorem we obtain

$$\begin{aligned}-\frac{\partial}{\partial s}U_{mt}(\xi,x,s)+rU_{mt}(\xi,x,s)-D_\xi U_{mt}(\xi,x,s).g(\xi,\hat{v}_{mt}(\xi,s))\\ -\mathrm{tr}(a(\xi,\hat{v}_{mt}(\xi,s))D_\xi^2U_{mt}(\xi,x,s))=0,\\ U_{mt}(\xi,x,T)=D^2F(\rho_{mt})\frac{\partial\rho_{mt}(x)}{\partial m}(\xi-\rho_{mt}).\end{aligned} \tag{43}$$

Using (43) together with (17), we obtain

$$\frac{d}{ds}e^{-r(s-t)}\int_{R^n}U_{mt}(\xi,x,s)m^{\hat{v}_{mt}(.)}(\xi,s)d\xi=0, \tag{44}$$

and we have also

$$\begin{aligned}&\int_{R^n}U_{mt}(\xi,x,T)m^{\hat{v}_{mt}(.)}(\xi,T)d\xi\\ &=D^2F(\rho_{mt})\frac{\partial\rho_{mt}(x)}{\partial m}(\int_{R^n}\xi m^{\hat{v}_{mt}(.)}(\xi,T)-\rho_{mt})=0.\end{aligned} \tag{45}$$

Integrating (44) for s between t and T and using (45), we obtain

$$\int_{R^n} U_{mt}(\xi, x, t) m(\xi) d\xi = 0, \tag{46}$$

which is (40) and thus completes the proof of (38). □

We turn now to the derivative in t of $V(m,t)$ denoted $\frac{\partial V}{\partial t}(m,t)$. We have the

Proposition 5.4. *The derivative in time is given*

$$\frac{\partial V}{\partial t}(m,t) = \int_{R^n} \frac{\partial u_{mt}}{\partial s}(x,t) m(x) dx. \tag{47}$$

Proof. We denote by $u'_{mt}(x,s)$ the derivative in t of $u_{mt}(x,s)$. To prove (47) amounts to proving that

$$\int_{R^n} u'_{mt}(x,t) m(x) dx = 0. \tag{48}$$

We proceed as in Proposition 5.3. We can differentiate (42) in t and obtain

$$\begin{aligned} -\frac{\partial}{\partial s} u'_{mt}(x,s) + r u'_{mt}(x,s) - D u'_{mt}(x,s).g(\xi, \hat{v}_{mt}(\xi,s)) \\ -\mathrm{tr}(a(\xi, \hat{v}_{mt}(\xi,s)) D^2 u'_{mt}(x,s)) = 0, \\ u'_{mt}(x,T)) = D^2 F(\rho_{mt}) \frac{\partial \rho_{mt}}{\partial t}(x - \rho_{mt}), \end{aligned} \tag{49}$$

and (48) follows like in Proposition 5.3. □

5.6. Bellman equation

We can now write the Bellman equation satisfied by $V(m,t)$. We want to prove the

Theorem 5.1. *The functional $V(m,t)$ satisfies the equation*

$$\begin{aligned} -\frac{\partial V(m,t)}{\partial t} + rV(m,t) \\ = \int_{r^n} H(x, D\frac{\partial V(m,t)}{\partial m}(x), D^2 \frac{\partial V(m,t)}{\partial m}(x)) m(x) dx, \\ V(m,T) = \int_{R^n} h(x) m(x) dx + F(\int_{R^n} x m(x) dx). \end{aligned} \tag{50}$$

Proof. We recall (42), written at $s = t$. We obtain, by multiplying by m and integration

$$-\int_{R^n} \frac{\partial}{\partial s} u_{mt}(x,t)m(x)dx + r\int_{R^n} u_{mt}(x,t)m(x)dx$$
$$= \int_{R^n} H(x, Du_{mt}(x,t), D^2u_{mt}(x,t))m(x)dx.$$

But, using (47), (39), (38) we can interpret this equation, as equation (50). On the other hand, we have, from (28),

$$\rho_{mT} = \int_{R^n} xm(x)dx;$$

hence

$$V(m,T) = \int_{R^n} u_{mT}(x,T)m(x)dx = \int_{R^n} h(x)m(x)dx + F(\rho_{mT}),$$

and thus the final condition (50) is also satisfied. This completes the proof. □

5.7. Verification theorem

We now claim

Theorem 5.2. *The functional $V(m,t)$ defined by (33) is the value function (16) and $\hat{v}_{mt}(x,s)$ is the optimal feedback.*

Proof. Let $v(x,s)$ be an admissible feedback and let $m_{mt}^{v(.)}(x,s)$ be the solution of (17), which we denote also, to save notation $m^{v(.)}(x,s)$, or $m^{v(.)}(s)$ for the function $x \to m^{v(.)}(x,s)$. We can compute

$$\frac{d}{ds}(V(m^{v(.)}(s),s)e^{-r(s-t)}) = (\frac{\partial}{\partial s}V(m^{v(.)}(s),s) - rV(m^{v(.)}(s),s))e^{-r(s-t)})$$
$$+e^{-r(s-t)}\int_{R^n} \frac{\partial V}{\partial m}(m^{v(.)}(s),s)(x)$$
$$\left(\sum_{ij=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}(x,v(x,s))m^{v(.)}(x,s)) - \text{div } (g(x,v(x,s))m^{v(.)}(x,s))\right) dx.$$

After integration by part in the last integral, we get

$$\frac{d}{ds}(V(m^{v(.)}(s),s)e^{-r(s-t)}) = (\frac{\partial}{\partial s}V(m^{v(.)}(s),s) - rV(m^{v(.)}(s),s))e^{-r(s-t)})$$
$$+e^{-r(s-t)}\int_{R^n}\Big(\mathrm{tr}(a(x,v(x,s))D^2\frac{\partial V}{\partial m}(m^{v(.)}(s),s)(x))$$
$$+ D\frac{\partial V}{\partial m}(m^{v(.)}(s),s)(x).g(x,v(x,s))\Big)m^{v(.)}(x,s)dx,$$

and from Bellman equation applied at the arguments $m^{v(.)}(s)$ and s, we obtain immediately

$$\frac{d}{ds}(V(m^{v(.)}(s),s)e^{-r(s-t)}) \le -e^{-r(s-t)}\int_{R^n} f(x,v(x,s))m^{v(.)}(x,s)dx. \tag{51}$$

Integrating between t and T, and using the final condition (50) we get immediately

$$V(m,t) \ge \int_t^T e^{-r(s-t)}\int_{R^n} f(x,v(x,s))m_{mt}^{v(.)}(x,s)dxds$$
$$+ e^{-r(T-t)}\Big(\int_{R^n} h(x)m_{mt}^{v(.)}(x,T)dx + F(\int_{R^n} xm_{mt}^{v(.)}(x,T)dx)\Big)$$
$$= J_{mt}(v(.)),$$

and, since this inequality holds for any $v(.)$, we get also

$$V(m,t) \ge \sup_{v(.)} J_{mt}(v(.)). \tag{52}$$

So $V(m,t)$ is larger than the value function. On the other hand, from (32), (33), $V(m,t) = J_{mt}(\hat{v}_{mt}(.))$. Therefore, $V(m,t)$ is equal to the value function and $\hat{v}_{mt}(.)$ is optimal. This concludes the proof. □

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