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Heap and ternary self-distributive cohomology

Mohamed Elhamdadi, Masahico Saito, and Emanuele Zappala

Department of Mathematics, University of South Florida, Tampa, Florida, USA

ABSTRACT

Heaps are algebraic structures endowed with para-associative ternary operations, bijectively exemplified by groups via the operation $(x, y, z) \mapsto xy^{-1}z$. They are also ternary self-distributive and have a diagrammatic interpretation in terms of framed links. Motivated by these properties, we define para-associative and heap cohomology theories and also a ternary self-distributive cohomology theory with abelian heap coefficients. We show that one of the heap cohomologies is related to group cohomology via a long exact sequence. Moreover, we construct maps between second cohomology groups of normalized group cohomology and heap cohomology, and show that the latter injects into the ternary self-distributive second cohomology group. We proceed to study heap objects in symmetric monoidal categories providing a characterization of pointed heaps as involutory Hopf monoids in the given category. Finally, we prove that heap objects are also “categorically” self-distributive in an appropriate sense.

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1. Introduction

Self-distributive magmas, called shelves, and their cohomology theories have been extensively studied in recent decades with applications to constructing invariants of classical knots and knotted surfaces [6–8]. Ternary self-distributivity (TSD) and its cohomology has been studied, for example, in [9, 11], and shown in [11] to have a diagrammatic interpretation in terms of framed links. Constructions of ternary self-distributive operations from binary ones are also given in [11], and it was shown that the (co)homology of ternary operations thus obtained and the (co)homology of the binary operations used for this construction are related through certain (co)chain maps.

A heap is an abstraction of a group endowed with the ternary operation $a \times b \times c \mapsto ab^{-1}c$, that allows to “forget” which element of the group is the unit. In fact, the operation just described extends to a functor that determines an equivalence between the category of pointed (i.e. an element is specified) heaps and the category of groups. More specific definitions will be given in Section 2. Heaps have been studied in algebra and algebraic geometry under the name of torsors. They appeared in knot theory in relation to region colorings as well (see, for example, [18, 19]). It was pointed out in [9] that heaps are ternary self-distributive, and consequently, the aforementioned diagrammatic interpretation of ternary self-distributive operations specializes to the case of heaps, as depicted in Figure 1 (cf. [11]).

Each of parallel arcs is colored by a group heap element. In the top left of the figure, a doubled crossing is colored by (x, x') and (y, z) at the top arcs. The left arc below the crossing traced from the arc colored x is colored by $xy^{-1}z$, the value of heap operation. The other arc is

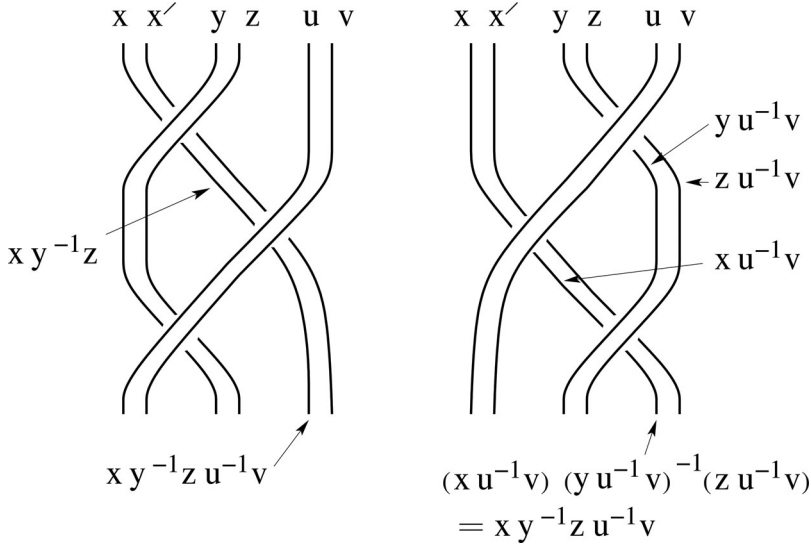


Figure 1. Diagrammatic interpretation of ternary self-distributivity of heaps as Reidemeister type III move for framed links.

similarly colored by $x'y^{-1}z$. At the bottom right of x -colored string, the left- and right-hand side of the Reidemeister type III move in Figure 1, the implications of the coloring condition are computed. The output corresponds to the TSD, as expected, for the heap operation.

In the present article, we introduce and develop a cohomology theory for heaps, a homology theory for TSD with heap coefficients, and present relations between them.

Specifically, the para-associativity comes in three types which we call type 0, 1, and 2, where type 0 takes the familiar form $[[x, y, z], u, v] = [x, y, [z, u, v]]$. In this case, we define a chain complex in a similar manner to group homology, and establish a long exact sequence relating the two homology theories. For the other types, however, such an analogue in general dimensions is elusive. Thus, we take an approach of defining low dimensional cochain maps from point of view of extensions, and with the goal of applying them to the TSD cohomology. In particular, the second cohomology group classifies isomorphism classes of heap extensions. We also provide relations between 2-cocycles for groups and para-associative (PA) 2-cocycles of types 1 and 2. Our motivation is to use this relation to construct TSD cocycles from group cocycles.

The main results of the article are introducing and studying a TSD homology theory with abelian group heap coefficients. This differs from [9, 11] in that the heap structure of the coefficient is essentially used in the definition of the chain complex. The definition again provides the classification of extensions with heap coefficients by the second TSD cohomology group. We then present an injective map from heap to TSD cohomology groups $H_H^2(X, A) \rightarrow H_{SD}^2(X, A)$ in dimension 2. Non-trivial examples are provided throughout.

Binary self-distributive operations have been studied in relation to the Yang-Baxter operators through tensor categories (e.g. [5]). In [11], a diagrammatic interpretation of TSD was given in terms of framed links, providing set-theoretic Yang-Baxter operations. It is, then, a natural question whether the constructions of TSD operations from heaps generalize to monoidal categories. For this goal, we introduce category versions of heaps and TSD operations, and prove that a heap object in symmetric monoidal category is also a TSD object.

The article is organized as follows. After a review of basic materials of heaps in Section 2, a cohomology theory and extensions by 2-cocycles are presented in Section 3. Using the bijection between pointed heaps and groups, constructions of heap 2-cocycles from group 2-cocycles, and vice versa, are discussed in Section 4. A cohomology theory of TSD operations with abelian group heap coefficients is introduced in Section 5, and the extension theory is built from 2-cocycles. A

construction of TSD 2-cocycles from heap 2-cocycles is given in [Section 6](#), and internalization in the symmetric monoidal category is discussed in [Section 7](#).

2. Basic review of heaps

In this section, we recall the definitions of PA structure on a set and introduce the nomenclature that will be followed throughout the rest of the article. Given a set with a ternary operation $[-]$, we call the equalities

$$[[x_1, x_2, x_3], x_4, x_5] = [x_1, x_2, [x_3, x_4, x_5]]$$

$$[[x_1, x_2, x_3], x_4, x_5] = [x_1, [x_4, x_3, x_2], x_5]$$

$$[x_1, x_2, [x_3, x_4, x_5]] = [x_1, [x_4, x_3, x_2], x_5]$$

the type 0, 1, 2 para-associativity (or simply para-associativity), respectively. Observe that any pair of equalities of type 0, 1, or 2 implies that the remaining one also holds. If a ternary operation satisfies all types of para-associativity, then it is called PA. We call the condition $[x, x, y] = y$ and $[x, y, y] = x$ the degeneracy (conditions).

Definition 2.1. A *semi-heap* is a non-empty set with a ternary operation satisfying para-associativity [10]. A *heap* is a semi-heap whose operation further satisfies the degeneracy conditions.

We mention that a structure satisfying the para-associativity conditions only, is called *semi-heap* in Chapter 8 of [15].

A typical example of a heap is a group G where the ternary operation is given by $[x, y, z] = xy^{-1}z$, which we call a *group heap*. If G is abelian, we call it an *abelian (group) heap*. Conversely, given a heap X with a fixed element e , one defines a binary operation on X by $x \cdot y = [x, e, y]$ which turns $(X, \cdot)$ into a group with e as the identity, and the inverse of x is $[e, x, e]$ for any $x \in X$.

Once this correspondence from a heap to a group is established, quandle structures on groups can be defined on heaps. For instance, *conjugation* and *core* quandles can be constructed via $x * y = y^{-1} \cdot x \cdot y$ and $x * y = y \cdot x^{-1} \cdot y$, respectively [13]. Relations between the heap structure and these quandle structures, along with their cohomology theories are of interest as well.

We refer the reader to the classical reference [2], chapter IV, which contains a short historical background and a description in terms of universal algebra. In [22], a quantum version of heap was introduced and it has been shown, in analogy to the “classical” case, that the category of quantum heaps is equivalent to the category of pointed Hopf algebras. Further developments of the thematics introduced in [22] can also be found in [16, 21]. Other sources include [3, 17]. We observe that the definition of quantum heap given in [22] is in some sense dual to the notion of heap object in a symmetric monoidal category, that we introduce in [Section 7](#). Our heap objects in symmetric monoidal categories are much in the same spirit as in the definition of non-commutative torsor treated in [3].

3. Heap cohomology

In this section, we introduce a heap cohomology. Let $(X, [-])$ be a semi-heap and A be an abelian group. The n -dimensional *cochain group* $C_{\text{PA}}^n(X, A)$, is the group of functions $\{f : X^{2n-1} \rightarrow A\}$ for $n = 1, 2$.

Definition 3.1. Let $(X, [-])$ be a semi-heap and A be an abelian group. Then the 1-dimensional coboundary map $\delta^1 : C_{\text{PA}}^1(X, A) \rightarrow C_{\text{PA}}^2(X, A)$ is defined for $f \in C_{\text{PA}}^1(X, A)$ by

$$\delta^1 f(x, y, z) = f([x, y, z]) - f(x) + f(y) - f(z).$$

The kernel $Z_{\text{PA}}^1(X, A)$ of δ^1 is called the 1-dimensional cocycle group. In this case, we define 1-dimensional cohomology group $H_{\text{PA}}^1(X, A)$ to be $Z_{\text{PA}}^1(X, A)$.

We observe that $f \in Z_{\text{PA}}^1(X, A)$ if and only if f is a PA homomorphism from X to A regarded as an abelian heap. We determine $Z_{\text{PA}}^1(X, A)$ for the following two examples.

Example 3.2. Consider $\mathbb{Z}_2$ with the abelian heap operation $[x, y, z] = x + y + z$. We compute the group $Z_{\text{PA}}^1(\mathbb{Z}_2, \mathbb{Z}_2)$. Given three variables x, y , and z , at least two of them need to coincide. Consider the case when $x = y$, the 1-cocycle condition becomes $f([x, x, z]) = f(z)$ which is satisfied. The other cases are analogous. It follows that $Z_{\text{PA}}^1(\mathbb{Z}_2, \mathbb{Z}_2) = C_{\text{PA}}^1(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. This operation is also ternary self-distributive, see Lemma 6.1.

Example 3.3. We proceed to compute $Z_{\text{PA}}^1(\mathbb{Z}_3, \mathbb{Z}_n)$, where $\mathbb{Z}_3$ is given the same ternary operation as before: $[x, y, z] = x + y + z$. Observe that this operation does not define a heap, but it is PA. Take $x = y = 1$ in the 1-cocycle condition. We obtain $f(z + 2) = f(z)$, which implies that f is the constant map. It follows that $Z_{\text{PA}}^1(\mathbb{Z}_3, \mathbb{Z}_n) \cong \mathbb{Z}_n$ for all n , including the coefficient group $A = \mathbb{Z}$.

Definition 3.4. Define $C_{\text{PA}(i)}^3(X, A)$ for $i = 0, 1, 2$ to be three isomorphic copies of the abelian group of functions $\{f : X^5 \rightarrow A\}$. The 2-dimensional coboundary map $\delta_{(i)}^2 : C_{\text{PA}}^2(X, A) \rightarrow C_{\text{PA}(i)}^3(X, A)$ of type $i = 0, 1, 2$, respectively, are defined by

$$\begin{aligned} \delta_{(0)}^2 \eta(x_1, x_2, x_3, x_4, x_5) &= \eta(x_1, x_2, x_3) + \eta([x_1, x_2, x_3], x_4, x_5) \\ &\quad - \eta(x_3, x_4, x_5) - \eta(x_1, x_2, [x_3, x_4, x_5]), \\ \delta_{(1)}^2 \eta(x_1, x_2, x_3, x_4, x_5) &= \eta(x_1, x_2, x_3) + \eta([x_1, x_2, x_3], x_4, x_5) \\ &\quad + \eta(x_4, x_3, x_2) - \eta(x_1, [x_4, x_3, x_2], x_5), \\ \delta_{(2)}^2 \eta(x_1, x_2, x_3, x_4, x_5) &= \eta(x_3, x_4, x_5) + \eta(x_1, x_2, [x_3, x_4, x_5]) \\ &\quad + \eta(x_4, x_3, x_2) - \eta(x_1, [x_4, x_3, x_2], x_5). \end{aligned}$$

Direct calculations give the following.

Lemma 3.5. If $(X, [-])$ is a semi-heap, then $\delta_{(i)}^2 \delta^1 = 0$ for $i = 0, 1, 2$.

Definition 3.6. Let $(X, [-])$ be a semi-heap and let A be an abelian group. Define $C_{\text{PA}}^3(X, A) = C_{\text{PA}(1)}^3(X, A) \oplus C_{\text{PA}(2)}^3(X, A)$. Then, $\delta^2 = \delta_{(1)}^2 \oplus \delta_{(2)}^2$ defines a homomorphism $C_{\text{PA}}^2(X, A) \rightarrow C_{\text{PA}}^3(X, A)$. Define the group of 2-cocycles $Z_{\text{PA}}^2(X, A)$ by $\ker(\delta^2)$. Define the second coboundary group $B_{\text{PA}}^2(X, A)$ by $\text{im}(\delta^1)$. Then the second cohomology group is defined as usual: $H_{\text{PA}}^2(X, A) = Z_{\text{PA}}^2(X, A)/B_{\text{PA}}^2(X, A)$.

Definition 3.7. Let $(X, [-])$ be a semi-heap and let A be an abelian group. A 2-cocycle $\eta \in Z_{\text{PA}}^2(X, A)$ is said to satisfy the *degeneracy condition* if the following holds for all $x, y \in X$: $\eta(x, x, y) = 0 = \eta(x, y, y)$.

We observe that 2-coboundaries $\delta^1 f$ satisfy the degeneracy condition.

Definition 3.8. Let X be a heap and A be an abelian group. The second heap cocycle group $Z_{\text{H}}^2(X, A)$ is defined as the subgroup of $Z_{\text{PA}}^2(X, A)$ consisting of 2-cocycles that satisfy the degeneracy conditions. The second heap cohomology group $H_{\text{H}}^2(X, A)$ is defined as the quotient $Z_{\text{H}}^2(X, A)/B_{\text{PA}}^2(X, A)$.

Example 3.9. Let $X = \mathbb{Z}_2$ with group heap operation and $A = \mathbb{Z}_2$. Computations show that $\eta \in Z_{\text{PA}}^2(X, A)$ if and only if η satisfies the following set of equations:

$$\begin{aligned}\eta(0, 0, 0) &= \eta(0, 0, 1) = \eta(1, 0, 0), \\ \eta(1, 1, 1) &= \eta(1, 1, 0) = \eta(0, 1, 1), \\ \eta(0, 0, 0) + \eta(1, 1, 1) + \eta(0, 1, 0) + \eta(1, 0, 1) &= 0.\end{aligned}$$

Express $\eta = \sum \eta(x, y, z) \chi_{(x, y, z)}$ by characteristic functions $\chi_{(x, y, z)}$. By setting $\eta(0, 0, 0) = a$, $\eta(1, 1, 1) = b$ and $\eta(0, 1, 0) = c$, the last equation above implies $\eta(1, 0, 1) = -(a + b + c)$. Then, η is expressed as

$$\begin{aligned}\eta &= a(\chi_{(0, 0, 0)} + \chi_{(0, 0, 1)} + \chi_{(1, 0, 0)} - \chi_{(1, 0, 1)}) \\ &\quad + b(\chi_{(1, 1, 1)} + \chi_{(1, 1, 0)} + \chi_{(0, 1, 1)} - \chi_{(1, 0, 1)}) \\ &\quad + c(\chi_{(0, 1, 0)} - \chi_{(1, 0, 1)}).\end{aligned}$$

Since the group of coboundaries is zero from Example 3.2, it follows that $H_{\text{PA}}^2(\mathbb{Z}_2, \mathbb{Z}_2) = Z_{\text{PA}}^2(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since the degeneracy condition implies $a = b = 0$, we have $H_H^2(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$.

Definition 3.10. Let X be a heap, A be an abelian group and $\eta : X \times X \times X \rightarrow A$ be a 2-cochain. We define the heap extension of X by the 2-cochain η with coefficients in A , denoted $X \times_\eta A$, as the cartesian product $X \times A$ with ternary operation given by:

$$[(x, a), (y, b), (z, c)] = ([x, y, z], a - b + c + \eta(x, y, z)).$$

Although direct computation gives the following lemma, it is one of the motivations of the definition of the heap differential maps.

Lemma 3.11. *The abelian extension $X \times_\eta A$ of the heap X by a 2-cochain η satisfies para-associativity of type 1, 2, and degeneracy if and only if η is a heap 2-cocycle of type 1, 2, and with degeneracy condition, respectively. In particular, a 2-cochain η defines a heap extension if and only if it satisfies para-associativity of type 1, 2, and degeneracy.*

It is not clear at this time whether extensions above can be defined for non-abelian case, or in other generalized settings as in the quandle extensions.

Example 3.12. The following is a common construction applied to the heap. Let $0 \rightarrow A \xrightarrow{i} E \xrightarrow{\pi} G \rightarrow 0$ be a short exact sequence of abelian groups, and $s : G \rightarrow E$ be a set-theoretic section ($\pi s = \text{id}$). Since s is a section, we have that $s(x) - s(y) + s(z) - s([x, y, z])$ is in the kernel of π for all $x, y, z \in G$, so that there is $\eta : G \times G \times G \rightarrow A$ such that

$$\eta(x, y, z) = s(x) - s(y) + s(z) - s([x, y, z]).$$

Then, computations of the two 2-cocycle conditions and the degeneracy conditions give the following.

Lemma 3.13. $\eta \in Z_H^2(G, A)$.

Example 3.14. For a positive integer $n > 0$, let $0 \rightarrow \mathbb{Z}_n \xrightarrow{i} \mathbb{Z}_{n^2} \xrightarrow{\pi} \mathbb{Z}_n \rightarrow 0$ be as above, where $s(x) \bmod(n^2) = x$, representing elements of $\mathbb{Z}_m$ by $\{0, \dots, m-1\}$. Then for all $x, y, z \in G = \mathbb{Z}_n$, $\eta(x, y, z)$ is divisible by n in $E = \mathbb{Z}_{n^2}$, so that the value of η is computed by $\eta(x, y, z) = \eta(x, y, z)/n$. For example, for $n=3$, $\eta(2, 0, 2) = [s(2) - s(0) + s(2) - s([2, 0, 2])]/3 = 1 \in \mathbb{Z}_3$. We will show in Example 3.22 that $[\eta] \neq 0$ and therefore $H_H^2(\mathbb{Z}_3, \mathbb{Z}_3)$ is nontrivial.

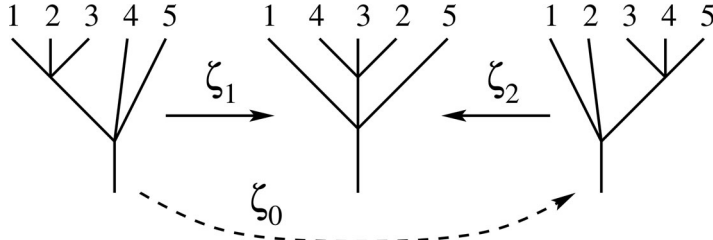


Figure 2. Heap 3-cocycle notations.

Definition 3.15. Let $X \times_{\eta} A$ and $X \times_{\eta'} A$ be two heap extensions with coefficients in an abelian group A , by two 2-cocycles η and η' of type 1, 2 and with degeneracy condition. We define a morphism of extensions, indicated by $\phi : X \times_{\eta} A \rightarrow X \times_{\eta'} A$, to be a morphism of heaps making the following diagram (of sets) commute.

$$\begin{array}{ccc}
 X \times A & \xrightarrow{\phi} & X \times A \\
 & \searrow & \swarrow \\
 & X &
 \end{array}$$

An invertible morphism of extensions is also called isomorphism of extensions, which induces an equivalence relation, and its equivalence classes are called isomorphism classes.

The following is one of the motivations and significance of our definition of the heap differential in dimension 2.

Proposition 3.16. *There is a bijective correspondence between isomorphism classes of heap extensions by A , and the second heap cohomology group $H_H^2(X; A)$.*

Proof. Standard arguments, similar to the group-theoretic case, give the result. $\square$

Lemma 3.17. *Let η_0 be a heap 2-cocycle of type 0 that satisfies the degeneracy condition. Then, the following equality holds*

$$\eta_0(x_1, x_2, x_3) + \eta_0([x_1, x_2, x_3], x_3, x_2) = 0.$$

Proof. The 2-cocycle condition applied to $\eta_0(x_1, x_2, x_3, x_3, x_2)$ becomes

$$\begin{aligned}
 \delta_{(0)}^2 \eta_0(x_1, x_2, x_3, x_3, x_2) &= \eta_0(x_1, x_2, x_3) + \eta_0([x_1, x_2, x_3], x_3, x_2) \\
 &\quad - \eta_0(x_3, x_3, x_2) - \eta_0(x_1, x_2, [x_3, x_3, x_2]).
 \end{aligned}$$

By applying the degeneracy condition and the degeneracy heap axiom, we obtain the result. $\square$

As we have mentioned, the heap operation has a diagrammatic representation by framed links as described in [11]. The TSD cohomology and its 2-cocycles in the form originally defined in [10, 14], and studied in [11], was used to construct a cocycle invariant in [23]. It is, however, not clear at this time whether 2-cocycles defined in this article using heap coefficients can be used for defining cocycle knot invariants, and it is a problem of interest.

Toward extending cohomology theory to general dimensions, we propose the following 3-differentials.

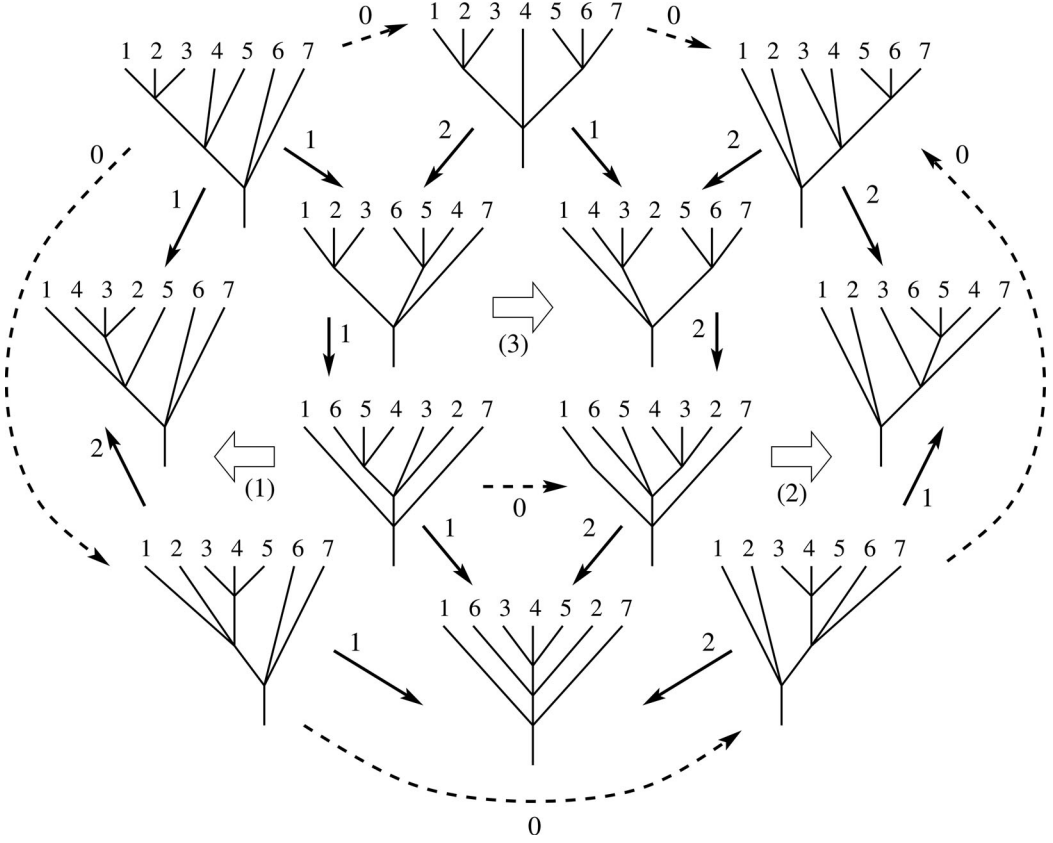


Figure 3. Heap 3-cocycle conditions.

Definition 3.18. Let X be a set with PA operation $[-]$ and let A be an abelian group. Let $C_{\text{PA}(i)}^4(X, A)$ be three isomorphic copies of the abelian group of functions $\{f : X^7 \rightarrow A\}$ for $i = 1, 2, 3$. Let $C_{\text{PA}}^4(X, A) = \oplus_{i=1,2,3} C_{\text{PA}(i)}^4(X, A)$. For $(\zeta_1, \zeta_2) \in C_{\text{PA}}^3(X, A) = C_{\text{PA}(1)}^3(X, A) \oplus C_{\text{PA}(2)}^3(X, A)$, define $\delta_{(i)}^3 : C_{\text{PA}}^3(X, A) \rightarrow C_{\text{PA}(i)}^4(X, A)$, for $i = 1, 2, 3$, as follows.

$$\begin{aligned}
 \delta_{(1)}^3(\zeta_1, \zeta_2)(x_1, x_2, x_3, x_4, x_5, x_6, x_7) &= \zeta_1([x_1, x_2, x_3], x_4, x_5, x_6, x_7) + \zeta_1(x_1, x_2, x_3, [x_6, x_5, x_4], x_7) \\
 &\quad - \zeta_1(x_6, x_5, x_4, x_3, x_2) - \zeta_1(x_1, x_2, x_3, x_4, x_5) \\
 &\quad + \zeta_2(x_1, x_2, x_3, x_4, x_5) - \zeta_1(x_1, x_2, [x_3, x_4, x_5], x_6, x_7), \\
 \delta_{(2)}^3(\zeta_1, \zeta_2)(x_1, x_2, x_3, x_4, x_5, x_6, x_7) &= \zeta_2(x_1, x_2, x_3, x_4, [x_5, x_6, x_7]) + \zeta_2(x_1, [x_4, x_3, x_2], x_5, x_6, x_7) \\
 &\quad - \zeta_2(x_6, x_5, x_4, x_3, x_2) - \zeta_2(x_3, x_4, x_5, x_6, x_7) \\
 &\quad + \zeta_1(x_3, x_4, x_5, x_6, x_7) - \zeta_2(x_1, x_2, [x_3, x_4, x_5], x_6, x_7), \\
 \delta_{(3)}^3(\zeta_1, \zeta_2)(x_1, x_2, x_3, x_4, x_5, x_6, x_7) &= \zeta_2([x_1, x_2, x_3], x_4, x_5, x_6, x_7) + \zeta_1(x_1, x_2, x_3, [x_6, x_5, x_4], x_7) \\
 &\quad + \zeta_1(x_6, x_5, x_4, x_3, x_2) - \zeta_1(x_1, x_2, x_3, x_4, [x_5, x_6, x_7]) \\
 &\quad - \zeta_2(x_1, [x_4, x_3, x_2], x_5, x_6, x_7) - \zeta_2(x_6, x_5, x_4, x_3, x_2).
 \end{aligned}$$

Then, define $\delta^3 := \oplus_{i=1,2,3} \delta_{(i)}^3 : C_{\text{PA}}^3(X, A) \rightarrow C_{\text{PA}}^4(X, A)$.

Let X be a heap, and let $x_i \in X$ for $i = 1, \dots, 5$. We utilize the following diagrammatic representations of heap 3-cocycles in [Theorem 3.19](#). In [Figure 2](#), 3-cocycles are associated to changes of diagrams. The three tree diagrams with top vertices labeled represent the elements in the equality

$$[[x_1, x_2, x_3], x_4, x_5] = [x_1, [x_4, x_3, x_2], x_5] = [x_1, x_2, [x_3, x_4, x_5]],$$

from left to right, respectively. The 3-cocycle $\zeta_1(x_1, x_2, x_3, x_4, x_5)$ (resp. $\zeta_2(x_1, x_2, x_3, x_4, x_5)$) is associated to the change from left to middle (resp. right to middle) tree diagrams as depicted by the solid arrows. The 3-cocycle $\zeta_0(x_1, x_2, x_3, x_4, x_5)$ is associated to the change from left to right, and depicted by the dotted arrow.

In [Figure 3](#), the 3-cocycle conditions are represented by diagrams with seven elements. In the figure, labeled arrows represent 3-cocycles as described above. In the middle, there is a hexagon formed by labeled arrows, and has double arrow labeled by (3). This hexagon represents the differential $\delta_{(3)}^3$. The definition of the differentials, as well as the proof of [Theorem 3.19](#), are aided by this figure.

Theorem 3.19. *The composition $\delta^3 \delta^2$ vanishes.*

Proof. This follows by proving, for $\eta \in C_{\text{PA}}^2(X, A)$ and $\zeta_i = \delta_{(i)}^2 \eta$ for $i = 1, 2$, that $\delta_{(j)}^3(\zeta_1, \zeta_2) = 0$ for $j = 1, 2, 3$. For $\delta_{(3)}^3(\zeta_1, \zeta_2) = 0$, first we compute positive terms:

$$\begin{aligned} & \zeta_2([x_1, x_2, x_3], x_4, x_5, x_6, x_7) \\ & + \zeta_1(x_1, x_2, x_3, [x_6, x_5, x_4], x_7) + \zeta_1(x_6, x_5, x_4, x_3, x_2) \\ & = \{ \eta(x_5, x_6, x_7) + \eta([x_1, x_2, x_3], x_4, [x_5, x_6, x_7]) \\ & \quad - \underline{\eta(x_6, x_5, x_4)} - \underline{\eta([x_1, x_2, x_3], [x_6, x_5, x_4], x_7)} \} \\ & + \{ \eta(x_1, x_2, x_3) + \underline{\eta([x_1, x_2, x_3], [x_6, x_5, x_4], x_7)} \\ & \quad - \underline{\eta([x_6, x_5, x_4], x_3, x_2)} - \eta(x_1, [[x_6, x_5, x_4], x_3, x_2], x_7) \} \\ & + \{ \underline{\eta(x_6, x_5, x_4)} + \underline{\eta([x_6, x_5, x_4], x_3, x_2)} \\ & \quad - \eta(x_3, x_4, x_5) - \eta(x_6, [x_3, x_4, x_5], x_2) \} \end{aligned}$$

where canceling terms are underlined. For the remaining terms, one computes

$$\begin{aligned} & \zeta_1(x_1, x_2, x_3, x_4, [x_5, x_6, x_7]) \\ & + \zeta_2(x_1, [x_4, x_3, x_2], x_5, x_6, x_7) + \zeta_2(x_6, x_5, x_4, x_3, x_2) \\ & = \{ \eta(x_1, x_2, x_3) + \eta([x_1, x_2, x_3], x_4, [x_5, x_6, x_7]) \\ & \quad - \underline{\eta(x_4, x_3, x_2)} - \underline{\eta(x_1, [x_4, x_3, x_2], [x_5, x_6, x_7])} \} \\ & + \{ \eta(x_5, x_6, x_7) + \underline{\eta(x_1, [x_4, x_3, x_2], [x_5, x_6, x_7])} \\ & \quad - \underline{\eta(x_6, x_5, [x_4, x_3, x_2])} - \eta(x_1, [x_6, x_5, [x_4, x_3, x_2]], x_7) \} \\ & + \{ \underline{\eta(x_4, x_3, x_2)} + \underline{\eta(x_6, x_5, [x_4, x_3, x_2])} \\ & \quad - \eta(x_3, x_4, x_5) - \eta(x_6, [x_3, x_4, x_5], x_2) \} \end{aligned}$$

and all terms cancel.

The conditions $\delta_{(1)}^3(\zeta_1, \zeta_2) = 0$ and $\delta_{(2)}^3(\zeta_1, \zeta_2) = 0$ follow similarly from direct computations. $\square$

For a type 0 condition, a chain complex is defined in a similar manner to the group homology as follows.

Definition 3.20. Let X be a set with a type 0 PA ternary operation $[-]$. The n th (type 0) PA chain group, denoted by $C_n^{\text{PA}}(X)$, is defined to be the free abelian group on tuples $(x_1, \dots, x_{2n-1}), x_i \in X$, and the boundary map $\partial_n^{(0)} : C_n^{\text{PA}}(X) \rightarrow C_{n-1}^{\text{PA}}(X)$ is defined by

$$\begin{aligned} \partial_n^{(0)}(x_1, \dots, x_{2n-1}) &= -(x_3, \dots, x_{2n-1}) \\ &\quad + \sum_{i=1}^n (-1)^{i+1} (x_1, \dots, x_{2i-2}, [x_{2i-1}, x_{2i}, x_{2i+1}], x_{2i+2}, \dots, x_{2n-1}) \\ &\quad + (-1)^{n+1} (x_1, \dots, x_{2n-3}) \end{aligned}$$

for $n \geq 2$ and $\partial_2^{(0)}(x_1, x_2, x_3) = ([x_1, x_2, x_3]) - (x_1) + (x_2) - (x_3)$.

It is straightforward to verify that the boundary maps defined above do indeed satisfy the differential condition and define therefore a chain complex. The dual cochain groups with coefficient group A and their dual differential maps coincide with those in Definitions 3.1 and 3.4 for (type 0) cochain maps.

Definition 3.21. The homology of the chain complex introduced in Definition 3.20 is called *type 0 PA homology*, and written $H_n^{(0)}(X)$.

We note that $\partial_2^{(0)}$ defined above is dual to δ^1 in Definition 3.1. Therefore, if ϕ is a 2-coboundary and α is a 2-cycle, then

$$\phi(\alpha) = \delta^1 f(\alpha) = f(\partial_2^{(0)} \alpha) = f(0) = 0.$$

Hence, the standard argument applies that if $\phi(\alpha) \neq 0$ for a 2-cycle α then ϕ is not null cohomologous.

Example 3.22. Consider $0 \rightarrow \mathbb{Z}_3 \xrightarrow{\iota} \mathbb{Z}_9 \xrightarrow{\pi} \mathbb{Z}_3 \rightarrow 0$ as in Example 3.14 and the corresponding η . The 2-chain $\alpha := (1, 0, 2) + (0, 1, 0) + (1, 2, 0)$ is easily seen to be a heap 2-cycle and $\eta(\alpha) = 1 \neq 0$. Hence, η is non-trivial. Therefore, $H_{\text{PA}}^2(\mathbb{Z}_3, \mathbb{Z}_3) \neq 0$.

4. From heap cocycles to group cocycles and back

The main purpose of this section is to elucidate connections between group (co)homology and heap (co)homology. For a group G , let denote the group chain complex by (G, ∂) . We consider the trivial action case.

4.1. Group homology and type 0 heap homology

In this section, we provide an explicit relation between type 0 heap homology and group homology.

Proposition 4.1. Let X be a heap, $e \in X$, and G be the associated group, so that $xy = [x, e, y]$ for all $x, y \in X$. Let $\Psi_n : C_n^G \rightarrow C_n^{(0)}(X)$ be the map on chain groups defined by

$$\Psi_n(x_1, \dots, x_n) = (x_1, e, x_2, e, \dots, e, x_n).$$

Then, $\Psi_\bullet$ is a chain map and therefore induces a well-defined map

$$\bar{\Psi}_n : H_n^G(X) \rightarrow H_n^{(0)}(X).$$

Proof. This is a direct computation, using the fact that $[x_{2i}, e, x_{2i+1}] = x_{2i} \cdot x_{2i+1}$ by definition. $\square$

Due to the complexity in computing the second (co)homology group of racks and quandles, there has been much attention devoted toward constructing chain maps from rack/quandle complexes to group homology complexes. Various approaches have been shown to be proficuous, as seen in [1, 12, 20]. [Proposition 4.1](#) gives a construction of this sort in the case of type 0 heap homology. It would be of interest to study a possible relation between this chain map and those given in [1, 12, 20] when considering quandle structures induced by the heap operation, as described in [Section 2](#).

Remark 4.2. By dualizing [Proposition 4.1](#), we obtain a cochain map between type 0 heap cohomology and group cohomology. In the specific case of the second cohomology group, we observe that [Proposition 4.1](#) corresponds to the construction of a group from a heap through extensions as follows. Let X be a heap, A an abelian group, $E = X \times A$ the heap extension defined in [Definition 3.10](#) with a heap 2-cocycle η . Let $(e, c) \in E$ be a fixed element. Then, the group structure on E defined from the heap structure on E is computed as

$$\begin{aligned} (x, a) \cdot (y, b) &= [(x, a), (e, c), (y, b)] \\ &= ([x, e, y], a - c + b + \eta(x, e, y)) \\ &= (xy, a + b + \theta(x, y)) \end{aligned}$$

giving rise to the relation $\theta(x, y) = \eta(x, e, y) - c$, a difference of a constant comparing to [Proposition 4.1](#).

Let X be a heap, and $e \in X$. We define chain subgroups $\hat{C}_n^{(0)}(X)$ to be the free abelian group generated by

$$\left\{ (x_1, \dots, x_{2n-1}) \in C_n^{(0)}(X) \mid x_{2i} = e, i = 1, 2, \dots, n-1 \right\},$$

and $\hat{C}_1^{(0)}(X) = C_1^{(0)}(X)$. It is checked by direct computation that $\partial_n^{(0)}(\hat{C}_n^{(0)}(X)) \subset \hat{C}_{n-1}^{(0)}(X)$, so that $\{\hat{C}_n^{(0)}(X), \partial_n^{(0)}\}$ forms a chain subcomplex. Let $\hat{H}_n^{(0)}(X)$ denote the homology groups of this subcomplex, and let $\tilde{H}_n^{(0)}(X)$ denote the relative homology groups for the quotient $C_n^{(0)}(X)/\hat{C}_n^{(0)}(X)$. We now have the following result.

Theorem 4.3. *In the same setting as in [Proposition 4.1](#), the map $\bar{\Psi}_n$ is an injection for all n . Furthermore, there is a long exact sequence of homology groups*

$$\dots \rightarrow H_n^G(X) \rightarrow H_n^{(0)}(X) \rightarrow \tilde{H}_n^{(0)}(X) \xrightarrow{\partial} H_{n-1}^G(X) \rightarrow \dots$$

Proof. The chain map Ψ_n gives an isomorphism between chain groups $C_n^G(X)$ and $\hat{C}_n^{(0)}(X)$ and commute with differentials, giving rise to an isomorphism of chain complexes. Through the map, $\bar{\Psi}_n, H_n^G(X)$ is identified with $\hat{H}_n^{(0)}(X)$.

The second statement follows from the short exact sequence of chain complexes

$$0 \rightarrow \hat{C}_\bullet^{(0)}(X) \rightarrow C_\bullet^{(0)}(X) \rightarrow C_\bullet^{(0)}(X)/\hat{C}_\bullet^{(0)}(X) \rightarrow 0$$

using the isomorphism $\Psi_\bullet$ and defining ∂ as the usual connecting homomorphism, via the Snake Lemma. □

Definition 4.4. The homology $\tilde{H}_\bullet^{(0)}(X)$ is called the type zero *essential* heap homology.

Remark 4.5. The essential homology of a group heap X is regarded as a measure of how far group homology is from being isomorphic to the type zero heap homology.

Example 4.6. We show that $\tilde{H}_\bullet^{(0)}(X)$ can be nontrivial. Consider the group heap corresponding to $\mathbb{Z}_2$. The 2-chain $(0, 1, 1)$ is easily seen to be a type zero 2-cycle. We show that the class $[(0, 1, 1)] \in \tilde{H}_2^{(0)}(X)$ is nontrivial. The 2-cochain $\eta(1, 1, 1) = \eta(1, 1, 0) = \eta(0, 1, 1) = 1$, and zero otherwise is a heap 2-cocycle, as seen in [Example 3.9](#). As previously observed, a heap 2-cocycle is also a type zero 2-cocycle. Furthermore, $\partial_1^{(0)}$ is dual to δ_{PA}^1 , so that η is nontrivial as a type zero heap cocycle. Suppose that $[(0, 1, 1)] = 0$ in $\tilde{H}_2^{(0)}(X)$. Then, there is a 3-chain α such that $\partial_3^{(0)}\alpha - (0, 1, 1) \in \hat{C}_2^{(0)}(X)$. Therefore, $\eta(\partial^{(0)}\alpha - (0, 1, 1)) = 0$, since by definition η vanishes on $\hat{C}_2^{(0)}(X)$. Since $\eta(\partial^{(0)}) = \delta^{(0)}\eta$ and η is a type zero 2-cocycle, we have obtained that $\eta(0, 1, 1) = 0$, in contradiction with the choice of η . Therefore, $[(0, 1, 1)]$ is nontrivial in $\tilde{H}_2^{(0)}(X)$.

4.2. From group cocycles to PA cocycles

In this section, we present a construction of PA 2-cocycles from group 2-cocycles. The following gives an answer to a natural question on how the relation between groups and heaps descends to relations in their homology theories. It also provides a construction of ternary self-distributive 2-cocycles from group 2-cocycles through heap 2-cocycles ([Section 6](#)). We recall that the group 2-cocycle condition [4] with trivial action on the coefficient group is written as

$$\theta(x, y) + \theta(xy, z) = \theta(y, z) + \theta(x, yz)$$

for all $x, y, z \in G$ of a group G . The *normalized* 2-cocycle satisfies $\theta(x, 1) = 0 = \theta(1, x)$, and it follows that normalized 2-cocycles satisfy $\theta(x, x^{-1}) = \theta(x^{-1}, x)$. Define the normalized 2-cochain group $\hat{C}_G^2(X)$ to consist of normalized 2-cochains, and the normalized 1-cochain group $\hat{C}_G^1(X)$ to consist of $f \in C_G^1(X)$ such that $f(1) = 0$. Then, these form a subcomplex up to dimension 2, and the corresponding 2-dimensional cohomology group is denoted by $\hat{H}_G^2(X)$.

Theorem 4.7. *Let G be a group, and X be its group heap. Let θ be a normalized group 2-cocycle with trivial action on the coefficient group A . Then,*

$$\eta(x, y, z) := \theta(x, y^{-1}) + \theta(xy^{-1}, z) - \theta(y, y^{-1})$$

is a PA 2-cocycle. This construction $\Phi_2(\theta) = \eta$ defines a cohomology map $\bar{\Phi}_2 : \hat{H}_G^2(X) \rightarrow H_{\text{PA}}^2(X)$.

Proof. First we note that for an extension group 2-cocycle θ , the condition $y^{-1}(zu^{-1}) = ((uz^{-1})y)^{-1}$ implies the following identity

$$\begin{aligned} & \theta(z, u^{-1}) + \theta(y^{-1}, zu^{-1}) - \theta(y, y^{-1}) - \theta(u, u^{-1}) \\ &= \theta(u, z^{-1}) + \theta(z, z^{-1}) + \theta(u^{-1}, uz^{-1}) - \theta(y, y^{-1}) \\ &= \theta(u, z^{-1}) + \theta(uz^{-1}, y) - \theta(z, z^{-1}) - \theta(uz^{-1}y, y^{-1}zu^{-1}), \end{aligned}$$

which we call the *product-inversion* relation. Observe that the normalization condition has been implicitly used to rewrite the term corresponding to $\theta(y^{-1}, y)$. For $\delta_{(1)}^2(\eta) = 0$, one computes

$$\begin{aligned}
& \eta(x, y, z) + \eta([x, y, z], u, v) \\
&= \underline{\theta(x, y^{-1})} + \underline{\theta(xy^{-1}, z)} - \theta(y, y^{-1}) \\
&\quad + \theta(xy^{-1}z, u^{-1}) + \theta(xy^{-1}zu^{-1}, v) - \theta(u, u^{-1}) \\
&= \theta(y^{-1}, z) + \underline{\theta(x, y^{-1}z)} - \theta(y, y^{-1}) \\
&\quad + \underline{\theta(xy^{-1}z, u^{-1})} - \theta(u, u^{-1}) + \theta(xy^{-1}zu^{-1}, v) \\
&= \underline{\theta(y^{-1}, z)} + \underline{\theta(y^{-1}z, u^{-1})} + \theta(x, y^{-1}zu^{-1}) \\
&\quad - \theta(y, y^{-1}) - \theta(u, u^{-1}) + \theta(xy^{-1}zu^{-1}, v) \\
&= \theta(z, u^{-1}) + \theta(y^{-1}, zu^{-1}) + \theta(x, y^{-1}zu^{-1}) \\
&\quad + \theta(xy^{-1}zu^{-1}, v) - \theta(y, y^{-1}) - \theta(u, u^{-1}) \\
&= \theta(u, z^{-1}) + \theta(uz^{-1}, y) - \theta(z, z^{-1}) \\
&\quad - \theta(uz^{-1}y, y^{-1}zu^{-1}) + \theta(x, y^{-1}zu^{-1}) + \theta(xy^{-1}zu^{-1}, v) \\
&= \eta(u, z, y) + \eta(x, [u, z, y], v),
\end{aligned}$$

where we have underlined the terms undergoing the group 2-cocycle relation at each step, and used the product-inverse relation in the penultimate equality. Similar computations show $\delta_{(2)}^2(\eta) = 0$.

To complete the proof, consider the maps $\Phi_1 := -\mathbb{1} : \hat{C}_G^1(X) \rightarrow C_{PA}^1(X)$, and $\Phi_2 : \hat{C}_G^2(X) \rightarrow C_{PA}^2(X)$, $\theta \mapsto \eta$, as in the previous part of the proof. It is easy to see that $\delta_G^1\Phi_2 = \Phi_1\delta_{PA}^1$, therefore showing that Φ is well defined on cohomology groups. $\square$

Remark 4.8. Extensions of groups and heaps, in this case, are related as in [Remark 4.2](#). The group extension is defined, for a group G and the coefficient abelian group A , by

$$(x, a) \cdot (y, b) = (xy, a + b + \theta(x, y))$$

for $x, y \in G$ and $a, b \in A$. For the heap $E = G \times A$ constructed from the group $E = G \times A$ defined above, one computes

$$\begin{aligned}
& [(x, a), (y, b), (z, c)] \\
&= (x, a)(y, b)^{-1}(z, c) \\
&= (x, a)(y^{-1}, -b - \theta(y, y^{-1}))(z, c) \\
&= (xy^{-1}z, a - b + c + \theta(x, y^{-1}) + \theta(xy^{-1}, z) - \theta(y, y^{-1}))
\end{aligned}$$

so that we obtain the correspondence

$$\eta(x, y, z) = \theta(x, y^{-1}) + \theta(xy^{-1}, z) - \theta(y, y^{-1}).$$

Let θ be a group 2-cocycle satisfying the *inverse property*: $\theta(x^{-1}, y^{-1}) = -\theta(y, x)$ for all $x, y \in X$. Then $\eta(x, y, z) := \theta(x, y^{-1}) + \theta(xy^{-1}, z)$ is a PA 2-cocycle.

5. Ternary self-distributive cohomology with heap coefficients

In this section, we introduce a cohomology theory of ternary self-distributive operations with abelian group heap coefficients, and investigate extension theory by 2-cocycles. A ternary operation T on a set X is called *ternary self-distributive* (TSD for short) if it satisfies

$$T(T(x, y, z), u, v) = T(T(x, u, v), T(y, u, v), T(z, u, v))$$

for all $x, y, z, u, v \in X$. Such operations have been widely studied (e.g. [9, 11] and references therein). The set X with TSD operation T , or the pair (X, T) , is also called a *ternary shelf*. In [9, 11], homology theories of ternary shelves are defined and studied. The theory introduced here differs in the use of heap structures.

Let (X, T) be a ternary shelf, and let us define the n th chain group of X with heap coefficients in A , denoted by $C_n^{\text{SD}}(X)$, to be the free abelian group on $(2n-1)$ -tuples X^{2n-1} . We set by definition $C_0^{\text{SD}}(X)$ to be the trivial group. We introduce maps $\partial_n^i : C_n^{\text{SD}}(X) \rightarrow C_{n-1}^{\text{SD}}(X)$, for all $i = 1, \dots, n$. The map $\partial_1^1 = 0$ by definition, while for $n \geq 2$ we distinguish case $i = 1$, where we set

$$\begin{aligned} \partial_n^1(x_1, \dots, x_{2n-1}) &= (x_1, x_4, \dots, x_{2n-1}) - (x_2, x_4, \dots, x_{2n-1}) \\ &\quad + (x_3, x_4, \dots, x_{2n-1}) - (T(x_1, x_2, x_3), x_4, \dots, x_{2n-1}), \end{aligned}$$

and case $i \geq 2$ where we set

$$\begin{aligned} \partial_n^i(x_1, \dots, x_{2n-1}) &= (x_1, \dots, \widehat{x_{2i}}, \widehat{x_{2i+1}}, \dots, x_{2n-1}) \\ &\quad - (T(x_1, x_{2i}, x_{2i+1}), \dots, T(x_{2i-1}, x_{2i}, x_{2i+1}), \widehat{x_{2i}}, \widehat{x_{2i+1}}, \dots, x_{2n-1}), \end{aligned}$$

where $\widehat{}$ denotes the deletion of that factor.

Proposition 5.1. *The maps ∂_n^i define a pre-simplicial module structure on the chain groups $\{C_n^{\text{SD}}(X)\}$.*

Proof. Recall that by definition, the maps ∂_n^i define a pre-simplicial module structure on $\{C_n^{\text{SD}}(X)\}$ if they satisfy the face map relations $\partial_{n-1}^{j-1} \partial_n^i = \partial_{n-1}^i \partial_n^j$ for $i < j$. To show that the pre-simplicial equations hold, the cases with $i \geq 2$ are standard, while the remaining cases $1 = i < j$ can be verified by a direct computation. We show, as an example, the computations for the case $i = 1, j = 2$. On the one hand, we have

$$\begin{aligned} \partial_{n-1}^1 \partial_n^2(x_1, \dots, x_{2n-1}) &= (x_1, x_6, \dots, x_{2n-1}) - (x_2, x_6, \dots, x_{2n-1}) \\ &\quad + (x_3, x_6, \dots, x_{2n-1}) - (T(x_1, x_2, x_3), x_6, \dots, x_{2n-1}) \\ &\quad - (T(x_1, x_4, x_5), x_6, \dots, x_{2n-1}) + (T(x_2, x_4, x_5), x_6, \dots, x_{2n-1}) \\ &\quad - (T(x_3, x_4, x_5), x_6, \dots, x_{2n-1}) \\ &\quad + (T(T(x_1, x_4, x_5), T(x_2, x_4, x_5), T(x_3, x_4, x_5)), x_6, \dots, x_{2n-1}). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \partial_{n-1}^1 \partial_n^1(x_1, \dots, x_{2n-1}) &= (x_1, x_6, \dots, x_{2n-1}) - (x_4, x_6, \dots, x_{2n-1}) \\ &\quad + (x_5, x_6, \dots, x_{2n-1}) - (T(x_1, x_4, x_5), x_6, \dots, x_{2n-1}) \\ &\quad - (x_2, x_6, \dots, x_{2n-1}) + (x_4, x_6, \dots, x_{2n-1}) \\ &\quad - (x_5, x_6, \dots, x_{2n-1}) + (T(x_2, x_4, x_5), x_6, \dots, x_{2n-1}) \\ &\quad + (x_3, x_6, \dots, x_{2n-1}) - (x_4, x_6, \dots, x_{2n-1}) \\ &\quad + (x_5, x_6, \dots, x_{2n-1}) - (T(x_3, x_4, x_5), x_6, \dots, x_{2n-1}) \\ &\quad - (T(x_1, x_2, x_3), x_6, \dots, x_{2n-1}) + (x_4, x_6, \dots, x_{2n-1}) \\ &\quad - (x_5, x_6, \dots, x_{2n-1}) + (T(T(x_1, x_2, x_3), x_4, x_5), x_6, \dots, x_{2n-1}). \end{aligned}$$

The two expressions coincide. □

Standard arguments imply that, for each n , the alternating sum $\partial_n = (-1)^n \sum_{i=1}^n (-1)^i \partial_n^i$ satisfies the equation $\partial^2 = 0$. It follows that $(C_n^{\text{SD}}(X), \partial)$ is a chain complex, and the following is well posed.

Definition 5.2. Let X be a ternary shelf and let $C_n^{\text{SD}}(X)$ be the n th chain group as defined above. The homology corresponding to the chain complex $(C_n^{\text{SD}}(X), \partial)$ is indicated by the symbol $H_n^{\text{SD}}(X)$. By dualization, given an abelian group A , we obtain a cochain complex with coefficients in the abelian heap A , $C_{\text{SD}}^n(X, A)$ whose cohomology is indicated by $H_{\text{SD}}^n(X, A)$.

Our focus is on significance and constructions of 2-cocycles in relation to heaps for this theory, so that we provide explicit cocycle conditions in low dimensions below.

Example 5.3. Let (X, T) be a ternary shelf, and A be an abelian group heap. Then cochain groups and differentials dual to Definition 5.2 in low dimensions are formulated as follows. The cochain groups $C_{\text{SD}}^n(X, A)$ are defined to be the abelian groups of functions $\{f : X^{2n-1} \rightarrow A\}$. The differentials $\delta^n = \delta_{\text{SD}}^n : C_{\text{SD}}^n(X, A) \rightarrow C_{\text{SD}}^{n+1}(X, A)$ are formulated for $n = 1, 2, 3$ as follows.

$$\begin{aligned} \delta^1 \zeta(x_1, x_2, x_3) &= \zeta(x_1) - \zeta(x_2) + \zeta(x_3) - \zeta(T(x_1, x_2, x_3)), \\ \delta^2 \eta(x_1, x_2, x_3, x_4, x_5) &= \eta(x_1, x_2, x_3) + \eta(T(x_1, x_2, x_3), x_4, x_5) \\ &\quad - \eta(x_1, x_4, x_5) + \eta(x_2, x_4, x_5) - \eta(x_3, x_4, x_5) \\ &\quad - \eta(T(x_1, x_4, x_5), T(x_2, x_4, x_5), T(x_3, x_4, x_5)), \\ \delta^3 \psi(x_1, x_2, x_3, x_4, x_5, x_6, x_7) &= \psi(x_1, x_2, x_3, x_4, x_5) + \psi(T(x_1, x_4, x_5), T(x_2, x_4, x_5), T(x_3, x_4, x_5), x_6, x_7) \\ &\quad + \psi(x_1, x_4, x_5, x_6, x_7) - \psi(x_2, x_4, x_5, x_6, x_7) + \psi(x_3, x_4, x_5, x_6, x_7) \\ &\quad - \psi(T(x_1, x_2, x_3), x_4, x_5, x_6, x_7) - \psi(x_1, x_2, x_3, x_6, x_7) \\ &\quad - \psi(T(x_1, x_6, x_7), T(x_2, x_6, x_7), T(x_3, x_6, x_7), T(x_4, x_6, x_7), T(x_5, x_6, x_7)). \end{aligned}$$

The case $n=0$ is defined by convention that $C_{\text{SD}}^0(X, A) = 0$.

We investigate properties of TSD 2-cocycles. We start with extensions by abelian group heaps.

Definition 5.4. Let (X, T) be a ternary self-distributive set, A an abelian group heap and $\eta : X \times X \times X \rightarrow A$ a 2-cocycle of X with values in A . We define the self-distributive cocycle extension of X with heap coefficients in A , by the cocycle η to be the cartesian product $X \times A$, endowed with the ternary operation T' given by

$$(x, a) \times (y, b) \times (z, c) \mapsto (T(x, y, z), a - b + c + \eta(x, y, z)).$$

In this situation, we denote the extension by $X \times_{\eta} A$.

Lemma 5.5. *The TSD 2-cocycle condition gives extension cocycles of TSDs with abelian group heap coefficients. Specifically, the ternary operation in Definition 5.4, corresponding to a 2-cocycle η satisfying the second condition $\delta^2 \eta = 0$ in 5.3, is ternary self-distributive.*

Definition 5.6. Given two extensions $X \times_{\eta} A$ and $X \times_{\eta'} A$, we define a morphism of extensions to be a morphism of ternary self-distributive sets making a commutative diagram identical to the one in Definition 3.15. An invertible morphism of extensions is called isomorphism.

Similarly to [Definition 3.15](#), this defines an equivalence relation and corresponding isomorphism classes. We have the following result.

Proposition 5.7. *There is a bijective correspondence between $H_{\text{SD}}^2(X, A)$ and equivalence classes of extensions.*

Proof. Similar to the group-theoretic case and [Proposition 3.16](#). □

Example 5.8. Let $X = \mathbb{Z}_2$ with the TSD operation $T(x, y, z) = x + y + z \in \mathbb{Z}_2$. This is in fact the abelian heap $\mathbb{Z}_2$ and by [Lemma 6.1](#) below, the same operation is self-distributive. In this example, we compute the first cohomology group $H_{\text{SD}}^1(X, \mathbb{Z}_2)$ and the second cohomology group $H_{\text{SD}}^2(X, \mathbb{Z}_2)$ with coefficients in the abelian heap $\mathbb{Z}_2$. For a function $f : X \rightarrow \mathbb{Z}_2$, a straightforward computation gives that $\delta^1(f) = 0$. This gives $H_{\text{SD}}^1(X, \mathbb{Z}_2) \cong C_{\text{SD}}^1(X, \mathbb{Z}_2)$. To compute the kernel of δ^2 , let us write an element $\phi : X^3 \rightarrow \mathbb{Z}_2$ in terms of characteristic functions as $\phi = \sum_{x,y,z} \phi(x, y, z) \chi_{(x,y,z)}$. Then, $\delta^2(\phi) = 0$ gives the following system of equations in $\mathbb{Z}_2$:

$$\begin{cases} \phi(1, 1, 1) + \phi(0, 0, 0) = 0 \\ \phi(1, 1, 0) + \phi(0, 0, 1) = 0 \\ \phi(1, 0, 1) + \phi(0, 1, 0) = 0 \\ \phi(1, 0, 0) + \phi(0, 1, 1) = 0 \end{cases}$$

implying that $\ker(\delta^2)$ is 4-dimensional with a basis $\chi_{(1,1,1)} + \chi_{(0,0,0)}$, $\chi_{(1,1,0)} + \chi_{(0,0,1)}$, $\chi_{(1,0,1)} + \chi_{(0,1,0)}$, and $\chi_{(1,0,0)} + \chi_{(0,1,1)}$. Since $\text{im}(\delta^1) = 0$, we then obtain that $H_{\text{SD}}^2(X, \mathbb{Z}_2) \cong \mathbb{Z}_2^{\oplus 4}$.

Example 5.9. In this example, we compute the first cohomology group $H_{\text{SD}}^1(X, \mathbb{Z}_3)$ and the second cohomology group $H_{\text{SD}}^2(X, \mathbb{Z}_3)$ for the same $X = \mathbb{Z}_2$ as above, with coefficients in the abelian heap $\mathbb{Z}_3$. For a function $f : X \rightarrow \mathbb{Z}_3$, a direct computation gives that $\delta^1(f)(1, 0, 1) = f(0) - f(1)$, $\delta^1(f)(0, 1, 0) = f(1) - f(0)$ and all other unspecified values of $\delta^1(f)(x, y, z)$ are zeros. This gives $H_{\text{SD}}^1(X, \mathbb{Z}_3) \cong \mathbb{Z}_3$. We continue to use the characteristic function notation. Then, hand computations give that $\ker(\delta^2)$ is 3-dimensional with a basis $\chi_{(1,1,1)} + \chi_{(0,0,0)} + \chi_{(1,0,0)} + \chi_{(0,1,1)}$, $\chi_{(1,1,0)} + \chi_{(0,0,1)} - \chi_{(0,1,0)}$, and $\chi_{(1,0,1)} - \chi_{(0,1,0)}$. Since $\text{im}(\delta^1)$ is generated by $\chi_{(1,0,1)} - \chi_{(0,1,0)}$, we then obtain that $H_{\text{SD}}^2(X, \mathbb{Z}_3) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

Example 5.10. Let X be a finite trivial TSD set, that is $T(x, y, z) = x$ for all $x, y, z \in X$, then the differentials δ^1 and δ^2 take the following simpler forms:

$$\begin{aligned} \delta^1 \xi(x, y, z) &= \xi(z) - \xi(y), \\ \delta^2 \eta(x, y, z, u, v) &= \eta(y, u, v) - \eta(z, u, v). \end{aligned}$$

This gives, for an abelian group A ,

$$\text{im}(\delta^1) = \{\eta : X^3 \rightarrow A, \eta(x, y, z) = \xi(z) - \xi(y), \text{ for some } \text{map } \xi : X \rightarrow A\}.$$

Thus, $H_{\text{SD}}^1(X, A) = Z_{\text{SD}}^1(X, A)$ is the group of constant functions, which is isomorphic to A . The kernel of δ^2 is given by

$$\ker(\delta^2) = \{\eta : X^3 \rightarrow A \mid \eta(x, y, z) = \eta(x', y, z), \forall x, x', y, z \in X\}$$

that are functions constant on the first variable. Hence, $Z_{\text{SD}}^2(X, A)$ is isomorphic to $A^{X \times X}$, the group of functions $A^{X \times X}$ from $X \times X$ to A . This group has the subgroup $B_{\text{SD}}^2(X, A) = \text{im}(\delta^1) = \{\eta : X^3 \rightarrow A \mid \eta(x, y, z) = \xi(z) - \xi(y), \xi \in A^X\}$.

Let $X = \mathbb{Z}_n$ be equipped with the trivial TSD operation, and A be cyclic, $\mathbb{Z}_m$ or $\mathbb{Z}$. We represent elements of $\mathbb{Z}_n$ by $\{0, \dots, n-1\}$. Then, an element $\eta \in Z_{\text{SD}}^2(X, A) \cong A^{X \times X}$ as a function $\eta(x, y, z)$ is constant on x and with variables (y, z) . Hence, $Z_{\text{SD}}^2(X, A)$ has basis of characteristic functions $\chi_{(y, z)} \in A^{X \times X}$.

Since $\delta^1(\xi)(y, z) = \xi(z) - \xi(y)$, we have $\delta^1(\chi_i) = \sum_{x \neq i} \chi_{(x, i)} - \sum_{x \neq i} \chi_{(i, x)}$ for $i = 0, \dots, n-1$. Since each term $\chi_{(x, y)}$, $x \neq y \in \mathbb{Z}_n$, appears exactly twice in $\sum_{i=0}^{n-1} \delta^1(\chi_i)$ with opposite signs, we have $\sum_{i=0}^{n-1} \delta^1(\chi_i) = 0$. Hence $B_{\text{SD}}^2(X, A)$ is spanned by $\delta^1(\chi_i)$, $i = 0, \dots, n-2$.

We show that $B_{\text{SD}}^2(X, A)$ is a direct summand of $Z_{\text{SD}}^2(X, A)$. Set $\chi_{\{x, y\}} = \chi_{(x, y)} - \chi_{(y, x)}$ for $x < y$. Then, $\{\chi_{\{x, y\}}, \chi_{(x, y)}, \chi_{(x, x)} \mid x < y, x, y \in \mathbb{Z}_n\}$ forms a basis of $Z_{\text{SD}}^2(X, A)$. One computes $\delta^1(\chi_i) = \sum_{x < i} \chi_{\{x, i\}} - \sum_{i < y} \chi_{\{i, y\}}$. The term $-\chi_{\{i, n-1\}}$ appears exactly once only in $\delta^1(\chi_i)$ for $i = 0, \dots, n-2$, and no other $\delta^1(\chi_j)$ for $j \neq i$. Hence, $\delta^1(\chi_i)$, $i = 0, \dots, n-2$, are linearly independent and form a direct summand. Thus, we obtained $Z_{\text{SD}}^2(X, A) \cong A^{n^2}$, $B_{\text{SD}}^2(X, A) \cong A^{n-1}$, and $H_{\text{SD}}^2(X, A) \cong A^{n^2-n+1}$.

The following provides an algebraic meaning of the TSD 3-cocycle condition.

Proposition 5.11. *The TSD 3-cocycle condition gives obstruction cocycles of TSDs for short exact sequences of coefficients. Specifically, let X be a TSD set and consider a short exact sequence of abelian groups,*

$$0 \rightarrow H \xrightarrow{i} E \xrightarrow{\pi} A \rightarrow 0,$$

where E is the extension heap corresponding to the 2-cocycle $\phi \in Z^2(X, A)$, and a section $s : A \rightarrow E$, such that $s(0) = 0$, the obstruction for $s\phi$ to satisfy the 2-cocycle condition is a 3-cocycle with heap coefficients in H .

Proof. We construct the mapping $\alpha : X^5 \rightarrow H$ by the equality

$$\begin{aligned} i\alpha(x_1, \dots, x_5) &= s\phi(x_1, x_2, x_3) - s\phi(T(x_1, x_4, x_5), T(x_2, x_4, x_5), T(x_3, x_4, x_5)) \\ &\quad + s\phi(T(x_1, x_2, x_3), x_4, x_5) - s\phi(x_1, x_4, x_5) \\ &\quad + s\phi(x_2, x_4, x_5) - s\phi(x_3, x_4, x_5). \end{aligned}$$

Since ϕ satisfies the 2-cocycle condition, we see that $\pi\alpha$ is the zero map, where $\pi : E \rightarrow A$ is the projection. It follows that there is $\alpha : X^5 \rightarrow H$ satisfying the above equality. It is proved that $\alpha : X^5 \rightarrow H$ so defined satisfies the 3-cocycle condition with heap coefficients in H , by a direct (though long) calculation. $\square$

Further study on homology groups with heap coefficients and their properties, such as existence of torsion, lower/upper bounds of their ranks and an interpretation of the rank of the 0-dimensional homology group, would be desirable and of interest.

6. From heap cocycles to TSD cocycles

In this section, we show that heaps and their 2-cocycles give rise to those for TSDs. Although a heap gives rise to a group, and $T(x, y, z) = xy^{-1}z$ gives a TSD operation, the following lemma provides a direct argument, which provides an idea for the proof of [Theorem 6.2](#).

Lemma 6.1. *A heap is ternary self-distributive.*

Proof. First, we note that for a heap operation it holds that

$$[[x, y, z], z, y] = [x, y, [z, z, y]] = [x, y, y] = x.$$

Then, one computes

$$\begin{aligned} & T(T(x_1, x_4, x_5), T(x_2, x_4, x_5), T(x_3, x_4, x_5)) \\ &= [[x_1, x_4, x_5], [x_2, x_4, x_5], [x_3, x_4, x_5]] \\ &= [x_1, [[x_2, x_4, x_5], x_5, x_4], [x_3, x_4, x_5]] \\ &= [x_1, x_2, [x_3, x_4, x_5]] \\ &= [[x_1, x_2, x_3], x_4, x_5] \\ &= T(T(x_1, x_2, x_3), x_4, x_5) \end{aligned}$$

as desired. The notation $T(x, y, z) = [x, y, z]$ was used for clarification. $\square$

Theorem 6.2. *Let X be a heap, with the operation regarded as a TSD operation by Lemma 6.1, and let A be an abelian group. Suppose that $\eta \in Z_H^2(X, A)$, that is, η satisfies $\delta_{(1)}^2 \eta = 0 = \delta_{(2)}^2 \eta$ and the degeneracy condition. Then η is a TSD 2-cocycle, $\eta \in Z_{SD}^2(X, A)$. This assignment induces an injection of $H_H^2(X, A)$ into $H_{SD}^2(X, A)$.*

Proof. We note that $\delta_{(1)}^2 \eta = 0 = \delta_{(2)}^2 \eta$ also implies $\delta_{(0)}^2 \eta = 0$, and the equality $[[x, y, z], z, y] = x$ from the proof of Lemma 6.1. One computes

$$\begin{aligned} & \underline{\eta(x_1, x_4, x_5)} - \eta(x_2, x_4, x_5) + \eta(x_3, x_4, x_5) \\ & \quad + \underline{\eta(T(x_1, x_4, x_5), T(x_2, x_4, x_5), T(x_3, x_4, x_5))} \\ &= -\eta([x_2, x_4, x_5], x_5, x_4) + \underline{\eta(x_1, [[x_2, x_4, x_5], x_5, x_4], [x_3, x_4, x_5])} \\ & \quad (= \eta(x_1, x_2, [x_3, x_4, x_5])) \\ & \quad - \eta(x_2, x_4, x_5) + \underline{\eta(x_3, x_4, x_5)} \\ &= \eta(x_1, x_2, x_3) + \eta([x_1, x_2, x_3], x_4, x_5) \\ & \quad - \underline{\eta([x_2, x_4, x_5], x_5, x_4)} - \underline{\eta(x_2, x_4, x_5)} \\ &= \eta(x_1, x_2, x_3) + \eta(T(x_1, x_2, x_3), x_4, x_5) \end{aligned}$$

as desired. The equalities follow from $\delta_{(1)}^2 \eta = 0$, $\delta_{(0)}^2 \eta = 0$, and Lemma 3.17, respectively, and the underlined terms indicate where they are applied. This proves that we have an inclusion $h : Z_H^2(X, A) \hookrightarrow Z_{SD}^2(X, A)$. Since we have the equality $C_H^1(X, A) = C_{SD}^1(X, A)$ and the first cochain differentials for heap and TSD cohomologies coincide up to sign, $\delta_H^1 = -\delta_{SD}^1$, we have $h(\delta_H^1(f)) = -\delta_{SD}^1(h(f))$ and $h(B_H^2(X, A)) \subset B_{SD}^2(X, A)$, so that h induces a homomorphism $\bar{h} : H_H^2(X, A) \rightarrow H_{SD}^2(X, A)$. Lastly, the map $\bar{h}$ is injective. Indeed, for $\eta \in Z_H^2(X, A)$, assume that $h(\eta) \in Z_{SD}^2(X, A)$ is null-cohomologous. Then, $h(\eta) = \delta_{SD}^1(\xi')$ for some $\xi' \in C_{SD}^1(X, A)$. For $\xi = -\xi' \in C_H^1$, we have $\eta = \delta_H^1(\xi)$, so that η is null-cohomologous in $Z_H^2(X, A)$. $\square$

Example 6.3. In Example 3.22, a nontrivial heap 2-cocycle η was given for $X = \mathbb{Z}_3 = A$. By Theorem 6.2, η is a non-trivial TSD 2-cocycle. Hence, we obtain $H_{SD}^2(X, A) \neq 0$.

Remark 6.4. The construction in Theorem 6.2 and taking extensions commute (c.f. Remarks 4.2 and 4.8). Indeed, for a heap X and an abelian heap A , the heap extension $X \times A$ by a heap A

cocycle η is defined by

$$[(x, a), (y, b), (z, c)] = ([x, y, z], a - b + c + \eta(x, y, z)),$$

and [Lemma 6.1](#) states that this heap operation gives a ternary shelf. On the other hand, this is the extension of a ternary shelf by a TSD 2-cocycle η with the heap coefficient A by [Definition 5.4](#).

7. Internalization

In this section, we generalize to monoidal categories, the construction of TSD structures from heaps. Throughout the section, all symmetric monoidal categories are strict (the associator $(A \boxtimes B) \boxtimes C \rightarrow A \boxtimes (B \boxtimes C)$, the right and left unitors $I \boxtimes X \rightarrow X$ and $X \boxtimes I \rightarrow X$ are all identity maps, where I is the unit object).

Let $(\mathcal{C}, \boxtimes)$ be a symmetric monoidal category, (X, Δ, ϵ) be a comonoid object in $\mathcal{C}$ and consider a morphism $\mu : X \boxtimes X \boxtimes X \rightarrow X$. We translate the heap axioms of [Section 2](#) into commutative diagrams in the category $\mathcal{C}$. The equalities of type 1 and 2 para-associativity are defined by the commutative diagram

$$\begin{array}{ccccc} X^{\boxtimes 3} & \xleftarrow{\mu \boxtimes 1^2} & X^{\boxtimes 5} & \xrightarrow{1^2 \boxtimes \mu} & X^{\boxtimes 3} \\ & \searrow \mu & \downarrow & \swarrow \mu & \\ & & X & & \end{array}$$

where the central arrow corresponds to the morphism $\mu(1 \boxtimes \mu \boxtimes 1)\tau_{321}$ and τ_{321} is defined by

$$\tau_{321} = (1 \boxtimes \tau \boxtimes 1^2)(1^2 \boxtimes \tau \boxtimes 1)(1 \boxtimes \tau \boxtimes 1^2).$$

The type 0 para-associativity is defined by

$$\begin{array}{ccc} X^{\boxtimes 5} & \xrightarrow{\mu \boxtimes 1^2} & X^{\boxtimes 3} \\ 1^2 \boxtimes \mu \downarrow & & \downarrow \mu \\ X^{\boxtimes 3} & \xrightarrow{\mu} & X \end{array}$$

and follows from those of types 1 and 2. The degeneracy conditions are formulated as commutativity of the following diagrams.

$$\begin{array}{ccc} X \boxtimes X & \xrightarrow{1 \boxtimes \Delta} & X \boxtimes X \boxtimes X \\ 1 \boxtimes \epsilon \downarrow & \swarrow \mu & \\ X & & \end{array} \quad \text{and} \quad \begin{array}{ccc} X \boxtimes X & \xrightarrow{\Delta \boxtimes 1} & X \boxtimes X \boxtimes X \\ \epsilon \boxtimes 1 \downarrow & \swarrow \mu & \\ X & & \end{array}$$

Definition 7.1. A heap object in a symmetric monoidal category is a comonoid object (X, Δ, ϵ) , where $\epsilon : X \rightarrow I$ is a counital morphism to the unit object I , endowed with a morphism of comonoids $\mu : X^{\boxtimes 3} \rightarrow X$ making all the diagrams above commute.

Example 7.2. A (set-theoretic) heap in the sense of [Section 2](#) is a heap object in the category of sets.

The following appeared implicitly in [\[11\]](#).

Example 7.3. Let H be an involutory Hopf algebra (i.e. $S^2 = \mathbb{1}$) over a field $\mathbb{k}$. Then, H is a heap object in the monoidal category of vector spaces and tensor products, with the ternary operation μ induced by the assignment

$$x \otimes y \otimes z \mapsto xS(y)z$$

for single tensors. Indeed, we have

$$\begin{aligned} \mu(\mu(x \otimes y \otimes z) \otimes u \otimes v) &= xS(y)zS(u)v \\ &= xS(y)S^2(z)S(u)v \\ &= xS(uS(z)y)v \\ &= \mu(x \otimes \mu(u \otimes z \otimes y) \otimes v) \end{aligned}$$

corresponding to the commutativity of the diagram representing equality of type 1. Observe that we have used the involutory hypothesis to obtain the second equality. We also have

$$\begin{aligned} \mu(\mathbb{1} \otimes \Delta)(x \otimes y) &= \mu(x \otimes y^{(1)} \otimes y^{(2)}) \\ &= xS(y^{(1)})y^{(2)} \\ &= \epsilon(y)x \end{aligned}$$

which shows the left degeneracy constraint. The rest of the axioms can be checked in a similar manner.

The opposite direction in the group-theoretic case is the assertion that a pointed heap generates a group by means of the operation $xy = [x, e, y]$. The following is a Hopf algebra version and can be obtained by calculations. More general statement of this can be found in [3] and below.

Proposition 7.4. Let $(X, [-])$ be a heap object in a coalgebra category, and let $e \in X$ be a group-like element (i.e. $\Delta(e) = e \otimes e$ and $\epsilon(e) = 1$). Then, X is an involutory Hopf algebra with multiplication $m(x \otimes y) := \mu_e(x \otimes y) := [x \otimes e \otimes y]$, unit e , and antipode $S(x) := [e \otimes x \otimes e]$.

Proof. We use Sweedler's notation $\Delta(x) = x^{(1)} \otimes x^{(2)}$. The associativity of m follows from the type 0 para-associativity of μ . A unit condition is computed by

$$m(e \otimes x) = \mu(e \otimes e \otimes x) = \mu(\Delta(e) \otimes x) = x$$

by the degeneracy condition and the assumption that e is group-like. The other condition $m(x \otimes e) = x$ is similar. The compatibility between m and Δ is computed as

$$\begin{aligned} \Delta m(x \otimes y) &= \Delta \mu(x \otimes e \otimes y) \\ &= \mu \tau(\Delta(x) \otimes \Delta(e) \otimes \Delta(y)) \\ &= \mu \tau(x^{(1)} \otimes x^{(2)} \otimes e \otimes e \otimes y^{(1)} \otimes y^{(2)}) \\ &= \mu(x^{(1)} \otimes e \otimes y^{(1)}) \otimes \mu(x^{(2)} \otimes e \otimes y^{(2)}) \\ &= m(x^{(1)} \otimes y^{(1)}) \otimes m(x^{(2)} \otimes y^{(2)}) \end{aligned}$$

as desired, where τ is an appropriate permutation that give the third equality, and the group-like assumption is used in the second equality. An antipode condition is computed as

$$\begin{aligned}
m(S \otimes \mathbb{1})\Delta(x) &= \mu\left(\mu(e \otimes x^{(1)} \otimes e) \otimes e \otimes x^{(2)}\right) \\
&= \mu\left(e \otimes x^{(1)} \otimes \mu(e \otimes e \otimes x^{(2)})\right) \\
&= \mu\left(e \otimes x^{(1)} \otimes \mu(\Delta(e) \otimes x^{(2)})\right) \\
&= \mu(e \otimes x^{(1)} \otimes x^{(2)})\epsilon(e) \\
&= e\epsilon(x)
\end{aligned}$$

as desired, where the group-like condition $\epsilon(e)$ and the degeneracy condition for μ were used. The other case $m(\mathbb{1} \otimes S)\Delta(x) = e\epsilon(x)$ is similar. This completes the proof. $\square$

Remark 7.5. Observe that S so defined, is involutory. This observation corroborates the necessity of including the involutory hypothesis in Example 7.3.

Remark 7.6. We observe a relation between a choice of a group-like element e in Proposition 7.4 and a *coaugmentation map* of a coalgebra. Let (X, Δ, ϵ) be a coalgebra. A coaugmentation is a coalgebra morphism $\eta : \mathbb{k} \rightarrow X$ (i.e. $\Delta\eta = (\eta \otimes \eta)j$, where $j : \mathbb{k} \rightarrow \mathbb{k} \otimes \mathbb{k}$ is the canonical isomorphism, $j(1) = 1 \otimes 1$) such that $\epsilon\eta = 1|_{\mathbb{k}}$. Let $e = \eta(1)$. We show that e is group-like. One computes $\Delta(e) = \Delta\eta(1) = \eta(1) \otimes \eta(1) = e \otimes e$, and $\epsilon(e) = \epsilon(\eta(1)) = 1$ as desired. Conversely, for any group element $e \in X$, let η be defined by $\eta(1) = e$. Then, one computes $\Delta\eta(1) = \Delta(e) = e \otimes e = \eta(e) \otimes \eta(e) = (\eta \otimes \eta)j(1)$ and $\epsilon\eta(1) = \epsilon(e) = 1$. We observe that an advantage of using coaugmentation map is that the desired condition can be stated by a map, without mention of particular elements, which becomes advantageous in categorical definitions as we see below.

We generalize Example 7.3 and Proposition 7.4 to symmetric monoidal categories as follows, using Remark 7.6. For this purpose, first we define a *coaugmentation* of a comonoid object (X, Δ, ϵ) in a symmetric monoidal category with a unit object I as a comonoidal morphism $\eta : I \rightarrow X$ such that $\epsilon\eta = \mathbb{1}$.

Definition 7.7. Let $\mathcal{C}$ be a symmetric monoidal category. We define the *category of heap objects* in $\mathcal{C}$, $\mathcal{H}_{\mathcal{C}}$, as follows. The objects of $\mathcal{H}_{\mathcal{C}}$ are the heap objects as in Definition 7.1. The morphisms are defined to be the morphisms of $\mathcal{C}$ commuting with the heap maps and the comonoidal structures. A heap object X , is called *pointed*, if it is endowed with a coaugmentation $\eta : I \rightarrow X$. The *category of pointed heap objects* in $\mathcal{C}$, $\mathcal{H}_{\mathcal{C}}^*$, is the category consisting of pointed heap objects over $\mathcal{C}$, and morphisms of heap objects commuting with the coaugmentations.

An *involutory Hopf* monoid (object) in a symmetric monoidal category is equipped with a monoidal product m , a unit object I , a comonoidal product Δ , an antipode S that is an antimorphism (i.e. $\Delta S = (S \boxtimes S)\tau\Delta$ and $Sm = m\tau(S \boxtimes S)$) satisfying $m(S \boxtimes \mathbb{1})\Delta = \eta\epsilon = m(\mathbb{1} \boxtimes S)\Delta$ and $S^2 = \mathbb{1}$, a unit morphism $\eta : I \rightarrow X$ that satisfies the left and right unital conditions $m(\eta \boxtimes \mathbb{1}) = \mathbb{1}$ and $m(\mathbb{1} \boxtimes \eta) = \mathbb{1}$, and a counit morphism $\epsilon : X \rightarrow I$ that satisfies the left and right counital conditions $(\epsilon \boxtimes \mathbb{1})\Delta = \mathbb{1}$, $(\mathbb{1} \boxtimes \epsilon)\Delta = \mathbb{1}$, and $\epsilon\eta = \mathbb{1}_I$.

Theorem 7.8. Let $\mathcal{C}$ be a symmetric monoidal category. There is an equivalence of categories between the category $\mathcal{H}_{\mathcal{C}}^*$ and the category of involutory Hopf monoids in $\mathcal{C}$, $sH_{\mathcal{C}}$.

Proof. We define a functor $\mathcal{F} : \mathcal{H}_{\mathcal{C}}^* \rightarrow sH_{\mathcal{C}}$ as follows. Let $(X, \eta, \mu, \epsilon, \Delta)$ be a pointed heap object in $\mathcal{C}$, define $\mathcal{F}(X) := (X, I, \eta, \lambda_{\eta}, \rho_{\eta}, m, \epsilon, \Delta, S)$, where multiplication $m := \mu \circ \mathbb{1} \boxtimes \eta \boxtimes \mathbb{1}$, antipode $S := \mu \circ \eta \boxtimes \mathbb{1} \boxtimes \eta$, the unit object I , the left and right unitors $\lambda_{\eta}, \rho_{\eta}$ by $\lambda_{\eta} := m(\eta \boxtimes \mathbb{1}) : I \boxtimes X \rightarrow$

X and $\rho_\eta := m(1 \boxtimes 1) : X \boxtimes I \rightarrow X$, and comonoidal structure unchanged. The functor $\mathcal{F}$ is defined to be the identity on morphisms. The fact that $\mathcal{F}(X)$ is a Hopf monoid in $\mathcal{C}$ is a translation of the computations in Example 7.3 in commutative diagrams or series of composite morphisms.

Specifically, defining conditions are verified as follows. The associativity of m follows from the para-associativity as before. The unit η and the counit ϵ are unchanged and a left unital condition is checked by

$$\begin{aligned} m(\eta \boxtimes 1) &= \mu(\eta \boxtimes \eta \boxtimes 1) \\ &= \mu(\Delta \eta \boxtimes 1) = (\epsilon \boxtimes 1)(\eta \boxtimes 1) = \epsilon \eta \boxtimes 1 = 1_I \boxtimes 1 = 1 \end{aligned}$$

as desired. The right unital condition is computed similarly. The compatibility between m and Δ is computed as

$$\begin{aligned} (m \boxtimes m)(\Delta \boxtimes \Delta) &= [(\mu(1 \boxtimes \eta \boxtimes 1) \boxtimes (\mu(1 \boxtimes \eta \boxtimes 1))](\Delta \boxtimes \Delta) \\ &= (\mu \boxtimes \mu)(\Delta \boxtimes \Delta \eta \boxtimes \Delta) = (\mu \boxtimes \mu)(\Delta \boxtimes \Delta \boxtimes \Delta)(1 \boxtimes \eta \boxtimes 1) \\ &= \Delta \mu(1 \boxtimes \eta \boxtimes 1) = \Delta m. \end{aligned}$$

We note that the compositions involving $\Delta \boxtimes \Delta$ contain appropriate permutations of factors of objects. The antipode condition is computed as

$$\begin{aligned} m(S \boxtimes 1)\Delta &= \mu(1 \boxtimes \eta \boxtimes 1)(\mu(\eta \boxtimes 1 \boxtimes \eta) \boxtimes 1)\Delta \\ &= \mu(\eta \boxtimes 1 \boxtimes 1)(\mu(\eta \boxtimes \eta \boxtimes 1) \boxtimes 1)\Delta \\ &= (\eta \boxtimes \epsilon)(\epsilon \eta \boxtimes 1)\Delta \\ &= \epsilon \eta. \end{aligned}$$

The other antipode condition is similar.

Similarly, we define a functor $\mathcal{G} : sH_{\mathcal{C}} \rightarrow \mathcal{H}_{\mathcal{C}}^*$ by the assignment on objects $\mathcal{G}(X, \eta, m, \epsilon, \Delta, S) := (X, \eta, \mu, \epsilon, \Delta)$, with $\mu := m(m \boxtimes 1)(1 \boxtimes S \boxtimes 1)$. Also, $\mathcal{G}$ is the identity on morphisms. The proof is obtained by sequences of equalities of composite morphisms mimicking the computations in Example 7.3. $\square$

Next we show that Lemma 6.1 holds for a coalgebra (i.e. a comonoid in the category of vector spaces). Although this is a special case of Theorem 7.12, we include its statement and proof here to illustrate and further motivate Theorem 7.12. For this goal, we slightly modify the definition of TSD maps in symmetric monoidal categories, given in [11].

Definition 7.9. Let (X, Δ, ϵ) be a comonoidal object in a symmetric monoidal category $\mathcal{C}$. A *ternary self-distributive object* $(X, \Delta, \epsilon, \mu)$ in $\mathcal{C}$ is a comonoidal object that satisfies the following condition:

$$\mu(\mu^{\boxtimes 3}) \text{III}_3[(\Delta' \boxtimes 1)\Delta \boxtimes (\Delta' \boxtimes 1)\Delta] = \mu(\mu \boxtimes 1^{\boxtimes 2})$$

where III_3 denotes the composition of switching maps corresponding to transpositions (24)(37)(68) and $\Delta' := \tau\Delta$.

This differs from the definition found in [11] only in the use of Δ' instead of Δ . The main examples, set theoretical ones and Hopf algebras, satisfy both definitions.

Proposition 7.10. Let H and μ be as in the Example 7.3. Then, μ defines a ternary self-distributive object in the category of vector spaces.

Proof. One proceeds as in the proof of Lemma 6.1 as follows. We use the Sweedler notation $\Delta(x) = x^{(1)} \otimes x^{(2)}$ and $(\Delta \otimes 1)\Delta(x) = x^{(11)} \otimes x^{(12)} \otimes x^{(2)}$. Then, one computes

$$\begin{aligned}
& \mu\left(\mu\left(x \otimes y^{(1)} \otimes z^{(1)}\right) \otimes z^{(2)} \otimes y^{(2)}\right) \\
&= \mu\left(x \otimes y^{(1)} \otimes \mu\left(z^{(1)} \otimes z^{(2)} \otimes y^{(2)}\right)\right) \\
&= \mu\left(x \otimes y^{(1)} \otimes y^{(2)}\right) \epsilon(z) = x \epsilon(y) \epsilon(z).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
& \mu\left(\mu\left(x_1 \otimes x_4^{(12)} \otimes x_5^{(12)}\right) \otimes \mu\left(x_2 \otimes x_4^{(11)} \otimes x_5^{(11)}\right) \otimes \mu\left(x_3 \otimes x_4^{(2)} \otimes x_5^{(2)}\right)\right) \\
&= \mu\left(x_1 \otimes \mu\left(\mu\left(x_2 \otimes x_4^{(11)} \otimes x_5^{(11)}\right) \otimes x_5^{(12)} \otimes x_4^{(12)}\right) \otimes \mu\left(x_3 \otimes x_4^{(2)} \otimes x_5^{(2)}\right)\right) \\
&= \mu\left(x_1 \otimes x_2 \epsilon\left(x_4^{(1)}\right) \epsilon\left(x_5^{(1)}\right) \otimes \mu\left(x_3 \otimes x_4^{(2)} \otimes x_5^{(2)}\right)\right) \\
&= \mu\left(\mu\left(x_1 \otimes x_2 \otimes x_3\right) \otimes \epsilon\left(x_4^{(1)}\right) \epsilon\left(x_5^{(1)}\right) \left(x_4^{(2)} \otimes x_5^{(2)}\right)\right) \\
&= \mu\left(\mu\left(x_1 \otimes x_2 \otimes x_3\right) \otimes x_4 \otimes x_5\right)
\end{aligned}$$

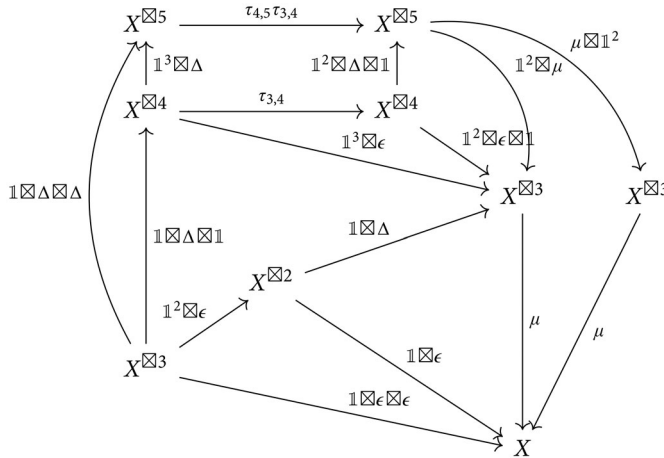
as desired. $\square$

Our goal, next, is to show that a more general version of [Lemma 6.1](#) and [Proposition 7.10](#) holds in an arbitrary symmetric monoidal category. We first have the following preliminary result.

Lemma 7.11. *Let $(X, \Delta, \epsilon, \mu)$ be a heap object in a symmetric monoidal category with tensor product $\boxtimes$ and switching morphism τ . Then, the following identity of morphisms holds*

$$\mu(\mu \boxtimes 1^{\boxtimes 2}) \tau_{4,5} \tau_{3,4} (1 \boxtimes \Delta \boxtimes \Delta) = 1 \boxtimes \epsilon \boxtimes \epsilon.$$

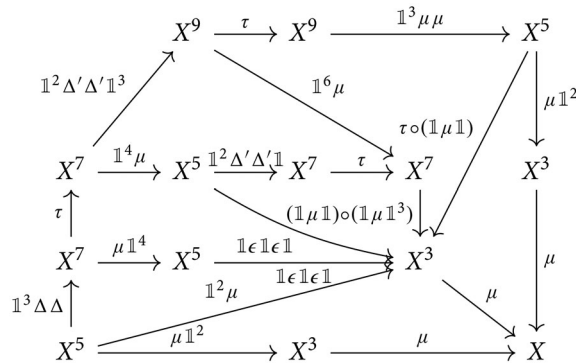
Proof. We observe that the following commutative diagram implies our statement.



where the rectangle on top, and the two triangles below commute because of naturality of the switching morphism, while the other parts of the diagram commute by heap and comonoid axioms. $\square$

Theorem 7.12. *Let $(X, \Delta, \epsilon, \mu)$ be a heap object in a symmetric monoidal category $\mathcal{C}$. Then, $(X, \Delta, \epsilon, \mu)$ is also a ternary self-distributive object in $\mathcal{C}$.*

Proof. Since (X, Δ, ϵ) is a comonoid in $\mathcal{C}$ by hypothesis, we just need to prove that TSD of μ . We use the following commutative diagram



where we have omitted the symbol $\boxtimes$ in the product of morphisms, omitted the subscripts corresponding to the switching morphisms τ , to slightly shorten the notation and, finally, we have used the notation $\circ$ to indicate the composition of morphisms. The leftmost map $\tau : X^7 \rightarrow X^7$ is the composition of symmetry constraints corresponding to the transposition $(5\ 6)(4\ 5)(5\ 6)(4\ 5)(3\ 4)$, proceeding clockwise, $\tau : X^9 \rightarrow X^9$ corresponds to $(3\ 4)(4\ 5)(2\ 3)$. The reader can easily find the correct compositions corresponding to the remaining τ 's by a diagrammatic approach. The triangles on the right and at the bottom, are instances of type 1 and type 0 axioms, respectively. The middle triangle commutes as a consequence of Lemma 7.11. The other diagrams can be seen to be commutative either by applying the comonoid axioms or naturality of the switching morphism. Finally, by direct inspection we see that the upper perimeter of the diagram corresponds to the LHS of TSD, as stated in Definition 7.9. This completes the proof. $\square$

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