

Higher arity self-distributive operations in Cascades and their cohomology

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We investigate constructions of higher arity self-distributive operations, and give relations between cohomology groups corresponding to operations of different arities. For this purpose we introduce the notion of mutually distributive n -ary operations generalizing those for the binary case, and define a cohomology theory labeled by these operations. A geometric interpretation in terms of framed links is described, with the scope of providing algebraic background of constructing 2-cocycles for framed link invariants. This theory is also studied in the context of symmetric monoidal categories. Examples from Lie algebras, coalgebras and Hopf algebras are given.

Keywords: Higher arity self-distributive operation; cohomology; internalization.

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1. Introduction

Self-distributivity of binary operations and its cohomology theories, motivated by knot theory, have been studied extensively in recent years, see for example [3, 5–7, 10, 21]. In particular, in [6], the authors have introduced a (co)homology theory of quandles and utilized the 2-cocycles with abelian coefficients to define a link invariant, called *cocycle invariant*. In [3], it has been given a non-abelian version of these results, and it has been shown that the cocycle invariant is a *quantum* invariant based on previous work of Graña, see [12]. Mutually distributive binary operations were investigated in [22] in which the author proposed a general framework to study homology of distributive structures. Ternary self-distributivity, which is a natural generalization of the binary case, and its cohomology theory have been studied in

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[9, 19]. In this paper, we consider higher arity self-distributivity and its cohomology, which provide an algebraic background suitable to the definition of 2-cocycle framed link invariants.

We generalize the notion of mutual distributive operations given in [22] to n -ary operations. We show that composing mutually distributive operations results in new higher arity self-distributive operations (Proposition 3.1). Specifically, for m -ary and n -ary self-distributive operations W_m and W_n on X , respectively, under the condition called *mutual distributivity* defined in Sec. 3, the composition $W : X^{m+n-1} \rightarrow X$ defined by

$$W(x, \mathbf{y}, \mathbf{z}) = W_n(W_m(x, \mathbf{y}), \mathbf{z})$$

is shown to be an $(m+n-1)$ -ary self-distributive operation. This procedure, in the specific case of mutually distributive binary operations, is particularly proficuous to describe colorings of framed link diagrams by means of ternary operations, paving the way for the possibility of introducing an analogue of the cocycle invariant in the context of framed links. We defer the study of such an invariant to a subsequent work.

We generalize both the n -ary distributive homology [9] and homology of distributive sets [22] to mutually distributive sets of general n -ary operations (Definition 4.1). The relation between this chain complex and the n -ary operations that result from mutually distributive sets by composition as in Definition 5.1, is given in the form of a chain map.

We also present constructions, called *doubling*, for mutually distributive sets, that are similar to the composition introduced in Sec. 3 but defined on the product $X \times X$. A geometric interpretation uses parallel strings and allows us to relate colorings of diagrams of links and colorings of diagrams of framed links. The cohomological counterpart of this procedure, also described in this paper, is expected to provide the algebraic context for the definition of a framed link cocycle invariant. The relation between the doubling and the composition in Sec. 3, as well as its implications to cohomology are discussed.

Higher arity self-distributivity is also investigated in the context of symmetric monoidal categories. We define the notion of n -ary self-distributive object in a symmetric monoidal category, providing therefore a higher arity version of the work in [2]. In particular, we show how to produce ternary self-distributive objects in the category of vector spaces, starting from binary self-distributive objects (Theorem 8.1). Specifically, let (X, Δ) be a comonoid in a (strict) symmetric monoidal category $\mathcal{C}$ (e.g. a coalgebra in the category of vector spaces). Let $q : X \boxtimes X \rightarrow X$ be a morphism such that (X, q) is a binary self-distributive object in $\mathcal{C}$. Then the pair (X, T) , where $T = q(q \boxtimes \mathbb{1})$, defines a ternary self-distributive object in $\mathcal{C}$. The construction defines a functor $\mathcal{F} : \mathcal{BSD} \rightarrow \mathcal{TSD}$, from the category of binary self-distributive objects in $\mathcal{C}$, to the category of ternary self-distributive objects in $\mathcal{C}$. This procedure of internalization of higher order self-distributivity indeed produce interesting examples of self-distributive objects among coalgebras. Examples from

Lie algebras, coalgebras and Hopf algebras are given. As in [2] this categorical version is expected to be related to the Yang–Baxter equation in tensors of vector spaces through Hopf algebras.

The paper is organized in the following manner. Section 2 gives the basics of binary and ternary racks with examples. In Sec. 3, we introduce the higher arity case and show that composing mutually distributive operations results in new higher arity self-distributive operations. In Sec. 4, we define a cohomology theory that generalizes those given in [9, 22] to mutually distributive operations in various arities. A chain map that relates this chain complex and the one for the operation resulted from composition is given in Sec. 5. In Sec. 6, we introduce functors (which we call doubling) in the binary and ternary mutually distributive rack categories. Section 7 shows the passages between binary mutually distributive racks and ternary self-distributive racks. We exhibit a direct construction of ternary cocycles from binary cocycles. We close the circle of functors relating binary and ternary operations by introducing a construction that brings back from ternary to binary. The relations among these functors are also discussed and a geometric interpretation is given. Section 8 is devoted to the development of a categorical point of view of n -ary self-distributivity, extending the previous results of [2]. We define self-distributive objects in symmetric monoidal categories and construct examples in the category of vector spaces. We describe a procedure to obtain higher order self-distributive operations from Lie algebras. Appendix A deals with some detailed computations related to Lie algebras. In Appendix B, we introduce a higher order analogue of augmented rack that enables us to produce Hopf algebra versions of group theoretic examples, such as the heap operation.

2. Preliminary

2.1. Basics of Racks

We review, for the convenience of the reader, some basic definitions of shelves, racks and quandles and give a few examples. This material can be found, for example, in [7, 10, 11, 21, 22].

Definition 2.1. A *shelf* X is a set with a binary operation $(a, b) \mapsto a * b$ such that for any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

If, in addition, the maps $R_y : x \mapsto x * y$ are bijections of X , for all $y \in X$, then $(X, *)$ is called a *rack*.

A *quandle* is an idempotent $(a * a = a, \text{ for all } a \in X)$ rack.

Example 2.1. The following are typical examples of quandles:

- A group G with conjugation as operation: $a * b = b^{-1}ab$, denoted by $X = \text{Conj}(G)$, is a quandle.
- A group G with the operation $a * b = ba^{-1}b$ is a quandle called the *core* quandle.

- Any $\Lambda(=\mathbb{Z}[t, t^{-1}])$ -module M is a quandle with $a * b = ta + (1 - t)b$, for $a, b \in M$, and is called an *Alexander quandle*.
- For any group G , and an automorphism $f \in \text{Aut}(G)$, the operation $x * y = f(xy^{-1})y$ defines a quandle structure on G , usually referred to as *generalized Alexander quandle*.

A rack *homomorphism* $f : (X, *) \rightarrow (X', *')$ is a map satisfying $f(x * y) = f(x) *' f(y)$ for all $x, y \in X$. The category of racks is denoted by $\mathcal{R}$.

Let $(X, *)$ be a rack and A be an abelian group. A function $\phi : X \times X \rightarrow A$ is said to be a (rack) *2-cocycle* if for all $x, y, z \in X$, the following holds:

$$\phi(x, y) + \phi(x * y, z) = \phi(x, z) + \phi(x * z, y * z).$$

Lemma 2.1 ([4]). *Let $(X, *)$ be a rack, A be an abelian group, and $\phi : X \times X \rightarrow A$ be a 2-cocycle. Define an operation $*$ on $X \times A$ by*

$$(x, a) * (y, b) := (x * y, a + \phi(x, y)).$$

*Then $(X \times A, *)$ is a rack.*

Rack and quandle 2-cocycles have been constructed from extensions [4], polynomial expressions [1, 18], determinants [20], and computer calculations [24].

2.2. Ternary distributive structures

Ternary racks and quandles were investigated in [9, 13, 19] and generalized further in [8]. Here, we review the basics of ternary racks and give some examples.

Definition 2.2. Let (X, T) be a set equipped with a ternary operation $T : X \times X \times X \rightarrow X$. The operation T is said to be (*right*) *distributive* if it satisfies the following condition for all $x, y, z, u, v \in X$:

$$T(T(x, y, z), u, v) = T(T(x, u, v), T(y, u, v), T(z, u, v)).$$

In this paper, we will consider distributivity from the right.

Definition 2.3. Let $T : X \times X \times X \rightarrow X$ be a ternary distributive operation on a set X . If for all $a, b \in X$, the map $R_{a,b} : X \rightarrow X$ given by $R_{a,b}(x) = T(x, a, b)$ is invertible, then (X, T) is said to be *ternary rack*.

Example 2.2. The following constructions are found in [9]:

- Let $(X, *)$ be a rack and define a ternary operation on X by $T(x, y, z) = (x * y) * z$, for all $x, y, z \in X$. It is straightforward to see that (X, T) is a ternary rack. Note that in this case $R_{a,b} = R_b \circ R_a$. We will say that this ternary rack is induced by a (binary) rack.

In particular, if $(X, *)$ is an Alexander quandle with $x * y = tx + (1 - t)y$, then the ternary rack coming from X has the operation

$$T(x, y, z) = t^2x + t(1 - t)y + (1 - t)z.$$

- Let M be any Λ -module where $\Lambda = \mathbb{Z}[t^{\pm 1}, s]$. The operation $T(x, y, z) = tx + sy + (1 - t - s)z$ defines a ternary rack structure on M . We call this an *affine* ternary rack.

In particular, consider $\mathbb{Z}_8$ with the ternary operation $T(x, y, z) = 3x + 2y + 4z$. This affine ternary rack given in [9] is not induced by an Alexander quandle structure as described in the preceding item since 3 is not a square in $\mathbb{Z}_8$.

- Any group G with the ternary operation $T(x, y, z) = xy^{-1}z$ gives a ternary rack. This operation is well known and called a *heap* (sometimes also called a *group*) of the group G .

A morphism of ternary racks is a map $f : (X, T) \rightarrow (X', T')$ such that

$$f(T(x, y, z)) = T'(f(x), f(y), f(z)).$$

A bijective ternary rack endomorphism is called ternary rack automorphism. We denote by $\mathcal{T}$ the category of ternary racks.

Let (X, T) be a ternary rack and A be an abelian group. A function $\psi : X \times X \times X \rightarrow A$ is said to be a *ternary 2-cocycle* if for all $x, y, z, u, v \in X$, the following hold:

$$\begin{aligned} & \psi(x, y, z) + \psi(T(x, y, z), u, v) \\ &= \psi(x, u, v) + \psi(T(x, u, v), T(y, u, v), T(z, u, v)). \end{aligned}$$

This equation is motivated by the following lemma, which is verified by calculations.

Lemma 2.2. *Let (X, T) be a ternary rack and A be an abelian group. Let $\phi : X \times X \times X \rightarrow A$ be a map. The set $X \times A$ with the ternary operation given by*

$$T((x, a), (y, b), (z, c)) = (T(x, y, z), a + \psi(x, y, z))$$

is a ternary rack if and only if the map ϕ satisfies the following ternary 2-cocycle condition:

$$\begin{aligned} & \phi(x, y, z) + \phi(T(x, y, z), u, v) \\ &= \phi(x, u, v) + \phi(T(x, u, v), T(y, u, v), T(z, u, v)). \end{aligned}$$

For a ternary distributive operation T on X , we also use the notation

$$x * \mathbf{y} := T(x, y_0, y_1),$$

where $\mathbf{y} = (y_0, y_1)$. Although strictly speaking $T(x, y_0, y_1)$ is not equal to $T(x, (y_0, y_1))$, no confusion is likely to arise by this convention. Furthermore, for $\mathbf{x} = (x_0, x_1)$, we use the notation $\mathbf{x} * \mathbf{y}$ to represent

$$(x_0 * \mathbf{y}, x_1 * \mathbf{y}) = (T(x_0, y_0, y_1), T(x_1, y_0, y_1)).$$

In this notation, the ternary distributivity can be written as

$$(x * \mathbf{y}) * \mathbf{z} = (x * \mathbf{z}) * (\mathbf{y} * \mathbf{z})$$

in analogy to the binary case.

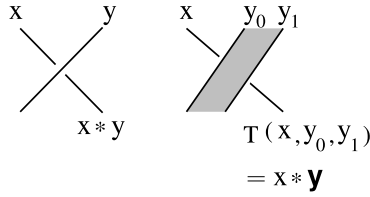


Fig. 1. Diagrammatic representations of a binary (left) and ternary (right) operations.

Figure 1 depicts diagrammatic representations of binary and ternary operations, on the left and on the right, respectively. See [7], for example, for more details on diagrammatics for racks and their knot colorings.

We also recall the definition of homology of ternary racks [9]. Define first $C_n(X)$ to be the free abelian group generated by $(2n+1)$ -tuples $(x_0, x_1, \dots, x_{2n})$ of elements of a ternary rack (X, T) . Define the differentials $\partial_n : C_n(X) \rightarrow C_{n-1}(X)$ as

$$\begin{aligned} \partial_n(x_0, x_1, \dots, x_{2n}) &= \sum_{i=1}^n (-1)^i [(x_0, \dots, \widehat{x_{2i-1}}, \widehat{x_{2i}}, \dots, x_{2n}) \\ &\quad - (T(x_0, x_{2i-1}, x_{2i}), \dots, T(x_{2i-2}, x_{2i-1}, x_{2i}), \widehat{x_{2i-1}}, \widehat{x_{2i}}, \dots, x_{2n})]. \end{aligned}$$

Definition 2.4. The n th homology group of the ternary rack X is defined to be:

$$H_n(X) = \ker \partial_n / \text{im} \partial_{n+1}.$$

By dualizing the chain complex given above, we get a cohomology theory for ternary racks.

Remark 2.1. Similar definitions give a homology and a cohomology theory for higher arity self-distributive operations.

3. Compositions of n -Ary Self-Distributive Operations

In this section, we generalize the notion of mutual distributive operations found in [22] to n -ary operations. The vector notation for ternary operations is directly generalized to the n -ary ones: Let (X, W) be an n -ary distributive set. Let $\mathbf{y} = (y_1, \dots, y_{n-1}) \in X^{n-1}$. Then the operation $W : X^n \rightarrow X$ is denoted by $W(x, y_1, \dots, y_{n-1}) = W(x, \mathbf{y})$. An n -ary operation is also denoted by $x * \mathbf{y} := W(x, \mathbf{y})$. Here the extra parentheses caused by the vector notation is ignored, i.e. for $\mathbf{y} = (y_1, \dots, y_{n-1})$ and $\mathbf{z} = (z_1, \dots, z_{n-1})$, the concatenation $(\mathbf{y}, \mathbf{z})$ or simply $\mathbf{y}, \mathbf{z}$ denotes $(y_1, \dots, y_{n-1}, z_1, \dots, z_{n-1})$. Furthermore, for $\mathbf{x} = (x_1, \dots, x_m) \in X^m$ and $\mathbf{y} \in X^{n-1}$, denote $(W(x_1, \mathbf{y}), \dots, W(x_m, \mathbf{y}))$ by $W(\mathbf{x}, \mathbf{y})$ or $\mathbf{x} * \mathbf{y}$.

Definition 3.1. Let W_m and W_n be m -ary and n -ary distributive operations on X , respectively. The two operations W_m and W_n are called *mutually distributive* if

they satisfy

$$\begin{aligned} W_n(W_m(x, \mathbf{y}), \mathbf{z}) &= W_m(W_n(x, \mathbf{z}), W_n(\mathbf{y}, \mathbf{z})), \\ W_m(W_n(x, \mathbf{u}), \mathbf{v}) &= W_n(W_m(x, \mathbf{v}), W_m(\mathbf{u}, \mathbf{v})) \end{aligned}$$

for all $x \in X$, $\mathbf{y}, \mathbf{v} \in X^{m-1}$ and $\mathbf{z}, \mathbf{u} \in X^{n-1}$.

Example 3.1. Let $(X, *_X)$, $(Y, *_Y)$ be racks. Define $*_0, *_1$ on $X \times Y$, respectively, by

$$\begin{aligned} (x_0, y_0) *_0 (x_1, y_1) &= (x_0 *_X x_1, y_0), \\ (x_0, y_0) *_1 (x_1, y_1) &= (x_0, y_0 *_Y y_1). \end{aligned}$$

Then computation shows that $(*_0, *_1)$ are mutually distributive.

Example 3.2. The following construction appears in [14] and provides examples of mutually distributive rack operations. Denote by $*^n$ the rack operation on X defined by n -fold leftmost product $x *^n y = (\cdots (x * y) * y) * \cdots * y$. Then $*_0 = *^m$ and $*_1 = *^n$ are mutually distributive for positive integers m and n .

More generally, the following appears in [15, 22]. Let X be a group, and let $f_0, f_1 \in \text{Aut}(X)$ be mutually commuting group automorphisms. Let $*_\epsilon$ be the generalized Alexander quandles with respect to f_ϵ for $\epsilon = 0, 1$. Thus $x *_\epsilon y = (xy^{-1})^{f_\epsilon} y$, where the action is denoted in exponential notation. Then computations show that $*_0$ and $*_1$ are mutually distributive.

There are mutually distributive operations with different arities, as the following example shows.

Example 3.3. Let X be a module over $\mathbb{Z}[u^{\pm 1}, t^{\pm 1}, s]$ and $*, T$ be affine binary and ternary rack operations, respectively, defined by

$$\begin{aligned} x * y &= ux + (1 - u)y, \\ T(x, y, z) &= tx + sy + (1 - t - s)z. \end{aligned}$$

Then computations show that $*$ and T are mutually distributive.

Remark 3.1. We note that for a group G , the core binary operation ($x * y = yx^{-1}y$) and the ternary operation heap ($x \hat{*} (y_0, y_1) = xy_0^{-1}y_1$) satisfy $(x * y) \hat{*} \mathbf{z} = (x \hat{*} \mathbf{z}) * (y \hat{*} \mathbf{z})$ but not $(x \hat{*} \mathbf{y}) * z = (x * z) \hat{*} (\mathbf{y} * z)$.

Next, we show that composing mutually distributive operations results in new higher arity self-distributive operations.

Proposition 3.1. Let W_m and W_n be mutually distributive m -ary and n -ary distributive operations on X . Then $W : X^{m+n-1} \rightarrow X$ defined by

$$W(x, \mathbf{y}, \mathbf{z}) = W_n(W_m(x, \mathbf{y}), \mathbf{z})$$

is an $(m + n - 1)$ -ary distributive operation.

Proof. We establish the equality

$$W(W(x, \mathbf{y}, \mathbf{z}), \mathbf{u}, \mathbf{v}) = W(W(x, \mathbf{u}, \mathbf{v}), W(\mathbf{y}, \mathbf{u}, \mathbf{v}), W(\mathbf{z}, \mathbf{u}, \mathbf{v})).$$

We replace $W_n(x, \mathbf{y})$ by the notation $x *_n \mathbf{y}$. Thus we have

$$W(x, \mathbf{y}, \mathbf{z}) := (x *_m \mathbf{y}) *_n \mathbf{z}.$$

Then we compute

$$\begin{aligned} W(W(x, \mathbf{y}, \mathbf{z}), \mathbf{u}, \mathbf{v}) &= [[(x *_m \mathbf{y}) *_n \mathbf{z}] *_m \mathbf{u}] *_n \mathbf{v} \\ &= (((x *_m \mathbf{y}) *_m \mathbf{u}) *_n [\mathbf{z} *_m \mathbf{u}]) *_n \mathbf{v} \\ &= [[[x *_m \mathbf{u}] *_m (\mathbf{y} *_m \mathbf{u})] *_n (\mathbf{z} *_m \mathbf{u})] *_n \mathbf{v} \\ &= [[[x *_m \mathbf{u}] *_m (\mathbf{y} *_m \mathbf{u})] *_n \mathbf{v}] *_n [(\mathbf{z} *_m \mathbf{u}) *_n \mathbf{v}] \\ &= [[[x *_m \mathbf{u}] *_n \mathbf{v}] *_m [(\mathbf{y} *_m \mathbf{u}) *_n \mathbf{v}]] *_n [(\mathbf{z} *_m \mathbf{u}) *_n \mathbf{v}] \\ &= W(W(x, \mathbf{u}, \mathbf{v}), W(\mathbf{y}, \mathbf{u}, \mathbf{v}), W(\mathbf{z}, \mathbf{u}, \mathbf{v})), \end{aligned}$$

where the second and the fifth equalities follow from the mutual distributivity of $*_m$ and $*_n$. This concludes the proof. $\square$

Remark 3.2. Let $(X, *_0, *_1)$ be mutually distributive binary operations. Let T be the TSD operation defined in Proposition 3.1. Then it is written as $T(x, y, z) = (x *_0 y) *_1 z$ for $x, y, z \in X$. We note that the two ternary structures $T(x, y, z) = (x *_0 y) *_1 z$ and $T'(x, y, z) = (x *_1 y) *_0 z$ may not be isomorphic in general as the following example shows.

Consider the set $\mathbb{Z}_3$ with the two binary operations $x *_0 y = x$ and $x *_1 y = 2y - x$. The induced ternary structures $T(x, y, z) = (x *_0 y) *_1 z$ and $T'(x, y, z) = (x *_1 y) *_0 z$ are not isomorphic. In fact, if $f : (\mathbb{Z}_3, T) \rightarrow (\mathbb{Z}_3, T')$ is an isomorphism then for all x, y, z in $\mathbb{Z}_3$, we have $f(T(x, y, z)) = T'(f(x), f(y), f(z))$. Then $f(2z - x) = 2f(y) - f(x)$. One obtains then a contradiction, for example, by setting $x = z = 0$.

Definition 3.2. Let $*_{n_j}$, $j = 1, \dots, k$, be distributive n_j -ary operations on X that are pairwise mutually distributive. Then we call $(X, \{*_n\}_{j=1}^k)$ a *mutually distributive set*.

4. Homology of Mutually Distributive Sets

We generalize both the n -ary distributive homology [9] and homology of distributive sets [22] to mutually distributive sets of general n -ary operations as follows. The relation between this chain complex and the n -ary operations that result from mutually distributive sets as in Proposition 3.1 will be given in Sec. 5 in the form of chain map.

Definition 4.1. Let $(X, \{*_n\}_{j=1}^k)$ be a mutually distributive set. Let $\vec{\epsilon} = (\epsilon_1, \dots, \epsilon_{n-1})$ be a vector such that $\epsilon_i \in \{n_j\}_{j=1}^k$ for $i = 1, \dots, n-1$. Let chain groups $C_n^{\vec{\epsilon}}(X)$ be defined by the free abelian group generated by tuples $\mathbf{x} = (x_0, (\mathbf{x}_1, \epsilon_1), \dots, (\mathbf{x}_{n-1}, \epsilon_{n-1}))$. Define $C_n(X) = \bigoplus_{\vec{\epsilon}} C_n^{\vec{\epsilon}}(X)$ where the direct sum ranges over all possible vectors $\vec{\epsilon}$. Define the differential $\partial_n^{\vec{\epsilon}} : C_n^{\vec{\epsilon}}(X) \rightarrow C_{n-1}(X)$ by

$$\begin{aligned} \partial_n^{\vec{\epsilon}}(\mathbf{x}) = & \sum_{i=1}^{n-1} (-1)^i [(x_0 *_{\epsilon_i} \mathbf{x}_i, (\mathbf{x}_1 *_{\epsilon_i} \mathbf{x}_i, \epsilon_1), \dots, (\mathbf{x}_{i-1} *_{\epsilon_i} \mathbf{x}_i, \epsilon_{i-1}), \widehat{(\mathbf{x}_i, \epsilon_i)}, \\ & (\mathbf{x}_{i+1}, \epsilon_{i+1}), \dots, (\mathbf{x}_{n-1}, \epsilon_{n-1})) \\ & - (x_0, (\mathbf{x}_1, \epsilon_1), \dots, \widehat{(\mathbf{x}_i, \epsilon_i)}, \dots, (\mathbf{x}_{n-1}, \epsilon_{n-1}))], \end{aligned}$$

and let

$$\partial_n = \sum_{\vec{\epsilon}} \partial_n^{\vec{\epsilon}} : C_n(X) \rightarrow C_{n-1}(X).$$

Lemma 4.1. Let $(X, \{*_n\}_{j=1}^k)$ be a mutually distributive set. Then the sequence $(C_n(X), \partial_n)$ defines a chain complex.

Proof. We define, for each vector $\vec{\epsilon}$ and $i = 1, \dots, n-1$, linear maps

$$\begin{aligned} \partial_n^{i\vec{\epsilon}}(\mathbf{x}) = & [(x_0 *_{\epsilon_i} \mathbf{x}_i, (\mathbf{x}_1 *_{\epsilon_i} \mathbf{x}_i, \epsilon_1), \dots, (\mathbf{x}_{i-1} *_{\epsilon_i} \mathbf{x}_i, \epsilon_{i-1}), \widehat{(\mathbf{x}_i, \epsilon_i)}, \\ & (\mathbf{x}_{i+1}, \epsilon_{i+1}), \dots, (\mathbf{x}_{n-1}, \epsilon_{n-1})) \\ & - (x_0, (\mathbf{x}_1, \epsilon_1), \dots, \widehat{(\mathbf{x}_i, \epsilon_i)}, \dots, (\mathbf{x}_{n-1}, \epsilon_{n-1}))]. \end{aligned}$$

Therefore by definition, $\partial_n^{\vec{\epsilon}} = \sum_i (-1)^i \partial_n^{i\vec{\epsilon}}$. It is enough to show now that the maps $\partial_n^{i\vec{\epsilon}}$ satisfy the pre-simplicial complex relation: $\partial_{n-1}^{i\vec{\epsilon}} \partial_n^{j\vec{\epsilon}} = \partial_{n-1}^{j\vec{\epsilon}} \partial_n^{(i+1)\vec{\epsilon}}$ for each $n \in \mathbb{N}$ whenever $j < i$.

Fix a vector $\vec{\epsilon} = (\epsilon_1, \dots, \epsilon_{n-1})$ and consider an element $(x_0, (x_1, \epsilon_1), \dots, (x_{n-1}, \epsilon_{n-1})) \in C_n^{\vec{\epsilon}}(X)$. Then we have

$$\begin{aligned} & \partial_{n-1}^i \partial_n^j (x_0, (\mathbf{x}_1, \epsilon_1), \dots, (\mathbf{x}_{n-1}, \epsilon_{n-1})) \\ &= ((x_0 *_{\epsilon_j} \mathbf{x}_j) *_{\epsilon_{i+1}} \mathbf{x}_{i+1}, ((\mathbf{x}_1 *_{\epsilon_j} \mathbf{x}_j) *_{\epsilon_{i+1}} \mathbf{x}_{i+1}, \epsilon_1), \dots, \widehat{(\mathbf{x}_j, \epsilon_j)}, \\ & (\mathbf{x}_{j+1} *_{\mathbf{x}_{i+1}}, \epsilon_{j+1}), \dots, \widehat{(\mathbf{x}_i, \epsilon_i)}, \dots, (\mathbf{x}_{n-1}, \epsilon_{n-1})) \\ & - (x_0, (\mathbf{x}_1, \epsilon_1), \dots, \widehat{(\mathbf{x}_j, \epsilon_j)}, \\ & (\mathbf{x}_{j+1}, \epsilon_{j+1}), \dots, \widehat{(\mathbf{x}_{i+1}, \epsilon_{i+1})}, \dots, (\mathbf{x}_{n-1}, \epsilon_{n-1})). \end{aligned}$$

Remark 4.4. For a given abelian group A , we obtain a labeled cochain complex with coefficients in A , upon dualizing the chain complex in Definition 4.1. We will write $C_L^n(X; A)$ and $H_L^n(X; A)$ to indicate the labeled n th cochain and cohomology groups with coefficients in A , respectively.

5. Chain Maps Under n -Ary Compositions

In this section, we show that the cohomology of an operation obtained by composing mutually distributive operations as in Proposition 3.1 and the cohomology of the operations themselves are related to the labeled cohomology of Definition 4.1 via chain maps as follows. The algebraic and geometric motivations and significance of the chain maps are explained later in this section.

Definition 5.1. Let $(X, *_0, *_1)$ be a distributive set, where $*_0$ and $*_1$ are operations of arity k and k' , respectively. Call W the $(k + k' - 1)$ -ary corresponding self-distributive operation. We define chain maps $\mathcal{F}_{\sharp, n} : C_n^W(X) \rightarrow C_n^L(X)$, from the $(k + k' - 1)$ -ary cochain complex relative to W , to the chain complex defined by Lemma 4.1 for $n = 1, 2, 3$. Explicitly

$$\begin{aligned}\mathcal{F}_{\sharp, 1} &= \mathbf{1}, \\ \mathcal{F}_{\sharp, 2}(x, \mathbf{y}_0, \mathbf{y}_1) &= (x, \mathbf{y}_0)_0 + (x *_0 \mathbf{y}_0, \mathbf{y}_1)_1, \\ \mathcal{F}_{\sharp, 3}(x, \mathbf{y}_0, \mathbf{y}_1, \mathbf{z}_0, \mathbf{z}_1) &= (x, \mathbf{y}_0, \mathbf{z}_0)_{00} + (x *_0 \mathbf{z}_0, \mathbf{y}_0 *_0 \mathbf{z}_0, \mathbf{z}_1)_{01} \\ &\quad + (x *_0 \mathbf{y}_0, \mathbf{y}_1, \mathbf{z}_0)_{10} + ((x *_0 \mathbf{y}_0) *_0 \mathbf{z}_0, \mathbf{y}_1 *_0 \mathbf{z}_0, \mathbf{z}_1)_{11},\end{aligned}$$

where we put the labels as a subscript and $\mathbf{y}_i, \mathbf{z}_i$ are vectors of appropriate lengths, according to the conventions explained in Sec. 3.

Definition 5.2. Let $(X, *_0, *_1)$ be a mutually distributive racks and A be an abelian group. Let $\mathcal{F}^{\sharp, n} : C_L^n(X, A) \rightarrow C_W^n(X, A)$ for $n = 2, 3$ be the maps obtained from $\mathcal{F}_{\sharp, n}$ by dualization.

Theorem 5.1. For $n = 2, 3$ the maps $\mathcal{F}_{\sharp, n}$ define chain maps. Therefore they define induced homomorphisms $\mathcal{F}_{*, n} : H_n^W(X, A) \rightarrow H_n^L(X, A)$ in homology and $\mathcal{F}^{*, n} : H_L^n(X, A) \rightarrow H_W^n(X, A)$ in cohomology.

Proof. We prove the statement in the case of two binary mutually distributive operations $*_0$ and $*_1$, resulting in a ternary self-distributive operation T . The general case being an application of the same procedure with vector notation. For a ternary 2-chain (x, y_0, y_1) we have

$$\partial \mathcal{F}_{\sharp, 2}(x, y_0, y_1) = -(x *_0 y_0) + (x) - ((x *_0 y_0) *_1 y_1) + (x *_0 y_0) = \partial_T(x, y_0, y_1).$$

By direct computation, we also have

$$\begin{aligned}
 \mathcal{F}_{\sharp,2}\partial_T(x, y_0, y_1, z_0, z_1) \\
 &= (x, z_0)_0 + (x *_0 z_0, z_1)_1 - (T(x, y_0, y_1), z_0)_0 \\
 &\quad - (T(x, y_0, y_1) *_0 z_0, z_1)_1 - (x, y_0)_0 - (x *_0 y_0, y_1)_1 \\
 &\quad + (T(x, z_0, z_1), T(y_0, z_0, z_1))_0 \\
 &\quad + (T(x, z_0, z_1) *_0 T(y_0, z_0, z_1), T(y_1, z_0, z_1))_1.
 \end{aligned}$$

On the other hand, the following holds:

$$\begin{aligned}
 \partial\mathcal{F}_{\sharp,3}(x, y_0, y_1, z_0, z_1) \\
 &= (x, z_0)_0 - (x *_0 y_0, z_0)_0 - (x, y_0)_0 \\
 &\quad + (x *_0 z_0, y_0 *_0 z_0)_0 + (x *_0 z_0, z_1)_1 - ((x *_0 z_0) *_0 (y_0 *_0 z_0), z_1)_1 \\
 &\quad - (x *_0 z_0, y_0 *_0 z_0)_0 + (T(x, z_0, z_1), T(y_0, z_0, z_1))_0 \\
 &\quad + (x *_0 y_0, z_0)_0 - (T(x, y_0, y_1), z_0)_0 \\
 &\quad - (x *_0 y_0, y_1)_1 + ((x *_0 y_0) *_0 z_0, y_1 *_0 z_0)_1 \\
 &\quad + ((x *_0 y_0) *_0 z_0, z_1)_1 - (((x *_0 y_0) *_0 z_0) *_1 (y_1 *_0 z_0), z_1)_1 \\
 &\quad - ((x *_0 y_0) *_0 z_0, y_1 *_0 z_0)_1 + (((x *_0 y_0) *_0 z_0) *_1 z_1, T(y_1, z_0, z_1))_1.
 \end{aligned}$$

The two quantities can be seen to be equal, making use of the identity:

$$T(x, z_0, z_1) *_0 T(y_0, z_0, z_1) = ((x *_0 y_0) *_0 z_0) *_1 z_1.$$

Therefore, we obtain $\mathcal{F}_{\sharp,2}\partial_T = \partial\mathcal{F}_{\sharp,3}$, which concludes the proof of the first statement. The second statement follows easily from the first one by standard arguments in homological algebra. $\square$

Remark 5.1. Let $(X, *_0, *_1)$ be a mutually distributive rack and let $C_L^2(X; A)$ be the second labeled cochain group with coefficients in A , then the labeled 2-cocycle conditions corresponding to $\delta^{(01)}\psi = 0$ and $\delta^{(10)}\psi = 0$ take the following form:

$$\begin{aligned}
 \phi_0(x, y) + \phi_1(x *_0 y, z) &= \phi_1(x, z) + \phi_0(x *_1 z, y *_1 z), \\
 \phi_1(x, y) + \phi_0(x *_1 y, z) &= \phi_0(x, z) + \phi_1(x *_0 z, y *_0 z).
 \end{aligned}$$

Definition 5.3. We call a pair (ϕ_0, ϕ_1) satisfying the preceding equations *mutually distributive*.

Observe that (ϕ_0, ϕ_1) being a labeled 2-cocycle means that it is mutually distributive, ϕ_0 is a 2-cocycle for the operation $*_0$ and ϕ_1 is a 2-cocycle for $*_1$.

Remark 5.2. Let $(X, *_0, *_1)$ be mutually distributive binary operations and T be the ternary self-distributive operation defined in Proposition 3.1. Let A be an

abelian group. By Theorem 5.1, $\mathcal{F}^{\sharp,2}(\phi_0, \phi_1) = \psi$ is a ternary 2-cocycle in $C_T^2(X, A)$ for a labeled 2-cocycle (ϕ_0, ϕ_1) . The explicit form of the ternary 2-cocycle

$$\psi(x, y, z) := \phi_0(x, y) + \phi_1(x *_0 y, z).$$

Remark 5.3. The case of mutually distributive binary operations whose composition gives a ternary operation is of particular interest to us since this is the algebraic counterpart of a diagrammatic doubling procedure particularly adapt to interpret colorings of framed tangles by ternary racks. The ternary 2-cocycles resulting from Theorem 5.1 can therefore be used to define cocycle invariants for framed tangles. This construction of cocycles corresponds to those in [14] for handlebody-links.

From this geometric point of view, we present a direct, geometric proof that ψ in Remark 5.2 satisfies the ternary 2-cocycle condition. We show

$$\begin{aligned} & \psi(x, y_0, y_1) + \psi(T(x, y_0, y_1), z_0, z_1) \\ &= \psi(x, z_0, z_1) + \psi(T(x, z_0, z_1), T(y_0, z_0, z_1), T(y_1, z_0, z_1)). \end{aligned}$$

The computations below are aided by diagrams shown in Fig. 3, where each equality is represented by a type III Reidemeister move. In the figure and the computations below, underlines highlight those terms to which the cocycle condition is applied.

$$\begin{aligned} \text{LHS} &= \phi_0(x, y_0) + \underline{\phi_1(x *_0 y_0, y_1)} + \phi_0(((x *_0 y_0) *_1 y_1), z_0) \\ &\quad + \phi_1(((x *_0 y_0) *_1 y_1) *_0 z_0, z_1) \\ &= \underline{\phi_0(x, y_0) + \phi_0(x *_0 y_0, z_0)} + \phi_1(((x *_0 y_0) *_1 y_1) *_0 z_0, z_1) \\ &\quad + \phi_1((x *_0 y_0) *_0 z_0, y_1 *_0 z_0) \\ &= \phi_0(x, z_0) + \phi_0(x *_0 z_0, y_0 *_0 z_0) + \underline{\phi_1(((x *_0 y_0) *_0 z_0) *_0 z_0, y_1 *_0 z_0)} \\ &\quad + \underline{\phi_1(((x *_0 y_0) *_1 y_1) *_0 z_0, z_1)} \\ &= \phi_0(x, z_0) + \underline{\phi_0(x *_0 z_0, y_0 *_0 z_0)} + \phi_1(((x *_0 y_0) *_0 z_0), z_1) \\ &\quad + \phi_1(((x *_0 y_0) *_0 z_0) *_1 z_1, (y_1 *_0 z_0) *_1 z_1) \\ &= \phi_0(x, z_0) + \phi_1(x *_0 z_0, z_1) \\ &\quad + \phi_1(((x *_0 y_0) *_0 z_0) *_1 z_1, (y_1 *_0 z_0) *_1 z_1) \\ &\quad + \phi_0(((x *_0 y_0) *_0 z_0) *_1 z_1, (y_1 *_0 z_0) *_1 z_1) = \text{RHS} \end{aligned}$$

Example 5.1. Let $(X, *_X)$, $(Y, *_Y)$ be racks, and $(*_0, *_1)$ be mutually distributive operations defined on $X \times Y$ in Example 3.1. Let ϕ_X and ϕ_Y be 2-cocycles of $(X, *_X)$ and $(Y, *_Y)$, respectively. Define 2-cocycles of $X \times Y$ corresponding to $*_0$, $*_1$, respectively, by $\phi_0((x_0, y_0), (x_1, y_1)) = \phi_X(x_0, x_1)$ and $\phi_1((x_0, y_0), (x_1, y_1)) = \phi_Y(y_0, y_1)$. Then computations show that (ϕ_0, ϕ_1) are mutually distributive.

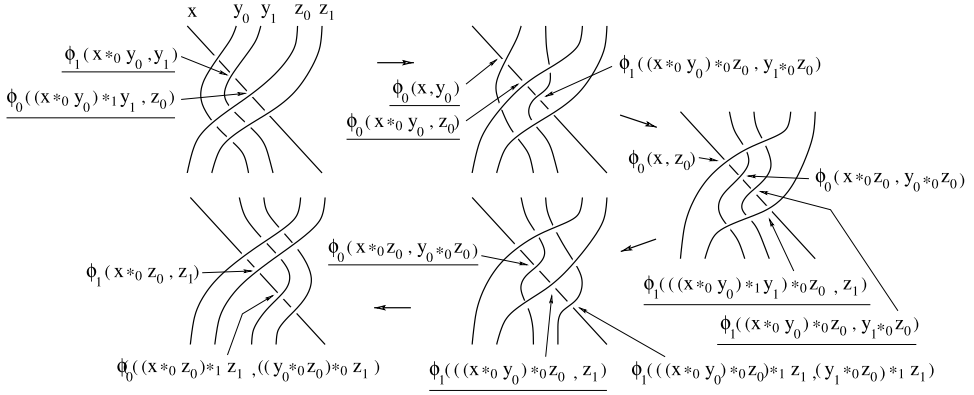


Fig. 3. Diagrammatic proof of 2-cocycle conditions.

Example 5.2. The following construction, found in [14], provides examples of mutually distributive 2-cocycles. Let $(X, *)$ be a rack, $\phi : X \times X \rightarrow A$ be a 2-cocycle, and $(E = X \times A, \tilde{*})$ be the corresponding extension. Recall that $*^n$ denotes the n -fold leftmost product $x *^n y = (\cdots (x * y) * y) * \cdots * y$. Then the function ϕ_n defined by

$$\phi_n(x, y) = \phi(x, y) + \phi(x * y, y) + \cdots + \phi(x *^{n-1} y, y)$$

is a 2-cocycle.

Let $(X, *_0 = *^m, *_1 = *^n)$ be the mutually distributive rack defined in Example 3.2, and let ϕ_m, ϕ_n be 2-cocycles defined above. Then ϕ_m and ϕ_n are mutually distributive. This is seen by the diagrammatic interpretation of parallel strings.

6. The Doubling Functor

In this section, we describe a construction called *doubling*, that is similar to the composition defined in Sec. 3 but defined on the product $X \times X$. A diagrammatic interpretation is to take parallel strings and provides a method of producing cocycle invariants for framed links by means of ternary cohomology. The relation between the doubling and the composition in Sec. 3 as well as implications to cohomology are discussed in the next section.

6.1. Doubling binary operations

Lemma 6.1. *Let $(X, *_0, *_1)$ be mutually distributive racks. Define the operation for $(x_0, x_1), (y_0, y_1) \in X \times X$ by*

$$(x_0, x_1) * (y_0, y_1) := ((x_0 *_0 y_0) *_1 y_1, (x_1 *_0 y_0) *_1 y_1).$$

*Then $(X \times X, *)$ is a rack.*

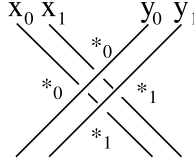


Fig. 4. Diagrammatic representation of doubling.

A diagrammatic representation of the preceding lemma is depicted in Fig. 4, and the computations in its proof are facilitated by the corresponding type III Reidemeister move with doubled strings.

Definition 6.1. Let $\mathcal{R}_M$ be the category defined as follows. The objects consist of $(X, *_0, *_1)$, where X is a set and $(*_0, *_1)$ is mutually distributive. For objects $(X, *_0, *_1)$ and $(X', *_0', *_1')$, a morphism f is a map $f : X \rightarrow X'$ that is a rack morphism for both $(*_0, *_0')$ and $(*_1, *_1')$.

We observe that if $f : X \rightarrow X'$ is a morphism in the sense of this definition, then f will automatically respect the mutual distributivity. Specifically, simple computations imply the following.

Lemma 6.2. *If $f : (X, *_0, *_1) \rightarrow (X', *_0', *_1')$ is a morphism in $\mathcal{R}_M$, then it holds that*

$$f((x *_0 y) *_1 z) = (f(x) *_1' f(z)) *_0' (f(y) *_1' f(z)).$$

Computations also show the following.

Lemma 6.3. *Let $(X, *_0, *_1)$ and $(X', *_0', *_1')$ be two mutually distributive racks, and $(X \times X, *)$ and $(X' \times X', *')$ be racks as in Lemma 6.1. If $f : (X, *_0, *_1) \rightarrow (X', *_0', *_1')$ is a morphism in $\mathcal{R}_M$, then the map $F : (X \times X, *) \rightarrow (X' \times X', *')$ defined by $F(x, y) = (f(x), f(y))$ is a rack morphism.*

Definition 6.2. The functor $\mathcal{D}_R$ from $\mathcal{R}_M$ to the category $\mathcal{R}$ of binary racks defined on objects by $\mathcal{D}_R(X, *_0, *_1) = (X \times X, *)$ through Lemma 6.1 and on morphisms by $\mathcal{D}_R(f) = f \times f$ through Lemma 6.3, is called the *doubling* functor.

Remark 6.1. The functor $\mathcal{D}_R$ is injective on objects and morphisms, but not surjective on either.

A direct computation gives the following lemma.

Lemma 6.4. *Let $(X, *_0, *_1)$ be a mutually distributive rack, and (ϕ_0, ϕ_1) be mutually distributive rack 2-cocycles. Let $(E, \tilde{*}_\epsilon)$ be abelian extensions of $(X, *_\epsilon)$ with respect to ϕ_ϵ ,*

$$(x, a) \tilde{*}_\epsilon (y, b) = (x *_\epsilon y, a + \phi_\epsilon(x, y))$$

for $\epsilon = 0, 1$. Then $(E, \tilde{}_0, \tilde{*}_1)$ is a mutually distributive rack.*

Theorem 6.1. *Let $(X, *_0, *_1)$ and $(X \times X, *)$ be as described in Lemma 6.1. Let ϕ_0, ϕ_1 be rack 2-cocycles of $(X, *_0)$ and $(X, *_1)$, respectively, that satisfy the mutually distributive rack 2-cocycle condition. Then*

$$\phi((x_0, x_1), (y_0, y_1)) = \phi_0(x_0, y_0) + \phi_1(x_0 *_0 y_0, y_1) + \phi_0(x_1, y_0) + \phi_1(x_1 *_0 y_0, y_1)$$

*is a rack 2-cocycle of $(X \times X, *)$. This assignment induces a well defined map $\Theta : H_L^2(X) \longrightarrow H_R^2(X \times X)$, where the subscript R indicates the binary rack cohomology.*

A proof will be given at the end of Sec. 7. The right-hand side corresponds to Fig. 4. We call ϕ the *doubled rack 2-cocycle*.

6.2. Doubling ternary operations

In this section, we give a doubling construction for ternary racks. The condition required for this construction differs from the mutual distributivity and defined as follows.

Definition 6.3. Let T_0 and T_1 be two ternary operations on a set X . We say that T_0 and T_1 are *compatible* if they satisfy

$$\begin{aligned} & T_0(T_0(x_0, y_0, y_1), z_0, z_1) \\ &= T_0(T_0(x_0, z_0, z_1), T_0(y_0, z_0, z_1), T_1(y_1, z_0, z_1)), \\ & T_1(T_1(x_1, y_0, y_1), z_0, z_1) \\ &= T_1(T_1(x_1, z_0, z_1), T_0(y_0, z_0, z_1), T_1(y_1, z_0, z_1)). \end{aligned}$$

A diagrammatic representation of the compatibility is depicted in Fig. 5. Observe that it corresponds to type III Reidemeister move for ribbons.

Example 6.1. Consider a Λ -module M where $\Lambda = \mathbb{Z}[t^{\pm 1}, t'^{\pm 1}, s, s']$. The following two ternary operation $T_0(x, y, z) = tx + sy + (1 - t - s)z$ and $T_1(x, y, z) = t'x + s'y + (1 - t' - s')z$ are compatible if and only if the following conditions hold:

$$\begin{cases} (1 - t - s)(t' - t) = 0, \\ (1 - t - s)(s' - s) = 0, \end{cases} \quad \text{and} \quad \begin{cases} (1 - t' - s')(t - t') = 0, \\ (1 - t' - s')(s - s') = 0. \end{cases}$$

For example, one can choose $M = \mathbb{Z}_8$ with $T_0(x, y, z) = 3x + 2y + 4z$ and $T_1(x, y, z) = -x + 2y$.

Definition 6.4. The category $\mathcal{T}_C$ of compatible ternary distributive racks is defined as follows. The objects consist of triples (X, T_0, T_1) where X is a set and (T_0, T_1) are compatible ternary operations on X . A morphism between two objects (X, T_0, T_1) and (Y, T'_0, T'_1) is a map $f : X \rightarrow Y$ which is morphism in the ternary category for both (T_0, T'_0) and (T_1, T'_1) .

We observe that if $f : X \rightarrow X'$ is a morphism in the sense of Definition 6.4, then it will automatically respect the mutual ternary distributivity. Specifically, computations imply the following.

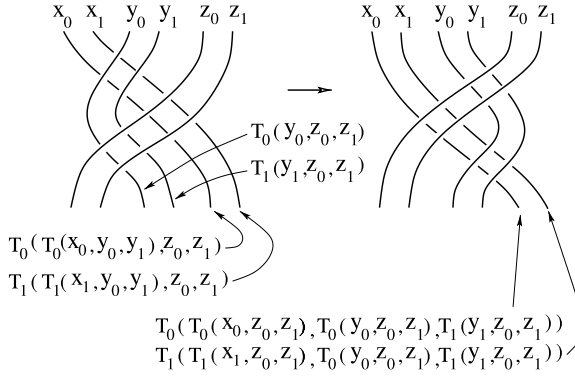


Fig. 5. Diagrammatic representation of compatible ternary rack operations.

Lemma 6.5. *If $f : (X, T_0, T_1) \rightarrow (X', T'_0, T'_1)$ is a morphism in $\mathcal{T}_C$, then it holds that*

$$\begin{aligned} & f(T_0(T_0(x_0, y_0, y_1), z_0, z_1)) \\ &= T'_0(T'_0(f(x_0), f(z_0), f(z_1)), T'_0(f(y_0), f(z_0), f(z_1)), T'_1(f(y_1), f(z_0), f(z_1))), \\ & f(T_1(T_1(x_0, y_0, y_1), z_0, z_1)) \\ &= T'_1(T'_1(f(x_0), f(z_0), f(z_1)), T'_0(f(y_0), f(z_0), f(z_1)), T'_1(f(y_1), f(z_0), f(z_1))). \end{aligned}$$

Theorem 6.2. *Let (T_0, T_1) be compatible ternary distributive operations on X . Then $T : X^2 \times X^2 \times X^2 \rightarrow X^2$ defined by*

$$\begin{aligned} & T((x_0, x_1), (y_0, y_1), (z_0, z_1)) \\ &= (T_0(T_0(x_0, y_0, y_1), z_0, z_1), T_1(T_1(x_1, y_0, y_1), z_0, z_1)) \end{aligned}$$

is a ternary distributive operation on X^2 .

Proof. It is enough to establish

$$\begin{aligned} & T(T((x_0, x_1), (y_0, y_1), (z_0, z_1)), (u_0, u_1), (v_0, v_1)) \\ &= T(T((x_0, x_1), (u_0, u_1), (v_0, v_1)), \\ & T((y_0, y_1), (u_0, u_1), (v_0, v_1)), T((z_0, z_1), (u_0, u_1), (v_0, v_1))). \end{aligned}$$

A diagrammatic representation of this equality is depicted in Fig. 6. This diagrammatic equality follows from a sequence of moves depicted in Fig. 5. Thus calculations are obtained by applications of defining relations of compatibility accordingly. $\square$

The following is analogous to Lemma 6.3 and is shown by direct computations.

Lemma 6.6. *Let (X, T_0, T_1) and (X', T'_0, T'_1) be sets with mutually distributive ternary operations, and $(X \times X, T)$ and $(X' \times X', T')$ be ternary distributive racks*

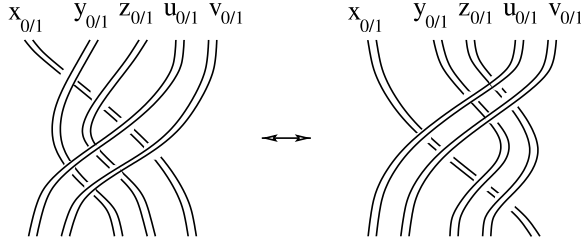


Fig. 6. Diagrammatic representation of Theorem 6.2.

constructed in Theorem 6.2. If $f : (X, T_0, T_1) \rightarrow (X', T'_0, T'_1)$ is a morphism in $\mathcal{T}_C$, then F defined from f by $f \times f$ is a morphism of $\mathcal{T}_C$.

Definition 6.5. We denote the functor from $\mathcal{T}_C$ to the category of ternary racks defined on objects by $\mathcal{D}_T(X, T_0, T_1) = (X \times X, T)$ and on morphisms by $\mathcal{D}_T(f) = f \times f$, and call it *doubling*.

Remark 6.2. The functor $\mathcal{D}_T$ is injective on both objects and morphisms, but is not surjective on either.

Definition 6.6. Let (T_0, T_1) be compatible ternary distributive operations on X . Let ψ_0, ψ_1 be 2-cocycles with respect to T_0 and T_1 , respectively. Then the following are called the compatibility conditions for ψ_0 and ψ_1 :

$$\begin{aligned} & \psi_0(x_0, y_0, y_1) + \psi_1(T_1(x_1, y_0, y_1), z_0, z_1) \\ &= \psi_1(x_1, z_0, z_1) + \psi_0(T_0(x_0, z_0, z_1), T_0(y_0, z_0, z_1), T_1(y_1, z_0, z_1)), \\ & \psi_1(x_1, y_0, y_1) + \psi_0(T_0(x_0, y_0, y_1), z_0, z_1) \\ &= \psi_0(x_0, z_0, z_1) + \psi_1(T_1(x_0, z_0, z_1), T_0(y_0, z_0, z_1), T_1(y_1, z_0, z_1)). \end{aligned}$$

Theorem 6.3. Let (T_0, T_1) be compatible ternary distributive operations on X . Let T be the doubled ternary operation defined in Theorem 6.2. Let ψ_0, ψ_1 be 2-cocycles with respect to T_0 and T_1 , respectively, that satisfy the compatibility condition defined in Definition 6.6. Then

$$\begin{aligned} & \psi((x_0, x_1), (y_0, y_1), (z_0, z_1)) \\ &= \psi_0(x_0, y_0, y_1) + \psi_1(x_1, y_0, y_1) \\ & \quad + \psi_0(T_0(x_0, y_0, y_1), z_0, z_1) + \psi_1(T_1(x_1, y_0, y_1), z_0, z_1) \end{aligned}$$

is a ternary rack 2-cocycle of $(X \times X, T)$.

A proof will be given at the end of Sec. 7. We call ψ the *doubled ternary rack 2-cocycle*.

7. From Binary Racks to Ternary Racks and Back

In this section, we provide relations among constructions of self-distributive operations discussed so far. To simplify the arguments, we focus on binary and ternary operations. Specifically, we observe that the doubling functors of binary (respectively, ternary) operations factor through ternary (respectively, binary) operations. This main result of the section is stated in Proposition 7.2. Furthermore corresponding constructions of 2-cocycles are given, and proofs of Theorems 6.1 and 6.3 are provided at the end of the section. We start with defining a functor for the construction given in Proposition 3.1.

Definition 7.1. The assignment of objects defined by Proposition 3.1 when W_n and W_m are binary operations (hence the obtained $W := T$ is ternary), is denoted $\mathcal{F}(X, *_0, *_1) = (X, T)$. This assignment on objects can be extended on morphisms as the identity, to define a functor $\mathcal{F} : \mathcal{R}_M \rightarrow \mathcal{T}$, from the category of mutually distributive binary racks (see Section 6), to the category of ternary racks, using Lemma 6.2.

By definition $\mathcal{F}$ is injective and surjective on morphisms. Computations give the following.

Lemma 7.1. *Let $\{*, *_0, *_1\}$ be a mutually distributive binary set. Let $(X, T) = \mathcal{F}(X, *_0, *_1)$. Then $\{*, T\}$ are mutually distributive.*

Next we define the opposite construction of binary from ternary operations.

Lemma 7.2. *Let T_0 and T_1 be a two compatible ternary rack operations. Then the binary operation on the cartesian product $X \times X$ defined by*

$$(x_0, x_1) * (y_0, y_1) := (T_0(x_0, y_0, y_1), T_1(x_1, y_0, y_1)) = (x_0 *_0 y, x_1 *_1 y)$$

*gives a rack structure $(X \times X, *)$.*

Definition 7.2. The functor defined by Lemma 7.2 is denoted by $\mathcal{G} : \mathcal{T}_C \rightarrow \mathcal{R}$, where $\mathcal{G}(X, T_0, T_1) = (X \times X, *)$ on objects, and $\mathcal{G}(f) = f \times f$ on morphisms.

Observe that $\mathcal{G}$ is injective on objects and on morphisms.

Proposition 7.1. *The functor $\mathcal{G}$ is not surjective on objects.*

Proof. Consider the binary rack structure on $\mathbb{Z} \times \mathbb{Z}$ defined by

$$(x_0, x_1) * (y_0, y_1) = (x_0 + x_1, x_1).$$

This rack is not in the image of $\mathcal{G}$ since the first entry depends on both x_0 and x_1 . □

Theorem 7.1. *Let (X, T_0, T_1) be an object in $\mathcal{T}_C$, and $(X \times X, *) = \mathcal{G}(X, T_0, T_1)$ be as in Lemma 7.2. Suppose ψ_0 and ψ_1 are compatible ternary 2-cocycles of*

respectively (X, T_0) and (X, T_1) . Then

$$\phi((x_0, x_1), (y_0, y_1)) := \psi_0(x_0, y_0, y_1) + \psi_1(x_1, y_0, y_1)$$

defines a 2-cocycle ϕ of $(X \times X, *)$.

Proof. We check that ϕ satisfies the following equation:

$$\begin{aligned} & \phi((x_0, x_1), (y_0, y_1)) + \phi((x_0, x_1) * (y_0, y_1), (z_0, z_1)) \\ &= \phi((x_0, x_1), (z_0, z_1)) + \phi((x_0, x_1) * (z_0, z_1), (y_0, y_1) * (z_0, z_1)). \end{aligned}$$

We have

$$\begin{aligned} \text{LHS} &= \psi_0(x_0, y_0, y_1) + \psi_1(x_1, y_0, y_1) \\ &\quad + \psi_0(T_0(x_0, y_0, y_1), z_0, z_1) + \psi_1(T_1(x_1, y_0, y_1), z_0, z_1), \\ \text{RHS} &= \psi_0(x_0, z_0, z_1) + \psi_1(x_1, z_0, z_1) \\ &\quad + \psi_0(T_0(x_0, z_0, z_1), T_0(y_0, z_0, z_1), T_1(y_1, z_0, z_1)) \\ &\quad + \psi_1(T_1(x_1, z_0, z_1), T_0(y_0, z_0, z_1), T_1(y_1, z_0, z_1)). \end{aligned}$$

The compatibility conditions of ψ_0 and ψ_1 show that LHS and RHS coincide. $\square$

The constructions are summarized as follows.

Proposition 7.2. *It holds that $\mathcal{G} \circ \mathcal{F} = \mathcal{D}_R$ and $\mathcal{F} \circ \mathcal{G} = \mathcal{D}_T$.*

Proof. Let $(X, *_0, *_1)$ be a set with mutually distributive rack operations. Let $(X, T) = \mathcal{F}(X, *_0, *_1)$. Then by definition $T(x, y_0, y_1) = (x *_0 y_0) *_1 y_1$. Lemma 7.2 implies that $(X \times X, *) = \mathcal{G}(X, T, T)$ is a rack, since T is mutually distributive over itself. One computes

$$\begin{aligned} \mathcal{G}(X, T, T) &= (x_0, x_1) * (y_0, y_1) \\ &= (T(x_0, y_0, y_1), T(x_1, y_0, y_1)) \\ &= ((x_0 *_0 y_0) *_1 y_1, (x_1 *_0 y_0) *_1 y_1) \\ &= \mathcal{D}_R(X, *_0, *_1) \end{aligned}$$

as desired.

Let (X, T_0, T_1) be a set with mutually distributive ternary rack operations. Let $(X \times X, *) = \mathcal{G}(X, T_0, T_1)$. Then by definition $(x_0, x_1) * (y_0, y_1) = (T_0(x_0, y_0, y_1), T_1(x_1, y_0, y_1))$. Since $*$ is mutually distributive over itself, we have

that $(X \times X, T) = \mathcal{F}(X \times X, *, *)$ is a rack, as in Definition 7.1. One computes

$$\begin{aligned} \mathcal{F}(X \times X, *, *) &= T((x_0, x_1), (y_0, y_1), (z_0, z_1)) \\ &= [(x_0, x_1) * (y_0, y_1)] * (z_0, z_1) \\ &= (T_0(x_0, y_0, y_1), T_1(x_1, y_0, y_1)) * (z_0, z_1) \\ &= (T_0(T_0(x_0, y_0, y_1), z_0, z_1), T_1(T_1(x_1, y_0, y_1), z_0, z_1)) \\ &= \mathcal{D}_T(X, T_0, T_1) \end{aligned}$$

as desired. $\square$

Proof of Theorem 6.1. Let $(*_0, *_1)$ be mutually distributive rack operations on X . Let $(X, T) = \mathcal{F}(X, *_0, *_1)$. We have that (X, T) is a ternary rack. Let ϕ_0, ϕ_1 be mutually distributive rack 2-cocycles of $(X, *_0)$ and $(X, *_1)$, respectively. Then by Theorem 7.1,

$$\psi(x, y_0, y_1) := \phi_0(x, y_0) + \phi_1(x *_0 y_0, y_1)$$

is a ternary rack 2-cocycle of (X, T) . Since T is compatible over itself,

$$\begin{aligned} (\mathcal{G} \circ \mathcal{F})(X, *_0, *_1)((x_0, x_1), (y_0, y_1), (z_0, z_1)) \\ &= \mathcal{G}(X \times X, T, T)((x_0, x_1), (y_0, y_1), (z_0, z_1)) \\ &= (T(T(x_0, y_0, y_1), z_0, z_1), T(T(x_1, y_0, y_1), z_0, z_1)) \end{aligned}$$

is a rack operation by Theorem 6.2. Then Theorem 7.1 applied to $(X \times X, T, T)$ with mutually distributive cocycles (ψ, ψ) implies that

$$\begin{aligned} \phi((x_0, x_1), (y_0, y_1)) \\ &= \psi(x_0, y_0, y_1) + \psi(x_1, y_0, y_1) \\ &= \phi_0(x, y_0) + \phi_1(x *_0 y_0, y_1) + \phi_0(x_1, y_0) + \phi_1(x_1 *_0 y_0, y_1) \end{aligned}$$

as desired. To show that the assignment $\Theta(\phi_0, \phi_1) = \psi$ passes to cohomology, it is enough to show that if $(\phi_0, \phi_1) = \delta_L^1 f$, we have that $\Theta(\delta_L^1 f) = \delta_R^1 g$, for some 1-cochain g . It is easy to see that the map $g(x_0, x_1) := f(x_0) + f(x_1)$ does indeed serve the purpose. $\square$

Proof of Theorem 6.3. Let (T_0, T_1) be compatible ternary distributive operations on X , and $(X \times X, *) = \mathcal{G}(X, T_0, T_1)$. By Lemma 7.2, $(X \times X, *)$ is a rack. Let ψ_0, ψ_1 be compatible ternary 2-cocycles of (X, T_0) and (X, T_1) , respectively. Then by Theorem 7.1,

$$\phi((x_0, x_1), (y_0, y_1)) := \psi_0(x_0, y_0, y_1) + \psi_1(x_1, y_0, y_1)$$

is a rack 2-cocycle of $(X \times X, *)$. Since $*$ is mutually distributive over itself,

$$\begin{aligned} & (\mathcal{F} \circ \mathcal{G})(X, T_0, T_1)((x_0, x_1), (y_0, y_1), (z_0, z_1)) \\ &= T((x_0, x_1), (y_0, y_1), (z_0, z_1)) \\ &= [(x_0, x_1) * (y_0, y_1)] * (z_0, z_1) \end{aligned}$$

is a ternary rack operation by Lemma 6.2. Then Theorem 5.1 applied to $(X \times X, *, *)$ with mutually distributive cocycles (ϕ, ϕ) implies that

$$\begin{aligned} & \psi((x_0, x_1), (y_0, y_1), (z_0, z_1)) \\ &= \phi((x_0, x_1), (y_0, y_1)) + \phi((x_0, x_1) * (y_0, y_1), (z_0, z_1)) \\ &= \phi((x_0, x_1), (y_0, y_1)) + \phi((T_0(x_0, y_0, y_1), T_1(x_1, y_0, y_1)), (z_0, z_1)) \\ &= \psi_0(x_0, y_0, y_1) + \psi_1(x_1, y_0, y_1) \\ &\quad + \psi_0((T_0(x_0, y_0, y_1), z_0, z_1) + \psi_1(T_1(x_1, y_0, y_1)), z_0, z_1) \end{aligned}$$

as desired. $\square$

8. Internalization of Higher Order Self-Distributivity

We begin this section with the definition of n -ary self-distributive object in a symmetric monoidal category, providing therefore a higher arity version of the work in [2]. We will use the symbol $\boxtimes$ to indicate the tensor product in the symmetric monoidal category $\mathcal{C}$, not to confuse the general setting with the standard tensor product in vector spaces, to be found in the examples. We remind the reader first, that a symmetric monoidal category is a monoidal category $\mathcal{C}$ together with a family of isomorphisms $\tau_{X,Y} : X \boxtimes Y \longrightarrow Y \boxtimes X$, natural in X and Y , satisfying the following conditions [17, Sec. 11]. The hexagon:

$$\begin{array}{ccccc} & & X \boxtimes (Y \boxtimes Z) & & \\ & \nearrow \alpha_{X,Y,Z} & & \searrow \tau_{X,Y \boxtimes Z} & \\ (X \boxtimes Y) \boxtimes Z & & & & (Y \boxtimes Z) \boxtimes X \\ \tau_{X,Y} \boxtimes \mathbb{1} \downarrow & & & & \downarrow \alpha_{Y,Z,X} \\ (Y \boxtimes X) \boxtimes Z & & & & Y \boxtimes (Z \boxtimes X) \\ & \searrow \alpha_{Y,X,Z} & & \nearrow \mathbb{1} \boxtimes \tau_{X,Z} & \\ & & Y \boxtimes (X \boxtimes Z) & & \end{array}$$

is commutative for all objects X, Y and Z in $\mathcal{C}$, where $\alpha_{X,Y,Z}$ indicates the associator of the monoidal category. We further have the following identity for all objects X and Y :

$$\tau_{Y,X} \tau_{X,Y} = \mathbb{1}_{X \boxtimes Y}.$$

For the sake of simplicity, we work on a strict symmetric monoidal category for the rest of the paper and therefore do not keep track of the bracketing. We recall also that a comonoid in a symmetric monoidal category is an object $X \in \mathcal{C}$ endowed with morphisms $\Delta : X \rightarrow X \boxtimes X$, called comultiplication or diagonal, and $\epsilon : X \rightarrow I$, called counit, where I is the unit object of the monoidal category. The comultiplication and the counit satisfy the usual coherence diagrams analogous to the coalgebra axioms. In virtue of the coassociative axiom we can inductively define an n -diagonal $\Delta_n : X \rightarrow X^{\boxtimes n}$ by the assignment: $\Delta_n = (\Delta \boxtimes \mathbb{1})\Delta_{n-1}$, for all $n \in \mathbb{N}$. Let us define the isomorphism $\tau_{i,i+1} : X^{\boxtimes n} \rightarrow X^{\boxtimes n}$ as $\tau_{i,i+1} = \mathbb{1}^{\boxtimes(i-1)} \boxtimes \tau_{X,X} \boxtimes \mathbb{1}^{\boxtimes(n-i-1)}$. It is easy to verify that the morphisms $\tau_{i,i+1}$ satisfy the relations of the transposition $(i, i+1)$ in $\mathbb{S}_n$, the symmetric group on n letters. We therefore obtain, for every object X , an action of $\mathbb{S}_n$ on $X^{\boxtimes n}$, by mapping $(i, i+1)$ to $\tau_{i,i+1}$, and extending to a homomorphism of groups between $\mathbb{S}_n$ and $\text{Aut}(X^{\boxtimes n})$, the automorphism group of $X^{\boxtimes n}$. In particular we will make use of the automorphism of $X^{\boxtimes n^2}$, corresponding to the permutation

$$\begin{aligned} \sqcup_n &= (2, n+1)(3, 2n+1) \cdots (n, (n-1)n+1) \\ &\quad (n+3, 2n+2)(n+4, 3n+2) \cdots (2n, (n-1)n+2) \\ &\quad \cdots ((n-2)n+n, (n-1)n+n-1). \end{aligned}$$

We are ready now to define n -ary self-distributive objects in a symmetric monoidal category $\mathcal{C}$.

Definition 8.1. An n -ary self-distributive object in a symmetric monoidal category $\mathcal{C}$ is a pair (X, W) , where X is a comonoid object in $\mathcal{C}$ and $W : X^{\boxtimes n} \rightarrow X$ is a morphism making the following diagram commute:

$$\begin{array}{ccccc} & & X^{\boxtimes n^2} & \xleftarrow{\mathbb{1}^{\boxtimes n} \boxtimes \Delta_n^{\boxtimes (n-1)}} & X^{\boxtimes (2n-1)} \\ & \swarrow \sqcup_n & & & \searrow W \boxtimes \mathbb{1}^{\boxtimes (n-1)} \\ X^{\boxtimes n^2} & & & & X^{\boxtimes n} \\ \downarrow W \boxtimes \cdots \boxtimes W & & & & \downarrow W \\ X^{\boxtimes n} & \xrightarrow{\quad W \quad} & & & X \end{array}$$

Example 8.1. Clearly, any n -ary rack is an n -ary self-distributive object in the symmetric monoidal category of sets, with τ and Δ defined in the obvious way.

In the rest of this section, we will make use of Sweedler notation in the following form: $\Delta(x) = x^{(1)} \otimes x^{(2)}$.

Example 8.2. Let H be an involutive Hopf algebra, i.e. $S^2 = \mathbb{1}$. Define a ternary operation $T : H \otimes H \otimes H \rightarrow H$ by the assignment $T(x \otimes y \otimes z) = xS(y)z$,

extended by linearity, where we use juxtaposition as a shorthand to indicate the multiplication μ of H and S is the antipode. By direct computation on tensor monomials we obtain, for the left-hand side of ternary self-distributivity:

$$\begin{aligned} & T(T(x \otimes y \otimes z) \otimes u \otimes z) \\ &= T(xS(y)z \otimes u \otimes v) \\ &= xS(y)zS(u)v. \end{aligned}$$

The right-hand side is

$$\begin{aligned} & TT^{\otimes 3} \sqcup_3 (\mathbb{1}^{\otimes 3} \otimes (\Delta \otimes \mathbb{1})\Delta \otimes (\Delta \otimes \mathbb{1})\Delta)(x \otimes y \otimes z \otimes u \otimes v) \\ &= TT^{\otimes 3}((x \otimes u^{(11)} \otimes v^{(11)} \otimes (y \otimes u^{(12)} \otimes v^{(12)}) \otimes (z \otimes u^{(2)} \otimes v^{(2)})) \\ &= T(xS(u^{(11)})v^{(11)}) \otimes yS(u^{(12)})v^{(12)} \otimes zS(u^{(2)})v^{(2)}) \\ &= xS(u^{(11)})v^{(11)}S(yS(u^{(12)})v^{(12)})zS(u^{(2)})v^{(2)} \\ &= xS(u^{(11)})v^{(11)}S(v^{(12)})S^2(u^{(12)})S(y)zS(u^{(2)})v^{(2)} \\ &= xS(u^{(11)})\epsilon(v^{(1)} \cdot 1)S^2(u^{(12)})S(y)zS(u^{(2)})v^{(2)} \\ &= xS(\epsilon(u^{(1)}) \cdot 1)S(y)zS(u^{(2)})\epsilon(v^{(1)})v^{(2)} \\ &= xS(y)zS(\epsilon(u^{(1)})u^{(2)})v \\ &= xS(y)zS(u)v. \end{aligned}$$

Note that we have used the fact that H is involutive in the sixth equality, to obtain $S(u^{(12)})u^{(11)} = u^{(1)}$. This ternary structure is the Hopf algebra analogue of the heap operation in group theory, which is known to be ternary self-distributive. We also observe that H being involutive is a parallel to the operation of taking inverses, obviously involutive as well.

In Fig. 7, a diagrammatic representation of categorical distributivity is depicted. It is read from top to bottom, where the top 3 end points of both sides represent

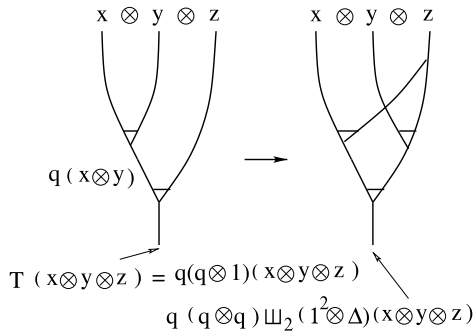


Fig. 7. Diagrammatic representation of categorical distributivity.

$x \otimes y \otimes z$, a trivalent vertex with a small triangle represents a self-distributive morphism $q : X \otimes X \rightarrow X$, and the left-hand side represents $T = q(q \otimes \mathbb{1})$.

Given a symmetric monoidal category $\mathcal{C}$, we define categories $n\mathcal{SD}$, for each $n \in \mathbb{N}$, as follows. The objects are n -ary self-distributive objects in $\mathcal{C}$, as in Definition 8.1. Given two objects (X, q) and (X', q') , we define the morphism class between them to be the class of morphism $f : X \rightarrow X'$ in $\mathcal{C}$, such that $f \circ q = q' \circ f^{\boxtimes n}$. In particular we define $\mathcal{BSD} = 2\mathcal{SD}$ and $\mathcal{TSD} = 3\mathcal{SD}$, $\mathcal{B}$ and $\mathcal{T}$ standing for binary and ternary, respectively.

We will make use of the following results in Theorem 8.1.

Lemma 8.1. *Let $\mathcal{C}$ be a strict symmetric monoidal category. Suppose (X, Δ, ϵ) is a comonoid in $\mathcal{C}$. Then the switching morphism and the multiplication commute. More specifically, we have: $\Delta \boxtimes \mathbb{1} \circ \tau_{X,Y} = \tau_{X,Y \boxtimes 2} \circ \mathbb{1} \boxtimes \Delta$.*

This lemma is represented in Fig. 8(a) below.

Proof. Consider the following diagram:

$$\begin{array}{ccccc}
 X \boxtimes Y & \xrightarrow{\tau_{X,Y}} & Y \boxtimes X & \xrightarrow{\Delta \boxtimes \mathbb{1}} & Y^{\boxtimes 2} \boxtimes X \\
 & \searrow \mathbb{1} \boxtimes \Delta & & \nearrow \tau_{X,Y \boxtimes 2} & \uparrow \mathbb{1} \boxtimes \tau_{X,Y} \\
 & & X \boxtimes Y^{\boxtimes 2} & \xrightarrow{\tau_{X,Y} \boxtimes \mathbb{1}} & Y \boxtimes X \boxtimes Y
 \end{array}$$

The outmost diagram commutes by naturality of switching map $\tau_{X,Y}$ with respect to X and Y . The lower right triangle commutes by the hexagon axiom:

$$\begin{array}{ccccc}
 & & X \boxtimes (Y \boxtimes Y) & & \\
 & \swarrow \tau_{X,Y \boxtimes 2} & & \searrow & \\
 (Y \boxtimes Y) \boxtimes X & & & & (X \boxtimes Y) \boxtimes Y \\
 \parallel & & & & \downarrow \tau_{X,Y} \boxtimes \mathbb{1} \\
 Y \boxtimes (Y \boxtimes X) & & & & (Y \boxtimes X) \boxtimes Y \\
 & \swarrow \mathbb{1} \boxtimes \tau_{X,Y} & & \searrow & \\
 & & Y \boxtimes (X \boxtimes Y) & &
 \end{array}$$

The assertion now follows. $\square$

Lemma 8.2. *Let (X, q) be a binary self-distributive object in a strict symmetric monoidal category $\mathcal{C}$. Then the switching morphism and the self-distributive operation commute. More specifically, we have: $\tau_{X,Y} \circ q \boxtimes \mathbb{1} = \mathbb{1} \boxtimes q \circ \tau_{X \boxtimes 2, Y}$.*

This lemma is represented in Fig. 8(b) below.

Proof. Similar to Lemma 8.1 and left to the reader. $\square$

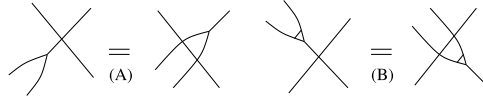


Fig. 8. The switching morphism commutes with comultiplication and binary self-distributive operation.

In general, the following result is useful to produce ternary self-distributive objects in the category of vector spaces, starting from binary self-distributive objects (see also [2]). Compare it to the construction of Sec. 3.

Theorem 8.1. *Let (X, Δ) be a comonoid in a (strict) symmetric monoidal category $\mathcal{C}$ (e.g. a coalgebra in the category of vector spaces). Let $q : X \boxtimes X \rightarrow X$ be a morphism such that (X, q) is a binary self-distributive object in $\mathcal{C}$. Then the pair (X, T) , where $T = q(q \boxtimes 1)$, defines a ternary self-distributive object in $\mathcal{C}$. The construction defines a functor $\mathcal{F} : \mathcal{BSD} \rightarrow \mathcal{TSD}$.*

Proof. We define $\mathcal{F}$ on objects as $\mathcal{F}(X, q) = (X, T)$ and as the identity on morphisms. To show that the map $T = q(q \boxtimes 1)$ is ternary self-distributive, we can proceed as in Fig. 9. In the left column of the figure, the part of the diagram representing each $T = q(q \boxtimes 1)$ are indicated by dotted circles. At each step we are using the definition of T , the binary self-distributivity of q and Lemmas 8.1 and 8.2. If $f : (X, q) \rightarrow (Y, q')$ is a morphism in $\mathcal{BSD}$, we can show that f is also a morphism in $\mathcal{TSD}$ between $(X, T = q(q \boxtimes 1))$ and $(Y, T' = q'(q' \boxtimes 1))$ via the following diagram:

$$\begin{array}{ccccc}
 X \boxtimes X \boxtimes X & \xrightarrow{q \boxtimes 1} & X \boxtimes X & \xrightarrow{q} & X \\
 \downarrow f \boxtimes f \boxtimes f & & \downarrow f \boxtimes f & & \downarrow f \\
 Y \boxtimes Y \boxtimes Y & \xrightarrow{q' \boxtimes 1} & Y \boxtimes Y & \xrightarrow{q'} & Y
 \end{array}$$

where the commutativity of the left and right squares is just a restatement of the fact that q is a morphism in $\mathcal{BSD}$. The consequent commutativity of the outer rectangle means that f is a morphism in $\mathcal{TSD}$ as well. It is also clear that $\mathcal{F}$ preserves composition of morphisms. $\square$

The following is a rephrased version of [2, Lemma 3.3], adapted to our language in this paper.

Lemma 8.3. *Let L be a Lie algebra over a ground field $\mathbb{k}$. Define $X = \mathbb{k} \oplus L$ and endow it with a comultiplication Δ , defined by $(a, x) \mapsto (a, x) \otimes (1, 0) + (1, 0) \otimes (0, x)$, and a counit ϵ , defined by $(a, x) \mapsto a$. Then (X, Δ, ϵ) is a comonoid in the symmetric monoidal category of vector spaces. The morphism $q : X \otimes X \rightarrow X$ defined by $(a, x) \otimes (b, y) \mapsto (ab, bx + [x, y])$ turns X into a binary self-distributive object.*

Proof. By direct computation making use of the Jacobi identity. This is done explicitly in [2, Lemma 3.3]. $\square$

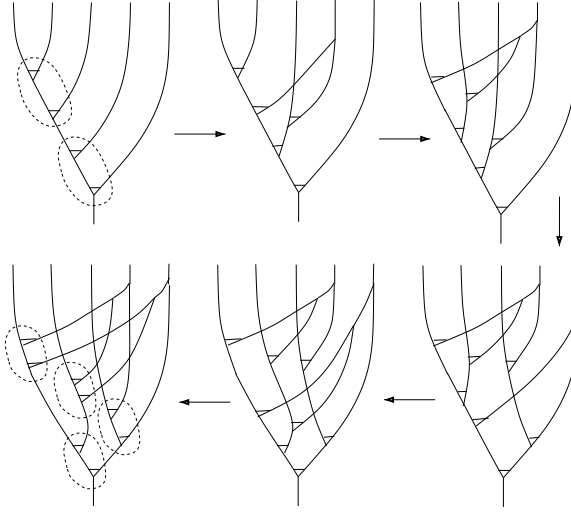


Fig. 9. Diagrammatic proof of doubling procedure.

Example 8.3. Let L be a Lie algebra and let $X = \mathbb{k} \oplus L$ be as in Lemma 8.3. The map $T : X \otimes X \otimes X \longrightarrow X$ defined by

$$(a, x) \otimes (b, y) \otimes (c, z) \longmapsto (abc, bcx + c[x, y] + b[x, z] + [[x, y], z]),$$

and extended by linearity, is such that (X, T) is a ternary self-distributive object in the category of vector spaces by an easy application of Theorem 8.1. An explicit, and tedious, computation that shows the self-distributivity of T directly, is postponed to Appendix A.

If H is a Hopf algebra, we can use the adjoint map to produce a ternary self-distributive map, as the following example shows.

Example 8.4. The map defined by $T(x \otimes y \otimes z) = S(z^{(1)})S(y^{(1)})xy^{(2)}z^{(2)}$ is ternary self-distributive, as an easy direct computation shows. This is the Hopf algebra analogue of the iterated conjugation quandle.

Remark 8.1. It is possible, *a priori*, to develop the theory of higher self-distributivity in braided monoidal categories, where the switching morphism satisfies the hexagon axiom but we do not require $\tau_{Y, X} \tau_{X, Y} = \mathbb{1}_{X \boxtimes Y}$. Similarly as above we have an action of the braid group on n strings on every object $X^{\boxtimes n}$ and the shuffle map $\sqcup_n$ takes now into account over passing and under passing of the strings.

Appendix A. Example 8.3 Revisited

In this appendix, we explicitly show that the map in Example 8.3 is indeed self-distributive. Each equality is obtained by applying the Jacobi identity as in the

proof of Lemma 3.3 in [2]. In fact, each step corresponds to one of the diagrams in the proof of Theorem 8.1 (cf. Fig. 9). Recall also the definition of the diagonal Δ , from Lemma 8.3, and the inductive definition for Δ_3 at the beginning of Sec. 8. Explicitly, we have for Δ_3 :

$$\Delta_3(a, x) = (a, x) \otimes (1, 0) \otimes (1, 0) + (1, 0) \otimes (0, x) \otimes (1, 0) + (1, 0) \otimes (1, 0) \otimes (0, x).$$

To make the steps easier for the reader, we declare the terms that are going to be replaced according to the Jacobi identity, and underline the replacing terms in the subsequent equality. We obtain therefore

$$\begin{aligned} T(T((a, x) \otimes (b_0, y_0) \otimes (b_1, y_1)) \otimes (c_0, z_0) \otimes (c_1, z_1)) \\ = (ab_0b_1c_0c_1, b_0b_1c_0c_1x + b_1c_0c_1[x, y_0] + b_0c_0c_1[x, y_1] \\ + c_0c_1[[x, y_0], y_1] + b_0b_1c_1[x, z_0] + b_1c_1[[x, y_0], z_0] \\ + b_0c_1[[x, y_1], z_0] + c_1[[[x, y_0], y_1], z_0] + b_0b_1c_0[x, z_1] \\ + b_1c_0[[x, y_0], z_1] + b_0c_0[[x, y_1], z_1] + c_0[[[x, y_0], y_1], z_1] \\ + b_0b_1[[x, z_0], z_1] + b_1[[[x, y_0], z_0], z_1] + b_0[[[x, y_1], z_0], z_1] \\ + [[[[x, y_0], y_1], z_0], z_1]). \end{aligned}$$

Applying the Jacobi identity to the terms $b_0c_1[[x, y_1], z_0]$, $c_1[[[x, y_0], y_1], z_0]$, $b_0[[[x, y_1], z_0], z_1]$ and $[[[[x, y_0], y_1], z_0], z_1]$ we obtain

$$\begin{aligned} = (ab_0b_1c_0c_1, b_0b_1c_0c_1x + b_1c_0c_1[x, y_0] + b_0b_1c_1[x, z_0] \\ + b_1c_1[[x, y_0], z_0] + b_0c_0c_1[x, y_1] + c_0c_1[[x, y_0], y_1] \\ + b_0c_1[[x, z_0], y_1] + \underline{c_1[[[x, y_0], z_0], y_1]} + \underline{b_0c_1[x, [y_1, z_0]]} \\ + \underline{c_1[[x, y_0], [y_1, z_0]]} + b_0b_1c_0[x, z_1] + b_1c_0[[x, y_0], z_1] \\ + b_0b_1[[x, z_0], z_1] + b_1[[[x, y_0], z_0], z_1] + b_0c_0[[x, y_1], z_1] \\ + c_0[[[x, y_0], y_1], z_1] + \underline{b_0[[[x, z_0], y_1], z_1]} + \underline{[[[[x, y_0], z_0], y_1], z_1]} \\ + \underline{b_0[[x, [y_1, z_0]], z_1]} + \underline{[[[x, y_0], [y_1, z_0]], z_1]}). \end{aligned}$$

We now apply the Jacobi identity to the term $b_1c_1[[x, y_0], z_0]$, $b_1[[[x, y_0], z_0], z_1]$, $c_1[[[x, y_0], z_0], y_1]$ and $[[[[x, y_0], z_0], y_1], z_1]$ to obtain

$$\begin{aligned} = (ab_0b_1c_0c_1, b_0b_1c_0c_1x + b_0b_1c_1[x, z_0] + b_1c_0c_1[x, y_0] \\ + \underline{b_1c_1[[x, z_0], y_0]} + b_0c_0c_1[x, y_1] + b_0c_1[[x, z_0], y_1] \\ + c_0c_1[[x, y_0], y_1] + \underline{c_1[[[x, z_0], y_0], y_1]} + \underline{b_1c_1[x, [y_0, z_0]]} \\ + \underline{c_1[[x, [y_0, z_0]], y_1]} + b_0c_1[x, [y_1, z_0]] + c_1[[x, y_0], [y_1, z_0]] \end{aligned}$$

$$\begin{aligned}
& + b_0 b_1 c_0 [x, z_1] + b_0 b_1 [[x, z_0], z_1] + b_1 c_0 [[x, y_0], z_1] \\
& + \underline{b_1 [[[x, z_0], y_0], z_1]} + b_0 c_0 [[x, y_1], z_1] + b_0 [[[x, z_0], y_1], z_1] \\
& + c_0 [[[x, y_0], y_1], z_1] + \underline{[[[[x, z_0], y_0], y_1], z_1]} + \underline{b_1 [[x, [y_0, z_0]], z_1]} \\
& + \underline{[[[x, [y_0, z_0]], y_1], z_1]} + b_0 [[x, [y_1, z_0]], z_1] + [[x, y_0], [y_1, z_0], z_1].
\end{aligned}$$

Next, we use the Jacoby identity on the terms $b_0 c_0 [[x, y_1], z_1]$, $b_0 [[[x, z_0], y_1], z_1]$, $b_0 [[x, [y_1, z_0]], z_1]$, $c_0 [[[[x, y_0], y_1], z_1]$, $[[[[x, z_0], y_0], y_1], z_1]$, $[[[x, [y_0, z_0]], y_1], z_1]$ and $[[[x, y_0], [y_1, z_0]], z_1]$:

$$\begin{aligned}
& = (ab_0 b_1 c_0 c_1, b_0 b_1 c_0 c_1 x + b_0 b_1 c_1 [x, z_0] + b_1 c_0 c_1 [x, y_0] \\
& + b_1 c_1 [[x, z_0], y_0] + b_0 b_1 c_0 [x, z_1] + b_0 b_1 [[x, z_0], z_1] \\
& + b_1 c_0 [[x, y_0], z_1] + b_1 [[[x, z_0], y_0], z_1] + b_0 c_0 c_1 [x, y_1] \\
& + b_0 c_1 [[x, z_0], y_1] + c_0 c_1 [[x, y_0], y_1] + c_1 [[x, [y_0, z_0]], y_1] \\
& + b_1 c_1 [x, [y_0, z_0]] + \underline{b_0 c_0 [[x, z_1], y_1]} + \underline{b_0 [[[x, z_0], z_1], y_1]} \\
& + \underline{c_0 [[[x, y_0], z_1], y_1]} + b_1 [[x, [y_0, z_0]], z_1] + \underline{[[[[x, z_0], y_0], z_1], y_1]} \\
& + \underline{[[[x, [y_0, z_0]], z_1], y_1]} + b_0 c_1 [x, [y_1, z_0]] + c_1 [[x, y_0], [y_1, z_0]] \\
& + \underline{b_0 [[x, z_1], [y_1, z_0]]} + \underline{[[[x, y_0], z_1], [y_1, z_0]]} + \underline{b_0 c_0 [x, [y_1, z_1]]} \\
& + \underline{b_0 [[x, z_0], [y_1, z_1]]} + c_1 [[x, [z_0], y_0], y_1] + \underline{c_0 [[x, y_0], [y_1, z_1]]} \\
& + \underline{[[[x, z_0], y_0], [y_1, z_1]]} + \underline{[[x, [y_0, z_0]], [y_1, z_1]]} + \underline{b_0 [x, [[y_1, z_0], z_1]]} \\
& + \underline{[x, y_0], [[y_1, z_0], z_1]}).
\end{aligned}$$

Lastly, making use of the Jacobi identity on the terms $b_1 c_0 [[x, y_0], z_1]$, $b_1 [[[x, z_0], y_0], z_1]$, $c_0 [[[[x, y_0], z_1], y_1]$, $b_1 [[x, [y_0, z_0]], z_1]$, $[[[[x, z_0], y_0], z_1], y_1]$, $[[[x, [y_0, z_0]], z_1], y_1]$ and $[[[x, y_0], z_1], [y_1, z_0]]$ we obtain

$$\begin{aligned}
& = (ab_0 b_1 c_0 c_1, b_0 b_1 c_0 c_1 x + b_0 b_1 c_1 [x, z_0] + b_0 b_1 c_0 [x, z_1] \\
& + b_0 b_1 [[x, z_0], z_1] + b_1 c_0 c_1 [x, y_0] + b_1 c_1 [[x, z_0], y_0] \\
& + \underline{b_1 c_0 [[x, z_1], y_0]} + \underline{b_1 [[[x, z_0], z_1], y_0]} + b_0 c_0 c_1 [x, y_1] \\
& + b_0 c_1 [[x, z_0], y_1] + b_0 c_0 [[x, z_1], y_1] + b_0 [[[x, z_0], z_1], y_1] \\
& + c_0 c_1 [[x, y_0], y_1] + c_1 [[x, [z_0], y_0], y_1] + \underline{c_0 [[[x, z_1], y_0], y_1]} \\
& + \underline{[[[[x, z_0], z_1], y_0], y_1]} + b_0 c_1 [x, [y_1, z_0]] + b_0 [[x, z_1], [y_1, z_0]] \\
& + c_1 [[x, y_0], [y_1, z_0]] + \underline{[[[x, z_1], y_0], [y_1, z_0]]} + b_1 c_1 [x, [y_0, z_0]] \\
& + \underline{b_1 [[x, z_1], [y_0, z_0]]} + c_1 [[x, [y_0, z_0]], y_1] + \underline{[[x, z_1], [y_0, z_0]], y_1}
\end{aligned}$$

$$\begin{aligned}
& + \underline{b_1 c_0[x, [y_0, z_1]]} + \underline{b_1[[x, z_0], [y_0, z_1]]} + \underline{c_0[[x, [y_0, z_1]], y_1]} \\
& + \underline{[[[x, z_0], [y_0, z_1]], y_1]} + \underline{b_1[x, [[y_0, z_0], z_1]]} + \underline{[[x, [[y_0, z_0], z_1]], y_1]} \\
& + \underline{[[x, [y_0, z_1]], [y_1, z_0]]} + b_0 c_0[x, [y_1, z_1]] + b_0[[x, z_0], [y_1, z_1]] \\
& + c_0[[x, y_0], [y_1, z_1]] + [[x, z_0], y_0], [y_1, z_1]] + [[x, [y_0, z_0]], [y_1, z_1]] \\
& + b_0[x, [[y_1, z_0], z_1]] + [[x, y_0], [[y_1, z_0], z_1]].
\end{aligned}$$

This last term can be seen to coincide with the right-hand side of the self-distributivity equation:

$$T(T^{\otimes 3}) \sqcup_3 (\mathbb{1}^3 \otimes \Delta_3^{\otimes 2})((a, x) \otimes (b_0, y_0) \otimes (b_1, y_1)) \otimes (c_0, z_0) \otimes (c_1, z_1)).$$

It follows therefore, that the map T turns X into a ternary self-distributive object in the category of vector spaces.

Appendix B. Augmented Ternary Racks for Sets and Hopf Algebras

It is of an independent interest how the concept of augmented racks generalize to ternary racks for both sets and monoidal categories in general. In this section, we propose such generalization and provide key motivational examples in heaps and Hopf algebras.

An *augmented rack* [11] (X, G) is a set X with a right group action by a group G and a map $p : X \rightarrow G$ satisfying the identity $p(x \cdot g) = g^{-1}p(x)g$ for all $x \in X$, $g \in G$. An augmented rack has a rack operation defined by $x * y = x \cdot p(y)$ for $x, y \in X$. The following definition can be considered a ternary analogue of an augmented rack [11].

Definition B.1. Let X be a set with a right G -action denoted by $X \times G \ni (x, g) \mapsto x \cdot g \in X$. Let G act on the right of $X \times X$ diagonally, $(y_0, y_1) \cdot g = (y_0 \cdot g, y_1 \cdot g)$ for $y_0, y_1 \in X$ and $g \in G$. A *(double) augmentation* of X is a map $p : X \times X \rightarrow G$ satisfying the condition

$$p((y_0, y_1) \cdot g) = g^{-1}p((y_0, y_1))g$$

for all $y_0, y_1 \in X$ and $g \in G$.

The following is a direct analogue of binary augmented rack and, therefore, the proof is omitted.

Lemma B.1. *Let X be a set with an augmentation $p : X \times X \rightarrow G$. Then the ternary operation $T : X^3 \rightarrow X$ defined by*

$$T(x, y_0, y_1) := x \cdot p((y_0, y_1))$$

is ternary self-distributive.

Definition B.2. Let X be a set with an augmentation $p : X^2 \rightarrow G$ and T be a ternary operation defined in Lemma B.1. Then (X, T) is called an *augmented ternary shelf*.

Example B.1. Heaps can be endowed with augmentation as follows. Let G be a group, with the TSD operation $T(x, y, z) = xy^{-1}z$. Consider the right multiplication as the right action of G on itself. Then consider the diagonal right action of G on $G \times G$, by $(x, y) \cdot z = (x \cdot z, y \cdot z)$. Let $p : G \times G \rightarrow G$ be defined by $p(y, z) = y^{-1}z$. Then we readily check the condition

$$p((y_0, y_1) \cdot g) = p((y_0 \cdot g, y_1 \cdot g)) = (y_0 g)^{-1}(y_1 g) = g^{-1}y_0^{-1}y_1 g = g^{-1}p((y_0, y_1))g.$$

Although more study on augmented ternary racks are desirable, we focus on the following further generalization to Hopf algebras. The point of interest is that the comultiplication plays the role of the diagonal map.

Definition B.3. Let X be a coalgebra, and let H be a Hopf algebra such that X is a right H -module, therefore $X^{\otimes 2}$ is also a right H -module via the comultiplication in H . The map of coalgebras $p : X^{\otimes 2} \rightarrow H$ is a ternary augmented shelf if, for all $z \in X^{\otimes 2}$ and $g \in H$, we have

$$p(z \cdot \Delta(g)) = S(g^{(1)})p(z)g^{(2)}.$$

This axiom is depicted diagrammatically in Fig. B.1, where solid lines refer to X , and dashed lines refer to H . We have used Δ , m and S to indicate comultiplication, multiplication and antipode in the Hopf algebra H , while μ stands for the action of H on X .

We have the following result.

Theorem B.1. Let $p : X^{\otimes 2} \rightarrow H$ be a ternary augmented shelf. Then the ternary operation defined on monomials via $x \otimes y \otimes z \mapsto x \cdot p(y \otimes z)$, and extended by linearity, is self-distributive.

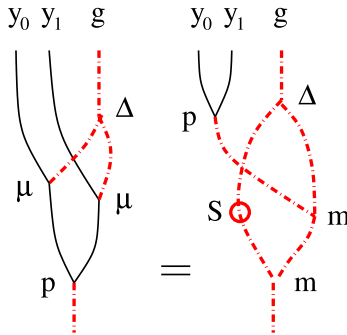


Fig. B.1. Augmented ternary shelf axiom.

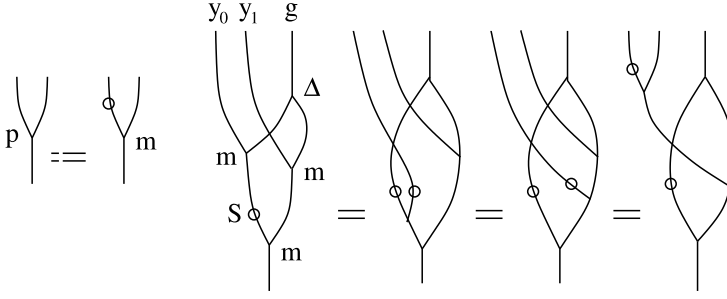


Fig. B.2. Hopf algebra heap as an augmented ternary shelf.

Proof. By direct computation we have, for the right-hand side of self-distributivity axiom:

$$\begin{aligned}
 & TT^{\otimes 3} \sqcup_3 (\mathbb{1}^{\otimes 3} \otimes (\Delta \otimes \mathbb{1})\Delta \otimes (\Delta \otimes \mathbb{1})\Delta)(x \otimes y_0 \otimes y_1 \otimes z_0 \otimes z_1) \\
 &= T(x \cdot p(z_0^{(1)} \otimes z_1^{(1)}) \otimes y_0 \cdot p(z_0^{(2)} \otimes z_1^{(2)}) \otimes y_1 \cdot p(z_0^{(3)} \otimes z_1^{(3)})) \\
 &= x \cdot (p(z_0^{(1)} \otimes z_1^{(1)})p(y_0 \cdot p(z_0^{(2)} \otimes z_1^{(2)}) \otimes y_1 \cdot p(z_0^{(3)} \otimes z_1^{(3)}))) \\
 &= x \cdot (p(z_0^{(1)} \otimes z_1^{(1)})p(y_0 \otimes y_1 \cdot \Delta p((z_0 \otimes z_1)^{(2)}))) \\
 &= x \cdot (p(z_0 \otimes z_1)^{(1)}S(p((z_0 \otimes z_1)^{(2)}))p(y_0 \otimes y_1)p((z_0 \otimes z_1)^{(3)})) \\
 &= x \cdot (\epsilon(p(z_0 \otimes z_1)^{(1)}) \cdot 1p(y_0 \otimes y_1)p((z_0 \otimes z_1)^{(2)})) \\
 &= x \cdot (p(y_0 \otimes y_1)p(z_0 \otimes z_1)),
 \end{aligned}$$

where we have used the fact that p is a coalgebra morphism in the third equality, the defining axiom for augmented ternary shelf in the fourth equality, the antipode and the counit axioms to obtain the fifth and sixth equations, respectively. It is easy to see that it coincide with the left hand side of self-distributivity. $\square$

Example B.2. Let H be an involutive Hopf algebra and let $X = H$. Then, H acts on X via multiplication. Define p to be the map given by $x \otimes y \mapsto S(x)y$ and extended by linearity. The ternary rack structure obtained is the one in Example 8.2. A diagrammatic proof that the given p satisfies the augmented ternary rack axiom is shown in Fig. B.2.

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