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RADICAL SUBGROUPS AND THE INDUCTIVE BLOCKWISE ALPERIN WEIGHT CONDITIONS FOR $\mathbf{PSp_4(q)}$

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We determine explicitly the 2-radical subgroups and their normalizers for the group $\mathrm{Sp}_4(q)$, where q is odd. We then show that the corresponding simple group $\mathrm{PSp}_4(q)$ satisfies the inductive blockwise Alperin weight conditions for the prime 2 and odd primes dividing q^2-1. When combined with existing literature, this completes the verification that $\mathrm{PSp}_4(q)$ satisfies the conditions for all primes and all choices of q.

1. Introduction

Given a prime ℓ, an ℓ-weight of a finite group G is a pair (R, μ), where R is an ℓ-radical subgroup and μ is a defect-zero character of $N_G(R)/R$. That is, R is an ℓ-subgroup such that $R = \boldsymbol{O}_\ell(N_G(R))$ and μ is an irreducible character with $\mu(1)_\ell = |N_G(R)/R|_\ell$. More generally, a weight for a block B of G is a pair (R, μ) as above, where μ further lies in a block b of $N_G(R)$ for which B is the induced block b^G. The Alperin weight conjecture (AWC) posits that if G is a finite group and ℓ is a prime dividing $|G|$, then the number of irreducible ℓ-Brauer characters of G equals the number of G-conjugacy classes of ℓ-weights of G. The blockwise Alperin weight conjecture (BAWC) refines the statement to say that the number of irreducible ℓ-Brauer characters belonging to a block B of G equals the number of G-conjugacy classes of ℓ-weights of B.

Navarro and Tiep [11] and Späth [16] reduced the AWC and BAWC, respectively, to simple groups. In particular, to verify these conjectures it suffices to show that certain more complicated "inductive" conditions hold for all finite nonabelian simple groups. Simple groups satisfying the inductive conditions for the AWC or BAWC are sometimes said to be "good" for the corresponding conjecture.

In this article, we deal especially with the simple groups $\mathrm{PSp}_4(q)$. It is shown in [11] and [16] that a simple group of Lie type defined in characteristic p satisfies the inductive BAWC conditions for the prime $\ell = p$. In [13], the second author has shown that when q is even, $\mathrm{Sp}_4(q)$ and $\mathrm{Sp}_6(q)$ satisfy the inductive conditions for all $\ell \neq 2$. Furthermore, in [9], S. Koshitani and B. Späth show that for ℓ odd, the inductive conditions hold whenever a Sylow ℓ-subgroup is cyclic.

Hence, to complete the proof that $\mathrm{PSp}_4(q)$ satisfies the inductive BAWC conditions (and therefore also the inductive AWC conditions), we must verify that these groups are good when $q \geq 5$ is odd for

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the prime $\ell = 2$ and for odd primes ℓ dividing $q^2 - 1$. (Note that $\mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2)$, and hence this group satisfies the inductive BAWC conditions for $q = 3$ and $\ell = 2$ by [11; 16].) Our main result is the following:

Theorem 1.1. *Let q be a power of an odd prime. Then the simple groups* $\mathrm{PSp}_4(q)$ *satisfy the inductive blockwise Alperin weight conditions* [16, Definition 4.1] *for any prime ℓ dividing $q^2 - 1$.*

This completes the statement that the simple groups $\mathrm{PSp}_4(q)$ are good for the BAWC for all primes ℓ and all choices of q.

We begin in Section 2 by explicitly describing all 2-radical subgroups of $\mathrm{Sp}_4(q)$ and their normalizer structures. We see that the situation here is much more complicated than for other choices of pairs (ℓ, q). In Section 3, we discuss some relevant defect-zero characters of these normalizers after summarizing results from [19] regarding the Brauer characters of $\mathrm{Sp}_4(q)$. Finally, we complete the proof of Theorem 1.1 in Section 4 by describing explicit bijections.

1.1. Notation. We write $\mathrm{Irr}(X)$ for the set of irreducible ordinary characters of a finite group X and $\mathrm{dz}(X) \subseteq \mathrm{Irr}(X)$ for the subset of those with defect zero. We further write $\mathrm{IBr}_\ell(X)$ for the irreducible ℓ-Brauer characters. When the characteristic ℓ is understood, we also write $\widehat{\chi}$ for the ℓ-Brauer character obtained from $\chi \in \mathrm{Irr}(X)$ by restriction to ℓ' elements. If a group X acts on a set Ω, then we write X_ω for the stabilizer in X of an element $\omega \in \Omega$.

Given an integer n, we write n_ℓ and $n_{\ell'}$ for the largest power of ℓ and largest number coprime to ℓ, respectively, dividing n. We write C_n for the cyclic group of size n and $X.n$ for an extension of a group X by C_n. The symmetric and alternating groups of degree n will be denoted by $\mathfrak{S}_n$ and $\mathfrak{A}_n$, respectively.

For the remainder of the article, let q be a power of an odd prime p and let $\ell \neq p$ be another prime. We will write e to denote the order of q^2 modulo ℓ and let $\epsilon \in \{\pm 1\}$ be such that $q \equiv \epsilon \pmod 4$ when $\ell = 2$, or $q^e \equiv \epsilon \pmod \ell$ when ℓ is odd. Let a be the positive integer such that $\ell^a = |q^e - \epsilon|_\ell$. In the case $\ell = 2$, note that this means $2^{a+1} = (q^2 - 1)_2$. Further, we remark that in Sections 3 and 4, we will be primarily interested in the case $e = 1$.

Throughout, G will denote the group $\mathrm{Sp}_4(q)$ and S the group $\mathrm{PSp}_4(q) = G/Z(G)$. Further, we will write $\widetilde{G}$ for the group $\mathrm{CSp}_4(q)$ and $\widetilde{S} = \widetilde{G}/Z(\widetilde{G})$ for the group of inner-diagonal automorphisms of S.

2. Radicals of $\mathrm{Sp}_4(q)$

To produce radical ℓ-subgroups for $G = \mathrm{Sp}_4(q)$, we make use of [4] for ℓ odd and [2] for $\ell = 2$. In particular we have the following theorem.

Theorem 2.1 ([2, 3A] and [4, 2D]). *Let R be an ℓ-radical subgroup of* $\mathrm{Sp}_{2n}(q) \cong \mathrm{Sp}(V)$. *Then there exists an orthogonal decomposition $V = V_0 + V_1 + V_2 + \cdots + V_t$ such that $R = R_0 \times R_1 \times R_2 \times \cdots \times R_t$. Here if $\ell = 2$, for each $i \geq 0$, either $R_i = \{\pm I_{V_i}\}$ or R_i is a basic subgroup of* $\mathrm{Sp}(V_i)$. *If ℓ is odd, then $R_0 = I_{V_0}$ and R_i is a basic subgroup of* $\mathrm{Sp}(V_i)$ *for $i \geq 1$.* (See Definitions 1 and 2 below.)

As symplectic groups are only defined over vector spaces of even dimension, each $\dim(V_i)$ must be even. Thus the aim is to study the basic subgroups of $\mathrm{Sp}_4(q)$ and $\mathrm{Sp}_2(q)$; see Definition 2 below for their construction. We will first consider the basic subgroups for ℓ odd, as the arguments are easier, before dealing with the more involved case $\ell = 2$.

2.1. $\boldsymbol{\ell}$-radical subgroups of $\mathbf{Sp_4(q)}$ for $\boldsymbol{\ell}$ odd. We follow the notation as given in [4]. For integers $\alpha, \gamma \geq 0$, let $V_{\alpha,\gamma}$ denote the symplectic or orthogonal space of dimension $2e\ell^{\alpha+\gamma}$, where $e = o(q^2)$ modulo ℓ. Recall that the integer $a \geq 1$ and $\epsilon \in \{\pm 1\}$ are defined by the equation $\ell^a = |q^e - \epsilon|_\ell$. Let $Z_\alpha := C_{\ell^{a+\alpha}}$ denote the cyclic group of order $\ell^{a+\alpha}$ and E_γ the extraspecial group of order $\ell^{2\gamma+1}$ and exponent ℓ. Set $R_{\alpha,\gamma}$ to be the image of the central product $Z_\alpha \circ E_\gamma$ under the natural embedding through $\mathrm{GL}_{\ell^\gamma}(\epsilon q^{e\ell^\alpha})$. (Here Z_α is mapped to $O_\ell(Z(\mathrm{GL}_{\ell^\gamma}(\epsilon q^{e\ell^\alpha}))$. For any integer $m \geq 1$, let $V_{m,\alpha,\gamma}$ denote the m-times orthogonal sum of copies of $V_{\alpha,\gamma}$, and let $R_{m,\alpha,\gamma}$ be the image of the natural m-fold diagonal embedding of $R_{\alpha,\gamma}$.

For a sequence of nonnegative integers $\boldsymbol{c} = \{c_1, \ldots, c_l\}$, set $|\boldsymbol{c}| = c_1 + \cdots + c_l$ and $V_{m,\alpha,\gamma,\boldsymbol{c}}$ to be the orthogonal sum of $\ell^{|\boldsymbol{c}|}$ copies of $V_{m,\alpha,\gamma}$. Denote by A_c the elementary abelian group of order ℓ^c and define $A_{\boldsymbol{c}} := A_{c_1} \wr \cdots \wr A_{c_l}$ and $R_{m,\alpha,\gamma,\boldsymbol{c}} := R_{m,\alpha,\gamma} \wr A_{\boldsymbol{c}}$.

Definition 1. For odd primes ℓ, the subgroups $R_{m,\alpha,\gamma,\boldsymbol{c}}$ are called the basic subgroups for $\mathrm{Sp}(V_{m,\alpha,\gamma,\boldsymbol{c}})$.

The basic subgroups $R_{m,\alpha,\gamma,\boldsymbol{c}}$ are uniquely determined up to conjugacy in $\mathrm{Sp}(V_{m,\alpha,\gamma,\boldsymbol{c}})$ and we have $\dim(V_{m,\alpha,\gamma,\boldsymbol{c}}) = \ell^{|\boldsymbol{c}|} m 2e\ell^{\alpha+\gamma}$. Let $u_{m,\alpha,\gamma,\boldsymbol{c}}$ denote the multiplicity of the basic subgroup $R_{m,\alpha,\gamma,\boldsymbol{c}}$ in the decomposition of R.

Proposition 2.2 [4, 2E]. *Let ℓ be an odd prime and R an ℓ-radical subgroup in $\mathrm{Sp}_{2n}(q)$ such that*

$$R = I_{V_0} \times \prod_{m,\alpha,\gamma,\boldsymbol{c}} R_{m,\alpha,\gamma,\boldsymbol{c}}^{u_{m,\alpha,\gamma,\boldsymbol{c}}}.$$

Then

$$N_{\mathrm{Sp}_{2n}(q)}(R)/R \cong \mathrm{Sp}(V_0) \times \prod_{m,\alpha,\gamma,\boldsymbol{c}} (N_{\mathrm{Sp}(V_{m,\alpha,\gamma,\boldsymbol{c}})}(R_{m,\alpha,\gamma,\boldsymbol{c}})/R_{m,\alpha,\gamma,\boldsymbol{c}}) \wr \mathfrak{S}_{u_{m,\alpha,\gamma,\boldsymbol{c}}}.$$

Since we see that wreath products play an integral role in these normalizers, we provide the following statement to understand radical subgroups with respect to wreath products.

Lemma 2.3. *Let H be a finite group, r a prime and n an integer. Assume that H contains an element of order coprime to r. Then $O_r(H \wr \mathfrak{S}_n) = O_r(H)^n$. In particular, for any finite group H, $O_r(H \wr \mathfrak{S}_n) = O_r(H)^n$ unless H is a 2-group and $(n, r) \in \{(2, 2), (4, 2)\}$, or H is a 3-group and $(n, r) = (3, 3)$.*

Proof. Let $N := O_r(H \wr \mathfrak{S}_n)$. Then $NH^n/H^n \leq O_r(\mathfrak{S}_n) = 1$ unless $(n, r) \in \{(2, 2), (4, 2), (3, 3)\}$. Thus the second statement clearly follows by proving the first statement.

Fix an element $g \in H$ whose order is coprime to r. Note that if $\{\underline{h}_1\sigma_1, \ldots, \underline{h}_m\sigma_m\}$ is a coset transversal of $O_r(H \wr \mathfrak{S}_n)$ over $O_r(H^n)$ with $\underline{h}_i \in H^n$ and $\sigma \in \mathfrak{S}_n$, then the σ_i must be distinct permutations. After suitable conjugation, we can assume that $\sigma_1(1) \neq 1$. Let $\underline{g} := (g, 1, \ldots, 1) \in H^n$. Then $(\underline{h}_1\sigma_1)^{\underline{g}} = (\underline{g}\underline{h}_1(\underline{g}^{\sigma_1})^{-1})\sigma_1$. Thus $(\underline{g}\underline{h}_1(\underline{g}^{\sigma_1})^{-1}) = \underline{h}\,\underline{h}_1$ with $\underline{h} \in O_r(H^n)$. However it now follows, by the choice of $\underline{g}$, that $g \in O_r(H)$, which is a contradiction. □

In particular, the following corollary is an immediate consequence.

Corollary 2.4. *Let ℓ be an odd prime not dividing q and let $R \leq \mathrm{Sp}_{2n}(q)$ be of the form*

$$R = \mathrm{Id}_{V_0} \times \prod_{m,\alpha,\gamma,\boldsymbol{c}} R_{m,\alpha,\gamma,\boldsymbol{c}}^{u_{m,\alpha,\gamma,\boldsymbol{c}}}.$$

Then R is a radical ℓ-subgroup of $\mathrm{Sp}_{2n}(q)$ if and only each $R_{m,\alpha,\gamma,\boldsymbol{c}}$ is radical in $\mathrm{Sp}(V_{m,\alpha,\gamma,\boldsymbol{c}})$.

Proof. In [4] it has been shown that 2 always divides

$$|N_{\mathrm{Sp}(V_{m,\alpha,\gamma,\boldsymbol{c}})}(R_{m,\alpha,\gamma,\boldsymbol{c}}):R_{m,\alpha,\gamma,\boldsymbol{c}}|$$

and therefore the result follows by combining the previous results. □

Observe that to construct the radical subgroups of $G = \mathrm{Sp}_4(q)$, we need only consider the basic subgroups with dimension 2 or 4. In particular, as $\dim(V_{m,\alpha,\gamma,\boldsymbol{c}}) = \ell^{|c|}m2e\ell^{\alpha+\gamma}$ it follows that for our cases, $\alpha = \gamma = 0$ and $\boldsymbol{c}$ is empty. Therefore, the following observation will deal with the basic subgroups of interest.

Lemma 2.5. *Let ℓ be an odd prime and let $R_{m,0,\gamma,\boldsymbol{c}}$ be a basic ℓ-subgroup of $\mathrm{Sp}(V_{m,0,\gamma,\boldsymbol{c}})$. Then $R_{m,0,\gamma,\boldsymbol{c}}$ is radical in $\mathrm{Sp}(V_{m,0,\gamma,\boldsymbol{c}})$.*

Proof. First note that by [4, Equation 2.5],

$$N_{\mathrm{Sp}(V_{m,\alpha,\gamma,\boldsymbol{c}})}(R_{m,\alpha,\gamma,\boldsymbol{c}})/R_{m,\alpha,\gamma,\boldsymbol{c}} \cong N_{\mathrm{Sp}(V_{m,\alpha,\gamma})}(R_{m,\alpha,\gamma}) \times \prod_{c_i \in \boldsymbol{c}} \mathrm{GL}_{c_i}(\ell),$$

and therefore it suffices to assume that $\boldsymbol{c}$ is empty.

Set $R = R_{m,0,\gamma}$, $C = C_{\mathrm{Sp}(V_{m,0,\gamma})}(R)$, and $N = N_{\mathrm{Sp}(V_{m,0,\gamma})}(R)$. Then for $N_0 := C_N(Z(R))$, page 12 of [4] yields that $N/N_0 \cong C_{2e\ell^\alpha}$, $N_0/CR \cong \mathrm{Sp}_{2\gamma}(\ell)$, and $C \cong \mathrm{GL}_m^\epsilon(q^{2e\ell^\alpha})$. Thus $O_\ell(N_0) = O_\ell(C)R = R$ and if $\alpha = 0$, it follows that N_0 has ℓ'-index in N as $e \le \ell - 1$ and therefore $O_\ell(N) = O_\ell(N_0)$. □

Corollary 2.6. *Let $\ell \neq p$ be odd primes and q a power of p. In addition, let R be a nontrivial ℓ-subgroup of $G = \mathrm{Sp}_4(q)$. Then R is ℓ-radical if and only if*

- *ℓ does not divide $q^2 - 1$ and $R = R_{1,0,0} \cong C_{\ell^a}$;*
- *ℓ divides $q^2 - 1$ and $R = \mathrm{Id}_2 \times R_{1,0,0} \cong C_{\ell^a}$, $R_{1,0,0} \times R_{1,0,0} \cong C_{\ell^a} \times C_{\ell^a}$ or $R_{2,0,0} \cong C_{\ell^a}$.*

Proof. By Theorem 2.1, R is either a basic subgroup of dimension 4 or $R = R_1 \times R_2$, where R_1, R_2 are basic subgroups of dimension 2 or trivial. As the dimension of $R_{m,\alpha,\gamma,\boldsymbol{c}}$ is $\ell^{|c|}m2e\ell^{\alpha+\gamma}$ it follows that $\alpha = \gamma = 0$ and $\boldsymbol{c}$ is empty for each basic subgroup of interested and thus by Lemma 2.5 and Corollary 2.4, R is radical in G. The result now follows by listing the basic subgroups.

If ℓ does not divide $q^2 - 1$, then $e = 2$ and hence $m = 1$ and the only basic subgroup is $R_{1,0,0}$. While, if ℓ divides $q^2 - 1$, then $e = 1$ and either $m = 1$ or 2 depending on whether the basic subgroup has dimension 2 or 4, respectively. This yields the basic subgroups $R_{1,0,0}$ and $R_{2,0,0}$. □

2.2. 2-radical subgroups of $\mathrm{Sp}_4(q)$. As with odd ℓ, to construct the 2-radical subgroups of G, we first need to construct the list of basic subgroups. As $\ell = 2$, we have that ℓ always divides $q^2 - 1$ and so let ϵ and $a \ge 2$ be defined so that $2^a = |q - \epsilon|_2$. This case requires some additional families of basic subgroups, which are obtained by taking the extra special group $E_\gamma = 2_-^{2\gamma+1}$ and replacing Z_α by a central product with $S_{2^{a+\alpha+1}}$, $D_{2^{a+\alpha+1}}$, or $Q_{2^{a+\alpha+1}}$: the semidihedral group, dihedral group, and generalized quaternion group of order $2^{a+\alpha+1}$, respectively.

We now turn our attention to summarizing the details of the required basic subgroups, taken from [2, Sections 1 and 2]. Note that as we are interested in the symplectic case, we have $\mathrm{Sym}(V) = -1$ and $\eta(V) = 1 = -\mathrm{Sym}(V)$ in the notation of [2]. We use $\circ$ to denote a central product.

$R^i_{\alpha,\gamma}$	isomorphism type	condition on α and γ	$\dim(V^i_{\alpha,\gamma})$
$R^0_{\alpha,\gamma}$	E_γ		2^γ
$R^1_{\alpha,\gamma}$	$E_\gamma \circ Z_\alpha$	$\alpha \geq 1$	$2^{\alpha+\gamma+1}$
$R^2_{\alpha,\gamma}$	$E_\gamma \circ S_{2^{a+\alpha+1}}$	$\alpha \geq 1$	$2^{\alpha+\gamma+1}$
$R^3_{\alpha,\gamma}$	$E_\gamma \circ D_{2^{a+\alpha+1}}$	$\alpha \geq 1$	$2^{\alpha+\gamma+2}$
$R^4_{\alpha,\gamma}$	$E_\gamma \circ Q_{2^{a+\alpha+1}}$	$\alpha \geq 1$ and $m \geq 2$ (see below)	$2^{\alpha+\gamma+1}$
		$\alpha = 0$	$2^{\gamma+1}$

Using the same construction as in the odd ℓ case, we then obtain the subgroups $R^i_{m,\alpha,\gamma,\boldsymbol{c}}$. Note that for the corresponding vector space $V^i_{m,\alpha,\gamma,\boldsymbol{c}}$, we have

$$\dim(V^i_{m,\alpha,\gamma,\boldsymbol{c}}) = 2^{|\boldsymbol{c}|} m \cdot \dim(V^i_{\alpha,\gamma}).$$

Definition 2. For the prime 2, the subgroups $R^i_{m,\alpha,\gamma,\boldsymbol{c}}$ are called the basic subgroups for $\mathrm{Sp}(V^i_{m,\alpha,\gamma,\boldsymbol{c}})$, excluding the case in which $i = \gamma = 0$ and $c_1 = 1$.

Throughout, for $R \leq G = \mathrm{Sp}_4(q)$ a 2-subgroup, we write $N := N_G(R)$ and $C := C_G(R)$. Let B_{2i} denote the set of basic subgroups of $\mathrm{Sp}_{2i}(q)$. Applying Theorem 2.1, a 2-radical subgroup of G is a member of one of the following:

$$\{\pm I_4\}, \quad \{\pm I_2\} \times \{\pm I_2\}, \quad \{\pm I_2\} \times B_2, \quad B_2 \times B_2, \quad B_4.$$

2.2.1. *The basic subgroups B_2.* In this case, the dimension of the underlying vector space $V^i_{m,\alpha,\gamma,\underline{c}}$ is equal to 2. Thus

$$2 = 2^{|c|} m \cdot \dim(V^i_{\alpha,\gamma}).$$

As $V^i_{\alpha,\gamma}$ is a symplectic space, it has even dimension. Therefore $m = 1$ and $\underline{c} = \varnothing$. In particular, the basic subgroups in B_2 are $R^0_{0,1} = E^{2+1}_{-} = Q_8$, $R^1_{0,0} = E_0 Z_0 \cong C_{2^a}$, and $R^4_{0,0} \cong Q_{2^{a+1}}$.

From this list we can in fact deduce the following well-known result. We note that this can be proven without the use of [2], however we shall use it here to help outline the details for the basic subgroups of $\mathrm{Sp}_4(q)$.

Theorem 2.7. *The radical* 2*-subgroups of* $\mathrm{Sp}_2(q) \cong \mathrm{SL}_2(q)$ *are given in Table 1.*

Proof. Let R be a radical 2-subgroup of $\mathrm{Sp}_2(q)$. Either R is from the list above or $R = \{\pm I_2\} = Z(\mathrm{Sp}_2(q))$. Thus assume R is a basic subgroup of $\mathrm{Sp}_2(q)$.

First consider $R = R^0_{0,1}$. Then $C_{\mathrm{Sp}_2(q)}(R) = Z(\mathrm{Sp}_2(q))$ and $N_{\mathrm{Sp}_2(q)}(R)/E \cong \mathfrak{S}_3$ or C_3 when $a \geq 3$ or $a = 2$ respectively, using [2, 1G]. Moreover, there is one conjugacy class when $a = 2$ and two conjugacy classes of subgroups when $a \geq 3$.

If $R = R^4_{0,0}$, then $C_{\mathrm{Sp}_2(q)}(R) = Z(\mathrm{Sp}_2(q))$ and $N_{\mathrm{Sp}_2(q)}(R)/R$ is trivial by [2, 2G], and R is determined uniquely up to conjugation.

Finally, if $R = R^1_{0,0}$, then $C_{\mathrm{Sp}_2(q)}(R) \cong \mathrm{GL}^\epsilon_1(q)$ and $N_{\mathrm{Sp}_2(q)}(R)/C_{\mathrm{Sp}_2(q)}(R) \cong C_2$, using [2, 1K]. However, it follows that any element in $N_{\mathrm{Sp}_2(q)}(R)$ not in $C_{\mathrm{Sp}_2(q)}(R)$ acts on $C_{\mathrm{Sp}_2(q)}(R)$ by inversion. Therefore, $O_2(N_{\mathrm{Sp}_2(q)}(R)) = R$ if and only if $C_{\mathrm{Sp}_2(q)}(R) \neq R$. Moreover, R is determined uniquely up to conjugation. $\square$

R	$C_{\mathrm{Sp}_2(q)}(R)$	$\mathrm{Out}_{\mathrm{Sp}_2(q)}(R)$	conditions
C_2	$\mathrm{Sp}_2(q)$	1	$q \geq 5$
Q_8	C_2	$\mathfrak{S}_3$	$a \geq 3$ (two classes)
Q_8	C_2	C_3	$a = 2$
C_{2^a}	$C_{(q-\epsilon)}$	C_2	$(q-\epsilon) \neq 2^a$
$Q_{2^{a+1}}$	C_2	1	$a \geq 3$

Table 1. The radical 2-subgroups of $\mathrm{Sp}_2(q)$.

We now have the following proposition.

Proposition 2.8. *Let $R = H_1 \times H_2$ with $H_i = \{\pm I_2\}$ or $H_i \in B_2$. Then R is a 2-radical subgroup of $G = \mathrm{Sp}_4(q)$ if and only if H_i is a 2-radical subgroup of $\mathrm{Sp}_2(q)$, unless $H_1 \cong H_2 \cong Q_{2^{a+1}}$ and $a \geq 3$.*

Proof. We have $Z(H_i) = \{\pm I_2\}$ unless $H_i = C_{2^a}$, in which case $Z(H_i) = H_i$. Furthermore, it can be assumed that $Z(H_i)$ consists only of diagonal matrices.

As $Z(R)$ is characteristic in R and $\mathrm{diag}(I_2, -I_2) \in Z(R)$, it follows that $\mathrm{diag}(I_2, -I_2)^g = \mathrm{diag}(I_2, -I_2)$ or $\mathrm{diag}(-I_2, I_2)$ for any $g \in N_G(R)$. Hence $g = \mathrm{diag}(A, D)$ or $h \cdot \mathrm{diag}(A, D)$, where $A, D \in \mathrm{Sp}_2(q)$ and

$$h = \begin{pmatrix} 0_2 & I_2 \\ I_2 & 0_2 \end{pmatrix}.$$

Thus

$$N_G(H_1 \times H_2) = N_{\mathrm{Sp}_2(q)}(H_1) \times N_{\mathrm{Sp}_2(q)}(H_2) \text{ or } N_{\mathrm{Sp}_2(q)}(H_1) \wr C_2.$$

Hence the result now follows by applying Lemma 2.3. □

In Table 2 we list all the radical 2-subgroups of $\mathrm{Sp}_4(q)$ that are not contained in B_4.

2.2.2. *The basic subgroups B_4.* In this case the dimension of the underlying vector space $V^i_{m,\alpha,\gamma,\boldsymbol{c}}$ is equal to 4. Thus

$$4 = 2^{|\boldsymbol{c}|} m \cdot \dim(V^i_{\alpha,\gamma}).$$

The following lemma deals with the case that $\boldsymbol{c}$ is nonempty. In particular, as $\dim(V^i_{\alpha,\gamma}) \geq 2$, it follows that $\boldsymbol{c} = \{1\}$.

Lemma 2.9. *Let $R = H \wr C_2$ for $H \in B_2$. Then R is radical in $G = \mathrm{Sp}_4(q)$ if and only if H is radical in $\mathrm{Sp}_2(q)$ and $H \neq Z(\mathrm{Sp}_2(q))$. Furthermore, the structure of C and N are given in Table 3.*

Proof. By [2, Equation 3.4], $R^i_{m,\alpha,\gamma,\boldsymbol{c}}$ is radical in $\mathrm{Sp}(V^i_{m,\alpha,\gamma,\boldsymbol{c}})$ if and only if $R^i_{m,\alpha,\gamma}$ is radical in $\mathrm{Sp}(V^i_{m,\alpha,\gamma})$, except when $i = \gamma = 0$ and $c_1 = 1$, in which case $R^0_{m,0,0,\boldsymbol{c}}$ is not radical as $R^0_{0,1}$ is not radical in $\mathrm{Sp}_2(q)$ by [2, 1J]. □

Thus, we can assume that $\boldsymbol{c}$ is empty. Next we deal with the case that $m = 2$ and $\dim(V^i_{\alpha,\gamma}) = 2$. In this case the basic subgroups are the 2-fold embeddings of basic subgroups in B_2. In particular, the relevant groups are:

$R^i_{m,\alpha,\gamma}$	$R^0_{2,0,1}$	$R^1_{2,0,0}$	$R^4_{2,0,0}$
isomorphism type	Q_8	C_{2^a}	$Q_{2^{a+1}}$

type	R	$C_G(R)$	$\mathrm{Out}_G(R) \cong \frac{N}{CR}$	conditions
$\{\pm I_4\}$	$Z(G) = C_2$	G	1	
$\{\pm I_2\} \times \{\pm I_2\}$	$C_2 \times C_2$	$\mathrm{SL}_2(q) \times \mathrm{SL}_2(q)$	C_2	$q \geq 5$
$\{\pm I_2\} \times B_2$	$C_2 \times C_{2^a}$	$\mathrm{SL}_2(q) \times C_{q-\epsilon}$	C_2	$q - \epsilon \neq 2^a$
	$C_2 \times Q_8$	$\mathrm{SL}_2(q) \times C_2$	C_3	$a = 2$ and $q \geq 5$
	$C_2 \times Q_8$	$\mathrm{SL}_2(q) \times C_2$	$\mathfrak{S}_3$	$a \geq 3$ two classes
	$C_2 \times Q_{2^{a+1}}$	$\mathrm{SL}_2(q) \times C_2$	1	$a \geq 3$
$B_2 \times B_2$	$C_{2^a} \times C_{2^a}$	$C_{q-\epsilon} \times C_{q-\epsilon}$	D_8	$q - \epsilon \neq 2^a$
	$C_{2^a} \times Q_8$	$C_{q-\epsilon} \times C_2$	C_6	$a = 2$ and $q \geq 7$
	$C_{2^a} \times Q_8$	$C_{q-\epsilon} \times C_2$	D_{12}	$a \geq 3$ and $q - \epsilon \neq 2^a$ two classes
	$C_{2^a} \times Q_{2^{a+1}}$	$C_{q-\epsilon} \times C_2$	C_2	$a \geq 3$ and $q - \epsilon \neq 2^a$
	$Q_8 \times Q_8$	$C_2 \times C_2$	$(C_3 \times C_3).2$	$a = 2$
	$Q_8 \times Q_8$	$C_2 \times C_2$	$(\mathfrak{S}_3 \times \mathfrak{S}_3).2$	$a \geq 3$ two classes
	$Q_8 \times Q_8$	$C_2 \times C_2$	$(\mathfrak{S}_3 \times \mathfrak{S}_3)$	$a \geq 3$
	$Q_8 \times Q_{2^{a+1}}$	$C_2 \times C_2$	$\mathfrak{S}_3$	$a \geq 3$ two classes

Table 2. The radical 2-subgroups of $G = \mathrm{Sp}_4(q)$ not of type B_4.

Proposition 2.10. *Let $R = R^0_{2,0,1} \cong Q_8$. Then R is a radical 2-subgroup of $G = \mathrm{Sp}_4(q)$ if and only if $q \geq 5$. In addition $C \cong D_{2(q+\epsilon)}$ and $\frac{N}{RC} \cong D_6$.*

Proof. Let $R := R^0_{2,0,1} \cong Q_8$. In this case we make use of [2, 1Jb]. Write $C := C_G(R)$, $N := N_G(R)$, and $N^1 := C_N(C)$. Then $N^1C = N^1 \circ_{Z(G)} C$ and $\frac{N^1}{R} \cong D_6$ or C_3 depending on $a \geq 3$ or $a = 2$ respectively. Thus $O_2(N^1C) = R$. As $m = 2$, R is determined uniquely up to conjugation. Furthermore, $C = O_2^{-\epsilon}(q) \cong D_{2(q+\epsilon)}$. Note that $O_2(D_{2(q+\epsilon)}) \neq Z(R)$ if and only if $D_{2(q+\epsilon)}$ is a 2-group, if and only if $q = 3$.

If $a \geq 3$, then $N = N^1C$ and so R is radical and $\frac{N}{RC} \cong \frac{N^1}{R} \cong D_6$. Thus assume $a = 2$. By [2, 1Jb] we have that $\frac{N}{RC} \cong D_6$. Thus it remains to show that R is radical. Consider $\frac{N}{R}$, which has a normal subgroup $\frac{N^1C}{R}$ of index 2. As $R = O_2(N^1C)$, it follows that $O_2\big(\frac{N^1C}{R}\big) = 1$. Therefore if $O_2\big(\frac{N}{R}\big) = H/R$, then $HCR \lhd N$ and HCR/CR is a nontrivial normal 2-subgroup in D_6, which is a contradiction. Thus $O_2(N) = R$. $\square$

Proposition 2.11. *Let $R = R^1_{2,0,0} \cong C_{2^a}$. Then R is a radical 2-subgroup of $G = \mathrm{Sp}_4(q)$. In addition, $C \cong \mathrm{GL}^\epsilon_2(q)$ and $\frac{N}{RC} \cong C_2$.*

Proof. Let

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

By [7], the subgroup C_{2^a} is generated by

$$w = \begin{cases} \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix} & \text{when } q \equiv 1 \text{ modulo } 4, \text{ or} \\ \begin{pmatrix} 0 & 1 \\ 1 & \eta + \eta^q \end{pmatrix}^2 & \text{when } q \equiv 3 \text{ modulo } 4, \end{cases}$$

where η has order 2^a in $\mathbb{F}_q^\times$ or $\mathbb{F}_{q^2}^\times$, respectively. Furthermore $w^J = w^{-1}$. As the image inside G is taken from the double embedding, we can take the symplectic form for G to be $J_2 := \operatorname{diag}(J, J)$.

Using eigenvalues, it follows that the image of w in G under conjugation must be either w or w^{-1}. Thus it follows that $N = \langle C, J_2 \rangle$. Furthermore, $O_2(N)$ equals either $O_2(C)$ or $\langle O_2(C), xJ_2 \rangle$ for some $x \in C$.

If xJ_2 is in $O_2(N)$, and $A \in N$, then $A(xJ_2)A^{-1} = zxJ_2$ for some $z \in O_2(C)$. As $C \cong \mathrm{GL}_2^\epsilon(q)$ by [2, 1Ka], it follows that $O_2(C) = R$ and $z \in R$. Furthermore, as J_2 is the symplectic form we have chosen, we see that $A^t J_2 = J_2 A^{-1}$. Hence $AxA^t = zx$. Assume

$$x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix},$$

with $x_i \in \mathrm{Mat}_2(\mathbb{F}_q)$. The element $\operatorname{diag}(I_2, -I_2) \in C$, so

$$\begin{bmatrix} I_2 & \\ & -I_2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} I_2 & \\ & -I_2 \end{bmatrix} = \begin{bmatrix} x_1 & -x_2 \\ -x_3 & x_4 \end{bmatrix} = \begin{bmatrix} z & \\ & z \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

for some $z \in R$ viewed as a subgroup of $\mathrm{Sp}_2(q)$. In particular, either $(z-1)x_1 = (z-1)x_4 = 0$ or $(z+1)x_2 = (z+1)x_3 = 0$. However either $x_1 = x_4 = 0$ or $x_2 = x_3 = 0$, as $(z-1)$ or $(z+1)$ is invertible. (Indeed, z is of the form $\operatorname{diag}(\lambda, \lambda^{-1})$ after possibly conjugating in $\mathrm{Sp}_2(\overline{\mathbb{F}}_q)$.) Thus

$$x = \begin{bmatrix} x_1 & \\ & x_4 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} & x_2 \\ x_3 & \end{bmatrix}.$$

However, in either case it now follows that at least one $x_i J$ lies in $O_2(N_{\mathrm{Sp}_2(q)}(R))$, which is a contradiction. Thus $O_2(N_{\mathrm{Sp}_2(q)}(R)) = O_2(C_{\mathrm{Sp}_2(q)}(R)) = R$. □

Proposition 2.12. *Let $R = R^4_{2,0,0} \cong Q_{2^{a+1}}$. Then R is not a radical 2-subgroup of $G = \mathrm{Sp}_4(q)$.*

Proof. Let J, J_2, and w be as in the proof of Proposition 2.11. Then $\langle J, w \rangle = Q_{2^{a+1}} \leq \mathrm{GL}_2(q)$, and J_2 is the symplectic form for G. Let $A \in C$. Then $A^{-1}J_2A = J_2$ and $A^tJA = J$. Therefore $A^t = A^{-1}$. Moreover, if

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

then each A_i is conjugate to a matrix of the form $\operatorname{diag}(a_i, a_i)$ in $\mathrm{GL}_2(q^2)$. Thus $C \cong O_2^\epsilon(q) \cong D_{2(q-\epsilon)}$. However $O_2(D_{2(q-\epsilon)}) \cong C_{2^a} > Z(R) = C_2$, and so R is not radical. □

It now only remains to consider the case that $\dim(V^i_{\alpha,\gamma}) = 4$. Here the groups of interest are:

$R^i_{\alpha,\gamma}$	$R^0_{0,2}$	$R^1_{0,1}$	$R^1_{1,0}$	$R^2_{1,0}$	$R^3_{0,0}$	$R^4_{0,1}$
isomorphism type	2^{1+4}_-	$Q_8 \circ C_{2^a}$	$C_{2^{a+1}}$	$S_{2^{a+2}}$	$D_{2^{a+1}}$	$Q_8 \circ Q_{2^{a+1}}$

Proposition 2.13. *Let $R = R^0_{0,2} \cong 2^{1+4}_-$. If $a = 2$, then there is a unique conjugacy class of subgroups isomorphic to R, while if $a \geq 3$ then there are two classes of subgroups isomorphic to R. Furthermore, R is a radical 2-subgroup of $G = \mathrm{Sp}_4(q)$, with $C \cong C_2$ and $\frac{N}{RC} \cong \mathfrak{A}_5$ when $a \geq 3$, or $\mathfrak{A}_5.2$ when $a = 2$.*

Proof. By [2, 1Jb], $C \cong C_2$ and

$$\frac{N}{CR} = \frac{N}{R} = \begin{cases} O_4^-(2) \cong \mathfrak{A}_5.2 & \text{if } a \geq 3, \\ \Omega_4^-(2) \cong \mathfrak{A}_5 & \text{if } a = 2, \end{cases}$$

so $O_2(N/R) = 1$ and R is a radical 2-subgroup. Moreover, if $a = 2$ then there is a unique class for R, while if $a \geq 3$ then there are two classes for R up to conjugacy. □

Proposition 2.14. *Let $R = R^1_{0,1} \cong C_{2^a} \circ Q_8$. Then R is a radical 2-subgroup of $\mathrm{Sp}_4(q)$ if and only if $q - \epsilon \neq 2^a$. In addition, $C \cong C_{q-\epsilon}$ and $\frac{N}{RC} \cong D_{12}$.*

Proof. In this case we use [2, 1K]. We obtain $R = Q_8 \circ C_{2^a}$ by the inclusion

$$R \leq \mathrm{GL}_2^\epsilon(q) \hookrightarrow \mathrm{Sp}_4(q),$$

where $C_{2^a} \leq Z(\mathrm{GL}_2(q))$. Let H denote the image of $N_{\mathrm{GL}_2^\epsilon(q)}(R)$ under this inclusion, so that H has index 2 in N and $C \cong \mathrm{GL}_1^\epsilon(q)$ is the image of $C_{\mathrm{GL}_2^\epsilon(q)}(R)$.

By [1, Lemma 1B] and [3, Lemma 1L], we have $O_2(H) = R$, since $O_2(CR) = R$. Then

$$O_2\left(\frac{H}{R}\right) \cong \frac{O_2\left(\frac{H}{R}\right)\frac{CR}{R}}{\frac{CR}{R}} \cong \frac{K}{CR} \lhd \frac{H}{CR} \cong D_6.$$

Moreover, $\frac{N}{RC}$ has a normal subgroup D_6 of index 2, so $\frac{N}{RC} \cong D_{12}$. (Indeed, this is the only group of order 12 containing a normal subgroup isomorphic to D_6.)

If $q - \epsilon = 2^a$, then $CR = R$ and so $O_2\left(\frac{N}{R}\right) > 1$ so R is not radical. On the other hand, if $q - \epsilon \neq 2^a$, then there exists $x \in C$ of odd order, and after the embedding is of the form $\mathrm{diag}(\eta, \eta, \eta^{-1}, \eta^{-1})$. Let $h \in O_2(N)$. If $x^h = x^{-1}$, then $h^x = x^{-2}h$, which implies $x^{-2} \in O_2(N)$, so $x = 1$. Therefore $x^h = x$ for all $h \in O_2(N)$. Thus $O_2(N) \leq C_N(C) \leq H$. In particular, it now follows that $O_2(N) = R$. □

Proposition 2.15. *Let $R = R^1_{1,0} \cong C_{2^{a+1}}$. Then R is a radical 2-subgroup of $G = \mathrm{Sp}_4(q)$ if and only if $q \geq 5$. In addition, $C \cong C_{q^2-1}$ and $\frac{N}{RC} \cong C_2 \times C_2$.*

Proof. The group $C_{2^{a+1}}$ is obtained via the embedding

$$\mathbb{F}_{q^2} \hookrightarrow \mathrm{GL}_2(q) \hookrightarrow \mathrm{Sp}_4(q).$$

If $\beta \in \mathbb{F}_q \setminus \mathbb{F}_q^2$, then the image of $\mathbb{F}_{q^2}$ in $\mathrm{GL}_2(q)$ is given by the subgroup

$$K := \left\{ \begin{bmatrix} \lambda & \beta\mu \\ \mu & \lambda \end{bmatrix} \;\middle|\; \lambda, \mu \in \mathbb{F}_q \text{ such that both } \lambda, \mu \neq 0 \right\} \cong C_{q^2-1},$$

while the embedding from $\mathrm{GL}_2(q)$ into $\mathrm{Sp}_4(q)$ is given by

$$A \mapsto \begin{bmatrix} A & \\ & (A^t)^{-1} \end{bmatrix}, \quad \text{with symplectic form } J = \begin{bmatrix} & I_2 \\ -I_2 & \end{bmatrix}.$$

Let g generate the subgroup $C_{2^{a+1}} \le \mathbb{F}_{q^2}^\times$. Then $C_{\mathrm{GL}_2(q)}(g) = K$. Moreover, since $\det(g) \neq 1$, we see that g and $(g^t)^{-1}$ have different eigenvalues, so the Sylvester matrix equation implies $C_G(g)$ is the image of K.

As the eigenvalues of g are $\lambda \pm \sqrt{\beta}\mu$, it follows that $N_{\mathrm{GL}_2(q)}(C_{2^{a+1}}) = \langle C_{\mathrm{GL}_2(q)}(g), X\rangle$ for

$$X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Furthermore, as the eigenvalues of $(g^t)^{-1}$ are $(\lambda \pm \sqrt{\beta}\mu)/(\lambda^2 - \beta\mu^2)$ it follows that any element in $N_G(C_{2^{a+1}})$ is a product of an element from H, the embedding of $N_{\mathrm{GL}_2(q)}(C_{2^{a+1}})$, with the element

$$Y = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix},$$

which acts on H by inversion on $C_{\mathrm{GL}_2(q)}(g)$ and sending the image of X to its negative. Thus $C := C_{\mathrm{Sp}_4(q)}(C_{2^{a+1}}) \cong C_{q^2-1}$ has index 2 in H and H has index 2 in $N := N_{\mathrm{Sp}_4(q)}(C_{2^{a+1}})$. Then if $q^2-1 = 2^{a+1}$, the group R is not radical.

Assume $q^2 - 1 \neq 2^{a+1}$, i.e., $q > 3$. If $O_2(H) > O_2(C)$, it follows that $cX \in O_2(H)$ for some $c \in C$. Let d be an element of odd order in C; then $[d, cX] = \det(d^{-1})d^2 \in O_2(C)$, which yields a contraction. If $O_2(N) > O_2(H)$, it follows that an element cY or cXY is in $O_2(N)$ for $c \in C$ and X is taken to be its image in N. Furthermore, $c^Y Y = c^{-1}Y$ and $(cX)^Y Y = -c^{-1}XY$ and therefore c^2 lies in $O_2(N)$, which provides a contradiction. Thus $O_2(N) = R$. □

Proposition 2.16. *Let $R = R^2_{1,0} \cong S_{2^{a+2}}$. Then R is a radical 2-subgroup of $G = \mathrm{Sp}_4(q)$ if and only if $q \ge 5$. In this case, $C \cong C_{q+\epsilon}$ and $\frac{N}{RC} \cong C_2$.*

Proof. When $\epsilon = 1$, we have $C \cong q+1$, $\frac{N}{RC} \cong C_{2^\alpha} = C_2$, and $R = O_2(N)$ by [2, 2Bd]. When $\epsilon = -1$, we have $C \cong q-1$, $\frac{N}{RC} \cong C_2$, and $R = O_2(N)$, unless $q = 3$ as in this case C is a 2-group and thus N is a 2-group by [2, 2Cc]. In each case, R is determined uniquely up to conjugation. □

Proposition 2.17. *Let $R = R^3_{0,0} \cong D_{2^{a+1}}$. Then R is a radical 2-subgroup of $G = \mathrm{Sp}_4(q)$ if and only if $q \ge 5$. In addition, $C \cong \mathrm{Sp}_2(q) \cong \mathrm{SL}_2(q)$ and $\frac{N}{RC} \cong C_2$.*

Proof. By [2, 2Ce], $R = O_2(N)$, $C_{\mathrm{Sp}_4(q)}(D_{2^{a+1}}) \cong \mathrm{Sp}_2(q) \cong \mathrm{SL}_2(q)$, and $\frac{N}{RC} \cong C_2$. Furthermore, R is determined uniquely up to conjugation. □

Proposition 2.18. *Let $R = R^4_{0,1} \cong Q_8 \circ Q_{2^{a+1}}$. Then R is a radical 2-subgroup of $G = \mathrm{Sp}_4(q)$. If $a = 2$, then $R \cong R^0_{0,2}$ and the structures of C and N are given in Proposition 2.13. If $a \ge 3$, then $C = Z(G) \cong C_2$ and $\frac{N}{RC} \cong \mathfrak{S}_3$.*

Proof. Let $R = R^4_{0,1} \cong Q_8 \circ Q_{2^{a+1}}$. When $a = 2$, $R = R^0_{0,2}$. Thus assume that $a \ge 3$. In this case, [2, 2G] yields that $C = Z(\mathrm{Sp}_4(q))$ and $\frac{N}{CR} = \frac{N}{R} \cong \mathrm{Sp}_{2^\gamma}(2) = \mathrm{Sp}_2(2) \cong \mathfrak{S}_3$. □

type for B_4	R	$C_G(R)$	$\mathrm{Out}_G(R) \cong \frac{N}{CR}$	conditions
$\underline{c} = 0$ and $m = 1$	2_-^{1+4}	C_2	$\Omega_4^-(2) \cong \mathfrak{A}_5$	$a = 2$
	2_-^{1+4}	C_2	$O_4^-(2) \cong \mathfrak{A}_5.2$	$a \geq 3$ two classes
	$C_{2^a} \circ_2 Q_8$	$C_{q-\epsilon}$	D_{12}	$q - \epsilon \neq 2^a$
	$C_{2^{a+1}}$	C_{q^2-1}	$C_2 \times C_2$	$q \geq 5$
	$S_{2^{a+2}}$	$C_{q+\epsilon}$	C_2	$q \geq 5$
	$D_{2^{a+1}}$	$\mathrm{SL}_2(q)$	C_2	$q \geq 5$
	$Q_8 \circ_2 Q_{2^{a+1}}$	C_2	$\mathfrak{S}_3$	$q \geq 5$, $a \geq 3$
$\underline{c} = 0$ and $m = 2$	Q_8	$O_2^{-\epsilon}(q) \cong D_{2(q+\epsilon)}$	$\mathfrak{S}_3$	$q \geq 5$
	C_{2^a}	$\mathrm{GL}_2^\epsilon(q)$	C_2	$q \geq 5$
$\underline{c} = 1$ and $m = 1$	$C_{2^a} \wr C_2$	$C_{q-\epsilon}$	C_2	$q - \epsilon \neq 2^a$
	$Q_8 \wr C_2$	C_2	C_3	$a = 2$
	$Q_8 \wr C_2$	C_2	$\mathfrak{S}_3$	$a \geq 3$ two classes
	$Q_{2^{a+1}} \wr C_2$	C_2	1	$a \geq 3$

Table 3. The radical 2-subgroups of $G = \mathrm{Sp}_4(q)$ of type B_4.

We finish by giving Table 3, which lists the radical 2-subgroups of type B_4.

3. Relevant characters for $\ell = 2$

3.1. Brauer characters of $\mathrm{Sp}_4(q)$. White [19] computed the 2-block distributions and 2-decomposition numbers for $G = \mathrm{Sp}_4(q)$. In an effort to keep this article self-contained, we summarize in Table 4 some of the relevant information.

Given $\chi \in \mathrm{Irr}(G)$, we write $\widehat{\chi}$ for the 2-Brauer character obtained by restricting χ to 2-regular elements of G. The notation for characters and indexing sets is taken from [17]. Further, the integer x in the description of the principal block characters is a number satisfying $0 \leq x \leq (q-1)/2$, and does not affect our work here. The indexing sets are defined as in [17] and [19], as follows:

The set T_1' is the set of multiples of $(q-1)_2$ in $\{1, \dots, (q-1)/2-1\}$. The set T_2' is the set of multiples of $(q+1)_2$ in $\{1, \dots, (q+1)/2-1\}$. We will further write T_ϵ' for T_1' when $\epsilon = 1$ and T_2' when $\epsilon = -1$. Similarly, $T_{-\epsilon}'$ denotes T_2' when $\epsilon = 1$ and T_1' when $\epsilon = -1$.

The set R_1' is comprised of the even integers in the equivalence classes of $\{1, \dots, q^2\} \setminus \{(q^2+1)/2\}$ under the equivalence relation $i \sim j$ when $i \equiv \pm j$ or $\pm qj \pmod{q^2+1}$.

The set R_2' is comprised of the set of multiples of $(q^2-1)_2$ in the equivalence classes of

$$\{1 \leq i \leq q^2 - 1 \mid (q+1) \nmid i;\ (q-1) \nmid i\}$$

under the equivalence relation $i \sim j$ when $i \equiv \pm j$ or $\pm qj \pmod{q^2-1}$.

3.2. Defect-zero characters of $N_G(R)/R$. In this section, we develop the notation to describe the defect-zero characters of $N_G(R)/R$ for the 2-radical subgroups R described in Tables 2 and 3. Recall that $\epsilon \in \{\pm 1\}$ is such that $q \equiv \epsilon \pmod 4$, so that $(q^2-1)_2 = 2(q-\epsilon)_2$. Throughout, let η, η', and θ denote fixed generators of the subgroups $C_{q-\epsilon}$, $C_{q+\epsilon}$, and C_{q^2-1} in $\mathbb{F}_{q^2}^\times$, respectively.

block B	Brauer characters $\mathrm{IBr}_2(B)$	indexing information	number of blocks
$b_1(r)$	$\widehat{\chi}_1(r)$	$r \in R_1'$	$\dfrac{q^2-1}{8}$
$b_2(r)$	$\widehat{\chi}_2(r)$	$r \in R_2'$	$\dfrac{((q-1)_{2'}-1)((q+1)_{2'}-1)}{4}$
$b_3(r,s)$	$\widehat{\chi}_3(r,s)$	$r, s \in T_1', r \neq s$	$\dfrac{((q-1)_{2'}-1)((q-1)_{2'}-3)}{8}$
$b_4(r,s)$	$\widehat{\chi}_4(r,s)$	$r, s \in T_2', r \neq s$	$\dfrac{((q+1)_{2'}-1)((q+1)_{2'}-3)}{8}$
$b_5(r,s)$	$\widehat{\chi}_5(r,s)$	$r \in T_2', s \in T_1'$	$\dfrac{((q-1)_{2'}-1)((q+1)_{2'}-1)}{4}$
$b_{67}(r)$	$\widehat{\chi}_6(r)$ $\widehat{\chi}_7(r)-\widehat{\chi}_6(r)$	$r \in T_2'$	$\dfrac{(q+1)_{2'}-1}{2}$
$b_{89}(r)$	$\widehat{\chi}_8(r)$ $\widehat{\chi}_9(r)-\widehat{\chi}_8(r)$	$r \in T_1'$	$\dfrac{(q-1)_{2'}-1}{2}$
$b_I(r)$	$\widehat{\xi}_1(r)$ $\widehat{\xi}'_{22}(r)$ $\widehat{\xi}'_{21}(r)$	$r \in T_2'$	$\dfrac{(q+1)_{2'}-1}{2}$
$b_{III}(r)$	$\widehat{\xi}_3(r)$ $\widehat{\xi}_{42}(r)-\widehat{\xi}_3(r)$ $\widehat{\xi}_{41}(r)-\widehat{\xi}_3(r)$	$r \in T_1'$	$\dfrac{(q-1)_{2'}-1}{2}$
b_0	$\varphi_0=\widehat{1}$ $\varphi_3=\widehat{\theta}_{12}-1$ $\varphi_6=\widehat{\theta}_{10}$ $\varphi_1=\widehat{\Phi}_3-x\widehat{\theta}_{10}-\widehat{\theta}_7$ $\varphi_2=\widehat{\Phi}_4-x\widehat{\theta}_{10}-\widehat{\theta}_8$ $\varphi_4=\widehat{\theta}_7$ $\varphi_5=\widehat{\theta}_8$		1

Table 4. The blocks and Brauer characters of $\mathrm{Sp}_4(q)$ for the prime 2; see [19].

We may embed the group $C_{q-\epsilon}.2$ into $\mathrm{SL}_2(q)$ naturally with $C_{q-\epsilon}$ realized as the subgroup generated by $\mathrm{diag}(\eta, \eta^{-1})$, up to $\mathrm{SL}_2(\overline{\mathbb{F}}_q)$-conjugation. Here, the C_2 factor maps $\eta \mapsto \eta^{-1}$. We will denote by $\tilde{\eta}_k$ the character of $C_{q-\epsilon}.2$ whose restriction to $C_{q-\epsilon}$ is $\bar{\eta}^k+\bar{\eta}^{-k}$, where $\bar{\eta}$ is a generator of $\mathrm{Irr}(C_{q-\epsilon})$ sending η to a fixed primitive $q-\epsilon$ root of unity in $\mathbb{C}$. The corresponding characters of $C_{(q-\epsilon)_{2'}}.2$ are the $\tilde{\eta}_k$ for $k \in T_\epsilon'$. We will use the same notation when $C_{q-\epsilon}$ is embedded into G via $\mathrm{SL}_2(q)\times\mathrm{SL}_2(q)$ as the subgroup generated by $\mathrm{diag}(\eta, \eta^{-1}, \eta, \eta^{-1})$.

Similarly, we may embed $C_{q+\epsilon}.2$ in G so that the $C_{q+\epsilon}$ factor is $\mathrm{Sp}_4(\bar{\mathbb{F}}_q)$-conjugate to

$$\mathrm{diag}(\eta', \eta'^{-1}, \eta', \eta'^{-1})$$

and the C_2 factor maps $\eta' \mapsto \eta'^{-1}$. Here $\tilde{\eta}'_k$ will denote the character of $C_{q+\epsilon}.2$ whose restriction to $C_{q+\epsilon}$ is $\bar{\eta}'^k + \bar{\eta}'^{-k}$, where $\bar{\eta}'$ is a generator of $\mathrm{Irr}(C_{q+\epsilon})$ sending η' to a fixed primitive $q+\epsilon$ root of unity in $\mathbb{C}$.

We may also embed the group $C_{q^2-1}.2^2$ in G so that the C_{q^2-1} factor is generated by the element $\mathrm{diag}(\theta, \theta^q, \theta^{-1}, \theta^{-q})$, up to $\mathrm{Sp}_4(\bar{\mathbb{F}}_q)$-conjugacy, with the two copies of C_2 in $C_2 \times C_2$ mapping $\theta \mapsto \theta^q$ and $\theta \mapsto \theta^{-1}$. We denote by $\tilde{\theta}_k$ the character of $C_{q^2-1}.2^2$ whose restriction to C_{q^2-1} is $\bar{\theta}^k + \bar{\theta}^{qk} + \bar{\theta}^{-k} + \bar{\theta}^{-qk}$, where $\bar{\theta}$ is a generator of $\mathrm{Irr}(C_{q^2-1})$, mapping θ to a fixed primitive q^2-1 root of unity in $\mathbb{C}$.

We will also require characters of $\mathrm{PSL}_2(q)$ of degree $q-\epsilon$. Specifically, when $\epsilon = 1$, we will denote by $\chi_\bullet(k)$ the family of characters $\chi_6(k)$ in CHEVIE notation of degree $q-1$, which may be indexed by $k \in T'_2$. We remark that this indexing is slightly different than that of CHEVIE; taking the indexing to be T'_2 yields the value $-\xi_1^{ik} - \xi_1^{-ik}$, where ξ_1 is a primitive $q+1$ root of unity (rather than $-\xi_1^{2ik} - \xi_1^{-2ik}$) on the class $C_5(i)$ in CHEVIE notation, since the indices are divisible by 2. Similarly, when $\epsilon = -1$, we will denote by $\chi_\bullet(k)$ the family $\chi_5(k)$ of characters of $\mathrm{PSL}_2(q)$ of degree $q+1$, which may be indexed by $k \in T'_1$, keeping similar considerations in mind. Note that under our notation, the indexing set for $\chi_\bullet$ is $T'_{-\epsilon}$. Finally, we let ψ denote the irreducible character of $\mathfrak{S}_3$ of degree 2, ν denote the irreducible character of $\mathfrak{A}_5$ of degree 4, and μ denote a fixed generator of $\mathrm{Irr}(C_3)$.

3.3. Defect groups. We begin by considering the normalizers of the radical subgroups that are defect groups (according to [19]) of blocks of $G = \mathrm{Sp}_4(q)$.

First, consider the radical subgroup $R \cong C_2 \times C_2$ of type $\{\pm I_2\} \times \{\pm I_2\}$. Here R is in fact the defect group of the block $B = b_3(r, s)$ when $\epsilon = -1$ and $b_4(r, s)$ when $\epsilon = 1$. The normalizer $N_G(R)$ is of the form $\mathrm{SL}_2(q) \wr C_2$, where the base subgroup $\mathrm{SL}_2(q)^2$, which is also the centralizer $C_G(R)$, can be viewed as being embedded blockwise in the natural way. Here $N_G(R)/R$ is of the form $\mathrm{PSL}_2(q) \wr C_2$, and $|N_G(R)/R|_2 = 2(q-\epsilon)_2^2$. Hence we see that $\mathrm{dz}(N_G(R)/R)$ is comprised of characters whose restriction to $\mathrm{PSL}_2(q)^2$ is $(\chi_\bullet(r) \times \chi_\bullet(s)) + (\chi_\bullet(s) \times \chi_\bullet(r))$ for $r \neq s$ in $T'_{-\epsilon}$. We will write $\chi_\bullet(r, s)$ for such a character.

Now let $R \cong C_2 \times C_{2^a}$ be the radical subgroup of type $\{\pm I_2\} \times B_2$. Then R is the defect group of the blocks of the form $b_5(r, s)$. Here $N_G(R) \cong \mathrm{SL}_2(q) \times C_{q-\epsilon}.2$ and $N_G(R)/R \cong \mathrm{PSL}_2(q) \times C_{(q-\epsilon)_{2'}}.2$. Hence a defect-zero character of $N_G(R)/R$ has degree $2(q-\epsilon)_2$, so must be $\chi_\bullet(k) \times \tilde{\eta}_t$ for some $(k, t) \in T'_{-\epsilon} \times T'_\epsilon$.

Let $R \cong C_2 \times Q_{2^{a+1}}$ be the radical subgroup of type $\{\pm I_2\} \times B_2$, which is the defect group of the blocks of the form $b_I(r)$ when $\epsilon = 1$ and $b_{III}(r)$ when $\epsilon = -1$. We remark that these blocks each contain three irreducible Brauer characters. We remark that in Section 4, we will define $\mathrm{IBr}_2(G|R)$ to contain just one of these from each block. Here $N_G(R) \cong \mathrm{SL}_2(q) \times Q_{2^{a+1}}$ and $N_G(R)/R \cong \mathrm{PSL}_2(q)$, whose defect-zero characters are those in the family $\chi_\bullet(k)$ for $k \in T'_{-\epsilon}$.

Let $R \cong C_{2^a} \times C_{2^a}$ be the radical subgroup of type $B_2 \times B_2$, which appears as the defect group of the blocks $b_4(r, s)$ when $\epsilon = -1$ and $b_3(r, s)$ when $\epsilon = 1$. Here $N_G(R) \cong C_{q-\epsilon}^2.D_8 \cong (C_{q-\epsilon}.2) \wr C_2$ and $N_G(R)/R \cong (C_{(q-\epsilon)_{2'}}.2) \wr C_2$. Since defect-zero characters of $N_G(R)/R$ have degree 8, for $k \neq t$ in T'_ϵ they must be of the form $(\tilde{\eta}_k \times \tilde{\eta}_t) + (\tilde{\eta}_t \times \tilde{\eta}_k)$ on restriction to $(C_{(q-\epsilon)_{2'}}.2)^2$. In this case, we will denote such a character of $N_G(R)/R$ by $\tilde{\eta}_{t,k}$.

When R is the radical subgroup of type $B_2 \times B_2$ of the form $C_{2^a} \times Q_{2^{a+1}}$, R is the defect group of a block of the form $b_{III}(r)$ when $\epsilon = 1$ and $b_I(r)$ when $\epsilon = -1$. Again, these blocks each contain three irreducible Brauer characters, and we will define $\mathrm{IBr}_2(G|R)$ in Section 4 below to contain just one of these from each block. Here $N_G(R) \cong C_{q-\epsilon}.2 \times Q_{2^{a+1}}$ and $N_G(R)/R \cong C_{(q-\epsilon)_{2'}}.2$, which has defect-zero characters $\tilde{\eta}_k$ for $k \in T'_\epsilon$.

The subgroups $R \cong C_{2^{a+1}}$ of type B_4 are the defect groups for the blocks of the form $b_2(r)$. Here we have $N_G(R) \cong C_{q^2-1}.2^2$ and $N_G(R)/R \cong C_{(q^2-1)_{2'}}.2^2$. Then $\mathrm{dz}(N_G(R)/R)$ is comprised of characters of degree 4, of the form $\tilde{\theta}_k$ for $k \in R'_2$.

Now let R be semidihedral of size 2^{a+2}. Here R is a defect group for a block of the form $b_{67}(r)$ when $\epsilon = 1$ and $b_{89}(r)$ when $\epsilon = -1$, which contain two Brauer characters in each block. We will define $\mathrm{IBr}_2(G|R)$ in Section 4 to contain one such character from each block. We have $N_G(R)/R \cong C_{(q+\epsilon)_{2'}}.2$ and the defect-zero characters are of the form $\tilde{\eta}'_k$ for $k \in T'_{-\epsilon}$.

Now let R be of type B_4 of the form $C_{2^a} \wr C_2$. Here we have R is a defect group for a block of the form $b_{89}(r)$ when $\epsilon = 1$ and $b_{67}(r)$ when $\epsilon = -1$, which contain two Brauer characters in each block. Again, we will define $\mathrm{IBr}_2(G|R)$ below to contain one such character from each block. We have $N_G(R)/R \cong C_{(q-\epsilon)_{2'}}.2$ and the defect-zero characters are of the form $\tilde{\eta}_k$ for $k \in T'_\epsilon$.

Let $R \in \mathrm{Syl}_2(G)$, so R is a defect group of b_0 and $N_G(R)/R$ is trivial when $a \geq 3$, which means there is a unique (trivial) defect-zero character. When $a = 2$, $N_G(R)/R \cong C_3$, and we have three defect-zero characters corresponding to the three members of $\mathrm{Irr}(C_3)$.

3.4. The remaining radical subgroups. We now address the radical subgroups that are not defect groups for any block of G.

For the radical subgroups of type B_4 of the form $R \cong D_{2^{a+1}}$ with $q \geq 5$, we have $N_G(R)/R \cong \mathrm{PSL}_2(q).2$, where the C_2 acts as the diagonal automorphism on $\mathrm{PSL}_2(q)$. Since $|N_G(R)/R|_2 = 2(q-\epsilon)_2$, a character of defect zero must be $\chi_\bullet(k)$ for some $k \in T'_{-\epsilon}$ when restricted to $\mathrm{PSL}_2(q)$. However, these characters are invariant under the diagonal automorphisms, as they extend to $\mathrm{PGL}_2(q)$. Hence we see that $\mathrm{dz}(N_G(R)/R)$ is empty in this case.

For the radical subgroups of type B_4 of the form $R \cong Q_8$ with $q \geq 5$, we have

$$N_G(R)/R \cong D_{2(q+\epsilon)}/Z(D_{2(q+\epsilon)}) \times \mathfrak{S}_3 \cong D_{q+\epsilon} \times \mathfrak{S}_3,$$

so that defect-zero characters have degrees whose 2-parts are 4. Hence these are $\psi_k \times \psi$ for $k \in T'_{-\epsilon}$, where ψ_k is the character of $D_{q+\epsilon}$ which takes values $2\cos(\pi k/(q+\epsilon))$ on the generating rotation.

When $R \cong C_{2^a}$ with $q \geq 5$, we have $N_G(R)/R$ is $(\mathrm{GL}_2^\epsilon(q)/C_{2^a}).2$, where $C_{2^a} \leq Z(\mathrm{GL}_2^\epsilon(q))$. Then a defect-zero character has 2-part $2(q^2-1)_2$, which is impossible, since the largest 2-part of a character of $\mathrm{GL}_2^\epsilon(q)$ is $(q-\epsilon)_2$. Hence there are no defect-zero characters in this case.

When $R \cong C_{2^a} \circ_2 Q_8$ with $(q-\epsilon)_{2'} \neq 1$, we have $N_G(R)/R$ is $C_{(q-\epsilon)_{2'}}.D_{12} \cong C_{(q-\epsilon)_{2'}}.2 \times \mathfrak{S}_3$. Then the defect-zero characters are of the form $\tilde{\eta}_k \times \psi$ for $k \in T'_\epsilon$.

For the remaining radical subgroups, the set $\mathrm{dz}(N_G(R)/R)$ (and sometimes R itself) depends on whether $a \geq 3$ or $a = 2$. We discuss the two situations separately.

3.4.1. *The case $a \geq 3$.* Recall that there are two classes of the form $R \cong C_2 \times Q_8$ when $a \geq 3$, from the two classes of radical subgroups Q_8 in $\mathrm{SL}_2(q)$. Here R is of type $\{\pm I_2\} \times B_2$, $N_G(R) \cong \mathrm{SL}_2(q) \times Q_8.\mathfrak{S}_3$, and $N_G(R)/R \cong \mathrm{PSL}_2(q) \times \mathfrak{S}_3$. Then $|N_G(R)/R|_2 = 2(q-\epsilon)_2$, and defect-zero characters are of the form $\chi_\bullet(k) \times \psi$ for $k \in T'_{-\epsilon}$.

There are also radical subgroups of the form $R \cong C_{2^a} \times Q_8$ with $a \geq 3$, $(q-\epsilon)_{2'} \neq 1$, coming from the two classes of radical subgroups Q_8 in $\mathrm{SL}_2(q)$. Here R is type $B_2 \times B_2$ and

$$N_G(R) \cong C_{q-\epsilon}.2 \times Q_8.\mathfrak{S}_3 \cong (C_{q-\epsilon} \times Q_8).D_{12}.$$

This yields $N_G(R)/R \cong C_{(q-\epsilon)_{2'}}.2 \times \mathfrak{S}_3$. Then the defect-zero characters have degree 4, and are of the form $\tilde{\eta}_k \times \psi$ for $k \in T'_\epsilon$.

Let $R \cong Q_8 \times Q_8$ with $a \geq 3$ be of type $B_2 \times B_2$, where the two copies of Q_8 are the same class in the respective $\mathrm{SL}_2(q)$, yielding two classes of radical subgroups like this and $N_G(R)/R \cong \mathfrak{S}_3 \wr C_2$. Here defect-zero characters have degree 8. However, this means that on restriction to the base subgroup $\mathfrak{S}_3 \times \mathfrak{S}_3$, the character must be $\psi \times \psi$. But this character is invariant under the C_2 action, and hence extends. Then in this case, $N_G(R)/R$ has no defect-zero characters.

When the two copies of Q_8 come from the distinct classes in $\mathrm{SL}_2(q)$, we get one additional class of radical subgroups for G, with $N_G(R)/R \cong \mathfrak{S}_3 \times \mathfrak{S}_3$. In this case, defect-zero characters of $N_G(R)/R$ have degree 4 and must be of the form $\psi \times \psi$.

When R is a member of one of the two classes of radical subgroups $Q_8 \times Q_{2^{a+1}}$ of the form $B_2 \times B_2$, we get $N_G(R)/R \cong \mathfrak{S}_3$. Here ψ is the only defect-zero character.

If $R \cong 2_-^{1+4}$, then $N_G(R)/R \cong \mathfrak{A}_5.2$. This has no defect-zero characters, as the degree would need to have 2-part 8, but $\mathfrak{A}_5$ has only one degree-4 character, which would extend to $N_G(R)/R$.

When $R \cong Q_8 \circ_2 Q_{2^{a+1}}$ or $Q_8 \wr C_2$ with $a \geq 3$, we have $N_G(R)/R \cong \mathfrak{S}_3$ and ψ is the only defect-zero character.

3.4.2. *The case $a = 2$ and $q \geq 5$.* When $a = 2$ and $q \geq 5$, there is one class of the form $R \cong C_2 \times Q_8$, with $N_G(R)/R \cong \mathrm{PSL}_2(q) \times C_3$. Here for each $k \in T'_{-\epsilon}$, there are three characters of defect zero corresponding to $\chi_\bullet(k) \times \mu^i$, where $\mu^i \in \mathrm{Irr}(C_3)$ with $i \in \{0, 1, 2\}$.

There is also one class of radical subgroups of the form $R \cong C_{2^a} \times Q_8$ with $a = 2$, $(q-\epsilon)_{2'} \neq 1$ (that is, $q \geq 7$), with $N_G(R)/R \cong C_{(q-\epsilon)_{2'}}.2 \times C_3$. Then for each $k \in T'_\epsilon$, there are three defect-zero characters of the form $\tilde{\eta}_k \times \mu^i$ corresponding to the three characters $\mu^i \in \mathrm{Irr}(C_3)$.

In this case, we have one class of the form $R \cong Q_8 \times Q_8$, with $N_G(R)/R \cong C_3 \wr C_2$. The defect-zero characters here have degree 2 and are of the form $(\mu^i \times \mu^j) + (\mu^j \times \mu^i)$ with $i \neq j \in \{0, 1, 2\}$ when restricted to the base subgroup C_3^2, which we will denote as μ_{ij}. This yields three characters in $\mathrm{dz}(N_G(R)/R)$.

We also have one class of the form $R \cong 2_1^{1+4}$ and such that $N_G(R)/R \cong \mathfrak{A}_5$. Then there is exactly one character in $\mathrm{dz}(N_G(R)/R)$, namely ν.

4. The inductive BAWC conditions for $\mathrm{PSp}_4(q)$

In this section, we prove the inductive conditions for the BAWC for $S = \mathrm{PSp}_4(q)$ when $q \geq 5$ is a power of an odd prime p and ℓ is a prime dividing $q^2 - 1$. Note that by [16, Remark 4.2], to show that S is BAWC-good, it suffices to show that S satisfies Conditions 4.1(ii)(3) and 4.1(iii)(4) of [16] in addition to being AWC-good in the sense of [11, Section 3].

4.1. The sets and bijections for $\ell = 2$. Since $|Z(G)| = 2$ when q is odd, note that $Z = 1$ and $S = X$ in the notation of [11, Section 3]. Hence for S to be AWC-good for the prime $\ell = 2$ in the sense of [11, Section 3], we require a partition $\bigcup \mathrm{IBr}_2(S|R)$ of Brauer characters of S, where the union is taken over classes of 2-radical subgroups R of S. There should then be an $\mathrm{Aut}(S)$-equivariant bijection $*_R : \mathrm{IBr}_2(S|R) \to \mathrm{dz}(N_S(R)/R)$ satisfying certain other properties. However, the fact that $|Z(G)| = 2$ also implies that the 2-blocks and 2-Brauer characters of G and $S = G/Z(G)$ can be identified, using [10, Theorem 7.6]. Further, the 2-radical subgroups of S are of the form $R/Z(G)$, where R is a 2-radical subgroup of G, and $N_S(R/Z(G))/(R/Z(G)) \cong N_G(R)/R$. Hence in what follows, we define the necessary sets $\mathrm{IBr}_2(G|R)$ and bijections $*_R$ for G rather than S.

Tables 5–6 and 7–8 describe the sets and bijections in the cases $a \geq 3$ and $a = 2$, respectively. We note that when R is the defect group of a block B of G, we have defined $\mathrm{IBr}_2(G|R)$ naturally as a subset of $\mathrm{IBr}_2(B)$. The indexing in the tables is taken as in Table 4. We remark that the condition $(q-\epsilon)_{2'} \neq 1$ for several of the radical subgroups is not restrictive, given the enumerations in 3.1, and that the discussions in Sections 3.1 and 3.2 yield that these do in fact define bijections.

Recall that $\widetilde{G}$ denotes the conformal symplectic group $\mathrm{CSp}_4(q)$, so that $\widetilde{G}$ contains an index-two subgroup $G \circ Z(\widetilde{G})$, which is a central product of G with $Z(\widetilde{G}) \cong \mathbb{F}_q^\times$. Then the outer automorphism group of S is isomorphic to $C_2 \times C_f$, where $q = p^f$. Here the C_2 component is induced by the action of $\widetilde{S}/S \cong \widetilde{G}/(G \circ Z(\widetilde{G}))$ and the C_f component is given by field automorphisms. We also remark that by [18, Theorem 16.2], we may choose a field automorphism ϕ generating the C_f component such that for $\chi \in \mathrm{Irr}(G)$, we have $(\widetilde{G} \rtimes \langle\phi\rangle)_\chi = \widetilde{G}_\chi \rtimes \langle\phi\rangle_\chi$. Throughout, let ϕ denote such a field automorphism and let δ denote a diagonal automorphism inducing the action of $\widetilde{S}/S \cong \widetilde{G}/(G \circ Z(\widetilde{G}))$.

Using [17] for the character table of G, arguments as in the proof of [13, Proposition 5.1] yield that the chosen maps are equivariant with respect to the field automorphism. Further, from [5] and the descriptions summarized in Section 3.1, we see that the action of $\widetilde{S}/S$ interchanges the following pairs of Brauer characters: $\{\varphi_1, \varphi_2\}$, $\{\varphi_4, \varphi_5\}$, $\{\hat{\xi}'_{22}(r), \hat{\xi}'_{21}(r)\}$, $\{\hat{\xi}_{42}(r) - \hat{\xi}_3(r), \hat{\xi}_{41}(r) - \hat{\xi}_3(r)\}$. The remaining irreducible Brauer characters are invariant under the action of $\widetilde{S}$.

On the other hand, when $a \geq 3$, the Brauer characters interchanged by δ correspond under our map to pairs of classes of radical subgroups which are also interchanged by δ. Indeed, note that δ induces a diagonal automorphism as well on $\mathrm{SL}_2(q)$, and that these pairs of classes come from pairs of classes of radical subgroups Q_8 in $\mathrm{SL}_2(q)$, which are fused in $\mathrm{GL}_2(q)$, by [14, Corollary 7.15]. When $a = 2$, it suffices to see that μ and μ^2 are interchanged by δ when C_3 is viewed as a subgroup of $\mathrm{SL}_2(q)$ inducing an automorphism of order 3 on Q_8 embedded into $\mathrm{SL}_2(q)$. Indeed, constructing Q_8 in the standard way in $\mathrm{SL}_2(q)$, for example as in [7], one can construct a generator for such an automorphism of Q_8 and see that there is an appropriate representative for δ that inverts it. This yields:

Proposition 4.1. *The sets* $\mathrm{IBr}_2(G|R)$ *and bijections* $*_R$ *defined in Tables 5–8 satisfy the partition and bijection conditions in* [11, Sections 3.1 and 3.2].

4.2. The sets and bijections for Sylow ℓ-subgroups, ℓ odd. From Section 2.1, we see that for ℓ an odd prime dividing $q^2 - 1$, the only noncyclic radical subgroups for G are the Sylow ℓ-subgroups. Hence applying the results of [9], in order to complete the proof of Theorem 1.1 when ℓ is odd, it suffices to consider the case that $R \in \mathrm{Syl}_\ell(G)$ and construct bijections from irreducible Brauer characters in blocks

R	$\theta \in \mathrm{IBr}_2(G\|R)$	$\theta^{*_R} \in \mathrm{dz}(N/R)$
$Z(G) = C_2$	$\widehat{\chi}_1(r)$	$\widehat{\chi}_1(r)$
$C_2 \times C_2$	$\widehat{\chi}_4(r,s)$	$\chi_6(r,s)$
$C_2 \times C_{2^a}$	$\widehat{\chi}_5(r,s)$	$\chi_6(r) \times \tilde{\eta}_s$
$C_2 \times Q_8$ (two classes)	$\widehat{\xi}'_{22}(r)$ $\widehat{\xi}'_{21}(r)$	$\chi_6(r) \times \psi$ $\chi_6(r) \times \psi$
$C_2 \times Q_{2^{a+1}}$	$\widehat{\xi}_1(r)$	$\chi_6(r)$
$C_{2^a} \times C_{2^a}$	$\widehat{\chi}_3(r,s)$	$\tilde{\eta}_{r,s}$
$C_{2^a} \times Q_8$ (two classes)	$\widehat{\xi}_{42}(r) - \widehat{\xi}_3(r)$ $\widehat{\xi}_{41}(r) - \widehat{\xi}_3(r)$	$\tilde{\eta}_r \times \psi$ $\tilde{\eta}_r \times \psi$
$C_{2^a} \times Q_{2^{a+1}}$	$\widehat{\xi}_3(r)$	$\tilde{\eta}_r$
$Q_8 \times Q_8$ (two classes), $N/R \cong (\mathfrak{S}_3 \times \mathfrak{S}_3).2$	empty	empty
$Q_8 \times Q_8$, $N/R \cong \mathfrak{S}_3 \times \mathfrak{S}_3$	φ_3	$\psi \times \psi$
$Q_8 \times Q_{2^{a+1}}$ (two classes)	φ_1 φ_2	ψ ψ
2_-^{1+4} (two classes)	empty	empty
$C_{2^a} \circ_2 Q_8$	$\widehat{\chi}_9(r) - \widehat{\chi}_8(r)$	$\tilde{\eta}_r \times \psi$
$C_{2^{a+1}}$	$\widehat{\chi}_2(r)$	$\tilde{\theta}_r$
$S_{2^{a+2}}$	$\widehat{\chi}_6(r)$	$\tilde{\eta}'_{2r}$
$D_{2^{a+1}}$	empty	empty
$Q_8 \circ_2 Q_{2^{a+1}}$	φ_6	ψ
Q_8	$\widehat{\chi}_7(r) - \widehat{\chi}_6(r)$	$\psi_r \times \psi$
C_{2^a}	empty	empty
$C_{2^a} \wr C_2$	$\widehat{\chi}_8(r)$	$\tilde{\eta}_{2r}$
$Q_8 \wr C_2$ (two classes)	φ_4 φ_5	ψ ψ
$Q_{2^{a+1}} \wr C_2$	1_S	1

Table 5. The sets and bijections for $\ell = 2$, $a \geq 3$: the case $\epsilon = 1$.

of maximal defect to $\mathrm{dz}(N_G(R)/R)$ satisfying Conditions 4.1(ii)(3) and 4.1(iii)(4) of [16] and those of [11, Section 3].

Let $\epsilon \in \{\pm 1\}$ be such that $\ell \mid (q-\epsilon)$. Note then that $R \cong C_{(q-\epsilon)_\ell} \times C_{(q-\epsilon)_\ell}$, that

$$N_G(R)/R \cong C_{(q-\epsilon)_{\ell'}}.2 \wr C_2,$$

R	$\theta \in \mathrm{IBr}_2(G\|R)$	$\theta^{*_R} \in \mathrm{dz}(N/R)$
$Z(G) = C_2$	$\widehat{\chi}_1(r)$	$\widehat{\chi}_1(r)$
$C_2 \times C_2$	$\widehat{\chi}_3(r, s)$	$\chi_5(r, s)$
$C_2 \times C_{2^a}$	$\widehat{\chi}_5(r, s)$	$\chi_5(s) \times \tilde{\eta}_r$
$C_2 \times Q_8$ (two classes)	$\widehat{\xi}_{42}(r) - \widehat{\xi}_3(r)$ $\widehat{\xi}_{41}(r) - \widehat{\xi}_3(r)$	$\chi_5(r) \times \psi$ $\chi_5(r) \times \psi$
$C_2 \times Q_{2^{a+1}}$	$\widehat{\xi}_3(r)$	$\chi_5(r)$
$C_{2^a} \times C_{2^a}$	$\widehat{\chi}_4(r, s)$	$\tilde{\eta}_{r,s}$
$C_{2^a} \times Q_8$ (two classes)	$\widehat{\xi}'_{22}(r)$ $\widehat{\xi}'_{21}(r)$	$\tilde{\eta}_r \times \psi$ $\tilde{\eta}_r \times \psi$
$C_{2^a} \times Q_{2^{a+1}}$	$\widehat{\xi}_1(r)$	$\tilde{\eta}_r$
$Q_8 \times Q_8$ (two classes), $N/R \cong (\mathfrak{S}_3 \times \mathfrak{S}_3).2$	empty	empty
$Q_8 \times Q_8$, $N/R \cong \mathfrak{S}_3 \times \mathfrak{S}_3$	φ_3	$\psi \times \psi$
$Q_8 \times Q_{2^{a+1}}$ (two classes)	φ_1 φ_2	ψ ψ
2_-^{1+4} (two classes)	empty	empty
$C_{2^a} \circ_2 Q_8$	$\widehat{\chi}_7(r) - \widehat{\chi}_6(r)$	$\tilde{\eta}_r \times \psi$
$C_{2^{a+1}}$	$\widehat{\chi}_2(r)$	$\tilde{\theta}_r$
$S_{2^{a+2}}$	$\widehat{\chi}_8(r)$	$\tilde{\eta}'_{2r}$
$D_{2^{a+1}}$	empty	empty
$Q_8 \circ_2 Q_{2^{a+1}}$	φ_6	ψ
Q_8	$\widehat{\chi}_9(r) - \widehat{\chi}_8(r)$	$\psi_r \times \psi$
C_{2^a}	empty	empty
$C_{2^a} \wr C_2$	$\widehat{\chi}_6(r)$	$\tilde{\eta}_{2r}$
$Q_8 \wr C_2$ (two classes)	φ_4 φ_5	ψ ψ
$Q_{2^{a+1}} \wr C_2$	1_S	1

Table 6. The sets and bijections for $\ell = 2$, $a \geq 3$: the case $\epsilon = -1$.

and that $\mathrm{dz}(N_G(R)/R) = \mathrm{Irr}(N_G(R)/R)$. Here we may embed $C_{q-\epsilon}^2$ through the block-diagonal embedding of $\mathrm{SL}_2(q)^2$ in G. With this identification, the C_2 components act on $C_{q-\epsilon}$ and on $(C_{(q-\epsilon)_{\ell'}}.2)^2$ via inversion and reversing components, respectively. These can be viewed as induced from the elements $x := \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ and $y := \left[\begin{smallmatrix} 0 & I_2 \\ -I_2 & 0 \end{smallmatrix}\right]$, respectively. We further remark that the nontrivial diagonal automorphism δ

R	$\theta \in \mathrm{IBr}_2(G\mid R)$	$\theta^{*_R} \in \mathrm{dz}(N_G(R)/R)$
$Z(G) = C_2$	$\widehat{\chi}_1(r)$	$\widehat{\chi}_1(r)$
$C_2 \times C_2$	$\widehat{\chi}_4(r, s)$	$\chi_6(r, s)$
$C_2 \times C_{2^a}$	$\widehat{\chi}_5(r, s)$	$\chi_6(r) \times \tilde{\eta}_s$
$C_2 \times Q_8$	$\widehat{\xi}'_{22}(r)$ $\widehat{\xi}'_{21}(r)$ $\widehat{\xi}_1(r)$	$\chi_6(r) \times \mu$ $\chi_6(r) \times \mu^2$ $\chi_6(r) \times 1_{C_3}$
$C_{2^a} \times C_{2^a}$	$\widehat{\chi}_3(r, s)$	$\tilde{\eta}_{r,s}$
$C_{2^a} \times Q_8$	$\widehat{\xi}_{42}(r) - \widehat{\xi}_3(r)$ $\widehat{\xi}_{41}(r) - \widehat{\xi}_3(r)$ $\widehat{\xi}_3(r)$	$\tilde{\eta}_r \times \mu$ $\tilde{\eta}_r \times \mu^2$ $\tilde{\eta}_r \times 1_{C_3}$
$Q_8 \times Q_8$	φ_3 φ_1 φ_2	μ_{12} μ_{01} μ_{02}
2_-^{1+4}	φ_6	ν
$C_{2^a} \circ_2 Q_8$	$\widehat{\chi}_9(r) - \widehat{\chi}_8(r)$	$\tilde{\eta}_r \times \psi$
$C_{2^{a+1}}$	$\widehat{\chi}_2(r)$	$\tilde{\theta}_r$
$S_{2^{a+2}}$	$\widehat{\chi}_6(r)$	$\tilde{\eta}'_{2r}$
$D_{2^{a+1}}$	empty	empty
Q_8	$\widehat{\chi}_7(r) - \widehat{\chi}_6(r)$	$\psi_r \times \psi$
C_{2^a}	empty	empty
$C_{2^a} \wr C_2$	$\widehat{\chi}_8(r)$	$\tilde{\eta}_{2r}$
$Q_8 \wr C_2$	φ_4 φ_5 1_S	μ μ^2 1_{C_3}

Table 7. The sets and bijections for $\ell = 2$, $a = 2$, $q \geq 5$: the case $\epsilon = 1$.

of S can be seen as induced by the matrix $\mathrm{diag}(-1, 1)$ on each $\mathrm{SL}_2(q)$ component, which fixes y and sends x to $-x$.

Let ± 1 denote the characters of $C_{(q-\epsilon)_{\ell'}}$ of order dividing 2. These characters are invariant under the action of the C_2 components of $C_{(q-\epsilon)_{\ell'}}.2$, and we denote by $\pm 1_a$ and $\pm 1_b$ the two corresponding extensions to $C_{(q-\epsilon)_{\ell'}}.2$. Then we obtain 14 characters of $N_G(R)/R$ whose restrictions to $C^2_{(q-\epsilon)_{\ell'}}$ are of the form $\pm 1 \times \pm 1$. In what follows, we will in most cases identify these characters by their restrictions to $(C_{(q-\epsilon)_{\ell'}}.2)^2$. However, for those that extend from $(C_{(q-\epsilon)_{\ell'}}.2)^2$ to $C_{(q-\epsilon)_{\ell'}}.2 \wr C_2$, we use subscripts a and b again to denote the two extensions. Further, we remark that δ fixes 1_a and 1_b and interchanges -1_a and -1_b. Table 9 describes the bijections for these characters. The corresponding blocks of maximal

R	$\theta \in \mathrm{IBr}_2(G\|R)$	$\theta^{*_R} \in \mathrm{dz}(N_G(R)/R)$
$Z(G) = C_2$	$\widehat{\chi}_1(r)$	$\widehat{\chi}_1(r)$
$C_2 \times C_2$	$\widehat{\chi}_3(r,s)$	$\chi_5(r,s)$
$C_2 \times C_{2^a}$	$\widehat{\chi}_5(r,s)$	$\chi_5(s) \times \tilde{\eta}_r$
$C_2 \times Q_8$	$\widehat{\xi}_{42}(r) - \widehat{\xi}_3(r)$ $\widehat{\xi}_{41}(r) - \widehat{\xi}_3(r)$ $\widehat{\xi}_3(r)$	$\chi_5(r) \times \mu$ $\chi_5(r) \times \mu^2$ $\chi_5(r) \times 1_{C_3}$
$C_{2^a} \times C_{2^a}$	$\widehat{\chi}_4(r,s)$	$\tilde{\eta}_{r,s}$
$C_{2^a} \times Q_8$	$\widehat{\xi}'_{22}(r)$ $\widehat{\xi}'_{21}(r)$ $\widehat{\xi}_1(r)$	$\tilde{\eta}_r \times \mu$ $\tilde{\eta}_r \times \mu^2$ $\tilde{\eta}_r \times 1_{C_3}$
$Q_8 \times Q_8$	φ_3 φ_1 φ_2	μ_{12} μ_{01} μ_{02}
2_-^{1+4}	φ_6	ν
$C_{2^a} \circ_2 Q_8$	$\widehat{\chi}_7(r) - \widehat{\chi}_6(r)$	$\tilde{\eta}_r \times \psi$
$C_{2^{a+1}}$	$\widehat{\chi}_2(r)$	$\tilde{\theta}_r$
$S_{2^{a+2}}$	$\widehat{\chi}_8(r)$	$\tilde{\eta}'_{2r}$
$D_{2^{a+1}}$	empty	empty
Q_8	$\widehat{\chi}_9(r) - \widehat{\chi}_8(r)$	$\psi_r \times \psi$
C_{2^a}	empty	empty
$C_{2^a} \wr C_2$	$\widehat{\chi}_6(r)$	$\tilde{\eta}_{2r}$
$Q_8 \wr C_2$	φ_4 φ_5 1_S	μ μ^2 1_{C_3}

Table 8. The sets and bijections for $\ell = 2$, $a = 2$, $q \geq 5$: the case $\epsilon = -1$.

defect and Brauer characters for G are obtained from [20]. (We remark that the number α is determined in [12] to be 1 or 2.) From [5], we see that the pairs $\{\Phi_i, \Phi_{i+1}\}$ and $\{\theta_i, \theta_{i+1}\}$ for $i = 1, 3, 5, 7$ are interchanged by δ and that Φ_9 and θ_j for $j = 9, 10, 11, 12, 13$ are fixed by δ. Since all characters listed are fixed by field automorphisms, we see that the bijections are $\mathrm{Aut}(S)$-equivariant. Further, by construction, $(\mathrm{IBr}_\ell(G|R) \cap \mathrm{IBr}_\ell(G|\nu))^{*_R} \subseteq \mathrm{Irr}(N_G(R)/R|\nu)$ for each $\nu \in \mathrm{Irr}(Z(G))$.

Now, consider the characters of $C_{(q-\epsilon)_{\ell'}}.2$ that are not ± 1 on restriction to $C_{(q-\epsilon)_{\ell'}}$. These characters are of the form $\tilde{\eta}_k$, where the notation is analogous to that in Section 3.2 for the case that $\ell = 2$ above. If $m = 2m_0 = (q-\epsilon)_{\ell'}$, this yields $(m-2)/2 = m_0 - 1$ characters of this form for $C_{(q-\epsilon)_{\ell'}}.2$. These characters are indexed by k in T'_ϵ, where, analogous to the case $\ell = 2$, the set T'_ϵ is the set of multiples of

ℓ-block B of G	$\theta \in \mathrm{IBr}_\ell(B)$ $\epsilon = 1$	$\epsilon = -1$	$\theta^{*_R} \in \mathrm{Irr}(N_G(R)/R)$
b_0	1_G	1_G	$(1_a \times 1_a)_a$
	$\widehat{\theta}_9$	$\widehat{\theta}_{10}$	$(1_a \times 1_a)_b$
	$\widehat{\theta}_{11}$	$\widehat{\theta}_{11} - 1_G$	$(1_b \times 1_b)_a$
	$\widehat{\theta}_{12}$	$\widehat{\theta}_{12} - 1_G$	$(1_b \times 1_b)_b$
	$\widehat{\theta}_{13}$	$\widehat{\theta}_{13} - \widehat{\theta}_{12} - \widehat{\theta}_{11} - \alpha\widehat{\theta}_{10} + 1_G$	$(1_a \times 1_b) + (1_b \times 1_a)$
b_1	$\widehat{\Phi}_5$	$\widehat{\Phi}_1$	$(-1_a \times 1_a) + (1_a \times -1_a)$
	$\widehat{\Phi}_6$	$\widehat{\Phi}_2$	$(-1_b \times 1_a) + (1_a \times -1_b)$
	$\widehat{\Phi}_7$	$\widehat{\Phi}_4 - \widehat{\Phi}_2$	$(-1_a \times 1_b) + (1_b \times -1_a)$
	$\widehat{\Phi}_8$	$\widehat{\Phi}_3 - \widehat{\Phi}_1$	$(-1_b \times 1_b) + (1_b \times -1_b)$
b_2	$\widehat{\theta}_3$	$\widehat{\theta}_3$	$(-1_a \times -1_a)_a$
	$\widehat{\theta}_4$	$\widehat{\theta}_4$	$(-1_b \times -1_b)_a$
	$\widehat{\Phi}_9$	$\widehat{\Phi}_9 - \widehat{\theta}_4 - \widehat{\theta}_3$	$(-1_a \times -1_b) + (-1_b \times -1_a)$
	$\widehat{\theta}_1$	$\widehat{\theta}_1 + \widehat{\theta}_3 - \widehat{\Phi}_9$	$(-1_a \times -1_a)_b$
	$\widehat{\theta}_2$	$\widehat{\theta}_2 + \widehat{\theta}_4 - \widehat{\Phi}_9$	$(-1_b \times -1_b)_b$

Table 9. The bijection for $R \in \mathrm{Syl}_\ell(G)$, ℓ odd, isolated blocks.

$(q-\epsilon)_\ell$ in $\{1, \ldots, (q-\epsilon)/2-1\}$. Given such a k, there are two characters of $N_G(R)/R$ that restrict to $\tilde{\eta}_k \times \tilde{\eta}_k$ on $(C_{(q-\epsilon)_{\ell'}}.2)^2$, which we again denote with an a and b. Further, for each $\varphi \in \{1_a, 1_b, -1_a, -1_b\}$, we have one character whose restriction is of the form $\tilde{\eta}_k \times \varphi + \varphi \times \tilde{\eta}_k$.

Finally, the characters of $N_G(R)/R$ whose restriction to neither component of $C_{(q-\epsilon)_{\ell'}}$ is ± 1 must be, on restriction to $(C_{(q-\epsilon)_{\ell'}}.2)^2$, of the form $(\tilde{\eta}_k \times \tilde{\eta}_t) + (\tilde{\eta}_t \times \tilde{\eta}_k)$ for $k \neq t$ in T'_ϵ. In this case, there are $(m_0-1)(m_0-2)/2$ characters of this form. Table 10 describes the bijections for these remaining characters. Again the Brauer character information is taken from [20], we have constructed the bijection so that $(\mathrm{IBr}_\ell(G|R) \cap \mathrm{IBr}_\ell(G|\nu))^{*_R} \subseteq \mathrm{Irr}(N_G(R)/R|\nu)$ for each $\nu \in \mathrm{Irr}(Z(G))$, and the discussion from above and arguments exactly as in [13, Propositions 5.1] yield that the bijection is $\mathrm{Aut}(S)$-equivariant.

Together, we have the following:

Proposition 4.2. *Let $R \in \mathrm{Syl}_\ell(G)$ and $\ell \mid (q^2-1)$ odd. The sets $\mathrm{IBr}_\ell(G|R)$ and bijections $*_R$ defined in Tables 9–10 satisfy the partition and bijection conditions in* [11, Sections 3.1 and 3.2].

4.3. The normally embedded conditions. In this section, let $G = \mathrm{Sp}_4(q)$ with q odd and let $\ell \mid (q^2-1)$ be a prime. Let R be any 2-radical subgroup in the case $\ell = 2$ or a member of $\mathrm{Syl}_\ell(G)$ if ℓ is odd, and fix $\theta \in \mathrm{IBr}_\ell(G|R)$, where $\mathrm{IBr}_\ell(G|R)$ is defined as in Tables 5–10. Notice that $\mathrm{Aut}(S)_\theta/S$ is cyclic unless q is a square and θ is fixed by δ and a field automorphism of order 2.

Lemma 4.3. *The characters $\theta \in \mathrm{IBr}_\ell(G|R)$ extend to their inertia groups in $\widetilde{G} \rtimes \langle\phi\rangle$. Further, the characters θ^{*_R} extend to their inertia groups in the normalizer of R in $\widetilde{G} \rtimes \langle\phi\rangle$, where $*_R$ is as defined in Tables 5–10.*

$\epsilon = 1$		$\epsilon = -1$		
ℓ-block B of G	$\theta \in \mathrm{IBr}_\ell(B)$	ℓ-block B of G	$\theta \in \mathrm{IBr}_\ell(B)$	$\theta^{*_R} \in \mathrm{Irr}(N_G(R)/R)$
$b_{89}(k)$	$\widehat{\chi}_8(k)$ $\widehat{\chi}_9(k)$	$b_{67}(k)$	$\widehat{\chi}_6(k)$ $\widehat{\chi}_7(k) - \widehat{\chi}_6(k)$	$(\tilde{\eta}_k \times \tilde{\eta}_k)_a$ $(\tilde{\eta}_k \times \tilde{\eta}_k)_b$
$b_{III}(k)$	$\widehat{\xi}_3(k)$ $\widehat{\xi}'_3(k)$	$b_I(k)$	$\widehat{\xi}_1(k)$ $\widehat{\xi}'_1(k) - \widehat{\xi}_1(k)$	$(\tilde{\eta}_k \times 1_a) + (1_a \times \tilde{\eta}_k)$ $(\tilde{\eta}_k \times 1_b) \times 1_b \times \tilde{\eta}_k)$
$b_{41}(k)$	$\widehat{\xi}_{41}(k)$ $\widehat{\xi}_{42}(k)$	$b_{21}(k)$	$\widehat{\xi}'_{21}(k)$ $\widehat{\xi}'_{22}(k)$	$(\tilde{\eta}_k \times -1_a) + (-1_a \times \tilde{\eta}_k)$ $(\tilde{\eta}_k \times -1_b) + (-1_b \times \tilde{\eta}_k)$
$b_3(k,t)$	$\widehat{\chi}_3(k,t)$	$b_4(k,t)$	$\widehat{\chi}_4(k,t)$	$(\tilde{\eta}_k \times \tilde{\eta}_t) + (\tilde{\eta}_t \times \tilde{\eta}_k)$

Table 10. The bijection for $R \in \mathrm{Syl}_\ell(G)$, ℓ odd, nonisolated blocks.

Proof. This is clear if $\mathrm{Aut}(S)_\theta/S$ is cyclic. Hence we may assume that $q \equiv 1 \pmod 8$ is a square and that θ is fixed by δ. In particular, $a \geq 3$ and $\epsilon = 1$ in the case $\ell = 2$. Note that θ and θ^{*_R} extend to $\widetilde{G}$ and $\widetilde{G}_R$, respectively. We claim that this extension can be chosen to be invariant under the same field automorphisms as θ, respectively θ^{*_R}.

Comparing the notations and values of the characters χ in [17] for the families χ_i for $1 \leq i \leq 9$, ξ_1, ξ'_1, ξ_3, and the unipotent characters of G fixed by δ to those of their extensions, using [5; 15], yields that each of these characters has an extension to $\widetilde{G}$ which is also invariant under the field automorphisms fixing χ. Hence each such χ extends to its inertia subgroup, and therefore so does θ.

Observing the character tables of $\mathrm{PSL}_2(q)$ and $\mathrm{PGL}_2(q)$, we see that the characters in the family χ_6 extend to characters of $\mathrm{PGL}_2(q)$ that are invariant under the same field automorphisms. Further, δ can be chosen to commute with the groups $C_{q\pm 1}$ and C_{q^2+1}, and modulo $Z(G)$, with $D_{2(q+1)}$ as well as the elements x and y introduced in Section 4.2. Then θ^{*_R} extends to a character of $\widetilde{G}_R$ invariant under the same field automorphisms as θ^{*_R}, except possibly in the cases that N/R contains $\mathfrak{S}_3$ as a factor. Since the only group containing $\mathfrak{S}_3$ with index 2 contains $\mathfrak{S}_3$ as a direct factor, we see that δ must act trivially on $\mathfrak{S}_3$, and hence the characters of N/R in the latter case also extend to characters of $\widetilde{G}_R$ invariant under the same field automorphisms. □

Corollary 4.4. *Let $G = \mathrm{Sp}_4(q)$ with q odd and let $\ell \mid (q^2-1)$ be a prime. Let R be a 2-radical subgroup of G if $\ell = 2$, or a Sylow ℓ-subgroup if ℓ is odd, and let* $\mathrm{IBr}_\ell(G|R)$ *and* $*_R$ *be defined as in Tables 5–10. Then the normally embedded conditions* [11, Section 3.3] *are satisfied.*

Proof. Fix $\theta \in \mathrm{IBr}_\ell(G|R)$ and write $\overline{G} := G/\ker(\theta|_{Z(G)})$. If θ is trivial on $Z(G)$, identify $S = G/Z(G)$ with $\mathrm{Inn}(S)$, so that we may write $\overline{G} = S \lhd \mathrm{Aut}(S)_\theta \lhd \mathrm{Aut}(S)$ and write $X := \mathrm{Aut}(S)_\theta$. If θ is nontrivial on $Z(G)$, let $X := \widetilde{G}_\theta \rtimes \langle\phi\rangle_\theta$. In any case, let $B := X_R$ be the subgroup of X stabilizing R. Then certainly, $\overline{G} \lhd X$, $Z(\overline{G}) \leq Z(X)$, θ is X-invariant, and B is exactly the set of automorphisms of $\overline{G}$ induced by the conjugation action of $N_X(R)$ on $\overline{G}$. Moreover, $C_X(\overline{G})$ is trivial and since θ and θ^{*_R} extend to X and B, respectively, by Lemma 4.3, their corresponding cohomology elements in $H^2(X/\overline{G}, \mathbb{F}_\ell^\times)$ are trivial. Hence the normally embedded conditions [11, Conditions 3.a–3.d] are satisfied, completing the proof. □

4.4. The block conditions. In this section, we consider Conditions 4.1(ii)(3) and 4.1(iii)(4) of [16]. Recall that to show that S is BAWC-good, it suffices by [16, Remark 4.2] to show that S satisfies these two conditions in addition to being AWC-good in the sense of [11, Section 3].

We will begin with an adaption of [16, Lemma 6.1] for our purposes. To do this, we consider a more general situation and set some notation. Let $\boldsymbol{G}$ be a simple, simply connected algebraic group over an algebraic closure of $\mathbb{F}_p$, and let F be a Frobenius morphism such that $\boldsymbol{G}^F$ is a finite group of Lie type, $Z(\boldsymbol{G}^F)$ is cyclic, and $\boldsymbol{G}^F/Z(\boldsymbol{G}^F)$ is simple. Further, let $\boldsymbol{G} \hookrightarrow \widetilde{\boldsymbol{G}}$ be a regular embedding as in [6, 15.1] and let D be the subgroup of $\mathrm{Aut}(\boldsymbol{G}^F)$ generated by field and graph automorphisms so that $\widetilde{\boldsymbol{G}}^F \rtimes D$ induces all automorphisms of $\boldsymbol{G}^F$.

Lemma 4.5. *Let ℓ be a prime and let G_0 be the universal ℓ' covering group of $\boldsymbol{G}^F/Z(\boldsymbol{G}^F)$ in the notation above. Let Q be a radical subgroup of G_0 and $\mathrm{IBr}_\ell(G_0|Q)$ and $*_Q$ be a subset of $\mathrm{IBr}_\ell(G_0)$ and map, respectively, satisfying the conditions of* [11, Section 3] *and* [16, Condition 4.1(ii)(3)]. *Further, assume that $\chi \in \mathrm{IBr}_\ell(G_0|Q)$ is such that the following hold when χ is viewed as a character of $\boldsymbol{G}^F$ by inflation*:

- $(\widetilde{\boldsymbol{G}}^F \rtimes D)_\chi = \widetilde{\boldsymbol{G}}^F_\chi \rtimes D_\chi$ *and* $(\widetilde{\boldsymbol{G}}^F_\chi \rtimes D_\chi)/\boldsymbol{G}^F$ *is abelian*;
- χ *extends to* $\widetilde{\boldsymbol{G}}^F_\chi \rtimes D_\chi$ *and* χ^{*_Q} *extends to* $(\widetilde{\boldsymbol{G}}^F \rtimes D)_{Q,\chi}$.

Then [16, Condition 4.1(iii)] *holds.*

Proof. By assumption, $*_Q$ is $\mathrm{Aut}(G_0)_R$-equivariant and χ and χ^{*_Q} lie in pseudo-corresponding blocks, in the sense of [16]. We largely follow and adapt the proof of [16, Lemma 6.1]. Let

$$\overline{G} := \boldsymbol{G}^F/\ker(\chi|_{Z(\boldsymbol{G}^F)}) \cong G_0/\ker(\chi|_{Z(G_0)}).$$

Write $A := \widetilde{\boldsymbol{G}}^F_\chi/\ker(\chi|_{Z(\boldsymbol{G}^F)}) \rtimes D_\chi$ and $A(\chi) := A/Z(A)_\ell$. Then because $\ell \nmid |Z(\overline{G})|$, our assumption $(\widetilde{\boldsymbol{G}}^F \rtimes D)_\chi = \widetilde{\boldsymbol{G}}^F_\chi \rtimes D_\chi$ yields that $A(\chi)$ has the properties of [16, Condition 4.1(iii)(1)]. Let $A_{\ell'}$ be such that $A_{\ell'}/\overline{G}$ is a Hall ℓ'-subgroup of $A(\chi)/\overline{G}$, which exists since by assumption $A(\chi)/\overline{G}$ is abelian.

Now, by assumption, χ extends to $A(\chi)$, and $\varphi := \chi^{*_Q}$ extends to $N_{A(\chi)}(Q)$. Then there is an extension of φ to $N_{A_{\ell'}}(Q)$. Let $\widetilde{\varphi} \in \mathrm{IBr}_\ell(N_{A_{\ell'}}(Q))$ denote the corresponding Brauer character extending $\widehat{\varphi}$.

Let $\widetilde{b}$ be the block of $N_{A_{\ell'}}(Q)$ containing $\widetilde{\varphi}$ and let B be the block of $\overline{G}$ containing χ. Then $\tilde{b}^{A_{\ell'}}$ is defined (see for example [10, Theorem 4.14]) and by observing the values of central characters, we see that $\tilde{b}^{A_{\ell'}}$ covers B, so that by [10, Theorem 9.4], we can choose an extension $\widetilde{\chi}$ of χ to $A_{\ell'}$ so that $\widetilde{\chi}$ is contained in $\tilde{b}^{A_{\ell'}}$. That is, $\widetilde{\chi}$ and $\widetilde{\varphi}$ lie in pseudo-corresponding blocks. Further, note that since $A(\chi)/\overline{G}$ is abelian, an application of Gallagher's theorem [8, Theorem 6.17] yields that every character of $A(\chi)$ above χ is an extension, and similarly for characters above φ in $N_{A(\chi)}(Q)$. It follows that $\widetilde{\chi}$ and $\widetilde{\varphi}$ may be extended to characters of $A(\chi)$ and $N_{A(\chi)}(Q)$, respectively. From here, arguing exactly as in the last two paragraphs of [16, Theorem 6.1] completes the proof. □

In particular, if [16, Condition 4.1(ii)(3)] holds, then [18, Theorem 16.2] and the observations from previous sections yield that Lemma 4.5 applies in the case that $\boldsymbol{G}^F = \mathrm{Sp}_4(q)$, $S = \boldsymbol{G}^F/Z(\boldsymbol{G}^F) = \mathrm{PSp}_4(q)$ for q a power of an odd prime, $Q = R$ is a nontrivial 2-radical subgroup of G when $\ell = 2$ or a Sylow ℓ-subgroup for $\ell \mid (q^2-1)$ odd, and $\mathrm{IBr}_\ell(G|R)$ and $*_R$ are as defined in Tables 5–10.

Lemma 4.6. *Let $G = \mathrm{Sp}_4(q)$ for q a power of an odd prime and let R be a nontrivial 2-radical subgroup of G when $\ell = 2$ or a Sylow ℓ-subgroup for $\ell \mid (q^2 - 1)$ odd. Let $\mathrm{IBr}_\ell(G|R)$ and $*_R$ be as defined in Tables 5–10. Then if B is the block of G containing $\theta \in \mathrm{IBr}_\ell(G|R)$ and b is the block of $N_G(R)$ containing θ^{*_R}, we have $b^G = B$. In particular,* [16, Condition 4.1(ii)(3)] *holds for $S = \mathrm{PSp}_4(q)$.*

Proof. Let $N := N_G(R)$ and $C := C_G(R)$. As $b \in \mathrm{Bl}(N)$, b^G is defined and $b^G = B$ if and only if $\lambda_B(\mathscr{K}^+) = \lambda_b((\mathscr{K} \cap C)^+)$ for all conjugacy classes $\mathscr{K}$ of G, where λ_B and λ_b are the central function corresponding to the blocks B and b respectively; see, for example, [8, Lemma 15.44]. Let $\chi \in \mathrm{Irr}(G|B)$. The central character ω_χ for G are available in [19] in the case $\ell = 2$ and can be computed in the relevant cases for ℓ odd from the information in [17]. The values of $\varphi \in \mathrm{Irr}(N|b)$ on C can be computed by their descriptions and using the character tables for $\mathrm{SL}_2(q)$ available in CHEVIE. Hence it remains only to determine the fusion of classes of C into G in order to compute $\omega_\varphi((\mathscr{K} \cap C)^+) = (1/\varphi(1)) \sum_{\mathscr{C} \subseteq \mathscr{K}} \varphi(g)|\mathscr{C}|$, where $g \in \mathscr{C}$ and the sum is taken over classes $\mathscr{C}$ of C which lie in $\mathscr{K}$, and compare the image of this under $*$ with $\omega_\chi(\mathscr{K}^+)^*$. (We note that $\omega_\chi(1^+) = 1 = \omega_\varphi((1 \cap C)^+)$ for all $\chi \in \mathrm{Irr}(G)$, $\varphi \in \mathrm{Irr}(N)$, so it suffices to consider nontrivial classes $\mathscr{K}$.) The considerations here, though tedious, are very similar to those in [13, Proposition 5.3], using the information in [17] for the classes of G. We omit the details. $\square$

Proof of Theorem 1.1. The theorem now follows by combining Lemmas 4.3, 4.5, and 4.6 with Propositions 4.1 and 4.2 and Corollary 4.4. $\square$

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