CONVOLUTIONAL BEAMSPACE FOR ARRAY SIGNAL PROCESSING

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ABSTRACT

A new type of beamspace for array processing is introduced called convolutional beamspace. It enjoys the advantages of traditional beamspace such as lower computational complexity, increased parallelism of subband processing, and improved resolution threshold for DOA estimation. But unlike traditional beamspace methods, it allows root-MUSIC and ESPRIT to be performed directly for ULAs without any overhead of preparation, as the Vandermonde structure and the shift-invariance are preserved under the transformation. The method produces more accurate DOA estimates than traditional beamspace methods, and for correlated sources it produces better estimates than element-space methods.

Index Terms— Beamspace, Convolution, Large Arrays, Dimension Reduction, MUSIC.

1. INTRODUCTION

The idea of beamforming prior to accurate estimation of directions of arrival (DOA) has been well-known in array signal processing, and is referred to as *beamspace* processing [1], [2], [7], [25], [26]. This topic continues to be of current research interest [5], [29], [30]. Given a N-sensor array with output $\mathbf{x} \in \mathbb{C}^N$, the beamspace transformation computes $\mathbf{y} = \mathbf{T}\mathbf{x} \in \mathbb{C}^B$ where $B < N$, and the smaller vector $\mathbf{y}$ is used to estimate DOAs. For example, the covariance of $\mathbf{y}$ can be estimated from its snapshots, and its noise eigenspace analyzed to perform DOA estimation as in MUSIC [14] and ESPRIT [11].

Due to dimensionality reduction ($B < N$), the $B \times B$ covariance of $\mathbf{y}$ has smaller size than that of $\mathbf{x}$. So the complexity of the eigenspace computation $O(B^3)$ is much smaller than $O(N^3)$ which is the complexity when using *element-space* directly ($\mathbf{T} = \mathbf{I}$). This is one of the major advantages of beamspace processing. If $\mathbf{T}$ is carefully chosen, then the DOAs which fall outside a chosen subband in $(-\pi/2, \pi/2)$ are attenuated by $\mathbf{T}$, so there are typically much fewer DOAs represented by $\mathbf{y}$ (compared to $\mathbf{x}$). One often uses a bank of transforms $\{\mathbf{T}_i\}$, which can be operated in *parallel*, to cover all DOAs in $(-\pi/2, \pi/2)$.

Besides low computation and parallelism, there are other advantages for beamspace. Beamspace methods tend to have smaller SNR threshold for resolution of closely spaced sources [7], [25], [28]. Beamspace estimates typically have smaller bias (and about the same mean square error) when compared with element-space estimates [31].

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However when performing root-MUSIC after the traditional beamspace transformation, one has to take elaborate steps to make sure the polynomial to be rooted has reduced degree, as explained with great clarity in [31], for the popular case where rows of $\mathbf{T}$ come from the DFT matrix. The transformation from $\mathbf{x}$ to $\mathbf{y}$ also destroys the geometrical (or shift-) invariance structure required by ESPRIT, and some effort is required to restore it, as explained in [27].

In this paper we introduce a new approach called the *convolutional beamspace method*, which allows root-MUSIC and ESPRIT to be directly applied after dimensionality reduction. This is an important advantage besides the usual computational benefit of going from $O(N^3)$ to $O(B^3)$. The method also produces more accurate DOA estimates than traditional beamspace methods, and for correlated sources it produces better estimates than element-space methods as well.

The use of convolution (digital filtering) prior to frequency estimation for time-domain sum-of-sinusoids was introduced many years ago by Silverstein, et al. [15], and studied in detail in [19]. But these methods, and many of the details in [19], are not directly applicable to spatial arrays. The purpose of this paper is to develop the appropriate formulation for spatial arrays. Crucial to the method is the extraction of a **steady-state** component from the convolution as we shall see. The basic idea is introduced in Sec. 2, and details of dimension reduction using uniform **decimation** are presented in Sec. 3. Noise whitening in the reduced space using Nyquist filter design is then discussed. Simulations in Sec. 4 demonstrate the performance of the new method.

Preliminaries. We consider uniform linear arrays (ULA) with sensor spacing $\lambda/2$, and monochromatic plane waves arriving from D directions. The array output equation is

$$\mathbf{x} = \mathbf{A}\mathbf{c} + \mathbf{e} \tag{1}$$

where $\mathbf{c}$ contains source amplitudes c_i, $\mathbf{e}$ contains additive noise terms, and $\mathbf{A} = \begin{bmatrix} \mathbf{a}(\omega_1) & \mathbf{a}(\omega_2) & \cdots & \mathbf{a}(\omega_D) \end{bmatrix}$ with $\mathbf{a}(\omega) = [1 \ e^{j\omega} \ e^{j2\omega} \ \ldots \ e^{j(N-1)\omega}]^T$, so that $\mathbf{A}$ is a Vandermonde matrix. Here $\omega = \pi \sin\theta$, with θ measured from the normal to the line of array. We assume $E[\mathbf{c}] = \mathbf{0}$, $E[\mathbf{e}] = \mathbf{0}$, $E[\mathbf{e}\mathbf{e}^H] = \sigma_e^2\mathbf{I}$, and $E[\mathbf{c}\mathbf{e}^H] = \mathbf{0}$, where superscript H denotes transpose conjugation.

2. THE CONVOLUTIONAL STEADY STATE

Let $x(n), 0 \leq n \leq N-1$ be the outputs of the N sensors of the ULA. Suppose we convolve this sequence with an FIR filter $h(n), 0 \leq n \leq L-1$ with $L < N$ to obtain the possibly nonzero output samples $y(n), 0 \leq n \leq N+L-2$. Of these, only

$$y(L-1), y(L), \cdots, y(N-1), \tag{2}$$

involve all the L filter coefficients, and can be considered *steady state* output samples:

$$\underbrace{\begin{bmatrix} y(L-1) \\ y(L) \\ \vdots \\ y(N-1) \end{bmatrix}}_{\text{call this } \mathbf{y}} = \mathbf{H} \underbrace{\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}}_{\text{call this } \mathbf{x}} \tag{3}$$

where $\mathbf{H}$ is a $(N-L+1) \times N$ banded Toeplitz matrix:

$$\mathbf{H} = \begin{bmatrix} h(L-1) & \cdots & h(0) & 0 & \cdots & 0 \\ 0 & h(L-1) & \cdots & h(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h(L-1) & \cdots & h(0) \end{bmatrix}$$

For example $y(L-2)$ does not contain $h(L-1)$ (as $x(-1) = 0$) and $y(N)$ does not contain $h(0)$ (as $x(N) = 0$). So these are not part of the steady state output (3). The steady state samples are obtained by sliding the weights $h(k)$ from left to right uniformly, as shown in Fig. 1, to obtain $\mathbf{y}$. We call $\mathbf{y}$ the **convolutional beamspace signal.** Compare this with traditional beamspace $\mathbf{y} = \mathbf{Tx}$ where $\mathbf{T}$ is a fat $(B \times N)$ matrix, but without any Toeplitz structure.

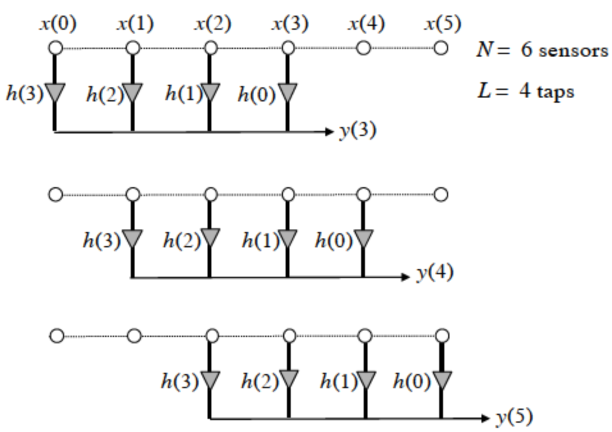

Fig. 1. The steady state convolutional beamspace signal $\mathbf{y} = [y(3)\ y(4)\ y(5)]^T$ generated by sliding the weights $h(k)$ over the sensors. Here $N = 6$ and $L = 4$.

Now assume we have a signal arriving from DOA θ so that $x(n) = e^{j\omega n}, 0 \leq n \leq N-1$ (up to some scale, which we ignore) where the normalized DOA $\omega = \pi \sin\theta$. Then from the steady state equation (3) we have

$$\mathbf{y} = e^{j(L-1)\omega} H(e^{j\omega}) \begin{bmatrix} 1 & e^{j\omega} & \ldots & e^{j(N-L)\omega} \end{bmatrix}^T \tag{4}$$

where $H(z) = \sum_{n=0}^{L-1} h(n) z^{-n}$. So the convolutional beamspace signal $\mathbf{y}$ in response to a single DOA is a **Vandermonde vector** just like the array output vector $\mathbf{x} = \mathbf{a}(\omega) = [1\ e^{j\omega}\ e^{j2\omega}\ \ldots\ e^{j(N-1)\omega}]^T$. Moreover, $\mathbf{y}$ is scaled by the filter frequency response $H(e^{j\omega})$. Thus if there are D sources with DOAs ω_k, then since $x(n) = \sum_{k=1}^{D} c_k e^{j\omega_k n}$, we have

$$\mathbf{y} = \sum_{k=1}^{D} c_k e^{j(L-1)\omega_k} H(e^{j\omega_k}) \mathbf{a}_L(\omega_k) \tag{5}$$

where $\mathbf{a}_L(\omega) = \begin{bmatrix} 1 & e^{j\omega} & \ldots & e^{j(N-L)\omega} \end{bmatrix}^T$. The arriving signals with DOAs ω_k are therefore filtered by the response $H(e^{j\omega})$. Thus the array equation (1) is replaced with

$$\mathbf{y} = \mathbf{A}_L \mathbf{d} + \mathbf{He} \tag{6}$$

where $\mathbf{A}_L$ is a Vandermonde matrix obtained from $\mathbf{A}$ by keeping the first $N-L+1$ rows, and $\mathbf{d}$ has elements $d_k = c_k e^{j(L-1)\omega_k} H(e^{j\omega_k})$. While the development is valid for any ULA, for large arrays (large N) we can make L large and design a sharp cutoff filter with good stop band. Then $\mathbf{y}$ contains only those DOAs that fall in the passband of $H(e^{j\omega})$, assuming signals in the stopband are not too strong:

$$\mathbf{y} \approx \sum_{k=1}^{D_0} c_k e^{j(L-1)\omega_k} H(e^{j\omega_k}) \mathbf{a}_L(\omega_k) = \mathbf{A}_{L,0} \mathbf{d}_0 + \mathbf{He} \tag{7}$$

where $\mathbf{A}_{L,0}$ has D_0 columns of $\mathbf{A}_L$ corresponding to the D_0 sources that fall in the passband of $H(e^{j\omega})$ and $\mathbf{d}_0$ has the corresponding D_0 rows of $\mathbf{d}$. Fig. 2 shows a typical filter response, with two out of six DOAs falling in the passband. Since $\omega = \pi \sin\theta$, the DOA range $-\pi/2 \leq \theta < \pi/2$ corresponds to $-\pi \leq \omega < \pi$; it can equivalently be taken as $0 \leq \omega < 2\pi$, as $H(e^{j\omega})$ has period 2π. The FIR filter $H(z)$ can be designed by any standard method such as the minimax or equiripple method, the window method, and so on [9]. If the filter does not have sharp cutoff, it is likely that a DOA falls in the transition band, which requires more careful consideration. Note that we can process the array output $x(n)$ with an entire **filter bank** $H_m(z), 0 \leq m \leq M-1$ to cover the full DOA range $0 \leq \omega < 2\pi$ (Fig. 3). The outputs of filters can be *processed in parallel* to estimate all D DOAs.

The DOA estimation procedure would be to first estimate the number of DOAs D_0 from $\mathbf{y}$, and identify these D_0 DOAs using standard methods. Since the filter output $\mathbf{y}$ is represented in terms of the Vandermonde matrix $\mathbf{A}_L$ just like the original array output $\mathbf{x}$, we can use root-MUSIC or ESPRIT without any further adjustment or processing to the data. This is an advantage of the proposed convolutional beamspace method compared to traditional beamspace methods, for which root-MUSIC requires some preprocessing [31] (due to loss of Vandermonde structure), and so does ESPRIT [27] (due to loss of shift-invariance). The method, as presented, works best for large ULAs which are receiving more attention these days [4], [18], but can also be extended to sparse arrays with relatively few sensor elements [3], which have difference coarrays with large ULA segments [10], [24].

3. DECIMATING THE FILTER OUTPUT

In traditional beamspace methods the complexity advantage is obtained because $B << N$. Similarly, there are many

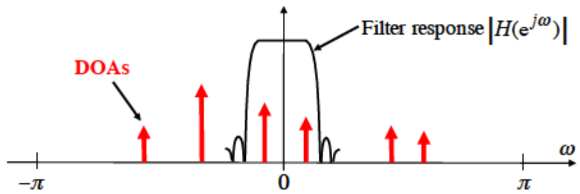

Fig. 2. Typical magnitude response $|H(e^{j\omega})|$, and example of DOA locations (red arrows). Two of the six DOAs are in passband.

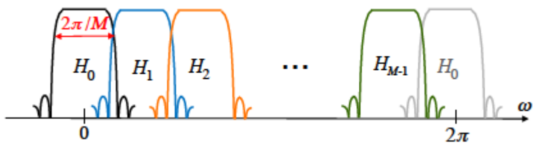

Fig. 3. Typical beamspace filter bank (magnitude responses).

ways to achieve dimensionality reduction for the convolutional beamspace output $\mathbf{y}$. One is to decimate $y(n)$ with a uniform downsampler. Since the passband of $H(z)$ has width $\approx 2\pi/M$ (Fig. 3) we can decimate $y(n)$ by the integer M. For larger arrays, L can be large, and the filters can be designed with sharp cutoff and good stopband attenuation to minimize aliasing due to decimation [23]. Let $v(n) = y(n+L-1)$ so that $\mathbf{y} = [v(0)\ v(1)\ \cdots\ v(N-L)]^T$. Define the decimated version $v_0(n) = v(Mn)$. The vector $\mathbf{y}$ is then replaced with the decimated vector $\mathbf{v}_0 = [v(0)\ v(M)\ v(2M)\ \cdots]^T$ which only has $J = \lceil (N-L+1)/M \rceil$ elements. We can estimate the $J \times J$ covariance of $\mathbf{v}_0$ from snapshots and estimate the D_0 DOAs in the passband, if $D_0 < J$. The complexity of eigenspace computation is now $O(J^3) << O(N^3)$. One might think that decimation leads to "waste" of hard-earned data, but we can make good use of *all* data while estimating the $J \times J$ covariance. Thus consider shifted versions $v(n+l)$ for $0 \leq l \leq M-1$ and define their decimated versions

$$v_l(n) = v(Mn+l). \tag{8}$$

These are the *polyphase components* of $v(n)$ [23]. Let $\mathbf{v}_l = [v_l(0)\ v_l(1)\ v_l(2)\ \cdots]^T$, that is,

$$\mathbf{v}_l = [v(l)\ v(l+M)\ v(l+2M)\ \cdots]^T. \tag{9}$$

We will estimate $J \times J$ covariances of $\mathbf{v}_l$ and average over all l to obtain a "coherent" estimate of the $J \times J$ covariance of decimated convolutional beamspace data.

3.1. The decimated covariance matrix

We can write $\mathbf{v}_l = \mathbf{D}_l\mathbf{y}$ where $\mathbf{D}_l$ is a decimation matrix (containing 0s and 1s). It retains the rows $l, l+M, l+2M, \cdots$ From (6) we have $\mathbf{v}_l = \mathbf{D}_l\mathbf{y} = \mathbf{D}_l\mathbf{A}_L\mathbf{d} + \mathbf{D}_l\mathbf{H}\mathbf{e}$. It can be verified that this simplifies to

$$\mathbf{v}_l = \mathbf{A}_J\mathbf{d}_l + \mathbf{D}_l\mathbf{H}\mathbf{e} \tag{10}$$

where

$$\mathbf{A}_J = \begin{bmatrix}\mathbf{a}_J(M\omega_1) & \mathbf{a}_J(M\omega_2) & \cdots & \mathbf{a}_J(M\omega_D)\end{bmatrix} \tag{11}$$

with $\mathbf{a}_J(\omega) = [1\ e^{j\omega}\ e^{j2\omega}\ \cdots\ e^{j(J-1)\omega}]^T$, and $\mathbf{d}_l$ is the $D \times 1$ vector

$$\mathbf{d}_l = \begin{bmatrix}c_1 e^{j(L-1+l)\omega_1}H(e^{j\omega_1}) & \cdots & c_D e^{j(L-1+l)\omega_D}H(e^{j\omega_D})\end{bmatrix}^T$$

Thus with $\mathbf{R}_{d_l} = E[\mathbf{d}_l\mathbf{d}_l^H]$, the covariance of $\mathbf{v}_l$ is

$$\mathbf{R}_{v_l} = E[\mathbf{v}_l\mathbf{v}_l^H] = \mathbf{A}_J\mathbf{R}_{d_l}\mathbf{A}_J^H + \sigma_e^2\mathbf{D}_l\mathbf{H}\mathbf{H}^H\mathbf{D}_l^H \tag{12}$$

The matrix $\mathbf{G} \triangleq \mathbf{H}\mathbf{H}^H$ is Hermitian and Toeplitz with first row $[g(0)\ g^*(1)\ g^*(2)\ \cdots\ g^*(N-L)]$, where

$$g(k) = \sum_n h(n)h^*(n-k) \tag{13}$$

is the deterministic autocorrelation of $h(n)$. Similarly it can be verified that the "decimated" matrix

$$\mathbf{G}_{dec} \triangleq \mathbf{D}_l\mathbf{H}\mathbf{H}^H\mathbf{D}_l \tag{14}$$

is $J \times J$ Hermitian and Toeplitz with first row

$$[g(0)\ g^*(M)\ g^*(2M)\ \cdots\ g^*((J-1)M)], \tag{15}$$

which is *independent* of l. Thus,

$$\mathbf{R}_{v_l} = \mathbf{A}_J\mathbf{R}_{d_l}\mathbf{A}_J^H + \sigma_e^2\mathbf{G}_{dec} \tag{16}$$

Thus, whereas $\mathbf{G}$ is the autocorrelation matrix of $h(n)$, the matrix $\mathbf{G}_{dec}$ is constructed from the decimated autocorrelation $g(Mk)$, and does not depend on l. The dependency of the first term of (16) on l can be averaged out:

$$\mathbf{R}_{ave} = \frac{1}{M}\sum_{l=0}^{M-1}\mathbf{R}_{v_l} = \mathbf{A}_J\breve{\mathbf{R}}_d\mathbf{A}_J^H + \sigma_e^2\mathbf{G}_{dec} \tag{17}$$

where $\breve{\mathbf{R}}_d$ is $\mathbf{R}_{d_l}$ averaged over l. (If J is not the same for all l we drop some samples at the end of some $\mathbf{v}_l$.) In practice we estimate $\mathbf{R}_{v_l}$ from snapshots for each l, and then estimate $\mathbf{R}_{ave}$. This is the estimated $J \times J$ covariance to be used for estimating DOAs in the filter passband. Since $\mathbf{v}_l$ for all l are used, *all* the $N-L+1$ components of the convolutional beamspace signal $\mathbf{y}$ are exploited, and no data is wasted.

Note that since the columns of $\mathbf{A}_J$ are $\mathbf{a}_J(M\omega_i)$ rather than $\mathbf{a}_J(\omega_i)$, we can only estimate $M\omega_i$, or equivalently

$$\omega_i + 2\pi s_i/M, \tag{18}$$

where the integers s_i are unknown, creating **ambiguity**. But since ω_i are known to be in the passband of $H(e^{j\omega})$ which has width $2\pi/M$, the ambiguities s_i can be resolved in most cases.

3.2. Spectral factors of Nyquist filters to whiten noise

The undecimated output of convolution (6) has covariance $\mathbf{R}_{\mathbf{yy}} = \mathbf{A}_L\mathbf{R}_d\mathbf{A}_L^H + \sigma_e^2\mathbf{G}$ where $\mathbf{G} = \mathbf{H}\mathbf{H}^H$ and $\mathbf{R}_d = E[\mathbf{d}\mathbf{d}^H]$. The noise term $\sigma_e^2\mathbf{G}$ cannot be a diagonal matrix unless the filter has the trivial form $H(z) = cz^{-n_0}$. But the decimated output (10) has covariance (16) for all l, so the corresponding noise term can be whitened by making $\mathbf{G}_{dec} = \mathbf{I}$, or equivalently

$$g(Mk) = \delta(k) \tag{19}$$

where $g(k)$ is as in (13). Eq. (19) is called the **Nyquist(M)** property of $g(k)$. Since $|H(e^{j\omega})|^2$ is the Fourier transform of $g(k)$, we say that $H(z)$ is a **spectral factor** of the Nyquist(M) filter $|H(e^{j\omega})|^2$. In short, by designing the FIR filter $H(z)$ to be a spectral factor of an FIR Nyquist(M) filter $G(z)$ with $G(e^{j\omega}) \geq 0$ we can ensure that the noise terms in the decimated versions $v(Mn + l)$ are white for all l. So $\mathbf{R}_{ave}$ becomes $\mathbf{R}_{ave} = \mathbf{A}_J \breve{\mathbf{R}}_d \mathbf{A}_J^H + \sigma_e^2 \mathbf{I}$ where $\mathbf{A}_J$ is as in (11). This makes it easy to find the noise eigenspace by computing eigenvectors of $\mathbf{R}_{ave}$, which is what we do in the next section on simulations. Spectral factors of Nyquist filters arise in digital communications and in filter bank theory [23]. There are many ways to design such filters [12], [13], [20], [22]. In fact any filter $H_k(e^{j\omega})$ in an *orthonormal* (equivalently paraunitary) filter bank is automatically a spectral factor of a Nyquist filter [23]. Many examples of good FIR designs with this property can be found in the literature [6], [8], [21], [22], [23]. In fact, if $H(e^{j\omega})$ is a "good" filter with total passband width $\approx 2\pi/M$ and ripples properly constrained, this Nyquist property (19) is approximately satisfied, that is,

$$\sum_{n \neq 0} |g(Mn)| << g(0) \quad \text{(nearly-Nyquist property)} \qquad (20)$$

For simplicity, this is what we use in the next section.

4. SIMULATIONS

Consider a ULA with $N = 96$ sensors receiving 6 sources with equal powers $p_k = 1$ at angles $-3°, 1.5°, 3°, 40°, 60°$, and $80°$. Let noise variance $\sigma_e^2 = 1$. We use a 24th order lowpass FIR $H(z)$ ($L = 25$) designed using the Parks-McClellan algorithm [9], with cutoff π/M for various M (where *cutoff* = average of pass and stop band edges). M is also the decimation ratio. Filter responses for some M are shown in Fig. 4(a), and satisfy (20). For all filters used, there are three sources in the passband ($-3°, 1.5°$ and $3°$) and three in the stopband ($40°, 60°$, and $80°$). Fig. 4(b) shows the RMS error in detected source angles in the passband using root-MUSIC, for various values of $1/M$ (cutoff normalized by π). The sources are assumed uncorrelated, 200 snapshots were used for covariance estimates, and 500 Monte Carlo runs averaged to get plots. The proposed convolutional beamspace method outperforms the traditional beamspace method [31], the poor performance of the latter being consistent with numerical sensitivity issues mentioned in [31] as the number of "beams" B (i.e., passband width in our notion) increases. Note that element-space root-MUSIC performs slightly better: it is well known that beamspace methods reduce complexity, increase parallelism, and improve resolution thresholds [7], but do not always improve mean square errors (compared to element-space), for *uncorrelated* sources [17].

Fig. 4(c) shows the performance for *correlated* sources: sources n and $n+3$ have a correlation coefficient $\rho = 0.85$ for $n = 1, 2, 3$. In this case the performance with the proposed convolutional beamspace is much better than in element-space. This does not contradict [17] because therein the signal subspace dimension in the beamspace is assumed to be the same as that in the element-space, but for convolutional beamspace, signal subspace dimension after filtering can be smaller. That is, in [17], all sources, including those in the stopband if any, still have to be estimated in the beamspace, while we only have to estimate in-band ones.

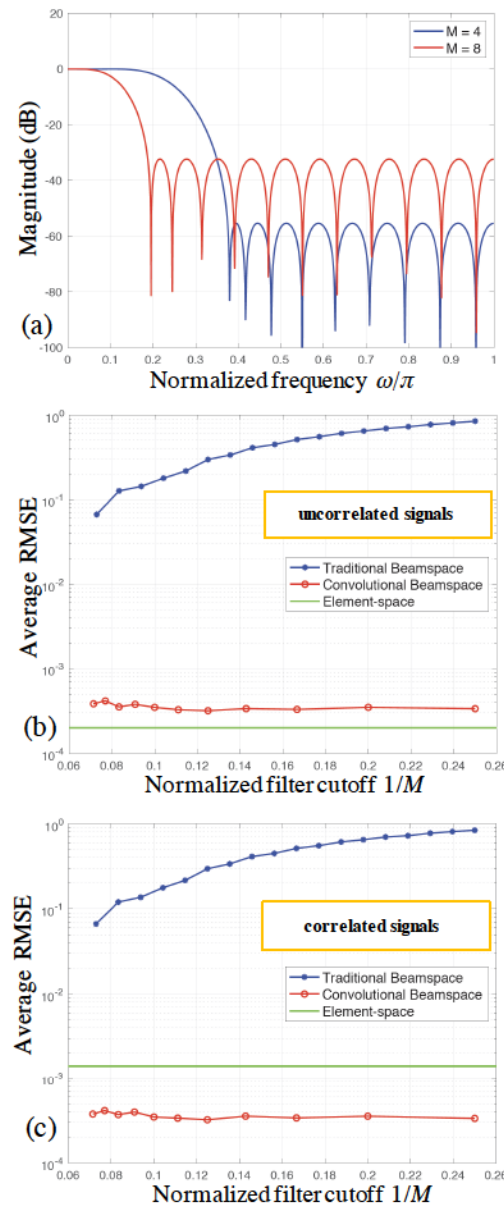

Fig. 4. Simulation example. (a) Responses of typical filters used. (b)-(c) RMSE for uncorrelated and correlated sources. For correlated sources, the proposed method is significantly better.

5. CONCLUDING REMARKS

We introduced a new beamspace method based on convolution of the ULA output with a filter. This allows the use of root-MUSIC and ESPRIT in beamspace without additional preparation (which is a major inconvenience in traditional beamspace). For correlated sources, in addition to the usual advantages of beamspace, the new method produces significantly better DOA estimates compared to element-space and traditional beamspace methods. This is due to effective filtering of out-of-band sources that might be correlated with in-band sources. While the results were presented for ULAs, they can be extended to sparse arrays like nested or coprime arrays [10], [24] with relatively fewer sensor elements [3].

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