Convolutional Beamspace and Sparse Signal Recovery for Linear Arrays

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***Abstract*—The convolutional beamspace (CBS) method for DOA estimation using dictionary-based sparse signal recovery is introduced. Beamspace methods enjoy lower computational complexity, increased parallelism of subband processing, and improved DOA resolution. But unlike classical beamspace methods, CBS allows root-MUSIC and ESPRIT to be performed directly for ULAs without additional preparation since the Vandermonde structure for ULAs are preserved in the CBS output. Due to the same reason, it is shown in this paper that sparse signal representation problems can also be directly formulated on the CBS output. Significant reduction in computational complexity and higher probability of resolution are obtained by using CBS. It is also shown how the regularization parameter involved in the method should be chosen.**[1]

***Index Terms*—Convolutional beamspace, DOA estimation, linear sensor arrays, sparse signal recovery, dictionaries.**

I. Introduction

The use of beamforming prior to direction-of-arrival (DOA) estimation, known as *beamspace* processing, is a well-known and still evolving technique in the literature of array signal processing [1]–[9]. In beamspace processing, given a N-sensor array with output $\mathbf{x} \in \mathbb{C}^N$, we compute a transformation $\mathbf{y} = \mathbf{T}\mathbf{x} \in \mathbb{C}^B$, where $B < N$, and estimate DOAs using $\mathbf{y}$. For instance, the $B \times B$ covariance of $\mathbf{y}$ can be estimated from its snapshots, and DOAs can be estimated via MUSIC [10] after the signal and noise eigenspaces are analyzed.

A major advantage of beamspace processing is the reduction in computation complexity since $B < N$. For instance, the complexity of the eigenspace computation for beamspace, $O(B^3)$, is much smaller than $O(N^3)$, which is the complexity when using *element-space* ($\mathbf{T} = \mathbf{I}$) directly. With careful selection of $\mathbf{T}$, the DOAs which fall outside a chosen subband in $[-\pi/2, \pi/2)$ are attenuated by $\mathbf{T}$, so there are typically much fewer DOAs represented by $\mathbf{y}$ than by $\mathbf{x}$. To cover all DOAs in $[-\pi/2, \pi/2)$, one can use a bank of transformations $\{\mathbf{T}_i\}$, which can be operated in parallel. Besides the advantages of low computation and parallelism, beamspace methods tend to have smaller SNR threshold for resolution of closely spaced sources [5], [11], [12]. Beamspace estimates also typically have smaller bias (and similar mean square error) when compared to element-space estimates [13], [14].

Classical beamspace methods compromise the Vandermonde structure in the output of a uniform linear array (ULA), so elaborate steps have to be taken to apply root-MUSIC [13] or ESPRIT [15]. By contrast, the *convolutional beamspace (CBS)* approach [9] allows root-MUSIC and ESPRIT to be performed directly for ULAs without additional preparation since the Vandermonde structure is preserved under the CBS transformation. CBS achieves this by convolving the ULA output with an FIR filter $H(z)$ and extracting the steady-state component. A uniform decimation (downsampling) is then applied to fulfill complexity reduction.

In this paper, we show how the CBS method can be integrated with dictionary-based sparse signal recovery. The use of sparse representation techniques for DOA estimation can be traced back to [16], where a dictionary of steering vectors (corresponding to a dense grid of potential DOAs) is used to represent the array output. An optimization problem involving the dictionary is then formulated, and the sparse solution to the problem reveals the DOAs. We will show that the dimension of the problem is greatly reduced by using the CBS method. The sparse signal representation problem can be directly formulated without additional preparation because the Vandermonde structure of ULAs is preserved in the CBS output. Besides its significant computational advantage, the method also produces higher probability of resolution with closely spaced sources.

This paper is organized as follows. The key operations of CBS, introduced in [9], are reviewed in Sec. II. The details of CBS for dictionary-based sparse signal recovery are developed in Sec. III. Simulations are given in Sec. IV, and Sec. V concludes the paper.

II. Review of Convolutional Beamspace

We consider a N-sensor ULA with sensor spacing $\lambda/2$, and monochromatic plane waves of wavelength λ arriving from D directions. The array output equation is

$$\mathbf{x} = \mathbf{A}\mathbf{c} + \mathbf{e}. \quad (1)$$

where $\mathbf{c}$ contains source amplitudes c_i, $\mathbf{e}$ contains additive noise terms, and $\mathbf{A} = [\mathbf{a}(\omega_1)\ \mathbf{a}(\omega_2)\ \cdots\ \mathbf{a}(\omega_D)]$ with $\mathbf{a}(\omega) = [1\ e^{j\omega}\ e^{j2\omega}\ \ldots\ e^{j(N-1)\omega}]^T$, so that $\mathbf{A}$ is a Vandermonde matrix. Here $\omega = \pi \sin\theta$, with DOA $\theta \in [-\pi/2, \pi/2)$ measured from the normal to the line of array. We assume $\mathrm{E}[\mathbf{c}] = \mathbf{0}$, $\mathrm{E}[\mathbf{e}] = \mathbf{0}$, $\mathrm{E}[\mathbf{e}\mathbf{e}^H] = \sigma_e^2\mathbf{I}$, and $\mathrm{E}[\mathbf{c}\mathbf{e}^H] = \mathbf{0}$, where $(\cdot)^H$ denotes Hermitian transpose.

A. Convolutional Beamspace

It is proposed in [9] that we convolve the sequence $x(n), 0 \le n \le N-1$, which is the N-sensor ULA output, with an FIR filter $H(z) = \sum_{n=0}^{L-1} h(n)z^{-n}$ with $L < N$ to get the possibly nonzero output samples $y(n), 0 \le n \le N+L-2$.

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Of these, only $y(L-1), y(L), \cdots, y(N-1)$ involve all the L filter taps, and can be considered *steady-state* output samples:

$$\mathbf{y} \triangleq \begin{bmatrix} y(L-1) \\ y(L) \\ \vdots \\ y(N-1) \end{bmatrix} = \mathbf{H} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} = \mathbf{Hx}, \qquad (2)$$

where $\mathbf{H}$ is a $(N-L+1) \times N$ banded Toeplitz matrix:

$$\mathbf{H} = \begin{bmatrix} h(L-1) & \cdots & h(0) & 0 & \cdots & 0 \\ 0 & h(L-1) & \cdots & h(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h(L-1) & \cdots & h(0) \end{bmatrix}.$$

We call $\mathbf{y}$ the *convolutional beamspace signal* [9].

The banded Toeplitz structure of $\mathbf{H}$ results in a Vandermonde structure in $\mathbf{y}$. When there are D sources with DOAs ω_k, i.e., $x(n) = \sum_{k=1}^{D} c_k e^{j\omega_k n}$, it can be shown that [9]

$$\mathbf{y} = \sum_{k=1}^{D} c_k e^{j(L-1)\omega_k} H(e^{j\omega_k}) \mathbf{a}_L(\omega_k) + \mathbf{He} \qquad (3)$$

where $\mathbf{a}_L(\omega) = \begin{bmatrix} 1 \ e^{j\omega} \cdots e^{j(N-L)\omega} \end{bmatrix}^T$. The arriving signals with DOAs ω_k are therefore filtered by the response $H(e^{j\omega})$. Thus the array equation (1) is replaced with

$$\mathbf{y} = \mathbf{A}_L \mathbf{d} + \mathbf{He} \qquad (4)$$

where $\mathbf{A}_L$ is a Vandermonde matrix obtained from $\mathbf{A}$ by keeping the first $N-L+1$ rows, and $\mathbf{d}$ has elements $d_k = c_k e^{j(L-1)\omega_k} H(e^{j\omega_k})$. Assuming signals in the stopband are not too strong so that $\mathbf{y}$ contains only those DOAs that fall in the passband of $H(e^{j\omega})$, we have

$$\mathbf{y} \approx \mathbf{A}_{L,0} \mathbf{d}_0 + \mathbf{He}. \qquad (5)$$

Here $\mathbf{A}_{L,0}$ has D_0 columns of $\mathbf{A}_L$ corresponding to the D_0 sources that fall in the passband of $H(e^{j\omega})$, and $\mathbf{d}_0$ has the corresponding D_0 rows of $\mathbf{d}$. Since the filter output $\mathbf{y}$ is represented in terms of the Vandermonde matrix $\mathbf{A}_L$ just like the original array output $\mathbf{x}$, we can use root-MUSIC or ESPRIT without any further adjustment or processing to the data. This is an advantage of the CBS method [9] compared to classical beamspace methods, for which some preprocessing is required for root-MUSIC [13] and ESPRIT [15]. The FIR filter $H(z)$ can be designed by any standard method such as the equiripple method and the window method [17].

B. Decimating the filter output

In classical beamspace methods, the complexity advantage is obtained because $B \ll N$. However, for CBS, $N-L+1 \approx N$ since $L \ll N$ in practice. To achieve the complexity reduction of beamspace methods, we simply *decimate* $y(n)$ with a uniform downsampler [9]. If the passband of $H(z)$ has width $\approx 2\pi/M$, we can decimate $y(n)$ by the integer M. Let $v(n) = y(n+L-1)$ so that $\mathbf{y} = [v(0) \ v(1) \ \cdots \ v(N-L)]^T$. Define the decimated version $v_0(n) = v(Mn)$. The vector $\mathbf{y}$ is then replaced by the decimated vector $\bar{\mathbf{v}}_0 = [v(0) \ v(M) \cdots v(J_0 M)]^T$, where $J_0 = \lceil (N-L+1)/M \rceil$. We can estimate the $J_0 \times J_0$ covariance of $\bar{\mathbf{v}}_0$ from snapshots and estimate the D_0 DOAs in the passband, if $D_0 < J_0$. The complexity of eigenspace computation is now $O(J_0^3) \ll O(N^3)$. To make good use of *all* the data, we estimate a $J \times J$ covariance, where $J = \lfloor (N-L+1)/M \rfloor$, by using all the *polyphase components* of $v(n)$ [18]. That is, we will estimate $J \times J$ covariances of

$$\mathbf{v}_l = [v(l) \ v(l+M) \ \cdots \ v(l+(J-1)M) \ ]^T \qquad (6)$$

and average over all l to obtain a "coherent" estimate of the $J \times J$ covariance of decimated CBS data.

We can write $\mathbf{v}_l = \mathbf{D}_l \mathbf{y}$, where $\mathbf{D}_l$ is a decimation matrix (containing 0s and 1s) such that it retains the rows $l, l+M, \ldots, l+(J-1)M$. It can be verified that [9]

$$\mathbf{v}_l = \mathbf{A}_J \mathbf{d}_l + \mathbf{D}_l \mathbf{He}, \qquad (7)$$

where

$$\mathbf{A}_J = [\mathbf{a}_J(M\omega_1) \ \mathbf{a}_J(M\omega_2) \ \cdots \ \mathbf{a}_J(M\omega_D)] \qquad (8)$$

with $\mathbf{a}_J(\omega) = [1 \ e^{j\omega} \ e^{j2\omega} \ \cdots \ e^{j(J-1)\omega}]^T$, and

$$\mathbf{d}_l = \begin{bmatrix} c_1 e^{j(L-1+l)\omega_1} H(e^{j\omega_1}) \ \cdots \ c_D e^{j(L-1+l)\omega_D} H(e^{j\omega_D}) \end{bmatrix}^T.$$

Eq. (7) has the same Vandermonde structure as in the ULA output (1), so root-MUSIC and ESPRIT can be applied directly via formulating the decimated covariance matrix [9]. Also, the noise term after decimation can be made white if $H(z)$ is a spectral factor of an FIR Nyquist(M) filter [9]. That is,

$$g(k) = \sum_n h(n) h^*(n-k), \qquad (9)$$

which is the deterministic autocorrelation of $h(n)$, satisfies

$$g(Mk) = \delta(k). \qquad (10)$$

Spectral factors of Nyquist filters arise in filter bank theory [18] and in digital communications [19]. See [18], [20]–[22] and references therein for design of such filters.

III. CONVOLUTIONAL BEAMSPACE AND SPARSE RECOVERY

We now show how CBS can be used in conjunction with sparse signal recovery. Sparse signal representation techniques for DOA estimation have been studied in the literature [16]. In this context, a dictionary $\mathbf{D}$ of steering vectors $\mathbf{a}(\omega_i)$ on a grid of potential DOAs $\{\omega_i\}_{i=1}^d$ is considered, and the goal is to find a sparse signal $\mathbf{q} = [q_1 \ q_2 \cdots q_d]^T$ that well represents the ULA output $\mathbf{x}$:

$$\mathbf{x} = \underbrace{[\mathbf{a}(\omega_1) \ \mathbf{a}(\omega_2) \ \cdots \ \mathbf{a}(\omega_d)]}_{\text{dictionary } \mathbf{D}} \mathbf{q} + \mathbf{e}, \qquad (11)$$

where the error term $\mathbf{e}$ should be "small". The number of dictionary atoms d is typically much larger than D, the number of sources. A popular technique to obtain the sparse vector $\mathbf{q}$ is the Lasso method [23] that solves the following problem:

$$\min_{\mathbf{q} \in \mathbb{C}^d} \quad \|\mathbf{q}\|_1 \qquad (12a)$$

$$\text{subject to} \ \ \|\mathbf{x} - \mathbf{Dq}\|_2^2 \leq \beta, \qquad (12b)$$

where $\beta \geq 0$ is a parameter. The l_1-norm objective (12a) serves as a surrogate for sparsity, and the l_2-norm constraint (12b) limits the search space to where the noise term is small.

A. Convolutional Beamspace and Dictionaries

As in (2), we convolve the sequence $x(n), 0 \leq n \leq N-1$ with an FIR filter $h(n), 0 \leq n \leq L-1$ with $L < N$, and extract the steady-state samples:

$$\mathbf{y} = \mathbf{H}\mathbf{x}. \tag{13}$$

As in (3), the response to a single DOA is (ignoring noise)

$$\mathbf{y} = \mathbf{H}\mathbf{a}(\omega) = e^{j(L-1)\omega} H(e^{j\omega}) \mathbf{a}_L(\omega). \tag{14}$$

Thus, (11) and (13) yield

$$\mathbf{y} = [\mathbf{a}_L(\omega_1)\ \mathbf{a}_L(\omega_2)\ \cdots\ \mathbf{a}_L(\omega_d)]\, \mathbf{\Lambda}_h \mathbf{q} + \mathbf{H}\mathbf{e}, \tag{15}$$

where $\mathbf{\Lambda}_h$ is a diagonal matrix with ith diagonal element $(\mathbf{\Lambda}_h)_{ii} = e^{j(L-1)\omega_i} H(e^{j\omega_i})$. In other words, the diagonal elements are the frequency responses of $h(n)$ evaluated at the dictionary frequencies (with some phase shift). If $H(e^{j\omega})$ is a good narrowband lowpass filter, then

$$\mathbf{y} \approx \underbrace{[\mathbf{a}_L(\omega_1)\ \mathbf{a}_L(\omega_2)\ \cdots\ \mathbf{a}_L(\omega_{d_0})]}_{\text{dictionary } \mathbf{D}_L} \mathbf{q}_0 + \mathbf{H}\mathbf{e}, \tag{16}$$

where $\omega_1, \omega_2, \ldots, \omega_{d_0}$ are the frequencies within the passband of $H(e^{j\omega})$, and $\mathbf{q}_0 \in \mathbb{C}^{d_0}$ is a much shorter vector than $\mathbf{q}$. Suppose $H(e^{j\omega})$ has passband width $\approx 2\pi/M$. Then, $d_0 \approx d/M$ (if ω_i is a uniform grid of frequencies in $[0, 2\pi)$). Thus, a Lasso problem can be formulated for the CBS signal $\mathbf{y}$ as

$$\min_{\mathbf{q}_0 \in \mathbb{C}^{d_0}} \quad \|\mathbf{q}_0\|_1 \tag{17a}$$

$$\text{subject to} \quad \|\mathbf{y} - \mathbf{D}_L \mathbf{q}_0\|_2^2 \leq \beta. \tag{17b}$$

Here, the original dictionary $\mathbf{D}$ in (12b) is replaced by the dictionary $\mathbf{D}_L$ for CBS. The complexity reduction is significant because the number of optimization variables is reduced by about M times.

B. Decimation for Dictionaries

To further reduce computational complexity, we can decimate the CBS signal $\mathbf{y}$ by M if $H(e^{j\omega})$ is a good filter with passband width $\approx 2\pi/M$. Let $v(n) = y(n+L-1)$ so that $\mathbf{y} = [v(0)\ v(1) \cdots v(N-L)]^T$. Let $v_0(n) = v(Mn)$ and $\mathbf{v}_0 = [v_0(0)\ v_0(1)\ \cdots v_0(J_0-1)]^T$, where $J_0 = \lceil (N-L+1)/M \rceil$, so the decimated version

$$\mathbf{v}_0 = [v(0)\ v(M)\ \cdots\ v((J_0-1)M)\ ]^T. \tag{18}$$

Then, we obtain the complexity-reduced problem

$$\min_{\mathbf{q}_0 \in \mathbb{C}^{d_0}} \quad \|\mathbf{q}_0\|_1 \tag{19a}$$

$$\text{subject to} \quad \|\mathbf{v}_0 - \mathbf{D}_{L,0} \mathbf{q}_0\|_2^2 \leq \beta, \tag{19b}$$

where $\mathbf{D}_{L,0}$ is the matrix obtained by retaining the rows $0, M, \ldots, (J_0-1)M$ of $\mathbf{D}_L$. The computational complexity of Problem (19) is much lower than the original problem (12).

C. Multiple Snapshots

The previous formulation is for a single snapshot. For multiple snapshots, we adopt the ℓ_1-SVD method proposed in [16]. Suppose we have K snapshots, $\mathbf{X} = [\mathbf{x}(1)\ \mathbf{x}(2) \cdots \mathbf{x}(K)]$. To reduce dimensionality, we take the SVD $\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$ and retain a $N \times k$ matrix containing most of the signal power: $\mathbf{X}_{SV} = \mathbf{U}\mathbf{\Sigma}\mathbf{J}_k = \mathbf{X}\mathbf{V}\mathbf{J}_k$, where $\mathbf{J}_k = [\mathbf{I}_k\ \mathbf{0}]^T$. We often take $k < K$ to be roughly the number of sources, and the original formulation of the ℓ_1-SVD method is then [16]

$$\min_{\mathbf{Q} \in \mathbb{C}^{d \times k}} \quad \|\mathbf{Q}\|_{1,2} \tag{20a}$$

$$\text{subject to} \quad \|\mathbf{X}_{SV} - \mathbf{D}\mathbf{Q}\|_F^2 \leq \beta, \tag{20b}$$

where $\|\mathbf{Q}\|_{1,2} = \sum_m \sqrt{\sum_n |\mathbf{Q}_{mn}|^2}$. That is, the ℓ_2-norm across singular vector samples is first computed for each spatial index, and then the ℓ_1-norm is computed across spatial samples for sparsity.

CBS can also be applied to the multiple snapshot scheme based on the ℓ_1-SVD method. We first convolve the spatial samples of each snapshot with a filter $h(n)$ of length L and extract the steady-state samples: $\mathbf{Y} = \mathbf{H}\mathbf{X}$, similar to (13). Then, we take the SVD $\mathbf{Y} = \mathbf{U}_Y \mathbf{\Sigma}_Y \mathbf{V}_Y^H$ and retain a $(N-L+1) \times k_0$ matrix containing most of the signal power: $\mathbf{Y}_{SV} = \mathbf{U}_Y \mathbf{\Sigma}_Y \mathbf{J}_{k_0} = \mathbf{Y}\mathbf{V}_Y \mathbf{J}_{k_0}$. A multiple-snapshot version of Problem (17) can then be formulated as

$$\min_{\mathbf{Q}_0 \in \mathbb{C}^{d_0 \times k_0}} \|\mathbf{Q}_0\|_{1,2} \tag{21a}$$

$$\text{subject to} \quad \|\mathbf{Y}_{SV} - \mathbf{D}_L \mathbf{Q}_0\|_F^2 \leq \beta, \tag{21b}$$

where we again assume $H(e^{j\omega})$ has passband width $\approx 2\pi/M$ so that $d_0 \approx d/M$. According to [16], to get adequate performance, we have to take k to be roughly the number of sources D in Problem (20). As only the D_0 sources in the passband are effective after filtering, we can take $k_0 \approx D_0$ in Problem (21). If the sources are roughly uniformly distributed, then $D_0 \approx D/M$, and $d_0 k_0 \approx dk/M^2$, so the number of optimization variables is reduced by about M^2 when we go from (20) to (21). This can be a very significant complexity reduction.

Likewise, a decimated version can be considered. Let $\mathbf{V}_0 = [\mathbf{v}_0(1)\ \mathbf{v}_0(2) \cdots \mathbf{v}_0(K)]$ be the multiple-snapshot counterpart of (18). Then, we take the SVD $\mathbf{V}_0 = \mathbf{U}_V \mathbf{\Sigma}_V \mathbf{V}_V^H$ and retain a $J_0 \times k_0$ matrix containing most of the signal power:

$$\mathbf{V}_{SV} = \mathbf{U}_V \mathbf{\Sigma}_V \mathbf{J}_{k_0} = \mathbf{V}_0 \mathbf{V}_V \mathbf{J}_{k_0}. \tag{22}$$

Then, a multiple-snapshot version of Problem (19) can be formulated as

$$\min_{\mathbf{Q}_0 \in \mathbb{C}^{d_0 \times k_0}} \|\mathbf{Q}_0\|_{1,2} \tag{23a}$$

$$\text{subject to} \quad \|\mathbf{V}_{SV} - \mathbf{D}_{L,0} \mathbf{Q}_0\|_F^2 \leq \beta. \tag{23b}$$

Besides reducing computation, Problem (23) can also yield a higher probability of resolution than Problem (20). See Sec. IV for numerical examples.

D. Selection of Parameter β

We follow the method in [16] to select the parameter β. Specifically, β is chosen large enough so that the probability that the constraint in each optimization problem is not satisfied is small. For example, consider Problem (23). The error term

$$\tilde{\mathbf{E}} \triangleq \mathbf{V}_{SV} - \mathbf{D}_{L,0}\mathbf{Q}_0 = \mathbf{D}_0\mathbf{H}\mathbf{E}\mathbf{V}_V\mathbf{J}_{k_0} \tag{24}$$

where $\mathbf{E} = [\mathbf{e}(1)\ \mathbf{e}(2)\cdots\mathbf{e}(K)]$ is the noise of K snapshots, $\mathbf{D}_0$ is a decimation matrix (containing 0s and 1s) such that it retains the rows $0, M, \ldots, (J_0-1)M$, and $\mathbf{V}_V$, $\mathbf{J}_{k_0}$ are as defined in (22). If the noise $\mathbf{e}$ is i.i.d. circularly symmetric complex Gaussian, and the filter $H(z)$ is a spectral factor of a Nyquist(M) filter, i.e., (10) is satisfied, then the entries of $\mathbf{D}_0\mathbf{H}\mathbf{E}$ are also i.i.d. circularly symmetric complex Gaussian. Hence, for moderate to high SNR, $\|\tilde{\mathbf{E}}\|_F^2$ is approximately χ^2 distributed with $2J_0k_0$ degrees of freedom upon normalization by the variance of $\mathbf{e}$. This holds only approximately because the SVD $\mathbf{V}_0 = \mathbf{U}_V\mathbf{\Sigma}_V\mathbf{V}_V^H$ depends on the noise, so $\mathbf{D}_0\mathbf{H}\mathbf{E}$ and $\mathbf{V}_V$ are dependent. Yet when noise is small, the signal term dominates, so the approximate χ^2 distribution is obtained. With this, we can select β large enough so that $\mathbb{P}[\|\tilde{\mathbf{E}}\|_F^2 > \beta]$ is small. However, we should also keep β not too large to prevent failures in detecting the DOAs. This will be demonstrated in the next section.

IV. Simulations

We compare the most complexity-reduced version of CBS (23) with element-space (20) under multiple snapshots. Consider a ULA with $N = 99$ sensors. For CBS, the filter $H(z)$ is designed to be lowpass using the Parks-McClellan algorithm [17], with passband edge $\pi/2M$ and stopband edge $3\pi/2M$, where $M = 4$ is the decimation ratio. This filter is a spectral factor of a nearly Nyquist filter, i.e., the deterministic autocorrelation (9) satisfies $\sum_{n\neq 0}|g(Mn)| \ll g(0)$. A dictionary of 200 points uniform in ω is used. There are two in-band sources (sources in the passband) with DOAs $\theta = -0.573°$, $0.573°$ with power 1, and one out-of-band source (source in the stopband) with DOA $\theta = 39.94°$ with power varied. Here all the DOAs are exactly on the dictionary grid for simplicity. Noise variance $\sigma_e^2 = 1$ is used. The reduced dimensions for the ℓ_1-SVD method in (20) and (23) are chosen as $k = k_0 = 3$. Following the method in Sec. III-D, we choose $\beta = \mu + 20\sigma = 221.7$ in (23b), where μ and σ are the mean and standard deviation of $\|\tilde{\mathbf{E}}\|_F^2$. In a similar way, we choose $\beta = 641.7$ in (20b). (The effect of choice of β will be discussed in Fig. 2.) The *dictionary power spectrum* serves as a performance measure for the method. For element-space, it is defined as $P(\omega_i) = \sum_n |\hat{\mathbf{Q}}_{in}|^2$ for $1 \leq i \leq d$, where $\hat{\mathbf{Q}}$ is the optimal solution of Problem (20). For CBS, it is similarly defined for Problem (23). We declare that there is a source at ω_i if there is a peak (local maximum) that is larger than a particular threshold: $P(\omega_i) \geq \epsilon$. Here $\epsilon = 0.1$ is used (with the spectrum normalized to have a maximum value 1). To compare CBS with element-space, we focus on in-band DOAs and ignore out-of-band DOAs. Fig. 1(a) shows the probability of resolving the correct number of in-band DOAs as a function of out-of-band source power. We use $K = 100$ snapshots and 100 Monte Carlo runs to get the plot. Due to good stopband attenuation, CBS uniformly has a higher probability of resolution than element-space. Fig. 1(b) shows a typical dictionary power spectrum of element-space when the out-of-band source power is 30 dB. The out-of-band source

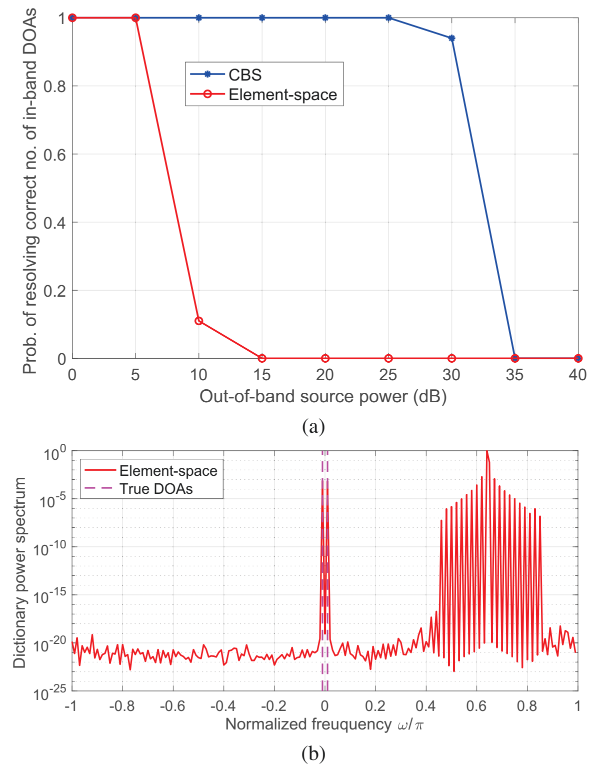
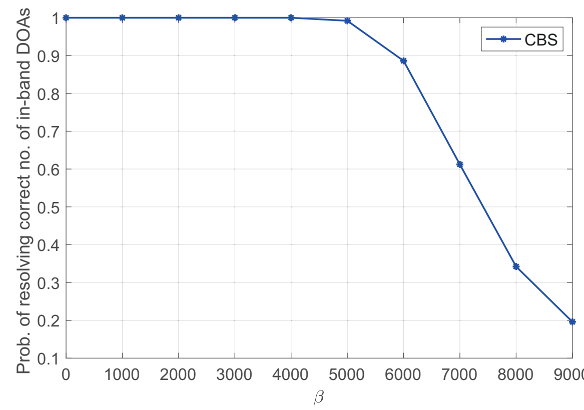


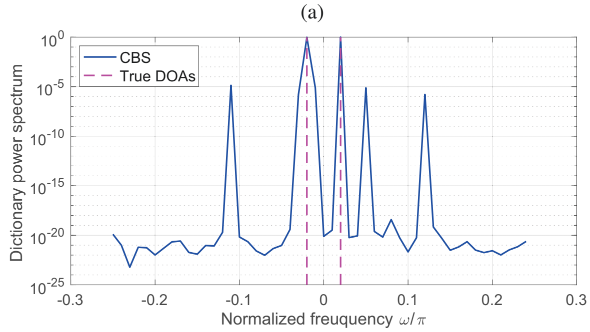
Fig. 1. Performance of CBS and element-space dictionaries when there are powerful out-of-band sources. (a) Probability of resolution. (b) Typical dictionary power spectrum of element-space for 30-dB out-of-band source power.

yields many false peaks, some of which have magnitudes comparable to the in-band peaks. This makes us unable to identify the in-band DOAs while rejecting the false peaks. MSE plots are not shown since the MSEs are all zero whenever the correct number of in-band DOAs are resolved for both CBS and element-space. The running time per Monte Carlo run is 0.61 seconds for CBS and 4.14 seconds for element-space. So, CBS offers a complexity reduction of 6.8.

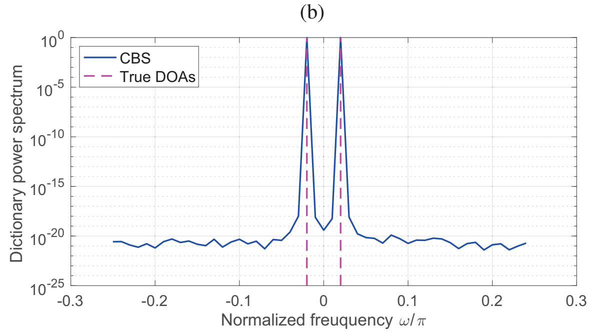
Next, we give an example to show how to implement the idea in Sec. III-D for parameter selection of β. Problem (23) for CBS is considered. The same ULA, filter $H(z)$, decimation ratio, and dictionary grid are used. There are two in-band sources with DOAs $\theta = -1.146°$, $1.146°$ and one out-of-band source with DOA $\theta = 39.94°$, all with power 1. Here all the DOAs are also on the dictionary grid. Noise variance is $\sigma_e^2 = 1$, the reduced dimension for the ℓ_1-SVD method in (23) is $k_0 = 2$, and the peak threshold for dictionary power spectra is $\epsilon = 0.1$. Fig. 2(a) shows the probability of resolving the correct number of in-band DOAs as a function of β. We use $K = 100$ snapshots and 500 Monte Carlo runs to get the plot. To understand the behavior of the plot, we show in Fig. 2(b)-(d) typical dictionary power spectra for $\beta = \mu + r\sigma$ for $r = -3, 20, 1228$, where μ and σ are the mean and standard deviation of $\|\tilde{\mathbf{E}}\|_F^2$. The selection $\beta = \mu - 3\sigma = 22.56$ is too small and may yield some false peaks. But interestingly, these false peaks have magnitudes smaller than ϵ, so the probability of resolution can still be 1. The selection $\beta = \mu + 20\sigma = 171.6$ is proper and yields



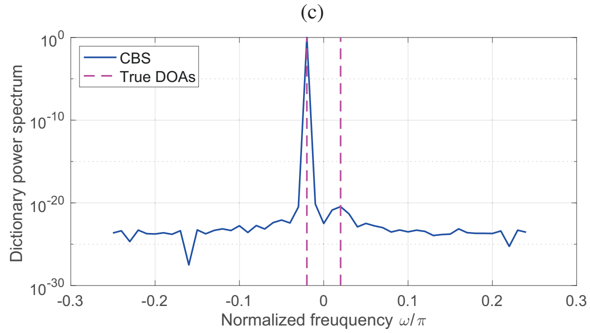
(a)



(b)



(c)



(d)

Fig. 2. Performance of CBS dictionaries for various β. (a) Probability of resolution. (b) Typical dictionary power spectrum when $\beta = \mu - 3\sigma = 22.56$. (c) Typical dictionary power spectrum when $\beta = \mu + 20\sigma = 171.6$. (d) Typical dictionary power spectrum when $\beta = \mu + 1228\sigma = 8000$.

exactly the two true peaks, so the probability of resolution is 1. The selection $\beta = \mu + 1228\sigma = 8000$ is too large and may reveal only one of the true DOAs, so the probability of resolution is lowered to 0.34. In the examples presented, we have noticed that $20 \leq r \leq 200$ is a good choice, although we do not have general guidelines at this time.

V. Conclusion

We showed how to apply the convolutional beamspace (CBS) method in the context of sparse signal representation based on dictionaries. Due to dimension reduction and effective filtering of out-of-band sources, CBS achieves much lower computational complexity and higher probability of resolution. We also addressed how to select the regularization parameter involved in the method.

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