# On the Zeros of Ramanujan Filters

Pranav Kulkarni, *Student Member, IEEE*, and P. P. Vaidyanathan, *Life Fellow, IEEE*

***Abstract*—Ramanujan filter banks have been used for identifying periodicity structure in streaming data. This letter studies the locations of zeros of Ramanujan filters. All the zeros of Ramanujan filters are shown to lie on or inside the unit circle in the $z$-plane. A convenient factorization appears as a corollary of this result, which is useful to identify common factors between different Ramanujan filters in a filter bank. For certain families of Ramanujan filters, further structure is identified in the locations of zeros of those filters. It is shown that increasing the number of periods of Ramanujan sums in the filter definition only increases zeros on the unit circle in $z$-plane. A potential application of these results is that by identifying common factors between Ramanujan filters, one can obtain efficient implementations of Ramanujan filter banks (RFB) as demonstrated here.**

***Index Terms*—Ramanujan sums, Ramanujan filter banks, period estimation, cyclotomic polynomials.**

## I. Introduction

Several authors have shown that Ramanujan sums can be used for signal processing applications [1]–[13]. A comprehensive analysis of Ramanujan sums in the context of signal processing for detecting periods in a discrete-time signal was developed in [3] and [4]. Dictionary approaches were also developed for detecting periods and a generalized framework of nested periodic dictionaries is presented in [5]. Ramanujan periodicity transform (RPT) is shown to be useful in applications such as robust detection of brain stimuli for brain computer interfaces [6]–[8] and removal of interference from ECG signals [9]–[11]. An overview of recent developments can be found in [14].

In order to detect periodicity in streaming data, where the periodicity structure in the signal can change over time, Ramanujan filter banks (RFB) were first proposed in [12] and further developed in [13]. Ramanujan filters have Ramanujan sums as their filter coefficients. RFB can be regarded as an analysis filter bank wherein the output is the filtered version of the input signal, filtered with FIR filters that correspond to different periods. In this letter we study the locations of zeros of Ramanujan filters. A very interesting structure is identified in the locations of zeros. A factorization formula is derived which helps identify the common factors between Ramanujan filters. An important application of these results is that one can obtain efficient implementations of Ramanujan filter banks (RFB) by sharing the common factors of different filters, as we shall demonstrate.

### A. Scope and Outline

In Section II we prove a lemma that characterizes the locations of zeros of Ramanujan filters in terms of critical points of cyclotomic polynomials by using Lucas's theorem. As a corollary, a factorization formula is obtained for the filters. In Section III we show that the plots of zeros are indeed in accordance with the derived result, and identify further structure in the locations of zeros for some specific families of Ramanujan filters. Zeros of Ramanujan filters in which multiple periods are used as filter coefficients are also characterized. In the Section IV, we illustrate a possible application of derived results in designing efficient structures for Ramanujan filter banks. Section V concludes the paper.

### B. Notation

Following notations are used throughout the paper:

1) The $W_q = e^{-j2\pi/q}$ is a $q^{th}$ complex root of unity.
2) The notation $q_i|q$ means that $q_i$ is a divisor of $q$.
3) The notation $(k, q)$ represents the gcd of the integers $k$ and $q$. So $(k, q) = 1$ means that $k$ and $q$ are coprime, that is, they have no common factor other than unity.
4) $\phi(q)$ is the Eulers totient function [15]. It is the number of integers $k$ in $1 \leq k \leq q$ satisfying $(k, q) = 1$.

### C. Preliminaries

Ramanujan sums are defined as follows [16]. For every integer $q > 0$, the $q^{th}$ Ramanujan sum is:

$$c_q(n) = \sum_{\substack{k=1\\(k,q)=1}}^{q} e^{j2\pi kn/q} = \sum_{\substack{k=1\\(k,q)=1}}^{q} W_q^{-kn} = \sum_{\substack{k=1\\(k,q)=1}}^{q} W_q^{kn} \quad (1)$$

It is well known [3] that $c_q(n)$ is periodic with period $q$. Throughout the paper, the $q$-th Ramanujan filter is defined by $C_q(z) = \sum_{n=0}^{q-1} c_q(n)z^{-n}$. In Section III-D will also define extended filters based on $l$ successive periods of $c_q(n)$, namely $C_q^{(l)}(z) = \sum_{n=0}^{ql-1} c_q(n)z^{-n}$ [12], [13].

## II. Locations of Zeros of $C_q(z)$

In this section we study the locations of zeros of $C_q(z)$. In order to make $C_q(z)$ a polynomial in $z$ with positive powers, we multiply it by $z^{q-1}$ to get $\widehat{C}_q(z)$:

$$\widehat{C}_q(z) = z^{q-1}C_q(z) = \sum_{n=0}^{q-1} c_q(n)z^{q-1-n} \quad (2)$$

Manuscript received February 26, 2020; revised April 8, 2020; accepted April 13, 2020. Date of publication April 27, 2020; date of current version May 21, 2020. This work was supported in part by the ONR under Grant N00014-18-1-2390, in part by the NSF under Grant CCF-1712633, and in part by the California Institute of Technology. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Jens Ahrens.
*(Corresponding author: Pranav Kulkarni.)*

The authors are with the Department of Electrical Engineering, California Institute of Technology, Pasadena, CA 91125 USA (e-mail: pkulkarn@caltech.edu; ppvnath@systems.caltech.edu).
Digital Object Identifier 10.1109/LSP.2020.2990610

Note that zeros of $\widehat{C}_q(z)$ and $C_q(z)$ which are not at the origin are the same. Also, $\widehat{C}_q(z)$ has no zero at $z=0$ for any positive integer $q$. Hence for simplicity, we study the location of zeros of $\widehat{C}_q(z)$ and discard the zeros at origin to get the zeros of $C_q(z)$.

As $c_q(0) \neq 0$ for any $q$, $\widehat{C}_q(z)$ is a degree $q-1$ polynomial in $z$. Hence it has $q-1$ (possibly complex) zeros. In this section, we will prove the following Lemma. The terms "cyclotomic polynomial," and "critical points" are explained after the lemma.

*Lemma 1 (Zeros of Ramanujan Filters):* Let $C_q(z) = \sum_{n=0}^{q-1} c_q(n) z^{-n}$ and let $\widehat{C}_q(z) = z^{q-1} C_q(z)$. Out of the $q-1$ zeros of $\widehat{C}_q(z)$

1) $\phi(q)-1$ zeros are located strictly inside the unit circle in $z$-plane and correspond to the critical points of the $q^{th}$ cyclotomic polynomial $F_q(z)$ defined in (3).
2) The remaining $q-\phi(q)$ zeros are located on the unit circle in $z$-plane at $W_q^k, (k,q) \neq 1$.

The zeros of $C_q(z)$ are the non-zero zeros of $\widehat{C}_q(z)$. ◇

The $q$-th cyclotomic polynomial $F_q(z)$ mentioned above is defined as

$$F_q(z) \triangleq \prod_{\substack{k=1 \\ (k,q)=1}}^{q} (z - W_q^k) \tag{3}$$

The term "critical points" of $F_q(z)$ refers to the zeros of $dF_q(z)/dz$.

Note that $F_q(z)$ is a degree $\phi(q)$ polynomial. Its zeros are located on the unit circle in $z$-plane at $z = W_q^k$, where $(k,q) = 1, 0 \leq k \leq q-1$. Also note

$$\dot{F}_q(z) \triangleq \frac{dF_q(z)}{dz} = F_q(z) \cdot \left( \sum_{\substack{k=1 \\ (k,q)=1}}^{q} \frac{1}{z - W_q^k} \right) \tag{4}$$

We will appeal to the following theorem by Lucas in the proof.

*Theorem 1 (Lucas, theorem (6,1) of [17]):* All the critical points of any non-constant polynomial $f(z)$ lie in the convex hull $H$ of zeros of $f(z)$. If the zeros of $f(z)$ are not collinear, no critical point of $f(z)$ lies on the boundary of $H$ unless it is a multiple zero of $f(z)$. ◇

*Proof of Lemma 1:* We can rewrite $C_q(z)$ as follows

$$C_q(z) = \sum_{n=0}^{q-1} c_q(n) z^{-n} = \sum_{n=0}^{q-1} \sum_{\substack{k=1 \\ (k,q)=1}}^{q} W_q^{kn} z^{-n} \tag{5}$$

Therefore, $\widehat{C}_q(z)$

$$= z^{q-1} \sum_{\substack{k=1 \\ (k,q)=1}}^{q} \sum_{n=0}^{q-1} (W_q^k z^{-1})^n$$

$$= z^{q-1} \sum_{\substack{k=1 \\ (k,q)=1}}^{q} \frac{1 - (W_q^k z^{-1})^q}{1 - W_q^k z^{-1}}$$

$$= z^{q-1}(1 - z^{-q}) \sum_{\substack{k=1 \\ (k,q)=1}}^{q} \frac{1}{1 - W_q^k z^{-1}}$$

$$= (z^q - 1) \sum_{\substack{k=1 \\ (k,q)=1}}^{q} \frac{1}{z - W_q^k}$$

$$= \frac{z^q - 1}{F_q(z)} \cdot \dot{F}_q(z) \tag{6}$$

A related expression for $\dot{F}_q(z)/F_q(z)$ was obtained in a different context for one sided $z$-transform of Ramanujan sums in [18]:

$$\sum_{n=0}^{\infty} c_q(n) z^{-n} = \frac{z dF_q(z)/dz}{F_q(z)} \tag{7}$$

The infinite sum in (7) converges to RHS only when $|z| > 1$, whereas no such assumption is required in the derivation (6).

Now from (6), we can see that either of the two factors in (6) can contribute to zeros of $\widehat{C}_q(z)$. For the first factor, the zeros of $F_q(z)$ are located at $z = W_q^k$, where $(k,q) = 1, 0 \leq k \leq q-1$ and the zeros of the numerator $(z^q - 1)$ are at $z = W_q^k, 0 \leq k \leq q-1$. Hence, after cancellation we are left with $q - \phi(q)$ zeros for $\widehat{C}_q(z)$ which are located at $W_q^k, (k,q) \neq 1$.

The remaining $\phi(q)-1$ zeros of $\widehat{C}_q(z)$ are contributed by the other factor $\dot{F}_q(z)$. Since $F_q(z)$ is a non-constant polynomial, by Lucas's theorem we have that the critical points of $F_q(z)$, namely the zeros of $\dot{F}_q(z)$, lie in the convex hull of the zeros of $F_q(z)$. Since the zeros of $F_q(z)$ are on the unit circle in $z$-plane, the convex hull of the zeros is strictly inside the unit circle, except at the zeros itself. Now note that $\dot{F}_q(z)$ cannot have a zero where $F_q(z)$ is zero, since none of the zeros of $F_q(z)$ are repeated. Hence all the zeros of $\widehat{C}_q(z)$ contributed by the term $\dot{F}_q(z)$ lie strictly inside the unit circle in $z$-plane. This completes the proof of Lemma 1. ■

Now we obtain an expression for $C_q(z)$ which enables efficient implementation of Ramanujan filter banks as explained in Section IV later. We have

$$C_q(z) = \frac{\widehat{C}_q(z)}{z^{q-1}} = \frac{z^q - 1}{z^{q-\phi(q)} F_q(z)} \cdot \frac{\dot{F}_q(z)}{z^{\phi(q)-1}} \tag{8}$$

Applying an identity for cyclotomic polynomials [19], [20]:

$$z^q - 1 = \prod_{q_k | q} F_{q_k}(z) \tag{9}$$

gives us

$$C_q(z) = \left( \prod_{\substack{q_k | q \\ q_k < q}} \widehat{F}_{q_k}(z) \right) \cdot \frac{\dot{F}_q(z)}{z^{\phi(q)-1}} \tag{10}$$

where

$$\widehat{F}_q(z) = z^{-\phi(q)} F_q(z) \tag{11}$$

is a causal version of the cyclotomic polynomial.

## III. Identifying Further Structure in the Locations of Zeros

Fig. 1 shows plots of zeros of $C_q(z)$ for some selected values of $q$. In each of these plots, the zeros are indeed in accordance with Lemma 1. We can see that the locations of zeros seem to exhibit a lot more structure than what is stated in Lemma 1. For

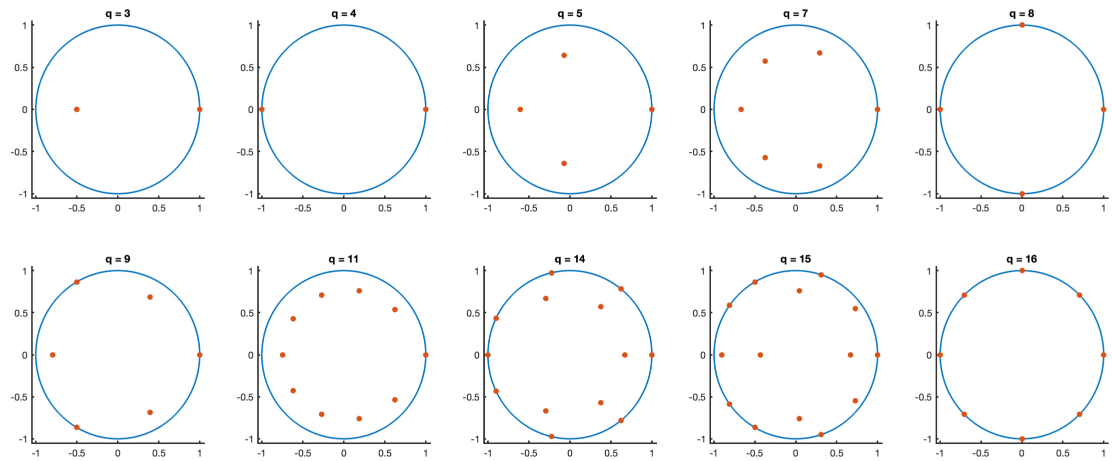


Fig. 1. Zeros of $C_q(z)$ for some selected values of $q$ plotted on the $z$-plane.

example, notice that $z = 1$ is always a zero of $C_q(z)$ for any $q > 1$. This follows from the fact that $\sum_{n=0}^{q-1} c_q(n) = 0$ for any $q$. In the following subsections, we consider special cases of $q$ and further derive properties for zeros of corresponding families of Ramanujan filters. We believe that these are of sufficient academic interest to merit inclusion here. We also obtain zeros for generalized filters with $l$ periods of Ramanujan sums as filter coefficients instead of just one.

### A. Case when $q$ is a power of two

For $q = 2^m$ where $m$ is a natural number, we have [3]

$$c_q(n) = \begin{cases} 0 & \text{if } 2^{m-1} \text{ does not divide } n \\ -2^{m-1} & \text{if } 2^{m-1} \text{ divides } n \text{ but} \\ & \quad 2^m \text{ does not divide } n \\ 2^{m-1} & \text{if } 2^m \text{ divides } n \end{cases}$$

Hence we have

$$\begin{aligned} \widehat{C}_q(z) &= 2^{m-1} z^{(2^m-1)} - 2^{m-1} z^{(2^{m-1}-1)} \\ &= 2^{m-1} z^{(2^{m-1}-1)} (z^{(2^{m-1})} - 1) \end{aligned} \quad (12)$$

Discarding the zeros at $z = 0$, the zeros of $C_q(z)$ are the $2^{m-1}$-th roots of unity, and lie on the unit circle in $z$-plane. No zeros lie inside the unit circle. We can indeed verify this result from Fig. 1. The zeros of $C_4(z)$ are the square roots of unity. Similarly the zeros of $C_q(z)$ for $q = 8$ and $q = 16$ are the fourth and the eighth roots of unity respectively.

### B. Case When $q$ is a Prime Number

For $q = p$, a prime, we have [3]

$$c_q(n) = \begin{cases} q-1 & \text{if } n \text{ is multiple of } q \\ -1 & \text{otherwise.} \end{cases} \quad (13)$$

So we have

$$\begin{aligned} \widehat{C}_q(z) &= (q-1)z^{q-1} - z^{q-2} - z^{q-3} - \cdots - 1 \\ &= (z-1)\left[(q-1)z^{q-2} + (q-2)z^{q-3} + \cdots + 2z + 1\right] \end{aligned}$$

Hence we have one zero located at $z = 1$ and other $q - 2$ zeros are the zeros of the special integer coefficient polynomial:

$$g_q(z) = (q-1)z^{q-2} + (q-2)z^{q-3} + \cdots + 2z + 1 \quad (14)$$

Since $q$ is prime, we know from Lemma 1 that the zeros of $g_q(z)$ are strictly inside the unit circle. We can verify this result for prime values of $q = 3, 5, 7, 11$ from the Fig. 1. Except at $z = 1$, all the zeros of $C_q(z)$ lie strictly inside the unit circle.

### C. Case When $q$ is a Power of a Prime

For $q = p^m$ where $p$ is a prime, we have [3]

$$c_q(n) = \begin{cases} 0 & \text{if } p^{m-1} \text{ does not divide } n \\ -p^{m-1} & \text{if } p^{m-1} \text{ divides } n \text{ but} \\ & \quad p^m \text{ does not divide } n \\ (p-1)p^{m-1} & \text{if } p^m \text{ divides } n \end{cases} \quad (15)$$

Hence we have

$$\widehat{C}_q(z) = (p-1)p^{m-1} z^{(p^m-1)} + \sum_{k=1}^{p-1} -p^{m-1} z^{(p^m-1-kp^{m-1})}$$

$$= p^{m-1} z^{(p^{m-1}-1)} \left[ (p-1) z^{((p-1)p^{m-1})} - \sum_{k=1}^{p-1} z^{(p-k-1)p^{m-1}} \right]$$

Simplifying, we get $\widehat{C}_q(z) = p^{m-1} z^{(p^{m-1}-1)} \widehat{C}_p(z^{p^{m-1}})$ (16)

Hence, discarding the zeros at $z = 0$, we have that the zeros of $\widehat{C}_q(z)$ are the $p^{m-1}$-th roots of zeros of $\widehat{C}_p(z)$.

To visualize the relation (16), compare the locations of zeros for $C_3(z)$ and $C_9(z)$ from Fig. 1. Here we have $p = 3, m = 2$ and $q = 3^2 = 9$. The zero of $C_3(z)$ at $z = 1$ gives rise to three zeros of $C_9(z)$ on the unit circle. These three zeros are the $p^{m-1}$-th i.e. $3^{rd}$ roots of $z = 1$. Similarly, the zero of $C_3(z)$ at $z = -1/2$ gives rise to three zeros of $C_9(z)$ that lie inside the unit circle, and correspond to cube roots of $z = -1/2$.

### D. Case When Multiple Periods of Ramanujan Sums are Used in the Filter Definition

When the filter impulse response has $l$ periods of $c_q(n)$ as filter coefficients, we have

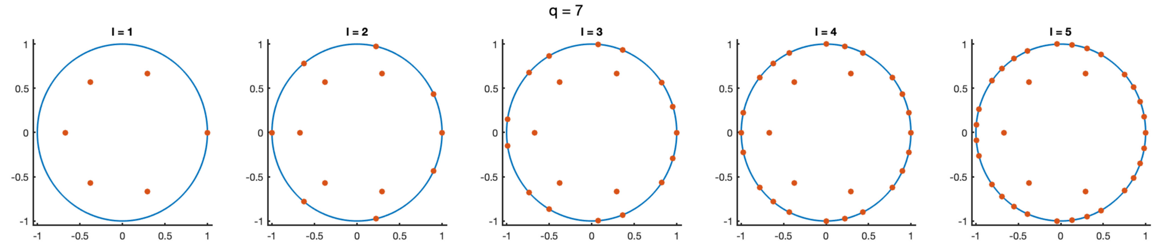


Fig. 2. Zeros of $C_7^{(l)}$ for $1 \leq l \leq 5$.

$$C_q^{(l)}(z) = \sum_{n=0}^{ql-1} c_q(n) z^{-n}$$

$$= \sum_{n=0}^{q-1} c_q(n) z^{-n} \left[1 + z^{-q} + \cdots z^{-(l-1)q}\right] \quad (17)$$

$$= \left(\sum_{n=0}^{q-1} c_q(n) z^{-n}\right)\left(\sum_{k=0}^{l-1} z^{-qk}\right) = C_q(z)\left(\frac{1-z^{-ql}}{1-z^{-q}}\right) \quad (18)$$

From this we can see that the $ql-1$ zeros of $C_q^{(l)}(z)$ are of two categories: (a) the zeros of $C_q(z)$, and (b) zeros that are $ql$-th roots of unity which are not also $q$-th roots of unity. There are $q(l-1)$ such zeros of the second category. Fig. 2 shows the locations of zeros for $C_7^{(l)}$ for $1 \leq l \leq 5$.

### E. A Special Case of $q$ as a Product of Two Numbers

One can ask whether we can characterize zeros of $C_q(z)$ where $q = p_1 p_2$, where $p_1$ and $p_2$ are some integers. We were not able to characterize this for general integers $p_1$ and $p_2$. However, in this section we consider a very specific case of this, where $p_1 = 2$ and $p_2$ is odd. Here we use the multiplicative property of the Ramanujan sums [3]:

$$c_{p_1 p_2}(n) = c_{p_1}(n) c_{p_2}(n) \text{ whenever } p_1 \text{ and } p_2 \text{ are coprime.} \quad (19)$$

When $p_1 = 2$ and $p_2$ is odd, the coprime condition is satisfied. Now note that $c_2(n) = \{1, -1\}$ in its first period. Hence for $q = 2p_2$ we have

$$C_q(z) = \sum_{n=0}^{q-1} c_2(n) c_{p_2}(n) z^{-n} = \sum_{n=0}^{q-1} (-1)^n c_{p_2}(n) z^{-n} \quad (20)$$

$$= \sum_{n=0}^{q-1} c_{p_2}(n)(-z)^{-n} = C_{p_2}^{(2)}(-z) \quad (21)$$

Hence the zeros of $C_q(z)$ for $q = 2p_2$, where $p_2$ is odd, are the negatives of the zeros of $C_{p_2}^{(2)}(z)$. As an example the zeros of $C_{14}(z)$ ($q = 14$ from Fig. 1) are the negatives of the zeros of $C_7^{(2)}(z)$ ($l = 2$ from Fig. 2).

## IV. Efficient Structure for Ramanujan Filter Banks

In this section, we show how the factorization (10) leads to a possible way of implementing the Ramanujan filter banks more efficiently. Firstly, note that cyclotomic polynomials have integer coefficients. The coefficients of $F_q(z)$ remain small even when $q$ is large. In particular, the first 104 cyclotomic polynomials have no coefficients other than 1, 0, or -1 [19], [20]. Hence the first 104 cyclotomic polynomials can be implemented without any multipliers!

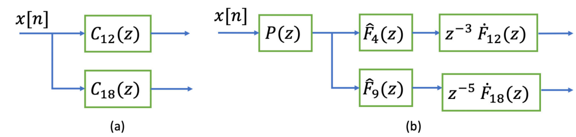


Fig. 3. (a) Standard implementation of two filters from RFB, (b) Equivalent efficient implementation, by extracting a common factor $P(z)$.

Now we present an example of an efficient implementation of a 2-filter RFB consisting of $C_{12}(z)$ and $C_{18}(z)$. We pull out a common filter factor from the first terms of the expression (10) for $q = 12$ and $q = 18$. Note that

$$C_{12}(z) = P(z)\widehat{F}_4(z) \cdot \left(z^{-3}\dot{F}_{12}(z)\right), \text{ and}$$

$$C_{18}(z) = P(z)\widehat{F}_9(z) \cdot \left(z^{-5}\dot{F}_{18}(z)\right) \quad (22)$$

where, $P(z)$ is the common factor given by

$$P(z) = \widehat{F}_1(z)\widehat{F}_2(z)\widehat{F}_3(z)\widehat{F}_6(z) \quad (23)$$

Hence instead of standard implementation as in Fig. 3(a), we can implement the two filters as in Fig. 3(b). Since the filter $\widehat{F}_q(z)$ has order of $\phi(q)$, $P(z)$ is a $6^{th}$ order filter (since $\sum_{q_k|q} \phi(q_k) = q$ [15]). Therefore the implementation as in Fig. 3(b) saves computations corresponding to a $6^{th}$ order filter, by reusing the common filter factor $P(z)$.

This saves a lot of repetitive convolutions of signals with impulse responses. This particular example was chosen as it well-illustrates the point of filters having common factors, owing to many common divisors of 12 and 18. In practice, an RFB usually has all filters from $C_1(z)$ to $C_N(z)$ for some integer $N$. When the filter bank has many filters, there are multiple ways in which common factors can be shared, and some are more efficient than others. An interesting problem for future would be to identify the most efficient way to exploit such common factors.

## V. Concluding Remarks

In this letter we have proved that all the zeros of Ramanujan filters lie on or inside the unit circle in $z$-plane. A general proof was based on Lucas's theorem. We considered different special cases of $q$, such as primes and powers of primes, and discovered interrelations between them. We also characterized the zeros of Ramanujan filters having $l$ periods of $c_q(n)$ instead of just one period. It is shown with an illustrative example that the factorization of filter transfer function in terms of cyclotomic polynomials and their derivatives opens up a possibility of efficient implementation of RFB.

## REFERENCES

[1] M. Planat, "Ramanujan sums for signal processing of low frequency noise," in *Proc. IEEE Int. Freq. Control Symp. PDA Exhib. (Cat. No.02CH37234)*, 2002, pp. 715–720.
[2] S. Pei and K. Lu, "Intrinsic integer-periodic functions for discrete periodicity detection," *IEEE Signal Process. Lett.*, vol. 22, no. 8, pp. 1108–1112, Aug. 2015.
[3] P. P. Vaidyanathan, "Ramanujan sums in the context of signal processing–Part I: Fundamentals," *IEEE Trans. Signal Process.*, vol. 62, no. 16, pp. 4145–4157, Aug. 2014.
[4] P. P. Vaidyanathan, "Ramanujan sums in the context of signal processing Part II: FIR representations and applications," *IEEE Trans. Signal Process.*, vol. 62, no. 16, pp. 4158–4172, Aug. 2014.
[5] S. V. Tenneti and P. P. Vaidyanathan, "Nested periodic matrices and dictionaries: New signal representations for period estimation," *IEEE Trans. Signal Process.*, vol. 63, no. 14, pp. 3736–3750, Jul. 2015.
[6] P. Saidi, G. Atia, and A. Vosoughi, "Detection of visual evoked potentials using Ramanujan periodicity transform for real time brain computer interfaces," in *Proc. IEEE Int. Conf. Acoust., Speech Signal Process.*, 2017, pp. 959–963.
[7] P. Saidi, G. Atia, and A. Vosoughi, "On robust detection of brain stimuli with Ramanujan periodicity transforms," in *Proc. 51st Asilomar Conf. Signals, Syst., Comput.*, 2017, pp. 729–733.
[8] P. Saidi, A. Vosoughi, and G. Atia, "Detection of brain stimuli using Ramanujan periodicity transforms," *J. Neural Eng.*, vol. 16, Apr. 2019, Art. no. 036021.
[9] B. Shah Shaik, V. K. Chakka, S. Goli, and A. S. Reddy, "Removal of narrowband interference (PLI in ECG signal) using Ramanujan periodic transform (RPT)," in *Proc. Int. Conf. Signal Process. Commun.*, 2016, pp. 233–237.
[10] B. S. Shaik and V. K. Chakka, "Joint reduction of baseline wander, PLI and its harmonics in ECG signal using Ramanujan periodic transform," in *Proc. IEEE Annual India Conf.*, 2016, pp. 1–5.
[11] B. S. Shaik, V. K. Chakka, and S. Goli, "Ramanujan and DFT mixed basis representation for removal of PLI in ECG signal," in *Proc. 4th Int. Conf. Signal Process. Integrated Netw.*, 2017, pp. 509–512.
[12] S. Tenneti and P. P. Vaidyanathan, "Ramanujan filter banks for estimation and tracking of periodicities," in *Proc. IEEE Int. Conf. Acoust., Speech Signal Process.*, 2015, pp. 3851–3855.
[13] P. P. Vaidyanathan and S. Tenneti, "Properties of Ramanujan filter banks," in *Proc. 23rd Euro. Signal Process. Conf.*, 2015, pp. 2816–2820.
[14] P. P. Vaidyanathan and S. Tenneti, "Srinivasa Ramanujan and signal-processing problems," *Philos. Trans. Roy. Soc. A*, vol. 378, no. 2163, Dec. 2019.
[15] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 4th ed. New York, NY, USA: Oxford, 1975.
[16] S. Ramanujan, "On certain trigonometrical sums and their applications in the theory of numbers," *Trans. Cambridge Phil. Soc.*, vol. 22, no. 13, pp. 259–276, 1918.
[17] M. Marden, *Geometry of Polynomials*, American Mathematical Society, 1949.
[18] S. Samadi, M. O. Ahmad, and M. N. S. Swamy, "Ramanujan sums and discrete Fourier transforms," *IEEE Signal Process. Lett.*, vol. 12, no. 4, pp. 293–296, Apr. 2005.
[19] T. Nagell, *Introduction to Number Theory*. New York, NY, USA: Chelsea, 1964.
[20] J. H. McClellan and C. M. Rader, *Number Theory in Digital Signal Processing*. Englewood Cliffs, NJ, USA: Prentice-Hall, 1979.