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# VARIATIONAL PRINCIPLES OF MICROMAGNETICS REVISITED*

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**Abstract.** We revisit the basic variational formulation of the minimization problem associated with the micromagnetic energy, with an emphasis on the treatment of the stray field contribution to the energy, which is intrinsically nonlocal. Under minimal assumptions, we establish three distinct variational principles for the stray field energy: a minimax principle involving magnetic scalar potential and two minimization principles involving magnetic vector potential. We then apply our formulations to the dimension reduction problem for thin ferromagnetic shells of arbitrary shapes.

**Key words.** micromagnetics, Maxwell's equations, stray field, minimizers, Γ-convergence

**AMS subject classifications.** 35Q61, 49J40, 49S05, 82D40

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## 1. Introduction.

Ferromagnetism is a striking and subtle phenomenon. Observable on the macroscopic scale, ferromagnetism has its origins from the two quintessentially quantum mechanical properties of matter, namely, the electron spin and the Pauli exclusion principle [1]. The quantum mechanical origin of ferromagnetism accounts for the existence of a multitude of intriguing spin textures, from macroscopic down to single nanometer scales [4, 22, 26, 32]. The small size of the magnetization patterns, along with the modest energy required to manipulate them, has produced and is continuing to lead to far-reaching applications in information technology [3, 6, 8, 47].

There is a well-established and extremely successful continuum theory of micromagnetism, the *micromagnetic variational principle*, that describes the equilibrium and dynamic magnetization configurations [10, 27, 32, 38, 39, 43]. In this theory, magnetization is described by a spatially varying vector field $\mathbf{M}$, and stable magnetization configurations correspond to global and local minimizers of the micromagnetic energy—a nonconvex, nonlocal functional involving multiple length scales. The mi-

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cromagnetic energy associated with the magnetization state of a ferromagnetic sample occupying three-dimensional bounded domain $\Omega$ ($\Omega \subset \mathbb{R}^3$) is [7, 32, 38]

$$E(\mathbf{M}) = \frac{A}{M_s^2}\int_\Omega |\nabla \mathbf{M}|^2\, d^3r + K\int_\Omega \Phi\left(\frac{\mathbf{M}}{M_s}\right) d^3r - \frac{\mu_0}{2}\int_\Omega \mathbf{H}_d\cdot\mathbf{M}\, d^3r - \mu_0\int_\Omega \mathbf{H}_a\cdot\mathbf{M}\, d^3r, \tag{1.1}$$

where $\mathbf{M} = (M_1, M_2, M_3)$ is the magnetization vector that satisfies $|\mathbf{M}| = M_s$ in $\Omega$ and $\mathbf{M} = \mathbf{0}$ in $\mathbb{R}^3\setminus\Omega$ (i.e., outside the domain $\Omega$), the positive constants $M_s$, $A$, and $K$ are the saturation magnetization and exchange and anisotropy constants, respectively, $\mathbf{H}_a$ is the applied magnetic field, and $\mu_0$ is the permeability of vacuum. Here we use the standard notation $|\nabla\mathbf{M}|^2 = |\nabla M_1|^2 + |\nabla M_2|^2 + |\nabla M_3|^2$ for the Euclidean norm of gradients of vectorial quantities. All physical quantities are assumed to be in SI units. The demagnetizing field $\mathbf{H}_d$ is determined via the magnetic induction $\mathbf{B} = \mathbf{B}_a + \mathbf{B}_d$, where $\mathbf{B}_a = \mu_0\mathbf{H}_a$ is the induction in the absence of the ferromagnet due to permanent external field sources, and

$$\mathbf{B}_d = \mu_0(\mathbf{H}_d + \mathbf{M}). \tag{1.2}$$

The pair $(\mathbf{H}_d, \mathbf{B}_d)$ solves the following system obtained from the time-independent Maxwell's equations:

$$\operatorname{div}\mathbf{B}_d = 0, \qquad \operatorname{curl}\mathbf{H}_d = \mathbf{0}, \tag{1.3}$$

where we noted that by definition $\operatorname{div}\mathbf{B}_a = 0$ in $\mathbb{R}^3$. In (1.1), the terms in the order of appearance are the exchange, $E_{ex}$, magnetocrystalline anisotropy, $E_a$, stray field, $E_s$, and Zeeman, $E_Z$, energies, respectively.

There exist several well-known representations of the stray field energy employed in the analysis of the micromagnetic energy [9]. Using (1.3), one can introduce the magnetic scalar potential $U_d : \mathbb{R}^3 \to \mathbb{R}$ associated with the demagnetizing field, such that $\mathbf{H}_d = -\nabla U_d$, and $U_d$ satisfies the following equation in the sense of distributions:

$$\Delta U_d = \operatorname{div}\mathbf{M}, \tag{1.4}$$

and vanishes at infinity. The stray field energy can be rewritten in terms of $U_d$ as [9]

$$E_s(\mathbf{M}) = \frac{\mu_0}{2}\int_\Omega \mathbf{M}\cdot\nabla U_d\, d^3r = \frac{\mu_0}{2}\int_{\mathbb{R}^3} |\nabla U_d|^2\, d^3r. \tag{1.5}$$

Using the fundamental solution of the Laplace equation in $\mathbb{R}^3$, one can also rewrite the stray field energy in the following way:

$$E_s(\mathbf{M}) = \frac{\mu_0}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\operatorname{div}\mathbf{M}(\mathbf{r})\operatorname{div}\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3r\, d^3r', \tag{1.6}$$

reflecting its nonlocal and singular nature. Note that since $\mathbf{M}$ has a jump at the boundary of domain $\Omega$, its divergence $\operatorname{div}\mathbf{M}$ has a singularity and, therefore, must be understood in a formal sense through its Fourier symbol.

Another way to represent the stray field energy is to employ the magnetic vector potential $\mathbf{A}$ satisfying $\mathbf{B} = \operatorname{curl}\mathbf{A} = \operatorname{curl}(\mathbf{A}_a + \mathbf{A}_d)$, where $\mathbf{A}_a$ and $\mathbf{A}_d$ are the contributions associated with $\mathbf{B}_a$ and $\mathbf{B}_d$, respectively. The magnetic vector potential is unobservable and not uniquely defined due to gauge invariance. However, this

potential is contained in the momentum operator for a charged particle and, therefore, plays a crucial role in the description of superconductivity and the Ehrenberg–Siday–Aharonov–Bohm effect underlying the method of electron holography [40]. In the Coulomb gauge one sets $\operatorname{div}\mathbf{A}_{\mathrm{a}} = \operatorname{div}\mathbf{A}_{\mathrm{d}} = 0$, leading to the following equation for $\mathbf{A}_{\mathrm{d}}$ understood in the sense of distributions [9]:

$$\operatorname{curl}(\operatorname{curl}\mathbf{A}_{\mathrm{d}}) = -\Delta\mathbf{A}_{\mathrm{d}} = \mu_0 \operatorname{curl}\mathbf{M}, \tag{1.7}$$

where we used the identity $\nabla(\operatorname{div}\mathbf{A}) - \operatorname{curl}(\operatorname{curl}\mathbf{A}) = \Delta\mathbf{A}$. In a similar way to the use of magnetostatic potential $U_{\mathrm{d}}$, we can rewrite the demagnetizing field $\mathbf{H}_{\mathrm{d}} = \mu_0^{-1}\operatorname{curl}\mathbf{A}_{\mathrm{d}} - \mathbf{M}$ to represent the stray field energy as

$$E_{\mathrm{s}}(\mathbf{M}) = \frac{1}{2}\int_{\Omega}\left(\mu_0|\mathbf{M}|^2 - \mathbf{M}\cdot\operatorname{curl}\mathbf{A}_{\mathrm{d}}\right)d^3r = \frac{1}{2\mu_0}\int_{\mathbb{R}^3}|\operatorname{curl}\mathbf{A}_{\mathrm{d}} - \mu_0\mathbf{M}|^2\,d^3r. \tag{1.8}$$

Again, using the fundamental solution of the Laplace equation in $\mathbb{R}^3$ we obtain another representation of the stray field energy:

$$E_{\mathrm{s}}(\mathbf{M}) = \frac{1}{2}\mu_0 M_{\mathrm{s}}^2|\Omega| - \frac{\mu_0}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\operatorname{curl}\mathbf{M}(\mathbf{r})\cdot\operatorname{curl}\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^3r\,d^3r', \tag{1.9}$$

where $|\Omega|$ is the volume of $\Omega$. Note that since $\mathbf{M}$ has a jump at the boundary of domain $\Omega$, $\operatorname{curl}\mathbf{M}$ has a singularity and, therefore, must again be understood in a formal sense through its Fourier symbol.

The multiscale complexity of the micromagnetic energy allows for a variety of distinct regimes characterized by different relations between material and geometrical parameters, and makes the micromagnetic theory very rich and challenging [16, 32]. One of the most powerful analytical approaches to study the equilibria of the micromagnetic energy is the investigation of its $\Gamma$-limits in various asymptotic regimes. To achieve this, one needs to obtain asymptotically matching lower and upper bounds for the micromagnetic energy. Typically, the construction of the upper bounds is done using appropriate test functions; the lower bound constructions are more difficult and require a careful analysis of the specific problem under consideration. We point out, however, that in the case of the stray field energy, even constructing the upper bounds might present a significant challenge due to the nonlocal and singular behavior of the demagnetizing field $\mathbf{H}_{\mathrm{d}}$.

In this paper, we revisit the variational formulation associated with the micromagnetic energy, emphasizing the treatment of the stray field energy to obtain efficient upper and lower bounds. To this aim, we formulate three distinct variational principles for local minimizers of the micromagnetic energy. The first variational principle can be stated as a minimax problem for the magnetization $\mathbf{M}$ and the scalar potential $U$. Specifically, for $\mathbf{M}$ fixed, the stray field energy may be expressed as

$$E_{\mathrm{s}}(\mathbf{M}) = \max_{U\in\mathring{H}^1(\mathbb{R}^3)}\mu_0\int_{\mathbb{R}^3}\left(\mathbf{M}\cdot\nabla U - \frac{1}{2}|\nabla U|^2\right)d^3r \tag{1.10}$$

and, therefore, yields convenient lower bounds on the stray field energy via the use of test functions for $U$ (recall that $\mathring{H}^1(\mathbb{R}^3)$ denotes the space of functions whose first derivatives are square integrable; see section 2 for the precise definitions of the function spaces).

The second variational principle is a joint minimization problem for the magnetization $\mathbf{M}$ and the vector potential $\mathbf{A}$ subject to the Coulomb gauge ($\operatorname{div}\mathbf{A} = 0$) with

the stray field energy expressed as

$$E_{\rm s}(\mathbf{M}) = \min_{\substack{\mathbf{A}\in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)\\ \operatorname{div}\mathbf{A}=0}} \frac{1}{2\mu_0}\int_{\mathbb{R}^3} |\operatorname{curl}\mathbf{A} - \mu_0\mathbf{M}|^2\, d^3r \tag{1.11}$$

and is useful in constructing upper bounds for the stray field energy via suitable test functions for $\mathbf{A}$.

Finally, we introduce the third variational principle closely linked to the second one that amounts to a joint minimization for the magnetization $\mathbf{M}$ and the vector potential $\mathbf{A}$ in the absence of the constraint on $\operatorname{div}\mathbf{A}$. It allows us to express the stray field energy in the form

$$E_{\rm s}(\mathbf{M}) = \frac{1}{2}\mu_0 M_{\rm s}^2 V + \min_{\mathbf{A}\in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)} \int_{\mathbb{R}^3} \left(\frac{1}{2\mu_0}|\nabla\mathbf{A}|^2 - \mathbf{M}\cdot\operatorname{curl}\mathbf{A}\right) d^3r. \tag{1.12}$$

This formula gives a novel representation of the magnetostatic energy, which is particularly convenient both for obtaining localized upper bounds for the micromagnetic energy and the numerical implementation of the stray field.

The variational principle in (1.10) leading to (1.5) is well known. In the context of micromagnetics, where one needs to minimize the energy in (1.1) with respect to $\mathbf{M}$ with $\mathbf{H}_{\rm d}$ determined by the unique solution of (1.3), it results in a minimax problem in terms of the pair $(\mathbf{M}, U)$. As such, this minimax principle has not been precisely formulated in the literature, although it has long existed in the micromagnetics folklore (see, e.g., [9, 10, 34]). Here we establish the validity of this variational principle under minimal assumptions that arise naturally in the context of micromagnetics.

Similarly, the minimization principles for the micromagnetic energy, in which the stray field energy is expressed through (1.11) or (1.12) appeared in some form in the engineering literature in the context of finite element discretization of the magnetostatic problems for ferromagnets. Specifically, the energy functional in (1.11) appeared in [5], and the associated problem is an extension of the well-known variational principles for Maxwell's equations [36, 42]. In [9, 12, 15, 50], the minimization principles rely on local constitutive relationships between the magnetic induction and the magnetic field, which in the context of micromagnetics may be obtained by first minimizing the micromagnetic energy written in terms of the pair $(\mathbf{M}, \mathbf{A})$ with respect to $\mathbf{M}$, provided the exchange energy is neglected [34, 45, 46]. However, in the full micromagnetics formulation the exchange energy plays a crucial role and, therefore, the variational formulation must include a joint minimization of $E$ in $(\mathbf{M}, \mathbf{A})$. Note that while in the case of (1.11) the minimization in $\mathbf{A}$ requires an additional constraint in the form of the Coulomb gauge, the minimization in (1.12) is unconstrained and automatically enforces the Coulomb gauge for the minimizers. In fact, if one were to minimize the expression in (1.12) within the class in (1.11), one would simply recover the problem in (1.11), since for $\operatorname{div}\mathbf{A} = 0$ the two energies coincide, as can be easily seen via an integration by parts [23]. On the other hand, the absence of the divergence-free constraint, first noted in [12], makes the formulation in (1.12) clearly more attractive than that in (1.11) and opens up a way for an efficient numerical treatment of minimizers of the micromagnetic energy. In this paper, we put the above variational principles on a rigorous footing under natural assumptions.

Finally, we illustrate the usefulness of our results for analytical studies of micromagnetics by applying the obtained variational principles to the problem of finding the $\Gamma$-limit of the micromagnetic energy in curved thin ferromagnetic shells. These problems are interesting due to intrinsic symmetry-breaking mechanisms coming from

the nonzero curvature of the shell generating surfaces (see [20, 41]; see also the recent review [51]). Some results on this problem have been previously obtained under technical assumptions on the geometry of the domain occupied by the ferromagnet; see [11, 18]. Here we show that using our approach these restrictions can be easily removed, resulting in a leading-order two-dimensional local energy functional in the spirit of Gioia and James [31] formulated on two-dimensional surfaces, in which the stray field energy reduces to the effective shape anisotropy term.

The paper is organized as follows. In section 2 we provide the mathematical setup of the problem defining appropriate functional spaces and proving some auxiliary results. In section 3 we prove Theorem 2, providing various characterizations of the stray field energy. Section 4 is devoted to the proof of Theorem 3, characterizing the $\Gamma$-limit of the micromagnetic energy of thin shells.

**2. Mathematical setup.** In this section, we introduce the definitions and some useful facts about the basic function spaces that will be needed in our analysis. We would like to point out that the vectorial nature of the problem associated with the demagnetizing field presents some technical issues in the treatment of stationary Maxwell's equations under minimal regularity assumptions on the magnetization. Although some of the problems we are interested in can be investigated in a potential-theoretic framework (see, e.g., [14, 28, 29, 48]), here we rely on their distributional formulations. Another technical issue has to do with the fact that the problem is considered in the whole space. For the sake of full generality, we consider the most general distributional solutions of (1.2) and (1.3) and show that the resulting solutions do indeed belong to the natural energy spaces, which is not obvious a priori.

We denote by $\mathcal{D}'(\mathbb{R}^3)$ the space of distributions on $\mathbb{R}^3$. Following [13, p. 230] and [14, pp. 117–118], we define the homogeneous Sobolev space

$$\mathring{W}^1(\mathbb{R}^3) := \{u \in \mathcal{D}'(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3;\mathbb{R}^3)\}. \tag{2.1}$$

It is straightforward to show that the quotient space

$$\mathring{H}^1(\mathbb{R}^3) := \mathring{W}^1(\mathbb{R}^3)/\mathbb{R} \tag{2.2}$$

is a Hilbert space for the $L^2$ gradient norm $u \in \mathring{H}^1(\mathbb{R}^3) \mapsto \|\nabla u\|_{L^2(\mathbb{R}^3)}$, and that $\mathring{H}^1(\mathbb{R}^3)$ is isometrically isomorphic to the weighted Sobolev space $\{u \in L^2_\omega(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3;\mathbb{R}^3)\}$, with

$$L^2_\omega(\mathbb{R}^3) := \left\{u \in L^1_{\mathrm{loc}}(\mathbb{R}) : \omega u \in L^2(\mathbb{R}^3)\right\}, \quad \omega(x) := \frac{1}{\sqrt{1+|x|^2}}. \tag{2.3}$$

In particular, up to an additive constant, every element of $\mathring{W}^1(\mathbb{R}^3)$ is in $L^2_\omega(\mathbb{R}^3) \subset L^1_{\mathrm{loc}}(\mathbb{R}^3)$. For further reference, we also define $L^2_{\omega^{-1}}(\mathbb{R}^3) := \{u \in L^1_{\mathrm{loc}}(\mathbb{R}) : \omega^{-1}u \in L^2(\mathbb{R}^3)\}$. The symbols $L^2_\omega(\mathbb{R}^3;\mathbb{R}^3)$ and $L^2_{\omega^{-1}}(\mathbb{R}^3;\mathbb{R}^3)$ denote the vector-valued analogs of the above spaces.

We denote by $\mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3)$ the space of vector-valued distributions on $\mathbb{R}^3$. Also we denote by $\mathring{W}^1(\mathbb{R}^3,\mathbb{R}^3)$ and $\mathring{H}^1(\mathbb{R}^3,\mathbb{R}^3) := \mathring{W}^1(\mathbb{R}^3,\mathbb{R}^3)/\mathbb{R}^3$, the vector-valued counterparts of $\mathring{W}^1(\mathbb{R}^3)$ and $\mathring{H}^1(\mathbb{R}^3)$, respectively, for which the same considerations hold. Observe that

$$\|\nabla \boldsymbol{a}\|^2_{L^2(\mathbb{R}^3)} = \|\operatorname{div} \boldsymbol{a}\|^2_{L^2(\mathbb{R}^3)} + \|\operatorname{curl} \boldsymbol{a}\|^2_{L^2(\mathbb{R}^3)} \qquad \forall \boldsymbol{a} \in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3), \tag{2.4}$$

which may be seen from the fact that for every $\boldsymbol{a} \in \mathcal{D}(\mathbb{R}^3;\mathbb{R}^3)$ we have

$$\|\nabla \boldsymbol{a}\|^2_{L^2(\mathbb{R}^3)} = -\int_{\mathbb{R}^3} \boldsymbol{a}\cdot\Delta\boldsymbol{a} = \int_{\mathbb{R}^3} \boldsymbol{a}\cdot\operatorname{curl}\operatorname{curl}\boldsymbol{a} - \int_{\mathbb{R}^3} \boldsymbol{a}\cdot\nabla\operatorname{div}\boldsymbol{a}, \tag{2.5}$$

and then arguing by density (see also [2, 33, 37]).

In the spirit of (2.1), we also define the homogeneous Sobolev space

$$\mathring{W}^1(\operatorname{curl},\mathbb{R}^3) := \left\{\boldsymbol{b}\in\mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3) : \operatorname{curl}\boldsymbol{b}\in L^2(\mathbb{R}^3;\mathbb{R}^3)\right\}. \tag{2.6}$$

Note that, $\mathring{W}^1(\operatorname{curl},\mathbb{R}^3)$ is a subspace of $\mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3)$, and that the functional

$$|\cdot|_{\operatorname{curl}} : \boldsymbol{b}\in\mathring{W}^1(\operatorname{curl},\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} |\operatorname{curl}\boldsymbol{b}|^2 \tag{2.7}$$

is a seminorm on $\mathring{W}^1(\operatorname{curl},\mathbb{R}^3)$. The kernel of $|\cdot|_{\operatorname{curl}}$ consists of all curl-free distributions. Therefore, by the Poincaré–de Rham lemma [49, p. 355],

$$\ker|\cdot|_{\operatorname{curl}} = \nabla\mathcal{D}'(\mathbb{R}^3) \;\equiv\; \left\{\boldsymbol{b}\in\mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3) : \boldsymbol{b}=\nabla v \text{ for some } v\in\mathcal{D}'(\mathbb{R}^3)\right\}. \tag{2.8}$$

We identify distributions which differ by a gradient field. The resulting quotient space

$$\mathring{H}^1(\operatorname{curl},\mathbb{R}^3) := \mathring{W}^1(\operatorname{curl},\mathbb{R}^3)/\nabla\mathcal{D}'(\mathbb{R}^3) \tag{2.9}$$

is a Hilbert space. Indeed, the following result holds.

PROPOSITION 1. *The pair* $(\mathring{H}^1(\operatorname{curl},\mathbb{R}^3),|\cdot|_{\operatorname{curl}})$ *forms a complete inner product space.*

*Proof.* Let $(\boldsymbol{b}_n)_{n\in\mathbb{N}}\in\mathring{H}^1(\operatorname{curl},\mathbb{R}^3)$ be a Cauchy sequence in $\mathring{H}^1(\operatorname{curl},\mathbb{R}^3)$. This means that $(\operatorname{curl}\boldsymbol{b}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R}^3;\mathbb{R}^3)$. Therefore, there exists $\boldsymbol{j}\in L^2(\mathbb{R}^3;\mathbb{R}^3)$ such that $\operatorname{curl}\boldsymbol{b}_n\to\boldsymbol{j}$ in $L^2(\mathbb{R}^3;\mathbb{R}^3)$. To prove completeness, it remains to show that $\boldsymbol{j}$ is in $\operatorname{curl}(\mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3))$. This is a consequence of the Poincaré–de Rham lemma [49, p. 355]. Indeed, as $\boldsymbol{j}\in L^2(\mathbb{R}^3;\mathbb{R}^3)$ we have, for every $\varphi\in\mathcal{D}(\mathbb{R}^3)$,

$$\langle\mathbf{div}\,\boldsymbol{j},\boldsymbol{\varphi}\rangle = \int_{\mathbb{R}^3}\boldsymbol{j}\cdot\nabla\varphi = \lim_{n\to\infty}\int_{\mathbb{R}^3}\operatorname{curl}\,\boldsymbol{b}_n\cdot\nabla\varphi = 0,$$

and therefore $\operatorname{div}\boldsymbol{j}=\mathbf{0}$. Hence, $\operatorname{curl}\boldsymbol{b}=\boldsymbol{j}$ for some $\boldsymbol{b}\in\mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3)$. ∎

We shall need the closed subspace of $\mathring{H}^1(\operatorname{curl},\mathbb{R}^3)$ generated by the limits of all divergence-free (solenoidal) and compactly supported vector fields. To this end, we set

$$\mathcal{D}_{\operatorname{sol}}(\mathbb{R}^3;\mathbb{R}^3) := \left\{\boldsymbol{a}\in\mathcal{D}(\mathbb{R}^3;\mathbb{R}^3) : \operatorname{div}\boldsymbol{a}\equiv 0\right\}. \tag{2.10}$$

*Remark* 2.1. Since the set of harmonic functions in $\mathcal{D}(\mathbb{R}^3;\mathbb{R}^3)$ reduces to the null function, it is natural to be concerned about the cardinality of $\mathcal{D}_{\operatorname{sol}}(\mathbb{R}^3;\mathbb{R}^3)$. In that regard, we observe that the vector space $\mathcal{D}_{\operatorname{sol}}(\mathbb{R}^3;\mathbb{R}^3)$ is infinite dimensional. Indeed, let $\rho:\mathbb{R}\to\mathbb{R}^+$ be in $\mathcal{D}(\mathbb{R})$ and suppose $\rho\equiv 1$ in a neighborhood of 0. Also, let $\boldsymbol{\xi}\in C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ and consider the vector field

$$\boldsymbol{a}(x) := \rho(|x|)(\boldsymbol{\xi}(x)\times x), \qquad x\in\mathbb{R}^3. \tag{2.11}$$

Clearly, $\boldsymbol{a}\in\mathcal{D}(\mathbb{R}^3;\mathbb{R}^3)$ and, moreover, $\operatorname{div}\boldsymbol{a}(x) = \rho(|x|)\operatorname{curl}\boldsymbol{\xi}(x)\cdot x + (\nabla[\rho(|x|)]\times\boldsymbol{\xi}(x))\cdot x$. Since $\nabla[\rho(|x|)]=0$ in a sufficiently small neighborhood of the origin, and outside

that neighborhood one has $\nabla[\rho(|x|)] = \rho'(|x|)x/|x|$, we get that $(\nabla[\rho(|x|)] \times \boldsymbol{\xi}(x)) \cdot x = 0$ everywhere in $\mathbb{R}^3$. It then follows that $\operatorname{div} \boldsymbol{a}(x) = \rho(|x|)\operatorname{curl} \boldsymbol{\xi}(x) \cdot x$. As a consequence, for any curl-free vector field $\boldsymbol{\xi} \in C^\infty(\mathbb{R}^3;\mathbb{R}^3)$, and any bump function $\rho$ we get $\operatorname{div} \boldsymbol{a} \equiv 0$. This proves that $\mathcal{D}_{\mathrm{sol}}(\mathbb{R}^3;\mathbb{R}^3)$ is infinite dimensional due to the arbitrary choices of $\rho$ and $\boldsymbol{\xi}$.

We denote by $\mathring{H}^1_{\mathrm{sol}}(\operatorname{curl},\mathbb{R}^3)$ the closure of $\mathcal{D}_{\mathrm{sol}}(\mathbb{R}^3;\mathbb{R}^3)$ in $\mathring{H}^1(\operatorname{curl},\mathbb{R}^3)$. We observe that, with $\omega(x) := (1+|x|^2)^{-1/2}$, the following inequality holds:

$$\int_{\mathbb{R}^3} |\boldsymbol{a}(x)|^2 \omega^2(x) \mathrm{d}x \leqslant 4 \int_{\mathbb{R}^3} |\operatorname{curl} \boldsymbol{a}(x)|^2 \mathrm{d}x \quad \forall \boldsymbol{a} \in \mathcal{D}_{\mathrm{sol}}(\mathbb{R}^3;\mathbb{R}^3). \tag{2.12}$$

Indeed, (2.4) and Hardy's inequality [25, p. 296] imply $\|\omega \boldsymbol{a}\|^2_{L^2(\mathbb{R}^3)} \leqslant 4 \|\nabla \boldsymbol{a}\|^2_{L^2(\mathbb{R}^3)}$.

Our first observation is a regularity result on the structure of $\mathring{H}^1_{\mathrm{sol}}(\operatorname{curl},\mathbb{R}^3)$. In what follows, we use the notation $[\boldsymbol{a}] \in \mathring{H}^1_{\mathrm{sol}}(\operatorname{curl},\mathbb{R}^3)$ to denote the equivalence class which has $\boldsymbol{a} \in \mathring{W}^1(\operatorname{curl},\mathbb{R}^3)$ as representative; in other words, $[\boldsymbol{a}] := \{\boldsymbol{a} + \nabla v\}_{v \in \mathcal{D}'(\mathbb{R}^3)}$.

THEOREM 1. *The following statements hold:*

(i) *Let $[\boldsymbol{a}] \in \mathring{H}^1_{\mathrm{sol}}(\operatorname{curl},\mathbb{R}^3)$. There exists a unique representative $\boldsymbol{a}^\star \in [\boldsymbol{a}] \cap \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)$ which is divergence-free. In particular, $\boldsymbol{a}^\star$ is the unique divergence-free representative of $[\boldsymbol{a}]$ that belongs to $L^2_\omega(\mathbb{R}^3;\mathbb{R}^3)$.*

(ii) *If $[\boldsymbol{a}] \in \mathring{H}^1_{\mathrm{sol}}(\operatorname{curl},\mathbb{R}^3)$ has a representative $\boldsymbol{\jmath} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$, then also $\boldsymbol{a}^\star$ belongs to $L^2(\mathbb{R}^3;\mathbb{R}^3)$. Precisely, $\boldsymbol{a}^\star$ can be decomposed into the form*

$$\boldsymbol{a}^\star = \boldsymbol{\jmath} + \nabla v_{\boldsymbol{\jmath}} \tag{2.13}$$

*with $v_{\boldsymbol{\jmath}}$ the unique solution in $\mathring{H}^1(\mathbb{R}^3)$ of the Poisson equation $-\Delta v_{\boldsymbol{\jmath}} = \operatorname{div} \boldsymbol{\jmath}$.*

(iii) *If $\boldsymbol{a}_\circ \in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)$ and $\operatorname{div} \boldsymbol{a}_\circ = 0$, then $[\boldsymbol{a}_\circ] \in \mathring{H}^1_{\mathrm{sol}}(\operatorname{curl},\mathbb{R}^3)$ and $\boldsymbol{a}_\circ = \boldsymbol{a}^\star$.*

*Proof.* (i) Let $[\boldsymbol{a}] \in \mathring{H}^1_{\mathrm{sol}}(\operatorname{curl},\mathbb{R}^3)$ and $\boldsymbol{a}_n \in \mathcal{D}_{\mathrm{sol}}(\mathbb{R}^3;\mathbb{R}^3)$ be such that $\boldsymbol{a}_n \to \boldsymbol{a}$ in $\mathring{H}^1(\operatorname{curl},\mathbb{R}^3)$. Clearly, $[\boldsymbol{a}] \in \mathring{H}^1_{\mathrm{sol}}(\operatorname{curl},\mathbb{R}^3)$ and $(\operatorname{curl} \boldsymbol{a}_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathring{H}^1_{\mathrm{sol}}(\operatorname{curl},\mathbb{R}^3)$. Since $L^2_\omega(\mathbb{R}^3;\mathbb{R}^3)$ is a complete space, by (2.12), there exists $\boldsymbol{a}^\star \in L^2_\omega(\mathbb{R}^3;\mathbb{R}^3)$ such that $\boldsymbol{a}_n \to \boldsymbol{a}^\star$ in $L^2_\omega(\mathbb{R}^3;\mathbb{R}^3)$. Therefore,

$$0 = \operatorname{div} \boldsymbol{a}_n \to \operatorname{div} \boldsymbol{a}^\star = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3), \tag{2.14}$$

$$\operatorname{curl} \boldsymbol{a}_n \to \operatorname{curl} \boldsymbol{a}^\star \quad \text{in } \mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3), \tag{2.15}$$

$$\operatorname{curl} \boldsymbol{a}_n \to \operatorname{curl} \boldsymbol{a} \quad \text{in } \mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3). \tag{2.16}$$

This means that $\operatorname{curl}(\boldsymbol{a}^\star - \boldsymbol{a}) = 0$ and, therefore, that in any equivalence class $[\boldsymbol{a}] \in \mathring{H}^1_{\mathrm{sol}}(\operatorname{curl},\mathbb{R}^3)$ there exists a divergence-free vector field $\boldsymbol{a}^\star \in L^2_\omega(\mathbb{R}^3;\mathbb{R}^3)$. Note that $\boldsymbol{a}^\star \in L^2_\omega(\mathbb{R}^3;\mathbb{R}^3)$ is then necessarily unique. Indeed, if $\boldsymbol{\jmath}^\star \in L^2_\omega(\mathbb{R}^3;\mathbb{R}^3)$ is another divergence-free representative, then $\operatorname{curl} \boldsymbol{a}^\star = \operatorname{curl} \boldsymbol{\jmath}^\star$ and $\operatorname{div} \boldsymbol{a}^\star = \operatorname{div} \boldsymbol{\jmath}^\star = 0$. This implies that

$$\mathbf{0} = \nabla(\operatorname{div}(\boldsymbol{a}^\star - \boldsymbol{\jmath}^\star)) - \operatorname{curl}(\operatorname{curl}(\boldsymbol{a}^\star - \boldsymbol{\jmath}^\star)) = \Delta(\boldsymbol{a}^\star - \boldsymbol{\jmath}^\star) \text{ in } \mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3), \tag{2.17}$$

and in view of $\boldsymbol{a}^\star - \boldsymbol{j}^\star \in L^2_\omega(\mathbb{R}^3;\mathbb{R}^3)$ we have $\Delta(\boldsymbol{a}^\star - \boldsymbol{\jmath}^\star) = \mathbf{0}$ in the sense of tempered distributions $\boldsymbol{\mathcal{S}}'(\mathbb{R}^3)$. Therefore, by Liouville's theorem [24, p. 41], it follows that $\boldsymbol{a}^\star - \boldsymbol{\jmath}^\star$ is a polynomial vector field. We conclude by observing that the only polynomial vector field in $L^2_\omega(\mathbb{R}^3;\mathbb{R}^3)$ is the zero vector field.

It remains to prove that $\boldsymbol{a}^\star \in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)$. We observe that since $\operatorname{div}\boldsymbol{a}^\star = 0$, if we set $\boldsymbol{b}^\star := \operatorname{curl}\boldsymbol{a}^\star$, then $\boldsymbol{a}^\star$ is a solution of the vector Poisson equation $-\Delta\boldsymbol{a} = \operatorname{curl}\boldsymbol{b}^\star$. Also, since $\boldsymbol{b}^\star \in L^2(\mathbb{R}^3;\mathbb{R}^3)$, we have that $\operatorname{curl}\boldsymbol{b}^\star$ generates a linear and continuous functional on $\mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)$ and, therefore, by the Riesz representation theorem, there exists a unique $\boldsymbol{a} \in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)$ such that $-\Delta\boldsymbol{a} = \operatorname{curl}\boldsymbol{b}^\star$. But this implies that $\boldsymbol{a}-\boldsymbol{a}^\star$ is a harmonic $L^2_\omega(\mathbb{R}^3;\mathbb{R}^3)$ vector field; therefore, necessarily $\boldsymbol{a}^\star = \boldsymbol{a} \in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)$.

(ii) If $\boldsymbol{\jmath} \in [\boldsymbol{a}] \cap L^2(\mathbb{R}^3;\mathbb{R}^3)$, then there exists $v_{\boldsymbol{\jmath}} \in \nabla\mathcal{D}'(\mathbb{R}^3)$ such that $\boldsymbol{\jmath}-\boldsymbol{a}^\star = -\nabla v_{\boldsymbol{\jmath}}$. Hence,

$$-\Delta v_{\boldsymbol{\jmath}} = \operatorname{div}(\boldsymbol{\jmath} - \boldsymbol{a}^\star) = \operatorname{div}\boldsymbol{\jmath}, \tag{2.18}$$

and the previous equation admits a unique solution $v_{\boldsymbol{\jmath}} \in \mathring{H}^1(\mathbb{R}^3)$ by the Riesz representation theorem for the dual of a Hilbert space.

(iii) Let $\boldsymbol{a}_\circ \in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)$ be such that $\operatorname{div}\boldsymbol{a}_\circ = 0$. The variational equation

$$\int_{\mathbb{R}^3} \operatorname{curl}\boldsymbol{a}\cdot\operatorname{curl}\boldsymbol{\varphi}^\star = \int_{\mathbb{R}^3} \operatorname{curl}\boldsymbol{a}_\circ\cdot\operatorname{curl}\boldsymbol{\varphi}^\star \quad \forall\boldsymbol{\varphi}^\star \in \mathring{H}^1_{\mathrm{sol}}(\operatorname{curl},\mathbb{R}^3) \tag{2.19}$$

has a unique solution $[\boldsymbol{a}] \in \mathring{H}^1_{\mathrm{sol}}(\operatorname{curl},\mathbb{R}^3)$ because $\operatorname{curl}\operatorname{curl}\boldsymbol{a}_\circ$ can be identified with an element of $\mathring{H}^{-1}_{\mathrm{sol}}(\operatorname{curl},\mathbb{R}^3)$. In particular, testing against functions of the type $\boldsymbol{\varphi}^\star := \operatorname{curl}\boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in \mathcal{D}(\mathbb{R}^3;\mathbb{R}^3)$, we get that

$$\operatorname{curl}(\operatorname{curl}\operatorname{curl}(\boldsymbol{a}-\boldsymbol{a}_\circ)) = \boldsymbol{0} \quad \text{in } \mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3). \tag{2.20}$$

At the same time, by the result in point (i) we have that $\boldsymbol{a}^\star \in L^2_\omega(\mathbb{R}^3;\mathbb{R}^3)$ is the unique divergence-free representative belonging to $[\boldsymbol{a}] \cap \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)$. This implies that

$$-\Delta(\operatorname{curl}(\boldsymbol{a}^\star - \boldsymbol{a}_\circ)) = \boldsymbol{0} \quad \text{in } \mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3) \tag{2.21}$$

with $\operatorname{curl}(\boldsymbol{a}^\star - \boldsymbol{a}_\circ) \in L^2(\mathbb{R}^3;\mathbb{R}^3)$. Therefore $\operatorname{curl}(\boldsymbol{a}^\star - \boldsymbol{a}_\circ) = \boldsymbol{0}$, which means $\boldsymbol{a}_\circ \in [\boldsymbol{a}^\star]$. Again, by the uniqueness of the divergence-free representative we conclude that $\boldsymbol{a}_\circ = \boldsymbol{a}^\star$. ∎

**3. Magnetostatics.** We begin by nondimensionalizing the micromagnetic energy, using the exchange length $\ell_{\mathrm{ex}} := \sqrt{2A/(\mu_0 M_{\mathrm{s}}^2)}$ as the unit of length. Introducing the normalized magnetization vector $\boldsymbol{m}(\mathbf{r}) := \mathbf{M}(\ell_{\mathrm{ex}}\mathbf{r})/M_{\mathrm{s}}$ depending on the dimensionless position vector $\mathbf{r}$, the quality factor $Q := 2K/(\mu_0 M_{\mathrm{s}}^2)$ associated with crystalline anisotropy, and

$$\boldsymbol{h}_{\mathrm{d}} = \frac{\mathbf{H}_{\mathrm{d}}}{M_{\mathrm{s}}}, \qquad \boldsymbol{h}_a = \frac{\mathbf{H}_{\mathrm{a}}}{M_{\mathrm{s}}}, \qquad \mathcal{E}(\boldsymbol{m}) = \frac{E(\mathbf{M})}{2A\ell_{\mathrm{ex}}}, \tag{3.1}$$

we can write the micromagnetic energy in dimensionless form as

$$\mathcal{E}(\boldsymbol{m}) := \frac{1}{2}\int_\Omega |\nabla\boldsymbol{m}|^2 + \frac{Q}{2}\int_\Omega \Phi(\boldsymbol{m}) - \int_\Omega \boldsymbol{h}_a\cdot\boldsymbol{m} - \frac{1}{2}\int_\Omega \boldsymbol{h}_{\mathrm{d}}\cdot\boldsymbol{m}, \tag{3.2}$$

where $\Omega$ was appropriately rescaled and the symbol $d^3r$ is omitted from all the integrals from now on for simplicity of presentation. The rescaled demagnetizing field $\boldsymbol{h}_{\mathrm{d}}$ and the associated rescaled magnetic induction $\boldsymbol{b}_{\mathrm{d}}$ solve

$$\operatorname{curl}\boldsymbol{h} = \boldsymbol{0} \qquad \text{in } \mathbb{R}^3, \tag{3.3}$$

$$\operatorname{div}\boldsymbol{b} = 0 \qquad \text{in } \mathbb{R}^3, \tag{3.4}$$

$$\boldsymbol{b} = \boldsymbol{h} + \boldsymbol{m} \quad \text{in } \mathbb{R}^3. \tag{3.5}$$

In turn, the corresponding rescaled scalar potential $\boldsymbol{u}_{\mathrm{d}}$ and vector potential $\boldsymbol{a}_{\mathrm{d}}$ are related to their unscaled counterparts via

$$\boldsymbol{u}_{\mathrm{d}}(\mathbf{r}) := \frac{U_{\mathrm{d}}(\ell_{\mathrm{ex}}\mathbf{r})}{M_{\mathrm{s}}\ell_{\mathrm{ex}}}, \qquad \boldsymbol{a}_{\mathrm{d}}(\mathbf{r}) := \frac{\mathbf{A}_{\mathrm{d}}(\ell_{\mathrm{ex}}\mathbf{r})}{\mu_0 M_{\mathrm{s}}\ell_{\mathrm{ex}}}, \tag{3.6}$$

so that $\boldsymbol{b}_{\mathrm{d}} = \operatorname{curl}\boldsymbol{a}_{\mathrm{d}}$ and $\boldsymbol{h}_{\mathrm{d}} = -\nabla\boldsymbol{u}_{\mathrm{d}}$. Finally, the rescaled stray field energy is

$$\mathcal{E}_{\mathrm{s}}(\boldsymbol{m}) := -\frac{1}{2}\int_{\mathbb{R}^3} \boldsymbol{h}_{\mathrm{d}}\cdot\boldsymbol{m}, \tag{3.7}$$

where $\boldsymbol{h}_{\mathrm{d}}$ is understood as a function of $\boldsymbol{m}$ uniquely determined by the solution of (3.3)–(3.5) (for a precise statement, see below).

Throughout the rest of this paper, we suppress the subscript "$d$" everywhere to avoid cumbersome notations. However, whenever needed we utilize the subscript $\boldsymbol{m}$ to explicitly indicate the dependence of the associated quantities on a given magnetization $\boldsymbol{m}$, so there should be no confusion. The main result of this section is Theorem 2. We remark that all the assumptions of this theorem are satisfied in the context of micromagnetics when the ferromagnet occupies a bounded domain.

THEOREM 2. *Let $\boldsymbol{m} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$. The following assertions hold:*

(i) *There exists a unique magnetic scalar potential $u_{\boldsymbol{m}} \in \mathring{H}^1(\mathbb{R}^3)$ such that*

$$\boldsymbol{h}_{\boldsymbol{m}} := -\nabla u_{\boldsymbol{m}}, \quad \boldsymbol{b}_{\boldsymbol{m}} := \boldsymbol{h}_{\boldsymbol{m}} + \boldsymbol{m}, \tag{3.8}$$

*is a solution of (3.3)–(3.5) in $L^2(\mathbb{R}^3;\mathbb{R}^3)\times L^2(\mathbb{R}^3;\mathbb{R}^3)$. The stray field energy is given through the following maximization problem:*

$$\mathcal{E}_{\mathrm{s}}(\boldsymbol{m}) = \max_{u\in\mathring{H}^1(\mathbb{R}^3)} \mathcal{W}(\boldsymbol{m},u), \qquad \mathcal{W}(\boldsymbol{m},u) := \int_{\mathbb{R}^3}\nabla u\cdot\boldsymbol{m} - \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2, \tag{3.9}$$

*whose unique solution coincides with $u_{\boldsymbol{m}}$. Moreover, if $\boldsymbol{m} \in L^2_{\omega^{-1}}(\mathbb{R}^3;\mathbb{R}^3)$, then $u_{\boldsymbol{m}} \in H^1(\mathbb{R}^3)$.*

(ii) *There exists a unique magnetic vector potential $[\boldsymbol{a}_{\boldsymbol{m}}] \in \mathring{H}^1(\operatorname{curl},\mathbb{R}^3)$ such that*

$$\boldsymbol{b}'_{\boldsymbol{m}} := \operatorname{curl}[\boldsymbol{a}_{\boldsymbol{m}}], \quad \boldsymbol{h}'_{\boldsymbol{m}} := \boldsymbol{b}'_{\boldsymbol{m}} - \boldsymbol{m}, \tag{3.10}$$

*is a solution of (3.3)–(3.5) in $L^2(\mathbb{R}^3;\mathbb{R}^3)\times L^2(\mathbb{R}^3;\mathbb{R}^3)$. The stray field energy is given through the following minimization problem:*

$$\begin{aligned} \mathcal{E}_{\mathrm{s}}(\boldsymbol{m}) &= \min_{[\boldsymbol{a}]\in\mathring{H}^1(\operatorname{curl},\mathbb{R}^3)} \mathcal{V}_{\operatorname{curl}}(\boldsymbol{m},[\boldsymbol{a}]),\\ \mathcal{V}_{\operatorname{curl}}(\boldsymbol{m},[\boldsymbol{a}]) &:= \frac{1}{2}\int_{\mathbb{R}^3}|\operatorname{curl}[\boldsymbol{a}]-\boldsymbol{m}|^2, \end{aligned} \tag{3.11}$$

*whose unique solution coincides with $[\boldsymbol{a}_{\boldsymbol{m}}]$.*
*Moreover, if $\boldsymbol{m} \in L^2_{\omega^{-1}}(\mathbb{R}^3;\mathbb{R}^3)$, then there exists a unique representative $\boldsymbol{a}^\star_{\boldsymbol{m}} \in [\boldsymbol{a}_{\boldsymbol{m}}]$ satisfying the Coulomb gauge conditions*

$$\boldsymbol{a}^\star_{\boldsymbol{m}} \in L^2(\mathbb{R}^3;\mathbb{R}^3), \quad \operatorname{div}\boldsymbol{a}^\star_{\boldsymbol{m}} = 0. \tag{3.12}$$

*The representative $\boldsymbol{a}^\star_{\boldsymbol{m}}$ belongs to $H^1(\mathbb{R}^3;\mathbb{R}^3)$ and can be characterized as the unique solution in $\mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)$ of the vector Poisson equation*

$$-\Delta\boldsymbol{a}^\star_{\boldsymbol{m}} = \operatorname{curl}\boldsymbol{m} \quad \text{in } \mathring{H}^{-1}(\mathbb{R}^3;\mathbb{R}^3). \tag{3.13}$$

*Equivalently, $\boldsymbol{a}^\star_{\boldsymbol{m}}$ can be characterized as the unique solution in $\mathring{H}^1_{\mathrm{sol}}(\mathrm{curl},\mathbb{R}^3)$ of the variational equation*

$$\int_{\mathbb{R}^3} \operatorname{curl} \boldsymbol{a}^\star_{\boldsymbol{m}} \cdot \operatorname{curl} \boldsymbol{\varphi}^\star = \int_{\mathbb{R}^3} \boldsymbol{m} \cdot \operatorname{curl} \boldsymbol{\varphi}^\star \quad \forall \boldsymbol{\varphi}^\star \in \mathring{H}^1_{\mathrm{sol}}(\mathrm{curl},\mathbb{R}^3). \tag{3.14}$$

1. *We have*

$$\boldsymbol{h}_{\boldsymbol{m}} = \boldsymbol{h}'_{\boldsymbol{m}}, \qquad \boldsymbol{b}_{\boldsymbol{m}} = \boldsymbol{b}'_{\boldsymbol{m}}, \qquad \mathcal{E}_{\mathrm{s}}(\boldsymbol{m}) = \frac{1}{2}\int_{\mathbb{R}^3} |\boldsymbol{h}_{\boldsymbol{m}}|^2. \tag{3.15}$$

2. *If $\boldsymbol{m} \in L^2_{\omega^{-1}}(\mathbb{R}^3;\mathbb{R}^3)$, the stray field energy admits the following representation:*

$$\begin{aligned} \mathcal{E}_{\mathrm{s}}(\boldsymbol{m}) &= \min_{\boldsymbol{a}\in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)} \mathcal{V}(\boldsymbol{m},\boldsymbol{a}), \\ \mathcal{V}(\boldsymbol{m},\boldsymbol{a}) &:= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \boldsymbol{a}|^2 + \frac{1}{2}\int_{\mathbb{R}^3} |\boldsymbol{m}|^2 - \int_{\mathbb{R}^3} \boldsymbol{m} \cdot \operatorname{curl} \boldsymbol{a}, \end{aligned} \tag{3.16}$$

*and the unique minimizer of $\mathcal{V}(\boldsymbol{m},\cdot)$ coincides with $\boldsymbol{a}^\star_{\boldsymbol{m}}$.*

*Proof.* (i) We start with an observation that holds under minimal regularity assumptions. Let $\boldsymbol{m} \in \mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3)$. If a solution $(\boldsymbol{h}_{\boldsymbol{m}}, \boldsymbol{b}_{\boldsymbol{m}}) \in \mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3) \times \mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3)$ of (3.3)–(3.5) exists, then $\operatorname{curl} \boldsymbol{h}_{\boldsymbol{m}} = \boldsymbol{0}$ distributionally. Therefore, according to the Poincaré–de Rham lemma [49, p. 355], there exists a magnetostatic potential $u_{\boldsymbol{m}} \in \mathcal{D}'(\mathbb{R}^3)$ such that $\boldsymbol{h}_{\boldsymbol{m}} = -\nabla u_{\boldsymbol{m}}$. But then, from (3.4) and (3.5), we get that $u_{\boldsymbol{m}}$ is a particular solution of the Poisson equation

$$\Delta u_{\boldsymbol{m}} = \operatorname{div} \boldsymbol{m} \quad \text{in } \mathcal{D}'(\mathbb{R}^3). \tag{3.17}$$

Conversely, if $u_{\boldsymbol{m}}$ is a particular solution of (3.17), then the general solution of the magnetostatic equations is given by

$$\boldsymbol{h}_{\boldsymbol{m}} := -\nabla u_{\boldsymbol{m}} + \nabla v_0, \quad \boldsymbol{b}_{\boldsymbol{m}} := \boldsymbol{h}_{\boldsymbol{m}} + \boldsymbol{m}, \tag{3.18}$$

for an arbitrary harmonic distribution $v_0 \in \mathcal{D}'(\mathbb{R}^3)$. Indeed, defining $\boldsymbol{h}_{\boldsymbol{m}} := -\nabla u_{\boldsymbol{m}}$ and $\boldsymbol{b}_{\boldsymbol{m}} := \boldsymbol{h}_{\boldsymbol{m}} + \boldsymbol{m}$ we have that $(\boldsymbol{h}_{\boldsymbol{m}}, \boldsymbol{b}_{\boldsymbol{m}})$ is a solution of (3.3)–(3.5), and any other demagnetizing field differs by a gradient distribution. Taking the divergence of the first equation in (3.18) we get that $v_0 \in \mathcal{D}'(\mathbb{R}^3)$ is necessarily harmonic.

Now, for $\boldsymbol{m} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ we have that $\operatorname{div} \boldsymbol{m}$ generates a linear continuous functional on $\mathring{H}^1(\mathbb{R}^3)$ and, therefore, by the Riesz representation theorem there exists a unique $u_{\boldsymbol{m}} \in \mathring{H}^1(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \nabla u_{\boldsymbol{m}} \cdot \nabla \varphi = \int_{\mathbb{R}^3} \boldsymbol{m} \cdot \nabla \varphi \qquad \forall \varphi \in \mathring{H}^1(\mathbb{R}^3). \tag{3.19}$$

Hence, setting

$$\boldsymbol{h}_{\boldsymbol{m}} := -\nabla u_{\boldsymbol{m}}, \quad \boldsymbol{b}_{\boldsymbol{m}} := \boldsymbol{h}_{\boldsymbol{m}} + \boldsymbol{m}, \tag{3.20}$$

we get a solution $(\boldsymbol{h}_{\boldsymbol{m}}, \boldsymbol{b}_{\boldsymbol{m}}) \in L^2(\mathbb{R}^3;\mathbb{R}^3) \times L^2(\mathbb{R}^3;\mathbb{R}^3)$ of (3.3)–(3.5). Also, note that $u_{\boldsymbol{m}}$ is the unique magnetostatic potential which gives a demagnetizing field in $L^2(\mathbb{R}^3;\mathbb{R}^3)$. Indeed, if $-\nabla u_{\boldsymbol{m}} + \nabla v_0 \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ with $v_0$ harmonic, then, according to Liouville's theorem, $\nabla v_0 = \boldsymbol{0}$. Finally, a standard argument gives that $u_{\boldsymbol{m}}$ coincides with the unique solution of the *maximization problem* (3.9).

Now, if $\boldsymbol{m} \in L^2_{\omega^{-1}}(\mathbb{R}^3;\mathbb{R}^3)$, then $\boldsymbol{m}$ generates a continuous linear functional on $\mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)$. Indeed, by Hardy's inequality, for every $\boldsymbol{\varphi} \in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)$ we have

$$\int_{\mathbb{R}^3} |\boldsymbol{m}\cdot\boldsymbol{\varphi}| \leqslant \|\omega^{-1}\boldsymbol{m}\|_{L^2(\mathbb{R}^3)}\,\|\omega\boldsymbol{\varphi}\|_{L^2(\mathbb{R}^3)} \leqslant 4\|\omega^{-1}\boldsymbol{m}\|_{L^2(\mathbb{R}^3)}\,\|\nabla\boldsymbol{\varphi}\|_{L^2(\mathbb{R}^3)}. \tag{3.21}$$

Therefore, by the Riesz representation theorem there exists a unique $\boldsymbol{\psi}_{\boldsymbol{m}} \in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)$ such that $-\Delta\psi_{\boldsymbol{m}} = \boldsymbol{m}$. We set $u_{\boldsymbol{m}} := -\operatorname{div}\psi_{\boldsymbol{m}}$. Note that $u_{\boldsymbol{m}} \in L^2(\mathbb{R}^3)$ and satisfies the equation

$$\Delta u_{\boldsymbol{m}} = -\operatorname{div}\nabla(\operatorname{div}\psi_{\boldsymbol{m}}) = -\operatorname{div}\Delta\psi_{\boldsymbol{m}} = \operatorname{div}\boldsymbol{m} \quad \text{in } \mathcal{D}'(\mathbb{R}^3). \tag{3.22}$$

This implies that $u_{\boldsymbol{m}} \in L^2(\mathbb{R}^3)\cap\mathring{H}^1(\mathbb{R}^3) = H^1(\mathbb{R}^3)$.

(ii) Once again, we start with an observation that is valid under minimal regularity assumptions. Let $\boldsymbol{m} \in \mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3)$. If a solution $(\boldsymbol{h}_{\boldsymbol{m}},\boldsymbol{b}_{\boldsymbol{m}}) \in \mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3)\times\mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3)$ of (3.3)–(3.5) exists, then $\operatorname{div}\boldsymbol{b}_{\boldsymbol{m}} = \boldsymbol{0}$ distributionally. Therefore, it follows from the Poincaré–de Rham lemma that there exists a vector potential $\boldsymbol{a}_{\boldsymbol{m}} \in \mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3)$ such that $\boldsymbol{b}_{\boldsymbol{m}} = \operatorname{curl}\boldsymbol{a}_{\boldsymbol{m}}$. But then, from (3.3) and (3.5), we get that $\boldsymbol{a}_{\boldsymbol{m}}$ is a particular solution of the double-curl equation

$$\operatorname{curl}\operatorname{curl}\boldsymbol{a}_{\boldsymbol{m}} = \operatorname{curl}\boldsymbol{m} \quad \text{in } \mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3). \tag{3.23}$$

Conversely, assume that $\bar{\boldsymbol{a}}_{\boldsymbol{m}}$ is a particular solution of (3.23). We claim that the general solution of (3.3)–(3.5) is given by

$$\boldsymbol{b}_{\boldsymbol{m}} := \operatorname{curl}\bar{\boldsymbol{a}}_{\boldsymbol{m}} + \nabla v_0, \quad \boldsymbol{h}_{\boldsymbol{m}} := \boldsymbol{b}_{\boldsymbol{m}} - \boldsymbol{m} \tag{3.24}$$

for an arbitrary harmonic distribution $v_0 \in \mathcal{D}'(\mathbb{R}^3)$. Indeed, the assignment $\bar{\boldsymbol{b}}_{\boldsymbol{m}} := \operatorname{curl}\bar{\boldsymbol{a}}_{\boldsymbol{m}}$ and $\bar{\boldsymbol{h}}_{\boldsymbol{m}} := \bar{\boldsymbol{b}}_{\boldsymbol{m}} - \boldsymbol{m}$ gives a particular solution of (3.3)–(3.5). Moreover, any other vector field $\boldsymbol{b}$ satisfying (3.3)–(3.5) must differ from $\bar{\boldsymbol{b}}_{\boldsymbol{m}}$ by a curl distribution, i.e., we have

$$\boldsymbol{b}_{\boldsymbol{m}} := \operatorname{curl}(\boldsymbol{a}_0 + \bar{\boldsymbol{a}}_{\boldsymbol{m}}), \quad \boldsymbol{h}_{\boldsymbol{m}} := \boldsymbol{b}_{\boldsymbol{m}} - \boldsymbol{m} = \operatorname{curl}(\boldsymbol{a}_0 + \bar{\boldsymbol{a}}_{\boldsymbol{m}}) - \boldsymbol{m} \tag{3.25}$$

for some $\boldsymbol{a}_0 \in \mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3)$. Taking the curl of the second equation in (3.25), we get

$$\operatorname{curl}\operatorname{curl}(\boldsymbol{a}_0 + \bar{\boldsymbol{a}}_{\boldsymbol{m}}) - \operatorname{curl}\boldsymbol{m} = \boldsymbol{0}, \tag{3.26}$$

and from the definition of $\bar{\boldsymbol{a}}_{\boldsymbol{m}}$ we obtain that $\operatorname{curl}\operatorname{curl}\bar{\boldsymbol{a}}_0 = \boldsymbol{0}$. It follows that $\operatorname{curl}\bar{\boldsymbol{a}}_0 = \nabla v_0$ for some $v_0 \in \mathcal{D}'(\mathbb{R}^3)$. In particular, $v_0$ is a harmonic distribution.

Now, for $\boldsymbol{m} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ we have that $\operatorname{curl}\boldsymbol{m}$ generates a linear continuous functional on $\mathring{H}^1(\operatorname{curl},\mathbb{R}^3)$ and, therefore, by the Riesz representation theorem there exists a unique $[\boldsymbol{a}_{\boldsymbol{m}}] \in \mathring{H}^1(\operatorname{curl},\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \operatorname{curl}[\boldsymbol{a}_{\boldsymbol{m}}]\cdot\operatorname{curl}\boldsymbol{\psi} = \int_{\mathbb{R}^3} \boldsymbol{m}\cdot\operatorname{curl}\boldsymbol{\psi} \quad \forall\boldsymbol{\psi} \in \mathring{H}^1(\operatorname{curl},\mathbb{R}^3). \tag{3.27}$$

Hence, setting

$$\boldsymbol{b}'_{\boldsymbol{m}} := \operatorname{curl}[\boldsymbol{a}_{\boldsymbol{m}}], \quad \boldsymbol{h}'_{\boldsymbol{m}} := \boldsymbol{b}'_{\boldsymbol{m}} - \boldsymbol{m}, \tag{3.28}$$

we get a solution $(\boldsymbol{h}'_{\boldsymbol{m}},\boldsymbol{b}'_{\boldsymbol{m}}) \in L^2(\mathbb{R}^3;\mathbb{R}^3)\times L^2(\mathbb{R}^3;\mathbb{R}^3)$ of (3.3)–(3.5). Note that $\boldsymbol{a}_{\boldsymbol{m}}$ is the unique magnetostatic potential which gives $\boldsymbol{b}_{\boldsymbol{m}} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$. Indeed, if

$\operatorname{curl} \boldsymbol{a_m} + \nabla v_0 \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ and $v_0$ is harmonic, then necessarily $\nabla v_0 = \mathbf{0}$. From the preceding considerations, it is clear that the variational characterization (3.11) holds.

Next, as in the proof of $(i)$, for $\boldsymbol{m} \in L^2_{\omega^{-1}}(\mathbb{R}^3;\mathbb{R}^3)$ there exists a unique $\boldsymbol{\psi_m} \in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)$ such that $-\Delta\psi_{\boldsymbol{m}} = \boldsymbol{m}$. We set $\boldsymbol{a^\star_m} := \operatorname{curl}\psi_{\boldsymbol{m}}$. Note that $\boldsymbol{a^\star_m} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ and, by construction, $\operatorname{div} \boldsymbol{a^\star_m} = 0$. Also, $\boldsymbol{a^\star_m}$ satisfies the equation

$$\operatorname{curl} \boldsymbol{a^\star_m} = \operatorname{curl}\operatorname{curl} \boldsymbol{\psi_m} = \boldsymbol{m} + \nabla \operatorname{div} \boldsymbol{\psi_m}. \tag{3.29}$$

But $\operatorname{div} \boldsymbol{\psi_m} \in L^2(\mathbb{R}^3)$ satisfies $-\Delta(\operatorname{div} \boldsymbol{\psi_m}) = \operatorname{div} \boldsymbol{m}$ and, therefore, $\nabla \operatorname{div} \boldsymbol{\psi_m} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$. Overall, from (3.29), we infer that $[\boldsymbol{a^\star_m}]$ is an element of $\mathring{H}^1(\operatorname{curl},\mathbb{R}^3)$ satisfying (3.27). It follows that $[\boldsymbol{a^\star_m}] = [\boldsymbol{a_m}]$ and $\operatorname{div} \boldsymbol{a^\star_m} = 0$. Also, from (3.29) we know that $\boldsymbol{a^\star_m}$ solves the equation $-\Delta \boldsymbol{a^\star_m} = \operatorname{curl} \boldsymbol{m}$ with data $\operatorname{curl} \boldsymbol{m}$ in $\mathring{H}^{-1}(\mathbb{R}^3;\mathbb{R}^3)$. Hence, $\boldsymbol{a^\star_m} \in H^1(\mathbb{R}^3;\mathbb{R}^3)$.

Finally, if $[\boldsymbol{a^{\star\star}_m}] \in \mathring{H}^1_{\mathrm{sol}}(\operatorname{curl},\mathbb{R}^3)$ is the unique solution of (3.14) and $\boldsymbol{a^{\star\star}_m} \in L^2_\omega(\mathbb{R}^3;\mathbb{R}^3)$ its unique divergence-free representative, testing against $\boldsymbol{\varphi^\star} = \operatorname{curl} \boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in \mathcal{D}(\mathbb{R}^3;\mathbb{R}^3)$ we get

$$\operatorname{curl}\operatorname{curl} \boldsymbol{a^{\star\star}_m} = \operatorname{curl} \boldsymbol{m} + \nabla v_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3) \tag{3.30}$$

for some harmonic polynomial $v_0$. Therefore, since $\boldsymbol{a^{\star\star}_m}$ is divergence-free, we have

$$-\Delta(\operatorname{curl}(\boldsymbol{a^{\star\star}_m} - \boldsymbol{a^\star_m})) = \mathbf{0}, \tag{3.31}$$

with $\operatorname{curl}(\boldsymbol{a^{\star\star}_m} - \boldsymbol{a^\star_m}) \in L^2(\mathbb{R}^3;\mathbb{R}^3)$. But this means that $\boldsymbol{a^{\star\star}_m} = \boldsymbol{a^\star_m} + \nabla v$ with $v$ harmonic and $\nabla v \in L^2_\omega(\mathbb{R}^3;\mathbb{R}^3)$. Therefore $\nabla v = \mathbf{0}$. This concludes the proof of (ii).

(iii) The first two equalities in (3.15) follow from the uniqueness of solutions of (3.3)–(3.5) in $L^2(\mathbb{R}^3;\mathbb{R}^3)$. The third equality in (3.15) follows from (3.8) and (3.9).

(iv) From (3.14) it is clear that

$$\mathcal{E}_{\mathrm{s}}(\boldsymbol{m}) = \min_{[\boldsymbol{a}^\star]\in \mathring{H}^1_{\mathrm{sol}}(\operatorname{curl},\mathbb{R}^3)} \mathcal{V}_{\operatorname{curl}}(\boldsymbol{m},[\boldsymbol{a}^\star]), \tag{3.32}$$

where we noted that the minimum above is attained because $\mathring{H}^1_{\mathrm{sol}}(\operatorname{curl},\mathbb{R}^3)$ is a closed subspace of the Hilbert space $\mathring{H}^1(\operatorname{curl},\mathbb{R}^3)$. Since $\mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)$ can be identified with a subset of $\mathring{H}^1(\operatorname{curl},\mathbb{R}^3)$, and (3.12) holds, it is sufficient to show that

$$\min_{\boldsymbol{a}\in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)} \mathcal{V}(\boldsymbol{m},\boldsymbol{a}) \leqslant \mathcal{E}_{\mathrm{s}}(\boldsymbol{m}). \tag{3.33}$$

To this end, we observe that if $[\boldsymbol{a^\star_m}] \in \mathring{H}^1_{\mathrm{sol}}(\operatorname{curl},\mathbb{R}^3)$ minimizes $\mathcal{V}_{\operatorname{curl}}(\boldsymbol{m},[\boldsymbol{a}^\star])$, then, without loss of generality, we can assume that $\boldsymbol{a^\star_m}$ is the unique representative satisfying the Coulomb gauge regularity conditions (3.12). But then, since $\operatorname{div} \boldsymbol{a^\star_m} = 0$, by (2.4) we have

$$\boldsymbol{a^\star_m} \in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3), \quad \mathcal{V}(\boldsymbol{m},\boldsymbol{a^\star_m}) = \mathcal{V}_{\operatorname{curl}}(\boldsymbol{m},[\boldsymbol{a^\star_m}]), \tag{3.34}$$

and this implies (3.33). ∎

*Remark* 3.1. The weight $\omega$ in the assumptions on $\boldsymbol{m}$ imposes the behavior at infinity of the magnetostatic potential $u_{\boldsymbol{m}}$. Note that in general $u_{\boldsymbol{m}}$ does not belong to $H^1(\mathbb{R}^3)$ if $\boldsymbol{m} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$. To see this consider $\boldsymbol{m} = -\nabla u$ with $u \in \mathring{H}^1(\mathbb{R}^3)\setminus H^1(\mathbb{R}^3)$. However, it is known that $u \in H^1(\mathbb{R}^3)$ provided $\boldsymbol{m} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ has compact support [34, 48]. The above theorem gives a generalization of this result to a wider class of functions $\boldsymbol{m} \in L^2_{\omega^{-1}}(\mathbb{R}^3;\mathbb{R}^3)$.

*Remark* 3.2. If $u_{\boldsymbol{m}'}$ is the unique weak solution of $\Delta u_{\boldsymbol{m}'} = \operatorname{div} \boldsymbol{m}'$ with $\boldsymbol{m}' \in L^2(\mathbb{R}^3;\mathbb{R}^3)$, then testing against $\varphi := u_{\boldsymbol{m}'}$ in the weak formulation of $\Delta u_{\boldsymbol{m}} = \operatorname{div} \boldsymbol{m}$, and testing against $\varphi := u_{\boldsymbol{m}}$ in the weak formulation of $\Delta u_{\boldsymbol{m}'} = \operatorname{div} \boldsymbol{m}'$, we get the so-called *reciprocity relations*

$$\int_{\mathbb{R}^3} \boldsymbol{h}_{\boldsymbol{m}} \cdot \boldsymbol{h}_{\boldsymbol{m}'} = -\int_{\mathbb{R}^3} \boldsymbol{m} \cdot \boldsymbol{h}_{\boldsymbol{m}'} = -\int_{\mathbb{R}^3} \boldsymbol{h}_{\boldsymbol{m}} \cdot \boldsymbol{m}'. \tag{3.35}$$

Thus, the operator $\mathcal{H} : \boldsymbol{m} \in L^2(\mathbb{R}^3;\mathbb{R}^3) \mapsto \boldsymbol{h}_{\boldsymbol{m}} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ is self-adjoint, and for $\boldsymbol{m} = \boldsymbol{m}'$ we recover the expression of $\mathcal{E}_{\mathrm{s}}(\boldsymbol{m})$ in (3.15). Furthermore, $\mathcal{H}$ has unit norm, as can be seen from

$$\|\boldsymbol{h}_{\boldsymbol{m}}\|_{L^2(\mathbb{R}^3)} \leqslant \|\boldsymbol{m}\|_{L^2(\mathbb{R}^3)} \qquad \forall \boldsymbol{m} \in L^2(\mathbb{R}^3;\mathbb{R}^3) \tag{3.36}$$

with equality achieved for all $\boldsymbol{m} = \nabla v$ with $v \in \mathring{H}^1(\mathbb{R}^3)$. Additionally, it is possible to prove that the spectrum of $\mathcal{H}$ is at most countable and contained in the interval $[0,1]$. Note that any element $\boldsymbol{m} \in \mathcal{D}_{\mathrm{sol}}(\mathbb{R}^3;\mathbb{R}^3)$, in particular, any configuration built as in Remark 2.1, belongs to the kernel of $\mathcal{H}$ (see [30] for a detailed analysis). Finally, we recall that $\mathcal{H}$ maps constant magnetizations in $\Omega$ (and zero outside) into constant magnetic fields in $\Omega$ (but not constant outside) if and only if $\Omega$ is an ellipsoid [17, 19, 35]. Thus, if $\Omega$ is an ellipsoid, the restriction of $\mathcal{H}$ to three-dimensional constant vector fields in $\Omega$ defines a finite-dimensional linear operator (the so called demagnetizing tensor), whose eigenvalues (the so-called demagnetizing factors) are among the most important quantities in ferromagnetism [44].

**4. Micromagnetics of curved thin shells.** We now illustrate the utility of the variational principles discussed in section 3 in the case of dimension reduction for thin ferromagnetic shells. Previously such results have been established under suitable technical assumptions on the geometry of the surface in the case of thin layers [11], and shells enclosing convex bodies [18]. Here we use Theorem 2 to give an elementary proof of the dimension reduction via $\Gamma$-convergence, which does not require convexity or other purely technical assumptions on the shape of the shell.

Let $\Omega$ be a bounded domain in $\mathbb{R}^3$. For any $\boldsymbol{m} \in H^1(\Omega,\mathbb{S}^2)$, the micromagnetic energy functional in (3.2) in the absence of crystalline anisotropy and the applied magnetic field, reads

$$\mathcal{G}_\Omega(\boldsymbol{m}) := \frac{1}{2}\int_\Omega \left(|\nabla \boldsymbol{m}|^2 - \boldsymbol{h}_{\boldsymbol{m}} \cdot \boldsymbol{m}\right), \tag{4.1}$$

where $\boldsymbol{h}_{\boldsymbol{m}}$ is the solution of (3.3)–(3.5) with $\boldsymbol{m}$ extended by zero outside $\Omega$. Taking into account Theorem 2, the following equivalent expressions arise:

$$\mathcal{G}_\Omega(\boldsymbol{m}) = \frac{1}{2}\int_\Omega |\nabla \boldsymbol{m}|^2 + \min_{\boldsymbol{a} \in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)} \mathcal{V}(\boldsymbol{m},\boldsymbol{a}), \tag{4.2}$$

$$\mathcal{G}_\Omega(\boldsymbol{m}) = \frac{1}{2}\int_\Omega |\nabla \boldsymbol{m}|^2 + \max_{u \in \mathring{H}^1(\mathbb{R}^3)} \mathcal{W}(\boldsymbol{m},u). \tag{4.3}$$

In particular, if we define

$$\mathcal{G}_\Omega(\boldsymbol{m},\boldsymbol{a}) := \frac{1}{2}\int_\Omega |\nabla \boldsymbol{m}|^2 + \mathcal{V}(\boldsymbol{m},\boldsymbol{a}), \quad \mathcal{G}_\Omega(\boldsymbol{m},u) := \frac{1}{2}\int_\Omega |\nabla \boldsymbol{m}|^2 + \mathcal{W}(\boldsymbol{m},u), \tag{4.4}$$

then

$$\min_{\boldsymbol{m}\in H^1(\Omega,\mathbb{S}^2)} \mathcal{G}_\Omega(\boldsymbol{m}) = \min_{\boldsymbol{m}\in H^1(\Omega,\mathbb{S}^2)} \min_{\boldsymbol{a}\in \mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)} \mathcal{G}_\Omega(\boldsymbol{m},\boldsymbol{a}), \tag{4.5}$$

$$\min_{\boldsymbol{m}\in H^1(\Omega,\mathbb{S}^2)} \mathcal{G}_\Omega(\boldsymbol{m}) = \min_{\boldsymbol{m}\in H^1(\Omega,\mathbb{S}^2)} \max_{u\in \mathring{H}^1(\mathbb{R}^3)} \mathcal{G}_\Omega(\boldsymbol{m},u). \tag{4.6}$$

Thus, the minimization problem for the micromagnetic energy functional $H^1(\Omega,\mathbb{S}^2)$ can be restated as a minimization problem on the product space $H^1(\Omega,\mathbb{S}^2)\times\mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)$, or as a minimax problem on the spaces $H^1(\Omega,\mathbb{S}^2)\times\mathring{H}^1(\mathbb{R}^3)$.

Let $S$ be a compact $C^2$ surface in $\mathbb{R}^3$. It is well known that $S$ is orientable and admits a tubular neighborhood (cf. [21, Prop. 1, p. 113]). Precisely, let $\boldsymbol{n}: S\to\mathbb{S}^2$ be the unit normal vector field associated with the choice of an orientation of $S$. For every $\xi\in S$, $\delta\in\mathbb{R}_+$, denote by $\ell_\delta(\xi) := \{\xi + t\boldsymbol{n}(\xi)\}_{|t|<\delta}$ the normal segment to $S$ having radius $\delta$ and centered at $\xi$. Then, there exists $\delta\in\mathbb{R}_+$ such that the following properties hold (cf. [21, p. 112]):

- For every $\xi_1,\xi_2\in S$ one has $\ell_\delta(\xi_1)\cap\ell_\delta(\xi_2)=\emptyset$ whenever $\xi_1\neq\xi_2$.
- The union $\Omega_\delta := \cup_{\xi\in S}\ell_\delta(\xi)$ is an open set of $\mathbb{R}^3$ containing $S$.
- For $I := (-1,1)$, set $\mathcal{M} := S\times I$. For every $\varepsilon\in I_\delta^+ := (0,\delta)$, the map

$$\psi_\varepsilon : (\xi,t)\in\mathcal{M}\mapsto \xi+\varepsilon t\boldsymbol{n}(\xi)\in\Omega_\varepsilon \tag{4.7}$$

  is a $C^1$ diffeomorphism of the product manifold $\mathcal{M}$ onto $\Omega_\varepsilon$. In particular, the nearest point projection $\pi:\Omega_\varepsilon\to S$, which maps any $x\in\Omega_\varepsilon$ onto the unique $\xi\in S$ such that $x\in\ell_\varepsilon(\xi)$, is a $C^1$ map. All integrals over $\mathcal{M}$ are with respect to the measure $\mathcal{H}^2\times\mathcal{L}^1$.

The open set $\Omega_\delta$ is then called a tubular neighborhood of $S$ of radius $\delta$. Note that $\Omega_\delta\equiv\psi_\delta(\mathcal{M})$.

In what follows, the symbols $\tau_1(\xi),\tau_2(\xi)$ denote the orthonormal basis of $T_\xi S$ made by the principal directions at $\xi\in S$. Also, we denote by $\sqrt{\mathfrak{g}_\epsilon}$ the metric factor which relates the volume form on $\Omega_\varepsilon$ to the volume form on $\mathcal{M}$, and by $\mathfrak{h}_{1,\varepsilon},\mathfrak{h}_{2,\varepsilon}$ the metric coefficients which transform the gradient on $\Omega_\varepsilon$ into the gradient on $\mathcal{M}$. A direct computation shows that

$$\sqrt{\mathfrak{g}_\varepsilon(\xi,t)} := |1+2\varepsilon tH(\xi)+\varepsilon^2t^2G(\xi)|,\quad \mathfrak{h}_{i,\varepsilon}(\xi,t) := (1+\varepsilon t\kappa_i(\xi))^{-1}\quad (i\in\mathbb{N}_2), \tag{4.8}$$

where $H(\xi)$ and $G(\xi)$ are, respectively, the mean and Gaussian curvature at $\xi\in S$, and $\kappa_1(\xi),\kappa_2(\xi)$ are the principal curvatures at $\xi\in S$. In what follows we always assume the thickness $\delta$ to be sufficiently small so that the quantities in (4.8) are uniformely bounded from both above and below by some positive constants depending only on $S$.

We denote by $H^1(\mathcal{M};\mathbb{R}^3)$ the Sobolev space of vector-valued functions defined on $\mathcal{M}$ endowed with the norm $\|\boldsymbol{m}\|^2_{H^1(\mathcal{M})} := \|\boldsymbol{m}\|^2_{L^2(\mathcal{M})} + \|\nabla_\xi\boldsymbol{m}\|^2_{L^2(\mathcal{M})} + \|\partial_t\boldsymbol{m}\|^2_{L^2(\mathcal{M})}$, where $\nabla_\xi\boldsymbol{m}$ stands for the tangential gradient of $\boldsymbol{m}$ on $S$. Finally, we write $H^1(\mathcal{M};\mathbb{S}^2)$ for the subset of $H^1(\mathcal{M};\mathbb{R}^3)$ consisting of functions taking values in $\mathbb{S}^2$.

Next, for every $\varepsilon\in I_\delta^+$ we consider the micromagnetic energy functional on $H^1(\Omega_\varepsilon,\mathbb{S}^2)$ which, after normalization, reads

$$\mathcal{G}_\varepsilon(\widetilde{\boldsymbol{m}}) := \frac{1}{2\varepsilon}\int_{\Omega_\varepsilon}|\nabla\widetilde{\boldsymbol{m}}|^2 + \frac{1}{2\varepsilon}\int_{\mathbb{R}^3}|\nabla u_{\widetilde{\boldsymbol{m}}}|^2 \tag{4.9}$$

with $u_{\widetilde{\boldsymbol{m}}}$ being the unique solution in $\mathring{H}^1(\mathbb{R}^3)$ of the Poisson equation $\Delta u_{\widetilde{\boldsymbol{m}}} = \operatorname{div}\widetilde{\boldsymbol{m}}$ with the understanding that $\widetilde{\boldsymbol{m}}$ is extended by zero outside of $\Omega_\varepsilon$. The change of

variables (4.7) allows for the following equivalent expression of the micromagnetic energy functional

$$\mathcal{F}_\varepsilon(\boldsymbol{m}) := \mathcal{E}_\varepsilon(\boldsymbol{m}) + \frac{1}{2\varepsilon}\int_{\mathbb{R}^3} |\nabla u_{\widetilde{\boldsymbol{m}}}|^2 \tag{4.10}$$

with $\boldsymbol{m}(\xi,t) := \widetilde{\boldsymbol{m}} \circ \psi_\varepsilon(\xi,t) \in H^1(\mathcal{M};\mathbb{S}^2)$ for $\widetilde{\boldsymbol{m}} \in H^1(\Omega_\varepsilon,\mathbb{S}^2)$, and $\mathcal{E}_\varepsilon$ the family of Dirichlet energies on $\mathcal{M}$ defined by

$$\mathcal{E}_\varepsilon(\boldsymbol{m}) := \frac{1}{2}\int_{\mathcal{M}} \sum_{i\in\mathbb{N}_2} |\mathfrak{h}_{i,\varepsilon}\partial_{\tau_i(\xi)}\boldsymbol{m}|^2\sqrt{\mathfrak{g}_\varepsilon} + \frac{1}{2\varepsilon^2}\int_{\mathcal{M}} |\partial_t \boldsymbol{m}|^2\sqrt{\mathfrak{g}_\varepsilon}. \tag{4.11}$$

We are interested in the limiting behavior of the minimizers of $\mathcal{F}_\varepsilon$ when $\varepsilon \to 0$. In that regard, we prove the following $\Gamma$-convergence result.

THEOREM 3. *As $\varepsilon \to 0$, the following statements hold:*

1. *If the sequence $(\boldsymbol{m}_\varepsilon) \subset H^1(\mathcal{M};\mathbb{S}^2)$ satisfies $\mathcal{F}_\varepsilon(\boldsymbol{m}_\varepsilon) \leqslant C$, then upon possible extraction of a subsequence there exists $\boldsymbol{m}_0 \in H^1(\mathcal{M};\mathbb{S}^2)$ such that $\boldsymbol{m}_\varepsilon \rightharpoonup \boldsymbol{m}_0$ weakly in $H^1(\mathcal{M};\mathbb{S}^2)$.*
2. *The family $(\mathcal{F}_\varepsilon)_{\varepsilon\in I_\delta^+}$ is equicoercive in the weak topology of $H^1(\mathcal{M};\mathbb{S}^2)$, and $(\mathcal{F}_\varepsilon)_{\varepsilon\in I_\delta^+}$ $\Gamma$-converges in that topology to the functional*

$$\mathcal{F}(\boldsymbol{m}) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{\mathcal{M}} \left[|\nabla_\xi \boldsymbol{m}|^2 + (\boldsymbol{m}\cdot\boldsymbol{n})^2\right] \mathrm{d}\xi & \text{if } \partial_t \boldsymbol{m} = 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.12}$$

3. *If $\boldsymbol{m}_\varepsilon$ are minimizers of $\mathcal{F}_\varepsilon$, then upon possible extraction of a subsequence $(\boldsymbol{m}_\varepsilon)$ converges strongly in $H^1(\mathcal{M};\mathbb{S}^2)$ to a minimizer of $\mathcal{F}$.*

*Proof.* The first statement is a direct consequence of the boundedness of the Dirichlet energy of $(\boldsymbol{m})_{\varepsilon\in I_\delta^+}$. The equicorecivity of the family $(\mathcal{F}_\varepsilon)_{\varepsilon\in I_\delta^+}$ is proved in [18], where it is also proved the $\Gamma$-convergence of the Dirichlet energies $\mathcal{E}_\varepsilon$ to the energy functional

$$\mathcal{E}_0 : \boldsymbol{m} \in H^1(\mathcal{M};\mathbb{S}^2) \mapsto \begin{cases} \dfrac{1}{2}\displaystyle\int_{\mathcal{M}} |\nabla_\xi \boldsymbol{m}|^2 \mathrm{d}\xi & \text{if } \partial_t \boldsymbol{m} = 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.13}$$

In particular, if $\boldsymbol{m} \in H^1(\mathcal{M};\mathbb{S}^2)$, $\boldsymbol{m}(\xi,\cdot)$ is not constant for a.e. $\xi \in S$, and $\boldsymbol{m}_\varepsilon \rightharpoonup \boldsymbol{m}$ weakly in $H^1(\mathcal{M};\mathbb{S}^2)$, then necessarily $\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\boldsymbol{m}_\varepsilon) = +\infty$. Therefore, without loss of generality, we can restrict our analysis to families $(\boldsymbol{m}_\varepsilon)_{\varepsilon\in I_\delta^+}$ in $H^1(\mathcal{M};\mathbb{S}^2)$ such that $\boldsymbol{m}_\varepsilon(\xi,s) \rightharpoonup \boldsymbol{m}_0(\xi)\chi_I(s)$ for some $\boldsymbol{m}_0 \in H^1(S,\mathbb{S}^2)$.

*Step* 1. $\Gamma$-liminf *inequality.* To shorten notation, it is convenient to introduce $\nabla_\varepsilon := (\mathfrak{h}_{1,\varepsilon}\partial_{\tau_1(\xi)}, \mathfrak{h}_{2,\varepsilon}\partial_{\tau_2(\xi)}, \varepsilon^{-1}\partial_t)$. Then, to every $\widetilde{\boldsymbol{m}_\varepsilon} \in H^1(\Omega_\varepsilon,\mathbb{S}^2)$, $\tilde{u} \in \mathring{H}^1(\mathbb{R}^3)$, we associate the vector field $\boldsymbol{m}_\varepsilon := \widetilde{\boldsymbol{m}_\varepsilon} \circ \psi_\varepsilon$ and the scalar potential $u_\varepsilon := \tilde{u} \circ \psi_\varepsilon$.

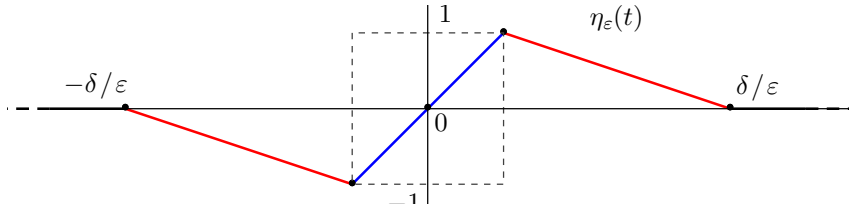


FIG. 1. *The function $\eta_\varepsilon$ used in the construction of the family of potentials.*

We use the characterization of the magnetostatic sef-energy given in Theorem 2 (cf. (3.9)). For every $\delta > 0$, we denote by $\mathcal{M}_\delta$ the product manifold $\mathcal{M}_\delta := S \times I_\delta$. We have, with the identification of $H^1_0(\Omega_\delta)$ as a subspace of $\mathring{H}^1(\mathbb{R}^3)$,

$$\begin{aligned}\frac{1}{2\varepsilon}\int_{\mathbb{R}^3}|\nabla u_{\widetilde{\boldsymbol{m}}_\varepsilon}|^2 &= \max_{\tilde u\in\mathring{H}^1(\mathbb{R}^3)}\frac{1}{\varepsilon}\left(\int_{\Omega_\varepsilon}\nabla\tilde u\cdot\widetilde{\boldsymbol{m}}_\varepsilon-\frac12\int_{\mathbb{R}^3}|\nabla\tilde u|^2\right)\\ &\geqslant \max_{\tilde u\in H^1_0(\Omega_\delta)}\frac{1}{\varepsilon}\left(\int_{\Omega_\varepsilon}\nabla\tilde u\cdot\widetilde{\boldsymbol{m}}_\varepsilon-\frac12\int_{\Omega_\delta}|\nabla\tilde u|^2\right)\\ &= \max_{\tilde u\in H^1_0(\Omega_\delta)}\left(\int_{\mathcal{M}}\nabla_\varepsilon[\tilde u\circ\psi_\varepsilon]\cdot\boldsymbol{m}_\varepsilon\sqrt{\mathfrak{g}_\varepsilon}-\frac12\int_{\mathcal{M}_{\delta/\varepsilon}}|\nabla_\varepsilon[\tilde u\circ\psi_\varepsilon]|^2\sqrt{\mathfrak{g}_\varepsilon}\right)\\ &\geqslant \int_{\mathcal{M}}\nabla_\varepsilon u_\varepsilon\cdot\boldsymbol{m}_\varepsilon\sqrt{\mathfrak{g}_\varepsilon}-\frac12\int_{\mathcal{M}_{\delta/\varepsilon}}|\nabla_\varepsilon u_\varepsilon|^2\sqrt{\mathfrak{g}_\varepsilon}\end{aligned}\tag{4.14}$$

for every $u_\varepsilon = \tilde u\circ\psi_\varepsilon$ with $\tilde u\in H^1_0(\Omega_\delta)$. Note that $u_\varepsilon$ is well defined on $\mathcal{M}_{\delta/\varepsilon}$. Next, we build the family of potentials (cf. Figure 1)

$$u_\varepsilon(\xi,t) := \varepsilon\eta_\varepsilon(t)(\boldsymbol{m}_0(\xi)\cdot\boldsymbol{n}(\xi)),\qquad \eta_\varepsilon(t) := \begin{cases} t & \text{if } |t|<1,\\ \frac{\delta-\varepsilon|t|}{\delta-\varepsilon} & \text{if } 1\leqslant|t|<\delta/\varepsilon,\\ 0 & \text{if } |t|>\delta/\varepsilon.\end{cases}\tag{4.15}$$

Note that $\eta_\varepsilon(t)=0$ if $|t|>\delta/\varepsilon>1$. Also we have

$$\eta_\varepsilon'(t)=1\quad\text{if } |t|<1,\qquad (\eta_\varepsilon'(t))^2=\frac{\varepsilon^2}{(\delta-\varepsilon)^2}\quad\text{if } 1<|t|<\delta/\varepsilon.\tag{4.16}$$

Hence, we have $\nabla_\xi u_\varepsilon(\xi,t)=\varepsilon\eta_\varepsilon(t)\nabla_\xi(\boldsymbol{m}_0(\xi)\cdot\boldsymbol{n}(\xi))$ and $\partial_t u_\varepsilon(\xi,t)=\varepsilon\eta_\varepsilon'(t)(\boldsymbol{m}_0(\xi)\cdot\boldsymbol{n}(\xi))$. It follows that $\|\nabla_\xi u_\varepsilon\|^2_{\mathcal{M}_{\delta/\varepsilon}}\to 0$ as $\varepsilon\to 0$. Therefore, from (4.14) and (4.15) we obtain

$$\liminf_{\varepsilon\to0}\frac{1}{2\varepsilon}\int_{\mathbb{R}^3}|\nabla u_{\widetilde{\boldsymbol{m}}_\varepsilon}|^2\ \geqslant\ \int_{\mathcal{M}}(\boldsymbol{m}_0\cdot\boldsymbol{n})^2-\frac12\limsup_{\varepsilon\to0}\int_{\mathcal{M}_{\delta/\varepsilon}}(\boldsymbol{m}_0\cdot\boldsymbol{n})^2(\eta_\varepsilon'(t))^2\mathrm{d}\xi\mathrm{d}t.\tag{4.17}$$

On the other hand, we have

$$\begin{aligned}&\int_{\mathcal{M}_{\delta/\varepsilon}}(\boldsymbol{m}_0(\xi)\cdot\boldsymbol{n})^2(\eta_\varepsilon'(t))^2\mathrm{d}\xi\mathrm{d}t\\ &\qquad=\left(1+\frac{\varepsilon}{\delta-\varepsilon}\right)\int_{\mathcal{M}}(\boldsymbol{m}_0\cdot\boldsymbol{n})^2\xrightarrow{\varepsilon\to0}\int_{\mathcal{M}}(\boldsymbol{m}_0\cdot\boldsymbol{n})^2.\end{aligned}\tag{4.18}$$

Summarizing, we get

$$\liminf_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{\mathbb{R}^3}|\nabla u_{\widetilde{\boldsymbol{m}}_\varepsilon}|^2 \;\geqslant\; \frac{1}{2}\int_{\mathcal{M}}(\boldsymbol{m}_0\cdot\boldsymbol{n})^2=\int_S(\boldsymbol{m}_0\cdot\boldsymbol{n})^2. \tag{4.19}$$

Taking into account (4.13), we conclude that for any $(\boldsymbol{m}_\varepsilon)_{\varepsilon\in I_\delta}$ in $H^1(\mathcal{M};\mathbb{S}^2)$ such that $\boldsymbol{m}_\varepsilon(\xi,s)\rightharpoonup \boldsymbol{m}_0(\xi)\chi_I(s)$ for some $\boldsymbol{m}_0\in H^1(S,\mathbb{S}^2)$, the following lower bound holds:

$$\liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(\boldsymbol{m}_\varepsilon)\geqslant\frac{1}{2}\int_{\mathcal{M}}|\nabla_\xi\boldsymbol{m}_0|^2+\frac{1}{2}\int_{\mathcal{M}}(\boldsymbol{m}_0\cdot\boldsymbol{n})^2. \tag{4.20}$$

*Step* 2. *Recovery sequence.* We now show that, for any $\boldsymbol{m}_0\in H^1(S,\mathbb{S}^2)$, the constant family of magnetizations given by $\boldsymbol{m}_\varepsilon(\xi,t):=\boldsymbol{m}_0(\xi)\chi_I(t)$ defines a recovery sequence. It is clear that such a family of functions works for the exchange energies $\mathcal{E}_\varepsilon$ due to (4.13). Therefore, we can focus on the magnetostatic self-energy. To shorten notation, it is convenient to introduce the symbol $\operatorname{curl}_\varepsilon\boldsymbol{a}^\star:=\operatorname{curl}_{\varepsilon,\xi}\boldsymbol{a}^\star+\varepsilon^{-1}\boldsymbol{n}\times\partial_t\boldsymbol{a}^\star$ with

$$\operatorname{curl}_{\varepsilon,\xi}\boldsymbol{a}^\star=\sum_{i=1}^{2}\mathfrak{h}_{i,\varepsilon}(\xi,t)\left(\boldsymbol{\tau}_i(\xi)\times\partial_{\tau_i(\xi)}\boldsymbol{a}^\star\right). \tag{4.21}$$

By the expression of the magnetostatic self-energy in terms of the magnetic vector potential given in Theorem 2 (cf. (3.16)), we have

$$\begin{aligned}\frac{1}{2\varepsilon}\int_{\mathbb{R}^3}|\nabla u_{\widetilde{\boldsymbol{m}}_\varepsilon}|^2&=\frac{1}{2\varepsilon}\min_{\tilde{\boldsymbol{a}}^\star\in\mathring{H}^1(\mathbb{R}^3;\mathbb{R}^3)}\int_{\mathbb{R}^3}\left(|\nabla\tilde{\boldsymbol{a}}^\star|^2+|\widetilde{\boldsymbol{m}}_\varepsilon|^2-2\operatorname{curl}\tilde{\boldsymbol{a}}^\star\cdot\widetilde{\boldsymbol{m}}_\varepsilon\right)\\&\leqslant\min_{\tilde{\boldsymbol{a}}^\star\in H^1_0(\Omega_\delta,\mathbb{R}^3)}\left(\frac{1}{2}\int_{\mathcal{M}_{\delta/\varepsilon}}|\nabla_\varepsilon[\tilde{\boldsymbol{a}}^\star\circ\psi_\varepsilon]|^2\sqrt{\mathfrak{g}_\varepsilon}\right.\\&\qquad\left.+\frac{1}{2}\int_{\mathcal{M}}(|\boldsymbol{m}_\varepsilon|^2-2\operatorname{curl}_\varepsilon[\tilde{\boldsymbol{a}}^\star\circ\psi_\varepsilon]\cdot\boldsymbol{m}_\varepsilon)\sqrt{\mathfrak{g}_\varepsilon}\right)\\&\leqslant\frac{1}{2}\int_{\mathcal{M}}\left(|\boldsymbol{m}_\varepsilon|^2-2\operatorname{curl}_\varepsilon\boldsymbol{a}^\star\cdot\boldsymbol{m}_\varepsilon\right)\sqrt{\mathfrak{g}_\varepsilon}+\frac{1}{2}\int_{\mathcal{M}_{\delta/\varepsilon}}|\nabla_\varepsilon\boldsymbol{a}^\star|^2\sqrt{\mathfrak{g}_\varepsilon}\end{aligned} \tag{4.22}$$

for every $\boldsymbol{a}^\star=\tilde{\boldsymbol{a}}^\star\circ\psi_\varepsilon$ with $\tilde{\boldsymbol{a}}^\star\in H^1_0(\Omega_\delta,\mathbb{R}^3)$. Next, we consider the family of potentials

$$\boldsymbol{a}^\star_\varepsilon(\xi,t):=\varepsilon\eta_\varepsilon(t)(\boldsymbol{m}_0(\xi)\times\boldsymbol{n}(\xi)) \tag{4.23}$$

with $\eta_\varepsilon$ given by (4.15). We get that

$$\nabla_\xi\boldsymbol{a}^\star_\varepsilon(\xi,t)=\varepsilon\eta_\varepsilon(t)\nabla_\xi(\boldsymbol{m}_0(\xi)\times\boldsymbol{n}(\xi)),\quad \partial_t\boldsymbol{a}^\star_\varepsilon(\xi,t)=\varepsilon\eta'_\varepsilon(t)(\boldsymbol{m}_0(\xi)\times\boldsymbol{n}(\xi)). \tag{4.24}$$

Hence, we have $\|\nabla_\xi\boldsymbol{a}^\star_\varepsilon\|^2_{\mathcal{M}_{\delta/\varepsilon}}\to 0$ as $\varepsilon\to 0$. Therefore

$$\begin{aligned}\limsup_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{\mathbb{R}^3}|\nabla u_{\widetilde{\boldsymbol{m}}_\varepsilon}|^2\leqslant\frac{1}{2}\int_{\mathcal{M}}|\boldsymbol{m}_0|^2-\int_{\mathcal{M}}[\boldsymbol{n}\times(\boldsymbol{m}_0\times\boldsymbol{n})]\cdot\boldsymbol{m}_0\\+\limsup_{\varepsilon\to 0}\left(\frac{1}{2}\int_{\mathcal{M}_{\delta/\varepsilon}}|(\boldsymbol{m}_0\times\boldsymbol{n})\eta'_\varepsilon(t)|^2\,\mathrm{d}t\right).\end{aligned} \tag{4.25}$$

Moreover, we have

$$\int_{\mathcal{M}_{\delta/\varepsilon}} |(\boldsymbol{m}_0 \times \boldsymbol{n})\eta_\varepsilon'(t)|^2 \,\mathrm{d}t = \left(1 + \frac{\varepsilon}{\delta - \varepsilon}\right) \int_{\mathcal{M}} |\boldsymbol{m}_0 \times \boldsymbol{n}|^2 \xrightarrow{\varepsilon \to 0} \int_{\mathcal{M}} |\boldsymbol{m}_0 \times \boldsymbol{n}|^2 . \tag{4.26}$$

Summarizing, we get

$$\begin{aligned} \limsup_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{\mathbb{R}^3} |\nabla u_{\widetilde{\boldsymbol{m}_\varepsilon}}|^2 &\leqslant \frac{1}{2} \int_{\mathcal{M}} |\boldsymbol{m}_0|^2 - \int_{\mathcal{M}} [\boldsymbol{n} \times (\boldsymbol{m}_0 \times \boldsymbol{n})] \cdot \boldsymbol{m}_0 + \frac{1}{2} \int_{\mathcal{M}} |\boldsymbol{m}_0 \times \boldsymbol{n}|^2 \\ &= \frac{1}{2} \int_{\mathcal{M}} (\boldsymbol{m}_0 \cdot \boldsymbol{n})^2. \end{aligned} \tag{4.27}$$

Strong convergence of minimizers $\boldsymbol{m}_\varepsilon \to \boldsymbol{m}_0$ in $H^1(\mathcal{M}; \mathbb{S}^2)$ follows from weak convergence in $H^1(\mathcal{M}; \mathbb{S}^2)$ and convergence of the norms

$$\int_{\mathcal{M}} \sum_{i \in \mathbb{N}_2} |\partial_{\tau_i(\xi)} \boldsymbol{m}_\varepsilon|^2 + \int_{\mathcal{M}} |\partial_t \boldsymbol{m}_\varepsilon|^2 \to \int_{\mathcal{M}} |\nabla_\xi \boldsymbol{m}_0|^2, \tag{4.28}$$

where the latter is a straightforward consequence of $\mathcal{E}_\varepsilon(\boldsymbol{m}_\varepsilon) \to \mathcal{E}_0(\boldsymbol{m})$ for a minimizing sequence $(\boldsymbol{m}_\varepsilon)$. This completes the proof. ∎

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