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GENERAL ARTICLE

The Computational Origin of Representation

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Abstract

Each of our theories of mental representation provides some insight into how the mind works. However, these insights often seem incompatible, as the debates between symbolic, dynamical, emergentist, sub-symbolic, and grounded approaches to cognition attest. Mental representations—whatever they are—must share many features with each of our theories of representation, and yet there are few hypotheses about how a synthesis could be possible. Here, I develop a theory of the underpinnings of symbolic cognition that shows how sub-symbolic dynamics may give rise to higher-level cognitive representations of structures, systems of knowledge, and algorithmic processes. This theory implements a version of conceptual role semantics by positing an internal universal representation language in which learners may create mental models to capture dynamics they observe in the world. The theory formalizes one account of how truly novel conceptual content may arise, allowing us to explain how even elementary logical and computational operations may be learned from a more primitive basis. I provide an implementation that learns to represent a variety of structures, including logic, number, kinship trees, regular languages, context-free languages, domains of theories like magnetism, dominance hierarchies, list structures, quantification, and computational primitives like repetition, reversal, and recursion. This account is based on simple discrete dynamical processes that could be implemented in a variety of different physical or biological systems. In particular, I describe how the required dynamics can be directly implemented in a connectionist framework. The resulting theory provides an "assembly language" for cognition, where high-level theories of symbolic computation can be implemented in simple dynamics that themselves could be encoded in biologically plausible systems.

Keywords Language of thought · Conceptual role semantics · Conceptual change · Combinatory · Logic

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1 Introduction

At the heart of cognitive science is an embarrassing truth: we do not know what mental representations are like. Many ideas have been developed, from the field's origins in symbolic AI (Newell and Simon 1976), to parallel and distributed representations of connectionism (Rumelhart and McClelland 1986; Smolensky and Legendre 2006), embodied theories that emphasize the grounded nature of the mental (Barsalou 2008, 2010), Bayesian accounts of structure learning and inference (Griffiths et al. 2010; Tenenbaum et al. 2011), theories of cognitive architecture (Newell 1994; Anderson et al. 1997, 2004), and accounts based on mental models, simulation (Craik 1967; Johnson-Laird 1983; Battaglia et al. 2013), or analogy (Gentner and Stevens 1983; Gentner and Forbus 2011; Gentner and Markman 1997; Hummel and Holyoak 1997). These research programs have developed in concert with foundational debates in the philosophy of mind about what kinds of things concepts may be (e.g. Margolis and Laurence 1999), with similarly diverse and seemingly incompatible answers.

This paper develops an approach that attempts to unify a variety of ideas about how mental representations may work. I argue that what is needed is an *intermediate* bridging theory that lives below the level of symbols and above the level of neurons or neural network nodes, and that in fact such a system can be found in a pre-existing formalism of mathematical logic. The system I present shows how it is possible to implement high-level symbolic constructs permitting essentially arbitrary (Turing-complete) computation while being simple, parallelizable, and addressing foundational questions about meaning. Although the particular instantiation I describe is an extreme simplification, its general principles, I'll argue, are likely to be close to the truth.

The theory I describe is a little unusual in that it is built almost exclusively out ideas that have been independently developed in fields adjacent to cognitive science. The logical formalism comes from mathematical logic and computer science in the early 1900s. The philosophical and conceptual framework has been well-articulated in prior debates. The inferential theory builds on work in theoretical AI and Bayesian cognitive science. The overall goal of finding symbolic cognition below the level of "symbols" comes from connectionism and other sub-symbolic accounts. An emphasis on the importance of getting symbols eventually comes from the remarkable properties of human symbolic thought and language, including centuries of argumentation in philosophy and mathematics about what kinds of formal systems may capture the core properties of humanlike thinking. What is new is the unification of these ideas into a framework that can be shown to learn complex symbolic processes and representations that are grounded in simple underlying dynamics without dodging key questions about meaning and representation. The resulting theory intentionally muddies the water between traditional symbolic and non-symbolic approaches by studying a system that is both simple to implement in neural systems, and in which it is simple to implement high-level cognitive processes. Such as system, I argue, represents a promising and concrete avenue for spanning the chasm between the neural and the cognitive.

The resulting theory formulates hypotheses about how fundamental *conceptual change* can occur, both computationally and philosophically. A pressing mystery is how children might start with whatever initial knowledge they are born with, and come

to develop the rich and sophisticated systems of representation we can find in adults (Rule et al. 2020). The problem is perhaps deepest when we consider simple logical capacities—for instance, the ability to represent boolean values, compute syllogisms, follow logical/deductive rules, use number, or process conditionals and quantifiers. If infants do not have some of these abilities, we are in need of a learning theory that can explain where such computational processes might come from. Yet, it is hard to imagine how a computational system that does *not* know these could function. From what starting point could learners possibly construct Boolean logic, for instance? It is hard to imagine what a computer would look like if it did not already have notions like `and`, `or`, `not`, `true`, and `false`. Is it even possible to "compute" without having concepts like logic or number? The answer provided in this paper is emphatically *yes*—computation is possible without any explicit form of these operations. In fact, learning systems can be made that construct and test these logical systems as hypotheses, genuinely without presupposing them. My goal here is to show how.

The paper is organized as follows: In the next section, I describe a leading example of symbolic cognitive science, Fodor's *Language of Thought* (LOT) theory (Fodor 1975, 2008). The LOT motivates the need for structured, compositional representations, but leaves at its core unanswered questions about how the symbols in these representations come to have meaning or may be implemented in neural systems. To address this, I then discuss *conceptual role semantics* (CRS) as an account of how meaningful symbols may arise. The problem with CRS is that it has no implementations, leaving its actual mechanics vague and unspecified. To address this, I connect the idea of CRS to formalizations in computer science that effectively use CRS-like methods to show how logical systems can encode other processes. Specifically, I describe a mathematical formalism, *combinatory logic*, and a key operation in it, *Church encoding*, which permits modeling of one system (e.g. world) within another (e.g. a mental logic). This reduces the problem of learning meaningful symbols to that of constructing the right structure in a mental language like combinatory logic. I then address the question of learning these representations, drawing on work in theoretical artificial intelligence. The inferential approach allows learners to acquire computationally complex representations, yet largely avoid the primary difficulty that faces learners who operate on spaces of computations: the halting problem. Providing a GPL-licensed implementation of the inference scheme, I then demonstrate how learners could acquire a large variety of mental representations found across human and non-human cognition. In each of these cases, the core symbolic aspect of representation is built out only out of extraordinarily simple and mechanistic dynamics from which the meanings emerge in an interconnected system of concepts. I argue that whatever mental representations are, they must be *like* these kinds of objects, where symbolic meanings for algorithms, structures, and relations arise out of the sub-symbolic dynamics that implement these processes. I then describe how the system can be implemented straightforwardly in existing connectionist frameworks, and discuss broader philosophical implications.

1.1 Representation in the Language of Thought

There is a lot going for the theory that human cognition uses, among other things, a structured symbolic system of representation analogous to language. The idea that something like a mental logic describes key cognitive processes dates back at least to Boole (1854), who described his logic as capturing "the laws of thought" and Gottfried Leibniz, who tried to systematize knowledge and reasoning in his own universal formal language, the *characteristica universalis*. As a psychological theory, the LOT reached prominence through the works of Jerry Fodor, who argued for a compositional system of mental representation that is analogous to human language called a *Language of Thought* (LOT) (Fodor 1975, 2008). The LOT has had numerous incarnations throughout the history of AI and cognitive science (Newell and Simon 1976; Penn et al. 2008; Fodor 2008; Kemp 2012; Goodman et al. 2015; Piantadosi et al. 2016; Rule et al. 2020; Chater and Oaksford 2013; Siskind 1996). Most recent versions focus on combining language-like—or program-like—representations with Bayesian probabilistic inference to explain concept induction in empirical tasks (Goodman et al. 2008a, b, 2015; Yildirim and Jacobs 2013; Erdogan et al. 2015; Piantadosi and Jacobs 2016; Rothe et al. 2017; Lake et al. 2015; Overlan et al. 2017; Romano et al. 2017, 2018; Amalric et al. 2017; Depeweg et al. 2018).[1]

If mentalese is like a program, its primitives are humans' most basic mental operations, a view of conceptual representation that has roots and branches in psychology and computer science. Miller and Johnson-Laird (1976) developed a theory of language understanding based on structured, program-like representations. Commonsense knowledge in domains like objects, beliefs, physical reasoning, and time may also draw on logical representations (Davis 1990) mirroring psychological theorizing about the LOT. Modern incarnations of conceptual representations can be found in programming languages like Church that aim to capture phenomena like the gradience of concepts through a semantics centered on probability and conditioning (Goodman et al. 2008a, 2015). In computer science, the program metaphor has been applied in computational semantics under the name *procedural semantics*, in which representations of linguistic meaning are taken to be programs that compute something about the meaning of the sentence (Woods 1968; Davies and Isard 1972; Johnson-Laird 1977; Woods 1981). For instance, the meaning of "How many US presidents have had a first name starting with the letter 'T'?" might be captured by a program that searches for an answer to this question in a database. This approach has been elaborated in a variety of modern machine learning models, many of which draw on logical tools closely akin to the LOT (e.g Zettlemoyer and Collins 2005; Wong and Mooney 2007; Liang et al. 2010; Kwiatkowski et al. 2010, 2012).

A LOT would explain some of the richness of human thinking by positing a combinatorial capacity through which a small set of built in cognitive operations can be combined to express new concepts. For instance, in the word learning model of (Siskind 1996) the meaning of the word "lift" might be captured as

[1] The LOT's focus on structural rules is consistent with many popular cognitive architectures that use production systems (Anderson et al. 1997, 2004; Newell 1994), although the emphasis on in most LOT models is on the learning and computational level of analysis, not the implementation or architecture.

```
lift(x,y) = CAUSE(x,GO(y,UP))
```

where CAUSE, GO and UP are "primitive" conceptual representations—possibly innate—that are composed to express a new word meaning. This compositionality allows theories to posit relatively little innate content, with the heavy lifting of conceptual development accomplished by combining existing operations *productively* in new ways. To discover what composition is "best" to explain their observed data, learners may engage in hypothesis testing or Bayesian inference (Siskind 1996; Goodman et al. 2008b; Piantadosi et al. 2012; Ullman et al. 2012; Mollica and Piantadosi 2015).

The content that a LOT assumes is distinctly symbolic, can be used compositionally to generate new thoughts, and obeys systematic patterns made explicit in the symbol structures. For example, it would be impossible to think that x was lifted without also thinking that x was caused to go up. Though the psychological tenability of systematicity (Johnson 2004) and compositionality (Clapp 2012) are debated, classically such compositionality, productivity, and systematicity have been argued to be desirable features of cognitive theories (Fodor and Pylyshyn 1988), and lacking in connectionism (for ensuing discussion, see e.g., Smolensky 1988, 1989; Chater and Oaksford 1990; Fodor and McLaughlin 1990; Chalmers 1990; Van Gelder 1990; Aydede 1997; Fodor 1997; Jackendoff 2002; Van Der Velde and De Kamps 2006; Edelman and Intrator 2003).

However, work on the LOT as a psychological theory has progressed despite a serious problem lurking at its foundation: how is it that symbols themselves come to have meaning? It is far from obvious what would make a symbol GO mean go and CAUSE mean cause. This is especially troublesome when we recognize that even ordinary concepts like these are notoriously difficult to formalize, perhaps even lacking definitions (Fodor 1975). Certainly there is nothing inherent in the symbol (the G and the O) itself that give it this meaning; in some cases the symbols don't even *refer* to anything external which could ground their meaning either, as is the case for most function words in language (e.g. "for", "too", "seven"). This problem appears even more pernicious when we consider how what meanings might be to a physical brain. If we look at neural spike trains, for instance, how would we find a meaning like CAUSE?[2]

1.2 Meaning Through Conceptual Role

The framework developed here builds off an approach to meaning in philosophy of mind and language known as *conceptual role semantics* (CRS) (Field 1977; Loar 1982; Block 1987, 1997; Harman 1987; Greenberg and Harman 2005). CRS which holds that mental tokens get their meaning through their relationship with other symbols, operations, and uses, an idea dating back at least to Newell and Simon (1976). There is nothing inherently disjunctive about your mental representation of the operation OR. What distinguishes it from AND is that the two interact differently with other mental tokens, in particular TRUE and FALSE. The idea extends to more ordinary concepts: a

[2] The problem is also faced by some connectionist models. For instance, Rogers and McClelland (2004), a connectionist model of semantics, builds in relations (e.g. IS-A, HAS, CAN) and observable attributes (e.g. PRETTY, FLY, SKIN) as activation patterns on individual nodes. There, the puzzle is: what precisely makes it the case that activation in one node *means* (whatever that means) HAS as opposed to IS-A?

concept of an "accordion" might be inherently about its role in a greater conceptual system, perhaps inseparable from the inferences it licenses about the player, its means of producing sound, its likely origin, etc. An example from Block (1987) is that of learning the system of concepts involved in physics:

> One way to see what the CRS approach comes to is to reflect on how one learned the concepts of elementary physics, or anyway, how I did. When I took my first physics course, I was confronted with quite a bit of new terminology all at once: 'energy', 'momentum', 'acceleration', 'mass', and the like. ... I never learned any definitions of these new terms in terms I already knew. Rather, what I learned was how to use the new terminology—I learned certain relations among the new terms themselves (e.g., the relation between force and mass, neither of which can be defined in old terms), some relations between the new terms and the old terms, and, most importantly, how to generate the right numbers in answers to questions posed in the new terminology.

Indeed, almost everyone would be hard-pressed to *define* a term like "force" in any rigorous way, other than appealing to other terminology like "mass" and "acceleration" (e.g., $f = m \cdot a$). This emphasis on the role of concepts in systems of other concepts leads CRS to be closely related to the Theory Theory in development (see Brigandt 2004) as well work in psychology emphasizing the role of entire systems of knowledge—*theories*—in conceptualization, categorization, and cognitive development (Carey 1985, 2009; Murphy and Medin 1985; Wellman and Gelman 1992; Wisniewski and Medin 1994; Gopnik and Meltzoff 1997; Kemp et al. 2006, 2010; Tenenbaum et al. 2006, 2007; Ullman et al. 2012; Bonawitz et al. 2012; Gopnik and Wellman 2012). Empirical studies of how human learners use systems of knowledge suggest that concepts cannot be studied in isolation—our inferences depend not only on simple perceptual factors, but the way in which our internal systems of knowledge interrelate.

Block (1987) further argues that CRS satisfies several key desiderata for a theory of mental representation, including its ability to handle truth, reference, meaning, compositionality, and the relativity of meaning. Some authors focus on the *inferential role* of concepts, meaning the way in which they can be used to discover new knowledge. For instance, the concept of conjunction AND may be defined by its ability to permit use of the "elimination rule" $P \& Q \rightarrow P$ (Sellars 1963). Here I will use the term CRS in a general way, but my implementation will make the specific meaning unambiguous later. In the version of CRS I describe, concepts will be associated with some, but perhaps not all, of these inferential roles, and some other relations that are less obviously inferential. The view that concepts are defined in terms of their relationships to other concepts has close connections to accounts of meaning given in early theories of intelligence (Newell and Simon 1976), as well as implicit assumptions of computer science. Operations in a computer only come to have meaning in virtue of how they interact with the architecture, memory, and other instructions. For example, nearly all modern computers represent negative numbers with a *two's complement* where a number can be negated by swapping the 1s and 0s and adding 1. For instance, a five-bit processor might represent 5 as 00101 and -5 as $11010 + 1 = 11011$. Then, -5 plus one 00001 is $11011 + 00001 = 11100$, which is the representation for -4. Use

of two-complement is only convention, and equally mathematically correct systems have been considered throughout the history of computer science, including using the first bit to represent sign, "ones complement" (swapping zeros and ones), and analog systems. If we just looked inside of an alien's computer and saw the bit pattern 11010, we could not form a good theory of what it meant unless we also understood how it was treated by operations like negation and addition. The meanings of symbols are inextricable from their use.

A primary shortcoming of conceptual role theories as cognitive accounts is that they lack a computational backbone, leaving vagueness about what a "role" might be. The lack of computational implementation has given rise to a variety of philosophical debates about what is possible for CRS, but as I argue, at least some of these issues become less problematic once we consider a concrete implementation. The lack of implementations also means that it is difficult to make progress on experimental psychology probing the particular representations and processes of a CRS because there are few ideas about what, formally, a role might be. A primary goal of this paper is to give the conceptual role semantics a computational footing by using a version of the LOT a framework where roles can be formalized, learned, and studied explicitly.

1.3 Isomorphism and Representation

Any CRS theory will have to start by saying *which* mental representations we create and why. Here, it will be assumed that the mental representations we construct are likely to correspond to evolutionarily relevant structures, relations, and dynamics present in the real world.[3] This notion of correspondence between mental representations and the world can be captured with the mathematical idea of *isomorphism*. Roughly, systems X and Y are *isomorphic* if operations in X do "the same thing" as the corresponding operations in Y and vice versa. For instance, the ticking of a second hand is isomorphic to the ticking of an hour hand: both take 60 steps and then loop around to the beginning. How one state leads to the next is "the same" even though the details are different since one ticks every second and the other every minute. Scientific theories form isomorphisms in that they attempt to construct formal systems which capture the key dynamics of the system under study. A simple case to have in mind in science is Newton's laws of gravity, where the behavior of a real physical object is captured by constructing an isomorphism into vectors of real numbers, which themselves represent position, velocity, etc. The dynamics of updating these numbers with Newton's equations is "*the same*" as updating the real objects, which is the whole reason why the equations are useful.[4]

The notion that the mind contains structures isomorphic to the world lies at the heart of many theories of mental content (Craik 1952; McNamee and Wolpert 2019; Gallistel 1998; Shepard and Chipman 1970; Hummel and Holyoak 1997). Shepard and Chipman (1970) emphasized that while mental representations need not be structurally similar to what they represent, the relationships between internal representations must

[3] Although this notion is controversial—see Hoffman et al. (2015) and the ensuing commentary.

[4] Curiously, an isomorphism into real number is not the only one possible for physics—it has been argued that physical theories could be stated without numbers at all (Field 2016).

be "parallel" to the relationships between the real world objects. Gallistel (1998) writes,

> A mental representation is a functioning isomorphism between a set of processes in the brain and a behaviorally important aspect of the world. This way of defining a representation is taken directly from the mathematical definition of a representation. To establish a representation in mathematics is to establish an isomorphism (formal correspondence) between two systems of mathematical investigation (for example, between geometry and algebra) that permits one to use one system to establish truths about the other (as in analytic geometry, where algebraic methods are used to prove geometric theorems).

In this case, mental representations could be used to establish truths about the world without having to alter the world. The notion of isomorphism is also deeply connected to the ability of the brain to usefully interact with the world. Conant and Ashby (1970) show that if a system X (e.g. the brain) wishes to control the dynamics of another system Y (e.g. the world), and X does so well (in a precise, information-theoretic sense), then X must have an isomorphism of Y (see Scholten 2010, 2011). This result developed out of cybernetics and control theory and is not well-known in cognitive science and neuroscience. Yet, the authors recognized its relevance, noting that the result "has the interesting corollary that the living brain, so far as it is to be successful and efficient as a regulator for survival, *must* proceed, in learning, by the formation of a model (or models) of its environment."

What is mysterious about the brain, though, is that we are able to encode a staggering array of different isomorphisms—from language, to social reasoning, physics, logical deduction, artistic expression, causal understanding, meta-cognition, etc. Such breadth suggests that our conceptual system can support essentially any computation or construct any isomorphism. Moreover, little of this knowledge could possibly be innate because it is so clearly driven by the right set of experiences. Yet, the question of how systems might encode, process, and learn isomorphisms *in general* has barely been addressed in cognitive science. Indeed, work on the LOT has typically made ad-hoc choices about what primitives should be considered in hypotheses in any given context, thus failing to provide a demonstrably generalized theory of learning that takes the breadth of human cognition seriously. The representational system below develops a *universal framework for isomorphism*, a mental system in which we can construct, in principle, a representation of anything else. Unsurprisingly, the existence of such a formalism is closely connected to the existence of universal computation.

1.4 The General Theory

We are now ready to put together some pieces. The overall setup is illustrated in Fig. 1. We assume that learners observe a structure in the world. In this case, the learner sees the circular structure of the **seasons**, where the successor (`succ`) of spring is summer, the successor of summer is fall, etc. Learners are assumed to have access to this relational information between tokens shown on the left. Their job is to *internalize* (mentally represent) each symbol and relation by mapping symbols

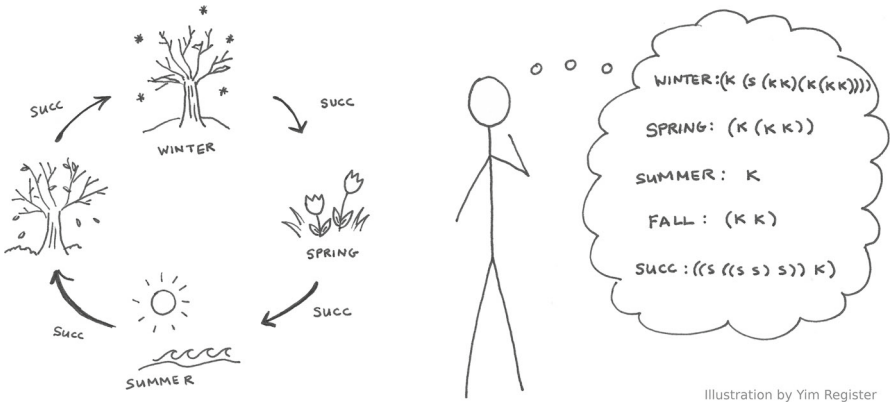


Fig. 1 Learners observe relations in the world, like the `succ`essor relationship between seasons. Their goal is to create an internal mental representation which obeys the same dynamics. This is achieved by mapping each observed token to a simple expression written in a universal mental language whose constructs/concepts specify interactions between elements. This system uses a logic for function compositions that is capable of universal computation

to expressions in their LOT that obey the right relations, as shown on the right. The mapping will effectively construct an internal isomorphism of the observations, written in the language of mentalese.

A little more concretely, the relations in Fig. 1 might be captured with the following facts,

```
(succ winter) → spring
(succ spring) → summer
(succ summer) → fall
(succ fall)   → winter
```

Here, I have written the relations as functions, where for instance the first line means that `succ` is a function applied to `winter`, that returns the value `spring`. To represent these, we must map each of these symbols to a mental LOT expression that obeys the same relational structure. So, if `succ`, `winter`, and `spring` get mapped to mental representations $\psi_{\texttt{succ}}$, $\psi_{\texttt{winter}}$, and $\psi_{\texttt{spring}}$ respectively, then the first fact means that

$$(\psi_{\texttt{succ}}\ \psi_{\texttt{winter}}) \rightarrow \psi_{\texttt{spring}}$$

also holds. Though this statement looks simple, it actually involves some subtlety. It says that whatever internal representation `succ` gets mapped to, this representation must also be able to be used internally as a function. The return value when this mental function is evaluated on $\psi_{\texttt{winter}}$ has to be the same as how `spring` is represented mentally. Each token participates in several roles and must simultaneously yield the correct answer in each, providing a full mental isomorphism of the observed relations.

One might immediately ask why we need anything other than the facts—isn't it enough to know that `(succ spring)` → `winter` in that purely symbolic form? A Prolog program might directly encode the relations (e.g. $\psi_{\texttt{spring}}$ is a symbol "`SPRING`") and look up facts in what is essentially a symbolic database. Such a program could even be able to answer questions like "The successor of which season is spring?" by compiling

these questions into the appropriate database query. Of course, if this worked well, good old fashioned AI would have yielded striking successes. Unfortunately, several limitations of such purely symbolic encoding are clear. First, it is not apparent how looked-up symbols get their meaning, a version of the problem highlighted by Searle (1980)'s Chinese Room. It is not enough to know these symbolic relationships; what matters is the semantic content that they correspond to, and there is no semantic content in a database lookup of the answer. Second, architectures for processing symbols *seem* decidedly unbiological, and the problem of how these symbols may be grounded in a biological or neural system has plagued theories of representation and meaning. Third, in many of the cases I'll consider, what matters is not the symbols themselves, but the computations they correspond to—what they *do* to other symbols. For instance, we might consider a case of a simple algorithm like **repetition**. Your internal concept of repetition must encode the process for repetition; but what mental format could do so? Fourth, our cognitive systems must go beyond memorization of facts—we are able to generalize beyond what we have observed, extracting regularities and abstract rules. What might representations be like such that they can allow us to deduce more than what we've already seen? Each of these four goals—meaning, implementation, computation, and induction—can be met with the logical system described below.

The core hypothesis developed in this paper is that the symbols like `succ` and `winter` get mapped to LOT expressions that are dynamical and computational objects supporting function composition. ψ_{succ} is represented as an object that, when applied via function composition to ψ_{winter}, gives us back the expression for ψ_{spring}. These meanings are specified in a language of pure computational dynamics, absent any additional primitives or meanings. This is shown with the minimalist set of primitives in Fig. 1, where each token is mapped to some structure built of **`s`** and **`k`**, special symbols whose "meanings" are discussed below. In this setup, symbols like `spring` come to have meaning as CRS supposes, by virtue of how the structures they are mapped to act on other symbols. Learners are able to derive *new* facts by applying their internal expressions to each other in novel ways. As I show, this can give rise to rich systems of knowledge that span classes of computations and permit learners to extend a few simple observations into the domain of richer cognitive theories.

2 Combinatory Logic as a Language for Universal Isomorphism

A mathematical system known as *combinatory logic* provides the formal tool we'll use to construct a universal isomorphism language as a hypothesized LOT. Combinatory logic was developed in the early- and mid-1900s in order to allow logicians to work with expressions that did not require variables like "x" and "y", yet had the same expressive power (Hindley and Seldin 1986). Combinatory logic's usefulness is demonstrated by the fact that it was invented at least three independent times by mathematicians, including Moses Schönfinkel, John von Neumann, and Haskell Curry (Cardone and Hindley 2006). The main advantages of combinatory logic are its simplicity (allowing us to posit very minimal built-in machinery) and its power (allowing us to model symbols, structures, and relations). In cognitive research, combinatory logic is primarily seen in formal theories of natural language semantics (Steedman

2001; Jacobson 1999), although its relevance has also been argued in other domains like motor planning (Steedman 2002). The use of combinatory logic as a representational substrate, moreover, fits with the idea that tree-like structures are fundamental to human-like thinking (e.g. Fitch 2014).

The next two sections are central to understanding the formalism used in the remainder of the paper and are therefore presented as a tutorial. This first section will illustrate how combinatory logic can write down a simple function. This illustrates only some of its basic properties, such as its simplicity (involving only two primitives), its ability to handle variables, and its ability to express arbitrary compositions of operations. The more powerful view of combinatory logic comes later, where I describe how we may use combinatory logic to create something essentially new.

2.1 A Very Brief Introduction to Combinatory Logic

To illustrate the basics of combinatory logic, consider the simple function definition,

$$f(x) = x + 1. \tag{1}$$

The challenge with expressions like (1) is that the use of a variable x adds bookkeeping to a computational system because one has to keep track of what variables are allowed where. Compare (1) to a function of two variables $g(x, y) = \ldots$. When we define `f`, we are permitted to use x. When we define g, we are permitted to use both x and y. But when we define `f`, it would be nonsensical to use y, assuming y is not defined elsewhere. Analogously in a programming language—or cognitive/logical theories that look like programs—we can only use variables that are defined in the appropriate context (scope). The syntax of what symbols are allowed changes in different places in a representation—and this creates a nightmare for the bookkeeping required in implementation. In combinatory logic's rival formalism, lambda calculus (Church 1936), most of the formal machinery is spent ensuring that variable names are distinct and only used in the appropriate places, and that substitution does not incorrectly handle variable names.

What logicians discovered was that this situation could be avoided by using *combinators* to glue together the primitive components +, and 1 without ever explicitly creating a variable x (e.g. Schönfinkel 1967). A combinator is a higher-order function (a function whose arguments are functions) that, essentially routes arguments to the correct places. For instance using := to denote a definition, let

```
f := (S + (K 1))
```

define `f` in terms of other functions **S**&**K**, in addition to the operator + and the number 1. Notably there is no x in the above expression for `f`, even though `f` does take an argument, as we will see. The functions **S**&**K** are just symbols, and when they are evaluated, they have very simple definitions:

```
(K x y)   → x
(S x y z)   → ((x z) (y z))
```

Here, the arrow (→) indicates a process of evaluation, or moving one step forward in the computation. The combinator **K** takes two arguments `x` and `y` and ignores `y`, a constant (**K**onstant) function. **S** is a function of three arguments, `x`, `y`, and `z`, that essentially passes `z` to each of `x` and `y` before composing the two results. In other notation, **S** might be written as $\mathbf{S}(x, y, z) := x(z, y(z))$. Note that both **S** and **K** both return expressions which are themselves structured compositions of whatever their arguments happened to be.

In this notation, if a function does not have enough arguments it may take the next one in line. For instance in `((K x) y)` the **K** only has one argument. But, because there is nothing else in the way, it can grab the `y` as its second argument, meaning that computation proceeds,

```
((K x) y) → (K x y) → x
```

This operation must respect the grouping of terms, so that `((K x) (y z))` becomes `(K x (y z))`. The capacity to take the next argument is known in logic as *currying*, although Curry attributed it to Schönfinkel, and it was more likely first invented by Frege (Cardone and Hindley 2006). Together, **S**&**K** and currying define a logical system that is much more powerful than it first appears.

To see how the combinator definition of `f` works, we can apply `f` to an argument. For instance, if we evaluate `f` on the number 7, we get can substitute in the definition of `f` into the expression `(f 7)`:

```
(f 7) =  ((S + (K 1)) 7)      ; Definition of f
→ (S + (K 1) 7)               ; Currying
→ ((+ 7) ((K 1) 7))           ; Definition of S
→ (+ 7 ((K 1) 7))             ; Currying
→ (+ 7 (K 1 7))               ; Currying
→ (+ 7 1)                     ; Definition of K
```

Essentially what has happened is that **S**&**K** have shuttled 7 around to the places where x would have appeared. They have done so merely by their compositional structure and definitions, without ever requiring the variable x in $f(x) = x + 1$ to be explicitly written. Schönfinkel—and other independent discoverers of combinatory logic—proved the non-obvious fact that *any* function composition could be expressed this way, meaning any structure with written variables has an equivalent combinatory logic expression without them.

The process of applying the rules of combinatory logic (shown in the gray box just above) is known as *reduction*. The question of whether a computation halts is equivalent to whether or not reduction leads to a *normal form* in which none of the combinators have enough arguments to continue reduction. In terms of computational power, combinatory logic is equivalent to lambda calculus (see Hindley and Seldin 1986), both of which are capable of expressing any computation through function composition (Turing 1937). This means that any typical program (e.g. in Python or C++) can be written as a composition of these combinators, and the combinators reduce to a normal form if and only if the program halts. Equivalently, any system that implements these very simple rules for **S**&**K** is, potentially, as powerful as any computer. This is a remarkable result in mathematical logic because it means that computation can be expressed with the simplest syntax imaginable, compositions of **S**&**K** with no

extra variables or syntactic operations. Evaluation is equally simple and requires no special machinery beyond the ability to perform **S**&**K**'s simple definitions, which are themselves just transformations of binary trees. It is this uniformity and simplicity of syntax that opens the door for implementation in physical or biological systems.

2.2 Church Encoding

The example above uses primitive operations like + and objects like the number 1. It therefore fits well within the traditional LOT view where mental representations correspond to compositions of intrinsically meaningful primitive functions. The primary point of this paper, however, is to argue that the right metaphor for mental representations is actually not structures like (1) or its combinator version, but rather structures without any cognitive primitives at all—that is, structures that contain only **S**&**K**.

The technique behind this is known as *Church encoding*. The idea is that if symbols and operations are encoded as pure combinator structures, they may act on each other *via the laws of combinatory logic alone* to produce equivalent algorithms to those that act on numbers, boolean operators, trees, or any other formal structure. As Pierce (2002) writes,

> [S]uppose we have a program that does some complicated calculation with numbers to yield a boolean result. If we replace all the numbers and arithmetic operations with [combinator]-terms representing them and evaluate the program, we will get the same result. Thus, in terms of their effects on the overall result of programs, there is no observable difference between the real numbers and their Church-[encoded]numeral representations.

A simple, yet philosophically profound, demonstration is to construct a combinator structure that implements **boolean logic**. One possible way to do this is to define

```
true  := (K K)
false := K
and   := ((S (S (S S))) (K (K K)))
or    := ((S S) (K (K K)))
not   := ((S ((S K) S)) (K K))
```

Defined this way, these combinator structures obey the laws of Boolean logic: `(not true)` → `false`, and `(or true false)` → `true`, etc. The meaning of mental symbols like `true` and `not` is given *entirely* in terms of how these little algorithmic chunks operate on each other. To illustrate, the latter computation would proceed as

```
(or true false) = (((S S) (K (K K)))  (K K)  K)
    ↪; Definition of or, true, false
→ (((S S) (K (K K)) (K K))  K)          ; Currying rule
→ ((S S (K (K K)) (K K)) K)             ; Currying rule twice
→ ((S (K K) ((K (K K)) (K K))) K)       ; Definition of S
→ ((S (K K) (K (K K) (K K))) K)         ; Currying
→ ((S (K K) (K K)) K)                   ; Definition of K
→ (S (K K) (K K) K)                     ; Currying
→ ((K K) K ((K K) K))                   ; Definition of S
→ ((K K K) ((K K) K))                   ; Currying
→ (K ((K K) K))                         ; Definition of K
→ (K (K K K))                           ; Currying
```

```
→ (K K)                                    ; Definition of K
```

resulting in an expression, `(K K)`, which is the same as the definition of true! Readers may also verify other relations, like that `(and true true)` → `true` and `(or (not false) false)` → `true`, etc.

The Church encoding has essentially tricked **S**&**K**'s boring default dynamics into doing something useful—implementing a theory of simple boolean logic. This is a CRS because the symbols have no intrinsic meaning beyond that which is specified by their dynamics and interaction. The meaning of each of these terms is, as in a CRS, critically dependent on the form of the others. The capacity to do this reflects a more general idea in dynamical systems—one which is likely central to understanding how minds represent other systems—which is that sufficiently powerful dynamical systems permit encoding or embedding of other computations (e.g. Sinha and Ditto 1998; Ditto et al. 2008; Lu and Bassett 2018). The ability to use **S**&**K** to perform useful computations is very general, allowing us to encode complex data types, operations, and a huge variety of other logical systems. Appendix A sketches a simple proof of conditions under which combinatory logic is capable of representing *any* consistent set of facts or relations, with a few assumptions, making it essentially a universal isomorphism language.

2.3 An Inferential Theory from the Probabilistic LOT

The capacity to represent anything is, of course, not enough. A cognitive theory must also have the ability to construct the right particular representations when data—perhaps partial data—is observed. The data that we will consider is sets of *base facts* like those shown in Fig. 1, `(succ winter)` → `spring`, etc. These facts may be viewed as structured or relational representations of perceptual observations—for instance, the observation that some season (`spring`) comes after (`succ`) another (`winter`). Note, though, that the meanings of these symbols are not specified by these facts; all we know is that `spring` (whatever that is) comes after (whatever that is) the season `winter` (whatever that is). Apart from any perceptual links, that knowledge is structurally no different from `(father jim)` → `billy`. Because these symbols do not yet have meanings, knowledge of the base facts is much like knowledge of a *placeholder structure* (Carey 2009), or concepts whose meanings yet to be filled in, even though some of their conceptual role is known.

The goal of the learner is to assign each symbol a combinator structure so that the base facts are satisfied.[5] For this one rule `(succ winter)` → `spring` we could assign `succ := (K S)`, `winter := K` and `spring := S` since then

```
(succ winter) := ((K S) K) → (K S K) → S = spring
```

Only some mappings of symbols to strings will be valid. For instance, if `spring := K` instead, we'd have that

[5] The general idea of finding a formal internal representation satisfying observed relations has close connections to model theory (Ebbinghaus and Flum 2005; Libkin 2013), as well as the solution of constraint satisfaction problems specified by logical formulas (*satisfiability modulo theories*) (Davis and Putnam 1960; Nieuwenhuis et al. 2006).

```
(succ winter) := ((K S) K) → (K S K) → S ≠ spring.
```

The real challenge a learner faces in each domain is to find a mapping of symbols to combinators that satisfies all of the facts simultaneously. Such a solution provides an internal model—a Church encoding—whose computational dynamics captures the relations you have observed under repeated reduction via **S**&**K**. Often, the mapping of symbols to combinators will often be required to be *unique*, meaning that we can always tell which symbol a combinator output stands for. In addition, once symbols are mapped to combinators satisfying the observed base facts, learners or reasoners may derive new generalizations that go beyond these facts.

The choice of which combinator each symbol should be mapped to is here made using ideas about smart ways of solving the problem of induction. In particular, our approach is motivated in large part by Solomonoff's theory of inductive inference, where learners observe data and try to find a concise Turing machine that describes the data (Solomonoff 1964a, b). Indeed, human learners prefer to induce hypotheses that have a shorter description length in logic (Feldman 2000, 2003a; Goodman et al. 2008b), with simplicity preferences perhaps a governing principle of cognitive systems (Feldman 2003b; Chater and Vitányi 2003). Simplicity-based preferences have been used to structure the priors in standard LOT models (Goodman et al. 2008b; Katz et al. 2008; Ullman et al. 2012; Piantadosi 2011; Piantadosi et al. 2012; Kemp 2012; Yildirim and Jacobs 2014; Erdogan et al. 2015), and has close connections to the idea of *minimum description lengths* (Grünwald 2007). Since **S**&**K** are equivalent in power to a Turing machine, then finding a concise **S**&**K** expression for a domain corresponds to finding a short program (up to an additive constant, as in Kolmogorov complexity (Li and Vitányi 2008)) that computes its dynamics; finding a **S**&**K** expression that evaluates quickly is tantamount to finding a fast-running program.

One problem with theories based on description length is that they can easily run into computability problems: short programs or logical expressions often do not halt[6] meaning that we may not be able to even evaluate every given logical hypothesis to see if it yields the correct answer, according to the base facts. A solution is to instead base our preferences in part on how quickly each combinator arrives at the correct answer. For instance, we can assign prior to a hypothesis h that is proportional to $(1/2)^{t(h)+l(h)}$ where $t(h)$ is the number of steps it takes h to reduce the base facts to normal form and $l(h)$ is the number of combinators in h (i.e. its description length). Similar time-based priors have been developed in theories of artificial intelligence (Levin 1973, 1984; Schmidhuber 1995, 2002, 2007; Hutter 2005) or as a measure of complexity (Bennett 1995). These priors allow inductive inference to avoid difficulties with the halting problem because, essentially, any finite amount of computation will allow us to *upper*-bound a hypothesis' probability, even if it does not halt. For instance, a machine that has not halted or a combinator that has not finished evaluation in 1000 steps will have a prior probability of at most $(1/2)^{1000}$. Thus, as a computation runs, its probability drops, meaning that long-running expressions can effectively be pruned out of searches without knowing whether they might eventually halt (thus, evaluation of all computations—halting or not—can be dovetailed). This fact can be used in Markov-Chain Monte-Carlo techniques like the Metropolis Hastings algorithm

[6] One simple "non-halting" combinator is (**S** (**S** **K** **K**) (**S** **K** **K**) (**S** (**S** **K** **K**) (**S** **K** **K**))).

to implement Bayesian inference over these spaces by rejecting proposed hypotheses once their probability drops too low (Piantadosi and Jacobs 2016). Here, we search over assignments of symbols in the base facts to combinators and disregard those that are too low probability either in complexity or in evaluation time when run on the given base facts.

Since so much other work has explored the probabilistic details of LOT theories, and I intend to provide a simple demonstration, I'll make two simplifying assumptions in this paper. First, I assume that learners want only the fastest-running combinator string which describes their data, ignoring the gradience of fully Bayesian accounts. Second, it will be assumed that only theories that are consistent with the data are considered. This will therefore assume that leaner's data is noise-free, although the general inferential mechanisms can readily be extended to noisy data (see LOT citations above).

2.4 Details of the Implementation

The problem of finding a concise mapping of the symbols to combinators that obey these laws is solved here using a custom implementation named *churiso* (pronounced like the sausage "chorizo") and available under a GPL license in both Scheme and Python.[7] The implementation was provided with base facts and searched for mappings from symbols to combinators that satisfies those constraints under the combinator dynamics defined above. Among all mappings of symbols to combinators that are consistent with the base facts, those with the fastest running time are preferred.

The implementation uses a backtracking algorithm that exhaustively tries pairing symbols and combinator structures (ordered in terms of increasing description-length complexity), rejecting a partial solution if it is found to violate a constraint. Several optimizations are provided in the implementation. First, the set of combinators considered for each symbol can be limited to those already in normal form to avoid re-searching equivalent combinators. Second, the algorithm uses a simple form of constraint propagation in order to rapidly reject assignments of symbols to combinator strings that would violate a later constraint. For instance, if a constraint says that `(f x)` must reduce to `y`, and `f` and `x` are determined, then the resulting value is pushed as the assignment for `y`. An order for searching is chosen which maximizes the number of constraints which can be propagated in this way. In order to explore the space, Churiso also allows us to define and include other combinators either as base primitives or as structures derived from **S**&**K**. The results in this paper use the search algorithm including several other standard combinators (**B**, **C**, **I**) to increase the search effectiveness, but each is converted to **S**&**K** in evaluating a potential hypothesis. Notably, however, search likely still scales exponentially in the number of symbols in base facts—trying to find how to assign c combinators to n symbols (assuming none can be determined through constraint propagation above) takes c^n search steps.[8] This form of backtracking shares much with, for example, implementations of the Prolog programming language (Bratko 2001, see, e.g.). However, it is important to

[7] https://github.com/piantado/pyChuriso.

[8] However, the general time complexity of this interface is not apparent to me at least.

Table 1 Church encoding inferred from the base facts that permit representation of several logical structures common in psychology

Domain	Facts	Representation
Seasons	(succ winter) → spring (succ spring) → summer (succ summer) → fall (succ fall) → winter	after := ((S ((S K) K)) (K (K (K K)))) fall := K summer := (K K) winter := (K (K (K K))) spring := (K (K K))
1, 2, Many	(succ one) → two (succ two) → many (succ many) → many	succ := ((S ((S S) K)) S) one := (K (K K)) two := K many := (S K)
Roshambo	(rock scissors) → win (rock rock) → draw (scissors scissors) → draw (scissors paper) → win (paper rock) → win (paper paper) → draw (rock paper) → lose (paper scissors) → lose (scissors rock) → lose	win := (K K) draw := K lose := (K (K K)) rock := ((S ((S K) K)) (K (K K))) paper := ((S ((S ((S K) K)) S)) (K (K K) ↪)) scissors := ((S ((S ((S K) K)) K)) (K K) ↪)
Family	(father sasha) → barack (father malia) → barack (mother sasha) → michelle (mother malia) → michelle (sister malia) → sasha (sister sasha) → malia (husband michelle) → barack (wife barack) → michelle	barack := ((S K) S) sasha := K malia := (K K) michelle := S father := (K ((S K) S)) mother := (K S) sister := ((S ((S K) K)) (K K)) husband := (S K) wife := ((S ((S K) K)) (K S))
Boston	downtown = (red braintree) park = (red downtown) alewife = (red park) park = (green riverside) government = (green park) northstation = (green ↪government) downtown = (orange foresthill) state = (orange downtown) northstation = (orange state) oakgrove = (orange ↪northstation) government = (blue bowdoin) state = (blue government) wonderland = (blue state)	downtown := ((S ((S K) K)) (K K)) braintree := (K ((S ((S K) K)) (K K))) park := (S K) alewife := ((S K) S) riverside := (K (S K)) government := ((S K) (K ((S K) K))) northstation := (K ((S K) K)) foresthill := ((S S) (K (K K))) state := K oakgrove := ((S K) K) bowdoin := (K ((S K) (K ((S K) K)))) wonderland := (K K) red := ((S ((S K) K)) (K S)) blue := ((S ((S K) K)) (K K)) orange := ((S ((S K) K)) (K ((S K) K))) green := ((S ((S K) K)) (K (K ((S K) K)) ↪))

note that we do not intend this backtracking search to be an *implementation*-level theory of how people themselves might find these representations. The implementation that people use to do something like Church encoding will depend on the specifics of the representation they possess. Instead, this algorithm is only intended to provide a *computational*-level theory (Marr and Poggio 1976; Marr 1982) that says *if* people choose representations of the base facts that are fast-evaluating and concise, then they will be able to represent and generalize similarly to people. Thus, our primary concern is whether this algorithm finds any solutions, not whether in doing so it searchers the space in a way similar to how biological organisms might.

In a testament to the simplicity and parsimony of combinatory logic, the full implementation requires only a few hundred lines of code, including algorithms for reading the base facts, searching for constraints, and evaluating combinator structures. Ediger (2011) provides an independent combinatory logic implementation that includes several abstraction algorithms and was used to validate implementation of combinatory logic in Churiso.

3 Applications to Cognitive Domains

This section presents a number of examples of using the inferential setup to discover combinator structures for a variety of domains. In each example, I will provide the base facts and then the fastest running combinator structure (Church encoding of the base facts) that was discovered by Churiso. The examples have been chosen to illustrate a variety of different domains that have been emphasized in human and animal psychology. The first section shows that theories can represent or encode relational structures. The second examines conceptual structures that involve *generalization*, meaning that we are primarily interested in how the combinator structure extends to compute new relations not in set of base facts. In each of these cases, the generalization fits simple intuitions about and permits derivation of new knowledge in the form of "correct" predictions about unobserved new structures. The third section look at combinatory logic to represent new *computations* in the form of components that could be used in new mental algorithms, and the fourth looks at formal languages.

3.1 Representation

Table 1 shows five domains with very different structural properties and how they may be represented with **S&K**. The middle column shows the base facts that were provided to Churiso and the right hand column shows the most efficient combinator structure. The **seasons** example show a circular system of labeling, where the successor (`succ`) of each season loops around in a Mod-4 type of system. The **1, 2, many** concept where there is similarly a successor, but the successor of any number above `two` is just the token `many`, a counting system found in many languages of the world. **Roshambo** (also called "Rock, paper, scissors") is an interesting case where each object represents a function that operates on each other object, and returns an outcome in a discrete set ("rock" beats "scissors", etc.). This game has been studied in primate neuroscience (Abe and Lee 2011). This illustrates a type of circular dominance structure, but one which is dependent on which object is considered the operator (e.g. first in the parentheses) and which is the operand. The **family** example shows a case where simple relations like `mother` and `husband` can be defined and are functions that yield the correct individual. Interestingly, realization of just these representations does *not* automatically correspond to representation of a "tree"—instead the requirement to represent only these relations yields a simpler composition of combinators without an explicit tree structure. Later examples (**tree**, **list**) show how trees themselves might be learned; an interesting challenge is to discover a latent tree structure from

examples like in **family** (for work on learning the general case with LOT theories, see (Katz et al. 2008; Mollica and Piantadosi 2015)). Note that in all examples, the combinator structures Churiso discovers shouldn't be intuitively obvious to us—these combinators structures are not the symbols we are used to thinking about (like `father` and `many`), certainly not in conscious awareness. Instead, the base facts should sound obvious; the **S&K** structures are the stuff out of which the symbols in the base facts are made. The **Boston** example shows encoding of a graph structure, a simplified version of Boston's subway map. Here, there are 4 relations `red`, `green`, `orange`, and `blue`, which map some stations to other stations. This structure, too, can be encoded into **S&K**.

One challenge for theories like this may be in learning multiple representations at the same time. It is indeed possible to do so, while enforcing uniqueness constraints among the symbols. To illustrate, for example, if we simultaneously try to encode **seasons**, **roshambo**, and **family** into a single set of facts, where symbols in each must be unique, Churiso finds the solution,

```
after := ((S ((S S) S)) K)
fall := K
summer := (K (K K))
winter := (K K)
spring := (K ((S (K K)) (K (K K))))

succ := ((S ((S S) K)) S)
one := (K (K (S (K K))))
two := (S (K K))
many := (K ((S (S (K K))) (S (K K))))

barack := S
father := (K S)
sasha := ((S K) S)
malia := (K ((S K) S))
michelle := ((S S) S)
mother := (K ((S S) S))
sister := ((S ((S K) S)) K)
husband := ((S (K (K S))) S)
wife := (S S)
```

This illustrates that while managing the complexity of multiple concepts may, in some situations, be tractable even with our simple search methods.

3.2 Generalization

The examples in Table 1 mainly shows how learners might memorize facts and relations using **S&K**. But equally or more important to cognitive systems is *generalization*: given some data, can we learn a representation whose properties allow us to infer new and unseen facts? What this means in the context of Churiso is that we are able to form new combinations of functions—those whose outcome is not specified in the base facts. Table 2 shows some examples. The first of these is a representation of a **singular/plural** system like those found in natural language. Here, there is a relation `marker` that takes some number of arguments `item`, `item`, etc. and returns `singular` if it receives one argument, `plural` if it receives more than one. This requires generalization

Table 2 **S&K** structures in domains involving interesting generalization, where the combinator structures allow deduction beyond the base facts

Domain	Facts	Representation

Singular/Plural	`(marker item) → singular`
	`(marker item item) → plural`
	`(marker item item item) → plural`
		`singular := (S (S K))`
		`marker := (K (S (S K)))`
		`item := ((S (S K)) (S (S K)))`
		`plural := ((S (S K)) ((S (S K)) (S (S K))))`

Number ($\mathbb{Z}$)	`(succ one) → two`
	`(succ two) → three`
	`(succ three) → four`
		`succ := S`
		`one := S`
		`two := (S S)`
		`three := (S (S S))`
		`four := (S (S (S S)))`

Dominance ($a \succ b \succ c \succ d$)	`True := (K K)`
	`(dom a b) → True`
	`(dom a c) → True`
	`; No information a,d relation`
	`(dom b c) → True`
	`(dom b d) → True`
	`(dom c d) → True`
	`(dom b a) ↛ True`
	`(dom c a) ↛ True`
	`(dom c b) ↛ True`
	`(dom d b) ↛ True`
	`(dom d c) ↛ True`
	`(dom b a) ↛ True`
	`(dom c b) ↛ True`
	`(dom d c) ↛ True`
		`dom := (S ((S S) S))`
		`a := (K (K K))`
		`b := ((S (S K)) K)`
		`c := K`
		`d := (K (K (K K)))`

Magnetism	`(attract p1 p2) ↛ True`
	`(attract p2 p1) ↛ True`
	`(attract p1 n1) → True`
	`(attract p1 n2) → True`
	`(attract p2 n1) → True`
	`(attract p2 n2) → True`
	`(attract n1 n2) ↛ True`
	`(attract n2 n1) ↛ True`
	`(attract n1 p1) → True`
	`(attract n1 p2) → True`
	`(attract n2 p1) → True`
	`(attract n2 p2) → True`
	`; and one single example`
	`(attract n1 x) → True`
	`True := (K K) ; fixed by design`
		`attract := ((S S) (S K))`
		`p1 := K`
		`p2 := K`
		`n1 := (K K)`
		`n2 := (K K)`
		`x1 := K`

because the base facts only show the output for up to four items. However, the learned representation is enough to go further to any number of items, outside of the base facts. For instance,

```
(marker item item item item item item item) → ((S (S K)) (S (S K))) = plural
```

Intuitively, the first time `marker` is applied to `item`, we get

```
(marker item) → ((S (K (S K K))) (S (S K))) = singular
```

When this is applied to another `item`, you get the expression for `plural`:

```
(marker item item) = ((marker item) item) → (singular item) → plural
```

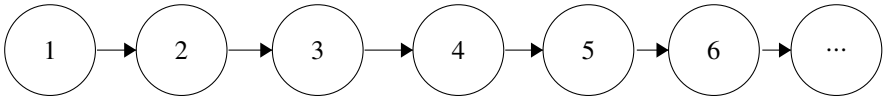
And then `plural` has the property that it returns itself when given one more item:

```
(plural item) → plural
```

So, `plural` is a "fixed point" for further applications of `item`, allowing it to generalize to any number of arguments. In other words, what Churiso discovers is a system that is functionally equivalent to a simple finite-state machine:

Note that this finite state machine is not an explicitly-specified hypothesis that learners have, but only emerges implicitly through the appropriate composition of **S**&**K**.

The next domain, **number**, shows a generalization that builds an infinite structure. Intuitively, the learner is given a `succ`essor relationship between the first few words. The critical question is whether this is enough to learn a **S**&**K** structure for `succ` that will continue to generalize *beyond* the meaning of `four`. The results show that it is: the form of `succ` that is learned is essentially one that builds a "larger" representation for each successive number. The way this works is extremely simple: the successor representation is a combinator that needs more arguments, and so nothing ever reduces. This means that each additional call of the successor function builds a larger structure, and therefore this structure is distinct from those that came before, generalizing infinitely. In this case, **S**&**K** create from the three base facts a structure isomorphic to the natural numbers,



The assumed base facts correspond to the kind of evidence that might be available to learners of counting (Carey 2009). This provides a theory related to Piantadosi et al. (2012)'s LOT model of number acquisition, which was created to solve the inductive riddle posed by Rips et al. (2006, 2008a, b) about what might constrain children's generalization in learning number. The difference is that Piantadosi et al. (2012)'s LOT representations were based in primitive cognitive content, like an ability to represent sets and perform operations on them. Here, the learning is not a counting algorithm, but rather an internal conceptual structure that is generative of the concepts themselves, providing a possible answer to the question of where the structure itself may come from (see Rips et al. 2013). It is interesting to contrast **number**, **1 2 many**, and **seasons**. In each of these, there is a "successor" function, but which function is learned depends on the structure of the base facts. This means that notions like "successorship" cannot be defined narrowly by the relationship between a few elements, but will critically rely on the role this concept plays in a larger collection of operations.

The **dominance** concept in Table 2 shows another interesting case of generalization. Dominance structures are common in animal cognition (see, e.g., Drews 1993). For instance, Grosenick et al. (2007) show that fish can make transitive inferences consistent with a dominance hierarchy: if they observe $a \succ b$ and $b \succ c$, then they know that $a \succ c$, where $\succ$ is a relation specifying who dominates in a pairwise interaction.

The base facts for the **dominance** encode slightly more information, corresponding to almost all of the dominance relationships between some mental tokens `a`, `b`, `c`, and `d`.[9] This example illustrates another feature of Churiso: we are able to specify constraints in terms of evaluations *not* yielding some relations. So for instance,

```
(dom b a) ↛ True
```

means that `(dom b a)` evaluates to something other than `True`. This relaxation of the constraints to only partially-specified often helps to learn more concise representations in **S**&**K**. Critically the relation between `a` and `d` is not specified in the base facts. Note that this relation *could* be anything and any possible value could be encoded by **S**&**K**. The simplicity bias of the inference, however, prefers combinator structures for these symbols such that the *unseen* relation `(dom a d)` → `True` but `(dom d a)` does not. Thus, the **S**&**K** encoding of the base facts gives learners an internal representation that automatically generalizes to an unseen relation.

The **magnetism** example is motivated by Ullman et al. (2012)'s model studying the learning of entire systems of knowledge (*theories*) in conceptual development. In magnetism, we know about different kinds of materials (positive, negative, non-magnetic) and that these follow some simple relationships, like that positives attract negatives and that positives repel each other, etc. The **magnetism** example provides base facts giving the pairwise interaction of two positive two positives (`p1`, `p2`) and two negatives (`n1`, `n2`). But from the point of view of the inference, these four symbols are not categorized into "positives" and "negatives", they are just arbitrary symbols. In this example, I have also dropped the uniqueness requirement to allow grouping of these symbols into "types", as shown by their learned combinator structures with the `pi` getting mapped to the same structure and the `ni` getting mapped to a different one. To test generalization, we can provide the model with one single additional fact, that `n1` and `x` attract each other. The model automatically infers that `x` has the same **S**&**K** structure as `p1` and `p2`, meaning that it learns from a single observation of it is a "positive", including all of the associated predictions such as that `(attract n1 x)` → `True`.

[9] Intuitively, more data is needed than in the simple Fish transitive inference cases because the **S**&**K** model does not inherently know it is dealing with a dominance hierarchy. Cases of dominance hierarchies in animal cognition may have a better chance of being innate, or at least higher prior than other alternatives.

Table 3 **S&K** structures that implement computational operations

Domain	Facts	Representation
Reversal	`(reverse x y) → (y x)`	`reverse := ((S (K (S ((S K) K)))) K)`

If-else	`True := (K K)`
	`False := K`
	`(ifelse True x y) → x`
	`(ifelse False x y) → y`
		`ifelse := ((S ((S K) K)) (S K))`

Identity	`(identity x) → x`	`identity := (S K K)`
Repetition	`(repeat f x) → (f (f x))`	`repeat := ((S ((S (K (S S))) K)) K)`
Recursion	`(Y f) --→ (f (Y f))`	`Y := ((S (K ((S ((S (K ((S (S S)) S))) ( ↪S K))) S))) K)`
Mutual recursion	`(Y* f g) --→ (f (g (Y* f g)))`	`Y* := ((S (K ((S S) ((S S) ((S S) ((S S) ↪ ((S S) S)))))))) ((S (K S)) K))`
Z-combinator	`(Z f g) --→ (f (Z f) g)))`	`Z := ((S (K ((S S) S))) ((S (S S)) ((S ↪(K K)) S)))`
Apply	`(apply f x ) → (f x)`	`apply = ((S K) K)`

Tree, List	`(first (pair x y)) → x`
	`(rest (pair x y)) → y`
		`first := ((((S ((S (K ((S (K S)) K))) S) ↪) (K K)) ((S K) K)) K)`
		`rest := ((S ((S S) (S K))) (K (K K)))`
		`pair := ((((S (K S)) K) ((S ((S (K ((S ( ↪K S)) K))) S)) (K K))) (((S ((S (K ↪ ((S (K S)) K))) S)) (K K)) (((S S ↪) (S K)) K)))`

3.3 Computational Process

The examples above respectively show computation and generalization, but they do not yet illustrate one of the most remarkable properties of thinking—we appear able to discover a wide variety of computational *processes*. The concepts in Table 3 are ones that implement some simple and intuitive algorithmic components. Here, I have introduced some new symbols to the base facts, f, x, and y. These are treated as universally quantified variables, meaning that the constraint must hold for *all* values (combinator expressions) they can take. The learning model's discovery of how to encode these facts corresponds to the creation of fundamental algorithmic representations using only the facts' simple description of what the algorithm must do.

An example is provided by **if-else**. A convenient feature of many computational systems it that when they reach a conditional branch ("if" statement), they only have to evaluate the corresponding branch of the program. The shown base facts make `if-else` return `x` if it receives a true first argument and `y` otherwise, regardless of what `x` and `y` happen to be. Even though conditional branching is a basic computation, it can be learned from even more primitive components **S**&**K**.

The **identity** example illustrates the distinction between implicit and explicit knowledge in the system. We can define `identity := (S K K)` so,

```
(identity x) = ((S K K) x) → (S K K x) → ((K x) (K x)) →
    ↪(K x (K x)) → x.
```

It may be surprising that we could construct a cognitive system without any notion of identity. Surely to even perform a computation on a representation `x`, the identity of `x`

must be respected! In **S&K**, this is true in one sense: the default dynamics of **S** and **K** do respect representational identity. But in another sense, such a system comes with no "built in" cognitive representation of a function which is the identity function. Instead, it can be learned.

A more complex example can be found in the example of **repetition**. Here, we seek a function `repeat` that takes two arguments f and x and calls f twice on x. Humans clearly have cognitive representations of a concept like *repeat*ing a computation; "again" is an early-learned word, and the general concept of repetition is often marked morphologically in the world's languages with reduplication. As is suggested by the preceding examples, the concept of repetition need not be assumed by a computational system.

Related to repetition, or doing an operation "again" is the ability to represent **recursion**, a computational ability that has been hypothesized to be the key defining characteristic of human cognition (Hauser et al. 2002) (see, e.g., Tabor (2011) for a study of recursion in neural networks). One example of how to implement recursion in combinatory logic is the *Y-combinator*,

```
Y = (S (K (S I I)) (S (S (K S) K) (K (S I I)))),
```

a function famous enough in mathematical logic to have been the target of at least one logician's arm tattoo. Like other concepts, the Y-combinator can be built only from **S&K**. It works by "making" a function recursive, passing the function to itself as an argument. The details of this clever mechanism are beyond the scope of this paper (see Pierce 2002). One challenge in learning `Y` is that by definition it has no normal form when applied to a function. To address this, we introduce a new kind of constraint `-->`, which holds true if the *partial* evaluation trace of the left and right hand sides yield expressions that are equal to a given fixed constant depth. To learn recursion, we require that applying `Y` to `f` is the same as applying `f` to this expression itself,

```
(Y f)   -->   (f (Y f))
```

Neither side reduces to a normal form, but the special arrow means that when we run both sides, we get out the same structure, which in this case happens to be the (infinite) recursion of f composed with itself,

```
(f (f (f (f ...))))
```

Churiso learns a slightly longer form than the typical Y-combinator due to the details of its search procedure (for the most concise recursive combinator possible, see Tromp 2007) (Note that in these searches, the runtime is ignored since the combinator does not stop evaluation). The ability to represent `Y` permits us to capture algorithms, some of which may never halt. For instance, if we apply `Y` top the definition of successor from the **number** example, we get back the concept of that counts forever, continuously adding one to its its result: `(Y succ)`. The ability to learn recursion as a computation from a simple constraint might be surprising to programmers and cognitive scientists alike, for whom recursion may seem like an aspect of computation that has to be "built in" explicitly. It need not be so, if what we mean by learning "recursion" is coming up with a Church encoding implementation of it.

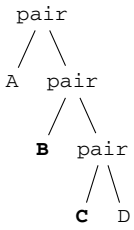
The **mutual recursion** case shows a recursive operator of two functions, known as the `Y*`-combinator, that yields an infinite alternating composition of f and g,

```
(f (g (f (g (f (g ...))))))
```

This is the analog of the Y-combinator but for mutually recursive functions–where f is defined in terms of g and g is defined in terms of f. This illustrates that even more sophisticated kinds of algorithmic processes can be discovered and implemented in **S**&**K**.

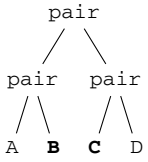
From surprisingly small base facts, Churiso is also able to discover primitives `first`, `rest`, and `pair`, corresponding to the structure-building operations with memory. The arguments for `pair` are "remembered" by the combinator structure until they are later accessed by either `first` or `rest`. As a result, they can build common data structures. For instance, a list may be constructed by combining `pair`:

```
L = (pair A (pair B (pair C D)))
```



Or, a binary tree may be encoded,

```
L = (pair (pair A B) (pair C D))
```



An element such as **C** may then be accessed `(first (rest T))`, the first element of the second grouping in the tree. These data structures are so foundational that they form the foundational built-in data type in programming languages like Scheme and Lisp (where they are called *car*, *cdr*, and *cons* for historical reasons), and thus support a huge variety of other data structures and algorithms (Abelson and Sussman 1996; Okasaki 1999). By showing how speakers might internalize these concepts, we can therefore demonstrate in principle how many algorithms and data structures could be represented as well.

3.4 Formal Languages

One especially interesting case to consider is how **S**&**K** handles concepts that correspond to (potentially) infinite sets of strings, or *formal languages*. Theoretical distinctions between classes of formal languages form the basis of computational theories of human language (e.g. Chomsky 1956, 1957) as well as computation itself (see Hopcroft et al. 1979). To implement each, Table 4 provides base facts giving the transitions between computational states for processing languages. The **regular**

Table 4 Church encoding of several formal language constructs

Language	Facts	Representation
Regular $((ab)^n)$	(a start) → state_a (b state_a) → accept (a accept) → state_a (b accept) → reject (a reject) → reject (b reject) → reject	start := ((S (K K)) (K K)) state_a := (K (K K)) accept := K reject := (K K) b := ((S ((S S) ((S K) K))) K) a := ((S (S (K K))) ((S K) K))
Context-free $(a^n b^n)$	(a start) → got_a (b got_a) → accept (a got_a) → got_aa (b got_aa) → want_b (b want_b) → accept (a got_aa) → got_aaa (b got_aaa) → want_bb (b want_bb) → want_b (a got_aaa) → got_aaaa (b got_aaaa) → want_bbb (b want_bbb) → want_bb	start := S a := K b := ((S (S S)) (K (S K)))
Existential $(\exists z \ldots)$	(start True) → accept (start False) → reject (reject True) → accept (reject False) → reject (accept True) → accept (accept False) → accept	start := ((S ((S K) K)) (K S)) accept := ((S ((S K) K)) (K K)) reject := ((S ((S K) K)) (K S)) True := (K ((S ((S K) K)) (K K))) False := ((S (K (S ((S K) K)))) K)
Finite $\{\text{a}, \text{man}, \text{am}, \text{an}, \text{mam}\}$	$\mathbb{S}$:= {_a, man, _am, _an, mam} For all s in $\{\_, m, a, n\}^3$ (check s) → valid if $s \in \mathbb{S}$ (check s) ↛ valid otherwise	valid := ((S ((S (K ((S (K S)) K))) S)) ↪(K K)) check := ((((S ((S (K ((S (K S)) K))) S) ↪) (K K)) ((S (K S)) K)) S) m := ((((S ((S (K ((S (K S)) K))) S)) (K ↪ K)) (((S S) (S K)) K)) K) a := (K ((((S ((S (K ((S (K S)) K))) S)) ↪ (K K)) ((S ((S (K ((S (K S)) K))) ↪ S)) (K K))) K)) n := (((S ((S (K ((S (K S)) K))) S)) (K ↪K)) (((S ((S (K ((S (K S)) K))) S) ↪) (K K)) (((S ((S (K ((S (K S)) K) ↪)) S)) (K K)) K))) _ := (((S (K S)) K) ((((S ((S (K ((S (K ↪S)) K))) S)) (K K)) ((S ((S (K ((S ↪ (K S)) K))) S)) (K K))) K))

language provides the transition table for a simple finite-state machine that recognizes the language $\{ab, abab, ababa, \ldots\}$. The **existential** one also describes a finite state machine that can implement first-order quantification, an association potentially useful in natural language semantics (van Benthem 1984; Mostowski 1998; Tiede 1999; Costa Florêncio 2002; Gierasimczuk 2007).

The most interesting example is provided by **context-free**, which is a language $\{ab, aabb, aaabbb, \ldots\}$ that provably cannot be expressed with a regular language (finite-state machine). Instead, the learned mapping essentially implements a computational device with an infinite number of states from the base facts. For instance, the state after observing 1, 2, and 3 `a`s are,

```
got_a    := ((S (K K)) S)
got_aa   := ((S (K K)) (S (K K) S))
got_aaa  := ((S (K K)) (S (K K) (S (K K) S)))
got_aaaa := ((S (K K)) (S (K K) (S (K K) (S (K K) S))))
```

Each additional `a` adds to this structure. Then, each incoming `b` removes from it

```
(b got_aaaa) = want_bbb = (K (K (K (K (S S)))))
(b got_aaa)  = want_bb  = (K (K (K (S S))))
(b got_aa)   = want_b   = (K (K (S S)))
(b want_b)   = accept   = (K (S S))
```

This works precisely like a *stack* in a parser, even though such a stack is not explicitly encoded into **S&K** or the base facts. Thus, this mapping generalizes infinitely to strings of arbitrary length, far beyond the input base facts' length of four (Note that the base facts and combinators ensure correct recognition, but do not guarantee correct rejection).

Finally, the **finite** example shows an encoding of the set of strings of the letters "m", "a", "n" and space ("_") that form valid English words, $\{a, man, am, an, mam\}$. These can be encoded by assigning each character a combinator structure, but the resulting structures are quite complex. Note, too, that these base facts do not guarantee correct generalization to longer character sequences. This example illustrates that while Church encoding can represent such information, it is unwieldy for rote memorization. Church encoding is more likely to be useful for algorithmic processes and conceptual systems with compressible patterns. Memorized facts (or sets) may instead rely on specialized systems of memory representation.

The ability to represent formal languages like these is important because they correspond to provably different levels of computational power, showing that a single system for learning and representation across these levels is a defining strength of this approach. For LOT work along these lines, see Yang and Piantadosi (in prep); for language learning on Turing-complete spaces in general, see (Chater and Vitányi 2007; Hsu and Chater 2010; Hsu et al. 2011). In the examples, we have taught Churiso the full algorithm by showing it a few steps from which it generalizes the appropriate algorithm. This ability demonstrates the induction of a novel dynamical system from a few simple observations, work in many ways reminiscent of encoding structure in continuous dynamical systems (Tabor et al. 1997; Tabor 2009, 2011; Lu and Bassett 2018).

3.5 Summary of Computational Results

The results of this section have shown that learners can *in principle* start with only representations of **S&K** and construct much richer types of knowledge. Not only can they represent structured knowledge, by doing so they permit derivation of fundamentally new knowledge and types of information processing. The ability of a simple search algorithm to actually discover these kinds of representations shows that the resulting representational and inductive system can "really work" on a wide variety of domains. However, the main contribution of this work are the general lessons that we can extract from considering systems like **S&K**.

4 Mental Representations are *like* Combinatory Logic (LCL)

My intention is not to claim that combinatory logic is *the* solution to mental representation—it would be pretty lucky if logicians of the early 19th century happened to hit on the right theory of how a biological system works. Rather, I see combinatory logic as a metaphor with some of the right properties. I will refer to the more general form of the theory as **like Combinatory-Logic**, or LCL, and describe some of its core components.

4.1 LCL Theories have no Cognitive Primitives

The primitives used in LCL theories (like $\mathbf{s}\&\mathbf{k}$) specify only the dynamical properties of a representation—how each structure interacts with any other. LCL therefore has no built-in *cognitive* representations, or symbols like `CAUSE` and `True`. This is the primary difference between LCL and LOT theories, whose bread and butter is meaningful components with intrinsic meaning. The lack of these operations is beneficial because LCL therefore leaves no lingering questions about how mental tokens may come to have meaning. The challenge for LCL is then to eventually specify how a token such as `CAUSE` comes to have its meaning by formalizing the necessary and sufficient relations to other concepts that fully characterize its semantics.

4.2 LCL Theories are Turing-Complete

Though it is not widely appreciated in cognitive science or philosophy of mind, humans excel at learning, internalizing, creating, and communicating algorithmic processes of incredible complexity (Rule et al. 2020). This is most apparent in domains of expertise—a good car mechanic or numerical analyst has a remarkable level of technical and computational knowledge, including not only domain-specific facts, but knowledge of specific algorithms, processes, causal pathways, and causal interventions. The developmental mystery is how humans start with what a baby knows and build the complex algorithms and representations that adults understand. The power of LCL systems come from starting with a small basis of computational elements that have the capacity to express arbitrary computations, and applying a powerful learning theory that can operate on such spaces.

Though combinatory logic does not differ from Turing machines in terms of computational power, it does differ in terms of qualitative character. There are several key differences that make combinatory logic a better architecture for thinking of cognition, even beyond its simplicity and uniformity. First, it is a formalism for computation which is entirely compositional, motivated here by the compositional nature of language and other human abilities. Turing machines are simply an architecture that Turing thought of in trying to formulate a theory of "effective procedures" and it does not seem particularly natural to connect biological neurons to Turing machines, efforts to do so in artificial neural network not withstanding (Graves et al. 2014; Trask et al. 2018).

4.3 LCL Theories are Compositional

The compositionality of natural language and natural thinking indicates that mental representations must themselves support composition (Fodor 1975). Semantic formalisms in language (e.g Montague 1973; Heim and Kratzer 1998; Steedman 2001; Blackburn and Bos 2005) rely centrally on compositionality, dating back to Frege (1892). It is no accident that these theories formalize meaning through function composition, using a system (λ-*calculus*) that is formally equivalent to combinatory logic. The inherently compositional architecture of LCL contrasts with Turing machines and von Neumann architectures, which have been dominant conceptual frameworks primarily because they are easy for us to conceptualize and implement in hardware. When we consider the apparent compositionality of thought, *a computational formalism based in composition* becomes a more plausible starting point.

4.4 LCL Theories are Structured

As with compositionality, the structure apparent in human language and thought seems to motivate a representational theory that incorporates structure. In **S**&**K**, for instance, `(S (K S))` is a different representation than `((S K) S)`, even though they are composed of the same elements in the same order. Structure-sensitivity is also a central feature of thought since thoughts can be composed of the same elements but differ in meaning due to their compositional structure (e.g. "Bill loves Mary" compared to "Mary loves Bill"). LCL's emphasis on structural isomorphism aligns it closely with the literature on *structure mapping* (Gentner 1983; Falkenhainer et al. 1986; Gentner and Markman 1997; French 2002), where the key operation is the construction of a correspondence between two otherwise separate systems. For instance, we might consider an alignment between the solar system and the Bohr model of the atom, where the sun corresponds to the nucleus and the planets to electrons. The correspondence is relation preserving in that a relation like `orbits(planets,sun)` holds true when its arguments are mapped into the domain of atoms, `orbits(electrons,nucleus)`. What structure mapping literature does not emphasize, however, is that the systems being aligned are sometimes dynamic and computational, rather than purely structural (isomorphism in dynamical systems theory is a mapping of dynamics).

LCL also shares much motivation and machinery with the literature on *structure learning*, which has aimed to explain how learners might discover latent structured representations to guide further inference and learning. For instance, Kemp and Tenenbaum (2008) show how learners could use statistical inference to discover the appropriate mental representation in a *universal* graph grammar capable of generating any structure. They show how learners could discover representations appropriate to many sub-domains such as phylogenetic trees for animal features, or the left-right spectrum seen in supreme court judgments. A limitation of that work is that it focuses on learning graph structures, not computational objects that can capture internal processes and algorithms.

Table 5 Several levels of abstraction for a function `f` and their corresponding combinator structures

Function	Equivalent combinatory logic structure
`(f)` → `(+ 1 4)`	`f := (+ 1 4)`
`(f x)` → `(+ x 1)`	`f := (S + (K 1))`
`(f x y)` → `(+ x y)`	`f := +`
`(f op x y)` → `(op x y)`	`f := (S K K)`

The combinator structures have no explicit variables (e.g. x, y, op). Note that if the constants or primitives `+`, `1`, and `4` were defined with a Church encoding, they too would be combinators, permitting us to translate everything into pure **S**&**K**

4.5 LCL Theories Handle Abstraction and Variables

Combinatory logic was created as a system to allow *abstraction* with a simple, uniform syntax that avoids difficulties with handling variables. Abstraction has long been a focus in cognitive science. For example, Marcus (2003) considers training data like "A rose is a rose", "A frog is a frog", "A blicket is a ___ ?" The intuition is that "blicket" is a natural response, even though we do not know what a blicket is. This means that we must have some system capable of remembering the symbol in the first slot of "A ___ is a ___" and filling it in the second slot; this problem more generally faces systems tasked with understanding and processing language (Jackendoff 2002). It may be natural to think of this problem as requiring variables. "An X is an X" might be a template pattern that learners induce from examples and then apply to the novel X="blicket". This makes intuitive sense when we talk on a high level, but variable binding is not as transparently solved by neural systems. Sub-symbolic approaches have explored a variety of architectural solutions to variable binding problems (Hadley 2009), including those based on temporal synchrony (Shastri et al. 1996), tensor products (Smolensky 1990; Smolensky and Legendre 2006), neural blackboard architectures (Van Der Velde and De Kamps 2006), and vector symbolic architectures (Gayler 2004, 2006). Marcus (2003) argues explicitly for variables in the sense of symbolic programming languages like Lisp; some work in the probabilistic LOT has explored how symbolic architectures might handle variables (Overlan et al. 2016) and how abstraction helps inductive inference (Goodman et al. 2011).

Unfortunately, the debate about the existence of variables has been completely misled by the notation that happens to be used in computer science and algebra. In fact, combinatory logic shows that systems may behave as though they have variables when in fact none are explicitly represented—this is precisely why it was invented. To illustrate this, Table 5 shows various levels of abstraction for a simple function f, none of which involve variables when expressed with **S**&**K**. The top row is a function of no arguments that always computes $1 + 4$. The next rows shows a function of one variable, x; the third adds its two arguments; the fourth row shows a highly abstract function that applies an operation (perhaps $+$) to its two arguments x and y. In none of these abstract functions do the arguments appear explicitly, meaning that abstraction can be captured without variables. This shows that when an appropriate formalism is used, cognitive theories need not explicitly represent variables, even though they may be there implicitly by how other operators act on symbols. This type of system is

especially convenient for thinking about how we might implement logic in artificial or biological neural networks because it means that the syntax of the representation does not need to change to accommodate new variables in an expression.

4.6 LCL Theories Permit Construction of Systems of Knowledge

It is absolutely central to LCL systems that the meaning of a representation can only be defined by the role it plays in an interconnected system of knowledge. There is no sense in which any of the combinator structures mean anything in isolation. Even for a single domain like **number**, the most efficient mapping to combinators will depend on on which operations must be easy and efficient (for comparison of encoding schemes, see Koopman et al. 2014). The idea that learners must create entire frameworks for understanding, or *theories*, comes from cognitive and developmental literature emphasizing the way in which concepts and internal mental representations relate, to create systems of knowledge (Carey 1985, 2009; Murphy and Medin 1985; Wellman and Gelman 1992; Gopnik and Meltzoff 1997; Ullman et al. 2012). Of course, these relationships must include a variety of levels of abstraction—specific representations, computational processes, abstract rules, new symbols and concepts, etc. LCL permits this by providing a uniform language for all the components that might comprise a theory. If this aspect of LCL is correct, that might help to explain why cognitive science—and its theories of conceptual representation in particular—seem so hard to figure out. Definitions, prototypes, associations, or simple logical expressions seem like they would license fairly straightforward investigation by experimental psychology. But if concepts are intrinsically linked to others by conceptual and inferential role, then it may not be easy or possible to study much in controlled isolation.

4.7 LCL Theories Come from a Simple Basis

Turing machines are simple when compared to modern microprocessors but they are not simple when compared to combinatory logic. A Turing machine has separate mechanisms for its state, memory, and update rules. Combinatory logic has only a few functions that always perform the same operation on a binary branching tree. Indeed, the combinators **S**&**K** are not even minimal. There exist *single* combinators from which **S**&**K** can be derived. Single-point systems have been studied primarily as curiosities in logic or computer science, or as objective ways to measure complexity (Stay 2005), but they suggest that the full complexity of human-like cognition may not be architecturally or genetically difficult to create.

The simplicity of **S**&**K** means that it is not implausible to imagine these operations emerging over evolutionary time. The idea of a neural implementation of a Turing machine might face a problem of irreducible complexity where the control structure of a Turing machine might be useless without the memory and state, and vice versa. However, combinators use only a single operation (function application) with a simple structure (binary branching structure), either of which may be useful on its own in cognitive systems evolved for prediction and representation or motor control (Steedman 2001).

4.8 LCL Theories are Dynamical

A popular view is that cognitive systems are best viewed not as symbolic, but rather dynamical (Van Gelder 1995, 1998; Beer 2000; Tabor 2009). It's always a curious claim because by definition, virtually everything is a dynamical system, even Turing machines. LCL theories are inherently dynamical since the meaning of abstract symbols comes from the underlying dynamics of combinator evaluation, executed mindlessly by the underlying machine architecture. In this way, LCL theories are very much in line with the idea of unifying symbolic and dynamical approaches to cognition (Dale and Spivey 2005; Edelman 2008a), and seeking a pluralistic approach to cognitive science (Edelman 2008b). Edelman (2008a), for instance, argues that we should be beware both the "older exclusively symbolic approach in the style of the 'language of thought' (Fodor 1975) and of the more recent 'dynamical systems' one (Port and van Gelder 1995) that aims to displace it, insofar as they strive for explanatory exclusivity while ignoring the multiple levels at which explanation is needed."

The emphasis on dynamics here emphasizes a key difference to traditional LOT theories. In most incarnations of the LOT the key part of having a concept would be building the structure (e.g. $CAUSE(x, GO(x, UP))$) and little attention is paid to the system by which this represents a computation that actually *runs*—what is the architecture of the evaluator, how does it run, and what makes those symbols mean what they do? In LCL theories, the important part of the representation is not only building the structure, but being able to run it or evaluate it with respect to other concepts. This, in some sense, puts the computational process itself back into "computational theory of mind"—we should not discuss symbols without discussing their computational roles, and in doing so we move closer to implementation.

For LCL, the important part of the dynamics is that it can be manipulated into capturing the relations present in any other system. Unlike most dynamical accounts in cognition, the dynamics are discrete in time and space (Jaeger 1999; Dale and Spivey 2005); research on discrete space systems is a subfield of dynamical systems research in itself (Lind and Marcus 1995) and is likely to hold many clues for how symbolic or symbolic-like processes may emerge out of underlying physics and biology. The general idea, for instance, of discretizing physical systems in order to provide adequate explanations lies at the heart of symbolic dynamics, including mathematical characterizations of general complex systems (Shalizi and Crutchfield 2001; Spivey 2008), as well as recent attempts to encode cognitive structures into dynamical systems (Lu and Bassett 2018). Theoretical work has also explored the tradeoff between symbolic and continuous systems, providing a formal account of when systems may become discretized (Feldman 2012).

4.9 LCL Meanings are Sub-symbolic and Emergent

While LCL dynamics do deal with discrete objects (like combinators), the *meaning* of these objects is not inherent in the symbols themselves. This is by design because LCL theories are intended to show how meaning, as cognitive scientists talk about it, might arise from lower-level dynamics. Cognitive symbols like `True` arise from the

sub-symbolic structures that `True` gets mapped to and the way these structures interact with other representations. This emphasis on sub-symbolic dynamics draws in part on connectionism, but also on theories that pre-date modern connectionism. Hofstadter (1985), for example, stresses the *active* nature of symbolic representations, which are "active, autonomous agents" (also Hofstadter 1980, 2008). He contrasts himself with Newell & Simon, for whom symbols are just "lifeless, dead, passive objects" objects that get manipulated. For Hofstadter, symbols are the objects that participate in the manipulating:

> A simple analogy from ordinary programming might help to convey the level distinction that I am trying to make here. When a computer is running a Lisp program, does it do function calling? To say "yes" would be unconventional. The conventional view is that functions call other functions, and the computer is simply the hardware that supports function-calling activity. In somewhat the same sense, although with much more parallelism, symbols activate, or trigger, or awaken, other symbols in a brain.
>
> The brain itself does not "manipulate symbols"; the brain is the medium in which the symbols are floating and in which they trigger each other. There is no central manipulator, no central program. There is simply a vast collection of "teams"—patterns of neural firings that, like teams of ants, trigger other patterns of neural firings. The symbols are not "down there" at the level of the individual firings; they are "up here" where we do our verbalization. We feel those symbols churning within ourselves in somewhat the same way as we feel our stomach churning; we do not do symbol manipulation by some sort of act of will, let alone some set of logical rules of deduction. We cannot decide what we will next think of, nor how our thoughts will progress.
>
> Not only are we not symbol manipulators; in fact, quite to the contrary, we are manipulated by our symbols!

Hofstadter's claim that sub-symbols are active representational elements that conspire to give rise to symbols is shared by LCL theories. His claim that the substrate on which sub-symbols live is an unstructured medium in which symbols are "floating" and interact with each other without central control, is not necessarily shared by LCL.

"Meaning" in this sense is *emergent* because it is not specified by any explicit feature of the design of the symbolic system. Instead, the meaning emerges out of the LCL combinators' dynamics as well as the constraints (the data) that say which specific structures are most appropriate in a given domain. Emergentism from dynamics rather than architecture may capture why emergent phenomena can be found in many types of models (e.g McClelland et al. 2010; Lee 2010). Recall that part of the motivation for LCL was that we wanted to formalize how symbols get meaning in order to better handle critiques like Searle's that "mere" symbol manipulation cannot explain the semantics of representation. As Chalmers (1992) argues, Searle's argument fails to apply to *sub*-symbolic systems like connectionist models because the computational (syntactic) elements are not intended to be meaningful themselves. The same logic saves LCL: **`s`**&**`k`** are not meant to, on their own, possess meaning other than dynamics,

Table 6 Learners who observe a correlation between dangerousness and size will generalize based on size (top)

Domain	Facts	Representation

Property Induction	`True = (small a)`
	`False = (small b)`
	`True = (dangerous a)`
	`False = (dangerous b)`
	
	`True = (small x)`
		`True := (K K)`
		`False := K`
		`dangerous := ((S K) S)`
		`small := ((S K) K)`
		`a := (K K)`
		`b := K`
		`x := (K K)`

so the question of how meaning arises is answered only at a level higher than the level at which symbols are manipulated.

4.10 LCL Theories are Parallelizable

Because combinators are compositional, they lend themselves to parallelization much more naturally than a program on a Turing machine would, a desirable property of cognitive theories. Functional programming languages, for instance, are based on logics like combinatory logic and can easily be parallelized precisely because of their uniform syntax and encapsulated components. Taking terminology from computer science, parallel execution is possible because LCL representations are *referentially transparent*, meaning that a combinator reduces in the same way, regardless of where it appears in the tree. As a result, two reductions in the same tree can happen in either order (a theorem known as the *Church-Rosser theorem* (Church and Rosser 1936)). A good implementation might do multiple reductions at the same time.

4.11 LCL Theories Eagerly Find Patterns and Use them in Generalization

An important part of generalization is the ability to infer unobserved properties from those that can be observed, a task studied as *property induction* in psychology (Rips 1975; Carey 1985; Gelman and Markman 1986; Osherson et al. 1990; Shipley 1993; Tenenbaum et al. 2006). Table 6 shows a simple example where Churiso is provided with two properties, `dangerous` and `small` applied to to some objects (`a`, `b`, and `x`). In this example, **property induction**, there is a perfect correlation between being dangerous and being small. In this case what we learn is a representation for these symbols where `(dangerous x)` $\rightarrow$ `True` even though all we know is that `x` is small. Being small, in fact, convinces the learner that `x` will have an identical conceptual role to `a`. This happens because in many cases, the easiest way for `x` to behave the correct way with respect to `small` is to make it the same as `a`. This illustrates a clear willingness to generalize, even from a small amount of data, a feature of human induction (Markman 1991; Li et al. 2006; Xu and Tenenbaum 2007; Salakhutdinov et al. 2010; Lake et al. 2015).

4.12 LCL Theories Supports Deduction and Simulation

The combinator structures that are learned are useful because they provide a way to derive new information. By combining operations in new ways (e.g. taking `(succ (succ (succ four)))`), learners are able to create the corresponding mental structures. This generative capacity is important in capturing the range of structures humans internalize. The ability can be viewed through two complementary lenses. The first is that LCL knowledge provides a deductive system which allows new knowledge to be proved. We can determine, for instance, whether

```
(succ (succ (succ three))) = (succ (succ four))
```

and thereby use our induced representations to learn about the world, since these representations are isomorphic to some structure in the world that matters. This view of knowledge is reminiscent of early AI attempts grounded in logic (see Nilsson 2009) and cognitive theories of natural reasoning through deduction (Rips 1989, 1994).

The second way to view the knowledge of an LCL system is as a means for mental simulation: one step forward of a combinator evaluation or one composition of two symbols corresponds to one step forward in a simulation of the relevant system. Simulation has received the most attention in physical understanding (e.g Hegarty 2004; Battaglia et al. 2013) and folk psychology (e.g Gordon 1986; Goldman 2006), both of which are controversial (Stone and Davies 1996; Davis and Marcus 2016). However, the literature on simulation has focused on simulations of particular (e.g. physical) processes and not on LCL's goal of capturing arbitrary, relational or algorithmic aspects of the world. In general, simulation may be the primary evolutionary purpose of constructing a representation, as it permits use in novel situations, a phenomenon with clear behavioral benefits (Bubic et al. 2010).

4.13 LCL Theories Support Learning

The mapping from symbols in base facts to combinators is solved by applying a bias for representational *simplicity* (Feldman 2003b; Chater and Vitányi 2003) and *speed* (Hutter 2005; Schmidhuber 2007). LCL theories inherit from the Bayesian LOT, and more generally work on program induction, a coherent computational-level theory of learning that provably works under a broad set of situations. Roughly, adopting the Bayesian setting, under basic conditions, with enough data learners will come to favor a hypothesis which is equivalent to the true generating one. In this setting where we consider hypotheses to be programs or systems of knowledge written in a LCL system, learners will be able to internalize dynamics they observe in the world.

As described above, a prior that favors hypotheses with short running times (e.g. $\exp(-\text{running time})$) permits learners to consider in principle hypotheses of arbitrary computational complexity. The trick is that the non-halting hypotheses have zero prior probability ($\exp(-\infty) = 0$) and this can be used to weed them out from a search or inference scheme that only needs upper-bounds on probability (like the Metropolis-Hastings algorithm). The ability to learn in such complex systems contrasts with arguments from the poverty of the stimulus in language learning and other areas of cognition. Note that this learning theory is stated as a computational theory, not an

algorithmic one. There is still a question about how learners actually navigate the space of theories and hypotheses.

Learnability is also a central question for conceptual theories (see Carey 2015). When considering cognitive development as a computational process, it might be tempting to think that the simpler algorithmic components must be "built in" for learners, a perspective shared by many LOT learning setups. But it may be that developmental studies will not bear this out—it is unlikely, for instance, that very young children have any ability to handle arbitrary boolean logical structures. Mody and Carey (2016) find that children under 3 years appear not to reason via disjunctive syllogism. If boolean logic is not innate, in what sense could it be learned? On one hand, boolean logic seems so simple that it is hard to see what capacity it could be derived from—this is why it is built in to virtually all programming languages. What LCL shows how it is possible to learn something so simple without presupposing it: what is built-in may be a general system for constructing isomorphisms, and learners may realize the particular structure of boolean logic (or logical deduction) only when it is needed to explain data they observe.

4.14 Features of the Implementation Which are not Part of the General LCL Theory

Because we have been required to make some (as of yet) under-determined choices in order to implement an LCL theory, it is important to also describe which of these specific choices are not critical to the general theory I am advocating. The implementation I describe chooses *particular* combinators $\mathbf{S}\&\mathbf{K}$, but there are infinitely many logically equivalent systems. In principle, different combinator bases may be distinguishable by the inductive biases they imply.

Moreover, because the brain is noisy and probabilistic, it is likely that the underlying representational system more naturally deals with probabilities than $\mathbf{S}\&\mathbf{K}$ as described here. It is also important to emphasize that many other formal systems have dynamics equivalent to LCL, some of which drop constraints such as compositionality or tree-structures. Cellular automata, for instance, rely only on local communication and computation, yet also are Turing-complete. Systems of cellular automata that implement useful conceptual roles would qualify as a system very much like combinatory logic (although not in all the ways described above). A lesson from complex systems research is that there are a huge variety of simple systems that are capable of universal computation (Wolfram 2002), suggesting that it would not be hard, in principle, for nature to implement CRS-like representations in any number of ways.

5 Towards a Connectionist Implementation

In order to encode $\mathbf{S}\&\mathbf{K}$ into a connectionist architecture, we primarily will require a means of representing binary tree structures and transformations of them. Many ways of representing tree structures or predicates in neural dynamics (e.g. Pollack 1990; Touretzky 1990; Smolensky 1990; Plate 1995; Van Der Velde and De Kamps 2006; Martin and Doumas 2018) or other spaces (Nickel and Kiela 2017) have been

considered, often with the appreciation that such representations will greatly increase the power of these systems. Related work has also explored how vector spaces might encode compositional knowledge or logic (Grefenstette 2013; Rocktäschel et al. 2014; Bowman et al. 2014a, a, 2015; Neelakantan et al. 2015; Gardner et al. 2015; Smolensky et al. 2016). Here, I will focus on Smolensky (1990)'s *tensor product* encoding (see Smolensky and Legendre 2006; Smolensky 2012). Section 3.7.2 of (Smolensky 1990) showed how the Lisp operations `first`, `rest`, and `pair` could be implemented in a tensor product system. These are the only operations needed in order to implement an evaluator for **S**&**K** (or something like it). Note that the idea of implementing `first`, `rest`, and `pair` in a connectionist network is quite distinct from implementing them in **S**&**K** above. The above construction in **S**&**K** is meant to explain cognitive manipulation of tree structures as a component of high-level thought. The implementation in tensor products could be considered to be part of the neural hardware which is "built in" to human brains—the underlying dynamics from which symbols **S**&**K** themselves emerge.

To implement **S**&**K**, we can define a function called `(reduce t)` that computes the next step of the dynamics (the "→" operation):

```
(reduce t) := (first (rest t))                                          if  (first t)  is  K
(reduce t) := (pair (pair (first t) (first (rest (rest t))))
                    (pair (first (rest t)) (first (rest (rest t)))))    if  (first t)  is  S
```

The left hand side of `(reduce t)` tell us that we are defining how the computational process of combinator evaluation may be carried out. The right hand side consists only of `first`, `rest`, and `pair` functions on binary trees, which, we assume, are here implemented in a connectionist network, following the methods of e.g. Smolensky (1990) or a similar approach. This function operates on data (combinators) that are themselves encoded with `pair`.

For instance, the structure `(K x y)` would be encoded in a connectionist network as `(pair K (pair x y))`. Then, following the definition of `reduce`,

```
(reduce (K x y)) = (reduce (pair K (pair x y)) → (first (
    ↪rest (pair K (pair x y)))) → x
```

The case of **S** is analogous.

To summarize, Figure 2 shows a schematic of the general setup: tensor product coding (or an alternative) can be used to encode a tree structure in a connectionist network. The `reduce` function can then be stated as tensor operations, and these implement the dynamics of **S**&**K**, or a system like it. Then, combinator structures can be built with `pair`. The way these structures interact through `reduce` can give rise to create structured, algorithmic systems of knowledge through the appropriate Church encoding. In fact, any of the encoding schemes that supports `first`, `rest`, and `pair` can implement LCL theories, thereby permitting a swath of symbolic AI and cognitive science to be implemented in neural systems. This, of course, treats connectionism as only an implementational theory of a CRS system. But this is, just a simplification for the present paper—there are certain to be areas where the parallel and distributed nature of connectionist theories are critically important (e.g. Rumelhart and McClelland 1986), particularly at the interfaces of sensation and perception.

It is notable that the move towards unifying high-level symbolic thought with theories of implementation has been almost entirely one-sided. There are many connectionist approaches that try to explain—or explain away—symbolic thought. However,

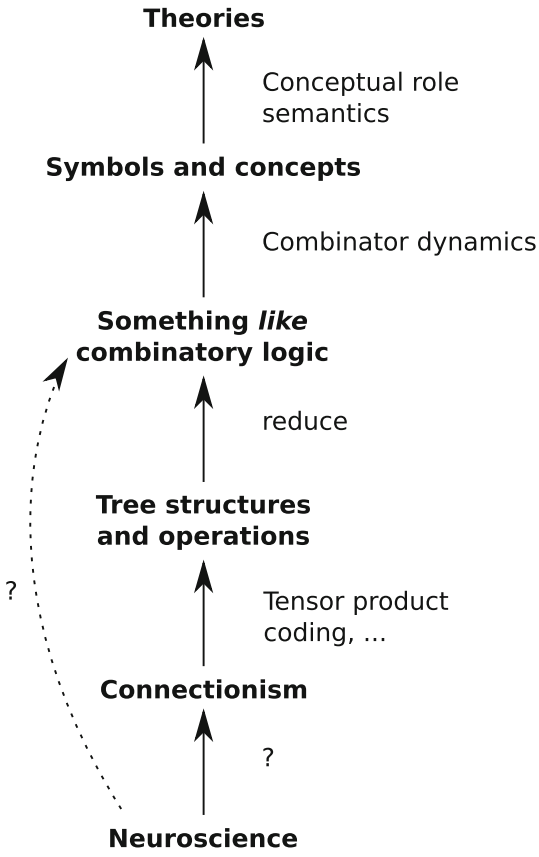


Fig. 2 An overview of the encoding of LCL dynamics into a connectionist architecture. Schemes like Smolensky (1990)'s tensor product encoding allow tree operations and structure (e.g. `first`, `rest`, `pair`), which can be used to implement the structures necessary for combinatory logic as well as the evaluator. The structures built in combinatory logic, as shown in this paper, create symbolic concepts which participate in theories and whose meaning is derived through conceptual role in those theories. It is possible that the intermediate levels below LCL are superfluous, and that dynamics *like* combinatory logic could be encoded directly in biological neurons (dotted line)

almost no work on the symbolic side has sought to push down towards representations that could more directly be implemented. Many symbolic modelers—myself included—have considered the problem of implementation to lie squarely in the neural modelers' domain: connectionist networks should strive to show where symbols could come from. However, if the view of this section is correct, the main sticking point has actually been on the symbolic side of thinking more carefully about how symbols might get their meaning.

6 Remaining Gaps to be Filled

The LCL theory I have presented has intentionally focused on a minimal representational theory for high-level logical concepts. I have done this because these areas of cognitive psychology seem to be in most desperate need of a link to implementation. However, it is important to discuss a few limitations of what I have presented in this paper.

6.1 LCL and the Interfaces

Missing from the description of LCL theories is a formalization of how such abstract operations might interface to perception and action. As Harnad (1990) argued, cognition must not get stuck in a symbolic "merry-go-round" where all that happens is symbol manipulation—at some point it must relate to the real world. Our examples have shown that in some cases a symbolic merry-go-round can be quite powerful, but it is true that even with an LOT-like representation, the way in which representations relate to the outside world is of central concern. Miller and Johnson-Laird (1976) write,

> A dictionary is a poor metaphor for a person's lexical knowledge. Dictionaries define words in terms of words. Such definitions, like some semantic theories, may provide plausible accounts of the intensional relations between words, but their circularity is vicious. There is no escape from the round of words. Language can apply to the nonlinguistic world, however, and this fact cannot be ignored by a theory of meaning. It is perhaps the single most important feature of language; a theory that overlooks it will provide simply a means of translating natural language into theoretical language. Although such a theory may be extremely useful as a device for formulating intensional relations, its ultimate value rests on a tacit appeal to its users' extensional intuitions.

The CRS itself has also been criticized for its circularity where meanings are defined in terms of each other (see Greenberg and Harman 2005; Whiting 2006). LCL embraces this circularity and shows that it is not inherently problematic to the construction of a working computational system (see also chapter 4 of Abelson and Sussman 1996). For a theory of concepts, the circularity is even desirable because it prevents us from pushing the problem of meaning off into someone else's field—linguistic philosophy, neuroscience, robotics, etc.

To avoid the circularity, many of the theories of where such high-level constructs come from are based on repeated abstraction of sensory-motor systems, perhaps maintaining tight links between conception and perception (e.g. Barsalou 1999, 2008, 2010; Sloutsky 2010). The challenge with this view is in understanding how concepts come to be distinct from perception, or more abstract, generalizing beyond immediate experience (Mahon 2015), a goal with some recent computational progress (Yildirim and Jacobs 2012). From the LCL point of view, the primary difficulty with theories closely tied to perception is that they do not engage with the computational richness of full human cognition—they do not explain how it is that we are able to carry out such a wide variety of computational processes and algorithms. A good example to consider might be *Bongard problems* (Bongard 1970). Solving these problems requires combining visual processes (perception, feature extraction, similarity computation, rotation, etc.) with high-level conceptions (e.g. search algorithms, knowledge about possible transformations, etc.) (Edelman and Shahbazi 2012; Depeweg et al. 2018), and so any theory of how humans solve these problems must bridge high- and low-level processes. Such high-level algorithms and knowledge are what LCL aims to capture.

On the flip-side, the theory I have described does not engage with perception and action, nor is it sensitive to the type of content its actions manipulate. However,

"two-factor" theories of CRS that more closely connect to perception and action have previously been proposed (Harman 1987; Block 1997). At the very least, the perceptual systems for shape must interface with high-level concepts–perhaps by speaking languages that are inter-translatable.[10] In the same way, a computer has a single representation language for its central processor; the various subsystems —graphics cards and hard drives—must speak languages that are translatable with the central processor so that they can coordinate complex computation. Consider some base facts for a **square** concept,

```
(number-of-edges square) → four
(number-of-vertices square) → four
(angles square) → (pair rt-angle (pair rt-angle (pair
   ↪rt-angle rt-angle)))
(rotate square 45) → diamond
(rotate diamond 45) → square
...
```

Even if this is the right conceptual role for square, it must *also* be the case that perception speaks this language. For instance, when `rotate` is called, we may rely on perceptual systems in order to execute the rotation so that `rotate` is not itself an operation in pure **S&K**. The key is that whatever dedicated hardware does do the rotation, it must send back symbols like `diamond` and `square` that are interpretable on a high level. To see this, consider the wide variety of computational processes that `square` can participate in and the other theories that it is related to. Here is one: Suppose I have a square-shaped ink stamp ■. I stamp it once, rotate the stamp by 45 degrees, ◆, stamp it again. If I do that forever, what shape will I have stamped out? What happens if I rotate it 90 degrees instead? The abstraction we readily handle is even more apparent in questions like, "If I rotate it 1 degree versus 2 degrees, which will give me a star with more points?" Likely, we can answer that question without forming a full mental image, relying instead on high-level properties of numbers, shapes, and their inter-relations. The situation can get even more complex: what happens if I rotate it $\sqrt{2}$ degrees each time? Solving this last problem in particular seems to require both an imagistic representation (to picture the stamping) as well as high-level logical reasoning about non-perceptual theories—in this case, the behavior of irrational numbers.

One feature of perception that is notably *un*like LCL is that small perceptual changes tend to correspond to small representational changes, a kind of principle of *continuity*.[11] For instance, the representation of a rotated symbol "$\mathcal{B}$" is likely to be very similar to "B". However, the representations of LCL—and logical theory learning more generally—appear not to obey this principle. Changing one of the constraints or one data point might have huge consequences for the underlying representation. In the same way, a single data point might lead a human to revise an intuitive theory or a scientific theory. As far as I am aware, there are not yet good general solutions to this

[10] Fodor and Pylyshyn (2014) notes there is no imagistic representation of a concept like "squareness", or the property that all squares, a problem that Berkeley (1709) struggled with in understanding the origin and form of abstract knowledge. Anderson (1978) shows how perceptual propositional codes might account for geometric concepts like the structure of the letter "R" and how imagistic and propositional codes can made similar behavioral predictions (see also, e.g. Pylyshyn 1973).

[11] A mathematical function, for instance mapping the world to a representation, is continuous if boundedly small changes in the input give rise to boundedly small changes in the output.

problem of finding logic-like representations that have good continuity properties. It may be reasonable to suspect that none exist—that no Turing-complete representational system will have semantics that are continuous in its syntax.

6.2 Extensions in Logic

I have presented a very simplified view of combinatory logic. Here I will discuss a few important points where prior mathematical work has charted out a useful roadmap for future cognitive theories.

First, as I have described it, any structure built from **S**&**K** is evaluable. In the **roshambo** example, for instance, we could consider a structure like `(win win)`. To us, this is semantically meaningless. The problem of incoherent compositions could be solved in principle by use of a *type system* that defines and forbids nonsensical constructs (see Hindley and Seldin 1986; Pierce 2002). Each symbol would be associated with a "type" that function composition would have to respect. For instance, since `rock`, `paper`, and `scissors` would be one type that is allowed to operate on itself; when it does so, it produces an outcome (`win`, `lose`, or `draw`) of a different type that should not be treated as a function or argument (thus preventing constructions like `(win win)`). The corresponding learning theory would have to create new types in each situation, but learning could be greatly accelerated by use of types to constrain the space of possible representations.

The second limitation of combinatory logic is one of expressivity. Combinatory logic is Turing-complete only in the sense that it can represent any computable function on natural numbers (Turing 1937). But there are well-defined operations that it cannot express, in particular about the nature of combinator structures themselves (see Theorem 3.1 of Jay and Given-Wilson 2011). There is no combinator that can express *equivalence* of two normal forms (Jay and Vergara 2014). That is, we cannot construct a combinator `E` such that `(E x y)` reduces to `True` if and only if `x` and `y` are the same combinator structure. There are, however, formal systems that extend combinatory logic that can solve this problem (Curry and Feys 1958; Wagner 1969; Kearns 1969; Goodman 1972; Jay and Kesner 2006; Jay and Given-Wilson 2011), and these provide promising alternatives for future cognitive work. These types of logical systems will likely be important to consider, given the centrality of notions like same/different in cognitive processes, with hallmarks in animal cognition (Martinho and Kacelnik 2016).

Finally, the semantics of combinatory logic is deterministic: evaluation always results in the same answer. However, probabilistic programming languages like Church (Goodman et al. 2008a) provide inherently stochastic meanings that support conditioning and probabilistic inference, with nice consequences for theories of conceptual representation (Goodman et al. 2015), including an ability to capture gradience, uncertainty, and statistical inference. Combinatory logic could similarly be augmented with probabilistic semantics, and this may be necessary to capture the statistical aspects of human cognition (Tenenbaum et al. 2011).

6.3 The Architecture Itself

The connectionist implementation described above only shows how evaluation can be captured in a connectionist network. However, it does not provide an implementation of the entire architecture. A full implementation must include other mechanisms.

First an implementation needs a means for observing base facts from the world and representing them. Churiso assumes base facts are given as symbolic descriptions of the world. There are many possibilities that might inform these facts, including basic observation, pedagogical considerations, knowledge of similarity and features, or transfer and analogy across domains. The mind will likely have to work with these varied types of information in order to learn the correct theories. Extending the LCL theory beyond explicit relational facts is an important direction for moving the theory from the abstract into more psychological implementations. Similarly, a fuller theory will require understanding the level of granularity for base facts. Unless you are a particle physicist, you don't have detailed models of particle physics in your head. If we consider an LCL encoding of even an ordinary object, there are many possibilities: we might have an LCL structure for the whole object (e.g. the `rock` in **roshambo**); we might map its component parts to combinators which obey the right relations; we might map its edges and surfaces to combinator structures; or we might go below the level of surfaces. The question of granularity for LCL is simply an empirical question, and a goal for cognitive psychology should be to discover what level of description is correct.

Second, an important next step would be to understand the connectionist or neural implementation of search and inference. Such a full model might build on recent work learning relations or graphs with neural networks (e.g Kipf et al. 2018, 2020; Grover et al. 2019; Zhang et al. 2020), control dynamics (e.g Watter et al. 2015; Levine et al. 2016), and components of computer architecture (Zylberberg et al. 2011; Graves et al. 2014; Trask et al. 2018; Rule et al. 2020). In addition, the full architecture will likely be connected to probabilities in order to characterize gradiations in learners' beliefs.

7 General Discussion

One reason for considering LCL theories is that they provide a new riff on a variety of important issues, each of which I will discuss in turn.

7.1 Isomorphism and Representation

Craik (1952) suggested that the best representations are those that mirror the causal processes in the world. However, it is a nontrivial question whether a given physical system—like the brain—is best described as isomorphic to a computational process. Putnam (1988) argues for instance that even a simple physical system like a rock can be viewed as implementing essentially any bounded computation because it has a huge number of states, meaning that we can draw an isomorphism between its states and the states of a fairly complex computational device (see also Searle 1990; Horsman

et al. 2014). LCL theories are kin to Chalmers' response to this argument highlighting the combinatorial nature of states which can be found in a real computational device, but not an inert physical system (Chalmers 1996, 1994). When we consider the brain, LCL theories predict that we should be able to find a small number of micro-level computational states that interact with each other in order to organize structures and dynamics which are isomorphic to the systems being represented. In this way, scientific theories of how mental representations work cannot be separated from an understanding of what problems a system solves or which aspects of reality are most important for it to internalize. When viewed as part of an integrated whole scientific enterprise, the scientific search for mental representations is therefore not just a search for isomorphisms between physical systems and abstract computing devices. It is a search for relations which are explanatory about empirical phenomena (e.g. behavior, neural dynamics) and which fit within the framework of knowledge developed elsewhere in biology, including our understanding of evolution. A good scientific theory of mental representation would only attribute isomorphic computational states to a rock if something explanatory could be gained.

7.2 Critiques of CRS

CRS is not without its critics (see Greenberg and Harman 2005; Whiting 2006). One argument from Fodor and Lepore (1992) holds that conceptual/inferential roles are not compositional, but meanings in language are. Therefore, meanings cannot be determined by their role. To illustrate, the meaning of "brown cow" is determined by "brown" and "cow". However, if we also know in our conceptual/inferential role that brown cows are dangerous (but white cows are not) then this is not captured compositionally. To translate this idea into LCL, imagine that composition is simply `pair`ing together `brown` and `cow` to form `(pair brown cow)`. How might our system know that the resulting structure is dangerous, if danger is not a part of the meanings (combinator structures) for `brown` or `cow`? A simple answer is that `(pair brown cow)` the information is encoded in the meaning of either `is-dangerous` or `pair` (not `brown` and `cow`, as Fodor assumes). More concretely, the following base facts roughly capture this example:

```
(is-dangerous cow )   → False
(is-dangerous brown) → False
(is-dangerous (pair brown cow)) → True ; brown cows are
   ↪dangerous
(is-brown brown) → True
(is-brown cow)    → False ; Cows are not generally brown
(is-brown (pair brown y))) → True ; anything brown is
   ↪brown
```

Given these facts and the standard definitions of `True`, `False`, and `pair`, Churiso finds

```
is-brown := ((S (S K K)) (S K))
cow      := ((S (S K K)) (K K))
brown    := (K (K K))
is-dangerous := ((S ((S S) K)) S)
```

Because these facts can be encoded into a compositional system like combinatory logic, this shows at a minimum that issues of compositionality and role are not so simple: when "role" is defined in a system like this, compositionality can become subtle. Contrary to the misleadingly informal philosophical debate, consideration of a real system shows that it is in principle straightforward for a system to satisfy the required compositional roles.

Of course, it is unlikely that memory speaks logic in this way. The reason is that it would be very difficult to add new information since doing so would require changing the internals of a concept's representational structure. If suddenly we learned that we were mistaken and brown cows were *not* dangerous, we'd have to alter the meaning of one of the symbols rather than church-encode a new fact. Doing so—and maintaining a consistent, coherent set of facts—may even be difficult when there are multiple interacting concepts or conceptual systems. Much better would be to have memory function as a look-up table, where combinator structures provide the index. In other words, `is-dangerous` might not be a combinator structure, but a special interface to a memory system (for an argument on the importance of memory architectures, see Gallistel and King 2009). In this way, `is-dangerous` might be a fundamentally different kind of thing than a statement which is true *because* of the inherent properties of `brown` and `cow` (like a predicate `is-brown`). The argument therefore requires that we accept that there different kinds of statements—some of which are true in virtue of their meaning ("Brown cows are brown") and some of which are true in virtue of how the world happens to be ("Brown cows are dangerous"). Famously, Quine (1951) rejected the distinction between these kinds of properties, arguing that the former are necessarily circularly defined. In their critique of CRS, Fodor and Lepore (1992) also reject this distinction (see (Block 1997) for a discussion of these issues in CRS and Fodor and Pylyshyn (2014) for more critiques). But to a psychologist, it's hard to see how a cognitive system that has both memory of the world and compositional meanings could be any other way; in fact, the mismatch between memory and compositional concepts is what drives us to learn and change conceptual representations themselves. Computers, too, certainly have some properties that can be derived from objects themselves *and* objects that can index facts in a database.

Fodor and Lepore (1992) also argue that CRS commits one to *holism*, where meaning depends critically on all other aspects of knowledge, since these other aspects factor into conceptual role. Holism is considered harmful in part because it would be unclear how two people could hold the same beliefs or knowledge since it is unlikely that all components of their inferential system are the same. The difficulty with learning very complex systems of knowledge is also made clear in Churiso: larger systems that have many symbols and relations tend to present a tougher constraint-satisfaction problem, although we have provided an example above of easily learning multiple domains at once. One solution to the general problem is to favor modularity: in a real working computational system, the set of defining relations for a symbol might be small and circumspect. The logical operators, for instance, may only be defined with respect to each other, and not to combinator structures that represent an entirely different domain. Even a single object—for instance a representation of a **square**—could have different sets of combinators to interface with different operations (e.g. rotation vs. flipping). Such modularity of roles and functions is a desirable feature of computational systems

in general, and the search to manage such complexity can be considered one of the defining goals of computer science (Abelson and Sussman 1996).

A third challenge to CRS is in its handling of meaning and reference. There is a titanic literature on the role of reference in language and cognition, including work arguing for its centrality in conceptual representation (e.g. Fodor and Pylyshyn 2014). To illustrate the importance of reference, Putnam (1975) considers a "twin earth" where there exists as substance that behaves in every way like water (H_2O) but is in fact composed of something else (XYZ). By assumption, XYZ plays the same role in conceptual systems as H_2O and yet is must be a different meaning since it refers to an entirely different substance. Any characterization of conceptual systems entirely by conceptual roles and relations will miss an important part of meaning. The problem leads others like Block (1997) to discuss a *two-factor* theory of CRS in which concepts are identified by both their conceptual role and their reference (see also Harman 1987). Critiques of the two-factor CRS are provided in Fodor and Lepore (1992) and discussed in Block (1997).[12]

7.3 The Origin of Novelty and Conceptual Change

One of the strangest arguments in philosophy of mind comes from Fodor (1975), who holds that there is an important sense in which most of our concepts must be innate. The argument goes, roughly, that the only way we learn new concepts is through composition of existing concepts. Thus, if we start with `GO` and `UP`, we can learn "lift" as `CAUSE(x, GO(y, UP))`. Fodor notes, however, that almost none of our concepts appear to have compositional formulations like these (see Margolis and Laurence 1999). He concludes, therefore, that learning cannot create most of our concepts. The only possibility then is that almost all of our concepts are innate, including famously concepts as obscure as *carburetor*. While some of taken this as a reductio ad absurdum of the LOT or of compositional learning, it's hard to ignore the suspicion that Fodor's argument is simply mistaken in some way. Indeed combinatory logic and other computational formalisms based on function composition show that it is: *any computational process can be expressed as a composition in a formal language or LOT* (see Piantadosi and Jacobs 2016). The present paper shows that the LOT need not have *any* innate meanings—just innate dynamics. This means that if a computational theory of mind is correct—computations are the appropriate description for concepts like *carburetor*—then these must be expressible compositionally and therefore can be learned in Fodor's sense. A compositional CRS like LCL solves, at least in principle, the problem of explaining how an organism could learn so many different computations without requiring innate content on a cognitive level.

[12] My inclination is that Putnam's argument tells us primarily about the meaning of the word "meaning" rather than anything substantive about the nature of mental representations (for a detailed cognitive view along these lines in a different setting, see Piantadosi 2015). It is true that intuitively the meaning of a term should include something about its referent; it is not clear that our intuitions about this word tell us anything about how brains and minds actually work. In other words, Putnam may just be doing lexical semantics, a branch of psychology, here—if his point is really about the physical/biological system of the brain, it would be good to know what evidence can be presented that convincingly shows so.

A corollary is that LCL theories can coherently formalize a notion of conceptual change, and that the processes of novel conceptual creation are inherently linked to the creation of systems of concepts, following Laurence and Margolis (2002) and Block (1987). One key question motivating this work is how learning could work if children lack knowledge of key computational domains like logic, number, or quantification. The idea that mental representations are like programs has an implicit assumption that the primitives of these programs are the genetic endowment that humans are born with.

LCL shows one way in which this metaphor can be revised to include the possibility that such fundamental logical abilities are not innate, but built through experience. Learners could come with an ability to execute only underlying dynamical operations like $\mathbf{S}_{\&}\mathbf{K}$, thus possessing a system for potentially building theories and representations. In terms of *cognitive* content, this system would be a blank slate. Knowledge of $\mathbf{S}_{\&}\mathbf{K}$ is not cognitive because they correspond to dynamical operations below the level of symbols, algorithms, and structures. As such, this work provides a representational foundation upon which one might construct a rational constructivist theory of development (Kushnir and Xu 2012; Xu 2019).

8 Conclusion: Towards a Synthesis

Cognitive science enjoys an embarrassment of riches. There are many seemingly incompatible approaches to understanding cognitive processes and no consensus view on which is right or even how they relate to each other. The major debates seem to be in large part disagreements of which *metaphor* is right for thinking about the brain. These debates have made two things clear: none of our metaphors are yet sufficient, and none of them is completely misguided. Cognitive systems are dynamical systems; they give rise to structured manipulation of symbols; they also are implemented in physical/biological systems whose dynamics are determined far below the level of mental algorithms. Our learning supports inference of broad classes of computations, yet clearly we have something built in that differs from other animals. The LCL can be thought of as a new metaphor—a sub-meaningful symbolic-dynamical system that gives rise straightforwardly to the types of structures, representations, and algorithms that permeate cognitive science, *and* that is implementable directly in neural architectures. In spanning these levels, it avoids dodging questions about meaning.

If anything like the resulting theory is correct, there are important consequences for theories of conceptual change as the LOT. Conceptual change from a strikingly minimal basis to arbitrarily systems of knowledge is possible if learners come with built-in dynamical objects and learning mechanisms. Theories of learning need not assume any cognitive content in principle, not even the basics of familiar computational systems, like logic and number. The key is in formalizing a theory of the meaning of mental content; if CRS is chosen, it permits construction of these systems of knowledge from much less. The framework I have described follows the LOT in positing a structured, internal language for mentalese. But it differs from most instantiations of the LOT in that the primitives of the language aren't cognitive at all. $\mathbf{S}_{\&}\mathbf{K}$ formalize the underlying neural (sub-cognitive) dynamics and it is only in virtue how structures built of these

dynamics interact that meaningful systems of thought can arise. Thus, the idea of a formal language for thinking was right; the idea that the language has primitives with intrinsic meaning—beyond their dynamics—was not.

A trope in cognitive science is that we need more constraints in order to narrow the space of possible theories. Each subfield chooses its own—architectural, rational, neural, computational etc. One idea that broading our view to consider LCL systems raises is that wanting more constraints might be premature. Additional constraints are useful when the pool of theories is too large and must be culled. But it might be the case that we have too *few* theories in circulation, in that none of our approaches satisfactorily and simultaneously handle all that we know about cognitive processes—meaning, reference, computation, structure, etc. In this case, our search might benefit from expanding its set of metaphors—fewer constraints—to consider new kinds of formal systems as possible cognitive theories. And in the case of LCL, useful kinds of theories have been developed in mathematical logic that might provide a good foundation for cognition. But of course, LCL is just one attempt, with clear strengths and clear weaknesses.

Perhaps the single greatest strength is that it unifies a variety of ideas in cognition. None of the formal machinery used here is original to this paper—all of it comes from allied fields. The motivating ideas then conspire to create a theory that is extraordinarily simple: a few elementary operations on trees are composed productively, giving rise to a huge variety of possible cognitive structures and operations—and computational richness that distinguishes human-like thinking. This paper made assumptions to show how a system could actually work, but the general theory is not about any particular logical system, representational formalism, or cognitive architecture. Instead, I have tried to present a system which captures some general properties of thought. This system suggests that mental representations, whatever they happen to be, will be like church encoding in combinatory logic in a few key ways.

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A Appendix A: Sketch of Universality

It may not be obvious that any statement about the relation between objects can be encoded into an LCL system. Here, I sketch a simple proof that this is possible when we are allowed to define what function composition means. My focus is on the high-

level logic of the proof while attempting to minimize the amount of notation required. Let's suppose that we are given an arbitrary base fact like,

```
(a (b x)) → (c (d e f))
```

We may re-write this into binary constraints, with a single variable on the left and a single function application on the right, by introducing "dummy" variables `D1`, `D2`, etc:

```
(d e)   → D1 ; right hand term is D1-D3
(D1 f)  → D2
(c D2)  → D3 ; D3 enforces the equality between the sides
(b x)   → D4 ; left hand term is D3-D4
(a D4)  → D3
```

This is akin to Chomsky normal form for a context-free grammar.

The challenge then is to find a mapping from symbols to combinators that satisfies these expressions. A difficulty to note is that some variables, like `D1`, may appear on the left *and* the right, meaning that their combinator structure must be the output of a function (appearing on the left) as well as a function that itself does something useful (on the right). To address this, the proof sketch here will assume that we are allowed to define the way functions are applied. For instance, instead of requiring `(d e)` → `D1`, we will replace the function application `(d e)` with our own custom one, `(evaluate d e)`. When `evaluate = I`, we are left with ordinary function application. I do not determine here if requiring `evaluate=I` permits universal isomorphism. But we can show that if we are free to choose `evaluate`, we can satisfy any constraints.

With this change, we can re-write our base facts as,

```
(evaluate d e) → D1 ; right hand term is D1-D3
(evaluate D1 f)→ D2
(evaluate c D2)  → D3 ; D3 enforces the equality between
   ↪the sides
(evaluate b x)  → D4 ; left hand term is D3-D4
(evaluate a D4)→ D3
```

With this addition, we can take each of the symbols (`a b c d e f x` and `D1 D2 D3` ↪ `D4`) and give them each an integer with Church encoding. Standard schemes for this can be found in Pierce (2002). Integers in Church encoding also support addition, subtraction, and multiplication. We may therefore view these facts as a set of integer-values, where `evaluate` is a function from two (integer) arguments to a single (integer) outcome:

```
(evaluate 4 5) → 8 ; (evaluate d e)
(evaluate 8 6) → 9 ; (evaluate D1 f)
(evaluate 3 9) → 10 ; (evaluate c D2)
(evaluate 2 7) → 11 ; (evaluate b x)
(evaluate 1 11) → 10; (evaluate a D4)
```

Note that at this point we may check if the facts are logically consistent—they may not state, for instance, that `(f x y)` → `z`, `(f x y)` → `w`, and `z` ≠ `w`.

Assuming consistency, we may then explicitly encode the facts by setting `evaluate` to be a polynomial which encodes these facts. To see how this is possible, suppose we have constraints

```
(evaluate α1 β1) → γ1
```

```
(evaluate α2 β2) → γ2
(evaluate α3 β3) → γ3
...
```

It is well-known that in one dimension, any set of x, y points can be approximated by a polynomial. The same holds for two dimensions, with a variety of available techniques. This means that we can set `evaluate` to be the combinator that implements the polynomial mapping each α_i, β_i to γ_i with the desired accuracy.

An alternative to 2D polynomials is to use Gödel numbering to convert the two-dimensional problem to a one-dimensional one. If `evaluate` first converts its arguments to a single integer, for instance $2^{\alpha_i}3^{\beta_i}$, then the problem of finding the right polynomial reduces to a one-dimensional interpolation problem. Explicit solutions then exist, such as this version of Lagrange's solution to the general problem,

$$(\texttt{evaluate}\ \alpha_i\ \beta_i) := \sum_{j=1}^{n} \gamma_j \prod_{\substack{1 \le m \le k \\ m \ne j}} \frac{2^{\alpha_i}3^{\beta_i} - 2^{\alpha_m}3^{\beta_m}}{2^{\alpha_j}3^{\beta_j} - 2^{\alpha_m}3^{\beta_m}}. \tag{2}$$

To check this, note that when $i = j$, the fractions inside the product cancel and the coefficient for γ_j becomes 1. However, when $i \neq j$, then there will be some numerator term which is zero, canceling out all of the other γ_m. Together, these give the output of `evaluate` as γ_i when given α_i and γ_i as input.

Note that this construction does not guarantee sensible generalizations when running `evaluate` on new symbols. The specific patterns of generalization will depend on how symbols are mapped to integers, but more problematically, polynomial interpolation famously exhibits chaotic or wild behavior on points other than those that are fixed, a fact known as *Runge's phenomenon* (Runge 1901). As a result, the polynomial mapping should be taken only as an existence proof that some mapping of combinators will be able to satisfy the base facts, or the combinatory logic can in principle encode any isomorphism when we define function application with `evaluate`.

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