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On odd rainbow cycles in edge-colored graphs

Andrzej Czygrinow [a,1], Theodore Molla [b,2], Brendan Nagle [b,3], Roy Oursler [a]

[a] School of Mathematical and Statistical Sciences, Arizona State University, Tempe, AZ 85281, USA
[b] Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA

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ABSTRACT

Let $G = (V, E)$ be an n-vertex edge-colored graph. In 2013, H. Li proved that if every vertex $v \in V$ is incident to at least $(n+1)/2$ distinctly colored edges, then G admits a rainbow triangle. We prove that the same hypothesis ensures a rainbow ℓ-cycle C_ℓ whenever $n \geq 432\ell$. This result is sharp for all odd integers $\ell \geq 3$, and extends earlier work of the authors for when ℓ is even.

1. Introduction

An *edge-colored graph* is a pair (G, c), where $G = (V, E)$ is a graph and $c : E \to P$ is a function mapping edges to some palette of colors P. A subgraph $H \subseteq G$ is a *rainbow* subgraph if the edges of H are distinctly colored by c. Rainbow subgraph problems are a well-studied area of graph theory (see, e.g., [1–13,16], and Section 1.1 below). Here, we consider degree conditions on (G, c) ensuring the existence of rainbow cycles C_ℓ of fixed length $\ell \geq 3$. To that end, a vertex $v \in V$ in an edge-colored graph (G, c) has *c-degree* $\deg_G^c(v)$ given by the number of distinct colors assigned by c to the edges $\{v, w\} \in E$. We set $\delta^c(G) = \min_{v \in V} \deg_G^c(v)$ for the minimum c-degree in G. The following result of H. Li [10] motivates our current work.

Theorem 1.1 (*H. Li [10], 2013*)**.** *Let (G, c) be an n-vertex edge-colored graph. If $\delta^c(G) \geq (n+1)/2$, then (G, c) admits a rainbow 3-cycle C_3.*

E-mail addresses: aczygri@asu.edu (A. Czygrinow), molla@usf.edu (T. Molla), bnagle@usf.edu (B. Nagle), Roy.Oursler@asu.edu (R. Oursler).

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A rainbow $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ establishes that Theorem 1.1 is best possible.

We prove an analogue of Theorem 1.1 for ℓ-cycles C_ℓ of fixed arbitrary length.

Theorem 1.2. *For every integer $\ell \geq 3$, every edge-colored graph (G, c) on $n \geq n_0(\ell)$ many vertices satisfying $\delta^c(G) \geq (n+1)/2$ admits a rainbow ℓ-cycle C_ℓ.*

A rainbow $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ also establishes that Theorem 1.2 is best possible for all odd integers ℓ. For even integers $\ell \geq 4$, the authors earlier proved in [4] a stronger form of Theorem 1.2.

Theorem 1.3 (*Czygrinow et al. [4]*). *For every even integer $\ell \geq 4$, every edge-colored graph (G, c) on $n \geq N_0(\ell)$ many vertices satisfying $\delta^c(G) \geq (n+5)/3$ admits a rainbow ℓ-cycle C_ℓ.*

It was shown in [4] that Theorem 1.3 is best possible for every even $\ell \not\equiv 0 \pmod 3$.

Theorem 1.1 holds non-vacuously when $n \geq 3$, and one may seek to quantify $n_0(\ell)$ and $N_0(\ell)$ in Theorems 1.2 and 1.3. The proof of Theorem 1.3 depends on an application of the Szemerédi Regularity Lemma [14,15], and therefore gives very poor bounds on $N_0(\ell)$. Our proof of Theorem 1.2 is elementary, and easily provides $n_0(\ell) = O(\ell^2)$. For the interested Reader, we provide a more detailed analysis in our final section which establishes that $n_0(\ell)$ is linear in ℓ.

Theorem 1.4. *The function $n_0(\ell)$ in Theorem 1.2 satisfies $n_0(\ell) \leq 432\ell$.*

The remainder of this paper is organized as follows. In Section 1.1, we discuss further results and context regarding rainbow cycle problems. In Section 2, we sketch Li's proof [10] of Theorem 1.1 and note the elements there which provide a basis for our approach here. In Section 3, we extend this proof to develop several tools useful for proving Theorems 1.2 and 1.4. In Section 4, we prove Theorem 1.2, and in Section 5, we prove Theorem 1.4. In the entirety of this paper, we employ the following observations.

Remark 1.5. We say that an edge-colored graph (G, c) is *edge-minimal* when every $e \in E(G)$ satisfies $\delta^c(G-e) < \delta^c(G)$. Every edge-colored graph (G, c) admits an edge-minimal spanning subgraph $H \subseteq G$ satisfying $\delta^c(G) = \delta^c(H)$, so in Theorems 1.1–1.4 it suffices to assume that (G, c) is already edge-minimal. As such, (G, c) admits no three commonly colored edges $\{u, v\}, \{v, w\}, \{w, x\} \in E(G)$, as removing $\{v, w\} \in E(G)$ violates edge-minimality. $\square$

1.1. Rainbow cycles and anti-Ramsey theory

Li et al. [11] extended Theorem 1.1 as follows: if the average c-degree $\alpha^c(G) = (1/n)\sum_{v\in V} \deg^c_G(v)$ satisfies $\alpha^c(G) \geq (n+1)/2$, then (G, c) admits a rainbow C_3; if the number of colors $|c(E)|$ used on $G = (V, E)$ satisfies $|E|+|c(E)| \geq \binom{n}{2}$, then (G, c) admits a rainbow C_3. These extensions relate to a classical anti-Ramsey[4] result of Erdős et al. [5] that any edge-coloring of $G = K_n$ with n colors admits a rainbow C_3. More generally, the following holds.

Theorem 1.6 (*Montellano-Ballesteros and Neumann-Lara [13]*). *For every integer $\ell \geq 3$, every edge-colored complete graph (K_n, c) satisfying*

$$|c(E(K_n))| \geq \left(\frac{\ell-2}{2} + \frac{1}{\ell-1}\right) n + O(1)$$

admits a rainbow C_ℓ.

Theorem 1.6 confirmed a conjecture in [5] whose sharpness was already noted there: let $n = q(\ell-1)+r$ for $q, r \in \mathbb{Z}$ satisfying $0 \leq r < \ell-1$; let $V(K_n) = V_1 \dot\cup \dots \dot\cup V_{q+1}$ be a partition satisfying $|V_1| = \cdots = |V_q| = \ell-1$ and $|V_{q+1}| = r$; let all pairs of $\bigcup_{1\leq i\leq q+1} \binom{V_i}{2}$ be given distinct colors; let all pairs crossing V_i and $V_{i+1}\dot\cup \dots \dot\cup V_{q+1}$ be given a new color ξ_i, where $1 \leq i \leq q$. This coloring is

[4] For a comprehensive survey of anti-Ramsey theory, see [7].

locally imbalanced, so one may seek bounds on $\delta^c(K_n)$ ensuring a rainbow C_ℓ in (K_n, c). For fixed $\ell \geq 3$, Axenovich et al. [2] proved that $\delta^c(K_n) \geq (1 + o(1))n/2$ ensures a rainbow C_ℓ, but that for $\ell = 3$ the bound $\delta^c(K_n) \geq (1 + o(1)) \log_2 n$ already suffices (where $\log_2 n$ is necessary). Thus, replacing K_n with an n-vertex host $G = (V, E)$ (see Theorem 1.1) significantly changes the nature of the problem.

2. Proof of Theorem 1.1

We recall Li's proof [10] of Theorem 1.1. Let (G, c) be an n-vertex edge-colored and edge-minimal graph with no rainbow triangle C_3. We show that $\delta^c(G) \leq n/2$. To that end, for a color $\alpha \in c(E)$ and a vertex $v \in V$, we define the *α-neighborhood*

$$N_\alpha(v) = \{u \in N(v) : \ c(\{u, v\}) = \alpha\},$$

where $N(v) = N_G(v) = \{u \in V : \{u, v\} \in E\}$ is the usual neighborhood of v in G, and $N[v] = \{v\} \cup N(v)$ is the *closed neighborhood* of v in G. We define

$$N_!(v) = \bigcup_{\alpha \in c(E)} \{N_\alpha(v) : \ |N_\alpha(v)| = 1\}$$

for the set of neighbors $u \in N(v)$ for which $c(\{u, v\})$ appears uniquely among $\{v, w\} \in E$. We define the *replication number* $R = R(G, c)$ of (G, c) by

$$R = R(G, c) = \max_{v \in V} \max_{\alpha \in c(E)} |N_\alpha(v)|. \tag{1}$$

For $v \in V$ and $U \subseteq V$, we denote by $\deg^c_G(v, U)$ the number of colors $c(\{u, v\})$ among $u \in N(v) \cap U$.

Fix $(z, \zeta) \in V \times c(E)$ for which $|N_\zeta(z)| = R$ (cf. (1)). If $N_!(z) = \emptyset$, then each color incident to z appears at least twice, so $\delta^c(G) \leq \deg^c_G(z) \leq (n-1)/2$ follows. Henceforth, we assume $N_!(z) \neq \emptyset$, and we define the directed graph $\vec{D} = (V_D, \vec{E}_D)$ on vertex set $V_D = N(z)$ by putting, for each edge $\{x, y\} \in E(G[N(z)])$, the arc $(x, y) \in \vec{E}_D$ if, and only if, $y \in N_!(z)$ and $c(\{x, z\}) = c(\{x, y\})$. Then

$$|V_D| = \deg_G(z) \geq \deg^c_G(z) + R - 1, \qquad \text{and} \qquad |\vec{E}_D| = \sum_{y \in N_!(z)} \deg^-_D(y) = \sum_{x \in N(z)} \deg^+_D(x), \tag{2}$$

where $\deg^+_D(x)$ denotes the *out-degree* of a vertex $x \in V_D$ in D, and $\deg^-_D(x)$ denotes the corresponding *in-degree*. We make three observations on D:

(*i*) Each $(x, y) \in \vec{E}_D$ places $x \in N_!(z)$, lest some $x \neq x' \in N(z) \setminus N_!(z)$ gives $c(\{x', z\}) = c(\{x, z\}) = c(\{x, y\})$ (cf. Remark 1.5);

(*ii*) Each $x \in N_!(z)$ with $\alpha = c(\{x, z\})$ satisfies $\deg^+_D(x) = |N_\alpha(x) \cap N_!(z)| \leq R - 1$ (cf. (1));

(*iii*) Each $y \in N_!(z)$ has $\deg^c_G(y, N[z]) \leq 1 + \deg^-_D(y)$, as $c(\{x, y\}) = \beta \neq \alpha = c(\{y, z\})$ for $x \in N(z)$ puts $(x, y) \in \vec{E}_D$, since $c(\{x, z\}) \neq \alpha$ by $y \in N_!(z)$ and $c(\{x, z\}) = \beta$ lest $\{x, y, z\}$ is rainbow.

Thus,

$$\sum_{y \in N_!(z)} (\deg^c_G(y, N[z]) - 1) \overset{(iii)}{\leq} \sum_{y \in N_!(z)} \deg^-_D(y) \overset{(2)}{=} \sum_{x \in N(z)} \deg^+_D(x) \overset{(i)}{=} \sum_{x \in N_!(z)} \deg^+_D(x) \overset{(ii)}{\leq} |N_!(z)|(R-1).$$

Averaging over $N_!(z)$ guarantees a vertex $y_0 \in N_!(z)$ for which

$$\deg^c_G(y_0, N[z]) \leq R. \tag{3}$$

Since $\deg^c_G(y_0, V \setminus N[z]) \geq \deg^c_G(y_0) - \deg^c_G(y_0, N[z])$, we conclude

$$\begin{aligned} n - 1 - \deg_G(z) &= n - |N[z]| \geq \deg^c_G(y_0, V \setminus N[z]) \\ &\geq \deg^c_G(y_0) - \deg^c_G(y_0, N[z]) \overset{(3)}{\geq} \deg^c_G(y_0) - R \\ &\implies n - 1 \geq \deg_G(z) + \deg^c_G(y_0) - R \overset{(2)}{\geq} \deg^c_G(z) + R - 1 + \deg^c_G(y_0) - R \end{aligned}$$

from which $2\delta^c(G) \leq \deg^c_G(z) + \deg^c_G(y_0) \leq n$ and $\delta^c(G) \leq n/2$ follow. □

3. Tools for proving Theorem 1.2

All tools of this section depend on the following concepts of *separation* and *restriction*.

Definition 3.1 (*Separates/restricts*). Let (G, c) be an edge-colored graph, and fix $v \in V = V(G)$ and $X \subseteq N(v)$. We say a color $\alpha \in c(E)$ will X-*separate* a vertex $y \in V$ from v when some $x \in N(y) \cap X$ satisfies $\alpha = c(\{x, y\}) \neq c(\{v, x\})$. If, additionally, $\alpha \neq c(\{w, y\})$ for all $w \in N(y) \setminus X$, then we say that (v, X) *restricts* the color α for y. We denote by $\sigma_{v,X}(y)$ the number of colors $\alpha \in c(E)$ which X-separate y from v, and we denote by $\rho_{v,X}(y)$ the number of colors $\alpha \in c(E)$ restricted for y by (v, X).

Every color $\alpha \in c(E)$ restricted for y by (v, X) also X-separates y from v, so $\sigma_{v,X}(y) \geq \rho_{v,X}(y)$ holds. The next result formally extends Theorem 1.1 (see Remark 3.3) by averaging these numbers.

Proposition 3.2. *Let (G, c) be an n-vertex edge-colored and edge-minimal (cf. Remark 1.5) graph with $R = R(G, c)$ from (1), and fix $v \in V$, $X \subseteq N(v)$, and $\emptyset \neq Y \subseteq V \setminus \{v\}$. Then*

$$\frac{1}{|Y|}\sum_{y\in Y}\sigma_{v,X}(y) \geq \frac{1}{|Y|}\sum_{y\in Y}\rho_{v,X}(y) \geq \delta^c(G) + |X| - n - (R-1)\frac{|X\cap N_!(v)|}{|Y|}.$$

Proof of Proposition 3.2. Let (G, c), R, v, X and Y be given as above, where it suffices to prove the rightmost inequality for $X \neq \emptyset$. Define the directed graph $D = (V_D, \vec{E}_D)$ on vertex set $V_D = X \cup Y$ by putting, for each edge $\{x, y\} \in E$ with $x \in X$ and $y \in Y$, the arc $(x, y) \in \vec{E}_D$ if, and only if, $c(\{x, y\}) = c(\{v, x\})$. Similarly to (i) and (ii) of Section 2, each $(x, y) \in \vec{E}_D$ gives $x \in N_!(v)$ and $\deg^+_D(x) \leq R - 1$, so

$$\sum_{y\in Y}\deg^-_D(y) = |\vec{E}_D| = \sum_{x\in X}\deg^+_D(x) = \sum_{x\in X\cap N_!(v)}\deg^+_D(x) \leq (R-1)|X\cap N_!(v)|. \quad (4)$$

Similarly to (iii) of Section 2, each $y \in Y$ admits at most $\deg^-_D(y) + \rho_{v,X}(y)$ many colors $\alpha \in c(E)$:

(a) $\alpha = c(\{x, y\})$ for some $x \in N(y) \cap X$;
(b) $\alpha \neq c(\{w, y\})$ for all $w \in N(y) \setminus X$.

Indeed, let $\alpha = c(\{x, y\})$ be such a color. If $\alpha = c(\{x, y\}) = c(\{v, x\})$, then $(x, y) \in \vec{E}_D$, and otherwise (v, X) restricts $\alpha = c(\{x, y\}) \neq c(\{v, x\})$ for y (cf. Definition 3.1). Consequently,

$$\begin{aligned} n - |X| &\geq \deg^c_G(y, V\setminus X) \geq \deg^c_G(y) - \deg^-_D(y) - \rho_{v,X}(y) \\ \Longrightarrow \quad & \deg^-_D(y) \geq \deg^c_G(y) - \rho_{v,X}(y) + |X| - n \geq \delta^c(G) - \rho_{v,X}(y) + |X| - n. \end{aligned} \quad (5)$$

Applying (5) to (4) renders the desired result. □

Remark 3.3. Proposition 3.2 implies Theorem 1.1: Let (G, c) be edge-minimal with no rainbow C_3, and fix (z, ζ) with $|N_\zeta(z)| = R$, $x \in N(z) = X$, and (if possible) $y \in N_!(z) = Y$. Then $\rho_{z,X}(y) = 0$ as $c(\{x, y\}) \neq c(\{x, z\})$ gives $c(\{y, z\}) = c(\{x, y\})$ with $z \notin X$, since $\{x, y, z\}$ is not rainbow and $c(\{y, z\}) = c(\{x, z\})$ violates $y \in N_!(z)$. Now, $\delta^c(G) \leq n - |X| + R - 1 \leq n - \delta^c(G)$ so $\delta^c(G) \leq n/2$. □

For (v, X) fixed, Proposition 3.2 shows that some vertices $y \in V$ may admit many colors which X-separate y from v. For relevant (G, c), Proposition 3.4 finds vertices $y \in V$ with few such colors.

Proposition 3.4. *Fix an integer $\ell \geq 3$, and let (G, c) be an edge-colored and edge-minimal[5] graph with no rainbow ℓ-cycle C_ℓ. Fix $v \in V = V(G)$ and $X \subseteq N(v)$, and let $\mathcal{C}_{\mathrm{rep}} = \mathcal{C}_{\mathrm{rep}}(v, X)$ be the colors that repeat on the edges between v and X. Let $Y = Y(v, \mathcal{C}_{\mathrm{rep}})$ be the vertices $y \in V$ to which there is an $(\ell - 1)$-vertex rainbow path P_{vy} from v, none of whose edges has a color in $\mathcal{C}_{\mathrm{rep}}$. Then every $y \in Y$ satisfies $\sigma_{v,X}(y) \leq 3\ell$.*

[5] Recall again Remark 1.5.

Proof of Proposition 3.4. Let (G, c), v, X, $\mathcal{C}_{\text{rep}}$, $y \in Y$, and P_{vy} be given as above. For a vertex $x \in N(y) \cap X$, the subgraph $P_{vy} + \{x, y\} + \{v, x\}$ is a rainbow ℓ-cycle in (G, c) unless:

(A) $x \in V(P_{vy})$; (B) $c(\{x, y\}) \in c(E(P_{vy}))$;
(C) $c(\{v, x\}) \in c(E(P_{vy}))$; or (D) $c(\{x, y\}) = c(\{v, x\})$.

At most $|N(y) \cap X \cap V(P_{vy})| \leq \ell - 3$ colors $c(\{x, y\})$ are given by a vertex $x \in N(y) \cap X \cap V(P_{vy})$ satisfying (A) and at most $|E(P_{vy})| \leq \ell - 2$ colors $c(\{x, y\})$ satisfy (B). At most $|E(P_{vy})| \leq \ell - 2$ colors $c(\{x, y\})$ satisfy (C) because $c(\{v, x\}) \notin \mathcal{C}_{\text{rep}}$. All remaining $c(\{x, y\})$ over $x \in N(y) \cap X$ satisfy (D), lest (G, c) admits a rainbow ℓ-cycle C_ℓ. □

3.1. Some corollaries

We now consider several useful corollaries of Propositions 3.2 and 3.4.

Corollary 3.5. *Fix an integer $\ell \geq 3$, and let (G, c) be an n-vertex edge-colored and edge-minimal graph with no rainbow ℓ-cycle C_ℓ. Let $(z, \zeta) \in V \times c(E)$ satisfy $|N_\zeta(z)| = R$ (cf. (1)), and let $Y = Y(z, \zeta) \subseteq V$ be the vertices $y \in V$ to which there is an $(\ell - 1)$-vertex rainbow path P_{zy} from z, none of whose edges is colored ζ. If $Y \neq \emptyset$,*

$$\delta^c(G) \leq \frac{n}{2} + \max\left\{0,\ 3\ell + (R - 1)\left(\frac{n+1}{2|Y|} - 1\right)\right\}.$$

Proof of Corollary 3.5. Let (G, c), z, ζ, R and $Y = Y(z, \zeta) \neq \emptyset$ be given as above, where for the sake of an argument we assume $\delta^c(G) > n/2$. Let $X \subseteq N(z)$ satisfy that $|X| = \lceil n/2 \rceil$, that $\zeta = c(\{x_0, z\})$ for some $x_0 \in X$, and that all $\{x, z\}$ with $x \in X$ are colored distinctly. Set $X^+ = X \cup N_\zeta(z)$, and set $\mathcal{C}_{\text{rep}} = \mathcal{C}_{\text{rep}}(z, X^+)$ to be the colors $\alpha = c(\{x, z\})$ repeating among $x \in X^+$. Then $\mathcal{C}_{\text{rep}} \subseteq \{\zeta\}$, which by hypothesis is forbidden on the path P_{zy} ending in $y \in Y = Y(z, \zeta)$. Proposition 3.4 guarantees that $\sigma_{z,X^+}(y) \leq 3\ell$ holds for each $y \in Y$, and Proposition 3.2 then renders

$$3\ell \geq \frac{1}{|Y|} \sum_{y \in Y} \sigma_{z,X^+}(y) \geq \delta^c(G) + |X^+| - n - \frac{(R-1)|X^+ \cap N_!(z)|}{|Y|},$$

and using $|X^+| = |X| + R - 1$ and $\lceil n/2 \rceil = |X| \geq |N_!(z) \cap X^+|$ completes the proof. □

In practice, the set $Y = Y(z, \zeta)$ in Corollary 3.5 will be large, and will guarantee the following result.

Corollary 3.6. *Fix an integer $\ell \geq 3$, and let (G, c) be an n-vertex edge-colored graph with no rainbow ℓ-cycle C_ℓ. Then $\delta^c(G) \leq (n/2) + 3\ell$.*

Proof of Corollary 3.6. Let (G, c) be given as above. For the sake of an argument, we assume $\delta^c(G) \geq (n/2) + 2\ell - 5$, and w.l.o.g. we assume (G, c) is edge-minimal. Let $(z, \zeta) \in V \times c(E)$ satisfy that $|N_\zeta(z)| = R$ (cf. (1)). For $1 \leq i \leq \ell - 1$, let $Y_i = Y_i(z, \zeta)$ be the set of vertices $y_i \in V$ to which there is an i-vertex rainbow path P_{zy_i} from z, none of whose edges is colored ζ. Inductively, these sets are non-empty as $Y_1 = \{z\}$, and for some $1 \leq j \leq \ell - 2$, a fixed $y_j \in Y_j$ and corresponding path P_{zy_j} provide

$$|Y_{j+1}| \geq \deg_G^c(y_j) - 1 - |E(P_{zy_j})| - (|V(P_{zy_j})| - 2) \geq \delta^c(G) - 2j + 2 \geq \delta^c(G) - 2\ell + 6 \geq \tfrac{n+1}{2} \quad (6)$$

with $\delta^c(G) \geq (n/2) + 2\ell - 5$. Corollary 3.5 now guarantees

$$\delta^c(G) \leq \frac{n}{2} + 3\ell + (R - 1)\left(\frac{n+1}{2|Y_{\ell-1}|} - 1\right) \overset{(6)}{\leq} \frac{n}{2} + 3\ell,$$

as desired. □

The following corollary describes sets similar to $Y(z, \zeta)$ which are also large.

Corollary 3.7. *Fix an integer $\ell \geq 3$, and let (G, c) be an edge-colored and edge-minimal graph with no rainbow ℓ-cycle C_ℓ. Let T be a triangle in G, let $v \in V(T)$, and let $\mathcal{C}_T \subseteq c(E)$ satisfy $\mathcal{C}_T \cap c(E(T)) = \emptyset$. Let $Y = Y(v, \mathcal{C}_T) = Y_{\ell-1}(v, \mathcal{C}_T)$ be the vertices $y \in Y$ to which there is an $(\ell-1)$-vertex rainbow path P_{vy} from v, none of whose edges has a color in $\mathcal{C}_T$. Then $|Y| \geq (3/2)(\delta^c(G) - |\mathcal{C}_T| - 4\ell)$.*

Proof of Corollary 3.7. Let (G, c), T, v and $\mathcal{C}_T$ be given as above, where for sake of argument we assume $\delta^c(G) \geq |\mathcal{C}_T| + 4\ell + 1$, and where we set $\hat{\mathcal{C}}_T = \mathcal{C}_T \cup c(E(T))$. Since T is not monochromatic (cf. Remark 1.5), we label $V(T) = \{v, x_1, x_2\}$ with $c(\{v, x_2\}) \neq c(\{x_1, x_2\})$. For $1 \leq i \leq \ell - 1$, let $W_i = W_i(x_1, \hat{\mathcal{C}}_T)$ be the set of vertices $w_i \in V$ to which there is an i-vertex rainbow path $P_{x_1 w_i}$ from x_1, none of whose edges has a color in $\hat{\mathcal{C}}_T$, and whose vertices meet $V(T)$ only in x_1. Inductively, these sets are non-empty as $W_1 = \{x_1\}$, and for some $1 \leq j \leq \ell - 2$, a fixed $w_j \in W_j$ and corresponding path $P_{x_1 w_j}$ provide that

$$|W_{j+1}| \geq \deg_G^c(w_j) - |\hat{\mathcal{C}}_T| - |E(P_{x_1 w_j})| - (|V(P_{x_1 w_j})| - 2) - |\{v, x_2\}| \geq \deg_G^c(w_j) - 2(j+1) - |\mathcal{C}_T| \quad (7)$$

is positive from $\delta^c(G) \geq |\mathcal{C}_T| + 4\ell + 1$. It is easy to see that

$$W_{\ell-3}(x_1, \hat{\mathcal{C}}_T) \cup W_{\ell-2}(x_1, \hat{\mathcal{C}}_T) = W_{\ell-3} \cup W_{\ell-2} \subseteq Y = Y_{\ell-1}(v, \mathcal{C}_T). \quad (8)$$

Indeed, if $w_{\ell-3} \in W_{\ell-3}$ is given by $P_{x_1 w_{\ell-3}}$, then the path $P_{x_1 w_{\ell-3}} + \{v, x_2\} + \{x_1, x_2\}$ places $w_{\ell-3} \in Y$, and if $w_{\ell-2} \in W_{\ell-2}$ is given by $P_{x_1 w_{\ell-2}}$, then the path $P_{x_1 w_{\ell-2}} + \{v, x_1\}$ places $w_{\ell-2} \in Y$. We bound (8) as follows. Let $\Gamma = G[W_{\ell-3}]$ be the edge-colored subgraph of G induced on $W_{\ell-3}$. Then Γ admits no rainbow ℓ-cycles C_ℓ, whence Corollary 3.6 guarantees a vertex $w_{\ell-3} \in W_{\ell-3}$ for which $\deg_\Gamma^c(w_{\ell-3}) \leq (1/2)|W_{\ell-3}| + 3\ell$. As such (see the last inequality of (7) to bound $|W_{\ell-2}|$),

$$|W_{\ell-2} \setminus W_{\ell-3}| \geq \deg_G^c(w_{\ell-3}) - 2(\ell-2) - |\mathcal{C}_T| - \deg_\Gamma^c(w_{\ell-3}) \geq \delta^c(G) - \tfrac{1}{2}|W_{\ell-3}| - 5\ell - |\mathcal{C}_T|, \quad (9)$$

and so

$$\begin{aligned} |Y| &\overset{(8)}{\geq} |W_{\ell-2} \cup W_{\ell-3}| = |W_{\ell-2} \setminus W_{\ell-3}| + |W_{\ell-3}| \overset{(9)}{\geq} \delta^c(G) + \tfrac{1}{2}|W_{\ell-3}| - 5\ell - |\mathcal{C}_T| \\ &\overset{(7)}{\geq} \tfrac{3}{2}\delta^c(G) - 6\ell - \tfrac{3}{2}|\mathcal{C}_T|, \end{aligned}$$

as promised. □

4. Proof of Theorem 1.2

Fix an integer[6] $\ell \geq 3$. Let (G, c) be an n-vertex edge-colored and edge-minimal graph (cf. Remark 1.5) satisfying $\delta^c(G) \geq (n+1)/2$. We assume that (G, c) admits no rainbow ℓ-cycle C_ℓ, and we bound $n \leq n_0(\ell)$ from above in the course of this proof. Fix $(z, \zeta) \in V \times c(E)$ with $|N_\zeta(z)| = R$ (cf. (1)). Let $X \subset N(z)$ satisfy that $|X| = \delta^c(G) - 1$ and that $c(\{x, z\}) \neq \zeta$ are distinct among $x \in X$. We distinguish two cases.

Case 1 ($\exists\, e_0 \in E(G[X]) :\ c(e_0) \neq \zeta$**).** By our choice of X, the following hold:

(I) ζ does not appear on the triangle $T = \{z\} \cup e_0$;
(II) ζ is the only color possibly repeating among $c(\{x, z\})$ for $x \in X^+ = X \cup N_\zeta(z)$.

As such, we set $\mathcal{C}_{\text{rep}} = \mathcal{C}_T \subseteq \{\zeta\}$ so that the set $Y = Y(z, \mathcal{C}_{\text{rep}}) = Y(z, \zeta) = Y(z, \mathcal{C}_T)$ commonly featured in each of Proposition 3.4 and Corollaries 3.5 and 3.7 has size, by the last of these,

$$|Y| \geq \tfrac{3}{2}(\delta^c(G) - 1 - 4\ell) \geq \tfrac{3}{2}\left(\tfrac{n+1}{2} - 1 - 4\ell\right) \overset{*}{\geq} \tfrac{2}{3}(n+1), \quad (10)$$

where $*$ holds when $n \geq 78\ell$, which we assume for the sake of an argument. Corollary 3.5 then yields

$$\frac{n+1}{2} \leq \delta^c(G) \leq \frac{n}{2} + 3\ell + (R-1)\left(\frac{n+1}{2|Y|} - 1\right) \overset{(10)}{\leq} \frac{n}{2} + 3\ell - \frac{1}{4}(R-1) \quad \Longrightarrow \quad R \leq 12\ell. \quad (11)$$

[6] By Theorem 1.3, it suffices to prove Theorem 1.2 for odd integers ℓ. However, most of the current argument is independent of parity considerations, so we make no distinction now.

Now, define the directed graph $F = (V, \vec{E}_F)$ on vertex set $V = V(G)$, where

$$\vec{E}_F = \left\{(x, y) \in X^+ \times V : \{x, y\} \in E = E(G) \text{ and } c(\{x, y\}) \neq c(\{x, z\})\right\}.$$

Note that for every $(x, y) \in \vec{E}_F$, the color $c(\{x, y\})$ does X^+-separate y from z (cf. Definition 3.1). On the one hand, every $x \in X^+$ clearly satisfies $\deg_F^+(x) \geq \deg_G^c(x) - 1$, and so

$$|\vec{E}_F| = \sum_{x \in X^+} \deg_F^+(x) \geq \sum_{x \in X^+} (\deg_G^c(x) - 1) \geq |X^+|(\delta^c(G) - 1). \tag{12}$$

On the other hand, with Y defined above,

$$|\vec{E}_F| = \sum_{y \in V} \deg_F^-(y) = \sum_{y \in V \setminus Y} \deg_F^-(y) + \sum_{y \in Y} \deg_F^-(y) \leq (n - |Y|)|X^+| + \sum_{y \in Y} \deg_F^-(y). \tag{13}$$

For a fixed $y \in Y$, we bound

$$\begin{aligned} \deg_F^-(y) &= \left|\left\{x \in N(y) \cap X^+ : c(\{x, y\}) \neq c(\{x, z\})\right\}\right| \\ &= \sum_{\alpha \in c(E)} \left| \left\{x \in N_\alpha(y) \cap X^+ : c(\{x, z\}) \neq \alpha\right\} \right|. \end{aligned}$$

Let $\mathcal{A} = \mathcal{A}_y$ be these colors $\alpha \in c(E)$ admitting some $x \in N_\alpha(y) \cap X^+$ with $c(\{x, z\}) \neq \alpha$ (where $\alpha = c(\{x, y\})$ from $x \in N_\alpha(y)$). Then $\mathcal{A}$ is precisely the set of colors which X^+-separate y from z, so $|\mathcal{A}| = \sigma_{z,X^+}(y)$ holds by Definition 3.1. Then

$$\deg_F^-(y) = \sum_{\alpha \in \mathcal{A}} \left| \left\{x \in N_\alpha(y) \cap X^+ : \alpha \neq c(\{x, z\})\right\} \right| \leq \sum_{\alpha \in \mathcal{A}} \left|N_\alpha(y) \cap X^+\right| \leq \sum_{\alpha \in \mathcal{A}} \left|N_\alpha(y)\right| \overset{(1)}{\leq} |\mathcal{A}|R. \tag{14}$$

Since $y \in Y = Y(z, \mathcal{C}_{\text{rep}})$, Proposition 3.4 guarantees that $|\mathcal{A}| = \sigma_{z,X^+}(y) \leq 3\ell$, and so

$$\deg_F^-(y) \overset{(14)}{\leq} |\mathcal{A}|R = \sigma_{z,X^+}(y) \cdot R \overset{\text{Prop. 3.4}}{\leq} 3\ell R \overset{(11)}{\leq} 36\ell^2. \tag{15}$$

Applying (15) to (13) yields

$$|\vec{E}_F| \leq (n - |Y|)|X^+| + \sum_{y \in Y} \deg_F^-(y) \leq (n - |Y|)|X^+| + 36\ell^2|Y|. \tag{16}$$

Comparing (12) and (16) yields $|X^+|(\delta^c(G) - 1) \leq (n - |Y|)|X^+| + 36\ell^2|Y|$, or equivalently,

$$n \geq \delta^c(G) - 1 + \left(1 - \frac{36\ell^2}{|X^+|}\right)|Y|.$$

Using $|X^+| = \delta^c(G) - 1 + R \geq \delta^c(G) \geq (n + 1)/2$, we infer

$$\tfrac{1}{2}(n + 1) \geq n - \delta^c(G) + 1 \geq \left(1 - \tfrac{72\ell^2}{n+1}\right)|Y| \overset{(10)}{\geq} \left(1 - \tfrac{72\ell^2}{n+1}\right) \times \tfrac{2}{3}(n + 1), \tag{17}$$

which implies $n \leq 288\ell^2 - 1$. □

Case 2 ($\forall e \in E(G[X]),\ c(e) = \zeta$). Set $Y = V \setminus (\{z\} \cup X)$. We first observe

$$\delta^c(G) - 2 \leq |Y| \leq \delta^c(G) - 1 \quad \text{and} \quad \tfrac{1}{2}(n - 1) \leq \delta^c(G) - 1 = |X| \leq \tfrac{1}{2}(n + 1). \tag{18}$$

Indeed, $|Y| = n - 1 - |X| = n - \delta^c(G) \leq \delta^c(G) - 1$ holds from $|X| = \delta^c(G) - 1 \geq (1/2)(n - 1)$. Now, fix $x \in X$ and $\{x, y\} \in E$ where $c(\{x, y\}) \neq \zeta$ and $c(\{x, y\}) \neq c(\{x, z\})$. Then $y \in Y$ and there are at least $\deg_G^c(x) - 2$ many such edges. Thus, $|Y| \geq \deg_G^c(x) - 2 \geq \delta^c(G) - 2$ holds and $|X| \leq (1/2)(n + 1)$ follows.

We now define two subsets of Y that we wish to later avoid. For that, let $H = G[X, Y]$ be the bipartite subgraph of G induced by the bipartition $X \cup Y$, and let D be the subgraph of H consisting of edges $\{x, y\} \in E(H)$ with $x \in X$, $y \in Y$, and $c(\{x, y\}) = c(\{x, z\})$. Let Y_H be the vertices $y \in Y$ sending

$\deg^c_H(y) \le (5/2)\ell$ many distinct colors to X, and let Y_D be the vertices $y \in Y$ sending $\deg_D(y) \ge 2$ many D-edges to X.

Claim 4.1. *$|Y_H| \le 11\ell$ and $|Y_D| \le |X|/2$.*

Proof of Claim 4.1. Let $\Gamma = G[A]$ be the edge-colored subgraph of G induced on $A = Y_H$. Since $G[A]$ has no rainbow ℓ-cycles C_ℓ, Corollary 3.6 guarantees $a \in A$ with $\deg^c_\Gamma(a) \le (1/2)|A| + 3\ell$. Since a sends at most $(5/2)\ell + 1$ distinct colors to $\{z\} \cup X$ and at most $|Y| - |A|$ distinct colors to $Y \setminus A$, we see

$$\tfrac{1}{2}|A|+3\ell \ge \deg^c_\Gamma(a) \ge \deg^c_G(a) - \tfrac{5}{2}\ell - 1 - |Y| + |A| \overset{(18)}{\ge} |A| - \tfrac{5}{2}\ell \qquad \Longrightarrow \qquad |Y_H| = |A| \le 11\ell.$$

Since each $x \in X$ sends to Y precisely $\deg_D(x)$ many D-edges and $\ge \deg^c_G(x) - 2$ many ζ-free $H \setminus D$-edges,

$$\begin{aligned} &\deg_D(x) + \deg^c_G(x) - 2 \le \deg_H(x) \le |Y| \overset{(18)}{\le} \delta^c(G) - 1 \quad \Longrightarrow \quad \deg_D(x) \le 1 \\ &\Longrightarrow \quad 2|Y_D| \le \sum_{y \in Y_D} \deg_D(y) \le \sum_{y \in Y} \deg_D(y) = |E(D)| = \sum_{x \in X} \deg_D(x) \le |X|, \end{aligned}$$

and so $|Y_D| \le |X|/2$ follows. □

Continuing with Case 2, set $Y_0 = Y \setminus (Y_H \cup Y_D)$, set $H[X, Y_0] = G[X, Y_0]$ to be the bipartite subgraph of H induced by the bipartition $X \cup Y_0$, and set $H_0 = H[X, Y_0] \setminus D$. For each $x \in X$, we already observed (cf. (18)) that x sends at least $\deg^c_G(x) - 2$ many non-ζ, non-$c(\{x, z\})$ colors into Y, and so

$$\forall\, x \in X, \qquad \deg^c_{H_0}(x) \ge \deg^c_G(x) - 2 - |Y \setminus Y_0| \overset{\text{Clm. 4.1}}{\ge} \delta^c(G) - 2 - 11\ell - \tfrac{1}{2}|X| \overset{(18)}{\ge} \tfrac{1}{4}(n+1) - 11\ell - 2. \tag{19}$$

To X, each $y \in Y_0$ sends $\deg^c_H(y) \ge (5/2)\ell + 1$ many colors and $\deg_D(y) \le 1$ many D-edges, and so

$$\forall\, y \in Y_0, \quad \deg^c_{H_0}(y) \ge \tfrac{5}{2}\ell. \tag{20}$$

To conclude Case 2, it is convenient to now distinguish between $\ell \pmod 2$.

Case 2A (ℓ is odd). With (z, ζ) fixed at the start, fix $y_0 \in N_\zeta(z)$ arbitrarily, where necessarily $y_0 \in Y$. The number of non-ζ colors that y_0 sends to X is at least $\deg^c_G(y_0) - 1 - (|Y| - 1) \ge \delta^c(G) - |Y| \ge 1$ by (18), so fix $x_1 \in X \cap N(y_0)$ to satisfy $c(\{x_1, y_0\}) \ne \zeta$. For an even integer $k \ge 2$, let $Q_{k-1} = (z, y_0, x_1, y_2, \dots, x_{k-1})$ be a rainbow path, where $x_1, \dots, x_{k-1} \in X$ and $y_2, \dots, y_{k-2} \in Y_0$. Then Q_{k-1} would be extended to a rainbow path $Q_k = (z, y_0, \dots, x_{k-1}, y_k)$ along at least

$$\deg^c_{H_0}(x_{k-1}) - |E(Q_{k-1})| - |\{y_0, \dots, y_{k-2}\}| = \deg^c_{H_0}(x_{k-1}) - k - \tfrac{k}{2} \overset{(19)}{\ge} \tfrac{1}{4}(n+1) - 11\ell - 2 - \tfrac{3}{2}k \tag{21}$$

many $y_k \in Y_0 \setminus \{y_0, \dots, y_{k-2}\}$, and Q_k would be extended to a rainbow path $Q_{k+1} = (z, y_0, \dots, y_k, x_{k+1})$ along at least

$$\deg^c_{H_0}(y_k) - |E(Q_k)| - |\{x_1, \dots, x_{k-1}\}| = \deg^c_{H_0}(y_k) - (k+1) - (k/2)$$

many $x_{k+1} \in X \setminus \{x_1, \dots, x_{k-1}\}$. More strongly, X was chosen with $c(\{x, z\})$ distinct among $x \in X$, so Q_k would be extended to a rainbow path $Q_{k+1} = (z, y_0, \dots, y_k, x_{k+1})$ along at least

$$\deg^c_{H_0}(y_k) - 2(k+1) - (k/2) \overset{(20)}{\ge} \tfrac{5}{2}(\ell - k - (4/5)) \tag{22}$$

many $x_{k+1} \in X \setminus \{x_1, \dots, x_{k-1}\}$ where additionally $c(\{x_{k+1}, z\}) \notin c(E(Q_k))$. Then Q_{k+1} bears the rainbow $(k+3)$-cycle $(z, y_0, x_1, \dots, y_k, x_{k+1}, z)$ since $c(\{x_{k+1}, z\}) \notin c(E(Q_k))$ holds and since $c(\{x_{k+1}, y_k\}) \ne c(\{x_{k+1}, z\})$ holds from $\{x_{k+1}, y_k\} \notin E(D)$. Since (G, c) has no rainbow ℓ-cycles C_ℓ, it must be that $k+3 \leq \ell - 1$. Since (22) is positive with $k = \ell - 4$, (21) must be non-positive, whence $n \leq 50\ell$. $\square$

Case 2B (ℓ is even). The argument above slightly simplifies. Choose $x_1 \in X$ arbitrarily. As before, we extend a rainbow path $\hat{Q}_{k-1} = (z, x_1, y_2, \ldots, x_{k-1})$ with $x_1, \ldots, x_{k-1} \in X$ and $y_2, \ldots, y_{k-2} \in Y_0$ to rainbow paths $\hat{Q}_k = (z, x_1, \ldots, x_{k-1}, y_k)$ and $\hat{Q}_{k+1} = (z, x_1, \ldots, y_k, x_{k+1})$ where $y_k \in Y_0 \setminus \{y_2, \ldots, y_{k-2}\}$ and $x_{k+1} \in X \setminus \{x_1, \ldots, x_{k-1}\}$, and where $c(\{x_{k+1}, z\}) \notin c(E(\hat{Q}_k))$. The paths $\hat{Q}_k$ and $\hat{Q}_{k+1}$ are respectively shorter than Q_k and Q_{k+1} above, so inequalities analogous to those in (21) and (22) still hold, and with $k + 1 < \ell - 1$, we similarly conclude $n \leq 50\ell$. $\square$

5. Proof of Theorem 1.4

Our proof of Theorem 1.4 follows that of Theorem 1.2, where we also use the following corollary of Propositions 3.2 and 3.4 from Section 3.

Corollary 5.1. *Fix an integer $\ell \geq 3$, and let (G, c) be an n-vertex edge-colored and edge-minimal graph with no rainbow ℓ-cycle C_ℓ and with $\delta^c(G) \geq 5R + 27\ell$ (cf. (1)). Then $\delta^c(G) < n/2$ or $\Delta(G) < \delta^c(G) + 4R + 3\ell$.*

Proof of Corollary 5.1. Let (G, c) be given as above. Assume for a contradiction that $\delta^c(G) \geq n/2$ and that some $v \in V = V(G)$ has $\deg_G(v) \geq \delta^c(G)+4R+3\ell$. Then $\deg_G(v) \geq (n+1)/2$ whence v is incident to some triangle $T = T_v$ in G, where we set[7] $c(E(T)) = \{\alpha, \beta, \gamma\}$. Since $|N_\alpha(v) \cup N_\beta(v) \cup N_\gamma(v)| \leq 3R$, some $X \subseteq N(v)$ has size $|X| = \delta^c(G)+R+3\ell$, where at least $\delta^c(G)$ of the colors $c(\{v, x\})$ are distinct among $x \in X$, and where $|N_\alpha(v) \cap X|, |N_\beta(v) \cap X|, |N_\gamma(v) \cap X| \leq 1$. Let $\mathcal{C}_T = \mathcal{C}_{\text{rep}}$ be the $\leq R + 3\ell$ colors $c(\{v, x\})$ repeating among $x \in X$, where $\mathcal{C}_T \cap c(E(T)) = \emptyset$. Let $Y = Y(v, \mathcal{C}_{\text{rep}}) = Y(v, \mathcal{C}_T)$ be the set commonly featured in each of Proposition 3.4 and Corollary 3.7. Corollary 3.7 guarantees

$$\begin{aligned} |Y| &\geq \tfrac{3}{2}(\delta^c(G) - |\mathcal{C}_T| - 4\ell) \geq \tfrac{3}{2}(\delta^c(G) - R - 7\ell) = \delta^c(G) + \tfrac{1}{2}\delta^c(G) - \tfrac{3}{2}(R + 7\ell) \\ &\geq \delta^c(G) + \tfrac{1}{2}(5R + 27\ell) - \tfrac{3}{2}(R + 7\ell) = \delta^c(G) + R + 3\ell = |X|. \end{aligned} \tag{23}$$

Proposition 3.2 then guarantees

$$\begin{aligned} \frac{1}{|Y|} \sum_{y \in Y} \sigma_{v,X}(y) &\geq \delta^c(G) + |X| - n - (R - 1)\frac{|X \cap N_!(v)|}{|Y|} \\ &\overset{(23)}{\geq} \delta^c(G) + |X| - n - (R - 1) = 2\delta^c(G) + R + 3\ell - n - (R - 1) \geq 3\ell + 1, \end{aligned} \tag{24}$$

which with $Y = Y(v, \mathcal{C}_{\text{rep}})$ contradicts Proposition 3.4. $\square$

5.1. Proof of Theorem 1.4

Let (G, c), (z, ζ), and $X \subset N(z)$ be given as in Section 4. In Case 2, we proved that $n \leq 50\ell$, but in Case 1 we proved only that $n \leq 288\ell^2$, which we now improve to $n \leq 432\ell - 1$. The bottleneck of Case 1 arises in (15), where a fixed $y \in Y$ satisfies $\deg^-_F(y) \leq 3\ell R \leq 36\ell^2$ (cf. (11)). We claim that

$$\deg^-_F(y) \leq 4R + 6\ell \overset{(11)}{\leq} 54\ell, \tag{25}$$

which if true updates (17) to say

$$\tfrac{1}{2}(n + 1) \geq \left(1 - \tfrac{108\ell}{n+1}\right) \times \tfrac{2}{3}(n + 1),$$

[7] The colors α, β, γ are not identical by Remark 1.5, but they need not all be distinct. These considerations, however, play no role in the current context.

which gives $n \le 432\ell - 1$. To see (25), recall from (14) that

$$\deg_F^-(y) \le \sum_{\alpha \in \mathcal{A}} |N_\alpha(y)|, \tag{26}$$

where $\mathcal{A} = \mathcal{A}_y$ is the set of colors $\alpha \in c(E)$ where some $x \in X^+$ satisfies $\alpha = c(\{x, y\}) \ne c(\{x, z\})$. In particular, $\mathcal{A}$ is precisely the set of colors which X^+-separate y from z, so $|\mathcal{A}| = \sigma_{z,X^+}(y)$ holds by Definition 3.1. Moreover, recall (cf. (15)) that Proposition 3.4 guarantees $|\mathcal{A}| = \sigma_{z,X^+}(y) \le 3\ell$. Now, let $\mathcal{B} = \mathcal{B}_y$ consist of all non-$\mathcal{A}$ colors incident to y, in which case

$$\sum_{\beta \in \mathcal{B}} |N_\beta(y)| \ge |\mathcal{B}| = \deg_G^c(y) - |\mathcal{A}| = \deg_G^c(y) - \sigma_{z,X^+}(y) \overset{\text{Prop. 3.4}}{\ge} \deg_G^c(y) - 3\ell \ge \delta^c(G) - 3\ell. \tag{27}$$

Then

$$\Delta(G) \ge \deg_G(y) = \sum_{\alpha \in \mathcal{A}} |N_\alpha(y)| + \sum_{\beta \in \mathcal{B}} |N_\beta(y)| \overset{(26)}{\ge} \deg_F^-(y) + \sum_{\beta \in \mathcal{B}} |N_\beta(y)| \overset{(27)}{\ge} \deg_F^-(y) + \delta^c(G) - 3\ell.$$

Corollary 5.1 concludes the proof: since $\delta^c(G) \ge (n+1)/2$ holds by hypothesis, $\Delta(G)$ must satisfy

$$\deg_F^-(y) \le \Delta(G) - \delta^c(G) + 3\ell \overset{\text{Cor. 5.1}}{<} \delta^c(G) + 4R + 3\ell - \delta^c(G) + 3\ell \le 4R + 6\ell.$$

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