J. Math. Pures Appl. 149 (2021) 135–185

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

Systems of quasilinear parabolic equations in $\mathbb{R}^n$ and systems of quadratic backward stochastic differential equations

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ARTICLE INFO

Article history:
Received 7 July 2020
Available online 14 January 2021

MSC:
35B37
49N70
60H15
91A06

Keywords:
Partial differential equations
Quasi-linear parabolic systems
Backward stochastic differential systems
Markov processes
Hamiltonian
Differential games

ABSTRACT

The objective of this paper is two-fold. The first objective is to complete the former work of Bensoussan and Frehse [2]. One big limitation of this paper was the fact that they are systems of PDE. on a bounded domain. One can expect solutions to be bounded, since one looks for smooth solutions. This is a very important property for the development of the method. It is true also that solutions which exist in a bounded domain may fail to exist on $\mathbb{R}^n$, because of the lack of bounds. We give conditions so that the results of [2] can be extended to $\mathbb{R}^n$. The second objective is to consider the BSDE (Backward stochastic differential equations) version of the system of PDE. This is the objective of a more recent work of Xing and Žitković [8]. They consider systems of BSDE with quadratic growth, which is a well-known open problem in the BSDE literature. Since the BSDE are Markovian, the problem is equivalent to the analytic one. However, because of this motivation the analytic problem is in $\mathbb{R}^n$ and not on a bounded domain. Xing and Žitković developed a probabilistic approach. The connection between the analytic problem and the BSDE is not apparent. Our objective is to show that the analytic approach can be completely translated into a probabilistic one. Nevertheless probabilistic concepts are also useful, after their conversion into the analytic framework. This is in particular true for the uniqueness result.

RÉSUMÉ

Cet article a deux objectifs. Le premier est de completer le travail de Bensoussan, Frehse (2002). Une très grande restriction était de se placer dans un ouvert borné. Dans ce cas on recherche des solutions bornées puisque régulières. Le fait d'avoir des solutions bornées était très important pour le développement de la méthode.

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[1] Alain Bensoussan is also with the School of Data Sciences, the City University of Hong Kong. Research supported by grant from the National Science Foundation 1905449 and grant from the Research Grants Council HKGRF-11303316.

[2] Phillip Yam acknowledges the financial supports from HKGRF-14300717 with the project title "New kinds of Forward–backward Stochastic Systems with Applications", HKGRF-14300319 with the project title "Shape-constrained Inference: Testing for Monotonicity". He also thanks Columbia University for the kind invitation to be a visiting faculty member in the Department of Statistics during his sabbatical leave. He also recalled the unforgettable moments and the happiness shared with his beloved father and used this work in memory of his father's brave battle against liver cancer.

https://doi.org/10.1016/j.matpur.2021.01.006

Cette absence de bornes peut empêcher de trouver des solutions. Nous donnons des conditions pour que ces résultats puissent être étendus à $\mathbb{R}^n$. Le second objectif est de considerer la version systèmes (BSDE) d'équations differentielles stochastiques rétrogrades du système d'EDP. Ceci est notamment l'objectif d'un travail beaucoup plus récent of Xing, Žitković (2018). Ils cherchent a résoudre des systèmes BSDE à croissance quadratique, ce qui est un probleme ouvert, bien connu dans la literature BSDE. Dans le cas markovien, ce problème est équivalent au problème analytique. Ce qui explique que le problème analytique doit être résolu dans $\mathbb{R}^n$ et non dans un domaine borné. Xing and Žitković developpent une approche probabiliste, en utilisant de nombreux resultats analytiques. Notre objectif est de donner deux approches, l'une completement analytique et l'autre completement probabiliste, sans aucun mélange, comme c'est le cas dans la formulation des deux problèmes.

1. Introduction

In this paper, we consider parabolic systems of quasilinear PDE of the following type

$$\begin{cases} -\dfrac{\partial u_\nu}{\partial t} + Au_\nu = H_\nu(x, u, Du),\ x \in \mathbb{R}^n,\ t \in [0, T], \\ \quad u_\nu(x, T) = h_\nu(x), \end{cases} \tag{1.1}$$

$$u_\nu \in L^d(0, T; W^{2,d}_{\text{loc}}(\mathbb{R}^n)),\ \ \frac{\partial u_\nu}{\partial t} \in L^d(0, T; L^d_{\text{loc}}(\mathbb{R}^n)), d > \frac{n}{2} + 1, \tag{1.2}$$

where $T > 0$ is a given constant and A is the second order differential operator (defined on smooth test functions $\varphi(\cdot)$)

$$A(t)\varphi(x) = -\sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x, t)\frac{\partial \varphi}{\partial x_j}\right), x \in \mathbb{R}^n, t \in [0, T]. \tag{1.3}$$

We write it in divergence form, for convenience. For the probabilistic interpretation, we shall consider the diffusion operator $A(t)$ defined (on smooth test functions $\varphi(\cdot)$) by

$$A(t)\varphi(x) = -\sum_i g_i(x, t)\frac{\partial \varphi}{\partial x_i} - \sum_{ij=1}^{n} a_{ij}(x, t)\frac{\partial^2 \varphi}{\partial x_i \partial x_j}, x \in \mathbb{R}^n, t \in [0, T]. \tag{1.4}$$

This operator corresponds to the stochastic differential equation

$$\begin{cases} dX(s) = g(X(s), s)ds + \sigma(X(s), s)dw(s),\ s > t, \\ \ X(t) = x, \end{cases} \tag{1.5}$$

where $w(s)$ is an n dimensional standard Wiener process, built on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We denote by $\mathcal{F}^s_t$ the σ-algebra generated by $w(\tau) - w(t)$, $t \leq \tau \leq s$. The process $X(s)$ will be denoted in the sequel $X^{xt}(s)$. Furthermore the matrix $a(x, t) = a_{ij}(x, t)$ is given by

$$a(x, t) = \frac{1}{2}\sigma(x, t)\sigma^*(x, t). \tag{1.6}$$

The diffusion operator (1.4) and the divergence form operator coincide when $g_i(x, t) = \sum_j \dfrac{\partial a_{ji}(x, t)}{\partial x_j}$. For simplicity of notation, we will avoid in the sequel to indicate the dependence in t. Also to simplify the probabilistic interpretation we shall assume that the two operators (1.4) and (1.5) coincide. This supposes the differentiability in x of a_{ij}, which will not be necessary in the analytic treatment (however necessary to obtain full regularity). The index ν varies from 1 to N. The functions $H_\nu(x,t,y,p)$ are defined on $\mathbb{R}^n \times [0,T] \times \mathbb{R}^N \times \mathbb{R}^{Nn}$, and will called Hamiltonians in the sequel. We represent p as an $N \times n$ matrix with line vectors $\begin{pmatrix} p_1 \\ \vdots \\ p_N \end{pmatrix}$ in $\mathbb{R}^n$. We will omit the time variable in the Hamiltonians in the sequel. In (1.1), $u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$ and $Du = \begin{pmatrix} Du_1 \\ \vdots \\ Du_N \end{pmatrix}$. These Hamiltonians are motivated by differential games, among other applications. We refer to [3] for details on this application. If we define the stochastic processes

$$Y^{xt}(s) = \begin{pmatrix} u_1(X^{xt}(s),s) \\ \vdots \\ u_N(X^{xt}(s),s) \end{pmatrix}, \quad Z^{xt}(s) = \begin{pmatrix} Du_1(X^{xt}(s),s) \\ \vdots \\ Du_N(X^{xt}(s),s) \end{pmatrix}, \tag{1.7}$$

which take value in $\mathbb{R}^N$, $\mathcal{L}(\mathbb{R}^n;\mathbb{R}^N)$ respectively, then an easy application of Ito's formula shows that the relation

$$Y_\nu^{xt}(s) = h_\nu(X^{xt}(T)) + \int_s^T H_\nu(X^{xt}(\tau), Y^{xt}(\tau), Z^{xt}(\tau))d\tau - \int_s^T Z_\nu^{xt}(\tau).\sigma(X^{xt}(\tau))dw(\tau), \ t \le s \le T, \tag{1.8}$$

holds. The system (1.8) is a system of Markovian BSDE; "Markovian" means that $H_\nu(x,y,p)$ and $h_\nu(x)$ are deterministic functions of the arguments. The solutions $Y^{xt}(s), Z^{xt}(s)$ are $\mathcal{F}_t^s$-measurable.

Our objective is to study the systems (1.1) and (1.8) independently, in the sense that (1.1) will rely on purely analytic techniques and (1.8) on probabilistic techniques.

2. Assumptions and results

2.1. Assumptions

We shall assume that there exists $\alpha > 0$ such that

$$\alpha|\xi|^2 \le a(x)\xi \cdot \xi \le M|\xi|^2, \text{ for all } \ \xi \in \mathbb{R}^n. \tag{2.1}$$

To get maximum smoothness we will assume that

$$a_{ij}(\cdot,\cdot) \text{ are } C^1(\mathbb{R}^n \times [0,T]) \text{ and globally Lipschitz continuous with respect to the variable } x \in \mathbb{R}^n, \tag{2.2}$$

$$\sum_j \frac{\partial a_{ji}(\cdot,\cdot)}{\partial x_j} \text{ are globally Lipschitz continuous in } x \text{ with respect to the variable } x \in \mathbb{R}^n;$$

$$h_\nu(\cdot) \in C^1(\mathbb{R}^n); \tag{2.3}$$

$$\text{for each } t \in [0,T], \ \ H_\nu(x,t,y,p) \text{ is continuous in } (x,y,p) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{Nn}. \tag{2.4}$$

To get upper bounds, we assume that there exist constants γ_ν and functions $\lambda_\nu, \nu \in \{0,1,\cdots,N\}$ such that

$$H_\nu(x,t,y,p) \le \lambda_\nu(x,t) + \gamma_\nu|p_\nu|^2, \gamma_\nu > 0, \nu \in \{1,\cdots,N\}, \tag{2.5}$$

$$\sum_{\nu=1}^N H_\nu(x,t,y,p) \ge -\lambda_0(x,t) - \gamma_0\left|\sum_{\nu=1}^N y_\nu\right|^2, \gamma_0 > 0, \tag{2.6}$$

where

$$\lambda_0, \lambda_1, \cdots, \lambda_N \ge 0. \tag{2.7}$$

We draw the attention to a risk of confusion on the index ν, which primarily refers to the index of Hamiltonians (and of equations in the system). Therefore $\nu \in \{1,\cdots,N\}$ for the Hamiltonians. We have added an index 0, which relates to inequality (2.6). Therefore, when we write $\gamma_\nu, \lambda_\nu, \nu$ runs from 0 to N.

An important assumption will be a growth assumption on the functions $\lambda_0, \lambda_1, \cdots, \lambda_N$ in combination with the functions $h_1, \cdots, h_N$. We also define $h_0(x) = -\sum_1^N h_\nu(x)$. The most intuitive way to express the assumption is through using probabilistic formulas. Define

$$\bar{z}_\nu(x,t) = \mathbb{E}\left[\exp\left(\frac{\gamma_\nu}{\alpha}\right)\left(h_\nu^+(X_{xt}(T)) + \int_t^T \lambda_\nu(X_{xt}(\tau),\tau)\,d\tau\right)\right], \tag{2.8}$$

for $\nu = 0, 1, \cdots, N$. We assume

$$\bar{z}_\nu(x,t) < +\infty, x \in \mathbb{R}^n, t \in [0,T], \nu \in \{0,1,\cdots,N\}. \tag{2.9}$$

This is indeed an assumption, because the right-hand side of (2.8) can well be $+\infty$. An analytic formulation of the assumption is to use the fact that each function $\bar{z}_\nu(\cdot,\cdot)$ is solution of the linear PDE

$$\begin{cases} -\dfrac{\partial \bar{z}_\nu}{\partial t} + A\bar{z}_\nu = \bar{z}_\nu \dfrac{\gamma_\nu}{\alpha}\lambda_\nu(x,t), \\ \quad \bar{z}_\nu(x,T) = \exp\left(\dfrac{\gamma_\nu}{\alpha} h_\nu^+(x)\right). \end{cases} \tag{2.10}$$

In spite of the linearity this equation may not have a solution satisfying (2.9). So the analytic formulation of the assumption is that there exists a solution of (2.10), which is finite, in the sense of (2.9).

We next make the following special structure assumptions for the Hamiltonians: there exist positive constants K_ν and K_μ^ν such that

$$\begin{gathered} H_\nu(x,t,y,p) = Q_\nu(x,t,y,p)\cdot p_\nu + H_\nu^0(x,t,y,p), \\ |Q_\nu(x,t,y,p)| \le K_\nu|p|, \text{ and } |H_\nu^0(x,t,y,p)| \le \sum_{\mu=1}^{\nu} K_\mu^\nu |p_\mu|^2 + k_\nu(x,t), \end{gathered} \tag{2.11}$$

with

$$k_\nu \in L^d(0,T;L^d_{\text{loc}}(\mathbb{R}^d)),\ d > \frac{n}{2}+1,\ k_\nu \ge 0. \tag{2.12}$$

2.2. Discussion about the special structure assumptions

The special structure assumptions imply an ordering of the equations. But this is of course a matter of convenience. We can always reorder the equations if necessary. There is a less stringent assumption than (2.11) which turns out to be equivalent. We assume

$$|H_N(x,t,y,p)| \leq K_N|p|^2 + k_N(x,t), \tag{2.13}$$

$$|H_\nu(x,t,y,p)| \leq K_\nu|p||p_\nu| + \sum_{\mu=1}^{\nu} K_\mu^\nu |p_\mu|^2 + k_\nu(x,t),\ \nu = 1,\cdots,N-1. \tag{2.14}$$

In particular, when $N = 1$, (2.14) is discarded and the Hamiltonian has general quadratic growth. This is consistent with classical results on parabolic scalar equations. For systems, it is known that the result does not carry over, counter examples are available. To check that the assumptions (2.13), (2.14) imply (2.11) we introduce the quantities (we delete the argument t)

$$\sigma_\nu(x,y,p) = \frac{H_\nu(x,y,p)}{K_\nu|p||p_\nu| + \sum_{\mu=1}^{\nu} K_\mu^\nu|p_\mu|^2 + k_\nu(x,t)},\ \nu = 1,\cdots N-1.$$

We then set

$$Q_\nu(x,y,p) = K_\nu\sigma_\nu(x,y,p)|p|\frac{p_\nu}{|p_\nu|},\ \nu = 1,\cdots N-1, \text{ and } Q_N(x,y,p) = Q_{N-1}(x,y,p),$$

and

$$H_\nu^0(x,y,p) = H_\nu(x,y,p) - Q_\nu(x,y,p)\cdot p_\nu,\ \nu = 1,\cdots,N.$$

It is then easy to check that all the properties (2.11) are satisfied with $K_\nu, K_\nu^\mu, \mu \leq \nu$, for $\nu = 1,\cdots,N-1$, and $K_N = K_{N-1}$, $K_N^\mu = K_N + \dfrac{K_{N-1}}{2}$, $\mu \leq N-1$, $K_N^N = K_N + K_{N-1}$.

2.3. Statements of results

We state the following results of this work

Theorem 1. *We make the assumptions (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), (2.9), (2.11), (2.12). Then there exists a solution $u_\nu, \nu = 1,\cdots,N$ of (1.1) such that*

$$u_\nu \in L^d\left(0,T;W_{loc}^{2,d}(\mathbb{R}^n)\right),\quad \frac{\partial u_\nu}{\partial t} \in L^d\left(0,T;L_{loc}^d\left(\mathbb{R}^d\right)\right). \tag{2.15}$$

We next state the stochastic counterpart of Theorem 1

Theorem 2. *Under the assumptions of Theorem 1, there exists a Markovian solution of the system of BSDE (1.8) in the sense*

$$Y_\nu^{xt}(s) = u_\nu(X^{xt}(s),s),\ Z_\nu^{xt}(s) = v_\nu(X^{xt}(s),s), \tag{2.16}$$

in which $u_\nu(x,t)$ is locally Hölderian and $v_\nu(x,t) = Du_\nu(x,t)$ in the sense of distributions. Moreover

$$\mathbb{E}\left[\int_t^T \Psi^2(X^{xt}(s),s)\sum_\nu |Z_\nu^{xt}(s)|^2 ds\right] \leq C(\Psi), \tag{2.17}$$

for any function Ψ which is $C^{2,1}$ and has a compact support. The constant $C(\Psi)$ depends only on the function Ψ and of the constants of the problem.

Remark 3. Of course, Theorem 2 is a consequence of Theorem 1, by simple application of Ito's formula, but the objective is not to use Theorem 1 and to give a probabilistic proof. ■

3. Preliminaries

3.1. Functions $y \mapsto X_\nu(y)$

We begin by constructing a sequence $X_\nu(y), y \in \mathbb{R}^N$, following [2]. We first set

$$\beta(x) := e^x - x - 1, x \in \mathbb{R}. \tag{3.1}$$

We pick positive constants $\gamma_1, \cdots, \gamma_N$ and define recursively

$$\begin{cases} X_N(y) = \exp[\beta(\gamma_N y_N) + \beta(-\gamma_N y_N)], \\ X_\nu(y) = \exp[\beta(\gamma_\nu y_\nu) + \beta(-\gamma_\nu y_\nu) + X_{\nu+1}(y)],\ \nu = N-1, \cdots, 1. \end{cases} \tag{3.2}$$

The following properties hold

$$\begin{cases} \dfrac{\partial X_\nu(y)}{\partial y_\mu} = 0,\ \text{if } \mu > \nu, \\ \dfrac{\partial X_\nu(y)}{\partial y_\mu} = \gamma_\mu X_\nu(y) \cdots X_\mu(y)(\beta'(\gamma_\mu y_\mu) + \beta'(-\gamma_\mu y_\mu),\ \text{if } \mu \le \nu. \end{cases} \tag{3.3}$$

We can summarize this formula as follows. Let

$$\begin{aligned} &P_\nu(y) := X_1(y) \cdots X_\nu(y),\ \nu = 1, \cdots N,\ P_0(y) = 1, \\ &A_\nu(y) := \gamma_\nu P_\nu(y)(\beta'(\gamma_\nu y_\nu) - \beta'(-\gamma_\nu y_\nu)); \end{aligned} \tag{3.4}$$

then we can write

$$\frac{\partial X_\nu(y)}{\partial y_\mu} = A_\mu(y) P_{\nu-1}^{-1}(y) \mathbb{1}_{\mu \ge \nu},\ \nu = 1, \cdots, N. \tag{3.5}$$

3.2. Lyapunov function

Following the terminology of [8], we call $L(y) = X_1(y) - X_1(0)$ a Lyapunov function. We note that $X_N(0) = 1, X_{N-1}(0) = e, X_{N-2}(0) = \exp(e), ..., X_1(0) = \exp(\exp(\cdots e))$ in which we take $N-1$ successive exponentiations. We first obtain from (3.5)

$$\frac{\partial L(y)}{\partial y_\nu} = A_\nu(y). \tag{3.6}$$

We want to compute $\dfrac{\partial^2 L(y)}{\partial y_\mu \partial y_\nu} = \dfrac{\partial A_\mu(y)}{\partial y_\nu}$. Using $\dfrac{\partial P_\mu}{\partial y_\nu} = P_\mu(y) A_\nu(y) \sum_{j=1}^{\mu\wedge\nu} P_j^{-1}(y)$, we easily get

$$\frac{\partial^2 L(y)}{\partial y_\mu \partial y_\nu} = A_\mu(y) A_\nu(y) \sum_{j=1}^{\mu\wedge\nu} P_j^{-1}(y) + \gamma_\mu^2 P_\mu(y)(\beta''(\gamma_\mu y_\mu) + \beta''(-\gamma_\mu y_\mu)) \mathbb{1}_{\mu=\nu}. \tag{3.7}$$

To explain the concept of Lyapunov function, we consider a matrix $p \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^N)$ with components $p_{\mu i}$. The vectors p_μ of components $p_{\mu i}$ are line vectors in $\mathbb{R}^n$. We consider next the line vectors $q_\mu = q_{\mu i}$ in $\mathbb{R}^n$ defined by $q_\mu = \sigma^* p_\mu$, and we also set

$$\eta_l(q)(y) = \sum_{\mu=l}^{N} A_\mu(y)q_\mu, l = 1, \cdots, N,$$

which are vectors in $\mathbb{R}^n$. We then have:

Lemma 4. *The following relation holds*

$$tr(D^2L(y)p\sigma\sigma^*p^*) = \sum_{l=1}^{N} P_l^{-1}(y)|\eta_l(q)(y)|^2 + \sum_{\mu} \gamma_\mu^2 P_\mu(y)(\beta''(\gamma_\mu y_\mu) + \beta''(-\gamma_\mu y_\mu))|q_\mu|^2. \tag{3.8}$$

Proof. We have

$$\begin{aligned}\mathrm{tr}(D^2L(y)p\sigma\sigma^*p^*) &= \sum_{\mu\nu} \frac{\partial^2 L(y)}{\partial y_\mu \partial y_\nu} \sum_k \left(\sum_i p_{\mu i}\sigma_{ik}\right)\left(\sum_j p_{\nu j}\sigma_{jk}\right) \\ &= \sum_{\mu\nu} \frac{\partial^2 L(y)}{\partial y_\mu \partial y_\nu} \sum_k q_{\mu k}q_{\nu k}\end{aligned}$$

and using (3.7)

$$= \sum_{\mu\nu} A_\mu(y)A_\nu(y) \sum_{l=1}^{\mu\wedge\nu} P_l^{-1}(y) \sum_k q_{\mu k}q_{\nu k} + \sum_\mu \gamma_\mu^2 P_\mu(y)(\beta''(\gamma_\mu y_\mu) + \beta''(-\gamma_\mu y_\mu))|q_\mu|^2,$$

and from the definition of $\eta_l(q)(y)$ we conclude easily (3.8). $\square$

Noting that

$$\frac{1}{2}|q_\mu|^2 = \sum_{ij} p_{\mu i}a_{ij}p_{\mu j} \geq \alpha|p_\mu|^2,$$

we can state the inequality

$$\frac{1}{2}\mathrm{tr}(D^2L(y)p\sigma\sigma^*p^*) \geq \frac{1}{2}\sum_{l=1}^{N} P_l^{-1}(y)|\eta_l(q)(y)|^2 + \alpha\sum_\mu \gamma_\mu^2 P_\mu(y)(\beta''(\gamma_\mu y_\mu) + \beta''(-\gamma_\mu y_\mu))|p_\mu|^2. \tag{3.9}$$

The Lyapunov property is expressed as follows

Proposition 5. *We assume the special structure of the Hamiltonians, see (2.11), (2.12). We then state*

$$\begin{aligned}&\frac{1}{2}tr\left(D^2L(y)p\sigma\sigma^*p^*\right) - \sum_\mu A_\mu(y)H_\mu(x,y,p) \\ &\geq \alpha\sum_\nu |p_\nu|^2 + \sum_\nu |p_\nu|^2 P_\nu(y)\Bigg\{\left(\frac{\alpha}{2}\gamma_\nu^2 - \gamma_\nu K_\nu^\nu\right)(\exp(\gamma_\nu y_\nu) + \exp(-\gamma_\nu y_\nu)) \\ &\quad - \frac{||a^{-1}||}{4}\left[\sum_{\mu=1}^{\nu}(K_\mu + K_{\mu-1})^2 + \sum_{\mu=\nu+1}^{N}(K_\mu + K_{\mu-1})^2 \prod_{\sigma=\nu+1}^{\mu} X_\sigma(y)\right] \\ &\quad - \sum_{\mu=\nu+1}^{N} \gamma_\mu K_\nu^\mu \left|\exp(\gamma_\mu y_\mu) - \exp(-\gamma_\mu y_\mu)\right| \prod_{\sigma=\nu+1}^{\mu} X_\sigma(y)\Bigg\} - \sum_\nu |A_\nu(y)|k_\nu(x,t),\end{aligned} \tag{3.10}$$

in which $K_0 = 0, K_{N+1} = 0,\ \sum_{N+1}^{N} = 0$ *by custom. The constants* $\gamma_\nu \geq 1$.

Proof. We have

$$\begin{aligned}
&\frac{1}{2}\mathrm{tr}(D^2L(y)p\sigma\sigma^*p^*) - \sum_\mu \frac{\partial L(y)}{\partial y_\mu} H_\mu(x,y,p) \\
&\geq \frac{1}{2}\sum_{l=1}^{N} P_l^{-1}(y)|\eta_l(q)(y)|^2 + \alpha\sum_\mu \gamma_\mu^2 P_\mu(y)(\beta''(\gamma_\mu y_\mu) + \beta''(-\gamma_\mu y_\mu))|p_\mu|^2 \\
&\quad - \sum_\mu A_\mu(y)(Q_\mu(x,y,p)\cdot p_\mu + H_\mu^0(x,y,p)).
\end{aligned} \tag{3.11}$$

We first write

$$\begin{aligned}
\sum_\mu A_\mu(y)Q_\mu(x,y,p)\cdot p_\mu &= (\sigma^*)^{-1}\sum_\mu A_\mu(y)Q_\mu(x,y,p)\cdot q_\mu \\
&= (\sigma^*)^{-1}\sum_\mu Q_\mu(x,y,p)\cdot(\eta_\mu(q)(y) - \eta_{\mu+1}(q)(y)),
\end{aligned}$$

setting $\eta_{N+1}(q)(y) = 0$. Similarly, setting $Q_0(x,y,p) = 0$, hence $K_0 = 0$, we obtain

$$\sum_\mu A_\mu(y)Q_\mu(x,y,p).p_\mu = (\sigma^*)^{-1}\sum_\mu \eta_\mu(q)(y)\cdot(Q_\mu(x,y,p) - Q_{\mu-1}(x,y,p)).$$

Hence

$$\begin{aligned}
\sum_\mu A_\mu(y)Q_\mu(x,y,p)\cdot p_\mu &\leq ||(\sigma^*)^{-1}||\sum_\mu (K_\mu + K_{\mu-1})|\eta_\mu(q)(y)||p| \\
&\leq \frac{1}{2}\sum_\mu P_\mu^{-1}(y)||\eta_\mu(q)(y)|^2 + \frac{1}{2}\sum_\mu ||(\sigma^*)^{-1}||^2(K_\mu + K_{\mu-1})^2 P_\mu(y)|p|^2.
\end{aligned}$$

Note that $1 \leq P_1(y) \leq \cdots \leq P_N(y)$. So

$$\sum_\mu (K_\mu + K_{\mu-1})^2 P_\mu(y)|p|^2 \leq \sum_\nu P_\nu(y)|p_\nu|^2\left(\sum_{\mu=1}^{\nu}(K_\mu + K_{\mu-1})^2 + \sum_{\mu=\nu+1}^{N}(K_\mu + K_{\mu-1})^2\prod_{\sigma=\nu+1}^{\mu} X_\sigma(y)\right).$$

Next,

$$\begin{aligned}
&\sum_\mu A_\mu(y)H_\mu^0(x,y,p) \\
&\leq \sum_\mu |A_\mu(y)|\left(\sum_{\nu=1}^{\mu} K_\nu^\mu|p_\nu|^2 + k_\mu(x,t)\right) \\
&\leq \sum_\nu |p_\nu|^2\sum_{\mu=\nu}^{N}|A_\mu(y)|K_\nu^\mu + \sum_\mu |A_\mu(y)|k_\mu(x,t) \\
&\leq \sum_\nu P_\nu(y)|p_\nu|^2\left[\gamma_\nu K_\nu^\nu|\beta'(\gamma_\nu y_\nu) - \beta'(-\gamma_\nu y_\nu)| + \sum_{\mu=\nu+1}^{N}\gamma_\mu K_\nu^\mu|\beta'(\gamma_\mu y_\mu) - \beta'(-\gamma_\mu y_\mu)|\prod_{\sigma=\nu+1}^{\mu} X_\sigma(y)\right] \\
&\quad + \sum_\mu |A_\mu(y)|k_\mu(x,t).
\end{aligned}$$

We note that $\left|\left|(\sigma*)^{-1}\right|\right|^2 = \dfrac{1}{2}||a^{-1}||$, in which $||\cdot||$ is a matrix norm. Combining the previous relations, we obtain the inequality (3.10). ■ □

The important observation is that the quantity in parentheses on the right hand side of inequality (3.10) depends only on y, the constants γ_μ and the constants of the assumptions, α, K_ν and K_ν^μ. If y is bounded, then we can adjust the constants γ_μ, depending on the bound of y, so that the quantity is positive. If y is not *a priori* bounded, this is not possible. This aspect is a major difficulty of the case $\mathbb{R}^n$, versus bounded domain, for the system of PDE (1.1).

Remark 6. Xing and Žitković [8] have considered a closely related, but different, Lyapunov function. Our choice allows for a full parallel between the analytic and the probabilistic methods. Our proof of Proposition 5 is strongly inspired by their method.

4. Upper bounds

4.1. General comments

We will proceed with *a priori* estimates. For both systems (1.1) and (1.8) we will assume that a solution exists with sufficient regularity properties. Then we prove *a priori* estimates. To prove existence, we construct an approximation, which will satisfy the same estimates. This will allow to pass to the limit and prove the existence. We begin by upper bounds, which replace the L^∞ bounds used in [2]. We will do it for both (1.1) and (1.8), using the assumptions (2.5), (2.6), (2.7).

4.2. Upper bounds for the solution of (1.1)

Define $u_0 = -\sum_{\nu=1}^N u_\nu$. Then

$$\begin{cases} -\dfrac{\partial u_0}{\partial t} + Au_0 = -\displaystyle\sum_\nu H_\nu(x,u,Du), \\ \qquad u_0(x,T) = h_0(x) = -\displaystyle\sum_\nu h_\nu(x). \end{cases}$$

From the assumption (2.6) we can write

$$-\frac{\partial u_0}{\partial t} + Au_0 \le \lambda_0(x) + \gamma_0|Du_0|^2. \tag{4.1}$$

We then define

$$z_0(x,t) = \exp\left(\frac{\gamma_0}{\alpha}u_0(x,t)\right),$$

and we check easily that

$$-\frac{\partial z_0}{\partial t} + Az_0 = z_0\frac{\gamma_0}{\alpha}\left(-\frac{\partial u_0}{\partial t} + Au_0 - \frac{\gamma_0}{\alpha}\,Du_0\cdot aDu_0\right). \tag{4.2}$$

But

$$-\frac{\partial u_0}{\partial t} + Au_0 - \frac{\gamma_0}{\alpha}\,Du_0\cdot aDu_0 \le -\frac{\partial u_0}{\partial t} + Au_0 - \gamma_0|Du_0|^2,$$

and from (4.1), it is less than $\lambda_0(x)$. Therefore from (4.2), we get

$$\begin{cases} -\dfrac{\partial z_0}{\partial t} + Az_0 \leq z_0 \lambda_0(x) \dfrac{\gamma_0}{\alpha}, \\ \quad z_0(x,T) = \exp\left(\dfrac{\gamma_0}{\alpha} h_0(x)\right). \end{cases}$$

We recall the definition of $\bar{z}_0(x,t)$, see (2.10) with $\nu = 0$. From the assumption (2.9) $\bar{z}_0(x,t)$ is finite, at each $(x,t) \in \mathbb{R}^n \times [0,T]$. By comparison $z_0(x,t) \leq \bar{z}_0(x,t)$. From the definition of $\bar{z}_0(x,t)$, we have

$$u_0(x,t) \leq \frac{\alpha}{\gamma_0} \log \bar{z}_0(x,t).$$

Therefore

$$\sum_{\nu=1}^{N} u_\nu(x,t) \geq -\frac{\alpha}{\gamma_0} \log \bar{z}_0(x,t). \tag{4.3}$$

In the same way, we obtain

$$u_\nu(x,t) \leq \frac{\alpha}{\gamma_\nu} \log \bar{z}_\nu(x,t). \tag{4.4}$$

Combining (4.3) and (4.4) we obtain the estimate:

Proposition 7. *Assuming (2.5), (2.6), (2.7) a solution of (1.1) satisfies*

$$|u_\nu(x,t)| \leq \Phi(x,t), x \in \mathbb{R}^n, t \in [0,T], \tag{4.5}$$

with

$$\Phi(x,t) = \frac{\alpha}{\gamma_0} \log \bar{z}_0(x,t) + \sum_{\nu=1}^{N} \frac{\alpha}{\gamma_\nu} \log \bar{z}_\nu(x,t), x \in \mathbb{R}^n, t \in [0,T]. \tag{4.6}$$

4.3. Upper bounds for the solution of (1.8)

The probabilistic equivalent of Proposition 7 is:

Proposition 8. *Assuming (2.5), (2.6), (2.7), a solution of (1.8) satisfies*

$$|Y_\nu^{xt}(s)| \leq \Phi(X^{xt}(s), s). \tag{4.7}$$

Proof. Define

$$\eta_\nu^{xt}(s) = \exp\left(\frac{\gamma_\nu}{\alpha} Y_\nu^{xt}(s)\right).$$

Then we have

$$d\eta_\nu^{xt}(s) = \eta_\nu^{xt}(s) \frac{\gamma_\nu}{\alpha} \Bigg[\left(-H_\nu(X^{xt}(s), Y^{xt}(s), Z^{xt}(s)) + \frac{1}{2} \frac{\gamma_\nu}{\alpha} Z_\nu^{xt}(s) \cdot a(X^{xt}(s)) Z_\nu^{xt}(s) \right) ds \\ + Z_\nu^{xt}(s) \cdot \sigma(X^{xt}(s)) dw(s) \Bigg].$$

From the assumption (2.5), we obtain

$$d\eta_\nu^{xt}(s) \geq -\eta_\nu^{xt}(s)\frac{\gamma_\nu}{\alpha}\lambda_\nu(X^{xt}(s))ds + \eta_\nu^{xt}(s)\frac{\gamma_\nu}{\alpha}Z_\nu^{xt}(s)\cdot\sigma(X^{xt}(s))dw(s),$$

hence

$$d\left[\eta_\nu^{xt}(s)\exp\left(\frac{\gamma_\nu}{\alpha}\right)\int_t^s \lambda_\nu(X^{xt}(\tau))\,d\tau\right] \geq \eta_\nu^{xt}(s)\frac{\gamma_\nu}{\alpha}\left(\exp\left(\frac{\gamma_\nu}{\alpha}\right)\int_t^s \lambda_\nu(X^{xt}(\tau))\,d\tau\right)Z_\nu^{xt}(s)\cdot\sigma(X^{xt}(s))dw(s),$$

from which it follows

$$\exp\left(\frac{\gamma_\nu}{\alpha}Y_\nu^{xt}(s)\right) \leq \mathbb{E}\left[\exp\left(\frac{\gamma_\nu}{\alpha}\right)\left(h_\nu^+(X^{xt}(T)) + \int_s^T \lambda_\nu(X^{xt}(\tau))\,d\tau\right)\middle|\mathcal{F}_t^s\right] \leq \bar{z}_\nu(X^{xt}(s),s).$$

Similarly, defining

$$Y_0^{xt}(s) = -\sum_\nu Y_\nu^{xt}(s),$$

we can prove that

$$\exp\left(\frac{\gamma_0}{\alpha}Y_0^{xt}(s)\right) \leq \bar{z}_0(X^{xt}(s),s),$$

and the conclusion (4.7) follows at once. □

5. Basic inequalities

5.1. Analytic part

We associate to the solution u_ν of (1.1) a constant c_ν to be defined later. We will write $\tilde{u}_\nu(x,t) = u_\nu(x,t) - c_\nu$, and $\tilde{u}$ is the vector of components $\tilde{u}_\nu$. We define

$$X_\nu(x,t) = X_\nu(\tilde{u}(x,t)), \tag{5.1}$$

and similarly $P_\nu(x,t)$ and $A_\nu(x,t)$. We have

$$\begin{cases} DX_\nu(x,t) = \displaystyle\sum_{\mu=\nu}^N A_\mu(x,t)P_{\nu-1}^{-1}(x,t)Du_\mu(x,t), \\ \dfrac{\partial X_\nu(x,t)}{\partial t} = \displaystyle\sum_{\mu=\nu}^N A_\mu(x,t)P_{\nu-1}^{-1}(x,t)\frac{\partial u_\mu(x,t)}{\partial t}. \end{cases}$$

We also define

$$L(x,t) = L(\tilde{u}(x,t)) = X_1(\tilde{u}(x,t)) - X_1(0).$$

We want to obtain an inequality, which bears similarities with (3.10).

Proposition 9. *We assume the special structure of the Hamiltonians, see (2.11), (2.12). Let $\Psi(x,t) \geq 0$, sufficiently smooth, with a compact support. Then, we have*

$$\begin{aligned}
&-\int_0^T\int_{\mathbb{R}^n} \Psi \frac{\partial L}{\partial t} dxdt + \int_0^T\int_{\mathbb{R}^n} \sum_{ij} a_{ij} \frac{\partial \Psi}{\partial x_i}\frac{\partial L}{\partial x_j} dxdt + \alpha \int_0^T\int_{\mathbb{R}^n} \Psi |Du|^2 dxdt \\
&+ \sum_\nu \int_0^T\int_{\mathbb{R}^n} \Psi |Du_\nu|^2 P_\nu \Bigg\{ \left(\frac{\alpha}{2}\gamma_\nu^2 - \gamma_\nu K_\nu^\nu\right)\left(\exp(\gamma_\nu \tilde{u}_\nu) + \exp(-\gamma_\nu \tilde{u}_\nu)\right) \\
&- \frac{||a^{-1}||}{4}\left[\sum_{\mu=1}^{\nu} (K_\mu + K_{\mu-1})^2 + \sum_{\mu=\nu+1}^{N} (K_\mu + K_{\mu-1})^2 \prod_{\sigma=\nu+1}^{\mu} X_\sigma\right] \\
&- \sum_{\mu=\nu+1}^{N} \gamma_\mu K_\nu^\mu \left|\exp(\gamma_\mu \tilde{u}_\mu) - \exp(-\gamma_\mu \tilde{u}_\mu)\right| \prod_{\sigma=\nu+1}^{\mu} X_\sigma \Bigg\} dxdt \\
\leq\ & \sum_\nu \int_0^T\int_{\mathbb{R}^n} \Psi |A_\nu(x,t)| k_\nu(x,t) dxdt,
\end{aligned} \tag{5.2}$$

with constants $\gamma_\nu \geq 1$.

Proof. We test (1.1) with $\Psi(x,t)A_\nu(x,t)$. We have first

$$\sum_\nu \int_0^T\int_{\mathbb{R}^n} \frac{\partial u_\nu(x,t)}{\partial t} \Psi(x,t) A_\nu(x,t) dxdt = \int_0^T\int_{\mathbb{R}^n} \Psi(x,t) \frac{\partial L(x,t)}{\partial t} dxdt. \tag{5.3}$$

We note that

$$\begin{aligned}
\int_{\mathbb{R}^n} \sum_\nu A u_\nu (\Psi A_\nu) dx &= \sum_{i,j} \int_{\mathbb{R}^n} a_{ij} \sum_\nu \frac{\partial u_\nu}{\partial x_j} \frac{\partial (\Psi A_\nu)}{\partial x_i} dx \\
&= \sum_{i,j} \int_{\mathbb{R}^n} a_{ij} \frac{\partial \Psi}{\partial x_i}\frac{\partial L}{\partial x_j} dx + \sum_{i,j} \int_{\mathbb{R}^n} a_{ij} \Psi \sum_\nu \frac{\partial u_\nu}{\partial x_j}\frac{\partial A_\nu}{\partial x_i} dx,
\end{aligned}$$

and from the definition of A_ν, see (3.4)

$$\begin{aligned}
\sum_\nu \frac{\partial u_\nu}{\partial x_j}\frac{\partial A_\nu}{\partial x_i} = \sum_\nu P_\nu \gamma_\nu^2 (\beta''(\gamma_\nu \tilde{u}_\nu) + \beta''(-\gamma_\nu \tilde{u}_\nu)) \frac{\partial u_\nu}{\partial x_j}\frac{\partial u_\nu}{\partial x_i} + \sum_\nu P_\nu \gamma_\nu (\beta'(\gamma_\nu \tilde{u}_\nu) \\
- \beta'(-\gamma_\nu \tilde{u}_\nu)) \frac{\partial u_\nu}{\partial x_j} \sum_{\mu=1}^{\nu} \frac{\partial F_\mu}{\partial x_i},
\end{aligned}$$

with $F_\mu = \log X_\mu$. In the last term, we interchange $\sum_{\nu=1}^N \sum_{\mu=1}^\nu = \sum_{\mu=1}^N \sum_{\nu=\mu}^N$, we write

$$\begin{aligned}
\sum_{\mu=1}^N \frac{\partial F_\mu}{\partial x_i} \sum_{\nu=\mu}^N P_\nu \gamma_\nu (\beta'(\gamma_\nu \tilde{u}_\nu) - \beta'(-\gamma_\nu \tilde{u}_\nu)) \frac{\partial u_\nu}{\partial x_j} = \sum_{\mu=1}^N \frac{\partial F_\mu}{\partial x_i} P_\mu \sum_{\nu=\mu}^N \frac{1}{X_\mu} \gamma_\nu (\beta'(\gamma_\nu \tilde{u}_\nu) \\
- \beta'(-\gamma_\nu \tilde{u}_\nu)) \frac{\partial u_\nu}{\partial x_j} \prod_{\sigma=\mu}^{\nu} X_\sigma.
\end{aligned}$$

But, from (5.2),

$$\frac{\partial X_\mu}{\partial x_j}=\sum_{\nu=\mu}^{N}\gamma_\nu(\beta'(\gamma_\nu\tilde{u}_\nu)-\beta'(-\gamma_\nu\tilde{u}_\nu))\frac{\partial u_\nu}{\partial x_j}\prod_{\sigma=\mu}^{\nu}X_\sigma.$$

Therefore

$$\sum_\nu P_\nu\gamma_\nu(\beta'(\gamma_\nu\tilde{u}_\nu)-\beta'(-\gamma_\nu\tilde{u}_\nu))\frac{\partial u_\nu}{\partial x_j}\sum_{\mu=1}^{\nu}\frac{\partial F_\mu}{\partial x_i}=\sum_\nu P_\nu\frac{\partial F_\nu}{\partial x_j}\frac{\partial F_\nu}{\partial x_i}.$$

Collecting results, we get

$$\begin{aligned}\sum_\nu\int_0^T\int_{\mathbb{R}^n}Au_\nu(\Psi A_\nu)dxdt=&\int_0^T\int_{\mathbb{R}^n}\sum_{i,j}a_{ij}\frac{\partial\Psi}{\partial x_i}\frac{\partial L}{\partial x_j}dxdt\\&+\sum_\nu\int_0^T\int_{\mathbb{R}^n}\Psi\sum_\nu P_\nu\gamma_\nu^2(\beta''(\gamma_\nu\tilde{u}_\nu)+\beta''(-\gamma_\nu\tilde{u}_\nu))\sum_{i,j}a_{ij}\frac{\partial u_\nu}{\partial x_j}\frac{\partial u_\nu}{\partial x_i}dxdt\\&+\sum_\nu\int_0^T\int_{\mathbb{R}^n}\Psi P_\nu\sum_{i,j}a_{ij}\frac{\partial F_\nu}{\partial x_j}\frac{\partial F_\nu}{\partial x_i}dxdt.\end{aligned}\tag{5.4}$$

We next consider

$$\sum_{\nu=1}^{N-1}\int_0^T\int_{\mathbb{R}^n}Q_\nu\cdot Du_\nu\Psi A_\nu dxdt=\sum_{\nu=1}^{N-1}\int_0^T\int_{\mathbb{R}^n}\Psi P_\nu\gamma_\nu(\beta'(\gamma_\nu\tilde{u}_\nu)-\beta'(-\gamma_\nu\tilde{u}_\nu))Du_\nu\cdot Q_\nu dxdt,$$

and we notice that

$$\begin{aligned}DF_\nu&=DX_{\nu+1}+\gamma_\nu(\beta'(\gamma_\nu\tilde{u}_\nu)-\beta'(-\gamma_\nu\tilde{u}_\nu))Du_\nu,\\P_\nu(DF_\nu-DX_{\nu+1})&=P_\nu DF_\nu-P_{\nu+1}DF_{\nu+1}.\end{aligned}$$

Setting $Q_0=0$, we get

$$\sum_{\nu=1}^{N-1}\int_0^T\int_{\mathbb{R}^n}Q_\nu\cdot Du_\nu\Psi A_\nu dxdt=\sum_{\nu=1}^{N-1}\int_0^T\int_{\mathbb{R}^n}\Psi P_\nu\tilde{Q}_\nu.DF_\nu dxdt-\int_0^T\int_{\mathbb{R}^n}\Psi P_NQ_{N-1}\cdot DF_Ndxdt,$$

where we have denoted $\tilde{Q}_\nu.=Q_\nu-Q_{\nu-1}$. Since

$$\int_0^T\int_{\mathbb{R}^n}Q_N\cdot Du_N\Psi A_Ndxdt=\int_0^T\int_{\mathbb{R}^n}\Psi P_NQ_N\cdot DF_Ndxdt,$$

we conclude that

$$\sum_{\nu=1}^{N}\int_0^T\int_{\mathbb{R}^n}Q_\nu\cdot Du_\nu\Psi A_\nu dxdt=\sum_{\nu=1}^{N}\int_0^T\int_{\mathbb{R}^n}\Psi P_\nu\tilde{Q}_\nu.DF_\nu dxdt.\tag{5.5}$$

Finally

$$
\begin{aligned}
&\sum_{\nu=1}^{N}\int_0^T\int_{\mathbb{R}^n} H_\nu^0(x,u,Du)\Psi A_\nu dxdt \\
&\leq \sum_{\nu=1}^{N}\int_0^T\int_{\mathbb{R}^n} \Psi A_\nu\left(\sum_{\mu=1}^{\nu} K_\mu^\nu|Du_\mu|^2+k_\nu(x,t)\right)dxdt \\
&\leq \sum_{\nu=1}^{N}\int_0^T\int_{\mathbb{R}^n} \Psi|Du_\nu|^2 P_\nu\Bigg[\gamma_\nu K_\nu^\nu(\exp(\gamma_\nu\tilde{u}_\nu)+\exp(-\gamma_\nu\tilde{u}_\nu)) \\
&\qquad + \sum_{\mu=\nu+1}^{N}\gamma_\mu K_\nu^\mu|\exp(\gamma_\mu\tilde{u}_\mu)-\exp(-\gamma_\mu\tilde{u}_\mu)|\prod_{\sigma=\nu+1}^{\mu}X_\sigma\Bigg]dxdt.
\end{aligned}
$$

Collecting results and making easy majorations, we obtain easily the inequality (5.2). □

We recall that Ψ has compact support. We define

$$\rho_\Psi=\sup_{x,t}\Phi(x,t)\mathbb{1}_{\Psi(x,t)>0}. \tag{5.6}$$

From (4.5), we have

$$|u_\nu(x,t)|\leq\rho_\Psi, \forall t, x\in \operatorname{dom}\Psi(\cdot,t). \tag{5.7}$$

We take the constants c_ν such that

$$|c_\nu|\leq\rho_\Psi. \tag{5.8}$$

The idea is now to fix the constants γ_ν so that

$$
\begin{aligned}
0\leq \alpha\gamma_\nu^2-2\gamma_\nu K_\nu^\nu-\frac{||a^{-1}||}{4}\left[\sum_{\mu=1}^{\nu}(K_\mu+K_{\mu-1})^2+\sum_{\mu=\nu+1}^{N}(K_\mu+K_{\mu-1})^2\prod_{\sigma=\nu+1}^{\mu}X_\sigma(x,t)\right] \\
-\sum_{\mu=\nu+1}^{N}\gamma_\mu K_\nu^\mu\big|\exp(\gamma_\mu\tilde{u}_\mu(x,t))-\exp(-\gamma_\mu\tilde{u}_\mu(x,t))\big|\prod_{\sigma=\nu+1}^{\mu}X_\sigma(x,t),\ \forall t,x\in\operatorname{dom}\Psi(\cdot,t).
\end{aligned}
$$

We define

$$c(\rho_\Psi)=\sup_{\max_\nu|y_\nu|<2\rho_\Psi}\max_\nu|A_\nu(y)|, \tag{5.9}$$

which implies

$$\sup_{x,t}\max_\nu|A_\nu(x,t)|\mathbb{1}_{\Psi(x,t)>0}\leq c(\rho_\Psi). \tag{5.10}$$

We can state the

Proposition 10. *We assume the Hamiltonians satisfy the special structure assumptions, (2.11), (2.12). Let $\Psi(x,t) \geq 0$ and sufficiently smooth, with compact support, and the constants c_ν satisfy (5.8). Then we have*

$$-\int_0^T \int_{\mathbb{R}^n} \Psi \frac{\partial L}{\partial t} dxdt + \int_0^T \int_{\mathbb{R}^n} \sum_{i,j} a_{ij} \frac{\partial \Psi}{\partial x_i} \frac{\partial L}{\partial x_j} dxdt + \alpha \int_0^T \int_{\mathbb{R}^n} \Psi |Du|^2 dxdt \leq c(\rho_\Psi) \sum_\nu \int_0^T \int_{\mathbb{R}^n} \Psi k_\nu(x,t) dxdt. \tag{5.11}$$

5.2. Probabilistic part

We now give the equivalent of Proposition 9, for the problem (1.8):

Proposition 11. *We assume the special structure of the Hamiltonians, see (2.11), (2.12). Let $\Psi(x,t)$, sufficiently smooth, with a compact support. Then, we have*

$$\begin{aligned}
&\frac{\alpha}{2}\mathbb{E}\int_t^T \sum_\nu \Psi^2(X^{xt}(s),s)|Z_\nu^{xt}(s)|^2 ds \\
&+ \mathbb{E}\int_t^T \sum_\nu \Psi^2(X^{xt}(s),s)|Z_\nu^{xt}(s)|^2 P_\nu(Y^{xt}(s)-c) \Bigg\{ \left(\frac{\alpha}{2}\gamma_\nu^2 - \gamma_\nu K_\nu^\nu\right)(\exp(\gamma_\nu(Y_\nu^{xt}(s)-c_\nu)) \\
&+ \exp(-\gamma_\nu(Y_\nu^{xt}(s)-c_\nu))) - \frac{||a^{-1}||}{4}\left[\sum_{\mu=1}^{\nu}(K_\mu+K_{\mu-1})^2 + \sum_{\mu=\nu+1}^{N}(K_\mu+K_{\mu-1})^2 \prod_{\sigma=\nu+1}^{\mu} X_\sigma(Y^{xt}(s)-c)\right] \\
&- \sum_{\mu=\nu+1}^{N} \gamma_\mu K_\nu^\mu |\exp(\gamma_\mu(Y_\mu^{xt}(s)-c_\mu)) - \exp(-\gamma_\mu(Y_\mu^{xt}(s)-c_\mu))| \prod_{\sigma=\nu+1}^{\mu} X_\sigma(Y^{xt}(s)-c) \Bigg\} ds \\
\leq\ & \mathbb{E}\int_t^T \left(-\frac{\partial \Psi^2}{\partial s} + A(s)\Psi^2\right)(X^{xt}(s),s) L(Y^{xt}(s)-c) ds + \frac{8}{\alpha}||a||^2 \mathbb{E}\int_t^T \sum_\nu A_\nu^2(Y^{xt}(s)-c)|D\Psi(X^{xt}(s),s)|^2 ds \\
&+ \mathbb{E}\left(\Psi^2(X^{xt}(T),T)L(Y^{xt}(T)-c)\right) + \mathbb{E}\int_t^T \sum_\nu \Psi^2(X^{xt}(s),s)|A_\nu(Y^{xt}(s)-c)| k_\nu(X^{xt}(s),s) ds.
\end{aligned} \tag{5.12}$$

Proof. We consider the process $\Psi^2(X^{xt}(s),s)L(Y^{xt}(s)-c)$ and use Ito's formula to obtain

$$\begin{aligned}
&d(\Psi^2(X^{xt}(s),s)L(Y^{xt}(s)-c)) \\
&= \Bigg[\left(\frac{\partial \Psi^2}{\partial s} - A(s)\Psi^2\right)(X^{xt}(s),s)L(Y^{xt}(s)-c) + \Psi^2(X^{xt}(s),s) \\
&\times \left(\frac{1}{2}\mathrm{tr} D^2 L(Y^{xt}(s)-c) Z^{xt}(s)\sigma\sigma^*(X^{xt}(s))(Z^{xt}(s))^* \right. \\
&\left. - \sum_\nu A_\nu(Y^{xt}(s)-c) H_\nu(X^{xt}(s), Y^{xt}(s), Z^{xt}(s)) \right)
\end{aligned}$$

$$+ 4\sum_\nu A_\nu(Y^{xt}(s)-c)Z_\nu^{xt}(s).a(X^{xt}(s))\Psi D\Psi(X^{xt}(s),s)\Bigg]ds$$

$$+\left(L(Y^{xt}(s)-c)D\Psi^2(X^{xt}(s),s)+\Psi^2(X^{xt}(s),s)\sum_\nu A_\nu(Y^{xt}(s)-c)Z_\nu^{xt}(s)\right)\cdot\sigma(X^{xt}(s))dw(s).$$

We then use (3.10) to estimate the term

$$\begin{aligned}
&\Psi^2(X^{xt}(s),s)\Bigg(\frac{1}{2}\mathrm{tr}D^2L(Y^{xt}(s)-c)Z^{xt}(s)\sigma\sigma^*(X^{xt}(s))(Z^{xt}(s))^*\\
&-\sum_\nu A_\nu(Y^{xt}(s)-c)H_\nu(X^{xt}(s),Y^{xt}(s),Z^{xt}(s))\Bigg)\\
\geq\ &\alpha\sum_\nu \Psi^2(X^{xt}(s),s)|Z_\nu^{xt}(s)|^2\\
&+\sum_\nu \Psi^2(X^{xt}(s),s)|Z_\nu^{xt}(s)|^2P_\nu(Y^{xt}(s)-c)\\
&\times\Bigg\{\left(\frac{\alpha}{2}\gamma_\nu^2-\gamma_\nu K_\nu^\nu\right)(\exp(\gamma_\nu(Y_\nu^{xt}(s)-c_\nu))+\exp(-\gamma_\nu(Y_\nu^{xt}(s)-c_\nu)))\\
&-\frac{||a^{-1}||}{4}\left[\sum_{\mu=1}^{\nu}(K_\mu+K_{\mu-1})^2+\sum_{\mu=\nu+1}^{N}(K_\mu+K_{\mu-1})^2\prod_{\sigma=\nu+1}^{\mu}X_\sigma(Y^{xt}(s)-c)\right]\\
&-\sum_{\mu=\nu+1}^{N}\gamma_\mu K_\nu^\mu|\exp(\gamma_\mu(Y_\mu^{xt}(s)-c_\mu))-\exp(-\gamma_\mu(Y_\mu^{xt}(s)-c_\mu))|\prod_{\sigma=\nu+1}^{\mu}X_\sigma(Y^{xt}(s)-c)\Bigg\}\\
&-\sum_\nu \Psi^2(X^{xt}(s),s)|A_\nu(Y^{xt}(s)-c)|k_\nu(X^{xt}(s),s).
\end{aligned}$$

Next, we use

$$\begin{aligned}
&4\sum_\nu A_\nu(Y^{xt}(s)-c)Z_\nu^{xt}(s).a(X^{xt}(s))\Psi D\Psi(X^{xt}(s),s)\\
\geq\ &-\frac{\alpha}{2}\sum_\nu \Psi^2(X^{xt}(s),s)|Z_\nu^{xt}(s)|^2-\frac{8||a||^2}{\alpha}\sum_\nu A_\nu^2(Y^{xt}(s)-c)|D\Psi(X^{xt}(s),s)|^2.
\end{aligned}\tag{5.13}$$

Collecting results, integrating between t and T, then taking the mathematical expectation, we obtain the inequality (5.12). □

We next give the equivalent of Proposition 10. We recall (4.7). Hence if $X^{xt}(s)\in\mathrm{dom}\ \Psi(\cdot,s)$, then $|Y_\nu^{xt}(s)|\leq\rho_\Psi$. We then choose the constants γ_ν like in the proof of Proposition 9, to obtain:

Proposition 12. *We assume the Hamiltonians satisfy the special structure assumptions, (2.11), (2.12). Let $\Psi(x,t)$, sufficiently smooth, with a compact support. Then, we have*

$$
\begin{aligned}
&\frac{\alpha}{2}\mathbb{E}\int_t^T \sum_\nu \Psi^2(X^{xt}(s),s)|Z_\nu^{xt}(s)|^2 ds \\
&\le \mathbb{E}\int_t^T \left(-\frac{\partial \Psi^2}{\partial s} + A(s)\Psi^2\right)(X^{xt}(s),s)L(Y^{xt}(s)-c)ds + \frac{8}{\alpha}||a||^2\mathbb{E}\int_t^T \sum_\nu A_\nu^2(Y^{xt}(s)-c)|D\Psi(X^{xt}(s),s)|^2 ds \\
&\quad + \mathbb{E}\left(\Psi^2(X^{xt}(T),T)L(Y^{xt}(T)-c)\right) + c(\rho_\Psi)\mathbb{E}\int_t^T \Psi^2(X^{xt}(s),s)\sum_\nu k_\nu(X^{xt}(s),s)ds,
\end{aligned}
\tag{5.14}
$$

with $c(\rho_\Psi)$ defined by (5.9). The constants c_ν must satisfy $|c_\nu| \le \rho_\Psi$.

Remark 13. Note that we use Ψ^2 and not Ψ in (5.14), unlike in (5.11). This is in order to obtaining the estimate (5.13). This trick is simpler than the concept of testable function introduced by Xing and Žitković [8].

Corollary 14. *We have the estimate*

$$
\mathbb{E}\int_t^T \sum_\nu \Psi^2(X^{xt}(s),s)|Z_\nu^{xt}(s)|^2 ds \le C(\Psi)\left(1 + \mathbb{E}\int_t^T \Psi^2(X^{xt}(s),s)\sum_\nu k_\nu(X^{xt}(s),s)ds\right), \tag{5.15}
$$

where the constant $C(\Psi)$ depends only on Ψ and the constants of the problem.

Proof. This is an easy consequence of Proposition 12. □

6. Markov properties of the solution of (1.8)

6.1. Discussion on Markov properties

In the method of *a priori* estimates, we assume that the solution of (1.8) is given by deterministic functions of the process X_s^{xt}, namely

$$
Y_\nu^{xt}(s) = u_\nu(X^{xt}(s),s), \;\; Z_\nu^{xt}(s) = v_\nu(X^{xt}(s),s). \tag{6.1}
$$

In fact, if we have a solution, we can set

$$
u_\nu(x,t) = Y_\nu^{xt}(t), \;\; v_\nu(x,t) = Z_\nu^{xt}(t), \tag{6.2}
$$

which are deterministic functions. Since $X^{xt}(s)$ is a Markov process, we have for $t < s < \tau$, $X^{xt}(\tau) = X^{X^{xt}(s),s}(\tau)$. Therefore $Y_\nu^{X^{xt}(s),s}(\tau)$ and $Z_\nu^{X^{xt}(s),s}(\tau)$ is also a solution of (1.8) on $s < \tau < T$. In case of uniqueness, this implies $Y_\nu^{X^{xt}(s),s}(\tau) = Y_\nu^{xt}(\tau)$ and $Z_\nu^{X^{xt}(s),s}(\tau) = Z_\nu^{xt}(\tau)$, on $s < \tau < T$. In particular $Y_\nu^{X^{xt}(s),s}(s) = Y_\nu^{xt}(s)$ and $Z_\nu^{X^{xt}(s),s}(s) = Z_\nu^{xt}(s)$, which implies immediately the property (6.1). We recall that in the philosophy of the method of *a priori* estimates, we assume sufficient smoothness, hence we may assume $u_\nu(x,t)$ continuous.

We shall also need properties of the probability distribution of $X^{xt}(s)$, which we denote by $G_{xt}(\xi,s)$ for $s > t$, defined by

$$
\mathbb{E}\varphi(X^{xt}(s),s) = \int G_{xt}(\xi,s)\varphi(\xi,s)d\xi,
$$

for any continuous and bounded test function $\varphi(\xi, s)$. We will call it a Green function, for coherence with the analytic framework. It is solution of the problem

$$\begin{cases} -\dfrac{\partial G_{xt}}{\partial s} + \displaystyle\sum_{ij} \dfrac{\partial}{\partial \xi_j}\left(a_{ij}\dfrac{\partial G_{xt}}{\partial \xi_i}\right) = 0, \\ G_{xt}(\xi, t) = \delta(\xi - x). \end{cases} \tag{6.3}$$

We shall use the classical estimates on the Green function, see Aronson [1]

$$k_1(s-t)^{-\frac{n}{2}} \exp\left(-\delta_1 \frac{|\xi - x|^2}{s-t}\right) \leq G_{xt}(\xi, s) \leq k_2(s-t)^{-\frac{n}{2}} \exp\left(-\delta_2 \frac{|\xi - x|^2}{s-t}\right), \tag{6.4}$$

where the constants depend only on the bound M on $||a(x,t)||$, see (2.1) and the ellipticity constant α.

This implies the following estimate, for the functions k_ν entering in the assumptions (2.11), (2.12).

$$\mathbb{E}\int_t^T k_\nu(X^{xt}(s), s)\mathbb{1}_{|X^{xt}(s)|<M} ds \leq C_M, \tag{6.5}$$

in which the constant C_M depends only on M, the assumption (2.12) on k_ν and the constants in (6.4). To prove this result, we notice that

$$\mathbb{E}\int_t^T k_\nu(X^{xt}(s), s)\mathbb{1}_{|X^{xt}(s)|<M} ds = \int_t^T \int_{B_M} k_\nu(\xi, s) G_{xt}(\xi, s) d\xi ds,$$

where B_M is the ball of center 0 and radius M. Using (6.4) we get it is bounded above by

$$||k_\nu||_{L^d(B_M \times (0,T))} ||G_{xt}||_{L^{d'}(\mathbb{R}^n \times (0,T))} \leq C_M,$$

as easily seen from the second inequality (6.4) and $d > \dfrac{n}{2} + 1$. From this estimate and (5.15) we obtain

$$\mathbb{E}\int_t^T \sum_\nu \Psi^2(X^{xt}(s), s)|Z_\nu^{xt}(s)|^2 ds \leq C(\Psi), \tag{6.6}$$

in which the constant $C(\Psi)$ depends only on the function Ψ and the constants of the problem.

6.2. Relation between v_ν and u_ν

Our objective is to prove the following result

Proposition 15. *We assume the Hamiltonians satisfy the special structure assumptions, (2.11), (2.12). A solution (6.1) of the system (1.8) with $u_\nu(x,t)$ continuous satisfies $v_\nu(x,t) = Du_\nu(x,t)$ in the sense of distributions on $\mathbb{R}^n$.*

Proof. We follow the proof of Xing and Žitković [8]. We consider a function $\Psi(x,t)$ with a compact support and sufficiently smooth. We have $|Y_\nu^{xt}(s)|\Psi^2(X^{xt}(s), s) \leq \rho_\Psi ||\Psi||^2$. We introduce

$$\hat{u}_\nu(x,t) = Y_\nu^{xt}(t)\Psi^2(x,t), \tag{6.7}$$

and $Y_\nu^{xt}(s)\Psi^2(X^{xt}(s),s) = \hat{u}_\nu(X^{xt}(s),s)$. The function $\chi_\nu(x,t;s)$, $t < s$ defined by $\chi_\nu(x,t;s) = \mathbb{E}\hat{u}_\nu(X^{xt}(s),s)$ satisfies the PDE.

$$\begin{cases} -\dfrac{\partial \chi_\nu}{\partial t} + A(t)\chi_\nu = 0, \\ \qquad \chi_\nu(x,s;s) = \hat{u}_\nu(x,s). \end{cases} \tag{6.8}$$

We then define, for ϵ fixed and $l > \dfrac{1}{\epsilon}$, and $t < T-\epsilon$,

$$\hat{u}_\nu^l(x,t) = l\int_t^{t+\frac{1}{l}} \mathbb{E}Y_\nu^{xt}(s)\Psi^2(X^{xt}(s),s)\,ds = l\int_t^{t+\frac{1}{l}} \chi_\nu(x,t;s)ds, \tag{6.9}$$

from which we obtain, thanks to the PDE (6.8).

$$-\frac{\partial \hat{u}_\nu^l}{\partial t} + A(t)\hat{u}_\nu^l = l\left[\hat{u}_\nu^l(x,t) - \chi_\nu\left(x,t;t+\frac{1}{l}\right)\right]. \tag{6.10}$$

Recalling that $\chi_\nu\left(x,t;t+\dfrac{1}{l}\right) = \mathbb{E}Y_\nu^{xt}\left(t+\dfrac{1}{l}\right)\Psi^2\left(X^{xt}\left(t+\dfrac{1}{l}\right), t+\dfrac{1}{l}\right)$, and using Ito's formula to compute this expression, we obtain, from (6.10),

$$\begin{aligned} -\frac{\partial \hat{u}_\nu^l}{\partial t} + A(t)\hat{u}_\nu^l = l\int_t^{t+\frac{1}{l}} \mathbb{E}\Big[&\Psi^2(X^{xt}(s),s)H_\nu(X^{xt}(s),Y^{xt}(s),Z^{xt}(s)) \\ &+ Y^{xt}(s)\left(-\frac{\partial \Psi^2}{\partial s} + A(s)\Psi^2\right)(X^{xt}(s),s) \\ &- 4Z_\nu^{xt}(s)\cdot a(X^{xt}(s))\Psi D\Psi(X^{xt}(s),s)\Big]ds. \end{aligned} \tag{6.11}$$

We can write $|H_\nu(X^{xt}(s),Y^{xt}(s),Z^{xt}(s))| \leq K|Z^{xt}(s)|^2 + k_\nu(X^{xt}(s),s)$, for a fixed constant K. Also $4|Z_\nu^{xt}(s)\cdot a(X^{xt}(s))\Psi D\Psi(X^{xt}(s),s)| \leq 2||a||(\Psi^2(X^{xt}(s),s)|Z^{xt}(s)|^2 + |D\Psi(X^{xt}(s),s)|^2)$. Therefore, from (6.11), we obtain

$$\left|\left(-\frac{\partial \hat{u}_\nu^l}{\partial t} + A(t)\hat{u}_\nu^l\right)(x,t)\right| \leq l\int_t^{t+\frac{1}{l}} \left[C_0(\Psi) + \mathbb{E}\Psi^2(X^{xt}(s),s)|Z^{xt}(s)|^2 + \mathbb{E}\Psi^2(X^{xt}(s),s)k_\nu(X^{xt}(s),s)\right]ds. \tag{6.12}$$

Then, from Markov properties

$$\begin{aligned} &\mathbb{E}\int_t^{T-\epsilon} \left|\frac{\partial \hat{u}_\nu^l}{\partial t} - A(t)\hat{u}_\nu^l\right|(X^{xt}(\tau),\tau)d\tau \\ &\leq C(\Psi) + l\mathbb{E}\int_t^{T-\epsilon}\left(\int_\tau^{\tau+\frac{1}{l}} \Psi^2(X^{X_{xt}(\tau),\tau}(s),s)\left[|Z^{X_{xt}(\tau),\tau}(s)|^2 + k_\nu(X^{X_{xt}(\tau),\tau}(s),s)\right]ds\right)d\tau \end{aligned}$$

$$= C(\Psi) + l\mathbb{E}\int_t^{T-\epsilon}\left(\int_\tau^{\tau+\frac{1}{l}} \Psi^2(X^{xt}(s),s)\left[|Z^{xt}(s)|^2 + k_\nu(X^{xt}(s),s)\right]ds\right)d\tau$$

$$= C(\Psi) + \mathbb{E}\int_t^T \Psi^2(X^{xt}(s),s)\left[|Z^{xt}(s)|^2 + k_\nu(X^{xt}(s),s)\right]ds,$$

and from (6.5) and (6.6) it follows

$$\mathbb{E}\int_t^{T-\epsilon}\left|\frac{\partial \hat{u}_\nu^l}{\partial t} - A(t)\hat{u}_\nu^l\right|(X^{xt}(s),s)ds \le C(\Psi),\ \forall l > \frac{1}{\epsilon}. \tag{6.13}$$

The next thing is to notice that $\hat{u}_\nu(x,t)$ is continuous and vanishes outside a compact, so it is uniformly continuous. Moreover,

$$\begin{aligned} &\hat{u}_\nu^l(x,t) - \hat{u}_\nu(x,t) \\ &= l\int_t^{t+\frac{1}{l}}\left(\mathbb{E}\hat{u}_\nu(X^{xt}(s),s) - \hat{u}_\nu(x,t)\right)ds \\ &= l\int_t^{t+\frac{1}{l}}\int_{\mathbb{R}^n}(\hat{u}_\nu(\xi,s) - \hat{u}_\nu(x,t))G_{xt}(\xi,s)d\xi ds \\ &\le \sup_{|\xi-x|\le(\frac{1}{l})^{\frac{1}{4}},|s-t|\le\frac{1}{l}}|\hat{u}_\nu(\xi,s) - \hat{u}_\nu(x,t)| + l\int_t^{t+\frac{1}{l}}\int_{\mathbb{R}^n}(\hat{u}_\nu(\xi,s) - \hat{u}_\nu(x,t))\mathbb{1}_{|x-\xi|\ge(\frac{1}{l})^{\frac{1}{4}}}G_{xt}(\xi,s)d\xi ds. \end{aligned} \tag{6.14}$$

We use, see (6.4),

$$\begin{aligned} &l\int_t^{t+\frac{1}{l}}\int_{\mathbb{R}^n}\mathbb{1}_{|x-\xi|\ge(\frac{1}{l})^{\frac{1}{4}}}G_{xt}(\xi,s)d\xi ds \\ &\le k_2 l\int_t^{t+\frac{1}{l}}\int_{\mathbb{R}^n}\mathbb{1}_{|x-\xi|\ge(\frac{1}{l})^{\frac{1}{4}}}\frac{1}{(s-t)^{\frac{n}{2}}}\exp\left(-\frac{\delta_2|\xi-x|^2}{s-t}\right)d\xi ds \\ &= k_2\int_0^1\int_{\mathbb{R}^n}\mathbb{1}_{|\eta|\ge\frac{l^{\frac{1}{4}}}{\sqrt{\tau}}}\exp\left(-\delta_2|\eta|^2\right)d\eta d\tau. \end{aligned}$$

Recalling that $\hat{u}_\nu$ is bounded and collecting estimates we obtain

$$|\hat{u}_\nu^l(x,t) - \hat{u}_\nu(x,t)| \to 0, \text{ as } l \to +\infty, \text{ uniformly in } \mathbb{R}^n \times [0, T-\epsilon]. \tag{6.15}$$

The next thing is to apply Ito's formula to $\frac{1}{2}\sum_\nu(\hat{u}_\nu^l(X^{xt}(s),s) - \hat{u}_\nu(X^{xt}(s),s))^2$. We get

$$\mathbb{E}\int_t^{T-\epsilon}\sum_\nu\left[D\hat{u}^l_\nu(X^{xt}(s),s)-Z^{xt}_\nu(s)\Psi^2(X^{xt}(s),s)-Y^{xt}_\nu(s)D\Psi^2(X^{xt}(s),s)\right]$$
$$\cdot\, a(X^{xt}(s)\left[D\hat{u}^l_\nu(X^{xt}(s),s)-Z^{xt}_\nu(s)\Psi^2(X^{xt}(s),s)-Y^{xt}_\nu(s)D\Psi^2(X^{xt}(s),s)\right]ds$$
$$=\frac{1}{2}\mathbb{E}\sum_\nu(\hat{u}^l_\nu(X^{xt}(T-\epsilon),T-\epsilon)-\hat{u}_\nu(X^{xt}(T-\epsilon),T-\epsilon))^2-\frac{1}{2}\sum_\nu(\hat{u}^l_\nu(x,t)-\hat{u}_\nu(x,t))^2$$
$$+\,\mathbb{E}\int_t^{T-\epsilon}\sum_\nu(\hat{u}^l_\nu(X^{xt}(s),s)-\hat{u}_\nu(X^{xt}(s),s))$$
$$+\left(\left(-\frac{\partial\hat{u}^l_\nu}{\partial s}+A(s)\hat{u}^l_\nu\right)(X^{xt}(s),s)-\Psi^2(X^{xt}(s),s)H_\nu(X^{xt}(s),Y^{xt}(s),Z^{xt}(s))\right.$$
$$\left.+\,Y^{xt}_\nu(s)\left(\frac{\partial\Psi^2}{\partial s}-A(s)\Psi^2\right)(X^{xt}(s),s)+2Z^{xt}_\nu(s).a(X^{xt}(s))D\Psi^2(X^{xt}(s),s)\right)ds.$$

Thanks to the uniform convergence (6.15) and to the bounds (6.13) and (6.6) we check that the right hand side goes to 0 as $l\to+\infty$. Therefore we have obtained, recalling the Markov properties (6.1)

$$\mathbb{E}\int_t^{T-\epsilon}\sum_\nu|D\hat{u}^l_\nu(X^{xt}(s),s)-v_\nu(X^{xt}(s),s)\Psi^2(X^{xt}(s),s)-u_\nu(X^{xt}(s),s)D\Psi^2(X^{xt}(s),s)|^2ds\to 0,\ \text{as } l\to+\infty. \tag{6.16}$$

This means also

$$\int_t^{T-\epsilon}\int_{\mathbb{R}^n}G_{xt}(\xi,s)\sum_\nu|D\hat{u}^l_\nu(\xi,s)-v_\nu(\xi,s)\Psi^2(\xi,s)-u_\nu(\xi,s)D\Psi^2(\xi,s)|^2d\xi ds\to 0,\ \text{as } l\to+\infty. \tag{6.17}$$

We apply this property with $t=0$ and $x=0$. From the left inequality (6.4) we see that $G_{00}(\xi,s)\geq m>0$, if $\epsilon<s<T$ and $|\xi|<M$. Therefore

$$\int_\epsilon^{T-\epsilon}\int_{B_M}\sum_\nu|D\hat{u}^l_\nu(\xi,s)-v_\nu(\xi,s)\Psi^2(\xi,s)-u_\nu(\xi,s)D\Psi^2(\xi,s)|^2d\xi ds\to 0,\ \text{as } l\to+\infty. \tag{6.18}$$

On the other hand, from the uniform convergence (6.15) we have also $D\hat{u}^l_\nu(\xi,s)\to D\hat{u}_\nu(\xi,s)$ in the sense of distributions on $\mathbb{R}^n$. Since

$$D\hat{u}_\nu(\xi,s)=Du_\nu(\xi,s)\Psi^2(\xi,s)+u_\nu(\xi,s)D\Psi^2(\xi,s)$$

comparing with (6.18), since M is arbitrarily large, we obtain $v_\nu(\xi,s)\Psi^2(\xi,s)=Du_\nu(\xi,s)\Psi^2(\xi,s)$, which implies the desired result. □

7. Local Hölder regularity

7.1. Preliminaries

Let $x_0\in\mathbb{R}^n$ and $t_0\in(0,T)$. We denote $z_0=(x_0,t_0)$. We consider $G_{x_0,t_0-\theta}(x,t)$ which we denote by $G_\theta(x,t)$, with $t_0-\theta>0$. We have from (6.4)

$$k_1(t-t_0+\theta)^{-\frac{n}{2}}\ \exp\left(-\delta_1\frac{|x-x_0|^2}{t-t_0+\theta}\right)\leq G_{x_0,t_0-\theta}(x,t)\leq k_2(t-t_0+\theta)^{-\frac{n}{2}}\ \exp\left(-\delta_2\frac{|x-x_0|^2}{t-t_0+\theta}\right). \quad (7.1)$$

We establish several useful inequalities. By first noting that $s^{-\frac{n}{2}}\exp\left(-\frac{\beta}{s}\right)$ attains its maximum for $s>0$ at $\hat{s}=\frac{2\beta}{n}$, we immediately obtain the estimate

$$G_\theta(x,t)\leq c|x-x_0|^{-n},\ \ c=k_2\left(\frac{2\delta_2}{n}\right)^{-\frac{n}{2}}. \quad (7.2)$$

We can improve this estimate, noticing that $s^{-\frac{n}{2}}\exp\left(-\frac{\beta}{s}\right)$ is increasing, if $s<\hat{s}$. We see easily that

$$t-t_0+\theta\leq\epsilon^2|x-x_0|^2\Longrightarrow G_\theta(x,t)\leq\delta(\epsilon)|x-x_0|^{-n}, \quad (7.3)$$

with $\delta(\epsilon)=k_2\epsilon^{-n}\exp\left(-\frac{\delta_2}{\epsilon^2}\right)\to 0$, as $\epsilon\to 0$. Finally, we have

$$\epsilon^2|x-x_0|^2<t-t_0+\theta<m^2|x-x_0|^2\Longrightarrow G_\theta(x,t)\geq\delta_0(\epsilon)|x-x_0|^{-n} \quad (7.4)$$

with $\delta_0(\epsilon)=k_1m^{-n}\exp\left(-\frac{\delta_1}{\epsilon^2}\right)\to 0$, as $\epsilon\to 0$.

7.2. Basic inequality

We introduce a smooth enough function $\tau:\mathbb{R}^n\to\mathbb{R}$, such that $0\leq\tau(x)\leq 1$, $\tau(x)=1$ if $|x|\leq 1$, $\tau(x)=0$, if $|x|\geq 2$. We introduce also a smooth function $\beta(t)$, such that $0\leq\beta(t)\leq 1$, $\beta(t)=1$, if $0\leq t\leq 1$, $\beta(t)=0$, if $t\geq 4$. We then define

$$\tau_{R,x_0}(x)=\tau\left(\frac{x-x_0}{R}\right),\ \beta_{R,t_0}(t)=\beta\left(\frac{t-t_0}{R^2}\right),$$
$$\eta_{R,z_0}(x,t)=\eta_R(x,t)=\tau_{R,x_0}(x)\beta_{R,t_0}(t).$$

Let $B_R(x_0)=B_R=\{x||x-x_0|<R\}$ and $Q_R(z_0)=B_R(x_0)\cap\{t|t_0<t<(t_0+R^2)\wedge T\}$. We consider an open bounded domain of $\mathbb{R}^n$ called $\mathcal{O}$ and $x_0\in\bar{\mathcal{O}}$.

The number $R<R_1$ (the interesting case is R very small), with $R_1^2>T$, and we introduce $\tilde{\mathcal{O}}=\cup_{x_0\in\bar{\mathcal{O}}}B_{2R_1}(x_0)$. We define

$$\Psi(x,t)=\begin{cases}\eta_R^2G_\theta(x,t), & \text{if } t>t_0,\\ 0, & \text{if } t\leq t_0.\end{cases} \quad (7.5)$$

The domain of $\Psi(x,t)\subset\tilde{\mathcal{O}},\forall t\in[0,T]$. Then $\rho_\Psi\leq\rho$ with

$$\rho=\sup_{xt}\Phi(x,t)\mathbb{1}_{\tilde{\mathcal{O}}}(x). \quad (7.6)$$

We then obtain from the inequality (5.11)

$$-\int_0^T\int_{\mathbb{R}^n}\Psi\frac{\partial L}{\partial t}dxdt+\int_0^T\int_{\mathbb{R}^n}\sum_{ij}a_{ij}\frac{\partial\Psi}{\partial x_i}\frac{\partial L}{\partial x_j}dxdt+\alpha\int_0^T\int_{\mathbb{R}^n}\Psi|Du|^2dxdt\leq c(\rho)\sum_\nu\int_0^T\int_{\mathbb{R}^n}\Psi k_\nu(x,t)dxdt. \quad (7.7)$$

Our basic inequality is

Proposition 16. *We make the assumptions of Theorem 1. We take constants $c_{\nu R}$ such that $|c_{\nu R}| < \rho$. We then state*

$$\begin{aligned}\frac{\alpha}{2}\int_{t_0}^{T\wedge(t_0+R^2)}\int_{B_R(x_0)}|Du|^2G_\theta(x,t)dxdt \leq C(\rho,R_1)&\int_{Q_{2R}(z_0)-Q_R(z_0)}\frac{|u(x,t)-c_R|^2}{R^2}G_\theta(x,t)dxdt\\&+C(\rho)\mathbb{1}_{T<t_0+4R^2}\int_{B_{2R}(x_0)}|h(x)-c_R|^2G_\theta(x,T)dx\\&+C(\rho)||k||_{L^d(Q_{2R}(z_0))}R^{2-\frac{n+2}{d}},\end{aligned} \tag{7.8}$$

where c_R is the vector of components $c_{\nu R}$. $C(\rho)$ is a generic constant depending only of ρ, and increases with ρ. The number ρ depends on R_1 and the constant $C(\rho,R_1)$ is increasing in both variables.

Proof. We compute

$$-\int_0^T\int_{\mathbb{R}^n}\Psi\frac{\partial L}{\partial t}dxdt+\int_0^T\int_{\mathbb{R}^n}\sum_{ij}a_{ij}\frac{\partial\Psi}{\partial x_i}\frac{\partial L}{\partial x_j}dxdt=-\int_{t_0}^T\int_{\mathbb{R}^n}\eta_R^2G_\theta\frac{\partial L}{\partial t}dxdt+\int_0^T\int_{\mathbb{R}^n}\sum_{ij}a_{ij}\frac{\partial(\eta_R^2G_\theta)}{\partial x_i}\frac{\partial L}{\partial x_j}dxdt.$$

Performing integration by parts in t for the first integral and in x for the 2nd, and using the PDE of G_θ, then reinserting in (7.7), we obtain

$$\begin{aligned}&\alpha\int_{t_0}^T\int_{\mathbb{R}^n}|Du|^2G_\theta\eta_R^2dxdt+\int_{t_0}^T\int_{\mathbb{R}^n}\frac{\partial\eta_R^2}{\partial t}L(x,t)G_\theta(x,t)dxdt+\int_{t_0}^T\int_{\mathbb{R}^n}\sum_{ij}\frac{\partial a_{ij}}{\partial x_i}\frac{\partial\eta_R^2}{\partial x_j}L(x,t)G_\theta(x,t)dxdt\\&+\int_{t_0}^T\int_{\mathbb{R}^n}\sum_{ij}a_{ij}\frac{\partial^2\eta_R^2}{\partial x_j\partial x_i}L(x,t)G_\theta(x,t)dxdt+2\int_{t_0}^T\int_{\mathbb{R}^n}\sum_{ij}a_{ij}\frac{\partial\eta_R^2}{\partial x_i}\frac{\partial L}{\partial x_j}G_\theta(x,t)dxdt\\&\leq c(\rho)\sum_\nu\int_{t_0}^T\int_{\mathbb{R}^n}k_\nu(x,t)\eta_R^2G_\theta(x,t)dxdt+\int_{\mathbb{R}^n}L(x,T)\eta_R^2G_\theta(x,T)dx-\int_{\mathbb{R}^n}L(x,t_0)\eta_R^2G_\theta(x,t_0)dx.\end{aligned} \tag{7.9}$$

We can estimate the term $-2\int_{t_0}^T\int_{\mathbb{R}^n}\sum_{i,j}a_{ij}\frac{\partial\eta_R^2}{\partial x_i}\frac{\partial L}{\partial x_j}G_\theta(x,t)dxdt$, recalling $\frac{\partial L}{\partial x_j}(x,t)=\sum_\nu A_\nu(u-c_R)\frac{\partial u_\nu}{\partial x_j}$ by

$$\begin{aligned}&-2\int_{t_0}^T\int_{\mathbb{R}^n}\sum_{ij}a_{ij}\frac{\partial\eta_R^2}{\partial x_i}\frac{\partial L}{\partial x_j}G_\theta(x,t)dxdt\\&\leq\frac{\alpha}{2}\int_{t_0}^T\int_{\mathbb{R}^n}|Du|^2G_\theta\eta_R^2dxdt+\frac{4}{\alpha}||a||^2\int_{t_0}^T\int_{\mathbb{R}^n}\sum_\nu A_\nu^2(u-c_R)|D\eta_R|^2G_\theta(x,t)dxdt.\end{aligned}$$

Combining with (7.9) we obtain

$$
\begin{aligned}
\frac{\alpha}{2}\int_{t_0}^{T}\int_{\mathbb{R}^n}|Du|^2G_\theta\eta_R^2dxdt \leq &\int_{t_0}^{T}\int_{\mathbb{R}^n}\left(\left|\frac{\partial\eta_R^2}{\partial t}\right|+|\mathrm{div}a|\left|D\eta_R^2\right|+\left|\mathrm{tra}D^2\eta_R^2\right|\right)L(x,t)G_\theta(x,t)dxdt \\
&+\frac{4}{\alpha}||a||^2\int_{t_0}^{T}\int_{\mathbb{R}^n}\sum_\nu A_\nu^2(u-c_R)|D\eta_R|^2G_\theta(x,t)dxdt \\
&+c(\rho)\sum_\nu\int_{t_0}^{T}\int_{\mathbb{R}^n}k_\nu(x,t)\eta_R^2G_\theta(x,t)dxdt+\int_{\mathbb{R}^n}L(x,T)\eta_R^2G_\theta(x,T)dx.
\end{aligned}
\tag{7.10}
$$

We next use $L(0)=0$, $\dfrac{\partial L}{\partial y_\nu}(0)=A_\nu(0)=0$ to write $L(y)=\int_0^1\int_0^1\beta\sum_{\nu\mu}\dfrac{\partial A_\nu}{\partial y_\mu}(\beta\lambda y)\,y_\mu y_\nu d\lambda d\beta$. Therefore

$$
\begin{aligned}
L(x,t)&=L(u-c_R)\leq C(\rho)|u-c_R|^2,\\
|A_\nu(u-c_R)|&\leq C(\rho)|u-c_R|,\text{if } \ x\in\tilde{\mathcal{O}}.
\end{aligned}
$$

Also

$$
\left|\frac{\partial\eta_R^2}{\partial t}\right|+|\mathrm{div}a|\cdot\left|D\eta_R^2\right|+\left|\mathrm{tra}D^2\eta_R^2\right|\leq\frac{C(R_1)}{R^2}.
$$

This is clear for the derivative in t and the second derivative in x. For the first derivative, we have $|D\eta_R|\leq\dfrac{C}{R}\leq\dfrac{CR_1}{R^2}$. Finally

$$
\begin{aligned}
\int_{t_0}^{T}\int_{\mathbb{R}^n}k_\nu(x,t)\eta_R^2G_\theta(x,t)dxdt &\leq \int_{t_0}^{(t_0+4R^2)\wedge T}\int_{B_{2R}(x_0)}k_\nu(x,t)G_\theta(x,t)dxdt \\
&\leq\left(\int_{0}^{T}\int_{Q_{2R}(z_0)}k_\nu^d(x,t)dxdt\right)^{\frac{1}{d}}\left(\int_{t_0}^{(t_0+4R^2)\wedge T}\int_{B_{2R}(x_0)}G_\theta^{d'}(x,t)dxdt\right)^{\frac{1}{d'}}\\
&\leq c||k||_{L^d(Q_{2R}(z_0))}R^{2-\frac{n+2}{d}}.
\end{aligned}
$$

Collecting results, we obtain the inequality (7.8). $\square$

Probabilistic proof. We give now a probabilistic proof of (7.8), based on the inequality (5.14). We use the Markov property of Proposition 15, to write (5.14) as

$$
\begin{aligned}
&\frac{\alpha}{2}\mathbb{E}\int_{t}^{T}\sum_\nu\Psi^2(X^{xt}(s),s)|Du_\nu(X^{xt}(s),s)|^2ds\\
&\leq\mathbb{E}\int_{t}^{T}\left(-\frac{\partial\Psi^2}{\partial s}+A(s)\Psi^2\right)(X^{xt}(s),s)L(Y^{xt}(s)-c)ds+\frac{8}{\alpha}||a||^2\mathbb{E}\int_{t}^{T}\sum_\nu A_\nu^2(Y^{xt}(s)-c)|D\Psi(X^{xt}(s),s)|^2ds\\
&\quad+\mathbb{E}\left(\Psi^2(X^{xt}(T),T)L(Y^{xt}(T)-c)\right)+c(\rho_\Psi)\mathbb{E}\int_{t}^{T}\Psi^2(X^{xt}(s),s)\sum_\nu k_\nu(X^{xt}(s),s)ds.
\end{aligned}
\tag{7.11}
$$

We apply this inequality with $t = t_0 - \theta, x = x_0$ and

$$\Psi(x,s) = \begin{cases} \eta_R(x,s), & \text{if } s > t_0, \\ 0, & \text{if } s \leq t_0. \end{cases}$$

We obtain

$$\begin{aligned} &\frac{\alpha}{2}\mathbb{E}\int_{t_0}^{T}\sum_\nu \eta_R^2(X^{x_0,t_0-\theta}(s),s)|Du_\nu(X^{x_0,t_0-\theta}(s),s)|^2 ds \\ &\leq \mathbb{E}\int_{t_0}^{T}\left(-\frac{\partial\eta_R^2}{\partial s} + A(s)\eta_R^2\right)(X^{x_0,t_0-\theta}(s),s)L(Y^{x_0,t_0-\theta}(s)-c_R)ds \\ &\quad + \frac{8}{\alpha}||a||^2\mathbb{E}\int_{t_0}^{T}\sum_\nu A_\nu^2(Y^{x_0,t_0-\theta}(s)-c_R)|D\eta_R(X^{x_0,t_0-\theta}(s),s)|^2 ds \\ &\quad + \mathbb{E}\left(\eta_R^2(X^{x_0,t_0-\theta}(T),T)L(Y^{x_0,t_0-\theta}(T)-c_R)\right) + c(\rho)\mathbb{E}\int_{t_0}^{T}\eta_R^2(X^{x_0,t_0-\theta}(s),s)\sum_\nu k_\nu(X^{x_0,t_0-\theta}(s),s)ds, \end{aligned} \tag{7.12}$$

which implies, with considerations similar to those of Proposition 16,

$$\begin{aligned} &\frac{\alpha}{2}\mathbb{E}\int_{t_0}^{T}\mathbb{1}_{X^{x_0,t_0-\theta}(s)\in Q_R(z_0)}|Du(X^{x_0,t_0-\theta}(s),s)|^2 ds \\ &\leq C(\rho,R_1)\mathbb{E}\int_{t_0}^{T}\mathbb{1}_{X^{x_0,t_0-\theta}(s)\in Q_{2R}(z_0)-Q_R(z_0)}\frac{|u(X^{x_0,t_0-\theta}(s),s)-c_R|^2}{R^2}ds \\ &\quad + C(\rho)\mathbb{E}\eta_R^2(X^{x_0,t_0-\theta}(T),T)|Y^{x_0,t_0-\theta}(T)-c_R|^2 + c(\rho)\mathbb{E}\int_{t_0}^{T}\eta_R^2(X^{x_0,t_0-\theta}(s),s)\sum_\nu k_\nu(X^{x_0,t_0-\theta}(s),s)ds. \end{aligned} \tag{7.13}$$

Since the probability distribution of $X^{x_0,t_0-\theta}(s)$ is $G_\theta(\xi,s)$, we obtain immediately the inequality (7.8).

7.3. Choice of the constants $c_{\nu R}$

First we can write, as an immediate consequence of (7.8)

$$\begin{aligned} \frac{\alpha}{2}\int_{Q_R(z_0)}|Du|^2 G_\theta(x,t)dxdt \leq C(\rho,R_1)&\int_{Q_{2R}(z_0)-Q_{R/2}(z_0)}\frac{|u(x,t)-c_R|^2}{R^2}G_\theta(x,t)dxdt \\ &+ C(\rho)\mathbb{1}_{T<t_0+4R^2}\int_{B_{2R}(x_0)}|h(x)-c_R|^2 G_\theta(x,T)dx \\ &+ C(\rho)||k||_{L^d(Q_{2R}(z_0))}R^{2-\frac{n+2}{d}}. \end{aligned} \tag{7.14}$$

We now explain the choice of the constants $c_{\nu R}$. We follow ideas and results of M. Struwe [9]. We introduce

$$\zeta(x)=\begin{cases}0, & \text{for } |x|\le \frac{1}{2},\\ \tau(x), & \text{for } |x|\ge 1,\end{cases}$$

and set

$$\zeta_R(x)=\zeta\left(\frac{x-x_0}{R}\right),\ \varphi_R(x,t)=\zeta_R(x)\beta_R(t),\ \ \varphi_R(x,t)=\eta_R(x,t) \text{ if } x\notin B_R(x_0). \tag{7.15}$$

In the sequel we write, to simplify notation,

$$\beta_0=2-\frac{n+2}{d}. \tag{7.16}$$

We define

$$u^{\zeta}_{\nu R;x_0t}=\frac{\int_{B_{2R}(x_0)}u_\nu(x,t)\zeta_R(x)dx}{\int_{B_{2R}(x_0)}\zeta_R(x)dx}. \tag{7.17}$$

We note that $|u^{\zeta}_{\nu R;x_0t}|\le\rho$. We also call $u^{\zeta}_{R;x_0t}$ the vector of components $u^{\zeta}_{\nu R;x_0t}$. It is easy to check also that

$$\int_{\mathbb{R}^n}\zeta_R(x)dx\ge c_0R^n, \tag{7.18}$$

where $c_0>0$ is a fixed constant. We begin with the

Lemma 17. *Let $t_0<s\le t<T\wedge(t_0+4R^2)$, then*

$$|u^{\zeta}_{R;x_0t}-u^{\zeta}_{R;x_0s}|^2\le C(\rho)R^{-n}\int_s^t\int_{B_{2R}(x_0)-B_{R/2}(x_0)}|Du(x,\tau)|^2dxd\tau+c||k||^2_{L^d(Q_{2R}(z_0))}R^{2\beta_0}. \tag{7.19}$$

Proof. We have

$$\begin{aligned}&\int_{B_{2R}(x_0)}u_\nu(x,t)\zeta_R(x)dx-\int_{B_{2R}(x_0)}u_\nu(x,s)\zeta_R(x)dx\\
&=\int_s^t\int_{B_{2R}(x_0)}\frac{\partial u_\nu}{\partial\tau}(x,\tau)dxd\tau\\
&=\int_s^t\int_{B_{2R}(x_0)}A(\tau)u_\nu(x,\tau)\zeta_R(x)dxd\tau-\int_s^t\int_{B_{2R}(x_0)}H_\nu(x,u,Du)\zeta_R(x)dxd\tau\\
&=\int_s^t\int_{B_{2R}(x_0)}\sum_{ij}a_{ij}(x)\frac{\partial u_\nu}{\partial x_j}\frac{\partial\zeta_R}{\partial x_i}dxd\tau-\int_s^t\int_{B_{2R}(x_0)}H_\nu(x,u,Du)\zeta_R(x)dxd\tau.\end{aligned}$$

Therefore,

$$\left|\int_{B_{2R}(x_0)} u_\nu(x,t)\zeta_R(x)dx - \int_{B_{2R}(x_0)} u_\nu(x,s)\zeta_R(x)dx\right|$$

$$\leq \frac{c}{R}\int_s^t \int_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du_\nu|dxd\tau + K\int_s^t \int_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du|^2dxd\tau + \int_s^t \int_{B_{2R}(x_0)} k_\nu(x,\tau)dxd\tau,$$

and $\int_s^t \int_{B_{2R}(x_0)} k_\nu(x,\tau)dxd\tau \leq ||k||_{L^d(Q_{2R}(z_0))}R^{n+\beta_0}$. Hence,

$$\left|\int_{B_{2R}(x_0)} u_\nu(x,t)\zeta_R(x)dx - \int_{B_{2R}(x_0)} u_\nu(x,s)\zeta_R(x)dx\right|$$

$$\leq cR^{\frac{n}{2}}\left(\int_s^t \int_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du_\nu|^2dxd\tau\right)^{\frac{1}{2}} + K\int_s^t \int_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du|^2dxd\tau + c||k||_{L^d(Q_{2R}(z_0))}R^{n+\beta_0},$$

from which it follows, recalling (7.18)

$$\left|u^\zeta_{R;x_0t} - u^\zeta_{R;x_0s}\right|$$

$$\leq cR^{-\frac{n}{2}}\left(\int_s^t \int_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du|^2dxd\tau\right)^{\frac{1}{2}} + cR^{-n}\int_s^t \int_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du|^2dxd\tau + c||k||_{L^d(Q_{2R}(z_0))}R^{\beta_0}.$$

Using the fact that $|u^\zeta_{R;x_0t}| \leq c\rho$, we deduce from the preceding inequality

$$|u^\zeta_{R;x_0t} - u^\zeta_{R;x_0s}|^2 \leq C(\rho)R^{-n}\int_s^t \int_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du|^2dxd\tau$$
$$+ |u^\zeta_{R;x_0t} - u^\zeta_{R;x_0s}|\left(cR^{-\frac{n}{2}}\left(\int_s^t \int_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du|^2dxd\tau\right)^{\frac{1}{2}} + c||k||_{L^d(Q_{2R}(z_0))}R^{\beta_0}\right),$$

and (7.19) follows easily. ■ □

The preceding proof uses the PDE (1.1). We want to give a probabilistic proof

Probabilistic proof of (7.19). It is sufficient to give a probabilistic proof of the relation

$$\begin{aligned}&\int_{B_{2R}(x_0)} u_\nu(x,t)\zeta_R(x)dx - \int_{B_{2R}(x_0)} u_\nu(x,s)\zeta_R(x)dx \\ &= \int_s^t \int_{B_{2R}(x_0)} \sum_{ij} a_{ij}(x)\frac{\partial u_\nu}{\partial x_j}\frac{\partial \zeta_R}{\partial x_i}dxd\tau - \int_s^t \int_{B_{2R}(x_0)} H_\nu(x,u,Du)\zeta_R(x)dxd\tau,\end{aligned} \tag{7.20}$$

in which of course $u_\nu(x,t) = Y^{xt}(t)$ and $Du_\nu(x,t) = Z_\nu^{xt}(t)$. The idea is to give a probabilistic interpretation of the left hand side of (7.20). Define $\chi_R(x,s) = \dfrac{\zeta_R(x)}{G_\theta(x,s)}$ then we can write

$$\begin{aligned}&\int_{B_{2R}(x_0)} u_\nu(x,t)\zeta_R(x)dx - \int_{B_{2R}(x_0)} u_\nu(x,s)\zeta_R(x)dx \\ &= \mathbb{E}u_\nu(X^{x_0,t_0-\theta}(t),t)\chi_R(X^{x_0,t_0-\theta}(t),t) - \mathbb{E}u_\nu(X^{x_0,t_0-\theta}(s),s)\chi_R(X^{x_0,t_0-\theta}(s),s).\end{aligned} \tag{7.21}$$

Using Ito's formula, we obtain

$$\begin{aligned}&\int_{B_{2R}(x_0)} u_\nu(x,t)\zeta_R(x)dx - \int_{B_{2R}(x_0)} u_\nu(x,s)\zeta_R(x)dx \\ &= \mathbb{E}\int_t^s \Big[- \chi_R(X^{x_0,t_0-\theta}(\tau),\tau)H_\nu(X^{x_0,t_0-\theta}(\tau), u(X^{x_0,t_0-\theta}(\tau),\tau), Du(X^{x_0,t_0-\theta}(\tau),\tau)) \\ &\quad + u_\nu(X^{x_0,t_0-\theta}(\tau),\tau)\left(\frac{\partial \chi_R}{\partial \tau} - A(\tau)\chi_R\right)(X^{x_0,t_0-\theta}(\tau),\tau) \\ &\quad + 2Du_\nu(X^{x_0,t_0-\theta}(\tau),\tau).a(X^{x_0,t_0-\theta}(\tau))D\chi_R(X^{x_0,t_0-\theta}(\tau),\tau)\Big]d\tau,\end{aligned}$$

which we can reinterpret as

$$\begin{aligned}&\int_{B_{2R}(x_0)} u_\nu(x,t)\zeta_R(x)dx - \int_{B_{2R}(x_0)} u_\nu(x,s)\zeta_R(x)dx \\ &= \int_t^s\int_{\mathbb{R}^n} \Big[- \zeta_R(x)H_\nu(x,u(x,\tau),Du(x,\tau)) \\ &\quad + G_\theta(x,\tau)u_\nu(x,\tau)\left(\frac{\partial \chi_R}{\partial \tau} - A(\tau)\chi_R\right)(x,\tau) + 2G_\theta(x,\tau)Du_\nu(x,\tau).a(x)D\chi_R(x,\tau)\Big]dxd\tau.\end{aligned} \tag{7.22}$$

We note that $G_\theta \dfrac{\partial \chi_R}{\partial \tau} = -\chi_R \dfrac{\partial G_\theta}{\partial \tau} = -\chi_R \sum_{i,j} \dfrac{\partial}{\partial x_j}\left(a_{ij}\dfrac{\partial G_\theta}{\partial x_i}\right)$, so

$$\begin{aligned}&\int_{B_{2R}(x_0)} u_\nu(x,t)\zeta_R(x)dx - \int_{B_{2R}(x_0)} u_\nu(x,s)\zeta_R(x)dx \\ &= \int_t^s\int_{\mathbb{R}^n} \Big[- \zeta_R(x)H_\nu(x,u(x,\tau),Du(x,\tau)) - \sum_{ij}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial G_\theta}{\partial x_i}\right)\chi_R(x,\tau)u_\nu(x,\tau) \\ &\quad + G_\theta(x,\tau)u_\nu(x,\tau)\sum_{ij}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial \chi_R}{\partial x_i}\right) + 2G_\theta(x,\tau)Du_\nu(x,\tau).a(x)D\chi_R(x,\tau)\Big]dxd\tau.\end{aligned}$$

After integration by parts and simplification we obtain immediately (7.20). The rest of the proof of Lemma 17 does not use the PDE (1.1) and thus is unchanged.

From now on, we use only inequality (7.14) and (7.19). So there is no difference between the analytic and the probabilistic proof. We choose the constants $c_{\nu R}$ as follows. We take $\theta < \beta R^2$ and consider two cases (ϵ will be a small constant, as small as we need)

$$\textbf{Case} \quad t_0 - \theta + \epsilon^2 R^2 \geq T: \ c_{\nu R} = u^{\zeta}_{\upsilon R;x_0 T} \tag{7.23}$$

$$\textbf{Case} \quad t_0 - \theta + \epsilon^2 R^2 < T: \ c_{\nu R} = \frac{\int_{t_0 \vee (t_0-\theta+\epsilon^2 R^2)}^{T\wedge(t_0+4R^2)} u^{\zeta}_{\upsilon R;x_0 t} dt}{\int_{t_0 \vee (t_0-\theta+\epsilon^2 R^2)}^{T\wedge(t_0+4R^2)} dt} \tag{7.24}$$

7.4. Hölder property

From the choice of the constants $c_{\nu R}$ we are going to derive from (7.14) the property

Proposition 18. *We make the assumptions of Theorem 1. We take $\theta < \beta R^2$, and we choose the constants $c_{\nu R}$ according to (7.23) or (7.24). Then we have*

$$\begin{aligned} \frac{\alpha}{2} \int\limits_{Q_R(z_0)} |Du|^2 G_\theta(x,t) dx dt \leq C(\rho) \int\limits_{Q_{2R}(z_0)-Q_{R/2}(z_0)} (K(\epsilon) G_\theta(x,t) + \delta(\epsilon) R^{-n}) |Du|^2 dx dt \\ + C(||k||_{L^d(Q_{2R}(z_0))}, ||Dh||_{B_{2R}(x_0)}, \rho, R_1) R^{\beta_0}, \end{aligned} \tag{7.25}$$

where $K(\epsilon) \to +\infty, \delta(\epsilon) \to 0$, as $\epsilon \to 0$, and $C(\rho)$ is a constant depending only of ρ and increases with ρ. The constant $C(||k||_{L^d(Q_{2R}(z_0))}, ||Dh||_{B_{2R}(x_0)}, \rho, R_1)$ is monotonically increasing with the arguments.

Proof. We first consider the case $t_0 - \theta + \epsilon^2 R^2 \geq T$. We have

$$c_{\nu R} = h^{\zeta}_{\nu R, x_0} = \frac{\int_{B_{2R}(x_0)} h_\nu(x) \zeta_R(x) dx}{\int_{B_{2R}(x_0)} \zeta_R(x) dx}.$$

Considering (7.14), we first estimate the term

$$\mathbb{1}_{T<t_0+4R^2} \int\limits_{B_{2R}(x_0)} |h(x) - c_R|^2 G_\theta(x,T) dx = \int\limits_{B_{2R}(x_0)} |h(x) - c_R|^2 G_\theta(x,T) dx,$$

since we may assume $\epsilon < 2$. But for $x \in B_{2R}(x_0)$, we have $|h(x) - c_R| \leq ||Dh||_{B_{2R}(x_0)} R$, since $h(x)$ is $C^1(\mathbb{R}^n)$ and $\int_{B_{2R}(x_0)} G_\theta(x,T) dx \leq C$. Therefore,

$$I = C(\rho) \mathbb{1}_{T<t_0+4R^2} \int\limits_{B_{2R}(x_0)} |h(x) - c_R|^2 G_\theta(x,T) dx \leq C(\rho) ||Dh||^2_{B_{2R}(x_0)} R^2,$$

calling by $C(\rho)$ all generic constants, depending only on ρ. We then consider

$$J = C(\rho) \int\limits_{Q_{2R}(z_0)-Q_{R/2}(z_0)} \frac{|u(x,t) - c_R|^2}{R^2} G_\theta(x,t) dx dt.$$

For $\dfrac{R}{2} < |x - x_0| < 2R$, we have $T - t_0 + \theta < \epsilon^2 R^2 < 4\epsilon^2 |x - x_0|^2$, hence from (7.3), $G_\theta(x,t) \leq \delta(2\epsilon)|x - x_0|^{-n} < \delta(2\epsilon) 2^n R^{-n}$, therefore

$$J \leq C(\rho) \int_{t_0}^{(t_0+4R^2)\wedge T} \int_{B_{2R}(x_0)-B_{R/2}(x_0)} \frac{|u(x,t)-c_R|^2}{R^2} G_\theta(x,t)dxdt$$
$$\leq C(\rho)\delta'(\epsilon)R^{-n} \int_{t_0}^{(t_0+4R^2)\wedge T} \int_{B_{2R}(x_0)-B_{R/2}(x_0)} \frac{|u(x,t)-c_R|^2}{R^2} dxdt.$$

But

$$\int_{t_0}^{(t_0+4R^2)\wedge T} \int_{B_{2R}(x_0)-B_{R/2}(x_0)} \frac{|u(x,t)-c_R|^2}{R^2} dxdt \leq 2 \int_{t_0}^{(t_0+4R^2)\wedge T} \int_{B_{2R}(x_0)-B_{R/2}(x_0)} \frac{|u(x,t)-u^\zeta_{R,x_0t}|^2}{R^2} dxdt$$
$$+ 2cR^{n-2} \int_{t_0}^{(t_0+4R^2)\wedge T} |u^\zeta_{R,x_0t} - u^\zeta_{R,x_0T}|^2 dt.$$

Since $t_0 < t < T < t_0 - \theta + \epsilon^2 R^2 < t_0 + 4R^2$, we can apply (7.19) to claim

$$|u^\zeta_{R,x_0t} - u^\zeta_{R,x_0T}|^2 \leq C(\rho)R^{-n} \int_t^T \int_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du(x,\tau)|^2 dxd\tau + c||k||^2_{L^d(Q_{2R}(z_0))} R^{2\beta_0}$$

$$R^{n-2} \int_{t_0}^{(t_0+4R^2)\wedge T} |u^\zeta_{R,x_0t} - u^\zeta_{R,x_0T}|^2 dt \leq C(\rho) \int_{t_0}^T \int_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du(x,t)|^2 dxdt + c||k||^2_{L^d(Q_{2R}(z_0))} R^{n+2\beta_0}.$$

Moreover, by an easy extension of Poincaré inequality (weighted Poincaré inequality in [8]) we have also

$$\int_{B_{2R}(x_0)-B_{R/2}(x_0)} \frac{|u(x,t)-u^\zeta_{R,x_0t}|^2}{R^2} dx \leq c \int_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du|^2 dx.$$

Collecting results we have

$$\int_{t_0}^{(t_0+4R^2)\wedge T} \int_{B_{2R}(x_0)-B_{R/2}(x_0)} \frac{|u(x,t)-c_R|^2}{R^2} dxdt$$
$$\leq C(\rho) \int_{t_0}^{(t_0+4R^2)\wedge T} \int_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du(x,t)|^2 dxdt + c||k||^2_{L^d(Q_{2R}(z_0))} R^{n+2\beta_0},$$

and thus

$$J \leq C(\rho)\delta'(\epsilon)R^{-n} \int_{Q_{2R}(z_0)-Q_{R/2}(z_0)} |Du|^2 dxdt + C(\rho)||k||^2_{L^d(Q_{2R}(z_0))} \delta'(\epsilon) R^{2\beta_0},$$

and thus from (7.14) we obtain that (7.25) is satisfied.

We next consider the case $t_0 - \theta + \epsilon^2 R^2 < T$. We consider again the terms I and J. For I we may assume $T < t_0 + 4R^2$. The constants $c_{\nu R}$ are defined by formula (7.24). So we have to evaluate the term

$\int_{B_{2R}(x_0)} |h(x) - c_R|^2 G_\theta(x,T)dx$. We begin by considering $c_R - u^\zeta_{R,x_0T} = c_R - h^\zeta_{R,x_0}$. We have since $T < t_0 + 4R^2$,

$$c_R - u^\zeta_{R,x_0T} = \frac{\int_{t_0\vee(t_0-\theta+\epsilon^2R^2)}^T \left(u^\zeta_{R,x_0t} - u^\zeta_{R,x_0T}\right) dt}{\int_{t_0\vee(t_0-\theta+\epsilon^2R^2)}^T dt},$$

hence

$$|c_R - h^\zeta_{R,x_0}|^2 \leq \frac{\int_{t_0\vee(t_0-\theta+\epsilon^2R^2)}^T |u^\zeta_{R,x_0t} - u^\zeta_{R,x_0T}|^2 dt}{\int_{t_0\vee(t_0-\theta+\epsilon^2R^2)}^T dt},$$

and from Lemma 17 we can assert that

$$|c_R - h^\zeta_{R,x_0}|^2 \leq C(\rho)R^{-n} \int\limits_{t_0\vee(t_0-\theta+\epsilon^2R^2)}^T \int\limits_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du(x,t)|^2 dxdt + c||k||^2_{L^d(Q_{2R}(z_0))} R^{2\beta_0}. \tag{7.26}$$

Now, from $t > t_0 - \theta + \epsilon^2 R^2$ and $\frac{R}{2} < |x - x_0| < 2R$, we get $t > t_0 - \theta + \epsilon^2 \frac{|x-x_0|^2}{4}$. On the other hand, $t - t_0 + \theta < 4R^2 + \beta R^2$, recalling $\theta < \beta R^2$. So $t - t_0 + \theta < (16 + 4\beta)|x - x_0|^2$. Therefore we can apply (7.4) to obtain $G_\theta(x,t) \geq \delta_0\left(\frac{\epsilon}{2}\right)|x - x_0|^{-n} \geq \delta_0\left(\frac{\epsilon}{2}\right)(2R)^{-n}$. Therefore also $R^{-n} \leq G_\theta(x,t)K'(\epsilon)$ where $K'(\epsilon) \to +\infty$ as $\epsilon \to 0$. From (7.26) we then have

$$|c_R - h^\zeta_{R,x_0}|^2 \leq C(\rho)K'(\epsilon) \int\limits_{t_0\vee(t_0-\theta+\epsilon^2R^2)}^T \int\limits_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du(x,t)|^2 G_\theta(x,t) dxdt + c||k||^2_{L^d(Q_{2R}(z_0))} R^{2\beta_0},$$

hence also

$$|c_R - h^\zeta_{R,x_0}|^2 \leq C(\rho)K'(\epsilon) \int\limits_{Q_{2R}(z_0)-Q_{R/2}(z_0)} |Du(x,t)|^2 G_\theta(x,t) dxdt + c||k||^2_{L^d(Q_{2R}(z_0))} R^{2\beta_0}. \tag{7.27}$$

Now, we take $x \in B_{2R}(x_0)$, we have

$$|h(x) - c_R|^2 \leq 2|c_R - h^\zeta_{R,x_0}|^2 + 2|h(x) - h^\zeta_{R,x_0}|^2,$$

and

$$h(x) - h^\zeta_{R,x_0} = \frac{\int_{B_{2R}(x_0)} (h(x) - h(\xi))\zeta_R(\xi)d\xi}{\int_{B_{2R}(x_0)} \zeta_R(\xi)d\xi},$$

so, for $x \in B_{2R}(x_0)$, $|h(x) - h^\zeta_{R,x_0}| \leq ||Dh||_{B_{2R}(x_0)} R$. Therefore also, from (7.27), for $x \in B_{2R}(x_0)$,

$$|c_R - h(x)|^2 \leq C(\rho)K'(\epsilon) \int\limits_{Q_{2R}(z_0)-Q_{R/2}(z_0)} |Du(x,t)|^2 G_\theta(x,t) dxdt + C(||k||_{L^d(Q_{2R}(z_0))}, ||Dh||_{B_{2R}(x_0)}) R^{2\beta_0}.$$

But then recalling that $\int_{B_{2R}(x_0)} G_\theta(x,T)dx \leq C$, we obtain

$$I \leq C(\rho)K'(\epsilon) \int\limits_{Q_{2R}(z_0)-Q_{R/2}(z_0)} |Du(x,t)|^2 G_\theta(x,t)dxdt + C(||k||_{L^d(Q_{2R}(z_0))}, ||Dh||_{B_{2R}(x_0)}, \rho)R^{2\beta_0}. \quad (7.28)$$

We next turn to

$$J = C(\rho) \int\limits_{Q_{2R}(z_0)-Q_R(z_0)} \frac{|u(x,t)-c_R|^2}{R^2} G_\theta(x,t)dxdt = C(\rho)\tilde{J}.$$

Then $\tilde{J} \leq 2(J_1 + J_2)$ with

$$J_1 = \int\limits_{Q_{2R}(z_0)-Q_{R/2}(z_0)} \frac{|u(x,t)-u^\zeta_{R,x_0t}|^2}{R^2} G_\theta(x,t)dxdt,$$

$$J_2 = \int\limits_{Q_{2R}(z_0)-Q_{R/2}(z_0)} \frac{|c_R-u^\zeta_{R,x_0t}|^2}{R^2} G_\theta(x,t)dxdt.$$

Noting that $Q_{2R}(z_0) - Q_{R/2}(z_0) = \left\{x,t \middle| \frac{R}{2} < |x-x_0| < 2R \text{ and } t_0 < t < (t_0+4R^2)\wedge T\right\} \cup \{x,t||x-x_0| < 2R \text{ and } (t_0+R^2)\wedge T < t < (t_0+4R^2)\wedge T\}$. Therefore,

$$J_1 = J_{11} + J_{12},$$

with

$$J_{11} = \int\limits_{t_0}^{(t_0+4R^2)\wedge T} \int\limits_{B_{2R}(x_0)-B_{R/2}(x_0)} \frac{|u(x,t)-u^\zeta_{R,x_0t}|^2}{R^2} G_\theta(x,t)dxdt,$$

$$J_{12} = \int\limits_{(t_0+R^2)\wedge T}^{(t_0+4R^2)\wedge T} \int\limits_{B_{2R}(x_0)} \frac{|u(x,t)-u^\zeta_{R,x_0t}|^2}{R^2} G_\theta(x,t)dxdt.$$

We still split J_{11} in two parts,

$$J_{11} = \int\limits_{t_0}^{t_0\vee(t_0-\theta+\epsilon^2R^2)} \int\limits_{B_{2R}(x_0)-B_{R/2}(x_0)} \frac{|u(x,t)-u^\zeta_{R,x_0t}|^2}{R^2} G_\theta(x,t)dxdt$$
$$+ \int\limits_{t_0\vee(t_0-\theta+\epsilon^2R^2)}^{(t_0+4R^2)\wedge T} \int\limits_{B_{2R}(x_0)-B_{R/2}(x_0)} \frac{|u(x,t)-u^\zeta_{R,x_0t}|^2}{R^2} G_\theta(x,t)dxdt.$$

In the first integral, we have $t - t_0 + \theta < \epsilon^2 R^2 < 4\epsilon^2|x-x_0|^2$ then, from (7.3), $G_\theta(x,t) \leq \delta(2\epsilon)|x-x_0|^{-n} \leq \delta(2\epsilon)2^n R^{-n}$, hence

$$\int\limits_{t_0}^{t_0\vee(t_0-\theta+\epsilon^2R^2)} \int\limits_{B_{2R}(x_0)-B_{R/2}(x_0)} \frac{|u(x,t)-u^\zeta_{R,x_0t}|^2}{R^2} G_\theta(x,t)dxdt$$

$$\leq \delta'(\epsilon)R^{-n} \int\limits_{t_0}^{t_0\vee(t_0-\theta+\epsilon^2R^2)} \int\limits_{B_{2R}(x_0)-B_{R/2}(x_0)} \frac{|u(x,t)-u^{\zeta}_{R,x_0t}|^2}{R^2}dxdt,$$

and by the weighted Poincaré inequality, it is further less than

$$c\delta'(\epsilon)R^{-n} \int\limits_{t_0}^{t_0\vee(t_0-\theta+\epsilon^2R^2)} \int\limits_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du|^2dxdt.$$

Consider next the second integral. We have $t > t_0-\theta+\epsilon^2R^2$ and $\frac{R}{2} < |x-x_0| < 2R$, hence $t-t_0+\theta > \epsilon^2R^2 > \epsilon^2\frac{|x-x_0|^2}{4}$. Also $t < t_0+4R^2$ implies $t-t_0+\theta < (4+\beta)R^2 < (16+4\beta)|x-x_0|^2$. Therefore, from (7.2), (7.3) we get $\delta_0\left(\frac{\epsilon}{2}\right)|x-x_0|^{-n} \leq G_\theta(x,t) \leq c|x-x_0|^{-n}$, hence also $\delta_0\left(\frac{\epsilon}{2}\right)2^{-n}R^{-n} \leq G_\theta(x,t) \leq c2^nR^{-n}$. Therefore

$$\int\limits_{t_0\vee(t_0-\theta+\epsilon^2R^2)}^{(t_0+4R^2)\wedge T} \int\limits_{B_{2R}(x_0)-B_{R/2}(x_0)} \frac{|u(x,t)-u^{\zeta}_{R,x_0t}|^2}{R^2}G_\theta(x,t)dxdt$$

$$\leq cR^{-n} \int\limits_{t_0\vee(t_0-\theta+\epsilon^2R^2)}^{(t_0+4R^2)\wedge T} \int\limits_{B_{2R}(x_0)-B_{R/2}(x_0)} \frac{|u(x,t)-u^{\zeta}_{R,x_0t}|^2}{R^2}dxdt$$

$$\leq cR^{-n} \int\limits_{t_0\vee(t_0-\theta+\epsilon^2R^2)}^{(t_0+4R^2)\wedge T} \int\limits_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du|^2dxdt$$

$$\leq cK'(\epsilon) \int\limits_{t_0\vee(t_0-\theta+\epsilon^2R^2)}^{(t_0+4R^2)\wedge T} \int\limits_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du|^2G_\theta(x,t)dxdt.$$

Therefore we have obtained, connecting the estimates of the two integrals

$$J_{11} \leq c \int\limits_{Q_{2R}(z_0)-Q_{R/2}(z_0)} |Du|^2(K'(\epsilon)G_\theta(x,t)+\delta'(\epsilon)R^{-n})dxdt.$$

For J_{12}, we may assume $t_0+R^2 < T$. Then for $t_0+R^2 < t < (t_0+4R^2)\wedge T$, we have $t-t_0+\theta > R^2$, hence $t-t_0+\theta > \frac{|x-x_0|^2}{4}$. Also $t < t_0+4R^2$, hence from the estimates (7.2), (7.3) we can assert that $cR^{-n} \leq G_\theta(x,t) \leq cR^{-n}$. Therefore, combining with Poincaré inequality, we get

$$J_{12} \leq c \int\limits_{(t_0+R^2)\wedge T}^{(t_0+4R^2)\wedge T} \int\limits_{B_{2R}(x_0)} |Du|^2G_\theta(x,t)dxdt,$$

and from the estimate of J_{11}, we can infer that

$$J_1 \leq c \int\limits_{Q_{2R}(z_0)-Q_{R/2}(z_0)} |Du|^2(K'(\epsilon)G_\theta(x,t)+\delta'(\epsilon)R^{-n})dxdt. \tag{7.29}$$

We turn to J_2. We have first

$$J_2 \leq c \int_{t_0}^{(t_0+4R^2)\wedge T} \frac{|c_R - u^\zeta_{R,x_0t}|^2}{R^2} dt,$$

and replacing c_R, then by an easy majoration

$$J_2 \leq \frac{c}{R^2} \int_{t_0}^{(t_0+4R^2)\wedge T} \frac{\int_{t_0\vee(t_0-\theta+\epsilon^2R^2)}^{T\wedge(t_0+4R^2)} |u^\zeta_{R,x_0t} - u^\zeta_{R,x_0s}|^2 ds}{\int_{t_0\vee(t_0-\theta+\epsilon^2R^2)}^{T\wedge(t_0+4R^2)} ds} dt,$$

which we write as $J_2 \leq J_{21} + J_{22}$, where

$$J_{21} = \frac{c}{R^2} \int_{t_0}^{t_0\vee(t_0-\theta+\epsilon^2R^2)} \frac{\int_{t_0\vee(t_0-\theta+\epsilon^2R^2)}^{T\wedge(t_0+4R^2)} |u^\zeta_{R,x_0t} - u^\zeta_{R,x_0s}|^2 ds}{\int_{t_0\vee(t_0-\theta+\epsilon^2R^2)}^{T\wedge(t_0+4R^2)} ds} dt,$$

$$J_{22} = \frac{c}{R^2} \int_{t_0\vee(t_0-\theta+\epsilon^2R^2)}^{(t_0+4R^2)\wedge T} \frac{\int_{t_0\vee(t_0-\theta+\epsilon^2R^2)}^{T\wedge(t_0+4R^2)} |u^\zeta_{R,x_0t} - u^\zeta_{R,x_0s}|^2 ds}{\int_{t_0\vee(t_0-\theta+\epsilon^2R^2)}^{T\wedge(t_0+4R^2)} ds} dt.$$

For J_{21}, we note that $t_0 < t < s < T \wedge (t_0 + 4R^2)$, therefore from Lemma 17 and $t - t_0 < \epsilon^2 R^2$, we can assert

$$J_{21} \leq \epsilon^2 \left[C(\rho) R^{-n} \int_{t_0}^{T\wedge(t_0+4R^2)} \int_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du(x,t)|^2 dxdt + c||k||^2_{L^d(Q_{2R}(z_0))} R^{2\beta_0} \right]$$
$$\leq C(\rho) \int_{Q_{2R}(z_0)-Q_{R/2}(z_0)} |Du(x,t)|^2 \delta(\epsilon) R^{-n} dxdt + c||k||^2_{L^d(Q_{2R}(z_0))} R^{2\beta_0}.$$

Consider then J_{22}. We use

$$|u^\zeta_{R,x_0t} - u^\zeta_{R,x_0s}|^2 \leq C(\rho) R^{-n} \int_{\min(s,t)}^{\max(s,t)} \int_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du|^2 dxd\tau + c||k||^2_{L^d(Q_{2R}(z_0))} R^{2\beta_0}.$$

But the variable τ lies in the interval $t_0 - \theta + \epsilon^2 R^2 < \tau < (t_0 + 4R^2) \wedge T$. Since also $\frac{R}{2} < |x - x_0| < 2R$, we have $G_\theta(x,\tau) \geq \delta_0 \left(\frac{\epsilon}{2}\right) 2^{-n} R^{-n}$. Therefore,

$$|u^\zeta_{R,x_0t} - u^\zeta_{R,x_0s}|^2 \leq C(\rho) \int_{\min(s,t)}^{\max(s,t)} \int_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du|^2 K'(\epsilon) G_\theta(x,\tau) dxd\tau + c||k||^2_{L^d(Q_{2R}(z_0))} R^{2\beta_0}$$
$$\leq C(\rho) \int_{t_0}^{(t_0+4R^2)\wedge T} \int_{B_{2R}(x_0)-B_{R/2}(x_0)} |Du|^2 K'(\epsilon) G_\theta(x,t) dxdt + c||k||^2_{L^d(Q_{2R}(z_0))} R^{2\beta_0},$$

and

$$J_{22} \leq C(\rho) \int\limits_{Q_{2R}(z_0)-Q_{R/2}(z_0)} |Du|^2 K'(\epsilon) G_\theta(x,t) dxdt + c||k||^2_{L^d(Q_{2R}(z_0))} R^{2\beta_0}.$$

Collecting results, we can state

$$J \leq C(\rho) \int\limits_{Q_{2R}(z_0)-Q_{R/2}(z_0)} |Du|^2 (K'(\epsilon) G_\theta(x,t) + \delta'(\epsilon) R^{-n}) dxdt + C(\rho)||k||^2_{L^d(Q_{2R}(z_0))} R^{2\beta_0}. \tag{7.30}$$

Collecting all estimates, we obtain again from (7.14) the inequality (7.25). The proof is complete. □

We can then state:

Proposition 19. *We make the assumptions of Theorem 1. Consider a bounded open subset $\mathcal{O}$ of $\mathbb{R}^n$ and $R_1 = diam\ \bar{\mathcal{O}} > \sqrt{T}$. Define $\tilde{\mathcal{O}} = \cup_{x_0 \in \bar{\mathcal{O}}} B_{2R_1}(x_0)$, then ρ by (7.6). There exists $\delta_\rho < \dfrac{\beta_0}{2}$, depending only of ρ such that*

$$|u(x_1,t_1) - u(x_2,t_2)| \leq C(||k||_{L^d(\tilde{\mathcal{O}}\times(0,T))}, ||Dh||_{\tilde{\mathcal{O}}}, \rho, R_1) \left(|x_1 - x_2|^{\delta_\rho} + |t_1 - t_2|^{\frac{\delta_\rho}{2}} \right), \tag{7.31}$$

for all (x_1,t_1) and (x_2,t_2) with $x_1, x_2 \in \mathcal{O}$ and $t_1, t_2 \in (0,T)$.

Proof. We first note that $G_{R^2}(x,t) \geq cR^{-n}$. This follows easily from the left inequality (7.1), using $|x-x_0| < 4R$, then $\dfrac{|x-x_0|^2}{t-t_0+R^2} < 16$ and from $t < t_0 + 4R^2$. Therefore, the inequality (7.25) implies, modifying the constant $C(\rho)$:

$$\begin{aligned} \alpha \int\limits_{Q_R(z_0)} |Du|^2 G_\theta(x,t) dxdt \leq C(\rho) \left[K(\epsilon) \int\limits_{Q_{2R}(z_0)-Q_{R/2}(z_0)} G_\theta(x,t) + \delta(\epsilon) \int\limits_{Q_{2R}(z_0)-Q_{R/2}(z_0)} G_{R^2}(x,t) \right] dxdt \\ + C(||k||_{L^d(Q_{2R}(z_0))}, ||Dh||_{B_{2R}(x_0)}, \rho, R_1) R^{\beta_0}. \end{aligned} \tag{7.32}$$

By the famous hole-filling technique, see [9], we can also write

$$\begin{aligned} (\alpha + C(\rho)K(\epsilon)) \int\limits_{Q_{R/2}(z_0)} |Du|^2 G_\theta(x,t) dxdt \leq C(\rho) K(\epsilon) \int\limits_{Q_{2R}(z_0)} |Du|^2 G_\theta(x,t) dxdt \\ + C(\rho)\delta(\epsilon) \int\limits_{Q_{2R}(z_0)} |Du|^2 G_{R^2}(x,t) dxdt \\ + C(||k||_{L^d(Q_{2R}(z_0))}, ||Dh||_{B_{2R}(x_0)}, \rho, R_1) R^{\beta_0}. \end{aligned}$$

Since $\theta < \beta R^2$ and β can be assumed larger than 1, we can assert that

$$\begin{aligned} \sup_{0<\theta\leq\beta R^2} \int\limits_{Q_{R/2}(z_0)} |Du|^2 G_\theta(x,t) dxdt \leq \frac{C(\rho)K(\epsilon) + C(\rho)\delta(\epsilon)}{\alpha + C(\rho)K(\epsilon)} \sup_{0<\theta\leq 16\beta R^2} \int\limits_{Q_{2R}(z_0)} |Du|^2 G_\theta(x,t) dxdt \\ + \frac{C(||k||_{L^d(Q_{2R}(z_0))}, ||Dh||_{B_{2R}(x_0)}, \rho, R_1)}{\alpha + C(\rho)K(\epsilon)} R^{\beta_0}. \end{aligned} \tag{7.33}$$

We choose δ (not to be confused with $\delta(\epsilon)$), to be defined precisely below, such that $2\delta < \beta_0$ and set

$$\varphi(R) = R^{-2\delta} \sup_{0<\theta\leq\beta R^2} \int_{Q_{R/2}(z_0)} |Du|^2 G_\theta(x,t)dxdt.$$

We deduce from (7.33)

$$\varphi(R) \leq \frac{C(\rho)K(\epsilon) + C(\rho)\delta(\epsilon)}{\alpha + C(\rho)K(\epsilon)} 4^{2\delta}\varphi(4R) + B,$$

with

$$B = \frac{C(||k||_{L^d(Q_{2R_1}(z_0))}, ||Dh||_{B_{2R_1}(x_0)}, \rho, R_1)}{\alpha} R_1^{\beta_0 - 2\delta}.$$

We then choose $\delta = \delta_\rho$ and $\epsilon = \epsilon_\rho$ sufficiently small so that $\frac{C(\rho)K(\epsilon_\rho) + C(\rho)\delta(\epsilon_\rho)}{\alpha + C(\rho)K(\epsilon_\rho)} 4^{2\delta_\rho} = \nu(\rho) < 1$. This is possible, since $K(\epsilon)$ and $\delta(\epsilon)$ are fixed functions depending only on ϵ and $\delta(\epsilon) \to 0$ as $\epsilon \to 0$; indeed, we must guarantee that

$$K(\epsilon)\left(4^{2\delta} - 1\right) + \delta(\epsilon)4^{2\delta} < \frac{\alpha}{C(\rho)}.$$

We first fix ϵ_ρ by setting $\delta(\epsilon_\rho)4^{\beta_0} < \frac{\alpha}{2C(\rho)}$. We then choose δ_ρ by setting $K(\epsilon_\rho)\left(4^{2\delta_\rho} - 1\right) < \frac{\alpha}{2C(\rho)}$ and also $\delta_\rho < \frac{\beta_0}{2}$. This choice of $\delta = \delta_\rho$ completes the definition of B. We then have

$$\varphi(R) \leq \nu(\rho)\varphi(4R) + B,\ \ R < R_1,$$

then $\varphi\left(\frac{R}{4}\right) \leq \nu(\rho)\varphi(R) + B \leq \nu^2(\rho)\varphi(4R) + (\nu(\rho) + 1)B$, and more generally

$$\varphi\left(\frac{R}{4^k}\right) \leq \nu^{k+1}(\rho)\varphi(4R) + \left(\nu^k(\rho) + \cdots + 1\right)B.$$

In particular, $\varphi\left(\frac{R_1}{4^k}\right) \leq \nu(\rho)\varphi(4R_1) + \frac{B}{1-\nu(\rho)}$. But for $\frac{R_1}{4^{k+1}} \leq R \leq \frac{R_1}{4^k}$, it follows from the definition of $\varphi(R)$, that $\varphi(R) \leq 4^{2\delta_\rho}\varphi\left(\frac{R_1}{4^k}\right)$. This clearly implies that

$$\sup_{0<R\leq R_1} \varphi(R) \leq 4^{2\delta_\rho}\left[\nu(\rho)\varphi(4R_1) + \frac{B}{1-\nu(\rho)}\right] =: C(||k||_{L^d(Q_{2R_1}(z_0))}, ||Dh||_{B_{2R_1}(x_0)}, \rho, R_1).$$

In particular, $R^{-2\delta_\rho}\int_{Q_{R/2}(z_0)} |Du|^2 G_{R^2}(x,t)dxdt \leq C(||k||_{L^d(Q_{2R_1}(z_0))}, ||Dh||_{B_{2R_1}(x_0)}, \rho, R_1)$, hence

$$\int_{Q_R(z_0)} |Du|^2 dxdt \leq C(||k||_{L^d(Q_{2R_1}(z_0))}, ||Dh||_{B_{2R_1}(x_0)}, \rho, R_1)R^{n+2\delta_\rho},\ \ R < R_1. \tag{7.34}$$

We set then $\tilde{B}_R(x_0) = B_R(x_0) \cap \mathcal{O}$ and $\tilde{Q}_R(z_0) = \tilde{B}_R(x_0) \cap \{t | t_0 < t < (t_0 + R^2) \wedge T\}$. Clearly from (7.34), we can write

$$\int_{\tilde{Q}_R(z_0)} |Du|^2 dxdt \leq C(||k||_{L^d(Q_{2R_1}(z_0))}, ||Dh||_{B_{2R_1}(x_0)}, \rho, R_1)R^{n+2\delta_\rho},\ \ R < R_1. \tag{7.35}$$

We next set $u_{R,z_0} = \dfrac{\int_{\tilde{Q}_R(z_0)} u(x,t)dxdt}{|\tilde{Q}_R(z_0)|}$. We can estimate $\int_{\tilde{Q}_{2R}(z_0)} |u(x,t) - u_{2R,z_0}|^2 dxdt$. We have

$$\int\limits_{\tilde{Q}_{2R}(z_0)} |u(x,t) - u_{2R,z_0}|^2 dxdt \leq \int\limits_{\tilde{Q}_{2R}(z_0)} |u(x,t) - u^{\zeta}_{R,z_0}|^2 dxdt \leq \int\limits_{Q_{2R}(z_0)} |u(x,t) - u^{\zeta}_{R,z_0}|^2 dxdt, \tag{7.36}$$

with $u^{\zeta}_{R,z_0} = \dfrac{\int_{t_0}^{(t_0+4R^2)\wedge T} u^{\zeta}_{R,x_0 t} dt}{(t_0+4R^2)\wedge T - t_0}$. The first inequality (7.36) follows from the properties of the mean. Then,

$$\begin{aligned}
&\int\limits_{Q_{2R}(z_0)} |u(x,t) - u^{\zeta}_{R,z_0}|^2 dxdt \\
\leq & 2 \int\limits_{Q_{2R}(z_0)} |u(x,t) - u^{\zeta}_{R,x_0 t}|^2 dxdt + 2 \int\limits_{Q_{2R}(z_0)} \left| u^{\zeta}_{R,x_0 t} - \frac{\int_{t_0}^{(t_0+4R^2)\wedge T} u^{\zeta}_{R,x_0 t} ds}{(t_0+4R^2)\wedge T - t_0} \right|^2 dxdt \\
\leq & cR^2 \int\limits_{Q_{2R}(z_0)} |Du|^2 dxdt + cR^n \frac{\int_{t_0}^{(t_0+4R^2)\wedge T} \int_{t_0}^{(t_0+4R^2)\wedge T} |u^{\zeta}_{R,x_0 t} - u^{\zeta}_{R,x_0 s}|^2 dtds}{(t_0+4R^2)\wedge T - t_0}.
\end{aligned}$$

Using then Lemma 17, we obtain

$$\int\limits_{Q_{2R}(z_0)} |u(x,t) - u^{\zeta}_{R,z_0}|^2 dxdt \leq C(\rho) R^2 \int\limits_{Q_{2R}(z_0)} |Du|^2 dxdt + c||k||^2_{L^d(Q_{2R}(z_0))} R^{n+2+2\beta_0},$$

and from (7.34)

$$\begin{aligned}
\int\limits_{Q_{2R}(z_0)} |u(x,t) - u^{\zeta}_{R,z_0}|^2 dxdt \leq & C(||k||_{L^d(Q_{2R_1}(z_0))}, ||Dh||_{B_{2R_1}(x_0)}, \rho, R_1) R^{n+2+2\delta_\rho} \\
& + ||k||^2_{L^d(Q_{2R}(z_0))} R^{n+2+2\beta_0}.
\end{aligned}$$

Since $2\delta_\rho < \beta_0$, and recalling (7.36) we can state, changing $2R$ into R,

$$\int\limits_{\tilde{Q}_R(z_0)} |u(x,t) - u_{R,z_0}|^2 dxdt \leq C(||k||_{L^d(Q_{2R_1}(z_0))}, ||Dh||_{B_{2R_1}(x_0)}, \rho, R_1) R^{n+2+2\delta_\rho}, \quad R < R_1, \tag{7.37}$$

and also

$$\int\limits_{\tilde{Q}_R(z_0)} |u(x,t) - u_{R,z_0}|^2 dxdt \leq C(||k||_{L^d(\tilde{\mathcal{O}}\times(0,T))}, ||Dh||_{\tilde{\mathcal{O}}}, \rho, R_1) R^{n+2+2\delta_\rho}, \ R < R_1. \tag{7.38}$$

We now use the characterization of the space $C^{\delta_\rho,\delta_\rho/2}(\bar{\mathcal{O}} \times [0,T])$ as a Campanato space. We have, see [4], [5], setting $Q = \mathcal{O} \times (0,T)$,

$$\sup_{z_0 \in \bar{Q}, R < \text{diam } \bar{\mathcal{O}}} \frac{\int_{\tilde{Q}_R(z_0)} |u(x,t) - u_{R,z_0}|^2 dxdt}{R^{n+2+2\delta_\rho}} = \sup_{x_1,x_2 \in \bar{\mathcal{O}}, t_1,t_2 \in [0,T]} \frac{|u(x_1,t_1) - u(x_2,t_2)|^2}{|x_1 - x_2|^{2\delta_\rho} + |t_1 - t_2|^{\delta_\rho}}, \tag{7.39}$$

which implies (7.31). The proof is complete. □

7.5. The uniform Hölder case

Suppose now that in addition to the assumptions of Theorem 1 we assume, see (2.5), (2.6), that

$$\lambda_0, \lambda_\nu \text{ are bounded, and } k_\nu \text{ is bounded,} \tag{7.40}$$

and

$$h_\nu \text{ is bounded and uniformly Lipschitz,} \tag{7.41}$$

then the function $\Phi(x,t)$ is bounded, see (4.6). Therefore, the number ρ can be taken as a fixed number. We have $d = +\infty$ and $\beta_0 = 2$ and the quantities $||k||_{L^d(\tilde{\mathcal{O}}\times(0,T))}, ||Dh||_{\tilde{\mathcal{O}}}$ can be replaced by quantities independent of $\mathcal{O}$, namely $||k||_{L^\infty}$ and $||Dh||_{L^\infty}$. It follows that the constants δ_ρ and $C(||k||_{L^d(\tilde{\mathcal{O}}\times(0,T))}, ||Dh||_{\tilde{\mathcal{O}}}, \rho, R_1)$ depend only on the diameter of $\bar{\mathcal{O}}$ which we can assume larger than $\sqrt{T}$, so we obtain from (7.31).

Proposition 20. *We make the assumptions of Theorem 1 and (7.40), (7.41). Then a solution of (1.1) satisfies*

$$|u(x_1,t_1) - u(x_2,t_2)| \le C_M \left(|x_1 - x_2|^\delta + |t_1 - t_2|^{\frac{\delta}{2}}\right), \tag{7.42}$$

for a fixed number $\delta < 1$ and $|x_1 - x_2| < M$. The constant C_M depends only on M.

8. Existence results

8.1. Full regularity estimates

We want to obtain the following estimates, following methods initiated for elliptic systems, see Frehse [7].

Proposition 21. *We make the assumptions of Theorem 1. Then the functions u_ν belong to $L^q(0,T;W^{2q}(\mathcal{O}))$, $\dfrac{\partial u_\nu}{\partial t}$ belongs to $L^q(Q)$, for $q \le d$.*

Proof. Instead of balls $B_R(x_0)$, we will need cubes centered in x_0 and side length R. We denote such cubes by $Q_R(x_0)$. We shall associate to these cubes the same types of function $\tau_R(x) = \tau_{R,x_0(x)}$ used in Section 7.2. In the context of cubes we consider a smooth function $\tau(x)$ such that

$$\tau(x) = \begin{cases} 1, & \text{on } Q_1(0), \\ 0, & \text{outside } Q_2(0), \end{cases}$$

and $\tau_R(x) = \tau_{R,x_0}(x) = \tau(\frac{x-x_0}{R})$. Going back to the equations (1.1) we have

$$-\frac{\partial(u_\nu \tau_R)}{\partial t} + A(u_\nu \tau_R) = \tau_R H_\nu(x,u,Du) - 2\sum_{ij} a_{ij}\frac{\partial u_\nu}{\partial x_j}\frac{\partial \tau_R}{\partial x_i} - u_\nu \sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial \tau_R}{\partial x_j}\right).$$

We can then apply the linear theory of parabolic equations thanks to the regularity of a_{ij} and the functions h_ν. This implies for $q \le d$ that

$$\left(\int_0^T \int_{Q_R(x_0)} \left|\frac{\partial u_\nu}{\partial t}\right|^q dxdt\right)^{\frac{1}{q}} + \left(\int_0^T \int_{Q_R(x_0)} |D^2 u_\nu|^q dxdt\right)^{\frac{1}{q}} \le K_q \left(\int_0^T \int_{Q_{2R}(x_0)} |Du|^{2q} dxdt\right)^{\frac{1}{q}} + K_{Rq}(\rho), \tag{8.1}$$

in which the constant $K_{Rq}(\rho)$ depends on the bound on u on $\tilde{\mathcal{O}} \times (0,T)$, as discussed in section 7. The set $\tilde{\mathcal{O}}$ is adapted to cubes instead of balls. This is not the case for the constant K_q. We note the algebraic inequality

$$\left(\int_0^T \int_{Q_R(x_0)} \left|\frac{\partial u}{\partial t}\right|^q dxdt\right)^{\frac{1}{q}} \le N^{\frac{1}{q}} \sum_\nu \left(\int_0^T \int_{Q_R(x_0)} \left|\frac{\partial u_\nu}{\partial t}\right|^q dxdt\right)^{\frac{1}{q}},$$

and $|D^2 u|^2 = \sum_\nu |D^2 u_\nu|^2$, which also implies that

$$\left(\int_0^T \int_{Q_R(x_0)} \left|D^2 u\right|^q dxdt\right)^{\frac{1}{q}} \le N^{\frac{1}{q}} \sum_\nu \left(\int_0^T \int_{Q_R(x_0)} |D^2 u_\nu|^q dxdt\right)^{\frac{1}{q}}.$$

Therefore, we obtain from (8.1) that

$$\left(\int_0^T \int_{Q_R(x_0)} \left|\frac{\partial u}{\partial t}\right|^q dxdt\right)^{\frac{1}{q}} + \left(\int_0^T \int_{Q_R(x_0)} \left|D^2 u\right|^q dxdt\right)^{\frac{1}{q}} \le K_q \left(\int_0^T \int_{Q_{2R}(x_0)} |Du|^{2q} dxdt\right)^{\frac{1}{q}} + K_{Rq}(\rho). \tag{8.2}$$

We will use the inequality

$$\int_0^T \int_{Q_{2R}(x_0)} |Du|^{2q} dxdt \le \int_0^T \int_{Q_{4R}(x_0)} |\tau_{2R} Du|^{2q} dxdt, \tag{8.3}$$

and we define

$$c_{\nu R}(t) = \frac{1}{2}\left(\min_{x \in B_{4R}(x_0)} u_\nu(x,t) + \max_{x \in B_{4R}(x_0)} u_\nu(x,t)\right).$$

We have

$$\begin{aligned}
&\int_0^T \int_{Q_{4R}(x_0)} |\tau_{2R} Du|^{2q} dxdt \\
&= \int_0^T \int_{Q_{4R}(x_0)} \tau_{2R}^{2q} |Du|^{2q-2} \sum_\nu |Du_\nu|^2 dxdt \\
&= -\int_0^T \int_{Q_{4R}(x_0)} \tau_{2R}^{2q} |Du|^{2q-2} \sum_\nu \triangle u_\nu (u_\nu - c_{\nu R}(t)) dxdt \\
&\quad - 2q \int_0^T \int_{Q_{4R}(x_0)} \tau_{2R}^{2q-1} |Du|^{2q-2} \sum_\nu (u_\nu - c_{\nu R}(t)) D\tau_{2R} \cdot Du_\nu dxdt
\end{aligned}$$

$$
\begin{aligned}
&-(2q-2)\int_0^T\int_{Q_{4R}(x_0)}\tau_{2R}^{2q}|Du|^{2q-4}\sum_\nu(u_\nu-c_{\nu R}(t))Du_\nu\cdot D^2u_\nu Du_\nu dxdt\\
&\le C(q)\int_0^T\int_{Q_{4R}(x_0)}\tau_{2R}^{2q}\left|D^2u\right|^q|u-c_R(t)|dxdt+C(q)\int_0^T\int_{Q_{4R}(x_0)}\tau_{2R}^{2q}|Du|^{2q}|u-c_R(t)|dxdt\\
&+C(q)\int_0^T\int_{Q_{4R}(x_0)}|D\tau_{2R}|^{2q}|u-c_R(t)|dxdt,
\end{aligned}
$$

we now use the Hölder property to claim that it is less than

$$
K_q(\rho)R^\delta\int_0^T\int_{Q_{4R}(x_0)}\tau_{2R}^{2q}|D^2u|^qdxdt+K_q(\rho)R^\delta\int_0^T\int_{Q_{4R}(x_0)}\tau_{2R}^{2q}|Du|^{2q}dxdt+K_{qR}(\rho).
$$

Therefore, we also have

$$
\left(1-K_q(\rho)R^\delta\right)\int_0^T\int_{Q_{4R}(x_0)}\tau_{2R}^{2q}|Du|^{2q}dxdt\le K_q(\rho)R^\delta\int_0^T\int_{Q_{4R}(x_0)}\tau_{2R}^{2q}\left|D^2u\right|^qdxdt+K_{qR}(\rho),
$$

hence

$$
\int_0^T\int_{Q_{4R}(x_0)}\tau_{2R}^{2q}|Du|^{2q}dxdt\le\frac{K_q(\rho)R^\delta}{1-K_q(\rho)R^\delta}\int_0^T\int_{Q_{4R}(x_0)}\tau_{2R}^{2q}|D^2u|^qdxdt+\frac{K_{qR}(\rho)}{1-K_q(\rho)R^\delta}. \tag{8.4}
$$

We assume naturally that R is sufficiently small so that $1-K_q(\rho)R^\delta>0$. From (8.2), (8.3) and (8.4), we obtain

$$
\int_0^T\int_{Q_R(x_0)}|D^2u|^qdxdt\le\frac{K_q'(\rho)R^\delta}{1-K_q(\rho)R^\delta}\int_0^T\int_{Q_{4R}(x_0)}|D^2u|^qdxdt+K_{qR}'(\rho).
$$

Define $\xi:=\sup_{x_0\in\mathcal{O},R<R_2}\int_0^T\int_{Q_R(x_0)}|D^2u|^qdxdt$. Since $\sup_{x_0\in\mathcal{O},R<R_2}\int_0^T\int_{Q_{4R}(x_0)}|D^2u|^qdxdt\le C\xi$, where C is a fixed constant. Therefore, we have obtained $\xi\le\dfrac{K_q''(\rho)}{1-K_q(\rho)R_2^\delta}R_2^\delta\xi+K_{qR}'(\rho)$. We may assume R_2 sufficiently small in order that $\dfrac{K_q''(\rho)}{1-K_q(\rho)R_2^\delta}R_2^\delta<1$. Then ξ is bounded. For any bounded domain $\mathcal{O}$, we can consider a finite covering by cubes Q_R, therefore we have $\int_0^T\int_{\mathcal{O}}|D^2u|^qdxdt<+\infty$. From the Equations (1.1), we then have $\int_0^T\int_{\mathcal{O}}\left|\dfrac{\partial u_\nu}{\partial t}\right|^qdxdt<+\infty$. This concludes the proof. $\square$

8.2. Proof of Theorem 1

We approximate the $H_\nu(x,t,y,p)$ by the sequence

$$
H_\nu^\epsilon(x,t,y,p)=\frac{H_\nu(x,t,y,p)}{1+\epsilon|H(x,t,y,p)|},
$$

where $H(x,t,y,p)$ is the vector of coordinates $H_\nu(x,t,y,p)$. We verify easily that the Hamiltonians $H^\epsilon_\nu(x,t,y,p)$ satisfy the same assumptions as $H_\nu(x,t,y,p)$ uniformly in ϵ. Moreover $|H^\epsilon_\nu(x,t,y,p)| \le \dfrac{1}{\epsilon}$. We also consider the final condition $h^\epsilon_\nu(x) = \dfrac{h_\nu(x)}{1+\epsilon|h(x)|}$. We can then solve the system of PDEs

$$\begin{cases} -\dfrac{\partial u^\epsilon_\nu}{\partial t} + Au^\epsilon_\nu = H^\epsilon_\nu(x,u^\epsilon,Du^\epsilon), \ x\in\mathbb{R}^n, \ t\in[0,T], \\ \qquad u^\epsilon_\nu(x,T) = h^\epsilon_\nu(x). \end{cases} \tag{8.5}$$

We can then apply the *a priori* estimates to the solution u^ϵ which thus remains bounded in $L^q(0,T;W^{2q}(\mathcal{O}))$ and $\dfrac{\partial u^\epsilon}{\partial t}$ remains bounded in $L^q(0,T;L^q(\mathcal{O})), \forall \mathcal{O}$ bounded subset of $\mathbb{R}^n$, with $q\le d$. We can take a sequence of balls $B_M = B_M(0)$. We can construct a subsequence of u^ϵ which converges pointwise to a limit u, which belongs to $L^q(0,T;W^{2q}(B_M))$ and $\dfrac{\partial u}{\partial t}$ belongs to $L^q(0,T;L^q(B_M))$, for any M. From the continuity of the Hamiltonians it is fairly easy to check that u is a solution of (1.1). This concludes the proof.

8.3. Proof of Theorem 2

Considering the Hamiltonians $H^\epsilon_\nu(x,y,p)$ and the functions $h^\epsilon_\nu(x)$ as before, we can consider the system of BSDEs

$$Y^{\epsilon xt}_\nu(s) = h^\epsilon_\nu(X^{xt}(T)) + \int_s^T H^\epsilon_\nu(X^{xt}(\tau),Y^{\epsilon xt}(\tau),Z^{\epsilon xt}(\tau))d\tau - \int_s^T Z^{\epsilon xt}_\nu(\tau)\cdot\sigma(X^{xt}(\tau))dw(\tau), \ t\le s\le T, \tag{8.6}$$

which has a Markovian solution:

$$Y^{\epsilon xt}_\nu(s) = u^\epsilon_\nu(X^{xt}(s),s), \ \ Z^{\epsilon xt}_\nu(s) = Du^\epsilon_\nu(X^{xt}(s),s).$$

In fact, we can take the functions $u^\epsilon_\nu(x,t)$ to be solutions of the system of PDE (8.5). We can also proceed directly since the Hamiltonians H^ϵ_ν and the functions h^ϵ_ν are bounded by $\dfrac{1}{\epsilon}$. Let $B_M = B_M(0)$. We denote by $\tau_M \ = \tau^{xt}_M \ = \inf\{s>t|X^{xt}(s)\notin B_M\}$. From now on, to simplify notation, we drop the indices x,t which remain fixed. We write $X(s)$, $Y^\epsilon_\nu(s)$, $Z^\epsilon_\nu(s)$. From (5.15) and (6.5), we state that

$$\mathbb{E}\int_t^T |Z^\epsilon(s)|^2 \mathbb{1}_{s\le\tau_M} ds \le C_M, \ \mathbb{E}\int_t^T k_\nu(X(s),s)\mathbb{1}_{s\le\tau_M} ds \le C_M, \tag{8.7}$$

which implies, from the majorations on H^ϵ,

$$\mathbb{E}\int_t^T |H^\epsilon(X(s),Y^\epsilon(s),Z^\epsilon(s))\mathbb{1}_{s\le\tau_M} ds \le C_M. \tag{8.8}$$

From Ito's formula

$$
\begin{aligned}
&\mathbb{E}\int_t^T \sum_\nu (Z_\nu^\epsilon(s) - Z_\nu^{\epsilon'}(s)) \cdot a(X(s))(Z_\nu^\epsilon(s) - Z_\nu^{\epsilon'}(s))\mathbb{1}_{s\leq\tau_M} ds + \mathbb{E}\left(\frac{1}{2}|Y^\epsilon(t) - Y^{\epsilon'}(t)|^2\right) \\
&= \mathbb{E}\left(\frac{1}{2}|Y^\epsilon(T\wedge\tau_M) - Y^{\epsilon'}(T\wedge\tau_M)|^2\right) \\
&\quad + \mathbb{E}\int_t^T \sum_\nu (Y_\nu^\epsilon(s) - Y_\nu^{\epsilon'}(s))(H_\nu^\epsilon(X(s), Y^\epsilon(s), Z^\epsilon(s)) - H_\nu^{\epsilon'}(X(s), Y^{\epsilon'}(s), Z^{\epsilon'}(s)))\mathbb{1}_{s<\tau_M} ds.
\end{aligned}
\tag{8.9}
$$

We next write

$$
\begin{aligned}
\sup_{t\leq s\leq T} |Y^\epsilon(s) - Y^{\epsilon'}(s)|\mathbb{1}_{s\leq\tau_M} &= \sup_{t\leq s\leq T} |u^\epsilon(X(s), s) - u^{\epsilon'}(X(s), s)|\mathbb{1}_{s\leq\tau_M} \\
&\leq \sup_{|x|\leq M, t\leq s\leq T} |u^\epsilon(x, s) - u^{\epsilon'}(x, s)| \to 0, \text{ as } \epsilon \text{ and } \epsilon' \to 0.
\end{aligned}
$$

It then follows, from (8.9), and the majoration (8.8), that

$$
\mathbb{E}\int_t^T |Z^\epsilon(s) - Z^{\epsilon'}(s)|^2 \mathbb{1}_{s\leq\tau_M} \to 0, \text{ as } \epsilon, \epsilon' \to 0. \tag{8.10}
$$

This implies that $Z^\epsilon(\cdot)$ converges in $L^2\left(\Omega, \mathcal{A}, \mathbb{P}; L^2\left(t, \tau_M \wedge T; \mathbb{R}^{Nn}\right)\right)$ for any M. The limit is necessary of the form $Z(s)\mathbb{1}_{s\leq\tau_M}$ and

$$
\mathbb{E}\int_t^T |Z^\epsilon(s) - Z(s)|^2 \mathbb{1}_{s\leq\tau_M} \to 0, \text{ as } \epsilon \to 0, \ \forall M. \tag{8.11}
$$

Also $u^\epsilon(x, s) \to u(x, s)$ uniformly on $B_M \times (t, T)$, hence $Y^\epsilon(s)\mathbb{1}_{s\leq\tau_M} \to Y(s)\mathbb{1}_{s\leq\tau_M}$, $\forall\, s$, a.s., $\forall\ M$. Thanks to (8.11) we have also, for a subsequence, $Z^\epsilon(s)\mathbb{1}_{s\leq\tau_M} \to Z(s)\mathbb{1}_{s\leq\tau_M}$, a.e. a.s. From the continuity of the Hamiltonians, we obtain

$$
H_\nu^\epsilon(X(s), Y^\epsilon(s), Z^\epsilon(s))\mathbb{1}_{s\leq\tau_M} \to H_\nu(X(s), Y(s), Z(s))\mathbb{1}_{s\leq\tau_M} \text{ a.e., a.s., } \forall M. \tag{8.12}
$$

We note that, for a subsequence

$$
\mathbb{E}\int_t^T \sup_\epsilon |Z^\epsilon(s)|^2 \mathbb{1}_{s\leq\tau_M} \leq C_M,
$$

which implies that

$$
\mathbb{E}\int_t^T \sup_\epsilon |H^\epsilon(X(s), Y^\epsilon(s), Z^\epsilon(s))| \mathbb{1}_{s\leq\tau_M} ds \leq C_M.
$$

Therefore,

$$
\mathbb{E}\int_t^T |H^\epsilon(X(s), Y^\epsilon(s), Z^\epsilon(s)) - H(X(s), Y(s), Z(s))| \mathbb{1}_{s\leq\tau_M} ds \to 0. \tag{8.13}
$$

Also,

$$\mathbb{E}\left|\int_t^T (Z_\nu^\epsilon(s) - Z_\nu(s))\mathbb{1}_{s\le\tau_M}.\sigma(X(s))dw(s)\right|^2 \to 0. \tag{8.14}$$

Now, from the BSDE (8.6), we also have

$$\mathbb{1}_{s<\tau_M}Y_\nu^\epsilon(s) = \mathbb{1}_{s<\tau_M}Y_\nu^\epsilon(T\wedge\tau_M) + \int_s^T \mathbb{1}_{\tau<\tau_M}H_\nu^\epsilon(X(\tau), Y^\epsilon(\tau), Z^\epsilon(\tau))d\tau - \int_s^T \mathbb{1}_{\tau<\tau_M}Z_\nu^\epsilon(\tau)\cdot\sigma(X(s))dw(s).$$

Thanks to (8.13) and (8.14), we can pass to the limit and write

$$\mathbb{1}_{s<\tau_M}Y_\nu(s) = \mathbb{1}_{s<\tau_M}Y_\nu(T\wedge\tau_M) \int_s^T \mathbb{1}_{\tau<\tau_M}H_\nu(X(\tau), Y(\tau), Z(\tau))d\tau - \int_s^T \mathbb{1}_{\tau<\tau_M}Z_\nu(\tau).\sigma(X(s))dw(s). \tag{8.15}$$

We then notice that almost surely there exists an M (which is random) such that $\tau_M \ge T$, which proves that the pair $(Y(s), Z(s))$ is a solution of the system (1.8). The fact that $Z_\nu(s) = Z_\nu^{xt}(s) = Du_\nu(X^{xt}(s), s)$ is proved as in Proposition 15. This completes the proof.

Remark 22. The proofs of Theorem 1 and Theorem 2 look quite different. In Theorem 1, we rely on regularity results of solutions on linear parabolic equations, which has no probabilistic analogue. In fact, it is possible to prove Theorem 1 in a way similar to Theorem 2. We do not include it, to reduce the size of the paper.

The developments of Section 7 are simpler than those done in the paper [2]. This is due to the fact that the proof in [2] does not use the smoothness of the functions a_{ij}.

9. Uniqueness

We want to state a uniqueness result for smooth solutions. As for existence, we will first develop an analytic theory, then a probabilistic theory, following [8]. The analytic theory, which is new, mimics closely the probabilistic theory.

9.1. Analytic part

We state the following

Theorem 23. *We make the assumptions of Theorem 1 and (7.40), (7.41). We also assume that $H_\nu(x, y, p) = H_\nu(x, p)$ independent of y with*

$$|H_\nu(x, p) - H_\nu(x, \tilde{p})| \le k(|p| + |\tilde{p}|)|p - \tilde{p}|. \tag{9.1}$$

Then there exists a unique smooth solution of (1.1) such that $u_\nu(x, t)$ is bounded and satisfies the Hölder property (7.42).

We begin by an existence and uniqueness result based on (9.1), but not on all the assumptions of Theorem 1. However, it will need that the horizon T be small. We will need a Banach space, called $H^2_{BMO} = H^2_{BMO}(\mathbb{R}^n \times (0, T))$ defined as follows. A function f defined on $\mathbb{R}^n \times [0, T]$ is the space H^2_{BMO} if

$$||f||^2_{H^2_{BMO}} = \sup_{x,t} \int_t^T \int_{\mathbb{R}^n} |f(\xi,s)|^2 G_{xt}(\xi,s)d\xi ds < +\infty, \tag{9.2}$$

where $G_{xt}(\xi,s)$ is the Green function defined by (6.3). We can consider the natural extension to $n-$ dimensional vector functions and also to multiple vector functions. In particular, for a family $u_\nu(x,t)$, we will write

$$||Du||^2_{H^2_{BMO}} = \sup_{x,t} \sum_\nu \int_t^T \int_{\mathbb{R}^n} |Du_\nu(\xi,s)|^2 G_{xt}(\xi,s)d\xi ds. \tag{9.3}$$

Theorem 24. *We assume (2.1). The Hamiltonians are independent of y and satisfy (9.1). We assume that $H_\nu(x,0)$ and $h_\nu(x)$ are $L^{+\infty}(\mathbb{R}^n)$. Also $h_\nu(x)$ is locally Hölder continuous in the sense of (7.42). When T is sufficiently small, there is one and only one solution of the system (1.1), such that u is bounded and Du is in H^2_{BMO} with appropriate bounds.*

We begin with a lemma for the linear equations

$$\begin{cases} -\dfrac{\partial \eta_\nu}{\partial t} + A\eta_\nu = H_\nu(x,0), \ x \in \mathbb{R}^n, \ t \in [0,T], \\ \qquad \eta_\nu(x,T) = h_\nu(x), \end{cases} \tag{9.4}$$

which are in fact independent equations. We call η the vector with components the functions η_ν.

Lemma 25. *The solution η of (9.4) is bounded and $D\eta$ is H^2_{BMO} with*

$$||D\eta||^2_{H^2_{BMO}} \leq CT^{\delta/2}, \tag{9.5}$$

where C is a constant independent of T except for an upper-bound, and depends only on the constants of the equations; δ is the Hölder constant of the functions h_ν.

Proof. First we have the formula

$$\eta_\nu(x,t) = \int_{\mathbb{R}^n} h_\nu(\xi)G_{xt}(\xi,T)d\xi + \int_t^T \int_{\mathbb{R}^n} H_\nu(\xi,0)G_{xt}(\xi,s)d\xi ds, \tag{9.6}$$

hence $|\eta(x,t)| \leq C$, where C depends only on the L^∞ norms of $h(x)$ and $H(x,0)$. Next an easy calculation leads to

$$\begin{cases} -\dfrac{\partial(\eta_\nu)^2}{\partial t} + A(\eta_\nu)^2 = -2D\eta_\nu.a(x)D\eta_\nu + 2\eta_\nu\left(-\dfrac{\partial\eta_\nu}{\partial t} + A\eta_\nu\right) = -2D\eta_\nu.a(x)D\eta_\nu + 2\eta_\nu H_\nu(x,0), \\ \qquad (\eta_\nu)^2(x,T) = h_\nu^2(x), \end{cases}$$

from which we can derive

$$\begin{aligned} &2\int_t^T \int_{\mathbb{R}^n} D\eta_\nu(\xi,s).a(\xi)D\eta_\nu(\xi,s)G_{xt}(\xi,s)d\xi ds \\ &= \int_{\mathbb{R}^n} h_\nu^2(\xi)G_{xt}(\xi,T)d\xi - (\eta_\nu)^2(x,t) + 2\int_t^T \int_{\mathbb{R}^n} \eta_\nu(\xi,s)H_\nu(\xi,0)G_{xt}(\xi,s)d\xi ds. \end{aligned} \tag{9.7}$$

Hence,

$$2\alpha \int_t^T \int_{\mathbb{R}^n} |D\eta|^2(\xi,s)d\xi ds \leq \int_{\mathbb{R}^n} |h(\xi)|^2 G_{xt}(\xi,T)d\xi - |\eta(x,t)|^2 + CT. \tag{9.8}$$

We need to estimate $I = \int_{\mathbb{R}^n} |h(\xi)|^2 G_{xt}(\xi,T)d\xi - |\eta(x,t)|^2$. But using (9.6), we see easily that

$$\begin{aligned} I &\leq \int_{\mathbb{R}^n} |h(\xi)|^2 G_{xt}(\xi,T)d\xi - \left| \int_{\mathbb{R}^n} h(\xi) G_{xt}(\xi,T)d\xi \right|^2 - 2\sum_\nu \int_{\mathbb{R}^n} h_\nu(\xi) G_{xt}(\xi,T)d\xi \int_t^T \int_{\mathbb{R}^n} H_\nu(\xi,0) G_{xt}(\xi,s)d\xi ds \\ &\leq \int_{\mathbb{R}^n} |h(\xi)|^2 G_{xt}(\xi,T)d\xi - \left| \int_{\mathbb{R}^n} h(\xi) G_{xt}(\xi,T)d\xi \right|^2 + CT. \end{aligned}$$

Now,

$$\begin{aligned} J &= \int_{\mathbb{R}^n} |h(\xi)|^2 G_{xt}(\xi,T)d\xi - \left| \int_{\mathbb{R}^n} h(\xi) G_{xt}(\xi,T)d\xi \right|^2 \\ &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (h(\theta) - h(\xi)) G_{xt}(\xi,T)d\xi \right|^2 G_{xt}(\theta,T)d\theta \\ &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |h(\theta) - h(\xi)|^2 G_{xt}(\xi,T) G_{xt}(\theta,T)d\xi d\theta. \end{aligned} \tag{9.9}$$

Using the estimate (6.4), we obtain also

$$\begin{aligned} J &\leq \frac{k_2^2}{(T-t)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |h(\theta) - h(\xi)|^2 \exp\left(-\frac{\delta_2}{T-t} \left(|\theta - x|^2 + |\xi - x|^2 \right) \right) d\xi d\theta \\ &= k_2^2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| h\left(x + \sqrt{T-t}u\right) - h\left(x + \sqrt{T-t}v\right) \right|^2 \exp\left(-\delta_2 \left(|u|^2 + |v|^2 \right) \right) dudv. \end{aligned}$$

From the local Hölder regularity of h and easy manipulations, we obtain

$$\begin{aligned} J \leq C(T-t)^{\frac{\delta}{2}} &\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u - v|^\delta \exp\left(-\delta_2(|u|^2 + |v|^2) \right) dudv \\ &+ C(T-t)^{\frac{1}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u - v| \exp\left(-\delta_2(|u|^2 + |v|^2) \right) dudv. \end{aligned}$$

At this stage all the constants are independent of T. We obtain (9.5) by collecting results and using an upper bound of T. This completes the proof.

We turn to the proof of Theorem 24: We first make a trivial change of unknown function. We set

$$\begin{aligned} \bar{u}_\nu(x,t) &= u_\nu(x,t) - \eta_\nu(x,t), \\ \bar{H}_\nu(x,p) &= H_\nu(x,p) - H_\nu(x,0), \end{aligned} \tag{9.10}$$

so, recalling the equation for $\eta_\nu(x,t)$, we can assert that $\bar{u}_\nu$ satisfies

$$\begin{cases} -\dfrac{\partial \bar{u}_\nu}{\partial t} + A\bar{u}_\nu = \bar{H}_\nu(x, Du), \\ \bar{u}_\nu(x,T) = 0. \end{cases} \tag{9.11}$$

Note also that

$$|\bar{H}_\nu(x,p)| \leq k|p|^2. \tag{9.12}$$

We want to solve (1.1), or equivalently (9.11), by a fixed point approach. We will work in the Banach space H^2_{BMO} and consider functions $v_\nu(x,t)$ such that

$$||Dv||_{H^2_{BMO}} \leq \frac{\beta}{k}, \ \sup_{x,t} |v_\nu(x,t) - \eta_\nu(x,t)| \leq \frac{\beta^2}{k}, \tag{9.13}$$

where β is a fixed number to be defined below. We also consider the Banach space $\mathcal{B}$ of functions v such that

$$\sup_{xt} |v(x,t)| + ||Dv||_{H^2_{BMO}} < +\infty,$$

and the set (9.13) is a closed subset of $\mathcal{B}$.

We next define the map $v \to \mathcal{T}(v) = u$ as follows

$$\begin{cases} -\dfrac{\partial \bar{u}_\nu}{\partial t} + A\bar{u}_\nu = \bar{H}_\nu(x, Dv), \\ \bar{u}_\nu(x,T) = 0, \end{cases}$$

and $u = \bar{u} + \eta$. Since

$$\bar{u}_\nu(x,t) = \int_t^T \int_{\mathbb{R}^n} \bar{H}_\nu(\xi, Dv(\xi,s)) G_{xt}(\xi,s) d\xi ds,$$

we have

$$\sup_{x,t} |\bar{u}_\nu(x,t)| \leq k||Dv||^2_{H^2_{BMO}} \leq \frac{\beta^2}{k},$$

and the second condition (9.13) is satisfied for u. Consider next $\bar{u}^2_\nu(x,t)$. We have

$$\begin{cases} -\dfrac{\partial}{\partial t}\bar{u}^2_\nu + A\bar{u}^2_\nu = -2D\bar{u}_\nu.aD\bar{u}_\nu + 2\bar{u}_\nu\bar{H}_\nu(x, Dv), \\ \bar{u}^2_\nu(x,T) = 0. \end{cases}$$

Hence,

$$\bar{u}^2_\nu(x,t) + 2\int_t^T \int_{\mathbb{R}^n} D\bar{u}_\nu.aD\bar{u}_\nu(\xi,s) G_{xt}(\xi,s) d\xi ds = 2\int_t^T \int_{\mathbb{R}^n} \bar{u}_\nu(\xi,s)\bar{H}_\nu(\xi, Dv(\xi,s)) G_{xt}(\xi,s) d\xi ds,$$

which implies that

$$\alpha \int_t^T \int_{\mathbb{R}^n} |D\bar{u}_\nu|^2(\xi,s)G_{xt}(\xi,s)d\xi ds \le k^2||Dv||^2_{H^2_{BMO}} \int_t^T \int_{\mathbb{R}^n} |Dv|^2(\xi,s)G_{xt}(\xi,s)d\xi ds. \tag{9.14}$$

Therefore, by easy majorations,

$$\frac{\alpha}{2} \int_t^T \int_{\mathbb{R}^n} |Du_\nu|^2(\xi,s)G_{xt}(\xi,s)d\xi ds \le k^2||Dv||^2_{H^2_{BMO}} \int_t^T \int_{\mathbb{R}^n} |Dv|^2(\xi,s)G_{xt}(\xi,s)d\xi ds$$
$$+ \alpha \int_t^T \int_{\mathbb{R}^n} |D\eta_\nu|^2(\xi,s)G_{xt}(\xi,s)d\xi ds,$$

hence finally the inequality

$$\frac{\alpha}{2}||Du||^2_{H^2_{BMO}} \le k^2 N||Dv||^4_{H^2_{BMO}} + \alpha||D\eta||^2_{H^2_{BMO}}. \tag{9.15}$$

From (9.13), it follows that

$$||Du||^2_{H^2_{BMO}} \le \frac{2N}{\alpha}\frac{\beta^4}{k^2} + 2||D\eta||^2_{H^2_{BMO}}.$$

We will have $||Du||_{H^2_{BMO}} \le \dfrac{\beta}{k}$ if

$$\frac{2N}{\alpha}\frac{\beta^4}{k^2} + 2||D\eta||^2_{H^2_{BMO}} \le \frac{\beta^2}{k^2}. \tag{9.16}$$

If we can find β such that (9.16) is satisfied, we obtain that the map $\mathcal{T}$ is a map from $\mathcal{B}$ to $\mathcal{B}$ which preserves the set (9.13). We now want to show that for a convenient choice of β the map $\mathcal{T}$ is a contraction. Consider two elements of the set (9.13) denoted by v^1, v^2 and $u^1 = \mathcal{T}v^1$, $u^2 = \mathcal{T}v^2$. Set $\tilde{u} = u^1 - u^2$, $\tilde{v} = v^1 - v^2$ then

$$\begin{cases} -\dfrac{\partial \tilde{u}_\nu}{\partial t} + A\tilde{u}_\nu = H_\nu(x, Dv^1) - H_\nu(x, Dv^2), \\ \qquad \tilde{u}_\nu(x,T) = 0, \end{cases}$$

and

$$\begin{cases} -\dfrac{\partial \tilde{u}^2_\nu}{\partial t} + A\tilde{u}^2_\nu = -2D\tilde{u}_\nu.aD\tilde{u}_\nu + \tilde{u}_\nu(H_\nu(x, Dv^1) - H_\nu(x, Dv^2)), \\ \qquad \tilde{u}^2_\nu(x,T) = 0. \end{cases}$$

Consequently, we have

$$\tilde{u}_\nu(x,t) = \int_t^T \int_{\mathbb{R}^n} (H_\nu(\xi, Dv^1(\xi,s)) - H_\nu(\xi, Dv^2(\xi,s)))G_{xt}(\xi,s)d\xi ds.$$

So,

$$\tilde{u}_\nu^2(x,t) + 2\int_t^T \int_{\mathbb{R}^n} D\tilde{u}_\nu(\xi,s).a(\xi)D\tilde{u}_\nu(\xi,s)G_{xt}(\xi,s)d\xi ds$$
$$= \int_t^T \int_{\mathbb{R}^n} \tilde{u}_\nu(\xi,s)(H_\nu(\xi,Dv^1(\xi,s)) - H_\nu(\xi,Dv^2(\xi,s)))G_{xt}(\xi,s)d\xi ds.$$

Hence,

$$\sup_{x,t} |\tilde{u}_\nu(x,t)| \leq \sup_{xt} \int_t^T \int_{\mathbb{R}^n} |H_\nu(\xi,Dv^1(\xi,s)) - H_\nu(\xi,Dv^2(\xi,s))|G_{xt}(\xi,s)d\xi ds,$$

and

$$\begin{aligned}
&2\alpha \int_t^T \int_{\mathbb{R}^n} |D\tilde{u}_\nu(\xi,s)|^2 d\xi ds \\
&\leq \sup_{xt} \left(\int_t^T \int_{\mathbb{R}^n} |H_\nu(\xi,Dv^1(\xi,s)) - H_\nu(\xi,Dv^2(\xi,s))|G_{xt}(\xi,s)d\xi ds \right)^2 \\
&\leq k^2 \sup_{xt} \left(\int_t^T \int_{\mathbb{R}^n} (|Dv^1(\xi,s)| + |Dv^2(\xi,s)|)|D\tilde{v}(\xi,s)|G(\xi,s)d\xi ds \right)^2 \\
&\leq 2k^2 \sup_{xt} \int_t^T \int_{\mathbb{R}^n} \left(|Dv^1(\xi,s)|^2 + |Dv^2(\xi,s)|^2\right) G(\xi,s)d\xi ds \sup_{xt} \int_t^T \int_{\mathbb{R}^n} |D\tilde{v}(\xi,s)|^2 G(\xi,s)d\xi ds \\
&\leq 2k^2 \left(||Dv^1||^2_{H^2_{BMO}} + ||Dv^2||^2_{H^2_{BMO}} \right) ||D\tilde{v}||^2_{H^2_{BMO}}.
\end{aligned}$$

Since v^1 and v^2 satisfy the first inequality (9.13), we obtain

$$2\alpha \int_t^T \int_{\mathbb{R}^n} |D\tilde{u}_\nu(\xi,s)|^2 d\xi ds \leq 4\beta^2 ||D\tilde{v}||^2_{H^2_{BMO}}.$$

Hence,

$$||D\tilde{u}||^2_{H^2_{BMO}} \leq \frac{2N\beta^2}{\alpha} ||D\tilde{v}||^2_{H^2_{BMO}}. \tag{9.17}$$

In addition,

$$\sup_{x,t} |\tilde{u}(x,t)|^2 \leq 4N\beta^2 ||D\tilde{v}||^2_{H^2_{BMO}}.$$

Therefore,

$$||\mathcal{T}v^1 - \mathcal{T}v^2||_{\mathcal{B}} \leq 2\sqrt{N}\beta \left(1 + \frac{1}{\alpha}\right) ||v^1 - v^2||_{\mathcal{B}},$$

and $\mathcal{T}$ is a contraction if

$$\beta < \frac{1}{2\sqrt{N}\left(1+\dfrac{1}{\alpha}\right)}. \tag{9.18}$$

So we must find β satisfying both conditions (9.16) and (9.18). We must first have $\dfrac{\alpha^2}{4N^2} - \dfrac{4k^2\alpha}{N}||D\eta||^2 > 0$. Then (9.16) implies that

$$\beta^2 < \frac{1}{2}\left(\frac{\alpha}{2N} - \sqrt{\frac{\alpha^2}{4N^2} - \frac{4k^2\alpha}{N}||D\eta||^2}\right),$$

and the right hand side will be smaller than $\dfrac{1}{4N\left(1+\frac{1}{\alpha}\right)^2}$ if, for instance,

$$||D\eta||^2_{H^2_{BMO}} \leq \frac{3}{32Nk^2\left(1+\frac{1}{\alpha}\right)^2}. \tag{9.19}$$

Taking account of the estimate (9.5), we conclude that it is sufficient that T be small enough, so that $CT^{\delta/2}$ is smaller than the right hand side of (9.19). This concludes the proof of Theorem 24. □

Proof of Theorem 23. Since the conditions of Theorem 24 are valid, there exists one and only one solution of the system (1.1) on an interval $(T-\epsilon, T]$ for ϵ sufficiently small, depending only on fixed constants but also on the Hölder constant of the final condition h. But since the conditions of Proposition 20 hold the solution satisfies a uniform local Hölder regularity property, like (7.42), as long as it exists, which is also satisfied by h. We take the corresponding Hölder constant to define ϵ. Therefore we can start from final time $T-\epsilon$ with a final condition which corresponds to the value of the solution at time $T-\epsilon$. Then Theorem 24 will apply again on the interval $(T-2\epsilon, T-\epsilon]$ and ϵ is fixed. Therefore we have a unique solution on $[0,T]$. This concludes the proof of Theorem 23.

9.2. Probabilistic part

In fact, our analytic part is a translation of the probabilistic part. So we will not give all details. To this end, the probabilistic part as given by [8] relies on results and methods of C. Frei [6]. We have adapted them in our analytic presentation. We then state

Theorem 26. *We make the assumptions of Theorem 23. Then there exists one and only one solution of the system of BSDE (1.8) with*

$$Y^{xt}_\nu(s) = u_\nu(X^{xt}(s), s), \;\; Z^{xt}_\nu(s) = Du_\nu(X^{xt}(s), s). \tag{9.20}$$

The functions u_ν are bounded, and satisfy a local uniform Hölder regularity property like (7.42).

Proof. As said, we do not give details of the proof. Recalling (4.7), we see that $Y^{xt}(s)$ is bounded. For Markov solutions, the Hölder regularity theory applies, including Proposition 20 and the property (7.42) are valid. We then begin to prove the existence and uniqueness of a solution of (1.8) when T is small. We need the equivalent of Theorem 24. We define $\eta^{xt}_\nu(s) = \eta_\nu(X^{xt}(s), s)$ and $\bar{Y}^{xt}_\nu(s) = Y^{xt}_\nu(s) - \eta^{xt}_\nu(s)$. We write $\zeta^{xt}_\nu(s) = D\eta_\nu(X^{xt}(s), s)$, then $\bar{Y}^{xt}_\nu(s)$ satisfies

$$\bar{Y}^{xt}_\nu(s) = \int_s^T \bar{H}_\nu(X^{xt}(\theta), Z^{xt}(\theta))d\theta - \int_s^T (Z^{xt}_\nu(\theta) - \zeta^{xt}_\nu(s)) \cdot \sigma(X^{xt}(s))dw(s). \tag{9.21}$$

We define $||\bar{Y}||_{S^\infty} = \sup_{xt}\sup_{s\geq t}||\bar{Y}^{xt}_\nu(s)||_{L^\infty}$, where the space $L^\infty = L^\infty(\Omega, \mathcal{A}, \mathbb{P})$. We also consider the norm

$$||Z||^2_{H^2_{BMO}} = \sup_{xt}\sup_{s>t}\left|\left|\mathbb{E}\left[\int_s^T |Z^{xt}(\theta)|^2 d\theta \middle| \mathcal{F}^s_t\right]\right|\right|_{L^\infty} < +\infty$$

which is the analogue of the H^2_{BMO} norm (9.2). We then solve (9.21) by a fixed point argument. For $U^{xt}(s)$ in H^2_{BMO}, define the solution of the BSDE:

$$\bar{Y}^{xt}_\nu(s) = \int_s^T \bar{H}_\nu(X^{xt}(\theta), U^{xt}(\theta))d\theta - \int_s^T (Z^{xt}_\nu(\theta) - \zeta^{xt}_\nu(s)) \cdot \sigma(X^{xt}(s))dw(s), \tag{9.22}$$

then we first have:

$$||\bar{Y}_\nu||_{S^\infty} \leq k||U||^2_{H^2_{BMO}}. \tag{9.23}$$

We next apply Ito's formula to $(\bar{Y}^{xt}_\nu(s))^2$ and after easy calculations, we obtain the estimate

$$||Z||^2_{H^2_{BMO}} \leq \frac{4N\beta^4}{\alpha k^2} + 2||\zeta||^2_{H^2_{BMO}} \leq \frac{\beta^2}{k^2}, \tag{9.24}$$

provided that β satisfies the first condition

$$\beta^4 - \frac{\alpha\beta^2}{4N} + \frac{\alpha k^2}{2N}||\zeta||^2_{H^2_{BMO}} \leq 0. \tag{9.25}$$

To check the contraction property, consider two elements $U^{1xt}(\theta)$ and $U^{2xt}(\theta)$ and the corresponding solutions $(\bar{Y}^{1xt}(s), Z^{1xt}(s))$ and $(\bar{Y}^{2xt}(s), Z^{2xt}(s))$ of (9.22). We set $\tilde{U}^{xt}(s) = U^{1xt}(s) - U^{1xt}(s)$, $\tilde{Y}^{xt}(s) = \bar{Y}^{1xt}(s) - \bar{Y}^{2xt}(s)$ and $\tilde{Z}^{xt}(s) = Z^{1xt}(s) - Z^{2xt}(s)$, then we get the equations:

$$\tilde{Y}^{xt}_\nu(s) = \int_s^T (H_\nu(X^{xt}(\theta), U^{1xt}(\theta)) - H_\nu(X^{xt}(\theta), U^{2xt}(\theta)))d\theta - \int_s^T \tilde{Z}^{xt}_\nu(s) \cdot \sigma(X^{xt}(s))dw(s).$$

So,

$$\begin{aligned}
&(\tilde{Y}^{xt}_\nu(s))^2 + \mathbb{E}\left[\int_s^T (\tilde{Z}^{xt}_\nu(\theta)).\, a(X^{xt}(\theta))\tilde{Z}^{xt}_\nu(\theta)d\theta \middle| \mathcal{F}^s_t\right] \\
&= 2\mathbb{E}\left[\int_s^T \tilde{Y}^{xt}_\nu(\theta)(H_\nu(X^{xt}(\theta), U^{1xt}(\theta)) - H_\nu(X^{xt}(\theta), U^{2xt}(\theta)))d\theta \middle| \mathcal{F}^s_t\right],
\end{aligned}$$

and we obtain

$$\begin{aligned}
\alpha||\tilde{Z}_\nu||^2_{H^2_{MBO}} &\leq 2\sup_{x,t;s>t}\left(\left|\left|\mathbb{E}\left[\int_s^T |H_\nu(X^{xt}(\theta), U^{1xt}(\theta)) - H_\nu(X^{xt}(\theta), U^{2xt}(\theta))|d\theta \middle| \mathcal{F}^s_t\right]\right|\right|_{L^\infty}\right)^2 \\
&\leq 2k^2\sup_{x,t;s>t}\left(\left|\left|\mathbb{E}\left[\int_s^T (|U^{1xt}(\theta)| + |U^{2xt}(\theta)|)|\tilde{U}^{xt}(\theta)|d\theta \middle| \mathcal{F}^s_t\right]\right|\right|_{L^\infty}\right)^2,
\end{aligned}$$

and like in the analytic part

$$\alpha||\tilde{Z}||^2_{H^2_{MBO}} \leq 8\beta^2 N||\tilde{U}||^2_{H^2_{MBO}}. \tag{9.26}$$

We obtain the contraction property if β satisfies the second condition

$$\frac{8\beta^2 N}{\alpha} < 1. \tag{9.27}$$

We need a smallness condition on $||\zeta||^2_{H^2_{BMO}}$ which is satisfied if T is small. The calculations and the rest of the proof are very similar to the analytic case. This concludes the proof. $\square$

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