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OPTIMAL PERIODIC REPLENISHMENT POLICIES FOR SPECTRALLY POSITIVE LÉVY DEMAND PROCESSES*

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Abstract. We consider a version of the stochastic inventory control problem for a spectrally positive Lévy demand process, in which the inventory can only be replenished at independent exponential times. We show the optimality of a periodic barrier replenishment policy that restocks any shortage below a certain threshold at each replenishment opportunity. The optimal policies and value functions are concisely written in terms of the scale functions. Numerical results are also provided.

Key words. inventory models, spectrally one-sided Lévy processes, scale functions, periodic observations

AMS subject classifications. 60G51, 93E20, 90B05

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1. Introduction.

The classical continuous-time inventory model aims to optimally control the inventory level to strike a balance between minimizing the inventory costs and replenishment costs. The inventory in the absence of control is typically assumed to follow a Brownian motion, a compound Poisson process, or a mixture of the two. Under the assumption that the inventory can be monitored continuously and replenishment can be made instantaneously, the existing results have shown the optimality of a *barrier* or an (s, S)-policy, depending on whether fixed (replenishment) costs are considered. For a comprehensive review and various inventory models, see [6].

In this study, we consider a new extension of the inventory model under the constraint that replenishment opportunities occur at the arrival times of an independent Poisson process. This is because, in reality, one can monitor the inventory only at intervals and, hence, barrier or (s, S) policies are difficult to implement in practice. Recently, similar extensions have been studied in the context of insurance applications [2, 16, 17].

Analytical solutions can be pursued under the assumption of Poissonian replenishment opportunities in which, thanks to the memoryless property, the waiting time until the next opportunity is always (conditionally) exponentially distributed. With other replenishment opportunity times, the state space must be expanded to make the problem Markovian, and, to our knowledge, one must resort to numerical approaches rather than analytical solutions.

One important motivation for considering the Poissonian interarrival model is its potential applications in approximating the constant interarrival time cases. In the mathematical finance literature, randomization techniques (see, e.g., [9]) are known

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as efficient in approximating constant maturity problems with those with Erlang-distributed maturities. In particular, for short maturity cases, it is known empirically that accurate approximations can be obtained by simply replacing the constant with exponential random variables [20].

Although the Poissonian assumption simplifies the considered problem, it is still significantly more challenging and interesting in comparison to the continuous monitoring case. The solutions depend directly on the rate of Poisson arrivals, and it is, therefore, of interest to study its sensitivity.

In this study, we focus on the discounted continuous-time model driven by a spectrally positive Lévy demand process. In other words, the inventory, in the absence of control, follows a Lévy process with only negative jumps. As is typically assumed in the literature, the inventory cost is modeled by a convex function, and the cost of replenishment is assumed to be proportional to the order amount. Under these assumptions, the classical continuous monitoring case admits a simple solution (see section 7 of [21]): it is optimal to reflect the inventory process at a suitably chosen barrier, and the value function is expressed concisely in terms of the so-called *scale function* (see also [7] and sections 4–6 of [21] for the cases with fixed costs).

This study aims to show the optimality of a *periodic barrier replenishment policy*, which restocks any shortage below a certain threshold at each replenishment opportunity. The corresponding controlled inventory process becomes the *Parisian reflected process* studied in [4, 18]. We show that a periodic barrier replenishment policy is indeed optimal over the set of all admissible policies.

We follow the classical *guess-and-verify* procedure to solve this stochastic control problem:

1. The first step is to compute the expected net present value (NPV) of replenishment and inventory costs under periodic barrier replenishment policies. Replenishment costs, which are the expected amount of total discounted Parisian reflection, have been computed in [4]. Inventory costs require the resolvent identity, which we compute using a similar method as in [4]. These admit semiexplicit expressions written in terms of the scale function.
2. In the second step, we select the optimal periodic barrier, which we call b^* in the current study. We choose its value so that the slope of the candidate value function at the barrier equals the negative of the unit replenishment cost.
3. In the final step, we confirm the optimality of the selected candidate optimal policy. To this end, we obtain a verification lemma (sufficient condition for optimality), which requires the value function to be sufficiently smooth and satisfy certain variational inequality. By taking advantage of the existing analytical properties of the scale function, as well as some fluctuation identities, we confirm that the candidate value function indeed satisfies these conditions.

One major advantage of applying these three steps is that one can solve the problem for a general spectrally positive Lévy demand process (of both bounded and unbounded variations) without specifying a particular type of Lévy measure. By reducing the problem to certain analyses on the scale function of the underlying Lévy process, we avoid the use of integro-differential equation techniques, which tend to be difficult, particularly when the Lévy measure has infinite activity.

The rest of the paper is organized as follows. In section 2, we model the problem considered. Section 3 gives the verification lemma. In section 4, we study the periodic barrier replenishment policy and compute the corresponding expected NPV of the total costs. In section 5, we select the candidate barrier. In section 6, the optimality of the selected policy is shown and confirmed numerically. Long proofs and technical results are deferred to the appendix. Throughout the paper, superscripts $x^+ := \max(x,0)$ and $x^- := \max(-x,0)$ are used to indicate the positive and negative parts of x, respectively. The left- and right-hand limits are written as $f(x-) := \lim_{y\uparrow x} f(y)$ and $f(x+) := \lim_{y\downarrow x} f(y)$, respectively, whenever they exist.

2. Inventory models with periodic replenishment opportunities. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a stochastic process $D = (D(t); t \geq 0)$ with $D(0) = 0$, modeling the aggregate demand of a single item, is defined. Under the conditional probability $\mathbb{P}_x$, for $x \in \mathbb{R}$, the initial level of inventory is given by x (in particular, we let $\mathbb{P} \equiv \mathbb{P}_0$). Hence, the inventory, in the absence of control, follows the stochastic process

$$X(t) := x - D(t), \quad t \geq 0.$$

We consider a scenario where the item can be replenished only at the arrival times $\mathcal{T}_r := (T(i); i \geq 0)$ of a Poisson process $N^r = (N^r(t); t \geq 0)$ with intensity $r > 0$, which is independent of X (and D). In other words, the interarrival times $T(i) - T(i-1)$, $i \geq 1$ (with $T(0) := 0$) are independent and exponentially distributed with mean $1/r$. Let $\mathbb{F} := (\mathcal{F}(t); t \geq 0)$ be the filtration generated by the process (X, N^r).

In this setting, an admissible policy, representing the cumulative amount of replenishment $\pi := (R^\pi(t); t \geq 0)$ is a nondecreasing, right-continuous, and $\mathbb{F}$-adapted process such that

$$R^\pi(t) = \int_{[0,t]} \nu^\pi(s) \mathrm{d}N^r(s), \qquad t \geq 0,$$

for a cáglád process ν^π. In particular, the replenishment at the ith replenishment opportunity $T(i)$ is given by $\nu^\pi(T(i))$ for each $i \geq 1$. The controlled inventory process U^π becomes

$$U^\pi(t) := X(t) + R^\pi(t) = X(t) + \sum_{i=1}^{\infty} \nu^\pi(T(i)) 1_{\{T(i) \leq t\}}, \qquad t \geq 0.$$

We fix a discount factor $q > 0$ and a unit cost/reward of controlling $C \in \mathbb{R}$. Associated with the policy $\pi \in \mathcal{A}$, the cost of inventory is modeled by $\int_0^\infty e^{-qt} f(U^\pi(t)) \mathrm{d}t$ for a measurable function $f : \mathbb{R} \to \mathbb{R}$ and that of controlling is given by $C \int_{[0,\infty)} e^{-qt} \mathrm{d}R^\pi(t)$. The problem is to minimize their expected sum

$$v_\pi(x) := \mathbb{E}_x \left[\int_0^\infty e^{-qt} f(U^\pi(t)) \mathrm{d}t + C \int_{[0,\infty)} e^{-qt} \mathrm{d}R^\pi(t) \right], \quad x \in \mathbb{R},$$

over the set of all admissible policies $\mathcal{A}$ that satisfy all the constraints described above and

$$\mathbb{E}_x \left[\int_{[0,\infty)} e^{-qt} \mathrm{d}R^\pi(t) \right] < \infty. \tag{2.1}$$

The problem is to compute the value function

$$v(x) := \inf_{\pi \in \mathcal{A}} v_\pi(x), \quad x \in \mathbb{R}, \tag{2.2}$$

and to obtain the optimal policy π^* that attains it, if such a policy exists.

2.1. Spectrally one-sided Lévy processes. We shall consider the case where the demand D follows a spectrally positive Lévy process, or equivalently X is a spectrally negative Lévy process. We exclude the case in which X is the negative of a subordinator so that it does not have monotone paths a.s. We denote the Laplace exponent of X by $\kappa : [0,\infty) \to \mathbb{R}$ such that $\mathbb{E}[e^{\theta X(t)}] = e^{t\kappa(\theta)}$ for $t, \theta \geq 0$, with its Lévy–Khintchine decomposition

$$\kappa(\theta) = \frac{\sigma^2}{2}\theta^2 + \gamma\theta + \int_{(-\infty,0)} [e^{\theta y} - 1 - \theta y 1_{\{y>-1\}}]\Pi(\mathrm{d}y), \quad \theta \geq 0.$$

Here, $\sigma \geq 0$, $\gamma \in \mathbb{R}$, and the Lévy measure Π satisfies $\int_{(-\infty,0)}(1 \wedge y^2)\Pi(\mathrm{d}y) < \infty$.

It is known (see, e.g., Lemma 2.12 of [13]) that X has paths of bounded variation if and only if $\sigma = 0$ and $\int_{(-1,0)} |y|\Pi(\mathrm{d}y) < \infty$. For the bounded variation case, X can be written as

$$X(t) = ct - S(t), \quad t \geq 0, \quad \text{where} \quad c := \gamma - \int_{(-1,0)} y\Pi(\mathrm{d}y),$$

and $(S(t); t \geq 0)$ is a driftless subordinator. Here, by the assumption that X is not the negative of a subordinator, necessarily we have $c > 0$.

2.2. Assumptions. We solve the problem (2.2) under the following standing assumptions on the Lévy process X and the running cost function f.

Assumption 2.1. We assume that there exists $\bar{\theta} > 0$ such that

$$\int_{(-\infty,-1]} \exp(\bar{\theta}|z|)\Pi(\mathrm{d}z) < \infty.$$

This guarantees that $\mathbb{E}[X(1)] = \kappa'(0+) > -\infty$.

Assumption 2.2.

(i) We assume that f is convex and has at most polynomial growth in the tail. That is to say, there exist $k_1, k_2, m > 0$ and $N \in \mathbb{N}$ such that $|f(x)| \leq k_1 + k_2|x|^N$ for all $x \in \mathbb{R}$ such that $|x| > m$.

(ii) We assume that $f'(-\infty) < -Cq < f'(\infty)$, where $f'(\infty) := \lim_{x\to\infty} f'(x) \in (-\infty, \infty]$ and $f'(-\infty) := \lim_{x\to-\infty} f'(x) \in [-\infty, \infty)$.

These assumptions are critical for our analysis, and similar assumptions are imposed in the existing literature (see, e.g., [7, 11]).

Remark 2.1. By Assumptions 2.1 and 2.2 we have that $\mathbb{E}_x\left[\int_0^\infty e^{-qt}|f(X(t))|\mathrm{d}t\right] < \infty$ for all $x \in \mathbb{R}$. For its proof, see the proof of Lemma 7.5 of [21].

3. Verification lemma. We first obtain the verification lemma for the considered problem. Throughout the paper, we call a measurable function g *sufficiently smooth* on $\mathbb{R}$ if g is $C^1(\mathbb{R})$ (resp., $C^2(\mathbb{R})$) when X has paths of bounded (resp., unbounded) variation. Let $\mathcal{L}$ be the operator acting on a sufficiently smooth function g, defined by

$$\mathcal{L}g(x) := \gamma g'(x) + \frac{\sigma^2}{2}g''(x) + \int_{(-\infty,0)} [g(x+z) - g(x) - g'(x)z\mathbf{1}_{\{-1<z<0\}}]\Pi(\mathrm{d}z).$$

Also, we define the operator $\mathcal{M}$ acting on a measurable function g,

$$\mathcal{M}g(x) := \inf_{l\geq 0}\{Cl + g(x+l)\}. \tag{3.1}$$

LEMMA 3.1 (verification lemma). *Suppose $\hat{\pi} \in \mathcal{A}$ is such that $w := v_{\hat{\pi}}$ is sufficiently smooth on $\mathbb{R}$, has polynomial growth (see Assumption 2.2), and satisfies*

$$(\mathcal{L} - q)w(x) + r(\mathcal{M}w(x) - w(x)) + f(x) = 0, \quad x \in \mathbb{R}. \tag{3.2}$$

Then $v(x) = w(x)$ for all $x \in \mathbb{R}$ and hence $\hat{\pi}$ is an optimal policy.

Remark 3.2. (1) The equality (3.2) can be intuitively explained by the Bellman's principle. For a small time interval Δ_t, the corresponding Bellman's equation is expected to be approximated as

$$v(x) = e^{-r\Delta_t}\mathbb{E}_x[e^{-q\Delta_t}v(X(\Delta_t))] + (1 - e^{-r\Delta_t})\mathbb{E}_x[e^{-q\Delta_t}\mathcal{M}v(X(\Delta_t)] + \mathbb{E}_x\left[\int_0^{\Delta_t} e^{-qs}f(X(s))\mathrm{d}s\right] + o(\Delta_t),$$

where $e^{-r\Delta_t}$ is the probability of no replenishment opportunities over $(0, \Delta_t)$, and $1 - e^{-r\Delta_t}$ is its complement. Hence, using Itô's formula, by dividing by Δ_t and taking $\Delta_t \downarrow 0$, we arrive at (3.2).

(2) Define the set $\mathcal{C} := \{x \in \mathbb{R} : (\mathcal{L} - q)v(x) + f(x) = 0\}$. Then, $\mathcal{C}$ can be understood as the continuation region, and $\mathcal{D} := \mathbb{R}\backslash\mathcal{C}$ as the control region at which replenishment is made whenever the replenishment opportunity arrives.

In this paper, we aim to show that $\mathcal{C} = [b^*, \infty)$ and $\mathcal{D} = (-\infty, b^*)$ for some $b^* \in \mathbb{R}$. This property is closely related to the convexity of v and its slope at b^*. To see this, if v is convex and $v'(b^*) = -C$, then necessarily we have $\mathcal{M}v(x) - v(x) = 0$ if and only if $x \geq b^*$.

(3) There are both similarities and differences with the classical singular control case and the version where the control process must be absolutely continuous with a bounded density (see (4.2) of [11]). While the forms of the variational inequalities differ, the convexity and the slope condition at the candidate barrier are the key elements needed as in the current paper.

Proof of Lemma 3.1. By the definition of v as an infimum, it follows that $w(x) \geq v(x)$ for all $x \in \mathbb{R}$. Hence, it suffices to show the opposite inequality.

Fix $x \in \mathbb{R}$ and $\pi \in \mathcal{A}$ with its corresponding inventory process U^π. Let $(T_n)_{n\in\mathbb{N}}$ be defined by $T_n := \inf\{t > 0 : |U^\pi(t)| > n\}$; here and throughout, let $\inf \varnothing = \infty$.

Because U^π is a semi-martingale and w is sufficiently smooth on $\mathbb{R}$, the change of variables/Itô's formula (see Theorems II.31 and II.32 of [22]) gives under $\mathbb{P}_x$ that

$$\begin{aligned}
&\mathrm{e}^{-q(t\wedge T_n)}w(U^\pi(t \wedge T_n)) - w(x) \\
&= -\int_0^{t\wedge T_n} \mathrm{e}^{-qs}qw(U^\pi(s-))\mathrm{d}s + \int_{[0,t\wedge T_n]} \mathrm{e}^{-qs}w'(U^\pi(s-))\mathrm{d}X(s) \\
&\quad + \frac{\sigma^2}{2}\int_0^{t\wedge T_n} \mathrm{e}^{-qs}w''(U^\pi(s-))\mathrm{d}s + \sum_{0\leq s\leq t\wedge T_n} \mathrm{e}^{-qs}\big[\Delta w\big(U^\pi(s-) + \nu^\pi(s)\big)\Delta N^r(s)\big] \\
&\quad + \sum_{0\leq s\leq t\wedge T_n} \mathrm{e}^{-qs}\big[\Delta w\big(U^\pi(s-) + \Delta X(s)\big) - w'(U^\pi(s-))\Delta X(s)\big] \\
&= \int_0^{t\wedge T_n} \mathrm{e}^{-qs}(\mathcal{L} - q)w(U^\pi(s-))\mathrm{d}s - C\int_{[0,t\wedge T_n]} \mathrm{e}^{-qs}\nu^\pi(s)\mathrm{d}N^r(s) \\
&\quad + \int_0^{t\wedge T_n} \mathrm{e}^{-qs}r\left[C\nu^\pi(s) + w\big(U^\pi(s-) + \nu^\pi(s)\big) - w(U^\pi(s-))\right]\mathrm{d}s + M(t \wedge T_n),
\end{aligned}$$

where we define, for $t \geq 0$, with $\tilde{\mathcal{N}}(\mathrm{d}s \times \mathrm{d}y) := \mathcal{N}(\mathrm{d}s \times \mathrm{d}y) - \Pi(\mathrm{d}y)\mathrm{d}s$,

$$\begin{aligned}
&M(t \wedge T_n) \\
&:= \int_0^{t\wedge T_n} \sigma \mathrm{e}^{-qs} w'(U^\pi(s-))\mathrm{d}B(s) + \lim_{\varepsilon \downarrow 0} \int_{[0,t\wedge T_n]} \int_{(-1,-\varepsilon)} \mathrm{e}^{-qs} w'(U^\pi(s-)) y \tilde{\mathcal{N}}(\mathrm{d}s \times \mathrm{d}y) \\
&\quad + \int_{[0,t\wedge T_n]} \int_{(-\infty,0)} \mathrm{e}^{-qs} \big[w(U^\pi(s-) + y) - w(U^\pi(s-)) \\
&\qquad\qquad\qquad - w'(U^\pi(s-)) y \mathbf{1}_{\{y \in (0,1)\}} \big] \tilde{\mathcal{N}}(\mathrm{d}s \times \mathrm{d}y) \\
&\quad + \int_{[0,t\wedge T_n]} \mathrm{e}^{-qs} \left[C\nu^\pi(s) + w\big(U^\pi(s-) + \nu^\pi(s)\big) - w(U^\pi(s-)) \right] \mathrm{d}(N^r(s) - rs).
\end{aligned}$$

Here, $(B(s); s \geq 0)$ is a standard Brownian motion and $\mathcal{N}$ is a Poisson random measure in the measure space $([0,\infty) \times (-\infty, 0), \mathcal{B}[0,\infty) \times \mathcal{B}(-\infty, 0), \mathrm{d}s \times \Pi(\mathrm{d}x))$. By the definition of $\mathcal{M}$ as in (3.1),

$$\begin{aligned}
w(x) \leq &- \int_0^{t\wedge T_n} \mathrm{e}^{-qs} \Big[(\mathcal{L} - q) w(U^\pi(s-)) + r\big(\mathcal{M}w(U^\pi(s-)) - w(U^\pi(s-))\big) \Big] \mathrm{d}s \\
&+ C \int_{[0,t\wedge T_n]} \mathrm{e}^{-qs} \nu^\pi(s) \mathrm{d}N^r(s) - M(t \wedge T_n) + \mathrm{e}^{-q(t\wedge T_n)} w(U^\pi(t \wedge T_n)).
\end{aligned}$$

Using the assumption (3.2), together with the fact that the process $(M(t \wedge T_n); t \geq 0)$ is a zero-mean $\mathbb{P}_x$-martingale (see Corollary 4.6 of [13]), after taking expectations, we obtain

$$\begin{aligned}
w(x) \leq \mathbb{E}_x \Big[&\int_0^{t\wedge T_n} \mathrm{e}^{-qs} f(U^\pi(s)) \mathrm{d}s \\
&+ C \int_{[0,t\wedge T_n]} \mathrm{e}^{-qs} \nu^\pi(s) \mathrm{d}N^r(s) + \mathrm{e}^{-q(t\wedge T_n)} w(U^\pi(t \wedge T_n)) \Big].
\end{aligned} \tag{3.3}$$

We shall now take $t, n \uparrow \infty$ in the above inequality to complete the proof. First, assumption (3.2) and the fact that $\mathcal{M}w \leq w$ imply that $(\mathcal{L} - q)w(y) + f(y) \geq 0$ for $y \in \mathbb{R}$. Because w is sufficiently smooth and is of polynomial growth, by Itô's formula together with dominated convergence, we have $w(x) \leq \mathbb{E}_x[\int_0^\infty e^{-qs} f(X(s))\mathrm{d}s]$ for all $x \in \mathbb{R}$ (for more details, see the proof of Lemma 7.5 of [21]). This, together with the strong Markov property, implies

$$\mathbb{E}_x \left[\mathrm{e}^{-q(t\wedge T_n)} w(U^\pi(t \wedge T_n)) \right] \leq \mathbb{E}_x \Big[\int_{t\wedge T_n}^\infty e^{-qs} f\big(R^\pi(t \wedge T_n) + X(s)\big) \mathrm{d}s \Big]. \tag{3.4}$$

Now, following the same steps as the proof of Theorem 7.1 of [21], we have

$$\begin{aligned}
&\mathbb{E}_x \Big[\int_{t\wedge T_n}^\infty e^{-qs} f\big(R^\pi(t \wedge T_n) + X(s)\big) \mathrm{d}s \Big] \\
&\quad \leq \mathbb{E}_x \Big[\int_{[t\wedge T_n, \infty)} e^{-qs} \Big(f(U^\pi(s)) \mathrm{d}s + C \mathrm{d}R^\pi(s) \Big) \Big] \\
&\qquad + \mathbb{E}_x \Big[\int_{t\wedge T_n}^\infty e^{-qs} \Big(f(X(s)) + CqX(s) \Big) \mathrm{d}s \Big].
\end{aligned}$$

By using this and (3.4) in (3.3), we obtain $w(x) \leq v_\pi(x) + \mathbb{E}_x[\int_{t\wedge T_n}^\infty e^{-qs}(f(X(s)) + CqX(s))\mathrm{d}s]$. Because $\mathbb{E}_x[\int_{t\wedge T_n}^\infty e^{-qs}|f(X(s)) + CqX(s)|\mathrm{d}s] < \infty$ (which holds by Remark 2.1), upon taking $t, n \uparrow \infty$ via monotone convergence, we have $w(x) \leq v_\pi(x)$, as desired. ∎

4. Periodic barrier replenishment policies. The objective of this paper is to show the optimality of the *periodic barrier replenishment policy* π^b, $b \in \mathbb{R}$, that pushes the inventory up to b at the observation times $\mathcal{T}_r$ whenever it is below b. The resulting inventory process is precisely the *Parisian reflected Lévy process* of [4].

We denote, by R_r^b and U_r^b, the aggregate sum of replenishment and the resulting inventory, respectively. More concretely, we have

$$U_r^b(t) = X(t) \quad \text{and} \quad R_r^b(t) = 0, \quad 0 \le t < T_b^-(1),$$

where $T_b^-(1) := \inf\{S \in \mathcal{T}_r : X(S-) < b\}$ is the first replenishment time. The inventory is then pushed up by the amount $\Delta R_r^b(T_b^-(1)) = b - X(T_b^-(1)-)$ so that $U_r^b(T_b^-(1)) = b$. For $T_b^-(1) \le t < T_b^-(2) := \inf\{S \in \mathcal{T}_r : S > T_b^-(1), U_r^b(S-) < b\}$, we have $U_r^b(t) = X(t) + (b - X(T_b^-(1)-))$ and $R_r^b(t) = R_r^b(T_b^-(1))$. The controlled inventory process can be constructed by repeating this procedure.

We have the following decomposition:

$$U_r^b(t) = X(t) + R_r^b(t), \quad t \ge 0,$$

with

$$R_r^b(t) = \sum_{i=1}^{\infty} (b - U_r^b(T_b^-(i)-)) 1_{\{T_b^-(i) \le t\}} = \int_{[0,t]} (b - U_r^b(s-))^+ \mathrm{d}N^r(s), \quad t \ge 0,$$

where the replenishment times $(T_b^-(n); n \ge 1)$ can be constructed inductively by $T_b^-(1)$ defined above and $T_b^-(n+1) := \inf\{S \in \mathcal{T}_r : S > T_b^-(n), U_r^b(S-) < b\}$ for $n \ge 1$. We will see by (4.12) that the policy $\pi^b := (R_r^b(t); t \ge 0)$ satisfies (2.1), and is hence admissible.

In this section, we compute, via the scale function, the expected NPV of the total costs under π^b:

$$v_b(x) := \mathbb{E}_x\Big[\int_0^\infty e^{-qt} f(U_r^b(t)) \mathrm{d}t + C \int_{[0,\infty)} e^{-qt} \mathrm{d}R_r^b(t) \Big], \quad b, x \in \mathbb{R}. \tag{4.1}$$

4.1. Scale functions. We fix $q, r > 0$. The scale function $W^{(q)} : \mathbb{R} \to [0, \infty)$ of X takes the value zero on $(-\infty, 0)$, and on $[0, \infty)$ it is a strictly increasing function, defined by its Laplace transform:

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x) \mathrm{d}x = \frac{1}{\kappa(\theta) - q}, \quad \theta > \Phi(q) := \sup\{\lambda \ge 0 : \kappa(\lambda) = q\}. \tag{4.2}$$

In addition, let, for $x \in \mathbb{R}$,

$$\overline{W}^{(q)}(x) := \int_0^x W^{(q)}(y) \mathrm{d}y, \quad Z^{(q)}(x) := 1 + q\overline{W}^{(q)}(x), \quad \overline{Z}^{(q)}(x) := \int_0^x Z^{(q)}(z) \mathrm{d}z.$$

Note that, for $x \le 0$, $\overline{W}^{(q)}(x) = 0$, $Z^{(q)}(x) = 1$, and $\overline{Z}^{(q)}(x) = x$. We also define, for $\theta \ge 0$ and $x \in \mathbb{R}$,

$$Z^{(q)}(x, \theta) := e^{\theta x} \left(1 + (q - \kappa(\theta)) \int_0^x e^{-\theta z} W^{(q)}(z) \mathrm{d}z \right). \tag{4.3}$$

In particular, for $x \in \mathbb{R}$, $Z^{(q)}(x,0) = Z^{(q)}(x)$ and

$$\begin{aligned} Z^{(q)}(x, \Phi(q+r)) &= e^{\Phi(q+r)x}\left(1 - r\int_0^x e^{-\Phi(q+r)z}W^{(q)}(z)\mathrm{d}z\right), \\ Z^{(q+r)}(x, \Phi(q)) &= e^{\Phi(q)x}\left(1 + r\int_0^x e^{-\Phi(q)z}W^{(q+r)}(z)\mathrm{d}z\right). \end{aligned} \tag{4.4}$$

Finally, let

$$Z^{(q,r)}(x) := \frac{r}{q+r}Z^{(q)}(x) + \frac{q}{q+r}Z^{(q)}(x, \Phi(q+r)), \quad x \in \mathbb{R},$$

and, for all $x, y \in \mathbb{R}$,

$$\begin{aligned} W_y^{(q,r)}(x) &:= W^{(q+r)}(x-y) - r\int_0^x W^{(q)}(x-z)W^{(q+r)}(z-y)\mathrm{d}z \\ &= W^{(q)}(x-y) + r\int_0^{-y} W^{(q)}(x-u-y)W^{(q+r)}(u)\mathrm{d}u, \end{aligned} \tag{4.5}$$

where the second equality holds by (7) of [15], and in particular $W_y^{(q,r)}(x) = W^{(q)}(x-y)$ for $y \geq 0$.

For the rest of this subsection, we list several fluctuation identities which we use later in the paper. For the spectrally negative Lévy process X, define

$$\tau_a^- := \inf\{t > 0 : X(t) < a\} \quad \text{and} \quad \tau_a^+ := \inf\{t > 0 : X(t) > a\}, \quad a \in \mathbb{R}.$$

By using identity (3.19) in [3], for $x \in \mathbb{R}$ and $\theta \geq 0$,

$$\begin{aligned} H^{(q+r)}(x,\theta) &:= \mathbb{E}_x\left[e^{-(q+r)\tau_0^- + \theta X(\tau_0^-)}1_{\{\tau_0^- < \infty\}}\right] \\ &= Z^{(q+r)}(x,\theta) - \frac{\kappa(\theta) - (q+r)}{\theta - \Phi(q+r)}W^{(q+r)}(x), \end{aligned} \tag{4.6}$$

where, in particular,

$$\begin{aligned} H^{(q+r)}(x,\Phi(q)) &= \mathbb{E}_x\left[e^{-(q+r)\tau_0^- + \Phi(q)X(\tau_0^-)}1_{\{\tau_0^- < \infty\}}\right] \\ &= Z^{(q+r)}(x,\Phi(q)) - \frac{rW^{(q+r)}(x)}{\Phi(q+r) - \Phi(q)}, \\ H^{(q+r)}(x) &:= H^{(q+r)}(x,0) = \mathbb{E}_x\left[e^{-(q+r)\tau_0^-}\right] = Z^{(q+r)}(x) - \frac{q+r}{\Phi(q+r)}W^{(q+r)}(x). \end{aligned} \tag{4.7}$$

For any Borel set $A \subset (-\infty, 0]$ and $x \leq 0$, by Theorem 2.7(ii) in [12],

$$\mathbb{E}_x\Big[\int_0^{\tau_0^+} e^{-(q+r)t}1_{\{X(t)\in A\}}\mathrm{d}t\Big] = \int_A \Theta^{(q+r)}(x,y)\mathrm{d}y, \tag{4.8}$$

where we define, for $x, y \in \mathbb{R}$,

$$\Theta^{(q+r)}(x,y) := e^{\Phi(q+r)x}W^{(q+r)}(-y) - W^{(q+r)}(x-y). \tag{4.9}$$

Remark 4.1. (i) For $x, y \le 0$, by the identity (4.8), $\Theta^{(q+r)}(x,y) \ge 0$.

(ii) On the other hand, for $x > 0$ and $y \le x$, $\Theta^{(q+r)}(x,y) \le 0$. Indeed, by (4.8),

$$0 \le \mathbb{E}\Big[\int_0^{\tau_x^+} e^{-(q+r)t} 1_{\{X(t)\in \mathrm{d}y\}} \mathrm{d}t\Big] = -e^{-\Phi(q+r)x}\Theta^{(q+r)}(x,y)\mathrm{d}y. \tag{4.10}$$

Let $\underline{X}$ be the running infimum process of X and $\mathbf{e}_{q+r}$ be an independent exponential random variable with parameter $q+r$. By Corollary 2.2 of [12], for Borel subsets on $[0,\infty)$,

$$\mathbb{P}\left(-\underline{X}(\mathbf{e}_{q+r}) \in \mathrm{d}y\right) = \frac{q+r}{\Phi(q+r)} W^{(q+r)}(\mathrm{d}y) - (q+r)W^{(q+r)}(y)\mathrm{d}y, \tag{4.11}$$

where $W^{(q+r)}(\mathrm{d}y)$ is the measure such that $W^{(q+r)}(y) = \int_{[0,y]} W^{(q+r)}(\mathrm{d}z)$ (see [13, (8.20)]).

Remark 4.2.

(1) By (8.26) of [13], the left- and right-hand derivatives of $W^{(q)}$ always exist on $\mathbb{R}\backslash\{0\}$. In addition, as in, e.g., [10, Theorem 3], if X is of unbounded variation or the Lévy measure is atomless, we have $W^{(q)} \in C^1(\mathbb{R}\backslash\{0\})$.

(2) As in Lemmas 3.1 and 3.2 of [12],

$$W^{(q)}(0) = \begin{cases} 0 & \text{if } X \text{ is of unbounded variation,} \\ \frac{1}{c} & \text{if } X \text{ is of bounded variation,} \end{cases}$$

$$W^{(q)\prime}(0+) = \begin{cases} \frac{2}{\sigma^2} & \text{if } \sigma > 0, \\ \infty & \text{if } \sigma = 0 \text{ and } \Pi(-\infty,0) = \infty, \\ \frac{q+\Pi(-\infty,0)}{c^2} & \text{if } \sigma = 0 \text{ and } \Pi(-\infty,0) < \infty. \end{cases}$$

(3) As in Lemma 3.3 of [12], $W_{\Phi(q)}(x) := e^{-\Phi(q)x}W^{(q)}(x) \nearrow \kappa'(\Phi(q))^{-1}$, as $x \uparrow \infty$.

4.2. The computation of v_b. We shall now write the expected NPV of total costs v_b as in (4.1). For the controlling cost, it has already been obtained in Corollary 3.2(iii) of [4] that, for $b, x \in \mathbb{R}$,

$$\mathbb{E}_x\left[\int_{[0,\infty)} e^{-qt}\mathrm{d}R_r^b(t)\right] = \frac{\Phi(q+r)-\Phi(q)}{\Phi(q+r)\Phi(q)} Z^{(q,r)}(x-b) - \frac{r}{q+r}\Big\{\overline{Z}^{(q)}(x-b) + \frac{\kappa'(0+)}{q}\Big\}. \tag{4.12}$$

Hence, it is left to compute the expected NPV of the inventory cost.

Recall $H^{(q+r)}$ as in (4.6), and in order to obtain a concise expression for v_b let us define, for $x, y \in \mathbb{R}$,

$$\Upsilon(x,y) := -\Theta^{(q+r)}(x,y) + r\int_0^x W^{(q)}(x-z)\Theta^{(q+r)}(z,y)\mathrm{d}z \tag{4.13}$$

$$= W_y^{(q,r)}(x) - Z^{(q)}(x, \Phi(q+r))W^{(q+r)}(-y), \tag{4.14}$$

where the second equality holds by (4.4) and (4.9).

Remark 4.3. (i) Using (4.3), for $x < 0$, $H^{(q+r)}(x,\theta) = e^{\theta x} > 0$ for $\theta \ge 0$.

(ii) Using (4.14) together with (4.5), we have that $\Upsilon(x,y) = W^{(q)}(x-y)$ for $y > 0$.

Remark 4.4. The function $\Upsilon(x,y)$ will be a key function for the rest of the analysis in this paper. It coincides with $-\Theta^{(q+r)}(x,y)$ when $x<0$ and with $W^{(q)}(x-y)$ for $y>0$ as in the above remark. To see further relationships with these functions, see (A.4) and Lemma A.1 in the appendix.

The proof of the following theorem is given in Appendix A.1.

THEOREM 4.5. *For $x,b\in\mathbb{R}$, and a positive bounded measurable function h on $\mathbb{R}$ with compact support,*

$$\mathbb{E}_x\left[\int_0^\infty e^{-qt}h(U_r^b(t))\mathrm{d}t\right]=\int_{-\infty}^\infty h(y)r_b^{(q,r)}(x,y)\mathrm{d}y, \tag{4.15}$$

where, for $x,y\in\mathbb{R}$,

$$r_b^{(q,r)}(x,y):=\frac{q+r}{qr}\frac{\Phi(q)(\Phi(q+r)-\Phi(q))}{\Phi(q+r)}Z^{(q,r)}(x-b)H^{(q+r)}(b-y,\Phi(q))\\ -\Upsilon(x-b,y-b).$$

Now using (4.12) and Theorem 4.5, as well as Lemma B.1 (given in the appendix), we obtain the expression for (4.1).

PROPOSITION 4.6. *For $x,b\in\mathbb{R}$, the function $v_b(x)$ is finite and can be written*

$$\begin{aligned} v_b(x)&=F(b)Z^{(q,r)}(x-b)-\int_{-\infty}^\infty f(y)\Upsilon(x-b,y-b)\mathrm{d}y\\ &\quad-\frac{Cr}{q+r}\left\{\overline{Z}^{(q)}(x-b)+\frac{\kappa'(0+)}{q}\right\}, \end{aligned}\tag{4.16}$$

where

$$F(b):=\frac{\Phi(q+r)-\Phi(q)}{\Phi(q+r)}\left[\frac{q+r}{qr}\Phi(q)\int_{-\infty}^\infty f(y)H^{(q+r)}(b-y,\Phi(q))\mathrm{d}y+\frac{C}{\Phi(q)}\right], \tag{4.17}$$

which is well-defined and finite by Lemma B.1 *and Remark* 4.3(i)*. In particular, for $x<b$, from* (4.13)*,*

$$\begin{aligned} v_b(x)&=F(b)\frac{r+qe^{\Phi(q+r)(x-b)}}{q+r}\\ &\quad+\int_{-\infty}^b f(y)\Theta^{(q+r)}(x-b,y-b)\mathrm{d}y-\frac{Cr}{q+r}\left\{x-b+\frac{\kappa'(0+)}{q}\right\}. \end{aligned}\tag{4.18}$$

Proof. By Theorem 4.5 and dominated convergence (due to Lemmas B.1 and B.2), identity (4.15) holds for $h=f$. By this and (4.12), the result holds after simplification. ∎

4.3. Polynomial growth of v_b. We conclude this section with the following property of v_b, which is required in the verification lemma (Lemma 3.1).

LEMMA 4.7. *For each $b\in\mathbb{R}$, $x\mapsto v_b(x)$ is of polynomial growth.*

Proof. Under $\mathbb{P}$ where $X(0)=0$ and $z\in\mathbb{R}$, let $U_r^{b,z}$ be the Parisian reflected process with barrier $b\in\mathbb{R}$ driven by $(X(t)+z;t\geq 0)$ and define $R_r^{b,z}$ similarly so that $U_r^{b,z}(t)=z+X(t)+R_r^{b,z}(t)$, $t\geq 0$. Then,

$$U_r^{b,y}(t)-U_r^{b,x}(t)=(y-x)+(R_r^{b,y}(t)-R_r^{b,x}(t)),\quad x<y. \tag{4.19}$$

We first show that, for $y > x$,

$$U_r^{b,y}(t) - U_r^{b,x}(t) \geq 0, \quad t \geq 0. \tag{4.20}$$

Let $\sigma := \inf\{t > 0 : U_r^{b,x}(t) > U_r^{b,y}(t)\}$, and assume (to derive a contradiction) that $\sigma < \infty$. Because the increments of $U_r^{b,x}$ and $U_r^{b,y}$ can differ only at the jump times of $R_r^{b,x}$ and $R_r^{b,y}$, we must have that $\Delta R_r^{b,x}(\sigma) > 0$ and $U_r^{b,x}(\sigma-) < b$. If $U_r^{b,y}(\sigma-) \leq b$, then $U_r^{b,x}(\sigma) = U_r^{b,y}(\sigma) = b$. If $U_r^{b,y}(\sigma-) > b$, then $U_r^{b,x}(\sigma) = b < U_r^{b,y}(\sigma-) = U_r^{b,y}(\sigma)$. In both cases, $U_r^{b,x}(\sigma) \leq U_r^{b,y}(\sigma)$ and the inequality holds until the next Poisson arrival time after σ, which contradicts the definition of σ. Hence, we must have $\sigma = \infty$ or equivalently (4.20).

On the other hand, letting $\sigma_0 := \inf\{t > 0 : U_r^{b,x}(t) = U_r^{b,y}(t)\}$, we have, for $i \geq 1$ with $T(i) \leq \sigma_0$,

$$\Delta R_r^{b,x}(T(i)) = (b - U_r^{b,x}(T(i)-))^+ \geq (b - U_r^{b,y}(T(i)-))^+ = \Delta R_r^{b,y}(T(i)),$$

while, for $t \geq \sigma_0$, we must have $\Delta R_r^{b,x}(t) = \Delta R_r^{b,y}(t)$. This together with (4.20) implies that

$$0 \leq R_r^{b,x}(t) - R_r^{b,y}(t) \leq y - x, \quad t \geq 0. \tag{4.21}$$

By (4.19) and (4.21), we also have

$$0 \leq U_r^{b,y}(t) - U_r^{b,x}(t) \leq y - x, \quad t \geq 0. \tag{4.22}$$

By these bounds and Assumption 2.2(i), we have that v_b is of polynomial growth. ∎

5. Selection of b^*. In this section, motivated by the discussion given in Remark 3.2(2), we pursue our candidate barrier b^* such that $v'_{b^*}(b^*) = -C$ and show its existence. The convexity of v_{b^*} is shown later in the paper.

We first obtain the following two lemmas, whose proofs are deferred to Appendices C.1 and C.2.

LEMMA 5.1. *Define, for $x, y \in \mathbb{R}$,*

$$\Psi(x, y) := W_y^{(q,r)}(x) - \frac{\Phi(q+r)}{q+r} Z^{(q)}(x, \Phi(q+r)) Z^{(q+r)}(-y). \tag{5.1}$$

Then, for $y < b$,

$$\begin{aligned} \frac{\partial}{\partial z} \Upsilon(z, y-b)\Big|_{z=(x-b)+} &= -\frac{\partial}{\partial z} \Psi(x-b, z)\Big|_{z=(y-b)-} \\ &= W^{(q+r)\prime}((x-y)+) - r \int_b^x W^{(q)}(x-z) W^{(q+r)\prime}(z-y) \mathrm{d}z \\ &\quad - \Phi(q+r) W^{(q+r)}(b-y) Z^{(q)}(x-b, \Phi(q+r)). \end{aligned} \tag{5.2}$$

Remark 5.2. By Lemma B.1 and Proposition 4.6, we must have

$$\lim_{y \downarrow -\infty} f(y) H^{(q+r)}(b-y, \theta) = 0$$

for $\theta \geq 0$ and $\lim_{y \downarrow -\infty} f(y) \Upsilon(x-b, y-b) = 0$. In addition because

$$\Psi(x-b, y-b) = \Upsilon(x-b, y-b) - \frac{\Phi(q+r)}{q+r} Z^{(q)}(x-b, \Phi(q+r)) H^{(q+r)}(b-y), \tag{5.3}$$

we also have $\lim_{y \downarrow -\infty} f(y) \Psi(x-b, y-b) = 0$.

LEMMA 5.3. *Fix $x, b \in \mathbb{R}$. We can choose $-M \le b \wedge x$ sufficiently small so that*

$$\frac{\partial}{\partial x}\int_{-\infty}^{-M} f(y)\Upsilon(x-b,y-b)\mathrm{d}y = \int_{-\infty}^{-M} f(y)\frac{\partial}{\partial x}\Upsilon(x-b,y-b)\mathrm{d}y.$$

Using Lemmas 5.1 and 5.3, we obtain the results regarding the first derivative of v_b.

LEMMA 5.4. *Fix $b, x \in \mathbb{R}$.* (i) *We have*

(5.4)
$$v_b'(x) = (qF(b)-f(b))\frac{\Phi(q+r)}{q+r}Z^{(q)}(x-b,\Phi(q+r)) - \int_b^x W^{(q)}(x-y)f'(y)\mathrm{d}y - \int_{-\infty}^b f'(y)\Psi(x-b,y-b)\mathrm{d}y - \frac{Cr}{q+r}Z^{(q)}(x-b).$$

(ii) *We have*

$$\mathbb{E}_x\Big[\int_0^\infty e^{-qt}f'(U_r^b(t))\mathrm{d}t\Big] - v_b'(x) = \left(Z^{(q,r)}(x-b)\frac{q+r}{q}\frac{\Phi(q)}{\Phi(q+r)} - Z^{(q)}(x-b,\Phi(q+r))\right)M^{(q,r)}(b),$$

where

(5.5)
$$M^{(q,r)}(b) := \frac{\Phi(q+r)-\Phi(q)}{r}\int_{-\infty}^\infty f'(y)H^{(q+r)}(b-y,\Phi(q))\mathrm{d}y + \frac{q}{q+r}\frac{\Phi(q+r)}{\Phi(q)}C.$$

Proof. (i) By integration by parts, for $x \neq b$,

$$\frac{\partial}{\partial x}\int_b^\infty f(y)W^{(q)}(x-y)\mathrm{d}y = f(b)W^{(q)}(x-b) + \int_b^x W^{(q)}(x-y)f'(y)\mathrm{d}y. \tag{5.6}$$

Differentiating (4.16) and using (5.6) and Lemma 5.3 (with which the derivative can be interchanged over the integral) and that $\Upsilon(x-b,y-b)|_{y=x+} - \Upsilon(x-b,y-b)|_{y=x-} = -W^{(q+r)}(0)$, for $x \neq b$,

$$v_b'(x) = F(b)Z^{(q,r)\prime}(x-b) - f(b)W^{(q)}(x-b) - \int_b^x W^{(q)}(x-y)f'(y)\mathrm{d}y - \int_{-\infty}^b f(y)\frac{\partial}{\partial x}\Upsilon(x-b,y-b)\mathrm{d}y - f(x)W^{(q+r)}(0)1_{\{x<b\}} - \frac{Cr}{q+r}Z^{(q)}(x-b).$$

By Lemma 5.1, Remark 5.2, and integration by parts and noting that $\Psi(x-b,y-b)|_{y=x+} - \Psi(x-b,y-b)|_{y=x-} = -W^{(q+r)}(0)$,

$$\int_{-\infty}^b f(y)\frac{\partial}{\partial x}\Upsilon(x-b,y-b)\mathrm{d}y = -\int_{-\infty}^b f(y)\frac{\partial}{\partial y}\Psi(x-b,y-b)\mathrm{d}y = -f(b)\Psi(x-b,0) + \int_{-\infty}^b f'(y)\Psi(x-b,y-b)\mathrm{d}y - f(x)W^{(q+r)}(0)1_{\{x<b\}},$$

where $\Psi(x-b,0) = W^{(q)}(x-b) - \frac{\Phi(q+r)}{q+r} Z^{(q)}(x-b,\Phi(q+r))$. This together with $Z^{(q,r)\prime}(x-b) = \frac{q}{q+r}\Phi(q+r)Z^{(q)}(x-b,\Phi(q+r))$ shows (5.4).

For the case $x=b$, following the same computation for the right- and left-hand derivatives, it can be confirmed that they both match with (5.4).

(ii) Integration by parts gives

$$\int_b^\infty f(y)e^{-\Phi(q)(y-b)}\mathrm{d}y = \Big(f(b) + \int_b^\infty f'(y)e^{-\Phi(q)(y-b)}\mathrm{d}y\Big)/\Phi(q), \tag{5.7}$$

and by noticing that $\overline{H}^{(q+r)}(z) := \big(H^{(q+r)}(z,\Phi(q)) - \frac{r}{q+r}\frac{\Phi(q+r)}{\Phi(q+r)-\Phi(q)}H^{(q+r)}(z)\big)/\Phi(q)$, $z\in\mathbb{R}$, is an antiderivative of $H^{(q+r)}(\cdot,\Phi(q))$ and by Remark 5.2,

$$\begin{aligned}\int_{-\infty}^b f(y)H^{(q+r)}(b-y,\Phi(q))\mathrm{d}y = -\frac{f(b)}{\Phi(q)}\left(1-\frac{r}{q+r}\frac{\Phi(q+r)}{\Phi(q+r)-\Phi(q)}\right)\\ + \int_{-\infty}^b f'(y)\overline{H}^{(q+r)}(b-y)\mathrm{d}y.\end{aligned}$$

Therefore, using the previous identities in (4.17) together with Remark 4.3(i), we obtain

$$\begin{aligned}F(b) &= \frac{1}{q}\Big(f(b) - \int_{-\infty}^b f'(y)H^{(q+r)}(b-y)\mathrm{d}y\Big)\\ &\quad + \frac{\Phi(q+r)-\Phi(q)}{\Phi(q+r)}\left(\frac{q+r}{qr}\int_{-\infty}^\infty f'(y)H^{(q+r)}(b-y,\Phi(q))\mathrm{d}y + \frac{C}{\Phi(q)}\right).\end{aligned} \tag{5.8}$$

Now using (5.4) and Theorem 4.5 together with (5.3) and Remark 4.3(ii), we obtain that

$$\begin{aligned}&\mathbb{E}_x\left[\int_0^\infty e^{-qt}f'(U_r^b(t))\mathrm{d}t\right] - v_b'(x)\\ &= \frac{Cr}{q+r}Z^{(q)}(x-b)\\ &\quad + \frac{q+r}{qr}\frac{\Phi(q)(\Phi(q+r)-\Phi(q))}{\Phi(q+r)}\int_{-\infty}^\infty f'(y)Z^{(q,r)}(x-b)H^{(q+r)}(b-y,\Phi(q))\mathrm{d}y\\ &\quad + \frac{\Phi(q+r)}{q+r}Z^{(q)}(x-b,\Phi(q+r))\Big(f(b) - \int_{-\infty}^b f'(y)H^{(q+r)}(b-y)\mathrm{d}y - qF(b)\Big),\end{aligned}$$

which shows (ii) by (5.8). ∎

From (5.3), $\Psi(0,y-b) = -\frac{\Phi(q+r)}{q+r}H^{(q+r)}(b-y)$. Hence using (5.4) and (5.8), for any $b\in\mathbb{R}$,

$$\begin{aligned}v_b'(b) &= -\frac{Cr}{q+r} + \frac{\Phi(q+r)}{q+r}\Big(-\int_{-\infty}^b f'(y)H^{(q+r)}(b-y)\mathrm{d}y + q\frac{\Phi(q+r)-\Phi(q)}{\Phi(q+r)}\Big(\frac{C}{\Phi(q)}\\ &\quad + \frac{q+r}{qr}\int_{-\infty}^\infty f'(y)H^{(q+r)}(b-y,\Phi(q))\mathrm{d}y\Big)\Big) + \frac{\Phi(q+r)}{q+r}\int_{-\infty}^b f'(y)H^{(q+r)}(b-y)\mathrm{d}y\\ &= M^{(q,r)}(b) - C.\end{aligned} \tag{5.9}$$

In view of this and Remark 3.2(2), our natural selection of the candidate barrier b^* is such that $M^{(q,r)}(b^*) = 0$. With this choice, the following is immediate by Lemma 5.4(ii).

LEMMA 5.5. *If $b^* \in \mathbb{R}$ is such that $M^{(q,r)}(b^*) = 0$, then*

$$v'_{b^*}(x) = \mathbb{E}_x \left[\int_0^\infty e^{-qt} f'(U_r^{b^*}(t)) \mathrm{d}t \right]$$

for $x \in \mathbb{R}$.

5.1. Existence of the optimal barrier b^*. We first show the following two lemmas. The proof of the first lemma is deferred to Appendix C.3.

LEMMA 5.6. *Fix $b \in \mathbb{R}$ and $\theta \geq 0$. We can choose $-M < b$ sufficiently small so that*

$$\frac{\partial}{\partial b} \int_{-\infty}^{-M} f(y) H^{(q+r)}(b-y,\theta) \mathrm{d}y = \int_{-\infty}^{-M} f(y) \frac{\partial}{\partial b} H^{(q+r)}(b-y,\theta) \mathrm{d}y.$$

LEMMA 5.7. *For all $b \in \mathbb{R}$ at which $f'(b)$ exists,*

$$(e^{-\Phi(q)b} M^{(q,r)}(b))' = -e^{-\Phi(q)b} \frac{\Phi(q+r)}{q+r} \left(Cq + \mathbb{E}\left[f'(\underline{X}(\mathbf{e}_{q+r}) + b) \right] \right).$$

Proof. Using Lemma 5.6 together with Remark 4.3(i) and (4.11),

$$\begin{aligned} &M^{(q,r)\prime}(b) \\ &= \frac{\Phi(q+r) - \Phi(q)}{r} \\ &\quad \times \Bigg[- f'(b) + \Phi(q) \int_b^\infty f'(y) e^{-\Phi(q)(y-b)} \mathrm{d}y + f'(b) \left(1 - \frac{r W^{(q+r)}(0)}{\Phi(q+r) - \Phi(q)} \right) \\ &\quad + \int_{-\infty}^b f'(y) \left(\Phi(q) Z^{(q+r)}(b-y,\Phi(q)) + r W^{(q+r)}(b-y) - \frac{r W^{(q+r)\prime}(b-y)}{\Phi(q+r) - \Phi(q)} \right) \mathrm{d}y \Bigg] \\ &= \Phi(q) M^{(q,r)}(b) - \frac{\Phi(q+r)}{q+r} \Big(qC + \mathbb{E}\left[f'(\underline{X}(\mathbf{e}_{q+r}) + b) \right] \Big). \end{aligned}$$

By this, the desired result is immediate. ∎

PROPOSITION 5.8. *There exists a unique b^* such that $M^{(q,r)}(b^*) = 0$.*

Proof. (i) First we note

$$e^{-\Phi(q)b} \int_{-\infty}^b |f'(y)| H^{(q+r)}(b-y,\Phi(q)) \mathrm{d}y = \int_{-\infty}^0 e^{-\Phi(q)b} |f'(y+b)| H^{(q+r)}(-y,\Phi(q)) \mathrm{d}y. \tag{5.10}$$

Because f' is nondecreasing and also of polynomial growth,

$$f'((y+b)+)^+ \leq \sum_{0 \leq m \leq N} C_m |y|^m b^{N-m} + K, \quad y \in \mathbb{R},$$

for some $N \in \mathbb{N}$ and $C_m, K > 0$; similar bounds can be obtained for $f'((y+b)+)^-$. Because $b^k e^{-\Phi(q)b}$ is bounded in $b > 0$ for each $k \geq 0$, we see that $e^{-\Phi(q)b} |f'((y+b)+)|$

is bounded by a polynomial of y (independent of b). This together with Lemma B.1 allows us to apply dominated convergence, and hence (5.10) vanishes as $b \to \infty$. Therefore, in view of (5.5), we obtain that

$$\lim_{b\to\infty} e^{-\Phi(q)b} M^{(q,r)}(b) = 0. \tag{5.11}$$

(ii) By Lemma 5.7 and Assumption 2.2(i), $b \mapsto l(b) := e^{\Phi(q)b}(e^{-\Phi(q)b}M^{(q,r)}(b))'$ is nonincreasing. In addition, monotone convergence and Assumption 2.2(ii) give

(5.12)
$$\lim_{b\downarrow-\infty} l(b) = -\frac{\Phi(q+r)}{q+r}\Big[Cq + f'(-\infty)\Big] > 0, \quad \lim_{b\uparrow\infty} l(b) = -\frac{\Phi(q+r)}{q+r}\Big[Cq + f'(\infty)\Big] < 0.$$

By the positivity of $\exp(\Phi(q)b)$, there exists $\bar{b} \in \mathbb{R}$ such that $(e^{-\Phi(q)b}M^{(q,r)}(b))' \geq 0$ a.e. on $(-\infty, \bar{b})$ and $(e^{-\Phi(q)b}M^{(q,r)}(b))' \leq 0$ a.e. on $(\bar{b}, \infty)$; equivalently $b \mapsto e^{-\Phi(q)b}M^{(q,r)}(b)$ is nondecreasing (resp., nonincreasing) on $(-\infty, \bar{b})$ (resp., $(\bar{b}, \infty)$). By this and (5.11), there exists $-\infty < b^* \leq \bar{b}$ such that $e^{-\Phi(q)b}M^{(q,r)}(b)$ (and hence $M^{(q,r)}(b)$ as well) is nonpositive on $(-\infty, b^*)$ and nonnegative on (b^*, ∞). By the continuity of $M^{(q,r)}(b)$, we must have $M^{(q,r)}(b^*) = 0$.

(iii) To conclude, we show the uniqueness of b^*. Because $b^* \leq \bar{b}$, (by the definition of $\bar{b}$) we must have $(e^{-\Phi(q)b}M^{(q,r)}(b))'|_{b=b^*+} \geq 0$. Hence it suffices to show that $(e^{-\Phi(q)b}M^{(q,r)}(b))'|_{b=b^*+} \neq 0$ (equivalently $l(b^*+) \neq 0$). Suppose $l(b^*+) = 0$. Then, because l is nonincreasing on (b^*, ∞), $l(b) \leq 0$ a.e. on (b^*, ∞) and hence $e^{-\Phi(q)b}M^{(q,r)}(b) \leq 0$ for $b \in [b^*, \infty)$. Because this is also nonnegative by how b^* was chosen, $e^{-\Phi(q)b}M^{(q,r)}(b) = 0$ uniformly on $[b^*, \infty)$, implying $(e^{-\Phi(q)b}M^{(q,r)}(b))' = 0$ a.e. on (b^*, ∞), or equivalently, by Lemma 5.7, $Cq + \mathbb{E}\left[f'(\underline{X}(\mathbf{e}_{q+r}) + b)\right] = 0$ for a.e. (b^*, ∞), which contradicts (5.12). ∎

Remark 5.9. Using identity (5.7) in (5.5) leads to

$$\begin{aligned}\frac{q+r}{\Phi(q+r)} M^{(q,r)}(b) &= \frac{Cq}{\Phi(q)} + \frac{q+r}{r}\frac{(\Phi(q+r)-\Phi(q))}{\Phi(q+r)} \\ &\quad \times \Big[-f(b) + \Phi(q)\int_b^\infty f(y)e^{-\Phi(q)(y-b)}\mathrm{d}y + \int_{-\infty}^b f'(y)H^{(q+r)}(b-y,\Phi(q))\mathrm{d}y\Big].\end{aligned}$$

Because monotone convergence and the expression (4.7) give

$$\lim_{r\to\infty}\int_{-\infty}^b |f'(y)|H^{(q+r)}(b-y,\Phi(q))\mathrm{d}y = 0$$

and $\lim_{r\to\infty}\Phi(q+r) = \infty$, we have

$$\lim_{r\to\infty}\frac{q+r}{\Phi(q+r)}M^{(q,r)}(b) = \tilde{M}^{(q)}(b) := \Phi(q)\int_b^\infty f(y)e^{-\Phi(q)(y-b)}\mathrm{d}y + \frac{Cq}{\Phi(q)} - f(b).$$

This is consistent with [21], where the optimal barrier for the classical case is the root of $\tilde{M}^{(q)}(b) = 0$.

6. Proof of optimality. With $b^* \in \mathbb{R}$ selected in the previous section, we will prove that our candidate value function v_{b^*} satisfies the conditions required in Lemma 3.1 and hence that the strategy π^{b^*} is optimal.

We first confirm the desired smoothness for v_{b^*}; we defer the proof to Appendix C.4.

LEMMA 6.1. *The function v_{b^*} is sufficiently smooth on $\mathbb{R}$.*

Now in order to verify the equality (3.2), we prove the following.

LEMMA 6.2. *The function v_{b^*} is convex, and $v'_{b^*}(b^*) = -C$.*

Proof. (i) By Assumption 2.2(i), f' is increasing Lebesgue-a.e. Hence, using Lemma 5.5, together with the monotonicity of $U_r^{b^*}$ in the starting point as in (4.20), we obtain, for $x < y$,

$$v'_{b^*}(x) = \mathbb{E}_x\Big[\int_0^\infty e^{-qt} f'(U_r^{b^*}(t))\mathrm{d}t\Big] \le \mathbb{E}_y\Big[\int_0^\infty e^{-qt} f'(U_r^{b^*}(t))\mathrm{d}t\Big] = v'_{b^*}(y).$$

Therefore, v_{b^*} is convex.

(ii) By how b^* is chosen so that $M^{(q,r)}(b^*) = 0$ and (5.9), $v'_{b^*}(b^*) = -C$. ∎

Next, by an application of Lemma 6.2, the following result is immediate.

PROPOSITION 6.3. *For $x \in \mathbb{R}$, we have*

$$\mathcal{M}v_{b^*}(x) - v_{b^*}(x) = \begin{cases} C(b^* - x) + v_{b^*}(b^*) - v_{b^*}(x) & \text{if } x \in (-\infty, b^*), \\ 0 & \text{if } x \in [b^*, \infty). \end{cases} \tag{6.1}$$

Now we show the following auxiliary result.

PROPOSITION 6.4. (i) *For $x < b^*$, we have*

$$\begin{aligned}(\mathcal{L} - q)v_{b^*}(x) + f(x) = &-\frac{qr}{q+r}\Big(F(b^*)\Big(1 - e^{\Phi(q+r)(x-b^*)}\Big) + C(b^* - x)\Big) \\ &+ r\int_{-\infty}^{b^*} f(y)\Theta^{(q+r)}(x - b^*, y - b^*)\mathrm{d}y.\end{aligned}$$

(ii) *For $x \ge b^*$, we have $(\mathcal{L} - q)v_{b^*}(x) + f(x) = 0$.*

Proof. (i) Suppose $x < b^*$. Direct computation gives $(\mathcal{L}-(q+r))e^{\Phi(q+r)(x-b^*)} = 0$, and hence

$$(\mathcal{L} - q)\Big(r + qe^{\Phi(q+r)(x-b^*)}\Big) = qr(e^{\Phi(q+r)(x-b^*)} - 1).$$

Let us define, for fixed $z \le b^*$,

$$G^{(q+r)}(z) := \mathbb{E}_z\Big[\int_0^{\tau_{b^*}^+} e^{-(q+r)t} f(X(t))\mathrm{d}t\Big] = \int_{-\infty}^{b^*} f(y)\Theta^{(q+r)}(z - b^*, y - b^*)\mathrm{d}y, \tag{6.2}$$

where the last equality holds by (4.8) and is well-defined and finite for all $z \le b^*$ by Remark 2.1. With $T_{(-N,b^*)} := \inf\{t > 0 : X(t) \notin [-N, b^*]\}$ for $-N < x$, define the processes

$$\begin{aligned} I(t) &:= e^{-(q+r)(t\wedge T_{(-N,b^*)})} G^{(q+r)}(X(t \wedge T_{(-N,b^*)})) \\ &\quad + \int_0^{t\wedge T_{(-N,b^*)}} e^{-(q+r)s} f(X(s))\mathrm{d}s, \quad t \ge 0, \\ I(\infty) &:= \lim_{t\to\infty} I(t) \\ &= e^{-(q+r)T_{(-N,b^*)}} G^{(q+r)}(X(T_{(-N,b^*)})) + \int_0^{T_{(-N,b^*)}} e^{-(q+r)s} f(X(s))\mathrm{d}s. \end{aligned}$$

Note by the strong Markov property that $G^{(q+r)}(x) = \mathbb{E}_x[I(\infty)]$.

With $(\mathcal{G}(t); t \geq 0)$ being the natural filtration of X, we define the $\mathbb{P}_x$-martingale: $\tilde{I}(t) := \mathbb{E}_x[I(\infty)|\mathcal{G}(t)]$, $t \geq 0$. For $x < b^*$ and $t > 0$, by the strong Markov property of X and because, on $\{t \geq T_{(-N,b^*)}\}$, $I(t) = I(\infty) = \tilde{I}(t)$, we can write

$$\tilde{I}(t) = 1_{\{t<T_{(-N,b^*)}\}}\Big\{e^{-(q+r)t}\mathbb{E}_{X(t)}[I(\infty)] + \int_0^t e^{-(q+r)s} f(X(s))\mathrm{d}s\Big\} + 1_{\{t \geq T_{(-N,b^*)}\}} I(t).$$

On the other hand, because $\mathbb{P}_x$-a.s.,

$$1_{\{t<T_{(-N,b^*)}\}} I(t) = 1_{\{t<T_{(-N,b^*)}\}}\Big\{e^{-(q+r)t}\mathbb{E}_{X(t)}[I(\infty)] + \int_0^t e^{-(q+r)s} f(X(s))\mathrm{d}s\Big\},$$

we have that $I = \tilde{I}$, meaning it is a $\mathbb{P}_x$-martingale.

By Lemma 6.1 together with the expressions (4.18) and (6.2), we have that $G^{(q+r)}$ is sufficiently smooth. Therefore, using this martingale property and Itô's formula we conclude that $(\mathcal{L} - q - r)G^{(q+r)}(x) = -f(x)$, or equivalently, using the last equality of (6.2),

$$(\mathcal{L} - q)\int_{-\infty}^{b^*} f(y)\Theta^{(q+r)}(x - b^*, y - b^*)\mathrm{d}y + f(x) = r\int_{-\infty}^{b^*} f(y)\Theta^{(q+r)}(x - b^*, y - b^*)\mathrm{d}y.$$

Finally, direct computation gives $(\mathcal{L} - q)\left(b^* - x - \frac{\kappa'(0+)}{q}\right) = -q(b^* - x)$. Hence putting the pieces together, we complete the proof for the case $x < b^*$.

(ii) Fix $x > b^*$. Similarly to I defined above, the process

$$e^{-q(t\wedge T_{(b^*,N)})} v_{b^*}(X(t \wedge T_{(b^*,N)})) + \int_0^{t\wedge T_{(b^*,N)}} e^{-qs} f(X(s))\mathrm{d}s, \qquad t \geq 0,$$

where $T_{(b^*,N)} := \inf\{t > 0 : X(t) \notin [b^*, N]\}$ with $N > x$, is a $\mathbb{P}_x$-martingale. Hence using the martingale property and Itô's formula (which we can use thanks to the fact that v_{b^*} is sufficiently smooth as in Lemma 6.1), we conclude that $(\mathcal{L} - q)v_{b^*}(x) + f(x) = 0$, as desired.

For the case $x = b^*$, because v_{b^*} is sufficiently smooth, we obtain the result upon taking $x \to b^*$. ∎

Now we are ready to show the main result of the paper.

THEOREM 6.5. *The policy* π^{b^*} *is optimal and the value function is given by* $v(x) = v_{b^*}(x)$ *for all* $x \in \mathbb{R}$.

Proof. In view of Lemma 6.1, it is sufficient to verify (3.2).

(i) Suppose $x < b^*$. Using Proposition 6.3 and (4.18), we have

$$\mathcal{M}v_{b^*}(x) - v_{b^*}(x) = \frac{q}{q + r}\left(C(b^* - x) + F(b^*)\left(1 - e^{\Phi(q+r)(x-b^*)}\right)\right) - \int_{-\infty}^{b^*} f(y)\Theta^{(q+r)}(x - b^*, y - b^*)\mathrm{d}y.$$

Hence using this and Proposition 6.4(i), we deduce (3.2) for $x < b^*$. (ii) For the case $x \geq b^*$, using Proposition 6.4(ii) and (6.1), we have (3.2) as well. ∎

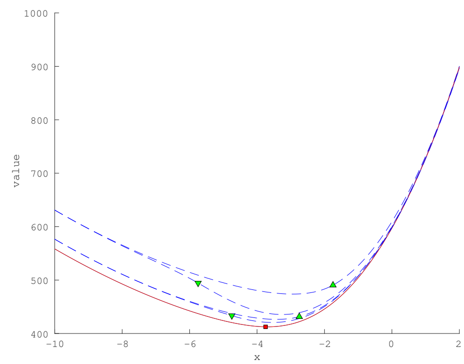

Fig. 1. *Plots of v_{b^*} (solid) in comparison to v_b for $b = b^* - 2, b^* - 1, b^* + 1, b^* + 2$ (dotted). The point $(b^*, v_{b^*}(b^*))$ is indicated by a square, while the points $(b, v_b(b))$ are indicated by down- and up-pointing triangles for $b < b^*$ and $b > b^*$, respectively.*

Remark 6.6. A natural extension of the considered problem is to allow additional *fixed ordering costs* incurred each time an order is made. In this case, a "periodic (s, S)-policy" is expected to be optimal. This policy replenishes the item up to the inventory level S at each observation time $\mathcal{T}_r$ whenever it is below the level s. This is an interesting and challenging problem and we leave it for further work.

6.1. Numerical examples. We now confirm numerically the obtained results using the quadratic inventory cost $f(x) = x^2$. In this case, a straightforward computation gives $b^* = \Phi(q + r)^{-1} - \Phi(q)^{-1} - \kappa'(0+)/(q + r) - qC/2$. We assume that $X(t) = X(0) + t + 0.2B(t) - \sum_{n=1}^{N(t)} Z_n$, for $0 \le t < \infty$. Here, B is a standard Brownian motion, N is a Poisson process with arrival rate 1, and $\{Z_n\}_{n\ge 1}$ is a sequence of independent and identically distributed phase-type random variables (whose parameters are given in [20]) approximating the Weibull distribution with shape and scale parameters 2 and 1, respectively. The corresponding scale function admits a closed form expression as in [8]. We set $q = 0.05$, $r = 0.5$, and $C = 1$, unless stated otherwise.

In Figure 1, we plot $x \mapsto v_b(x)$ for $b = b^*$ and for $b \neq b^*$ along with the points $(b, v_b(b))$. It is confirmed that v_{b^*} is indeed convex (as in Lemma 6.2) and minimizes over b uniformly in x.

In Figure 2, we show v_{b^*} for various values of the unit replenishment cost/reward C and the rate of Poisson arrivals r, along with those in the continuous monitoring case [21]. For the former, as C increases, the value function v_{b^*} increases (uniformly in x) while b^* decreases. On the other hand, as r increases, both v_{b^*} and b^* decrease. As $r \to \infty$, the convergence to the case [21] is also confirmed.

Appendix A. Proof of Theorem 4.5. Recall as in Corollaries 8.7 and 8.8 of [13] that for any Borel set A on $[0, \infty)$ and on $\mathbb{R}$, respectively,

$$\text{(A.1)}\quad \mathbb{E}_x\Big[\int_0^{\tau_0^-} e^{-qt} 1_{\{X(t)\in A\}} \mathrm{d}t\Big] = \int_A \Big[e^{-\Phi(q)y} W^{(q)}(x) - W^{(q)}(x - y)\Big] \mathrm{d}y, \quad x \ge 0,$$

(A.2)

$$\mathbb{E}_x\left[\int_0^{\infty} e^{-(q+r)t} 1_{\{X(t)\in A\}} \mathrm{d}t\right] = \int_A \left[\frac{e^{\Phi(q+r)(x-y)}}{\kappa'(\Phi(q + r))} - W^{(q+r)}(x - y)\right] \mathrm{d}y, \quad x \in \mathbb{R}.$$

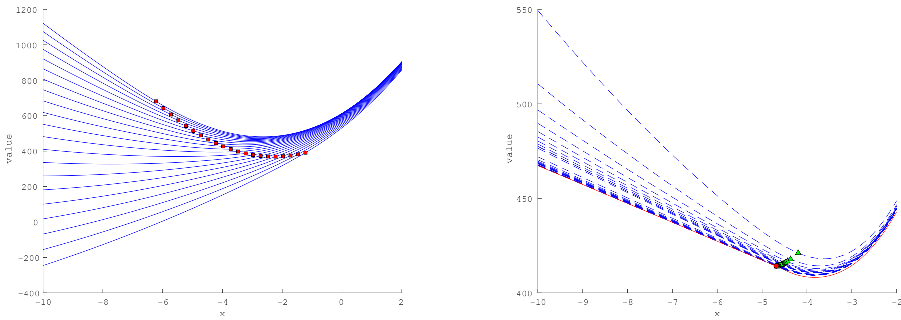

FIG. 2. *(Left) Plots of v_{b^*} for $C = -100, -90, \ldots, 90, 100$ with $(b^*, v_{b^*}(b^*))$ indicated by squares. (Right) Plots of v_{b^*} (dotted) for $r = 0.1, 0.2, \ldots, 0.9, 1, 2, \ldots, 9, 10, 20, \ldots, 90, 100, 200, \ldots, 900, 1000$ with $(b^*, v_{b^*}(b^*))$ indicated by triangles, along with the continuous monitoring case (solid) with the point at the optimal barrier indicated by a square.*

A.1. Proof of Theorem 4.5. For $x \in \mathbb{R}$, let us denote the left-hand side of (4.15) by $g_b(x)$ and in particular $g(x) := g_0(x)$. We will prove the result for $b = 0$; the general case follows because the spatial homogeneity of the Lévy process implies that $g_b(x) = \mathbb{E}_{x-b}\left[\int_0^\infty e^{-qt} h(U_r^0(t) + b)\mathrm{d}t\right]$.

(i) For $x \in \mathbb{R}$, by the strong Markov property,

$$g(x) = \mathbb{E}_x\Big[\int_0^{\tau_0^-} e^{-qt} h(X(t))\mathrm{d}t\Big] + \mathbb{E}_x\left[e^{-q\tau_0^-} g(X(\tau_0^-))1_{\{\tau_0^- < \infty\}}\right]. \tag{A.3}$$

In particular, for $x < 0$, again by the strong Markov property and because $U_r^0 = X$ on $[0, T(1) \wedge \tau_0^+)$,

$$g(x) = A(x)g(0) + B(x),$$

where, for $x \le 0$,

$$A(x) := \mathbb{E}_x\left[e^{-q(\tau_0^+ \wedge T(1))}\right] = \frac{r}{q+r} + \frac{q}{q+r} e^{\Phi(q+r)x},$$

$$B(x) := \mathbb{E}_x\Big[\int_0^{\tau_0^+} e^{-qt} 1_{\{t < T(1)\}} h(X(t))\mathrm{d}t\Big] = \int_{-\infty}^0 h(y)\Theta^{(q+r)}(x, y)\mathrm{d}y.$$

Here, the second equality of the former holds by the fact that $T(1)$ is an independent exponential random variable with parameter r and Theorem 3.12 of [13]. The second equality of the latter holds by (4.8).

Now applying identity (3.19) in [3],

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-q\tau_0^-} A(X(\tau_0^-))1_{\{\tau_0^- < \infty\}}\right] \\
&\quad = \frac{r}{q+r}\left(Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x)\right) + \frac{q}{q+r}\left(Z^{(q)}(x, \Phi(q+r)) - \frac{rW^{(q)}(x)}{\Phi(q+r) - \Phi(q)}\right) \\
&\quad = Z^{(q,r)}(x) - \frac{qr}{q+r}\frac{\Phi(q+r)}{\Phi(q)(\Phi(q+r) - \Phi(q))} W^{(q)}(x).
\end{aligned}$$

In addition, using identity (5) in [1] together with Lemma 2.1 in [15], we obtain for

$c > x$,

$$\mathbb{E}_x\left[e^{-q\tau_0^-}B(X(\tau_0^-))1_{\{\tau_0^-<\tau_c^+\}}\right] = \int_{-\infty}^0 h(y)\mathbb{E}_x\left[e^{-q\tau_0^-}\Theta^{(q+r)}(X(\tau_0^-),y)1_{\{\tau_0^-<\tau_c^+\}}\right]\mathrm{d}y$$

$$= -\int_{-\infty}^0 h(y)\Upsilon(x,y)\mathrm{d}y + \frac{W^{(q)}(x)}{W^{(q)}(c)}\int_{-\infty}^0 h(y)\Upsilon(c,y)\mathrm{d}y. \tag{A.4}$$

By (4.4) and Remark 4.2(3), for $y \in \mathbb{R}$, $\lim_{x\to\infty} W_y^{(q,r)}(x)/W^{(q)}(x) = Z^{(q+r)}(-y,\Phi(q))$. Also following the proof of Corollary 3.2(iii) in [4] we have $\lim_{x\to\infty} Z^{(q)}(x,\Phi(q+r))/W^{(q)}(x) = r/(\Phi(q+r)-\Phi(q))$. Hence by taking $c \uparrow \infty$ in (A.4) and using these limits, we get

$$\mathbb{E}_x\left[e^{-q\tau_0^-}B(X(\tau_0^-))1_{\{\tau_0^-<\infty\}}\right]$$
$$= -\int_{-\infty}^0 h(y)\Upsilon(x,y)\mathrm{d}y + W^{(q)}(x)\int_{-\infty}^0 h(y)H^{(q+r)}(-y,\Phi(q))\mathrm{d}y.$$

Substituting these in (A.3) and by (A.1) and Remark 4.3,

$$g(x) = g(0)\left\{Z^{(q,r)}(x) - \frac{qr}{q+r}\frac{\Phi(q+r)}{\Phi(q)(\Phi(q+r)-\Phi(q))}W^{(q)}(x)\right\} + W^{(q)}(x)\int_{-\infty}^{\infty} h(y)H^{(q+r)}(-y,\Phi(q))\mathrm{d}y - \int_{-\infty}^{\infty} h(y)\Upsilon(x,y)\mathrm{d}y. \tag{A.5}$$

(ii) On the other hand, by the strong Markov property, we can also write

$$g(0) = \gamma_1 + \gamma_2 g(0) + \gamma_3, \tag{A.6}$$

where

$$\gamma_1 := \mathbb{E}\Big[\int_0^{T(1)} e^{-qt}h(X(t))\mathrm{d}t\Big],$$
$$\gamma_2 := \mathbb{E}\Big[e^{-qT(1)}1_{\{X(T(1))\le 0\}}\Big],$$
$$\gamma_3 := \mathbb{E}\Big[e^{-qT(1)}g(X(T(1))1_{\{X(T(1))>0\}}\Big],$$

whose values are to be computed below.

(1) We get $\gamma_1 = \mathbb{E}\left[\int_0^\infty 1_{\{t<T(1)\}}e^{-qt}h(X(t))\mathrm{d}t\right] = \mathbb{E}[\int_0^\infty e^{-(q+r)t}h(X(t))\mathrm{d}t]$.

(2) Using (A.2), we obtain

$$\gamma_2 = r\Big(\frac{1}{q+r} - \mathbb{E}\left[\int_0^\infty e^{-(q+r)s}1_{\{X(s)\ge 0\}}\mathrm{d}s\right]\Big) = r\Big(\frac{1}{q+r} - \frac{1}{\kappa'(\Phi(q+r))}\frac{1}{\Phi(q+r)}\Big).$$

(3) Again by (A.2),

$$\gamma_3 = r\mathbb{E}\Big[\int_0^\infty e^{-(q+r)s}g(X(s))1_{\{X(s)>0\}}\mathrm{d}s\Big] = \frac{r}{\kappa'(\Phi(q+r))}\int_0^\infty e^{-\Phi(q+r)y}g(y)\mathrm{d}y, \tag{A.7}$$

which we shall compute using the expression of g as in (A.5). First, by integration by parts,

$$\int_0^\infty e^{-\Phi(q+r)y} Z^{(q)}(y)\mathrm{d}y = \frac{1}{\Phi(q+r)}\left(1 + q\int_0^\infty e^{-\Phi(q+r)u} W^{(q)}(u)\mathrm{d}u\right)$$
$$= \frac{1}{\Phi(q+r)}\frac{q+r}{r}.$$

Because (4.2) and (4.4) give $e^{-\Phi(q+r)y} Z^{(q)}(y, \Phi(q+r)) = r\int_y^\infty e^{-\Phi(q+r)z} W^{(q)}(z)\mathrm{d}z$,

(A.8)
$$\int_0^\infty e^{-\Phi(q+r)y} Z^{(q)}(y, \Phi(q+r))\mathrm{d}y = r\int_0^\infty \int_y^\infty e^{-\Phi(q+r)z} W^{(q)}(z)\mathrm{d}z\mathrm{d}y$$
$$= r\int_0^\infty z e^{-\Phi(q+r)z} W^{(q)}(z)\mathrm{d}z = \frac{\kappa'(\Phi(q+r))}{r},$$

where the second equality holds by the change of variables and the last holds because monotone convergence and (4.2) give that

$$\int_0^\infty z e^{-\theta z} W^{(q)}(z)\mathrm{d}z = -\frac{\partial}{\partial\theta}\int_0^\infty e^{-\theta z} W^{(q)}(z)\mathrm{d}z.$$

LEMMA A.1. *For $y \in \mathbb{R}$,*

$$\int_0^\infty e^{-\Phi(q+r)x}\Upsilon(x,y)\mathrm{d}x = [e^{-\Phi(q+r)y} - W^{(q+r)}(-y)\kappa'(\Phi(q+r))]/r.$$

Proof. We have for $\theta > \Phi(q+r)$, by the convolution theorem,

$$\int_0^\infty e^{-\theta x} W_y^{(q,r)}(x)\mathrm{d}x = \Big(\int_0^\infty e^{-\theta x} W^{(q+r)}(x-y)\mathrm{d}x\Big)\Big(1 - r\int_0^\infty e^{-\theta x} W^{(q)}(x)\mathrm{d}x\Big)$$
$$= \Big(\frac{e^{-\theta y}}{\kappa(\theta)-q-r} - \int_{y\wedge 0}^0 e^{-\theta x} W^{(q+r)}(x-y)\mathrm{d}x\Big)\frac{\kappa(\theta)-q-r}{\kappa(\theta)-q} \xrightarrow{\theta\downarrow\Phi(q+r)} \frac{e^{-\Phi(q+r)y}}{r}.$$

This together with (A.8) completes the proof. ∎

By this lemma, Fubini's theorem, and (A.2),

$$\int_0^\infty e^{-\Phi(q+r)y}\int_{-\infty}^\infty h(z)\Upsilon(y,z)\mathrm{d}z\mathrm{d}y = \frac{\kappa'(\Phi(q+r))}{r}\mathbb{E}\left[\int_0^\infty e^{-(q+r)t} h(X(t))\mathrm{d}t\right].$$

Substituting these and with the help of (A.5) in (A.7),

$$\gamma_3 = g(0)\frac{r}{\kappa'(\Phi(q+r))}\Big[\frac{1}{\Phi(q+r)} + \frac{q}{q+r}\frac{\kappa'(\Phi(q+r))}{r} - \frac{q}{q+r}\frac{\Phi(q+r)}{\Phi(q)(\Phi(q+r)-\Phi(q))}\Big]$$
$$- \mathbb{E}\left[\int_0^\infty e^{-(q+r)t} h(X(t))\mathrm{d}t\right] + \frac{1}{\kappa'(\Phi(q+r))}\int_{-\infty}^\infty h(y) H^{(q+r)}(-y, \Phi(q))\mathrm{d}y.$$

Now substituting the computed values of γ_1, γ_2, and γ_3 in (A.6) and after simplification, we have

$$g(0) = g(0) - \frac{rq}{q+r}\frac{\Phi(q+r)}{\Phi(q)(\Phi(q+r)-\Phi(q))}\frac{g(0)}{\kappa'(\Phi(q+r))}$$
$$+ \frac{1}{\kappa'(\Phi(q+r))}\int_{-\infty}^\infty h(y) H^{(q+r)}(-y, \Phi(q))\mathrm{d}y,$$

and hence, solving for $g(0)$ we obtain

$$g(0) = \frac{q+r}{qr}\frac{\Phi(q)(\Phi(q+r)-\Phi(q))}{\Phi(q+r)}\int_{-\infty}^{\infty} h(y)H^{(q+r)}(-y,\Phi(q))\mathrm{d}y.$$

Substituting this back in (A.5), we have (4.15) for $b = 0$, as desired.

Appendix B. Integrability results.

LEMMA B.1. *Consider $g : \mathbb{R} \to \mathbb{R}$ that satisfies Assumption* 2.2(i). *Then, for any $b \in \mathbb{R}$ and $\theta \geq 0$, we have $\int_{-\infty}^{b} |g(y)|H^{(q+r)}(b-y,\theta)\mathrm{d}y < \infty$.*

Proof. By identity (4.6),

$$\int_{-\infty}^{b} |g(y)|H^{(q+r)}(b-y,\theta)\mathrm{d}y = \int_{-\infty}^{b} |g(y)|\mathbb{E}_{b-y}\left[e^{-(q+r)\tau_0^- + \theta X(\tau_0^-)}1_{\{\tau_0^-<\infty\}}\right]\mathrm{d}y$$
$$\leq \int_{-\infty}^{b} |g(y)|\mathbb{P}(-\underline{X}(\mathbf{e}_{q+r}) > b-y)\mathrm{d}y = \int_{-\infty}^{b} |g(y)|\int_{b-y}^{\infty}\mathbb{P}(-\underline{X}(\mathbf{e}_{q+r}) \in \mathrm{d}z)\mathrm{d}y$$
$$= \int_{0}^{\infty} |g(b-u)|\int_{u}^{\infty}\mathbb{P}(-\underline{X}(\mathbf{e}_{q+r}) \in \mathrm{d}z)\mathrm{d}u = \int_{0}^{\infty}\mathbb{P}(-\underline{X}(\mathbf{e}_{q+r}) \in \mathrm{d}z)\int_{0}^{z}|g(b-u)|\mathrm{d}u.$$

Here, as in (3.11) of [11] (using Assumption 2.1), we have $\mathbb{E}[e^{-\theta\underline{X}(\mathbf{e}_{q+r})}] < \infty$ for $0 < \theta < \bar{\theta}$. This together with the polynomial growth of g as in Assumption 2.2(i) implies that the above is finite. ∎

LEMMA B.2. *Fix any $b \in \mathbb{R}$.* (i) *For any $x \geq b$, $\sup_{z\in[0,x-b]}\int_{-\infty}^{b}|f(y)||\Theta^{(q+r)}(z,y-b)|\mathrm{d}y < \infty$.* (ii) *For any $x \in \mathbb{R}$, $\int_{-\infty}^{b}|f(y)||\Upsilon(x-b,y-b)|\mathrm{d}y < \infty$.*

Proof. (i) Recall Remark 4.1. For $z \in [0, x-b]$, because $b \leq z+b$, by (4.8) and following similar arguments as in (4.10),

$$\int_{-\infty}^{b}|f(y)||\Theta^{(q+r)}(z,y-b)|\mathrm{d}y \leq \int_{-\infty}^{z+b}|f(y)||\Theta^{(q+r)}(z,y-b)|\mathrm{d}y$$
$$= e^{\Phi(q+r)z}\mathbb{E}_b\Big[\int_0^{\tau_{z+b}^+} e^{-(q+r)t}|f(X(t))|\mathrm{d}t\Big] \leq e^{\Phi(q+r)(x-b)}\mathbb{E}_b\Big[\int_0^{\tau_x^+} e^{-(q+r)t}|f(X(t))|\mathrm{d}t\Big],$$

and hence we have the result by Remark 2.1.

(ii) Fix $x < b$. Then by Remark 4.1(i) and (4.13), we have for $y < b$ that $|\Upsilon(x-b,y-b)| = \Theta^{(q+r)}(x-b,y-b)$. Hence

$$\int_{-\infty}^{b}|f(y)||\Upsilon(x-b,y-b)|\mathrm{d}y = \int_{-\infty}^{b}|f(y)|\Theta^{(q+r)}(x-b,y-b)\mathrm{d}y$$
$$= \mathbb{E}_x\Big[\int_0^{\tau_b^+} e^{-(q+r)t}|f(X(t))|\mathrm{d}t\Big],$$

which is finite by Remark 2.1.

On the other hand, for $x \geq b$, we note that, by an application of Fubini's theorem and (i),

$$\begin{aligned}
&\int_{-\infty}^{b} |f(y)| \int_0^{x-b} W^{(q)}(x-b-z)|\Theta^{(q+r)}(z, y-b)|\mathrm{d}z\mathrm{d}y \\
&= \int_0^{x-b} W^{(q)}(x-b-z) \int_{-\infty}^{b} |f(y)||\Theta^{(q+r)}(z, y-b)|\mathrm{d}y\mathrm{d}z \\
&\leq \overline{W}^{(q)}(x-b) \sup_{z\in[0,x-b]} \int_{-\infty}^{b} |f(y)||\Theta^{(q+r)}(z, y-b)|\mathrm{d}y < \infty.
\end{aligned}$$

In view of the form of Υ as in (4.13) and (i), the proof is complete. ∎

Appendix C. Other proofs.

C.1. Proof of Lemma 5.1. For $y < b$, because

$$\frac{\partial}{\partial u} W_{u-b}^{(q,r)}(x-b)\Big|_{u=y-} = -W^{(q+r)\prime}((x-y)+) + r\int_b^x W^{(q)}(x-z)W^{(q+r)\prime}(z-y)\mathrm{d}z,$$

we have that $-\frac{\partial}{\partial z}\Psi(x-b, z)|_{z=(y-b)-}$ reduces to the right-hand side of (5.2).

On the other hand, we obtain by integration by parts

$$\begin{aligned}
\frac{\partial}{\partial z} W_{y-b}^{(q,r)}(z)\Big|_{z=(x-b)+} &= W^{(q+r)\prime}((x-y)+) - rW^{(q)}(x-b)W^{(q+r)}(b-y) \\
&\quad - r\int_b^x W^{(q)}(x-z)W^{(q+r)\prime}(z-y)\mathrm{d}z. \qquad \text{(C.1)}
\end{aligned}$$

Using $\frac{\partial}{\partial z} Z^{(q)}(z, \Phi(q+r)) = \Phi(q+r)Z^{(q)}(z, \Phi(q+r)) - rW^{(q)}(z)$ and (C.1) in (4.14), we have that $\frac{\partial}{\partial z}\Upsilon(z, y-b)|_{z=(x-b)+}$ equals the right-hand side of (5.2).

C.2. Proof of Lemma 5.3. (i) Fix $y < b \wedge x$ and $\varepsilon > 0$. With $W_{\Phi(q+r)}$ defined as in Remark 4.2(3),

$$\begin{aligned}
&\frac{\Theta^{(q+r)}(x-b+\varepsilon, y-b) - \Theta^{(q+r)}(x-b, y-b)}{\varepsilon} = \frac{e^{\Phi(q+r)\varepsilon} - 1}{\varepsilon}\Theta^{(q+r)}(x-b, y-b) \\
&\quad - e^{\Phi(q+r)(x+\varepsilon-y)}\left(\frac{W_{\Phi(q+r)}(x+\varepsilon-y) - W_{\Phi(q+r)}(x-y)}{\varepsilon}\right). \qquad \text{(C.2)}
\end{aligned}$$

Here we note that $\varepsilon \mapsto (e^{\Phi(q+r)\varepsilon} - 1)/\varepsilon$ is bounded in compact sets on $(0, \infty)$ and that $\int_{-\infty}^b \big|f(y)||\Theta^{(q+r)}(x-b, y-b)\big|\mathrm{d}y < \infty$ by Lemma B.2(i).

As in Appendix A.1 of [11, p. 1150] we have that

$$y \mapsto |f(y)|e^{-\Phi(q+r)y}\left|\frac{W_{\Phi(q+r)}(u+\varepsilon-y) - W_{\Phi(q+r)}(u-y)}{\varepsilon}\right|$$

is bounded in $\varepsilon > 0$ by a function integrable over $(-\infty, -M)$ for some $-M < b \wedge x$. Therefore, dominated convergence gives

$$\begin{aligned}
&\frac{\partial}{\partial x}\int_{-\infty}^{-M} f(y)\Theta^{(q+r)}(x-b, y-b)\mathrm{d}y \\
&= \lim_{\varepsilon\downarrow 0}\int_{-\infty}^{-M} f(y)\frac{\Theta^{(q+r)}(x-b+\varepsilon, y-b) - \Theta^{(q+r)}(x-b, y-b)}{\varepsilon}\mathrm{d}y \\
&= \int_{-\infty}^{-M} f(y)\frac{\partial}{\partial x}\Theta^{(q+r)}(x-b, y-b)\mathrm{d}y. \qquad \text{(C.3)}
\end{aligned}$$

(ii) Fix $x > b$ and consider the second term of Υ in (4.13). We take $\delta > 0$ small enough so that $x - b - \delta > 0$. By Fubini's theorem

$$\int_{-\infty}^{-M} f(y) \int_0^{x-b} W^{(q)}(x-b-z)\Theta^{(q+r)}(z, y-b)\mathrm{d}z\mathrm{d}y \\ = \int_0^{x-b} W^{(q)}(x-b-z) \int_{-\infty}^{-M} f(y)\Theta^{(q+r)}(z, y-b)\mathrm{d}y\mathrm{d}z.$$

On the other hand, by the mean value theorem and Lemma B.2(i), for $0 < z < x - b - \delta$ and $0 < \varepsilon < \bar{\varepsilon}$,

$$\Big|\frac{W^{(q)}(x-b-z+\varepsilon) - W^{(q)}(x-b-z)}{\varepsilon} \int_{-\infty}^{-M} f(y)\Theta^{(q+r)}(z, y-b)\mathrm{d}y\Big| \\ \le \sup_{u\in[\delta, x-b+\bar{\varepsilon}]} W^{(q)\prime}(u+) \sup_{z\in[0,x-b]} \Big|\int_{-\infty}^{-M} f(y)\Theta^{(q+r)}(z, y-b)\mathrm{d}y\Big| < \infty.$$

This and dominated convergence imply

$$\lim_{\varepsilon\downarrow 0} \int_0^{x-b-\delta} \frac{W^{(q)}(x-b-z+\varepsilon) - W^{(q)}(x-b-z)}{\varepsilon} \int_{-\infty}^{-M} f(y)\Theta^{(q+r)}(z, y-b)\mathrm{d}y\mathrm{d}z \\ = \int_0^{x-b-\delta} W^{(q)\prime}(x-b-z) \int_{-\infty}^{-M} f(y)\Theta^{(q+r)}(z, y-b)\mathrm{d}y\mathrm{d}z.$$

On the other hand,

$$\Big|\int_{x-b-\delta}^{x-b} \frac{W^{(q)}(x-b+\varepsilon-z) - W^{(q)}(x-b-z)}{\varepsilon} \int_{-\infty}^{-M} f(y)\Theta^{(q+r)}(z, y-b)\mathrm{d}y\mathrm{d}z\Big| \\ \le \Big(\sup_{z\in[0,x-b]} \int_{-\infty}^{-M} |f(y)||\Theta^{(q+r)}(z, y-b)|\mathrm{d}y\Big) \\ \times \int_{x-b-\delta}^{x-b} \frac{W^{(q)}(x-b+\varepsilon-u) - W^{(q)}(x-b-u)}{\varepsilon}\mathrm{d}u,$$

which vanishes as $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$ because l'Hopital's rule gives

$$\int_{x-b-\delta}^{x-b} \frac{W^{(q)}(x-b+\varepsilon-z) - W^{(q)}(x-b-z)}{\varepsilon}\mathrm{d}z = \frac{\overline{W}^{(q)}(\varepsilon+\delta) - \overline{W}^{(q)}(\delta) - \overline{W}^{(q)}(\varepsilon)}{\varepsilon} \\ \xrightarrow{\varepsilon\downarrow 0} W^{(q)}(\delta) - W^{(q)}(0).$$

Putting the pieces together we obtain

$$\begin{aligned}A_1 &:= \lim_{\varepsilon\downarrow 0}\int_0^{x-b}\frac{W^{(q)}(x-b+\varepsilon-z)-W^{(q)}(x-b-z)}{\varepsilon}\\ &\qquad\times\int_{-\infty}^{-M} f(y)\Theta^{(q+r)}(z,y-b)\mathrm{d}y\mathrm{d}z\\ &= \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\int_0^{x-b-\delta}\frac{W^{(q)}(x-b+\varepsilon-z)-W^{(q)}(x-b-z)}{\varepsilon}\\ &\qquad\times\int_{-\infty}^{-M} f(y)\Theta^{(q+r)}(z,y-b)\mathrm{d}y\mathrm{d}z\\ &+\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\int_{x-b-\delta}^{x-b}\frac{W^{(q)}(x-b+\varepsilon-z)-W^{(q)}(x-b-z)}{\varepsilon}\\ &\qquad\times\int_{-\infty}^{-M} f(y)\Theta^{(q+r)}(z,y-b)\mathrm{d}y\mathrm{d}z\\ &= \int_0^{x-b} W^{(q)\prime}(x-b-z)\int_{-\infty}^{-M} f(y)\Theta^{(q+r)}(z,y-b)\mathrm{d}y\mathrm{d}z.\end{aligned}$$

On the other hand, by (C.2) the mapping $z\mapsto\int_{-\infty}^{-M} f(y)\Theta^{(q+r)}(z,y-b)\mathrm{d}y$ is continuous, and hence

$$\begin{aligned}A_2 &:= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{x-b}^{x-b+\varepsilon} W^{(q)}(x-b+\varepsilon-z)\int_{-\infty}^{-M} f(y)\Theta^{(q+r)}(z,y-b)\mathrm{d}y\mathrm{d}z\\ &= W^{(q)}(0)\int_{-\infty}^{-M} f(y)\Theta^{(q+r)}(x-b,y-b)\mathrm{d}y\mathrm{d}z.\end{aligned}$$

Therefore,

$$\begin{aligned}\text{(C.4)}\quad \frac{\partial}{\partial x}\int_{-\infty}^{-M} f(y)\int_0^{x-b} W^{(q)}(x-b-z)\Theta^{(q+r)}(z,y-b)\mathrm{d}z\mathrm{d}y &= A_1+A_2\\ &= \int_{-\infty}^{-M} f(y)\frac{\partial}{\partial x}\int_0^{x-b} W^{(q)}(x-b-z)\Theta^{(q+r)}(z,y-b)\mathrm{d}z\mathrm{d}y.\end{aligned}$$

We now conclude the proof by identities (C.3), (C.4), and (4.13).

C.3. Proof of Lemma 5.6. We have

$$\frac{\partial}{\partial z}H^{(q+r)}(z,\theta)\big|_{z=(b-y)+} = \theta H^{(q+r)}(b-y,\theta)-\frac{\kappa(\theta)-(q+r)}{\theta-\Phi(q+r)}\underline{r}^{(q+r)}((b-y)+),$$

where $\underline{r}^{(q+r)}(x) := W^{(q+r)\prime}(x)-\Phi(q+r)W^{(q+r)}(x) > 0$, $x>0$ with $(q+r)\underline{r}^{(q+r)}(x)/\Phi(q+r)$ being the density function of $-\underline{X}(\mathbf{e}_{q+r})$ as in (4.11), and hence

$$\Big|\frac{\partial}{\partial z}H^{(q+r)}(z,\theta)\big|_{z=(b-y)+}\Big| \le \theta H^{(q+r)}(b-y,\theta)+\left|\frac{\kappa(\theta)-(q+r)}{\theta-\Phi(q+r)}\right|\underline{r}^{(q+r)}((b-y)+).$$

Let us suppose that $b\in[b_1,b_2]$ with $b_1>-M$. First, by (4.6), we have that $H^{(q+r)}(b-y,\theta)\le\mathbb{E}_{b_1-y}[e^{-(q+r)\tau_0^-}]$. On the other hand, using the fact that $x\mapsto$

$W^{(q+r)\prime}(x+)/W^{(q+r)}(x)$ is decreasing as in Remark 3.1(3) of [11], the mapping $x \mapsto \underline{r}^{(q+r)}(x+)/W^{(q+r)}(x)$ is also decreasing. Therefore,

$$\begin{aligned}\underline{r}^{(q+r)}((b-y)+) &\leq \frac{W^{(q+r)}(b-y)}{W^{(q+r)}(b_1-y)}\underline{r}^{(q+r)}((b_1-y)+)\\ &\leq \frac{W^{(q+r)}(b_2-y)}{W^{(q+r)}(b_1-y)}\underline{r}^{(q+r)}((b_1-y)+).\end{aligned}$$

Because $W^{(q+r)}(b_2-y)/W^{(q+r)}(b_1-y)$ converges as $y \to -\infty$ by Remark 4.2(3), for $-M$ small enough, there exists a constant $K(b_1, b_2)$ dependent only on b_1, b_2 such that $W^{(q+r)}(b_2-y)/W^{(q+r)}(b_1-y) \leq K(b_1, b_2)$ for all $y \leq -M$. Hence

$$\begin{aligned}&\left|\frac{\partial}{\partial z}H^{(q+r)}(z,\theta)\big|_{z=(b-y)+}\right|\\ &\qquad\leq \theta\mathbb{E}_{b_1-y}\left[e^{-(q+r)\tau_0^-}\right] + \left|\frac{\kappa(\theta)-(q+r)}{\theta-\Phi(q+r)}\right| K(b_1,b_2)\underline{r}^{(q+r)}((b_1-y)+).\end{aligned}$$

Here by Lemma B.1 and the polynomial growth of f as in Assumption 2.2(i),

$$\int_{-\infty}^{-M} |f(y)|\mathbb{E}_{b_1-y}[e^{-(q+r)\tau_0^-}]\mathrm{d}y < \infty.$$

For the second term we have, by the density function of $-\underline{X}(\mathbf{e}_{q+r})$ as in (4.11),

$$\begin{aligned}\int_{-\infty}^{-M} |f(y)|\underline{r}^{(q+r)}(b_1-y)\mathrm{d}y &= \int_{b_1+M}^{\infty} |f(b_1-u)|\underline{r}^{(q+r)}(u)\mathrm{d}u\\ &\leq \frac{\Phi(q+r)}{q+r}\mathbb{E}\left[|f(b_1+\underline{X}(\mathbf{e}_{q+r}))|\right] < \infty,\end{aligned}$$

where the finiteness holds as in the proof of Lemma B.1. Hence, by Corollary 5.9 in [5], the derivative can be interchanged over the integral and the proof is complete.

C.4. Proof of Lemma 6.1. In view of the expression of Lemma 5.5, by monotone convergence (noting that f' is monotone) and (4.22), v_{b^*}' is continuous for all $x \in \mathbb{R}$.

Therefore, it just remains to show that v_{b^*}'' is continuous for the case in which X has paths of unbounded variation, where $W^{(q+r)}(0) = W^{(q)}(0) = 0$ by Remark 4.2(2).

Using the expression of Lemma 5.5 together with Theorem 4.5, we obtain after differentiation that

$$\begin{aligned}v_{b^*}''(x) = \frac{\Phi(q)}{r}&(\Phi(q+r)-\Phi(q))Z^{(q)}(x,\Phi(q+r))\int_{-\infty}^{\infty} f'(y)H^{(q+r)}(b-y,\Phi(q))\mathrm{d}y\\ &-\int_{b^*}^{x} f'(y)W^{(q)\prime}(x-y)\mathrm{d}y - \frac{\partial}{\partial x}\int_{-\infty}^{b^*} f'(y)\Upsilon(x-b^*, y-b^*)\mathrm{d}y.\end{aligned}$$

Because $\Upsilon(x-b^*, y-b^*)$ is continuous for the case of unbounded variation and by Lemma 5.3,

$$\frac{\partial}{\partial x}\int_{-\infty}^{b^*} f'(y)\Upsilon(x-b^*,y-b^*)\mathrm{d}y = \int_{-\infty}^{b^*} f'(y)\frac{\partial}{\partial x}\Upsilon(x-b^*,y-b^*)\mathrm{d}y, \quad x \in \mathbb{R}.$$

Using (5.2), we can write, for $x \neq y$,

$$\frac{\partial}{\partial x}\Upsilon(x-b^*, y-b^*) = A(x,y,b^*) - r\int_{b^*}^{x} W^{(q)}(x-z)A(z,y,b^*)\mathrm{d}z,$$

where

$$A(x,y,b^*) := \Big(W^{(q+r)\prime}(x-y) - \Phi(q+r)W^{(q+r)}(x-y)\Big) - \Phi(q+r)\Theta^{(q+r)}(x-b^*, y-b^*).$$

Because $A(x,y,b^*) = W^{(q)\prime}(x-y)$ for $y > b^*$,

$$\begin{aligned} K(x,b^*) &:= \int_{b^*}^{x} f'(y)W^{(q)\prime}(x-y)\mathrm{d}y + \int_{-\infty}^{b^*} f'(y)A(x,y,b^*)\mathrm{d}y \\ &= \int_{-\infty}^{b^*\vee x} f'(y)A(x,y,b^*)\mathrm{d}y. \end{aligned}$$

For $x \leq b^*$, recalling that $W^{(q+r)}(0) = 0$ as in Remark 4.2(2) for the case of unbounded variation,

$$K(x,b^*) = \frac{\Phi(q+r)}{q+r}\mathbb{E}\left[f'(\underline{X}(\mathbf{e}_{q+r})+x)\right] - \Phi(q+r)\mathbb{E}_x\Big[\int_0^{\tau_{b^*}^+} e^{-(q+r)t}f'(X(t))\mathrm{d}t\Big].$$

Similarly, for $x > b^*$, by Remark 4.1(ii),

$$\begin{aligned} K(x,b^*) &= \frac{\Phi(q+r)}{q+r}\mathbb{E}\left[f'(\underline{X}(\mathbf{e}_{q+r})+x)\right] \\ &\quad + \Phi(q+r)e^{\Phi(q+r)(x-b^*)}\mathbb{E}_{b^*}\Big[\int_0^{\tau_x^+} e^{-(q+r)t}f'(X(t))\mathrm{d}t\Big]. \end{aligned}$$

(1) The function $x \mapsto \mathbb{E}\left[f'(\underline{X}(\mathbf{e}_{q+r})+x)\right]$ is continuous by monotone convergence in view of Assumption 2.2(i). (2) By Assumption 2.2(i), for $\underline{x} \leq x \leq \overline{x}$, under $\mathbb{P}$,

$$\int_0^{\tau_{b^*-x}^+} e^{-(q+r)t}|f'(X(t)+x)|\mathrm{d}t \leq \int_0^{\infty} e^{-(q+r)t}(|f'(X(t)+\underline{x})| + |f'(X(t)+\overline{x})|)\mathrm{d}t,$$

which are integrable by Remark 2.1. Hence, by dominated convergence,

$$x \mapsto \mathbb{E}_x\left[\int_0^{\tau_{b^*}^+} e^{-(q+r)t}f'(X(t))\mathrm{d}t\right]$$

is continuous. (3) The function $x \mapsto \mathbb{E}_{b^*}[\int_0^{\tau_x^+} e^{-(q+r)t}f'(X(t))\mathrm{d}t]$ is continuous by again dominated convergence because the absolute value of the integrand is dominated by $\int_0^{\infty} e^{-(q+r)t}|f'(X(t))|\mathrm{d}t$. In sum, $K(x,b^*)$ is continuous in x.

For the case $x > b^*$, we have by Fubini's theorem that

(C.5)
$$\int_{-\infty}^{b^*} f'(y)\int_{b^*}^{x} W^{(q)}(x-z)A(z,y,b^*)\mathrm{d}z\mathrm{d}y = \int_{b^*}^{x} W^{(q)}(x-z)\int_{-\infty}^{b^*} f'(y)A(z,y,b^*)\mathrm{d}y\mathrm{d}z.$$

Here, for $\underline{x} \leq x \leq \overline{x}$,

$$W^{(q)}(x-z)\left|\int_{-\infty}^{b^*} f'(y)A(z,y,b^*)\mathrm{d}y\right|$$
$$\leq W^{(q)}(\overline{x})\frac{\Phi(q+r)}{q+r}\Bigg[\mathbb{E}\big[|f'(\underline{X}(\mathbf{e}_{q+r})+\underline{x})|+|f'(\underline{X}(\mathbf{e}_{q+r})+\overline{x})|\big]$$
$$+(q+r)e^{\Phi(q+r)(\overline{x}-b^*)}\mathbb{E}\left[\int_0^\infty e^{-(q+r)t}\big(|f'(X(t)+\underline{x})| \ +|f'(X(t)+\overline{x})|\big)\mathrm{d}t\right]\Bigg].$$

Hence, by bounded convergence, the term defined in (C.5) is also continuous in x. This concludes the proof of the continuity of v''_{b^*}.

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