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The special fiber of the motivic deformation of the stable homotopy category is algebraic

by

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Contents

1. Introduction

Motivic homotopy theory, introduced by Voevodsky and Morel [52], [54], [55], [69]–[72], is a successful application of abstract homotopy theory to solve problems in number theory and algebraic geometry (see [57], [67], [73], for example).

Over $\operatorname{Spec}\mathbb{C}$, one may view the p-completed stable motivic homotopy category as a deformation of the p-completed classical stable homotopy category. The parameter of the deformation is given by an element τ in $\pi_{0,-1}$ of the p-completed motivic sphere spectrum, which can be intuitively viewed as the standard coordinate $t\mapsto e^{2\pi it}$ on $\mathbb{G}_m$. Formally speaking, following Hu–Kriz–Ormsby [28], the element τ is the inverse limit of the Bockstein preimages of the Morel classes [53] of roots of unity. Dugger–Isaksen [14] have identified the generic fiber "$\tau=1$" with the classical stable homotopy category, and the first main result of this paper identifies the special fiber "$\tau=0$" with the derived category of $\mathrm{BP}_*\mathrm{BP}$-comodules that are concentrated in even degrees, which is entirely algebraic in nature. Moreover, under this identification, the motivic Adams–Novikov spectral sequence for the motivic sphere spectrum corresponds to the τ-Bockstein spectral sequence. This deformation induces a deformation of motivic Adams spectral sequences. The second main result of this paper identifies the motivic Adams spectral sequence for the motivic sphere spectrum at the special fiber "$\tau=0$" with the algebraic Novikov spectral sequence for the classical sphere spectrum, which is again entirely algebraic.

This deformation makes it possible for Isaksen, the second and third authors [32], [33] to compute classical stable homotopy groups of spheres at least to the 90-stem, with ongoing computations into even higher dimensions.

1.1. Main results

In this paper, we prove two results in the stable motivic homotopy theory over $\operatorname{Spec}\mathbb{C}$, with connections to chromatic homotopy theory and applications to classical homotopy theory.

The first result identifies the special fiber "$\tau=0$" of the motivic deformation with the derived category of $\mathrm{BP}_*\mathrm{BP}$-comodules. We prove an ∞-category version of a conjecture due to the first author and Isaksen in 2016 [19]. The derived category in the following Theorem 1.1 is understood as a stable ∞-category in the sense of Lurie in the book Higher Algebra [41, §1.3.2].

THEOREM 1.1. (Theorem 1.13) *At each prime p, there is an equivalence of stable ∞-categories equipped with t-structures,*

$$\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}}) \simeq \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{harm}},$$

between the bounded derived category of p-completed $\mathrm{BP}_*\mathrm{BP}$-comodules that are concentrated in even degrees, and the category of harmonic motivic left-module spectra over $\widehat{S^{0,0}}/\tau$, whose MGL*-homology has bounded Chow–Novikov degree, with morphisms the $\widehat{S^{0,0}}/\tau$-linear maps.*

Here, $\widehat{S^{0,0}}/\tau$ is a motivic E_∞-ring spectrum, which is also known as the cofiber of τ. The motivic spectrum MGL is the algebraic cobordism spectrum introduced by Voevodsky [69] and studied by Levine–Morel [39], Panin–Pimenov–Röndigs [58] and many others. A motivic left-module spectrum over $\widehat{S^{0,0}}/\tau$ is *harmonic*, if it is $\widehat{S^{0,0}}/\tau$-cellular and the map to its MGL-completion induces an isomorphism on $\pi_{*,*}$. See Definition 1.6 for a precise definition. The Chow–Novikov degree is the topological degree minus twice the motivic weight.

The derived category of p-completed $\mathrm{BP}_*\mathrm{BP}$-comodules that are concentrated in even degrees is also known as the derived category of quasi-coherent sheaves on the moduli stack of formal groups over $\mathbb{Z}_p$-algebras. This connection is foundational to chromatic homotopy theory, and is due to Quillen [61] and Morava [51] (see also Goerss–Hopkins [20], [23]). Our theorem further connects these categories to motivic homotopy theory.

The equivalence of stable ∞-categories in Theorem 1.1 is in fact symmetric monoidal. See Remark 4.15 for more details.

By an Ind-object argument, we have an unbounded version of Theorem 1.1 that connects to Hovey's [24] derived category $\mathbf{Stable}(\mathrm{BP}_*\mathrm{BP})$. Since every $\mathrm{BP}_*\mathrm{BP}$-comodule splits into its even-graded and odd-graded parts, the underlying stable ∞-category of Hovey's unbounded derived category $\mathbf{Stable}(\mathrm{BP}_*\mathrm{BP})$ splits accordingly:

$$\mathbf{Stable}(\mathrm{BP}_*\mathrm{BP}) \simeq \mathbf{Stable}(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}}) \times \mathbf{Stable}(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{odd}}).$$

COROLLARY 1.2. *There is an equivalence of stable ∞-categories at each prime p,*

$$\mathbf{Stable}(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}}) \simeq \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}},$$

between Hovey's unbounded derived category of $\mathrm{BP}_*\mathrm{BP}$-comodules that are concentrated in even degrees and the category of cellular motivic left-module spectra over $\widehat{S^{0,0}}/\tau$.

After the announcement of Theorem 1.1, alternative proofs of certain versions of Corollary 1.2 have appeared in work of Krause [35] and Pstrągowski [60].

The second result identifies the motivic Adams spectral sequence at the special fiber "τ=0" with the algebraic Novikov spectral sequence. It can be used to systematically compute a huge number of classical Adams differentials that are hard to obtain by other methods.

It is known to Isaksen [29, Proposition 6.2.5] and the first author [18, Corollary 3.14] that there is an isomorphism between the motivic homotopy groups of $\widehat{S^{0,0}}/\tau$ and the classical Adams–Novikov E_2-page. Our second result shows that there is an isomorphism of spectral sequences that converge to them.

THEOREM 1.3. (Theorem 1.17) *For each prime p, there is an isomorphism of spectral sequences between the motivic Adams spectral sequence for $\widehat{S^{0,0}}/\tau$ and the algebraic Novikov spectral sequence for the classical sphere spectrum $\widehat{S^0}$.*

Based on Theorem 1.3, Isaksen, the second and third authors [32], [33] have computed classical stable stems at least to the 90-stem, with ongoing computations into even higher dimensions. Computations of many historically difficult differentials in the range up to the 45-stem are included in the appendix.

In contrast to the original motivations of motivic homotopy theory, Isaksen and his collaborators [29]–[30], [34] have recently begun to reverse the information flow and applied stable motivic homotopy theory to obtain computational results in the classical stable homotopy theory. Our Theorems 1.1 and 1.3 have the same spirit and further deepen the connections to chromatic homotopy theory. Using motivic homotopy theory, we build up a new connection between the classical Adams spectral sequence and the Adams–Novikov spectral sequence, that allows us to compute stable stems in a much larger range than was previously possible.

Remark 1.4. Our Theorems 1.1 and 1.3 actually hold for any algebraically closed field of characteristic zero. In fact, it was clear in Dugger–Isaksen [14] that all related computations in algebraic closed field of characteristic zero work in the same way as over $\mathbb{C}$, which are based on Voevodsky's computation of the motivic Steenrod algebra (see [72, Theorem 4.47] and [70, §11]).

1.2. The stable ∞-category of motivic spectra over $\widehat{S^{0,0}}/\tau$

We work with the stable ∞-category of motivic spectra over $\operatorname{Spec}\mathbb{C}$, denoted by

$$\mathbb{C}\textbf{-mot-Spectra}.$$

This is a symmetric monoidal ∞-category in the sense of Lurie [41, §2.1.2].

There are several approaches for the construction of this category $\mathbb{C}$-**mot-Spectra**. For example, we can take the category of $\mathcal{S}$-modules constructed by Hu [26], which is a symmetric monoidal model category, and we take $\mathbb{C}$-**mot-Spectra** as the underlying ∞-category of Hu's model category. There is another construction entirely in ∞-categorical terms by Robalo [63]. In fact, any symmetric monoidal stable ∞-category satisfying the universal properties of Corollary 2.39 in [63] would serve our purposes.

For a fixed prime p, Voevodsky (see [72, §3]) constructed the mod-p motivic Eilenberg–Mac Lane spectrum that represents the mod-p motivic cohomology. We denote it by $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$. By arguments in Dundas–Röndigs–Østvær [15, Example 3.4], $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$ is an E_∞-algebra in $\mathbb{C}$-**mot-Spectra**.

Its value at a point is

$$(\mathrm{H}\mathbb{F}_p^{\mathrm{mot}})_{*,*} = \mathbb{F}_p[\tau],$$

where τ is in bi-degree $(0,-1)$.

We denote by $S^{0,0}$ the motivic sphere spectrum. For the grading, we denote by $S^{1,0}$ the suspension spectrum of the simplicial sphere S^1, and by $S^{1,1}$ the suspension spectrum of the multiplicative group $\mathbb{G}_m = \mathbb{A}^1\setminus\{0\}$.

Let $\widehat{S^{s,w}}$ be the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-completed motivic sphere spectrum in bi-degree (s,w). It is a theorem of Hu–Kriz–Ormsby [27], [28] that $\widehat{S^{0,0}}$ and the usual p-completion of the motivic sphere spectrum have isomorphic motivic homotopy groups. Moreover, $\widehat{S^{0,0}}$ is an E_∞-algebra in the symmetric monoidal ∞-category $\mathbb{C}$-**mot-Spectra**. See §7 for more details regarding this fact and discussion on the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-completion.

We denote by

$$\widehat{S^{0,0}}\textbf{-Mod}$$

the stable ∞-category of motivic module spectra over $\widehat{S^{0,0}}$.

The class τ can be lifted to a map between $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-completed motivic sphere spectra

$$\tau\colon \widehat{S^{0,-1}} \longrightarrow \widehat{S^{0,0}}$$

that induces a non-zero map on mod-p motivic homology. The reader should be warned that τ does not further lift to a map between uncompleted motivic sphere spectra. See Dugger–Isaksen [14] and Hu–Kriz–Ormsby [28] for more details. We denote by $\widehat{S^{0,0}}/\tau$ the cofiber of τ:

$$\widehat{S^{0,-1}} \xrightarrow{\tau} \widehat{S^{0,0}} \longrightarrow \widehat{S^{0,0}}/\tau \longrightarrow \widehat{S^{1,-1}}$$

Convention 1.5. All smash products without subscript $\wedge$ in this paper are understood taken over the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-completed sphere spectrum $\widehat{S^{0,0}}$. We may still write $\wedge_{\widehat{S^{0,0}}}$ in a few places to emphasize that the smash product is taking over $\widehat{S^{0,0}}$.

We have suspension functors

$$\Sigma^{s,w}(-) = \widehat{S^{s,w}} \wedge_{\widehat{S^{0,0}}} -$$

in the category $\widehat{S^{0,0}}$-**Mod** for any $s, w \in \mathbb{Z}$. In particular, the suspension functor $\Sigma^{1,0}$ gives the translation automorphism (in the sense of Lurie [41, §1.3.2]) of the stable ∞-category $\widehat{S^{0,0}}$-**Mod**.

Given an E_∞-algebra $R \in \widehat{S^{0,0}}$-**Mod**, denote by

$$R\text{-}\mathbf{Mod}$$

the stable ∞-category of left modules over R in $\widehat{S^{0,0}}$-**Mod**.

Following Dugger–Isaksen [13, Definition 2.10], denote by

$$R\text{-}\mathbf{Mod}_{\mathrm{cell}}$$

the smallest stable subcategory containing R that is closed under arbitrary small colimits and suspension by $\widehat{S^{s,w}}$ for all $p, q \in \mathbb{Z}$. We say that an object in R-**Mod** is *R-cellular* if it is contained in R-$\mathbf{Mod}_{\mathrm{cell}}$.

Recall from Lurie [41, Definition 1.1.3.2] that a stable subcategory of a stable ∞-category is a full subcategory containing a zero object and stable under the formation of fibers and cofibers. We warn the reader that not all motivic spectra in $\widehat{S^{0,0}}$-**Mod** are weakly equivalent to a cellular object.

It is a theorem of the first author [18] that $\widehat{S^{0,0}}/\tau$ is an E_∞-algebra in $\widehat{S^{0,0}}$-**Mod** at all primes p. In fact, the first author carried out all details in [18] for the $p{=}2$ case, using the vanishing regions of $\pi_{*,*}\widehat{S^{0,0}}/\tau$. It is straightforward to use the same arguments for all primes p. See [18, §1.2, Theorem 1.1 and explanations] for more details.

We therefore have defined stable ∞-categories

$$\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod} \quad \text{and} \quad \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}.$$

We can view the ring map

$$\widehat{S^{0,0}} \longrightarrow \widehat{S^{0,0}}/\tau$$

to exhibit $\widehat{S^{0,0}}/\tau$ as the special fiber of the deformation parameterized by τ. The generic fiber of this deformation is $\tau^{-1}\widehat{S^{0,0}}$.

Let MGL be the cellular motivic algebraic cobordism spectrum introduced by Voevodsky [69] and studied by Levine-Morel [39], Panin–Pimenov–Röndigs [58] and many others. It is an E_∞-algebra in $\mathbb{C}$-**mot-Spectra** (See [26, Theorem 14.2] for example). We define

$$\mathrm{MU}^{\mathrm{mot}} := \mathrm{MGL}\wedge_{S^{0,0}}\widehat{S^{0,0}}.$$

It is therefore an E_∞-algebra in $\widehat{S^{0,0}}$-$\mathbf{Mod}_{\mathrm{cell}}$. There is a natural map

$$\mathrm{MU}^{\mathrm{mot}} = \mathrm{MGL}\wedge_{S^{0,0}}\widehat{S^{0,0}} \longrightarrow \mathrm{MGL}^{\wedge}_{\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}}$$

to the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-completion of MGL. As we will explain in Proposition 7.2, this map induces an isomorphism on $\pi_{*,*}$. Their motivic homotopy groups are computed by Hu–Kriz–Ormsby [28] and Dugger–Isaksen [14, §8.3]:

$$\pi_{*,*}\mathrm{MU}^{\mathrm{mot}} = \pi_{*,*}\mathrm{MGL}^{\wedge}_{\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}} = \mathbb{Z}_p[\tau][x_1, x_2, \dots].$$

Here, $\mathbb{Z}_p$ is the p-adic integers and x_i is in bi-degree $(2i, i)$. Since $\pi_{*,*}\mathrm{MGL}$ is much more complicated, we will mostly work with $\mathrm{MU}^{\mathrm{mot}}$ instead of MGL in our paper.

For any $X \in \widehat{S^{0,0}}$-$\mathbf{Mod}_{\mathrm{cell}}$, we define the $\mathrm{MU}^{\mathrm{mot}}$-homology of X as

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}X = \pi_{*,*}(\mathrm{MU}^{\mathrm{mot}}\wedge_{\widehat{S^{0,0}}}X).$$

By adjunction, it is clear that the $\mathrm{MU}^{\mathrm{mot}}$-homology of X equals to $\mathrm{MGL}_{*,*}X$ when taking X as its underlying motivic spectrum in $\mathbb{C}$-**mot-Spectra**.

The spectrum $\mathrm{MU}^{\mathrm{mot}}/\tau := \widehat{S^{0,0}}/\tau\wedge_{\widehat{S^{0,0}}}\mathrm{MU}^{\mathrm{mot}}$ is an E_∞-algebra in $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}_{\mathrm{cell}}$. Its motivic homotopy groups are

$$\pi_{*,*}(\mathrm{MU}^{\mathrm{mot}}/\tau) = \mathbb{Z}_p[x_1, x_2, \dots] = \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau.$$

Forgetting the motivic weight, the bigraded ring $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$ can be identified as the singly graded ring MU_* completed at the prime p.

Definition 1.6. Let X be a motivic spectrum in $\widehat{S^{0,0}}/\tau$-**Mod**. We say that X is *harmonic*, if X is $\widehat{S^{0,0}}/\tau$-cellular and the map

$$X \longrightarrow X^{\wedge}_{\mathrm{MGL}}$$

induces an isomorphism on $\pi_{*,*}$. We denote by

$$\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}_{\mathrm{harm}}$$

the full stable ∞-subcategory of harmonic $\widehat{S^{0,0}}/\tau$-module spectra.

Here the MGL-nilpotent completion $X^{\wedge}_{\mathrm{MGL}}$ is understood taken in $\mathbb{C}$-**mot-Spectra**. For a precise definition, see §7 and [14], [28]. One could also define the $\mathrm{MU}^{\mathrm{mot}}$-completion $X^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}$ in $\widehat{S^{0,0}}$-**Mod** for any X in $\widehat{S^{0,0}}$-$\mathbf{Mod}_{\mathrm{cell}}$. By adjunction, it is clear that the two completions $X^{\wedge}_{\mathrm{MGL}}$ and $X^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}$ are equivalent:

$$X^{\wedge}_{\mathrm{MGL}} \xrightarrow{\simeq} X^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}.$$

So we may equivalently define a cellular $\widehat{S^{0,0}}/\tau$-module to be *harmonic*, if the map to its $\mathrm{MU}^{\mathrm{mot}}$-completion induces an isomorphism on $\pi_{*,*}$.

It is clear that the spectrum $\mathrm{MU}^{\mathrm{mot}}/\tau$ is harmonic. See §4.1 for more examples and non-examples.

We will define t-structures on certain stable ∞-categories of motivic spectra, such as $\mathrm{MU}^{\mathrm{mot}}/\tau$-$\mathbf{Mod}_{\mathrm{cell}}$ and $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}_{\mathrm{harm}}$. Recall that, by Lurie's [41, Definition 1.2.1.4], a t-structure on a stable ∞-category is a t-structure on its homotopy category, which is a triangulated category. To describe these t-structures, we define the Chow–Novikov degree of an element that belongs to the bigraded homotopy groups of a motivic spectrum.

Definition 1.7. For any motivic spectrum X, consider its bigraded motivic homotopy groups

$$\pi_{s,w}X.$$

Here, s is the topological degree under the Betti realization, and w is the motivic weight. The *Chow–Novikov degree* of an element in $\pi_{s,w}X$ is defined as $s-2w$.

We say that $\pi_{*,*}X$ is *concentrated in Chow–Novikov degrees* I, where I is a set of integers, if all non-zero elements in $\pi_{*,*}X$ are concentrated in Chow–Novikov degrees belonging to I.

For example, the homotopy groups of $\mathrm{MU}^{\mathrm{mot}}/\tau$ are concentrated in Chow–Novikov degree zero, while the homotopy groups of $\mathrm{MU}^{\mathrm{mot}}$ are concentrated in non-negative even Chow–Novikov degrees.

Definition 1.8. (1) We define

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{cell}}$$

as the stable full subcategory of $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}$ spanned by objects whose homotopy groups are concentrated in bounded Chow–Novikov degrees.

(2) We define

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b,\geqslant 0}_{\mathrm{cell}},$$
$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b,\leqslant 0}_{\mathrm{cell}},$$
$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$$

as the full subcategories of $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{cell}}$ spanned by objects whose homotopy groups are concentrated in non-negative, non-positive and zero Chow–Novikov degrees, respectively.

(3) We define

$$\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{harm}}$$

as the stable full subcategory of $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}_{\mathrm{harm}}$ spanned by objects whose $\mathrm{MU}^{\mathrm{mot}}$-homology groups are concentrated in bounded Chow–Novikov degrees.

(4) We define

$$\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b,\geqslant 0}_{\mathrm{harm}},$$
$$\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b,\leqslant 0}_{\mathrm{harm}},$$
$$\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}}$$

as the full subcategories of $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{harm}}$ spanned by objects whose $\mathrm{MU}^{\mathrm{mot}}$-homology groups are concentrated in non-negative, non-positive and zero Chow–Novikov degrees respectively.

Definition 1.9. We define

$$\mathrm{MU}_*\text{-}\mathbf{Mod}^{\mathrm{ev}}$$

as the abelian category of graded modules that are concentrated in even degrees over the p-completed ring MU_*, and

$$\mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}^{\mathrm{ev}},$$
$$\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}}$$

as the abelian categories of graded comodules that are concentrated in even degrees over the p-completed Hopf algebroids $\mathrm{MU}_*\mathrm{MU}$ and $\mathrm{BP}_*\mathrm{BP}$, respectively.

We define

$$\mathcal{D}^b(\mathrm{MU}_*\text{-}\mathbf{Mod}^{\mathrm{ev}}),$$
$$\mathcal{D}^b(\mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}^{\mathrm{ev}}),$$
$$\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}})$$

as their bounded derived categories.

PROPOSITION 1.10. ([51, Proposition 1.2.3]) *At each prime p, the categories*

$$\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}} \quad \textit{and} \quad \mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}^{\mathrm{ev}}$$

are equivalent as abelian categories.

The abelian categories of modules over p-completed MU_* and BP_* are *not* equivalent. However, Proposition 1.10 states that the abelian categories of even comodules over p-completed $\mathrm{MU}_*\mathrm{MU}$ and $\mathrm{BP}_*\mathrm{BP}$ are equivalent. We will work with MU and $\mathrm{MU}^{\mathrm{mot}}$ since they are E_∞-algebras in the corresponding categories while BP is not, due to a recent result of Lawson [38].

THEOREM 1.11. (1) *The full subcategories*

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b,\geqslant 0}_{\mathrm{cell}} \quad \textit{and} \quad \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b,\leqslant 0}_{\mathrm{cell}}$$

define a t-structure on $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{cell}}$.

(2) *The functor*

$$\pi_{*,*}\colon \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}} \longrightarrow \mathrm{MU}_*\text{-}\mathbf{Mod}^{\mathrm{ev}}$$

is an equivalence.

(3) *There exists an equivalence of stable ∞-categories*

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{cell}} \longrightarrow \mathcal{D}^b(\mathrm{MU}_*\text{-}\mathbf{Mod}^{\mathrm{ev}}),$$

that preserves the given t-structures and extends the functor $\pi_{*,*}$ on the heart.

Remark 1.12. The functor $\pi_{*,*}$ naturally lands in the category of bigraded modules over the bigraded ring $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$. Since all elements of this bigraded ring are concentrated in Chow–Novikov degree zero, it can be identified as the single graded ring MU_* by forgetting the motivic weight. A similar comment applies to the following theorem as well.

THEOREM 1.13. (1) *The full subcategories*

$$\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b,\geqslant 0}_{\mathrm{harm}} \quad \textit{and} \quad \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b,\leqslant 0}_{\mathrm{harm}}$$

define a t-structure on $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{harm}}$.

(2) *The functor*

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}\colon \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}} \longrightarrow \mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}^{\mathrm{ev}}$$

is an equivalence.

(3) *There exists an equivalence of stable* ∞*-categories*

$$\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{harm}} \longrightarrow \mathcal{D}^b(\mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}^{\mathrm{ev}})$$

that preserves the given t-structures and extends the functor $\mathrm{MU}^{\mathrm{mot}}_{*,*}$ *on the heart.*

Remark 1.14. The statements in Theorems 1.11 and 1.13 can be connected by the following commutative diagram of stable ∞-categories with t-structures:

$$\begin{array}{ccc} \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{harm}} & \longrightarrow & \mathcal{D}^b(\mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}^{\mathrm{ev}}) \\ \downarrow{\scriptstyle -\wedge_{\widehat{S^{0,0}}/\tau}\mathrm{MU}^{\mathrm{mot}}/\tau} & & \downarrow \\ \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{cell}} & \longrightarrow & \mathcal{D}^b(\mathrm{MU}_*\text{-}\mathbf{Mod}^{\mathrm{ev}}) \end{array}$$

The vertical functor on the right is the forgetful functor.

Remark 1.15. From a deformation perspective, our Theorem 1.13 gives a new connection between the moduli stack of formal groups and the classical stable homotopy theory.

From the deformation

$$\widehat{S^{0,0}}/\tau \xleftarrow[\text{fiber}]{\text{special}} \widehat{S^{0,0}} \xrightarrow[\text{fiber}]{\text{generic}} \tau^{-1}\widehat{S^{0,0}}$$

parameterized by τ, we have two adjunctions of stable ∞-categories:

$$(\widehat{S^{0,0}}\text{-}\mathbf{Mod})_{\tau=0} \rightleftarrows \widehat{S^{0,0}}\text{-}\mathbf{Mod} \rightleftarrows (\widehat{S^{0,0}}\text{-}\mathbf{Mod})_{\tau=1}.$$

We call this deformation a "motivic deformation" intuitively.

By Dugger–Isaksen [14], on the generic fiber, the full subcategory of cellular objects in

$$(\widehat{S^{0,0}}\text{-}\mathbf{Mod})_{\tau=1} := \tau^{-1}\widehat{S^{0,0}}\text{-}\mathbf{Mod}$$

is equivalent to the classical stable homotopy category at the prime p. In fact, Dugger–Isaksen showed that the motivic homotopy groups of the τ-inverted sphere spectrum are isomorphic to that of the classical sphere spectrum. By an inductive argument, one can show that a similar statement is true for all finite cellular objects. This shows that τ-inverted Betti realization functor is fully faithful. It is also essentially surjective, since the Betti realization functor admits a section with constant weight zero. An Ind-object argument (similar to the proof of Corollary 1.2) gives us the claim.

Our main theorem shows that, on the special fiber, the full subcategory of harmonic objects in the category

$$(\widehat{S^{0,0}}\text{-}\mathbf{Mod})_{\tau=0} := \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}$$

is equivalent to the derived category of comodules that are concentrated in even degrees over the p-completed Hopf algebroid $\mathrm{MU}_*\mathrm{MU}$. By Quillen's theorem [61], the latter can be identified with the derived category of quasi-coherent sheaves on the moduli stack of formal groups over $\mathbb{Z}_p$-algebras.

Remark 1.16. In our proof of Theorem 1.13, we set up a strongly convergent motivic Adams–Novikov spectral sequence in the category $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{cell}}$,

$$\mathrm{Ext}^{*,*,*}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y) \Longrightarrow [\Sigma^{*,*}X, Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}]_{\widehat{S^{0,0}}/\tau}\,.$$

This is stated as Theorem 5.6 in §5. Classically, the Adams–Novikov spectral sequence is set up in such a way that the first variable is the sphere spectrum. Our construction could be generalized to an abstract setting and applied to the classical situation when the first variable X is arbitrary. We will discuss this case in a general framework in future work.

1.3. The motivic Adams spectral sequence and the algebraic Novikov spectral sequence

The following Theorem 1.17 establishes an isomorphism between the algebraic Novikov spectral sequence and the motivic Adams spectral sequence for $\widehat{S^{0,0}}/\tau$.

THEOREM 1.17. *At each prime p, there is an isomorphism of tri-graded spectral sequences: the motivic Adams spectral sequence for $\widehat{S^{0,0}}/\tau$, which converges to the motivic homotopy groups of $\widehat{S^{0,0}}/\tau$, and the regraded algebraic Novikov spectral sequence, which converges to the Adams–Novikov E_2-page for the sphere.*

The indexes are indicated in the following diagram:

$$\begin{array}{ccc}
\mathrm{Ext}^{s,2w}_{\mathrm{BP}_*\mathrm{BP}/I}(\mathbb{F}_p, I^{a-s}/I^{a-s+1}) & \xrightarrow{\cong} & \mathrm{Ext}^{a,2w-s+a,w}_{A^{\mathrm{mot}}_{*,*}}(\mathbb{F}_p[\tau], \mathbb{F}_p) \\
\Vert & & \Vert \\
\text{Algebraic Novikov SS} & & \text{Motivic Adams SS} \\
\Downarrow & & \Downarrow \\
\mathrm{Ext}^{s,2w}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) & \xrightarrow{\cong} & \pi_{2w-s,w}(\widehat{S^{0,0}}/\tau).
\end{array}$$

Here, $I=(p, v_1, v_2, \dots)$ *is the augmentation ideal of* BP_* *and* $A^{\mathrm{mot}}_{*,*}$ *is the motivic mod* p *dual Steenrod algebra.*

Recall that both of the two spectral sequences in Theorem 1.17 are multiplicative, so there are multiplicative filtrations on the abutments.

THEOREM 1.18. *There is an isomorphism between*

$$\mathrm{Ext}^{*,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) \quad \textit{and} \quad \pi_{*,*}(\widehat{S^{0,0}}/\tau)$$

that preserves the multiplicative filtrations, composition products and higher compositions in the respective categories.

Proof. The multiplicative structure on the abutments comes from composition of morphisms in both categories

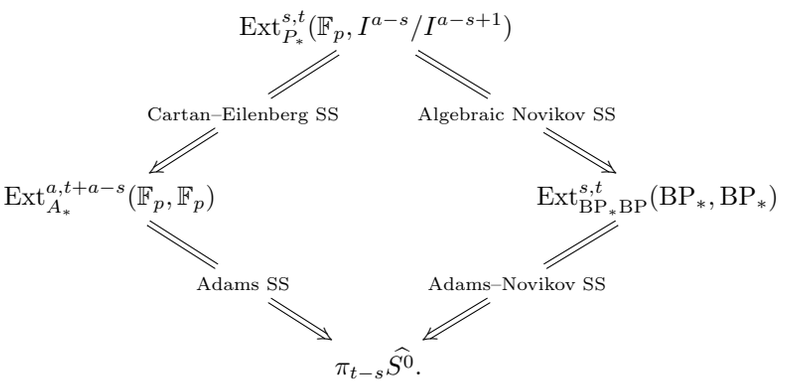
$$\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{harm}} \quad \text{and} \quad \mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}}).$$

The isomorphism on abutments is induced by the equivalence of categories in Theorem 1.13, and in particular respects compositions. □

The isomorphism between the abutments is known to Isaksen [29, Proposition 6.2.5] and the first author [18, Corollary 3.14]. Our Theorem 1.18 further states that the isomorphism preserves the multiplicative filtrations on the abutments. We do not prove that the group isomorphism on the E_2-pages is also a ring isomorphism. We shall prove it in future work.

There has been huge interest in obtaining information on the stable homotopy groups of spheres by comparing the Adams spectral sequence with the Adams–Novikov spectral sequence. In fact, this is a dream entertained by Novikov [56, §12]. See also [46], [47], [62], for example. An important connection and technique of studying both spectral sequences

is the following Miller square [47]:



By a change-of-ring isomorphism, the E_2-page of the Cartan–Eilenberg spectral sequence, which computes the Adams E_2-page, is isomorphic to the algebraic Novikov spectral sequence, which computes the Adams–Novikov E_2-page. For p odd, the Cartan–Eilenberg spectral sequence collapses for degree reasons.

To explore this square, Miller [47] smashes together the Adams resolution and the Adams–Novikov resolution, and gets a comparison theorem on the d_2-differentials in the algebraic Novikov spectral sequence and the Adams spectral sequence. The following theorem is due to Novikov [56], Miller [47, Theorem 4.2] and Andrews–Miller [1, Theorem 9.3.3].

THEOREM 1.19. *Let z' be an element in $\mathrm{Ext}_{A_*}^{a,t+a-s}(\mathbb{F}_p, \mathbb{F}_p)$ with Cartan–Eilenberg filtration s. Then, $d_2^{\mathbf{ASS}} z'$ has higher Cartan–Eilenberg filtration.*

Moreover, if z' is detected in the Cartan–Eilenberg spectral sequence by z in

$$\mathrm{Ext}_{P_*}^{s,t}(\mathbb{F}_p, I^{a-s}/I^{a-s+1}),$$

then

$$d_2^{\mathbf{ASS}} z' \quad \textit{is detected by} \quad d_2^{\mathbf{algNSS}} z,$$

where $d_2^{\mathbf{algNSS}} z$ is in

$$\mathrm{Ext}_{P_*}^{s+1,t}(\mathbb{F}_p, I^{a-s+1}/I^{a-s+2}).$$

Remark 1.20. We regraded the algebraic Novikov spectral sequence so our $d_2^{\mathbf{algNSS}}$ is $d_1^{\mathbf{algNSS}}$ in [1, Theorem 9.3.3].

Based on the Miller square and Theorem 1.19, Miller [47] proves the telescope conjecture at chromatic height 1 at odd primes.

To understand the connection between *higher* differentials in the Adams and algebraic Novikov spectral sequences, it would be desirable to establish new connections between them.

For example, suppose in general that we have two spectral sequences

$$E_2 \Longrightarrow E_\infty \quad \text{and} \quad E'_2 \Longrightarrow E'_\infty$$

that are not necessarily connected by a homomorphism of spectral sequences. To compare them, it would be useful to have a third spectral sequence

$$E''_2 \Longrightarrow E''_\infty$$

making a zig-zag diagram of spectral sequences:

$$\begin{array}{ccccc} E_2 & \longleftarrow & E''_2 & \longrightarrow & E'_2 \\ \Downarrow & & \Downarrow & & \Downarrow \\ E_\infty & \longleftarrow & E''_\infty & \longrightarrow & E'_\infty. \end{array}$$

This is the one of the major techniques used by the second and third authors in [74] to explore the Mahowald square [42] and compute differentials in the Adams spectral sequences.

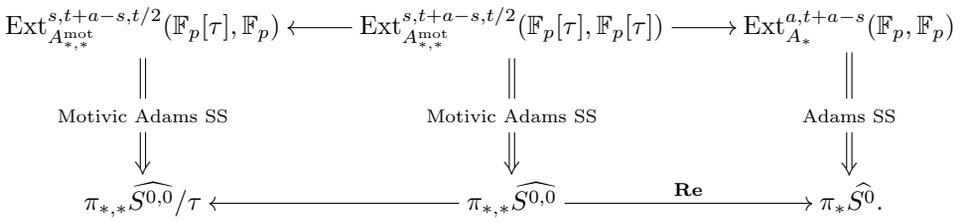
Following this philosophy, for the Miller square [47], a basic question would be the following: Which spectral sequence can we put in between these two spectral sequences and have a zig-zag diagram? Namely,

$$\begin{array}{ccccc} \mathrm{Ext}_{P_*}^{s,t}(\mathbb{F}_p, I^{a-s}/I^{a-s+1}) & \longleftarrow & ? & \longrightarrow & \mathrm{Ext}_{A_*}^{a,t}(\mathbb{F}_p, \mathbb{F}_p) \\ \| & & \| & & \| \\ \text{Algebraic Novikov SS} & & & & \text{Adams SS} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}^{s,t}(\mathrm{BP}_*, \mathrm{BP}_*) & \longleftarrow & ? & \longrightarrow & \pi_* \widehat{S^0}. \end{array}$$

Our Theorem 1.17 shows that we can achieve a zig-zag diagram in the motivic world.

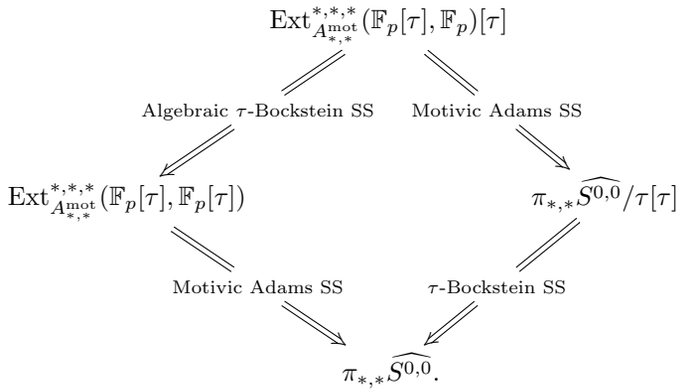
In fact, consider the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-completed motivic sphere spectrum $\widehat{S^{0,0}}$. Inverting τ, we get the classical p-completed sphere $\widehat{S^0}$ by Dugger–Isaksen [14], in the sense that the corresponding Adams and Adams–Novikov spectral sequences have equivalent data. On the other hand, reducing mod τ, we get $\widehat{S^{0,0}}/\tau$. Then, the naturality of the Adams

spectral sequences gives us the following zig-zag diagram:



By Theorem 1.17, the left side spectral sequence, which is the motivic Adams spectral sequence for $\widehat{S^{0,0}}/\tau$, is isomorphic to the algebraic Novikov spectral sequence.

More generally, we have the following motivic square:



Let us compare the motivic square with the Miller square.

For the lower-right side, it is proved by Isaksen [29] that the motivic Adams–Novikov spectral sequence for $\widehat{S^{0,0}}$ is isomorphic to the τ-Bockstein spectral sequence, and that it is rigid, in the sense that it contains the same information as the classical Adams–Novikov spectral sequence. Each non-trivial differential in the classical Adams–Novikov spectral sequence corresponds to a family of non-trivial differentials in the motivic Adams–Novikov spectral sequence, that are connected to each other by multiplication by τ. We can recover all non-zero differentials in the motivic Adams–Novikov spectral sequence by knowing all non-zero differentials in the classical Adams–Novikov spectral sequence, and vice versa.

We would like to point out that the above isomorphism between the motivic Adams–Novikov spectral sequence for the motivic sphere and the τ-Bockstein spectral sequence for the motivic sphere does not come from a map between two towers. In particular, the Chow–Novikov degree compresses motivic Adams–Novikov d_{2r+1}-differentials to τ-Bockstein d_r-differentials, and the motivic Adams–Novikov E_2-page is isomorphic to the

τ-Bockstein E_1-page, which is isomorphic to the homotopy groups of a motivic spectrum. Moreover, the edge map of the τ-Bockstein spectral sequence has the advantage of being induced by an actual map between motivic spectra, so naturality applies to motivic homotopy groups. Naturality also gives us a map of the motivic Adams spectral sequences from the motivic sphere to $\widehat{S^{0,0}}/\tau$, where the latter is isomorphic to the classical algebraic Novikov spectral sequence.

For the upper-left side, the relation of the two spectral sequences in the motivic square and the Miller square is the same as the relation on the lower-right side. The algebraic τ-Bockstein spectral sequence can be thought as a motivic version of the Cartan–Eilenberg spectral sequence, and contains the same information, in the same sense as the lower-right-side situation.

For the upper-right side, our Theorem 1.17 says that the two spectral sequences are isomorphic.

Therefore, for three out of the four sides, the motivic square contains exactly the same information as the ones in the Miller square.

For the remaining lower-left side, Dugger–Isaksen [14] show that the τ-inverted motivic Adams spectral sequence is isomorphic to the τ-inverted classical Adams spectral sequence. This means that the difference between the motivic square and the Miller square lies in the τ-torsion information. Therefore, when comparing the higher differentials in the classical and motivic Adams spectral sequences, the τ-torsion information is necessary to make the zig-zag strategy work.

Now, to compute a non-trivial classical Adams differential, for any r, start with an algebraic Novikov d_r-differential. Theorem 1.17 gives us a motivic Adams d_r-differential for $\widehat{S^{0,0}}/\tau$. Pulling back to the bottom cell of $\widehat{S^{0,0}}/\tau$ of the source element gives us a motivic Adams $d_{r'}$-differential for the motivic sphere with $r' \leqslant r$. Using the Betti realization functor, we then obtain a classical Adams $d_{r'}$-differential!

In practice, Isaksen, the second and the third authors [32], [33] extend the computation of classical and motivic stable stems into a large range using the following steps.

(1) Use a computer to carry out the entirely algebraic computation of the cohomology of the $\mathbb{C}$-motivic Steenrod algebra. These groups serve as the input to the $\mathbb{C}$-motivic Adams spectral sequence.

(2) Use a computer to carry out the entirely algebraic computation of the algebraic Novikov spectral sequence that converges to the cohomology of the Hopf algebroid $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$. This includes all differentials, and the multiplicative structure of the cohomology of $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$.

(3) Use Theorem 1.17 to identify the algebraic Novikov spectral sequence with the motivic Adams spectral sequence that computes the homotopy groups of $\widehat{S^{0,0}}/\tau$. This includes an identification of the cohomology of $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$ with the homotopy groups of $\widehat{S^{0,0}}/\tau$.

(4) Use the inclusion of the bottom cell and the projection to the top cell to pull back and push forward Adams differentials for $\widehat{S^{0,0}}/\tau$ to Adams differentials for the motivic sphere.

(5) Apply a variety of ad-hoc arguments to deduce additional Adams differentials for the motivic sphere. The most important method involves shuffling Toda brackets.

(6) Use a long exact sequence in homotopy groups to deduce hidden τ-extensions in the motivic Adams spectral sequence for the sphere.

(7) Invert τ to obtain the classical Adams spectral sequence and the classical stable homotopy groups.

We would like to highlight a few consequences of our stem-wise computations.

Example 1.21. Consider the following four differentials in the classical Adams spectral sequence for the 2-completed sphere.

(1) There is a d_3-differential in the 15-stem:

$$d_3(h_0h_4) = h_0d_0.$$

This is proved by May and Mahowald–Tangora in [43] , [44] by comparing with Toda's unstable computations [68].

(2) There is a d_4-differential in the 38-stem:

$$d_4(h_3h_5) = h_0x.$$

This is proved in Mahowald-Tangora [43] by an ad-hoc method using a certain finite CW spectrum.

(3) There is a d_3-differential in the 38-stem:

$$d_3(e_1) = h_1t.$$

This is proved by Bruner in [8] by power operations in the Adams spectral sequence.

(4) There is a d_3-differential in the 61-stem:

$$d_3(D_3) = B_3.$$

This is proved by the second and third authors [74] using the $\mathbb{R}\mathrm{P}^\infty$-technique. The proof of this differential in [74] is a significant part of the proof that the 61-sphere has a unique smooth structure.

It turns out that all these four differentials can be proved by our method. They all correspond to non-trivial differentials in the algebraic Novikov spectral sequence with the same length, and therefore are all consequences of purely algebraic computations and our Theorem 1.17.

Remark 1.22. For some of the differentials computed by Isaksen, the second and the third authors [32], [33] using Theorem 1.17, our method gives the only known proof. For example, we prove an Adams d_3-differential in the 68-stem

$$d_3(d_2) = h_0^2 Q_3,$$

which shows the non-existence of the homotopy class κ_2 in π_{68}. As another example, we prove an Adams d_5-differential in the 92-stem

$$d_5(g_3) = h_6 d_0^2,$$

which shows the non-existence of the homotopy class $\overline{\kappa}_3$ in π_{92}. Since both the elements d_2 and g_3 lie in a non-zero Sq^0-family in the 4-line of the classical Adams E_2-page, the two new non-trivial differentials serve as new evidence of Minami's new Doomsday conjecture.

Remark 1.23. Theorem 1.17 can also be used to compute non-trivial extensions and Toda brackets. For example, there is an η-extension from $h_3 d_1$ to N in the 46-stem. This is proved by the second and third authors [75, Proposition 1.3 (2)] using the $\mathbb{R}\mathrm{P}^\infty$-technique. As another example, there is a Toda bracket

$$\langle \theta_4, 2, \sigma^2 \rangle$$

in the 45-stem. It is computed by Isaksen in [29, Lemma 4.2.91] by ad-hoc methods. This Toda bracket computation is crucial in the third author's proof [77] that

$$2\theta_5 = 0$$

in the 62-stem. Both the non-trivial η-extension and the Toda bracket computations are present in the motivic homotopy groups of $\widehat{S^{0,0}}/\tau$. By Theorem 1.17, they can be computed by the product and Massey product structure on the classical Adams–Novikov E_2-page. In particular, the corresponding 3-fold Massey product can be verified in the algebraic Novikov spectral sequence using May's convergence theorem [45]. Therefore, both the non-trivial η-extension and the Toda bracket computations are consequences of purely algebraic computations and our Theorem 1.17.

1.4. Organization

This paper is organized in two parts.

In Part 1, we prove the equivalence of stable ∞-categories in Theorems 1.11 and 1.13. Our proofs use a theorem of Lurie in Higher Algebra [41] on the relation between a stable ∞-category with a t-structure and the derived category of its heart. We recall Lurie's theorem in §2, and prove Theorems 1.11 and 1.13 in §3 and §4, respectively. We also prove Corollary 1.2 in the end of §4. We introduce the absolute Adams–Novikov spectral sequence in the category

$$\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{harm}}$$

in §5, which is necessary for our proof of Theorem 1.13. We propose some further questions in §6. We discuss the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-completion in §7.

In Part 2, we prove the isomorphism of spectral sequences in Theorem 1.17. In §9, we check that, through the equivalence of stable ∞-categories in Theorem 1.13, the algebraic Novikov tower in the derived category of $\mathrm{BP}_*\mathrm{BP}$-comodules corresponds to the motivic Adams tower of $\widehat{S^{0,0}}/\tau$ in the category of $\widehat{S^{0,0}}/\tau$-modules. In Appendix A, we recompute certain low filtration and historically more difficult differentials in the range up to the 45-stem at the prime 2, as an illustration of the power of the isomorphism of spectral sequences in Theorem 1.17.

1.5. Conventions and notation

All colimits and limits in a stable ∞-category of spectra mean homotopy colimit and homotopy limit in the classical sense.

All modules over graded rings are graded modules.

Here is a summary of a list of the notation we use in this paper.

(1) $S^{1,0}$: the simplicial sphere S^1.
(2) $S^{1,1}$: the multiplicative group $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$.
(3) $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$: the mod-$p$ motivic Eilenberg–Mac Lane spectrum.
(4) $\widehat{S^{0,0}}$: the motivic $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-completed sphere spectrum.
(5) $\widehat{S^{s,w}}$: the motivic $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-completed sphere spectrum in bi-degree (s,w).
(6) $\widehat{S^{0,0}}/\tau$: the cofiber of τ.
(7) $\mathbb{C}$-**mot-Spectra**: the stable ∞-category of motivic spectra over $\operatorname{Spec}\mathbb{C}$.
(8) $\widehat{S^{0,0}}$-**Mod**: the stable ∞-category of motivic module spectra over $\widehat{S^{0,0}}$.
(9) $\Sigma^{s,w}(-)$: the suspension functor $\widehat{S^{s,w}} \wedge_{\widehat{S^{0,0}}} -$.

(10) R-**Mod**: the stable ∞-category of left modules over R in $\widehat{S^{0,0}}$-**Mod**, for an E_∞-algebra $R \in \widehat{S^{0,0}}$-**Mod**.

(11) R-$\mathbf{Mod}_{\mathrm{cell}}$: the stable ∞-category of cellular objects in R-**Mod**.

(12) MGL: the cellular motivic algebraic cobordism spectrum.

(13) $\mathrm{MU}^{\mathrm{mot}}$: the E_∞-algebra $\mathrm{MGL} \wedge_{S^{0,0}} \widehat{S^{0,0}}$ in $\widehat{S^{0,0}}$-$\mathbf{Mod}_{\mathrm{cell}}$.

(14) $\mathrm{MU}^{\mathrm{mot}}/\tau$: the E_∞-algebra $\widehat{S^{0,0}}/\tau \wedge_{\widehat{S^{0,0}}} \mathrm{MU}^{\mathrm{mot}}$ in $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}_{\mathrm{cell}}$.

(15) $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}_{\mathrm{harm}}$: the stable ∞-category of harmonic $\widehat{S^{0,0}}/\tau$-module spectra.

(16) The Chow–Novikov degree of an element in $\pi_{s,w}X$: $s-2w$, for X a motivic spectrum.

(17) $\mathrm{MU}^{\mathrm{mot}}/\tau$-$\mathbf{Mod}^{b}_{\mathrm{cell}}$: the stable ∞-category of cellular $\mathrm{MU}^{\mathrm{mot}}/\tau$-modules whose homotopy groups are concentrated in bounded Chow–Novikov degrees.

(18) $\mathrm{MU}^{\mathrm{mot}}/\tau$-$\mathbf{Mod}^{b,\geqslant 0}_{\mathrm{cell}}$, $\mathrm{MU}^{\mathrm{mot}}/\tau$-$\mathbf{Mod}^{b,\leqslant 0}_{\mathrm{cell}}$ and $\mathrm{MU}^{\mathrm{mot}}/\tau$-$\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$: full subcategories of $\mathrm{MU}^{\mathrm{mot}}/\tau$-$\mathbf{Mod}^{b}_{\mathrm{cell}}$ spanned by objects whose homotopy groups are concentrated in non-negative, non-positive and zero Chow–Novikov degrees, respectively.

(19) $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}^{b}_{\mathrm{harm}}$: the stable ∞-category of harmonic $\widehat{S^{0,0}}/\tau$-modules whose $\mathrm{MU}^{\mathrm{mot}}$-homology groups are concentrated in bounded Chow–Novikov degrees.

(20) $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}^{b,\geqslant 0}_{\mathrm{harm}}$, $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}^{b,\leqslant 0}_{\mathrm{harm}}$ and $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}}$: full subcategories of $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}^{b}_{\mathrm{harm}}$ spanned by objects whose $\mathrm{MU}^{\mathrm{mot}}$-homology groups are concentrated in non-negative, non-positive and zero Chow–Novikov degrees, respectively.

(21) MU_*-$\mathbf{Mod}^{\mathrm{ev}}$: the abelian category of graded modules that are concentrated in even degrees over the p-completed ring MU_*.

(22) $\mathrm{MU}_*\mathrm{MU}$-$\mathbf{Comod}^{\mathrm{ev}}$: the abelian category of graded comodules that are concentrated in even degrees over the p-completed Hopf algebroid $\mathrm{MU}_*\mathrm{MU}$.

(23) $\mathrm{BP}_*\mathrm{BP}$-$\mathbf{Comod}^{\mathrm{ev}}$: the abelian category of graded comodules that are concentrated in even degrees over the p-completed Hopf algebroid $\mathrm{BP}_*\mathrm{BP}$.

(24) $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$-**Mod**: the abelian category of graded left modules over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$.

(25) $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$-$\mathbf{Mod}^{0}$: the abelian category of graded left modules over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$ that are concentrated in Chow–Novikov degree zero.

(26) $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$-**Comod**: the abelian category of graded left comodules over the Hopf algebroid $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$.

(27) $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$-$\mathbf{Comod}^{0}$: the abelian category of graded left comodules over the Hopf algebroid $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$ that are concentrated in Chow–Novikov degree zero.

(28) $\mathcal{D}^b(\mathcal{A})$: the bounded derived category of an abelian category $\mathcal{A}$ as a stable ∞-category.

(29) **Stable**$(\mathrm{BP}_*\mathrm{BP})$: the underlying stable ∞-category of Hovey's unbounded derived category of $\mathrm{BP}_*\mathrm{BP}$-comodules.

1.6. Acknowledgements

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Part 1. Equivalence of stable ∞-categories

The question of when the homotopy category of module spectra over a certain ring spectrum is equivalent to the derived category of an abelian category as a triangulated category has been studied in many context by many people. For example, Schwede and Shipley [65] studied the case for the Eilenberg–Mac Lane spectrum HR, where R is a commutative ring, Patchkoria [59] studied the case for the complex periodic K-theory localized at an odd prime, Greenlees [21] studied the case for the rational S^1-equivarant sphere spectrum, and Deligne and Goncharov [11] studied the case for the rational motivic Eilenberg–Mac Lane spectrum $\mathrm{H}\mathbb{Q}^{\mathrm{mot}}$. The answers are positive in these cases. On the other hand, Schwede [64] showed that the classical stable homotopy category is not a derived category.

The goal of Part 1 is to prove that the homotopy category of harmonic $\widehat{S^{0,0}}/\tau$-spectra whose $\mathrm{MU}^{\mathrm{mot}}$-homology are concentrated in bounded Chow–Novikov degrees is equivalent to the bounded derived category of $\mathrm{MU}_*\mathrm{MU}$-comodules that are concentrated in even degrees. In fact, we will prove Theorem 1.13, which states that there exists an

equivalence of stable ∞-categories that preserves the given t-structures:

$$\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{harm}} \longrightarrow \mathcal{D}^{b}(\mathrm{MU}_{*}\mathrm{MU}\text{-}\mathbf{Comod}^{\mathrm{ev}}).$$

We apply a theorem of Lurie in Higher Algebra [41, Proposition 1.3.3.7] on the relation between a stable ∞-category with a t-structure and the derived category of its heart. As a warm-up, we will prove Theorem 1.11, which states that there exists an equivalence of stable ∞-categories that preserves the given t-structures:

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{cell}} \longrightarrow \mathcal{D}^{b}(\mathrm{MU}_{*}\text{-}\mathbf{Mod}^{\mathrm{ev}}).$$

2. Lurie's theorem on t-structures

In [41, Proposition 1.3.3.7], Lurie proves a theorem on the relation between a stable ∞-category with a t-structure and the derived category of its heart. In this section, we state a corollary of Lurie's theorem as Proposition 2.11, and its dual version Proposition 2.12. Both propositions are used in §3 and §4. We will first recall relevant definitions and Lurie's theorem and then prove Proposition 2.11.

Let $\mathcal{C}$ be an ∞-category. Denote by $h\mathcal{C}$ its homotopy category, and by $[-,-]_{\mathcal{C}}$ the abelian group of homotopy classes of maps in $\mathcal{C}$. When it is clear from the context, we will also denote it by $[-,-]$. If $\mathcal{C}$ is further a stable ∞-category, denote by Σ its translation automorphism.

We recall from [41, Definition 1.2.1.4] that a t-structure on a stable ∞-category $\mathcal{C}$ is defined as a t-structure on its homotopy category $h\mathcal{C}$, which is a triangulated category. More precisely, we have the following definition.

Definition 2.1. A *t-structure* on a stable ∞-category $\mathcal{C}$ is a pair of two full subcategories $\mathcal{C}_{\geqslant 0}$ and $\mathcal{C}_{\leqslant 0}$ that are stable under equivalences, satisfying the following three properties:

(1) for $X \in \mathcal{C}_{\geqslant 0}$ and $Y \in \Sigma^{-1}\mathcal{C}_{\leqslant 0}$, we have $[X,Y]_{\mathcal{C}}=0$;
(2) there are inclusions $\Sigma\mathcal{C}_{\geqslant 0} \subseteq \mathcal{C}_{\geqslant 0}$ and $\Sigma^{-1}\mathcal{C}_{\leqslant 0} \subseteq \mathcal{C}_{\leqslant 0}$;
(3) for any $X \in \mathcal{C}$, there exists a fiber sequence

$$X_{\geqslant 0} \longrightarrow X \longrightarrow X_{\leqslant -1},$$

with $X_{\geqslant 0} \in \mathcal{C}_{\geqslant 0}$ and $X_{\leqslant -1} \in \Sigma^{-1}\mathcal{C}_{\leqslant 0}$.

As in [41], we use homological indexing convention.

Definition 2.2. Let $\mathcal{C}$ and $\mathcal{C}'$ be stable ∞-categories equipped with t-structures. We say that an exact functor $f:\mathcal{C}\to\mathcal{C}'$ is *right t-exact*, if it carries $\mathcal{C}_{\geqslant 0}$ to $\mathcal{C}'_{\geqslant 0}$. An exact functor $f:\mathcal{C}\to\mathcal{C}'$ is *left t-exact*, if it carries $\mathcal{C}_{\leqslant 0}$ to $\mathcal{C}'_{\leqslant 0}$. A functor is *$t$-exact* if it is both left and right t-exact.

Definition 2.3. Denote by $\mathcal{C}_{\geqslant n}$ and $\mathcal{C}_{\leqslant n}$ the ∞-categories $\Sigma^n\mathcal{C}_{\geqslant 0}$ and $\Sigma^n\mathcal{C}_{\leqslant 0}$, respectively. For every integer n, the subcategories $\mathcal{C}_{\geqslant n}$ and $\mathcal{C}_{\leqslant n}$ sit in adjunctions

$$\mathcal{C}_{\geqslant n} \underset{\tau_{\geqslant n}}{\overset{}{\rightleftarrows}} \mathcal{C} \quad \text{and} \quad \mathcal{C} \overset{\tau_{\leqslant n}}{\underset{}{\rightleftarrows}} \mathcal{C}_{\leqslant n}$$

where $\tau_{\geqslant n}$ and $\tau_{\leqslant n}$ are called the nth *truncation functors*.

Sometimes the truncation functors are post-composed with the inclusion functors, so they land in $\mathcal{C}$.

Definition 2.4. Denote by $\mathcal{C}^+$ and $\mathcal{C}^-$ the stable full subcategories spanned by *left-bounded* and *right-bounded* objects in $\mathcal{C}$, respectively:

$$\mathcal{C}^+ = \bigcup_{n\geqslant 0} \mathcal{C}_{\leqslant n} \quad \text{and} \quad \mathcal{C}^- = \bigcup_{n\geqslant 0} \mathcal{C}_{\geqslant -n},$$

and by

$$\mathcal{C}^{\mathrm{b}} := \mathcal{C}^+ \cap \mathcal{C}^-$$

the stable subcategory of *bounded objects*. We say that the t-structure is *left-bounded*, *right-bounded* or *bounded*, if the inclusion of $\mathcal{C}^+$, $\mathcal{C}^-$ or $\mathcal{C}^{\mathrm{b}}$, respectively, in $\mathcal{C}$, is an equivalence.

The intersection

$$\mathcal{C}^{\heartsuit} = \mathcal{C}_{\geqslant 0} \cap \mathcal{C}_{\leqslant 0}$$

is called the *heart* of the t-structure.

The ∞-category $\mathcal{C}^{\heartsuit}$ is always equivalent to (the nerve of) its homotopy category $h\mathcal{C}^{\heartsuit}$, which is an abelian category (see [41, Remark 1.2.1.12]). Following [41], we abuse the notation by identifying $\mathcal{C}^{\heartsuit}$ with the abelian category $h\mathcal{C}^{\heartsuit}$.

Definition 2.5. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. We define the *left completion* $\widehat{\mathcal{C}}$ of $\mathcal{C}$ to be the limit of the tower

$$\ldots \longrightarrow \mathcal{C}_{\leqslant 2} \xrightarrow{\tau_{\leqslant 1}} \mathcal{C}_{\leqslant 1} \xrightarrow{\tau_{\leqslant 0}} \mathcal{C}_{\leqslant 0} \xrightarrow{\tau_{\leqslant -1}} \ldots .$$

We say that $\mathcal{C}$ is *left-complete* if the functor $\mathcal{C}\longrightarrow\widehat{\mathcal{C}}$ is an equivalence.

By [41, Proposition 1.2.1.17], the left completion $\widehat{\mathcal{C}}$ is again a stable ∞-category, inherits a t-structure from $\mathcal{C}$, and is left-complete.

Two important examples of stable ∞-categories with t-structures are the ∞-category of spectra (as discussed in [41, §1.4]) and the derived ∞-category of an abelian category (as discussed in [41, §1.3]).

Example 2.6. Denote by **Spectra** the ∞-category of spectra and the two full subcategories

$$\mathbf{Spectra}_{\geqslant 0} = \{X \in \mathbf{Spectra} : \pi_n X = 0 \text{ for } n < 0\},$$
$$\mathbf{Spectra}_{\leqslant 0} = \{X \in \mathbf{Spectra} : \pi_n X = 0 \text{ for } n > 0\}.$$

define a t-structure. Left- and right-bounded objects correspond to connective and coconnective spectra, respectively, and its heart can be identified with the abelian category of abelian groups. Moreover, as proved in [41, Proposition 1.4.3.6], it is left complete.

Example 2.7. Suppose that $\mathcal{A}$ is an abelian category with enough projective objects. There exists an associated *derived ∞-category* $\mathcal{D}^-(\mathcal{A})$, whose objects can be identified with right-bounded chain complexes with values in $\mathcal{A}$. This ∞-category $\mathcal{D}^-(\mathcal{A})$ is stable and its homotopy category $h\mathcal{D}^-(\mathcal{A})$ can be identified as the usual derived category as triangulated categories.

It admits a natural t-structure defined as follows:

- $\mathcal{D}^-(\mathcal{A})_{\geqslant 0}$ is the full subcategory spanned by the complexes whose homology vanishes in negative degrees;
- $\mathcal{D}^-(\mathcal{A})_{\leqslant 0}$ is the full subcategory spanned by the complexes whose homology vanishes in positive degrees.

As proved in [41, Proposition 1.3.3.16], this t-structure is left complete and right bounded. Also, as proved in [41, Proposition 1.3.3.12], the derived ∞-category $\mathcal{D}^-(\mathcal{A})$ has a universal property in the sense that if $\mathcal{C}$ is any stable ∞-category equipped with a left-complete t-structure, then any right exact functor $\mathcal{A} \to \mathcal{C}^\heartsuit$ extends (in an essentially unique way) to a right t-exact functor $\mathcal{D}^-(\mathcal{A}) \to \mathcal{C}$.

We have the following recognition criterion due to Lurie [41, Proposition 1.3.3.7].

PROPOSITION 2.8. *Let $\mathcal{C}$ be a stable ∞-category equipped with a left-complete t-structure, whose heart $\mathcal{A} = h\mathcal{C}^\heartsuit$ has enough projective objects. Then, there exists an essentially unique t-exact functor*

$$F : \mathcal{D}^-(\mathcal{A}) \longrightarrow \mathcal{C}$$

extending the inclusion $N(\mathcal{A}) \simeq \mathcal{C}^\heartsuit \subseteq \mathcal{C}$. Here, $N(\mathcal{A})$ is the nerve of the abelian category $\mathcal{A}$.

Moreover, the following two conditions are equivalent:

- *the functor F is fully faithful*;
- *for any pair of objects $X, Y \in \mathcal{A}$, if X is projective, then the abelian groups*

$$[\Sigma^{-i}X, Y]_{\mathcal{C}}$$

vanish for $i>0$.

If the conditions are satisfied, then the essential image of F is the full subcategory

$$\mathcal{C}^{-} = \bigcup_{n} \mathcal{C}_{\geqslant n}$$

of right-bounded objects in $\mathcal{C}$.

Remark 2.9. It is clear that if we restrict the functor F on the bounded stable subcategory $\mathcal{D}^b(\mathcal{A})$, then it gives an equivalence of stable ∞-categories

$$F\colon \mathcal{D}^b(\mathcal{A}) \longrightarrow \mathcal{C}^b$$

that preserves t-structures.

Remark 2.10. Lurie's theorem is exactly the reason we are working with stable ∞-categories instead of triangulated categories. Given a triangulated category equipped with a t-structure, there in general does not exist a functor from the derived category of the heart to the original triangulated category extending the identity functor on the heart (see [17, Remark IV.4.13] for a more detailed explanation for example). However, if the triangulated category comes from the homotopy category of a stable ∞-category, then such a functor always exists. Moreover, Lurie's theorem gives us a recognition criterion in terms of homological algebra to see when such a functor is also an equivalence and preserves t-structures.

We now use Lurie's theorem to prove the main result of this section.

PROPOSITION 2.11. *Let $\mathcal{C}$ be a stable ∞-category with a given bounded t-structure. Suppose that the following conditions hold*:

(1) *the abelian category $\mathcal{A}=h\mathcal{C}^{\heartsuit}$ has enough projective objects*;
(2) *for any pair of objects $X, Y \in \mathcal{A}$, if X is projective, then the abelian groups*

$$[\Sigma^{-i}X, Y]_{\mathcal{C}}$$

vanish for $i>0$.

Then, there exists an equivalence of stable ∞-categories

$$F\colon \mathcal{D}^b(\mathcal{A}) \longrightarrow \mathcal{C}$$

extending the inclusion $N(\mathcal{A})\simeq\mathcal{C}^{\heartsuit}\subseteq\mathcal{C}$, and which preserves t-structures. Here, $N(\mathcal{A})$ is the nerve of the abelian category $\mathcal{A}$ and $\mathcal{D}^b(\mathcal{A})$ is the bounded derived category of $\mathcal{A}$.

Proof. As explained in [41, Remark 1.2.1.18], for any stable ∞-category $\mathcal{C}$ with a t-structure, the functor $\mathcal{C}\to\widehat{\mathcal{C}}$ induces an equivalence

$$\mathcal{C}^{+}\longrightarrow(\widehat{\mathcal{C}})^{+}.$$

For the stable ∞-category $\mathcal{C}$ with a bounded t-structure in the statement of Proposition 2.11, we consider its left completion $\widehat{\mathcal{C}}$, so that we could apply Proposition 2.8. Therefore, the equivalence in the statement of Proposition 2.11 comes from the following zigzag of equivalences:

$$\mathcal{C}\longleftarrow\mathcal{C}^{+}\longrightarrow(\widehat{\mathcal{C}})^{+}\longleftarrow(\widehat{\mathcal{C}})^{b}\longleftarrow\mathcal{D}^b(\mathcal{A}),$$

where the first equivalence comes from the fact that the t-structure on $\mathcal{C}$ is bounded, the third equivalence comes from the fact that the t-structure on $\widehat{\mathcal{C}}$ is right bounded, since $\mathcal{C}$ is, and the last equivalence comes from Lurie's theorem and Remark 2.9. □

Considering the opposite category, we have the following dual version of Proposition 2.11.

PROPOSITION 2.12. *Let $\mathcal{C}$ be a stable ∞-category with a given bounded t-structure. Suppose that the following conditions hold*:

(1) *the abelian category $\mathcal{A}=h\mathcal{C}^{\heartsuit}$ has enough injective objects*;

(2) *for any pair of objects $X,Y\in\mathcal{A}$, if Y is injective, then the abelian groups*

$$[\Sigma^{-i}X,Y]_{\mathcal{C}}$$

vanish for $i>0$.

Then, there exists an equivalence of stable ∞-categories

$$G\colon \mathcal{D}^b(\mathcal{A}) \longrightarrow \mathcal{C}$$

extending the inclusion $N(\mathcal{A})\simeq\mathcal{C}^{\heartsuit}\subseteq\mathcal{C}$, and which preserves t-structures. Here, $N(\mathcal{A})$ is the nerve of the abelian category $\mathcal{A}$ and $\mathcal{D}^b(\mathcal{A})$ is the bounded derived category of $\mathcal{A}$.

3. An algebraic model for cellular $\mathrm{MU}^{\mathrm{mot}}/\tau$-modules

In this section, we use Proposition 2.11 to prove Theorem 1.11. Namely, there exists a t-exact equivalence of stable ∞-categories

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{cell}} \longrightarrow \mathcal{D}^{b}(\mathrm{MU}_{*}\text{-}\mathbf{Mod}^{\mathrm{ev}}),$$

whose restriction on the heart is given by

$$\pi_{*,*}\colon \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}} \longrightarrow \mathrm{MU}_{*}\text{-}\mathbf{Mod}^{\mathrm{ev}}.$$

In §3.1, we first recall the universal coefficient spectral sequence in the category

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}},$$

which is constructed by Dugger–Isaksen [13]. This is stated as Theorem 3.2. Using this spectral sequence, we prove the equivalence on the heart as Proposition 3.5 in §3.2. Then, using this spectral sequence again, we show in §3.3 that the full subcategories

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b,\geqslant 0}_{\mathrm{cell}} \quad \text{and} \quad \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b,\leqslant 0}_{\mathrm{cell}}$$

define a t-structure. In the end of this section, we prove the equivalence of stable ∞-categories as Theorem 3.8.

We will use in §5 the above equivalence of stable ∞-categories to construct enough motivic spectra to build $\mathrm{MU}^{\mathrm{mot}}/\tau$-based Adams resolutions in the category

$$\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}.$$

3.1. The category $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}$ and the universal coefficient spectral sequence

We begin with two adjunctions. The first adjunction

$$\widehat{S^{0,0}}\text{-}\mathbf{Mod}_{\mathrm{cell}} \underset{U}{\overset{\widehat{S^{0,0}}/\tau\wedge -}{\rightleftarrows}} \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}} \tag{3.1}$$

between cellular $\widehat{S^{0,0}}$-modules and cellular $\widehat{S^{0,0}}/\tau$-modules is induced by the E_∞-ring map

$$\widehat{S^{0,0}} \longrightarrow \widehat{S^{0,0}}/\tau.$$

Since $\mathrm{MU}^{\mathrm{mot}}/\tau$ is an E_∞-algebra that is cellular over $\widehat{S^{0,0}}/\tau$, the above adjunction (3.1) extends to

$$\widehat{S^{0,0}}\text{-}\mathbf{Mod}_{\mathrm{cell}} \underset{U}{\overset{\widehat{S^{0,0}}/\tau\wedge -}{\rightleftarrows}} \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}} \underset{U}{\overset{\mathrm{MU}^{\mathrm{mot}}\wedge -}{\rightleftarrows}} \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}. \tag{3.2}$$

Definition 3.1. Denote by

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}$$

the abelian category of graded left modules over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$, and by

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^{0}$$

the full subcategory of $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}$ spanned by all graded modules $M_{*,*}$ that are concentrated in Chow–Novikov degree zero, i.e., $M_{s,w}=0$ whenever $s\neq 2w$.

We thus have a commutative diagram

$$\begin{array}{ccc}
\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}} & \xrightarrow{\pi_{*,*}} & \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod} \\
\uparrow & & \uparrow \\
\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}} & \xrightarrow{\pi_{*,*}} & \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^{0}.
\end{array}$$

Since $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$ is concentrated in Chow–Novikov degree zero, forgetting the motivic weight we have an equivalence

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^{0} \cong \mathrm{MU}_{*}\text{-}\mathbf{Mod}^{\mathrm{ev}}.$$

To show that the restriction of $\pi_{*,*}$ to the heart induces an equivalence

$$\pi_{*,*}\colon \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}} \xrightarrow{\cong} \mathrm{MU}_{*}\text{-}\mathbf{Mod}^{\mathrm{ev}},$$

we recall the universal coefficient spectral sequence constructed by Dugger–Isaksen [13]. This spectral sequence is our main tool to compute homotopy classes of maps in the stable ∞-category $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}$.

THEOREM 3.2. (Universal coefficient spectral sequence) *For any*

$$X, Y \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}},$$

there is a conditionally convergent spectral sequence

$$E_2^{s,t,w} = \mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}(\pi_{*,*}X, \pi_{*,*}Y) \Longrightarrow [\Sigma^{t-s,w}X, Y]_{\mathrm{MU}^{\mathrm{mot}}/\tau}.$$

Moreover, if both $\pi_{*,*}X$ and $\pi_{*,*}Y$ are concentrated in bounded Chow–Novikov degrees, then the spectral sequence converges strongly and collapses at a finite page.

Proof. We refer to [13, Propositions 7.7 and 7.10] for the precise construction of the spectral sequence and the proof of conditional convergence. For the second statement of the theorem, we recall a few facts from the proof of [13, Propositions 7.7 and 7.10].

The E_1-page arises from a free resolution over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$:

$$0 \longleftarrow \pi_{*,*}X \twoheadleftarrow \pi_{*,*}F_0 \longleftarrow \pi_{*,*}F_1 \longleftarrow \ldots,$$

and is given by

$$E_1^{s,t,w} := \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}(\pi_{*,*}(\Sigma^{t,w}F_s), \pi_{*,*}Y).$$

The E_2-page is the cohomology of this chain complex, giving the claimed Ext groups.

Suppose that $\pi_{*,*}X$ and $\pi_{*,*}Y$ are concentrated in Chow–Novikov degrees $[a,b]$ and $[c,d]$, respectively, where $a \leqslant b$ and $c \leqslant d$. As $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$ is concentrated in Chow–Novikov degree zero, we can choose all $\pi_{*,*}(F_s)$ such that they are concentrated in Chow–Novikov degrees $[a,b]$. Therefore, $\pi_{*,*}(\Sigma^{t,w}F_s)$ is concentrated in Chow–Novikov degrees

$$[a+(t-2w), b+(t-2w)]$$

for all $s \geqslant 0$.

In order for the group $E_1^{s,t,w}$ to be non-zero, we must have

$$c \leqslant b+(t-2w) \quad \text{and} \quad d \geqslant a+(t-2w).$$

For a fixed weight w, this gives that

$$t \in [c-b+2w, d-a+2w].$$

Since later pages $E_r^{s,t,w}$ are iterated subquotients of $E_1^{s,t,w}$, their t-degrees are all concentrated in $[c-b+2w, d-a+2w]$.

Recall that the d_r-differential has the form

$$E_r^{s,t,w} \xrightarrow{d_r} E_r^{s+r,t+r-1,w}.$$

In particular, it changes the t-degrees by $r-1$. Since the t-degrees of all possible non-zero elements in the E_1-page satisfy $t \in [c-b+2w, d-a+2w]$, we must have $d_r = 0$ when

$$r-1 > (d-a+2w)-(c-b+2w) = (b-a)+(d-c)$$

for degree reasons. In other words, the spectral sequence collapses at the page

$$E_{(b-a)+(d-c)+2}.$$

Therefore, under the condition that both $\pi_{*,*}X$ and $\pi_{*,*}Y$ are concentrated in bounded Chow–Novikov degrees, this spectral sequence convergences strongly and collapses at a finite page. □

Recall from Definition 1.8 that

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b,\geqslant 0}_{\mathrm{cell}}, \quad \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b,\leqslant 0}_{\mathrm{cell}} \quad \text{and} \quad \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$$

are the full subcategories of $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{cell}}$ that are spanned by objects whose homotopy groups are concentrated in non-negative, non-positive and zero Chow–Novikov degrees, respectively.

COROLLARY 3.3. *Let*

$$X \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b,\geqslant 0}_{\mathrm{cell}} \quad \textit{and} \quad Y \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b,\leqslant 0}_{\mathrm{cell}}.$$

The abelian group of homotopy classes of bi-degree $(0,0)$ *can be computed algebraically by the isomorphism*

$$[X,Y]_{\mathrm{MU}^{\mathrm{mot}}/\tau} \longrightarrow \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}{}_{*,*}/\tau}(\pi_{*,*}X, \pi_{*,*}Y)$$

that is induced by applying $\pi_{*,*}$.

Proof. Consider the E_2-page of the universal coefficient spectral sequence. The tri-degrees that converge to the bi-degree $(0,0)$ are of the form $(t,t,0)$, which correspond to $E_2^{s,t,w} = E_2^{t,t,0}$, for $t \geqslant 0$.

By the proof of Theorem 3.2, the t-degrees of all possible non-zero elements in the E_1-page, and therefore E_2-page, satisfy $t \leqslant d-a+2w = d-a$. Since $\pi_{*,*}X$ and $\pi_{*,*}Y$ are concentrated in non-negative and non-positive bounded Chow–Novikov degrees, we have $d=a=0$. Therefore, we have $t \leqslant 0$.

Combining both facts, we have established that the only possible non-zero elements in the E_2-page that converge to the bi-degree $(0,0)$ are in

$$E_2^{0,0,0} = \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}{}_{*,*}/\tau}(\pi_{*,*}X, \pi_{*,*}Y).$$

To show that all elements in $E_2^{0,0,0}$ survive in the spectral sequence, first note that they are not targets of any non-zero differentials, since they are in s-degree zero. Second, all d_r-differentials for $r \geqslant 2$ increase the t-degree. Since the t-degrees of all non-zero elements are non-positive, the elements in $E_2^{0,0,0}$ do not support non-zero differentials. This completes the proof. □

3.2. The equivalence on the heart

We are now ready to show that the functor $\pi_{*,*}$ induces an equivalence on the heart. The following is a special case of Corollary 3.3.

COROLLARY 3.4. *The functor*

$$\pi_{*,*}\colon \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}} \longrightarrow \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^{0}$$

is fully faithful. Here, the right-hand side is understood as a discrete ∞-category.

As a consequence, Corollary 3.4 shows that $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$ is also a discrete ∞-category.

Proof. For $n\geqslant 0$ and two objects $X, Y \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$, by Corollary 3.3, the edge homomorphism

$$\left[\Sigma^{n,0}X, Y\right]_{\mathrm{MU}^{\mathrm{mot}}/\tau} \xrightarrow{\pi_{*,*}} \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*}/\tau}(\pi_{*,*}\Sigma^{n,0}X, \pi_{*,*}Y)$$

is an isomorphism. When $n>0$, the bigraded module $\pi_{*,*}\Sigma^{n,0}X$ is concentrated in positive Chow–Novikov degree. So the right-hand side of the above isomorphism is concentrated in the case $n=0$. This shows that $\pi_{*,*}$ is fully faithful on $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$. □

To show the equivalence on the heart, we only need to show the essential surjectivity of $\pi_{*,*}$.

PROPOSITION 3.5. *The functor*

$$\pi_{*,*}\colon \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}} \longrightarrow \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^{0}$$

is an equivalence of ∞-categories.

Proof. We need to show that any object $M \in \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^{0}$ can be realized as the homotopy groups of an object in $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$.

Suppose that M is a free $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$-module that is concentrated in Chow–Novikov degree zero:

$$M \cong \bigoplus_{i\in I} \Sigma^{2k_i,k_i}\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau.$$

Here, $\Sigma^{2k_i,k_i}\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$ is a free bigraded rank-1 module over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$ with a generator in bi-degree $(2k_i, k_i)$. We can realize M as the homotopy groups of the wedge

$$\bigvee_{i\in I} \Sigma^{2k_i,k_i}\mathrm{MU}^{\mathrm{mot}}/\tau$$

with the same index set, which is cellular.

For an arbitrary $M \in \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^{0}$, we can pick a free resolution

$$0 \longleftarrow M \longleftarrow F_0 \xleftarrow{f_1} F_1 \xrightarrow{f_2} F_2 \longleftarrow \ldots \tag{3.3}$$

in $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^{0}$.

Each F_i can be realized by

$$Z_i \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$$

and by Corollary 3.4, each map f_i can be realized by a map $g_i \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$ as in

$$Z_0 \xleftarrow{g_1} Z_1 \xleftarrow{g_2} Z_2 \longleftarrow \dots .$$

Using the standard method of lifting an Adams resolution to an Adams tower, we claim that we can construct a tower

$$X_1 \longrightarrow X_2 \longrightarrow \dots,$$

with the property that the homotopy groups of X_i are given by the following groups

$$\bigoplus_{l=-\infty}^{+\infty} \pi_{2l+k,l}(X_i) = \begin{cases} M = \mathrm{Coker} f_1, & \text{if } k = 0, \\ \Sigma^{i,0}\mathrm{Ker} f_i, & \text{if } k = i, \\ 0, & \text{otherwise.} \end{cases}$$

and that the maps in this tower induce an isomorphism on the Chow–Novikov-degree-zero part of these homotopy groups, which is M.

We prove this claim inductively.

In fact, we can choose X_1 to be the cofiber of

$$g_1 \colon Z_1 \longrightarrow Z_0.$$

This gives us a long exact sequence on homotopy groups

$$\dots \longrightarrow \pi_{*+1,*}X_1 \longrightarrow \pi_{*,*}Z_1 \xrightarrow{f_1} \pi_{*,*}Z_0 \longrightarrow \pi_{*,*}X_1 \longrightarrow \dots .$$

Since both $\pi_{*,*}Z_0$ and $\pi_{*,*}Z_1$ are concentrated in Chow–Novikov degree zero, we must have that $\pi_{*,*}X_1$ is concentrated in Chow–Novikov degrees zero and 1. We can compute directly from the long exact sequence that the Chow–Novikov degree-zero and degree-1 parts of $\pi_{*,*}X_1$ are isomorphic to $M = \mathrm{Coker} f_1$ and $\Sigma^{1,0}\mathrm{Ker} f_1$, respectively.

Suppose now that we have constructed the tower up to X_i. We have a homomorphism

$$\pi_{*,*}Z_{i+1} \cong F_{i+1} \twoheadrightarrow \mathrm{Im} f_{i+1} \cong \mathrm{Ker} f_i \hookrightarrow \pi_{*,*}(\Sigma^{-i,0}X_i)$$

in $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^{0}$. Here, the first map is induced by f_{i+1} and the second map corresponds to the Chow–Novikov-degree-i part of $\pi_{*,*}X_i$.

By Corollary 3.3, this homomorphism can be realized as a map

$$Z_{i+1} \longrightarrow \Sigma^{-i,0}X_i.$$

Define X_{i+1} as the $\Sigma^{i,0}$-suspension of its cofiber, so we have a cofiber sequence

$$\Sigma^{i,0}Z_{i+1} \longrightarrow X_i \longrightarrow X_{i+1}.$$

By the associated long exact sequence in homotopy groups, we have that $\pi_{*,*}X_{i+1}$ is concentrated in Chow–Novikov degrees zero and $i+1$. The Chow–Novikov-degree-zero part is isomorphic to M, and the Chow–Novikov degree $i+1$ part is isomorphic to

$$\Sigma^{i+1,0}\mathrm{Ker}f_{i+1},$$

as required.

Having the tower

$$X_1 \longrightarrow X_2 \longrightarrow ...,$$

we define X as its colimit

$$X := \mathrm{colim}\,(X_1 \longrightarrow X_2 \longrightarrow ...\,).$$

The homotopy groups of X are computed by the colimit

$$\pi_{*,*}X \cong \mathrm{colim}\,(\pi_{*,*}X_1 \longrightarrow \pi_{*,*}X_2 \longrightarrow ...\,) = M,$$

and are in particular concentrated in Chow–Novikov degree zero.

Therefore, we have proved that any module $M \in \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^0$ can be realized as a spectrum $X \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$. □

3.3. The t-structure, and the equivalence of categories

We prove in this subsection that the full subcategories previously defined satisfy the required axioms for the t-structure.

We first prove a general proposition regarding the existence of a t-structure.

PROPOSITION 3.6. *Let $\mathcal{C}$ be a stable ∞-category, and $\mathcal{C}^{\geqslant 0}$ and $\mathcal{C}^{\leqslant 0}$ be a pair of full subcategories of $\mathcal{C}$. Let $\mathcal{C}^{\geqslant n} = \Sigma^n\mathcal{C}^{\geqslant 0}$ and $\mathcal{C}^{\leqslant n} = \Sigma^n\mathcal{C}^{\leqslant 0}$. Suppose that the following conditions hold*:

(1) *$\mathcal{C}^{\geqslant 0}$ is closed under extensions and suspensions. $\mathcal{C}^{\leqslant 0}$ is closed under extensions and desuspensions*;

(2)

$$\mathcal{C} = \bigcup_{n\in\mathbb{Z}} \mathcal{C}^{\geqslant n};$$

(3) *for any* $X \in \mathcal{C}^{\geqslant 0}$ *and* $Y \in \mathcal{C}^{\leqslant -1}$, *we have*

$$[X, Y]_{\mathcal{C}} = 0;$$

(4) *for any* $X \in \mathcal{C}^{\geqslant 0}$, *there is an object* $X_0 \in \mathcal{C}^{\geqslant 0} \cap \mathcal{C}^{\leqslant 0}$ *and a morphism*

$$X \longrightarrow X_0$$

such that its fiber lies in $\mathcal{C}^{\geqslant 1}$.

Then, this pair of subcategories $\mathcal{C}^{\geqslant 0}$ *and* $\mathcal{C}^{\leqslant 0}$ *defines a* t*-structure on* $\mathcal{C}$.

Proof. In view of Definition 2.1, it remains to check that, for any $X \in \mathcal{C}$, there exists a fiber sequence

$$X_{\geqslant 0} \longrightarrow X \longrightarrow X_{\leqslant -1}$$

such that $X_{\geqslant 0} \in \mathcal{C}^{\geqslant 0}$ and $X_{\leqslant -1} \in \mathcal{C}^{\leqslant -1}$.

By the second assumption, there exists n such that $X \in \mathcal{C}^{\geqslant n}$.

If $n \geqslant 0$, we can take the fiber sequence to be

$$X \longrightarrow X \longrightarrow *.$$

If $n < 0$, using the fourth assumption, there exits an object $X_n \in \mathcal{C}^{\geqslant n} \cap \mathcal{C}^{\leqslant n}$ and a fiber sequence

$$X_{\geqslant n+1} \longrightarrow X \longrightarrow X_n$$

such that $X_{\geqslant n+1} \in \mathcal{C}^{\geqslant n+1}$.

For $n < 0$, iterating this process, we get a finite sequence of morphsims

$$X_{\geqslant 0} \longrightarrow X_{\geqslant -1} \longrightarrow \ldots \longrightarrow X_{\geqslant n+1} \longrightarrow X$$

such that $X_{\geqslant i} \in \mathcal{C}^{\geqslant i}$ and that the cofiber X_{i-1} of the morphism

$$X_{\geqslant i} \longrightarrow X_{\geqslant i-1}$$

lies in $\mathcal{C}^{\geqslant i-1} \cap \mathcal{C}^{\leqslant i-1}$, where $i \in [n+1, 0]$.

We can now take $X_{\leqslant -1}$ to be the cofiber of the morphism

$$X_{\geqslant 0} \longrightarrow X.$$

The object $X_{\leqslant -1}$ can be build up by finite extensions from the objects X_{i-1} for $i \in [n+1, 0]$. Therefore, by the first assumption, we have $X_{\leqslant -1} \in \mathcal{C}^{\leqslant -1}$. □

PROPOSITION 3.7. *The pair of full subcategories*

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\geqslant 0} \quad \textit{and} \quad \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\leqslant 0}$$

defines a bounded t-structure on

$$\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b}.$$

Proof. We check the four conditions in Proposition 3.6.

The first two conditions follow directly from the definition of the Chow–Novikov degree.

For the third condition, we need to show that

$$[X,Y]_{\mathrm{MU}^{\mathrm{mot}}/\tau} = 0$$

for any objects

$$X \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\geqslant 0} \quad \text{and} \quad Y \in \Sigma^{-1,0}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\leqslant 0}.$$

By Corollary 3.3, we have that

$$[X,Y]_{\mathrm{MU}^{\mathrm{mot}}/\tau} \cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}(\pi_{*,*}X, \pi_{*,*}Y)$$

As $\pi_{*,*}X$ is concentrated in non-negative Chow–Novikov degrees and $\pi_{*,*}Y$ is concentrated in negative Chow–Novikov degrees, the right-hand side is zero.

For the fourth condition, we need to show that, for any

$$X \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\geqslant 0},$$

there exists a fiber sequence

$$X_{\geqslant 1} \longrightarrow X \longrightarrow X_0$$

such that

$$X_0 \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{\heartsuit} \quad \text{and} \quad X_{\geqslant 1} \in \Sigma^{1,0}(\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{b,\geqslant 0}).$$

Consider the Chow–Novikov-degree-zero part of $\pi_{*,*}(X)$, namely

$$\pi_{*,*}(X)^{=0} := \bigoplus_k \pi_{2k,k}(X).$$

By Proposition 3.5, there is a spectrum

$$X_0 \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}^{\heartsuit}$$

realizing this bigraded $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$-module

$$\pi_{*,*}X_0 \cong \pi_{*,*}(X)^{=0}.$$

Consider the projection map

$$\pi_{*,*}(X) \twoheadrightarrow \pi_{*,*}(X)^{=0} \cong \pi_{*,*}X_0.$$

By Corollary 3.3, the projection map can be realized by a map

$$X \longrightarrow X_0.$$

Denote by $X_{\geqslant 1}$ its fiber. Then from the long exact sequence in homotopy groups, we have that

$$X_{\geqslant 1} \in \Sigma^{1,0}(\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b,\geqslant 0}_{\mathrm{cell}}). \qquad \square$$

Having this t-structure on $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{cell}}$, the main result of this section follows from Proposition 2.11.

THEOREM 3.8. *There is a t-exact equivalence of stable ∞-categories*

$$\mathcal{D}^b(\mathrm{MU}_*\text{-}\mathbf{Mod}^{\mathrm{ev}}) \xrightarrow{\cong} \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{cell}}.$$

Proof. By Proposition 3.7, the t-structure is bounded. By Proposition 3.5 and the equivalence

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^{0} \cong \mathrm{MU}_*\text{-}\mathbf{Mod}^{\mathrm{ev}},$$

the heart can be identified as modules over MU_*. Therefore, it has enough projective objects.

It remains to show that, for any two objects,

$$X, Y \in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}},$$

with $\pi_{*,*}X$ projective over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$, we have that

$$[\Sigma^{-i,0}X, Y]_{\mathrm{MU}^{\mathrm{mot}}/\tau} = 0$$

for $i>0$.

We apply the universal coefficient spectral sequence in Theorem 3.2, namely

$$\mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}(\pi_{*,*}X, \pi_{*,*}Y) \Longrightarrow [\Sigma^{t-s,w}X, Y]_{\mathrm{MU}^{\mathrm{mot}}/\tau}.$$

Since $\pi_{*,*}X$ is projective over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$, the E_2-page of the spectral sequence is concentrated on the line $s{=}0$, and therefore collapses at the E_2-page.

Moreover, since both $\pi_{*,*}X$ and $\pi_{*,*}Y$ are concentrated in Chow–Novikov degree zero, the E_2-page is also concentrated in Chow–Novikov degree zero, namely $t{-}2w{=}0$ in this case.

We are interested in the case when $t{-}s{=}{-}i{<}0$ and $w{=}0$. By the above analysis, the corresponding tri-degrees in the E_2-page are all zero in our case. Therefore, we must have that

$$[\Sigma^{-i,0}X, Y]_{\mathrm{MU}^{\mathrm{mot}}/\tau} = 0.$$

This completes the proof. □

4. An algebraic model for harmonic $\widehat{S^{0,0}}/\tau$-modules

After the warmup in §3, we use Proposition 2.12 to prove Theorem 1.13. Namely, there exists a t-exact equivalence of stable ∞-categories

$$\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{harm}} \longrightarrow \mathcal{D}^b(\mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}^{\mathrm{ev}}),$$

whose restriction on the heart is given by

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}\colon \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}} \longrightarrow \mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}^{\mathrm{ev}}.$$

The structure of this section is similar to that of §3.

In §4.1, we discuss the category of harmonic $\widehat{S^{0,0}}/\tau$-modules. We will also recall certain facts on the category of $\mathrm{MU}_*\mathrm{MU}$-comodules, such as Landweber's filtration theorem. Instead of using the universal coefficient spectral sequence in the category $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}$, we will use the absolute Adams–Novikov spectral sequence in the category of harmonic $\widehat{S^{0,0}}/\tau$-modules. This spectral sequence is constructed in §5. Using this spectral sequence, we prove the equivalence on the heart as Proposition 4.11 in §4.2. Then, using again this spectral sequence, we show in §4.3 that the full subcategories

$$\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b,\geqslant 0}_{\mathrm{harm}} \quad \text{and} \quad \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b,\leqslant 0}_{\mathrm{harm}}$$

define a t-structure, and conclude the equivalence of stable ∞-categories as Proposition 4.12 and Theorem 4.13.

4.1. The categories $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}_{\mathbf{harm}}$ and $\mathbf{MU_*MU}$-$\mathbf{Comod}^{\mathbf{ev}}$

We first recall from Definition 1.6 that a $\widehat{S^{0,0}}/\tau$-module spectrum Y is *harmonic* if it is $\widehat{S^{0,0}}/\tau$-cellular and the natural map

$$Y \longrightarrow Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}$$

is an isomorphism on $\pi_{*,*}$. As pointed out after Definition 1.6, the two completions $X^{\wedge}_{\mathrm{MGL}}$ in the category $\mathbb{C}$-**mot-Spectra** and $X^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}$ in the category $\widehat{S^{0,0}}$-**Mod** are equivalent for any X in $\widehat{S^{0,0}}$-$\mathbf{Mod}_{\mathrm{cell}}$. It is clear that in the category $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}_{\mathrm{cell}}$, being harmonic is closed under taking suspensions, finite products and fibers. The category of harmonic $\widehat{S^{0,0}}/\tau$-module spectra is denoted by $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}_{\mathrm{harm}}$.

We have the following examples and non-examples of harmonic $\widehat{S^{0,0}}/\tau$-module spectra.

Example 4.1. (1) Any finite cellular object in $\widehat{S^{0,0}}/\tau$-**Mod** is harmonic. In fact, by discussion in §7 of Dugger–Isaksen [14], the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-completed sphere $\widehat{S^{0,0}}$ is MGL-complete, and is therefore $\mathrm{MU}^{\mathrm{mot}}$-complete. Then, the claim follows from an induction argument.

(2) Any finite cellular object in $\mathrm{MU}^{\mathrm{mot}}/\tau$-**Mod** is harmonic.

(3) The η-inverted cofiber of τ is $\widehat{S^{0,0}}/\tau$-cellular but *not* harmonic.

Here, η is the Hopf map in $\pi_{1,1}\widehat{S^{0,0}}$. Post-composing with the unit map $\widehat{S^{0,0}} \to \widehat{S^{0,0}}/\tau$, we also denote its Hurewicz image in $\pi_{1,1}\widehat{S^{0,0}}/\tau$ by η. It is non-nilpotent in the ring $\pi_{*,*}\widehat{S^{0,0}}/\tau$. One way to see this fact is to identify $\pi_{*,*}\widehat{S^{0,0}}/\tau$ as $\mathrm{Ext}^{*,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*)$ by Gheorghe–Isaksen, and to use the fact from Miller–Ravenel–Wilson [50] that the element that detects η in $\mathrm{Ext}^{*,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*)$ is non-nilpotent. The η-inverted cofiber of τ,

$$\eta^{-1}\widehat{S^{0,0}}/\tau := \mathrm{colim}(\widehat{S^{0,0}}/\tau \longrightarrow \Sigma^{-1,-1}\widehat{S^{0,0}}/\tau \longrightarrow \Sigma^{-2,-2}\widehat{S^{0,0}}/\tau \longrightarrow \dots),$$

is a cellular object in $\widehat{S^{0,0}}/\tau$-**Mod**. Since η maps to zero in $\pi_{*,*}\mathrm{MU}^{\mathrm{mot}}$, we have that

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}(\eta^{-1}\widehat{S^{0,0}}/\tau) = 0.$$

Therefore, the completion $(\eta^{-1}\widehat{S^{0,0}}/\tau)^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}$ is contractible. This shows that the spectrum $\eta^{-1}\widehat{S^{0,0}}/\tau$ is not harmonic.

The following Lemma 4.2 will be used in the proof of Proposition 4.11. We postpone the proof of Lemma 4.2 until the end of §5.

LEMMA 4.2. *Suppose that* $\{Y_\alpha\}_\alpha$ *is a filtered system in* $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$ *such that each* Y_α *is harmonic. Then, the colimit of* $\{Y_\alpha\}_\alpha$ *in* $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$ *is also harmonic.*

Recall that, for a $\widehat{S^{0,0}}/\tau$-module X, its $\mathrm{MU}^{\mathrm{mot}}$-homology can also be described as

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}X = \pi_{*,*}(\mathrm{MU}^{\mathrm{mot}} \wedge_{\widehat{S^{0,0}}} X) \cong \pi_{*,*}(\mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{\widehat{S^{0,0}}/\tau} X).$$

Following computations of $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}$ from Hu–Kriz–Ormsby [28] and Dugger–Isaksen [14], we have the following $\mathrm{MU}^{\mathrm{mot}}$-homology of $\mathrm{MU}^{\mathrm{mot}}/\tau$:

$$\pi_{*,*}(\mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{\widehat{S^{0,0}}/\tau} \mathrm{MU}^{\mathrm{mot}}/\tau) \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau[b_1, b_2, \dots] \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau,$$

where $|b_i| = (2i, i)$, and is in Chow–Novikov degree zero. Since τ can be realized as a map $\widehat{S^{0,-1}} \to \widehat{S^{0,0}}$, it is primitive in $\mathrm{MU}^{\mathrm{mot}}$. Therefore, $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$ is a Hopf algebroid.

Definition 4.3. Denote by

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}$$

the abelian category of graded left comodules over the Hopf algebroid $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$, and by

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}^0$$

its full subcategory spanned by all graded comodules whose underlying $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$-modules are concentrated in Chow–Novikov degree zero.

We therefore have a commutative diagram

$$\begin{array}{ccc} \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}_{\mathrm{harm}} & \xrightarrow{\mathrm{MU}^{\mathrm{mot}}_{*,*}} & \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod} \\ \uparrow & & \uparrow \\ \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}} & \xrightarrow{\mathrm{MU}^{\mathrm{mot}}_{*,*}} & \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}^0. \end{array}$$

Forgetting the motivic weight, we have the equivalence

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}^0 \cong \mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}^{\mathrm{ev}}.$$

Recall that we have the adjunction between modules and comodules

$$U\colon \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod} \rightleftarrows \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}\colon \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} -. \quad (4.1)$$

The forgetful functor is a *left* adjoint, while the tensor-up functor is a *right* adjoint. We refer to [24, §1.1] for more details.

Using the ring map $\widehat{S^{0,0}}/\tau \longrightarrow \mathrm{MU}^{\mathrm{mot}}/\tau$, we can form the commutative diagram

$$\begin{array}{ccc} \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}} & \xrightarrow{\pi_{*,*}} & \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod} \\ \uparrow{\scriptstyle \mathrm{MU}^{\mathrm{mot}}/\tau \wedge_{\widehat{S^{0,0}}/\tau} -} & & \uparrow{\scriptstyle \mathrm{U}} \\ \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}} & \xrightarrow{\mathrm{MU}^{\mathrm{mot}}_{*,*}} & \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}. \end{array}$$

For the category of comodules over $\mathrm{MU}_*\mathrm{MU}$, we recall the Landweber's filtration theorem. Recall from [36] and [37] that there are elements $v_n \in \mathrm{MU}_*$, with $v_0 = p$, giving the invariant prime ideals $I_n = (v_0, ..., v_n) \trianglelefteq \mathrm{MU}_*$. Moreover, these elements satisfy the formula

$$\eta_R(v_n) \equiv v_n \mod I_{n-1},$$

and so MU_*/I_n is canonically a comodule over $\mathrm{MU}_*\mathrm{MU}$. This gives a short exact sequence of comodules

$$0 \longrightarrow \mathrm{MU}_*/I_n \xrightarrow{\cdot v_n} \mathrm{MU}_*/I_n \longrightarrow \mathrm{MU}_*/I_{n+1} \longrightarrow 0,$$

for every $n \geqslant 0$. Landweber's filtration theorem [36], [37] states that any comodule M over $\mathrm{MU}_*\mathrm{MU}$ whose underlying MU_*-module is finitely presented, can be reconstructed by finitely many extensions of suspensions of MU_*/I_n's.

THEOREM 4.4. (Landweber's filtration theorem) *Suppose that S is a class of comodules over $\mathrm{MU}_*\mathrm{MU}$ such that the following conditions hold*:

(1) *S contains MU_* and MU_*/I_n for all $n \geqslant 0$;*

(2) *S is closed under suspensions and extensions.*

Then, S contains all comodules over $\mathrm{MU}_*\mathrm{MU}$ whose underlying MU_*-modules are finitely presented.

There are two more facts that we will use on the category of comodules over $\mathrm{MU}_*\mathrm{MU}$. The first one is the following lemma. For a proof, see Miller–Ravenel [49, Lemma 2.11] and Hovey [24] for example.

LEMMA 4.5. *Any comodule over $\mathrm{MU}_*\mathrm{MU}$ is a filtered colimit of finitely presented comodules.*

The second one is a consequence of the fact that the forgetful functor from the category $\mathrm{MU}_*\mathrm{MU}$-$\mathbf{Comod}^{\mathrm{ev}}$ to MU_*-$\mathbf{Mod}^{\mathrm{ev}}$ has a right adjoint. For a precise argument, see the proof of Lemma 5.4 for its motivic analogue.

LEMMA 4.6. *The category* $\mathrm{MU}_*\mathrm{MU}$-$\mathbf{Comod}^{\mathrm{ev}}$ *has enough injective objects.*

We will construct the absolute Adams–Novikov spectral sequence, namely, for any two objects X and Y in this category, there is a strongly convergent spectral sequence that collapses at a finite page:

$$\mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y) \Longrightarrow [\Sigma^{t-s,w}X, Y]_{\widehat{S^{0,0}}/\tau},$$

with differentials

$$d_r \colon E_r^{s,t,w} \longrightarrow E_r^{s+r,t+r-1,w}.$$

The existence of this absolute Adams–Novikov spectral sequence in $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}^{b}_{\mathrm{harm}}$ is proved as Theorem 5.7 in §5.

Using the absolute Adams–Novikov spectral sequence, we will prove the following Corollaries 4.7 and 4.8 in §5.3.

COROLLARY 4.7. *For* $X \in \widehat{S^{0,0}}/\tau$-$\mathbf{Mod}^{b,\geqslant 0}_{\mathrm{harm}}$ *and* $Y \in \widehat{S^{0,0}}/\tau$-$\mathbf{Mod}^{b,\leqslant 0}_{\mathrm{harm}}$, *the following map induced by applying the functor* $\mathrm{MU}^{\mathrm{mot}}_{*,*}$ *is an isomorphism*:

$$[X, Y]_{\widehat{S^{0,0}}/\tau} \longrightarrow \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y).$$

COROLLARY 4.8. *Given* $X, Y \in \widehat{S^{0,0}}/\tau$-$\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}}$, *for any bi-degree* (t, w) *there is an isomorphism*

$$[\Sigma^{t,w}X, Y]_{\widehat{S^{0,0}}/\tau} \cong \mathrm{Ext}^{2w-t,2w,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y).$$

4.2. The equivalence on the heart

We are now ready to show that the functor $\mathrm{MU}^{\mathrm{mot}}_{*,*}$ induces an equivalence on the heart. The following is a special case of Corollary 4.7.

COROLLARY 4.9. *The functor*

$$\mathrm{MU}^{\mathrm{mot}}_{*,*} \colon \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}} \longrightarrow \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}^{0}$$

is fully faithful. Here, the right-hand side is understood as a discrete ∞*-category.*

As a consequence, Corollary 4.9 shows that $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}}$ is also a discrete ∞-category.

Proof. For $n \geqslant 0$ and two objects $X, Y \in \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}}$, by Corollary 4.7, the edge homomorphism

$$[\Sigma^{n,0}X, Y]_{\widehat{S^{0,0}}/\tau} \xrightarrow{\mathrm{MU}^{\mathrm{mot}}_{*,*}} \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}\Sigma^{n,0}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y)$$

is an isomorphism. When $n>0$, the bigraded module $\mathrm{MU}^{\mathrm{mot}}_{*,*}\Sigma^{n,0}X$ is concentrated in positive Chow–Novikov degree. So, the right-hand side of the above isomorphism is concentrated in the case $n=0$. This shows that $\mathrm{MU}^{\mathrm{mot}}_{*,*}$ is fully faithful on $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}}$. □

To show the equivalence on the heart, we only need to show the essential surjectivity of $\mathrm{MU}^{\mathrm{mot}}_{*,*}$.

Unlike the case for modules over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$, we do not have free resolutions for comodules over $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$. We will instead use Landweber's filtration theorem to realize all comodules that are finitely presented, and then extend the result using filtered colimits. In particular, all Smith–Toda complexes exist in $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}$.

We start with the following two-out-of-three lemma.

LEMMA 4.10. *Consider any short exact sequence in* $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}^0$,

$$0 \longrightarrow M' \xrightarrow{f'} M \xrightarrow{f''} M'' \longrightarrow 0. \tag{4.2}$$

If any two of the three comodules M', M *and* M'' *are realizable in* $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}}$, *then so is the third.*

Proof. There are three cases that we need to prove.

(1) Suppose that both comodules M' and M are realizable by

$$M' \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}X' \quad \text{and} \quad M \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}X.$$

By Corollary 4.9, the algebraic map f' is also realizable as the $\mathrm{MU}^{\mathrm{mot}}_{*,*}$-homology of a map

$$X' \xrightarrow{F'} X.$$

Since $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{harm}}$ is closed under taking cofibers, we can realize the comodule M'' by the $\mathrm{MU}^{\mathrm{mot}}_{*,*}$-homology of the cofiber of F'. In fact, the associated long exact sequence on the $\mathrm{MU}^{\mathrm{mot}}_{*,*}$-homology tells us that

$$M'' \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}X'',$$

where X'' is the the cofiber of F'.

(2) Suppose that both comodules M and M'' are realizable. Then, we realize the algebraic map and take the fiber instead. The same argument shows that it realizes M'.

(3) Suppose that both comodules M' and M'' are realizable by

$$M' \cong \mathrm{MU}^{\mathrm{mot}}_{*,*} X' \quad \text{and} \quad M'' \cong \mathrm{MU}^{\mathrm{mot}}_{*,*} X''.$$

In this case, the short exact sequence (4.2) corresponds to an element in

$$\mathrm{Ext}^{1,0,0}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(M'', M').$$

By Corollary 4.8, this algebraic element can be realized by a map

$$F\colon \Sigma^{-1,0} X'' \longrightarrow X'.$$

Define X to be the cofiber of the map F. We claim that X realizes M. In fact, the map F has Adams–Novikov filtration 1, so it induces the zero homomorphism on $\mathrm{MU}^{\mathrm{mot}}_{*,*}$. Therefore, the cofiber sequence that defines X induces a short exact sequence on $\mathrm{MU}^{\mathrm{mot}}_{*,*}$. Since the isomorphism in Corollary 4.8 comes from the collapse of the absolute Adams–Novikov spectral sequence on the E_2-page, this shows that this short exact sequence on $\mathrm{MU}^{\mathrm{mot}}_{*,*}$ is isomorphic to the the short exact sequence (4.2). Therefore,

$$M \cong \mathrm{MU}^{\mathrm{mot}}_{*,*} X.$$

This completes the proof. □

We now prove the equivalence on the heart.

PROPOSITION 4.11. *The functor*

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}\colon \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}} \xrightarrow{\cong} \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}^{0}$$

is an equivalence of categories.

Proof. We only need to show that the functor $\mathrm{MU}^{\mathrm{mot}}_{*,*}$ is essentially surjective. In other words, for any comodule $M \in \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}^{0}$, we show that it can be realized as a harmonic $\widehat{S^{0,0}}/\tau$-module X, whose $\mathrm{MU}^{\mathrm{mot}}_{*,*}$-homology is M. This follows from Lemmas 4.10, 4.5, 4.2 and Landweber's filtration theorem via the equivalence

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}^{0} \cong \mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}^{\mathrm{ev}}.$$

In fact, MU_* corresponds $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$, and is thus realized by $\widehat{S^{0,0}}/\tau$. By Lemma 4.10, we can inductively realize comodules MU_*/I_n for all $n \geqslant 0$. Then, by Landweber's filtration theorem and Lemma 4.10, we can realized all finitely presented comodules.

For any comodule $M \in \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}^0$, or equivalently a comodule over $\mathrm{MU}_*\mathrm{MU}$ that is concentrated in even degrees, by Lemma 4.5, we can write it as a filtered colimit of finitely presented ones M_α:

$$M \cong \operatorname{colim} M_\alpha.$$

By the above discussion, we can realize each M_α by $X_\alpha \in \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}}$. Moreover, by Corollary 4.9, we can realize the whole filtered system $\{M_\alpha\}_\alpha$ by a filtered system $\{X_\alpha\}_\alpha$. Taking the colimit, we define

$$X := \operatorname{colim} X_\alpha.$$

By Lemma 4.2, X is harmonic. Since $\mathrm{MU}^{\mathrm{mot}}_{*,*}$ commutes with filtered colimits, we have that the comodule M is realized by X. This completes the proof. □

4.3. The t-structure and the equivalence of categories

We prove that two full subcategories satisfy the required axioms for the t-structure.

PROPOSITION 4.12. *The pair of full subcategories*

$$\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b,\geqslant 0}_{\mathrm{harm}} \quad \textit{and} \quad \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b,\leqslant 0}_{\mathrm{harm}}$$

defines a bounded t-structure on $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{harm}}$.

Proof. The proof is exactly analogous to the proof of Proposition 3.7, with Corollary 4.7 replacing Corollary 3.3, and Proposition 4.11 replacing Proposition 3.5. □

Having this t-structure on $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{harm}}$, the main result of this section follows from Proposition 2.12.

THEOREM 4.13. *There is a t-exact equivalence of stable ∞-categories*

$$\mathcal{D}^b(\mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}^{\mathrm{ev}}) \xrightarrow{\cong} \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{harm}}.$$

Proof. The proof is analogous to the proof of Theorem 3.8. It is clear that the t-structure is bounded. By Proposition 4.11, and the equivalence

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}^0 \cong \mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}^{\mathrm{ev}},$$

the heart can be identified as comodules over $\mathrm{MU}_*\mathrm{MU}$. By Lemma 4.6, it has enough injective objects.

It remains to show that, for objects

$$X, Y \in \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}},$$

with $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ injective over $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$, we have that

$$[\Sigma^{-i,0}X, Y]_{\widehat{S^{0,0}}/\tau} = 0$$

for any $i>0$.

We apply the absolute Adams–Novikov spectral sequence

$$\mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y) \Longrightarrow [\Sigma^{t-s,w}X, Y]_{\widehat{S^{0,0}}/\tau}$$

in the category $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{harm}}$, as in Corollary 5.7.

Since $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ is an injective $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$-comodule, the E_2-page of the spectral sequence is concentrated on the line $s=0$, and therefore collapses at the E_2-page.

Moreover, since both $\mathrm{MU}^{\mathrm{mot}}_{*,*}X$ and $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ are concentrated in Chow–Novikov degree zero, the E_2-page is also concentrated in Chow–Novikov degree zero, namely

$$t-2w = 0$$

in this case.

We are interested in the case where $t-s=-i<0$ and $w=0$. By the above analysis, the corresponding tri-degrees in the E_2-page are all zero in our case. Therefore, we must have that

$$[\Sigma^{-i,0}X, Y]_{\widehat{S^{0,0}}/\tau} = 0.$$

This completes the proof. □

Remark 4.14. For the equivalence of stable ∞-categories in Theorem 4.13, we comment on its bi-grading through some examples.

(1) It is clear that $\widehat{S^{0,0}}/\tau$ corresponds to MU_* in the derived category of $\mathrm{MU}_*\mathrm{MU}$-comodules.

(2) Consider $\Sigma^{2,1}\widehat{S^{0,0}}/\tau$. As its $\mathrm{MU}^{\mathrm{mot}}$-homology is concentrated in Chow–Novikov degree zero, it lives in the heart. Therefore, by the t-exactness, it corresponds to a cochain complex that is concentrated in cohomological degree zero. A direct computation shows that it corresponds to $\Sigma^2\mathrm{MU}_*$. We also denote this object in the category

$$\mathcal{D}^b(\mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}^{\mathrm{ev}})$$

by $\Sigma^{2,1}\mathrm{MU}_*$.

(3) Consider $\Sigma^{1,0}\widehat{S^{0,0}}/\tau$. Its $\mathrm{MU}^{\mathrm{mot}}$-homology is concentrated in Chow–Novikov degree 1. By the t-exactness, it corresponds to the cochain complex that is concentrated in cohomological degree -1, with the comodule MU_* in that cohomological degree. We also denote this object by $\Sigma^{1,0}\mathrm{MU}_*$.

(4) In general, denote by $\Sigma^{m,n}\mathrm{MU}_*$ the object in the category

$$\mathcal{D}^b(\mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}^{\mathrm{ev}})$$

that $\Sigma^{m,n}\widehat{S^{0,0}}/\tau$ corresponds to. Then $\Sigma^{m,n}\mathrm{MU}_*$ is a cochain complex that is concentrated in cohomological degree $2n-m$, with the comodule $\Sigma^{2n}\mathrm{MU}_*$ in that cohomological degree.

By Proposition 1.10, we have that there exists an exact equivalence of categories between $\mathrm{BP}_*\mathrm{BP}$-$\mathbf{Comod}^{\mathrm{ev}}$ and $\mathrm{MU}_*\mathrm{MU}$-$\mathbf{Comod}^{\mathrm{ev}}$. Therefore, Theorem 4.13 implies Theorem 1.1.

Remark 4.15. The equivalence of stable ∞-categories in Theorem 1.1 is actually symmetric monoidal. In fact, the equivalence preserves colimits, so we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{D}^-(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}})_{\geqslant 0} & \longrightarrow & (\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b,\geqslant 0}_{\mathrm{harm}})^{\wedge} \\ \uparrow{\scriptstyle F_1} & & \uparrow{\scriptstyle F_2} \\ \mathbf{s}(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}}_{\mathrm{rel\ proj}}) & \longrightarrow & \mathbf{s}(\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}}) \end{array}.$$

Here, $(\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b,\geqslant 0}_{\mathrm{harm}})^{\wedge}$ is the left completion of $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b,\geqslant 0}_{\mathrm{harm}}$ with respect to its t-structure, $\mathbf{s}(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}}_{\mathrm{rel\ proj}})$ is the category of simplicial objects of relative projective $\mathrm{BP}_*\mathrm{BP}$-comodules that are concentrated in even degrees (see [24, Definition 2.1.2]), and $\mathbf{s}(\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}})$ is the category of simplicial objects in the category $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}}$. The horizontal arrows are induced by the equivalence in Theorem 1.1, and the vertical arrows are geometric realizations.

By [40, Proposition 5.5.9.14], geometric realization functors are symmetric monoidal. The lower-horizontal arrow is also symmetric monoidal, since it is level-wise symmetric monoidal, which is implied by Proposition 4.11 and the universal coefficient theorem (see [12, Proposition 7.10]).

Let W_1 and W_2 denote the class of morphisms in the ∞-categories

$$\mathbf{s}(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}}_{\mathrm{rel\ proj}}) \quad \text{and} \quad \mathbf{s}(\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}})$$

that are sent to equivalences by the functors F_1 and F_2. Let

$$\mathbf{s}(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}}_{\mathrm{rel\ proj}})[W_1^{-1}] \quad \text{and} \quad \mathbf{s}(\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}})[W_2^{-1}]$$

denote the localizations with respect to W_1 and W_2. The above commutative diagram factors through the following one:

$$\begin{array}{ccc} \mathcal{D}^-(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}})_{\geqslant 0} & \longrightarrow & (\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b,\geqslant 0}_{\mathrm{harm}})^{\wedge} \\ \uparrow & & \uparrow \\ \mathbf{s}(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}}_{\mathrm{rel\ proj}})[W_1^{-1}] & \longrightarrow & \mathbf{s}(\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}})[W_2{}^{-1}]. \end{array}$$

By the Dold–Kan correspondence and [24, Lemma 1.4.6], the left vertical arrow is an equivalence.

By [41, Proposition 2.2.1.9], localization functors preserve symmetric monoidal structures. Thus, the upper-horizontal arrow is symmetric monoidal. Restricting it to the bounded subcategory and by the universal property of stablization [63, Theorem 2.14], this gives us the claim.

We now prove Corollary 1.2.

Proof of Corollary 1.2. Let $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}_{\mathrm{fin}}$ be the category of finite cellular motivic left-module spectra over $\widehat{S^{0,0}}/\tau$, and $\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}})_{\mathrm{fin}}$ be the full subcategory of $\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}})$ consisting of objects generated by BP_*, under finite colimits and shifts by both homological and even internal degrees.

Since $\widehat{S^{0,0}}/\tau$ is harmonic, the category $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}_{\mathrm{fin}}$ is the full subcategory of $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{harm}}$ consisting of objects generated by $\widehat{S^{0,0}}/\tau$, under finite colimits and shifts by both the topological degree and the motivic weight.

Since $\widehat{S^{0,0}}/\tau$ corresponds to BP_* under the equivalence in Theorem 1.1, we have an equivalence of stable ∞-categories with given t-structures at each prime p:

$$\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}})_{\mathrm{fin}} \simeq \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}_{\mathrm{fin}}.$$

By Theorem 5.3.5.11 of Lurie's higher topos theory [40], if $\mathcal{D}$ is an ∞-category that admits filtered colimits, and $\mathcal{C}$ is an essentially small full subcategory of $\mathcal{D}$, whose elements are compact, and generate $\mathcal{D}$ under filtered colimits, then $\mathcal{D}$ is equivalent to the ∞-category $\mathrm{Ind}(\mathcal{C})$ of Ind-objects of $\mathcal{C}$.

It follows that

$$\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}} \simeq \mathrm{Ind}(\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}_{\mathrm{fin}}).$$

On the other hand, BP_* generates $\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}})_{\mathrm{fin}}$ under finite colimits. Moreover, it is proved by Hovey in [24, §6] that objects in $\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}})_{\mathrm{fin}}$ are compact, and generate $\mathbf{Stable}(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}})$ under filtered colimits. It then follows from [40, Theorem 5.3.5.11] that

$$\mathbf{Stable}(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}}) \simeq \mathrm{Ind}(\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}})_{\mathrm{fin}}).$$

Therefore, we have an equivalence of stable ∞-categories at each prime p

$$\mathbf{Stable}(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}}) \simeq \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}. \qquad \square$$

5. The absolute Adams–Novikov spectral sequence

In §3, we used the universal coefficient spectral sequence

$$E_2^{s,t,w} = \mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}(\pi_{*,*}X, \pi_{*,*}Y) \Longrightarrow [\Sigma^{t-s,w}X, Y]_{\mathrm{MU}^{\mathrm{mot}}/\tau}$$

of Theorem 3.2 to compute homotopy classes of maps in $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{cell}}$. This is a very convenient tool since both the t-structure on $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{cell}}$ and the E_2-page of the universal coefficient spectral sequence are defined in terms of homotopy groups. The bounds in the t-structure correspond to vanishing areas in the spectral sequence.

For the category $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{harm}}$, the t-structure is defined in terms of $\mathrm{MU}^{\mathrm{mot}}$-homology. We therefore need a version of the motivic Adams–Novikov spectral sequence that computes $\widehat{S^{0,0}}/\tau$-linear maps.

Recall from Dugger–Isaksen [14, §8] or Hu–Kriz–Ormsby [28] the usual $\mathrm{MU}^{\mathrm{mot}}$-based motivic Adams–Novikov spectral sequence

$$\mathrm{Ext}^{*,*,*}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}}(\mathrm{MU}^{\mathrm{mot}}_{*,*}\widehat{S^{0,0}}, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y) \Longrightarrow \pi_{*,*}Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}.$$

This spectral sequence is not what we need. We need a spectral sequence of the form

$$\mathrm{Ext}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y) \Longrightarrow [X, Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}]_{\widehat{S^{0,0}}/\tau},$$

for the following two reasons. Note that $Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}$ has the natural structure of being a $\widehat{S^{0,0}}/\tau$-module. (See property (3) after Definition 7.1 for this fact.)

First, we need a spectral sequence computing homotopy classes of maps in the category $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}$, instead of homotopy classes of maps between the underlying motivic spectra.

Second, we need the first variable X to be a general cellular $\widehat{S^{0,0}}/\tau$-module than just the unit object $\widehat{S^{0,0}}/\tau$. Classically, when the first variable X is the sphere spectrum,

we can use the standard cosimplicial cobar Adams–Novikov resolution for the second variable $Y \in \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}$ to set up this spectral sequence. This is done in [62, Chapter 2] classically, and in [14, §8] and [28] motivically. Such a resolution induces a resolution of $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ by relative injective comodules. It computes the E_2-page as an Ext-group only when the first variable $\mathrm{MU}^{\mathrm{mot}}_{*,*}X$ is a projective module over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$ [62, Corollary A1.2.12]. Since our first variable X is arbitrary, the E_2-page in general does not have a description as a relative Ext-group.

Instead of using the canonical Adams–Novikov tower that produces a resolution of $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ by relative injectives, we construct an **absolute** Adams–Novikov tower that produces a resolution of $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ by **absolute** injectives. The first step is to establish Lemmas 5.1 and 5.3, where we produce enough $\widehat{S^{0,0}}/\tau$-modules whose $\mathrm{MU}^{\mathrm{mot}}$-homology are injective comodules. The second step is Lemma 5.4, where we show that we can algebraically resolve comodules in $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}$ by these injective comodules. The third step is Proposition 5.5, where we topologically realize the algebraic construction to produce an absolute Adams–Novikov tower in the category $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{cell}}$. Finally, in Theorem 5.6, we construct the absolute Adams–Novikov spectral sequence and analyze its convergence.

5.1. The absolute Adams–Novikov tower

We construct the absolute Adams–Novikov tower in this subsection.

Recall that the forgetful functor from the abelian category $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Comod}$ to the category of abelian groups reflects monomorphisms, epimorphisms and exactness.

The following Lemma 5.1 is a consequence of Proposition 3.5 and the homology version of Dugger–Isaksen's the universal coefficient spectral sequence [13, Propositions 7.7 and 7.10].

LEMMA 5.1. *For any injective module* $N \in \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau\text{-}\mathbf{Mod}^0$, *the following statements hold*:

(1) $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} N$ *is an injective* $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$*-comodule*;

(2) *there exists* I *in* $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$ *such that*

$$\pi_{*,*}I \cong N;$$

(3) *for any such an object* I,

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}I \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} N.$$

Proof. (1) is straightforward (see [62, Lemma A1.2.2] for example). Statement (2) follows directly from Proposition 3.5.

For (3), we have the equivalences

$$\begin{aligned}\mathrm{MU}^{\mathrm{mot}}\wedge I &\simeq \mathrm{MU}^{\mathrm{mot}}/\tau\wedge_{\widehat{S^{0,0}}/\tau} I\\ &\simeq \mathrm{MU}^{\mathrm{mot}}/\tau\wedge_{\widehat{S^{0,0}}/\tau}(\mathrm{MU}^{\mathrm{mot}}/\tau\wedge_{\mathrm{MU}^{\mathrm{mot}}/\tau} I)\\ &\simeq (\mathrm{MU}^{\mathrm{mot}}/\tau\wedge_{\widehat{S^{0,0}}/\tau}\mathrm{MU}^{\mathrm{mot}}/\tau)\wedge_{\mathrm{MU}^{\mathrm{mot}}/\tau} I.\end{aligned}$$

Since $\mathrm{MU}^{\mathrm{mot}}/\tau$ is $\widehat{S^{0,0}}/\tau$-cellular, the homotopy groups of the last term can be computed by the following homology version of Dugger–Isaksen's universal coefficient spectral sequence [13, Proposition 7.10]:

$$\mathrm{Tor}^{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}_{s,t,w}(\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau, \pi_{*,*}I) \Longrightarrow \pi_{t+s,w}(\mathrm{MU}^{\mathrm{mot}}/\tau\wedge_{\widehat{S^{0,0}}/\tau}\mathrm{MU}^{\mathrm{mot}}/\tau\wedge_{\mathrm{MU}^{\mathrm{mot}}/\tau} I)$$

in the category $\mathrm{MU}^{\mathrm{mot}}/\tau$-$\mathbf{Mod}_{\mathrm{cell}}$.

Since $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$ is free, the spectral sequence is concentrated on the line $s{=}0$ and collapses at the E_2-page. This proves statement (3). □

Remark 5.2. In statement (3) we do not require that N is injective.

Lemma 5.1 is our source of motivic $\widehat{S^{0,0}}/\tau$-modules whose $\mathrm{MU}^{\mathrm{mot}}/\tau$-homology is injective as a comodule.

LEMMA 5.3. *Suppose that I is an object in $\mathrm{MU}^{\mathrm{mot}}/\tau$-$\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$ such that $\pi_{*,*}I$ is an injective $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$-module.*

Then, for any $X\in\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}^{b}_{\mathrm{cell}}$, we have

$$[X, I]_{\widehat{S^{0,0}}/\tau} \cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}I).$$

Proof. The lemma follows from the following isomorphisms:

$$\begin{aligned}[X, I]_{\widehat{S^{0,0}}/\tau} &\cong [\mathrm{MU}^{\mathrm{mot}}/\tau\wedge_{\widehat{S^{0,0}}/\tau} X, I]_{\mathrm{MU}^{\mathrm{mot}}/\tau}\\ &\cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \pi_{*,*}I)\\ &\cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, N)\\ &\cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} N)\\ &\cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}I).\end{aligned}$$

In fact, the first isomorphism follows from the adjunction (3.2) between $\widehat{S^{0,0}}/\tau$-modules and $\mathrm{MU}^{\mathrm{mot}}/\tau$-modules. The third and last isomorphisms follow from Lemma 5.1. The

fourth isomorphism follows from a change-of-ring isomorphism. It remains to show the second isomorphism.

Since both I and $\mathrm{MU}^{\mathrm{mot}}/\tau\wedge_{\widehat{S^{0,0}}/\tau}X$ belong to $\mathrm{MU}^{\mathrm{mot}}/\tau$-$\mathbf{Mod}_{\mathrm{cell}}$, the set of homotopy classes of maps

$$[\mathrm{MU}^{\mathrm{mot}}/\tau\wedge_{\widehat{S^{0,0}}/\tau}X, I]_{\mathrm{MU}^{\mathrm{mot}}/\tau}$$

can be computed by the universal coefficient spectral sequence of Theorem 3.2, that is

$$\mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \pi_{*,*}I) \Longrightarrow [\Sigma^{t-s,w}\mathrm{MU}^{\mathrm{mot}}/\tau\wedge_{\widehat{S^{0,0}}/\tau}X, I]_{\mathrm{MU}^{\mathrm{mot}}/\tau}.$$

As $\pi_{*,*}I\cong N$ is an injective $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$-module, the spectral sequence is concentrated on the line $s=0$ and collapses at the E_2-page. This gives the second isomorphism. □

LEMMA 5.4. *For any $M\in\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$-$\mathbf{Comod}$ that is concentrated in Chow–Novikov degree k, there exists a monomorphism*

$$M\hookrightarrow \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}N,$$

where N is injective in $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$-$\mathbf{Mod}$ and is concentrated in Chow–Novikov degree k.

Proof. Since there are enough injective objects in the category $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$-$\mathbf{Mod}$, we may choose an embedding $M\hookrightarrow N$ into an injective object in the category $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$-$\mathbf{Mod}$. Then, the induced comodule map

$$M\hookrightarrow \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau\otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau}N,$$

is also a monomorphism. □

PROPOSITION 5.5. *For any $Y\in\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}^b_{\mathrm{cell}}$, there exists a tower of the following form*

$$\begin{array}{ccccccccc} Y & = & Y_0 & \longleftarrow & Y_1 & \longleftarrow & Y_2 & \longleftarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & I_0 & & I_1 & & I_2 & & \end{array}$$

in the category $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}^b_{\mathrm{cell}}$, such that the following statements hold:

(1) *each map $Y_s\longrightarrow Y_{s-1}$ induces the zero homomorphism in $\mathrm{MU}^{\mathrm{mot}}$-homology*;

(2) *each cofiber I_s is a finite product of suspensions of objects I in $\mathrm{MU}^{\mathrm{mot}}/\tau$-$\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$ such that $\pi_{*,*}I$ is an injective $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$-module.*

We call such a tower an absolute *Adams–Novikov tower.*

Moreover, any map $f\colon X\longrightarrow Y$ in $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}^b_{\mathrm{cell}}$ can be lifted to a map of absolute Adams–Novikov towers.

Proof. Suppose that $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ is concentrated in Chow–Novikov degrees $[a,b]$, namely

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}Y \cong \bigoplus_{k=a}^{b} \bigoplus_{l=-\infty}^{+\infty} \mathrm{MU}^{\mathrm{mot}}_{2l+k,l}Y.$$

By Lemma 5.4, for every $k\in[a,b]$, there exists a monomorphism

$$\bigoplus_{l=-\infty}^{+\infty} \mathrm{MU}^{\mathrm{mot}}_{2l+k,l}(Y) \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{-k,0}Y)^{=0} \hookrightarrow \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} N_{0,k},$$

where $N_{0,k}$ is injective module that is concentrated in Chow–Novikov degree zero. By Lemma 5.1, there exists a spectrum $I_{0,k}\in \mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{cell}}$ such that

$$\pi_{*,*}I_{0,k} \cong N_{0,k}$$

and

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}I_{0,k} \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} N_{0,k}.$$

By Lemma 5.3, we have that

$$\begin{aligned}
&[\Sigma^{-k,0}Y, I_{0,k}]_{\widehat{S^{0,0}}/\tau} \\
&\quad\cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{-k,0}Y), \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} N_{0,k}) \\
&\quad\cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{-k,0}Y)^{=0}, \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} N_{0,k}).
\end{aligned}$$

The second isomorphism follows from the fact that $N_{0,k}$ is concentrated in Chow–Novikov degree zero. Therefore, the algebraic map of comodules

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{-k,0}Y) \twoheadrightarrow \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{-k,0}Y)^{=0} \hookrightarrow \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} N_{0,k},$$

where the first map is the project map to the Chow–Novikov-degree-zero part, can be realized as a $\widehat{S^{0,0}}/\tau$-linear map

$$\Sigma^{-k,0}Y \longrightarrow I_{0,k}.$$

Combining these maps for all $k\in[a,b]$, we obtain a map

$$Y \longrightarrow \prod_{k=a}^{b} \Sigma^{k,0}I_{0,k}.$$

This map induces a monomorphism in $\mathrm{MU}^{\mathrm{mot}}$-homology.

Denote the finite product by

$$I_0 := \prod_{k=a}^{b} \Sigma^{k,0} I_{0,k},$$

and the fiber of the map $Y \to I_0$ by Y_1, as in

$$\begin{array}{ccc} Y & \longleftarrow & Y_1 \\ \downarrow & & \\ I_0. & & \end{array}$$

By the associated long exact sequence in $\mathrm{MU}^{\mathrm{mot}}$-homology, the map $Y_1 \longrightarrow Y$ induces the zero map in $\mathrm{MU}^{\mathrm{mot}}$-homology, and $\mathrm{MU}^{\mathrm{mot}}_{*,*} Y_1$ is concentrated in Chow–Novikov degrees $[a-1, b-1]$. So, in particular, we have

$$Y_1 \in \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{cell}}.$$

We can repeat the procedure, producing an absolute Adams–Novikov tower

$$\begin{array}{ccccccc} Y & \longleftarrow & Y_1 & \longleftarrow & Y_2 & \longleftarrow & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \\ I_0 & & I_1 & & I_2 & & \end{array}$$

satisfying the desired properties.

We now prove the second claim of the theorem. For any $\widehat{S^{0,0}}/\tau$-linear map

$$f_0 \colon X_0 \longrightarrow Y_0,$$

we may assume that $\mathrm{MU}^{\mathrm{mot}}_{*,*} X_0$ and $\mathrm{MU}^{\mathrm{mot}}_{*,*} Y_0$ are both concentrated in Chow–Novikov degrees $[a, b]$. Denote the first step of their tower by

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ \downarrow & & \downarrow \\ I_0 & & J_0, \end{array}$$

where I_0 and J_0 are the finite products of suspensions of objects that satisfy the conclusions of Lemma 5.1. Applying $\mathrm{MU}^{\mathrm{mot}}_{*,*}$, we have the following diagram of $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$-comodules:

$$\begin{array}{ccc} \mathrm{MU}^{\mathrm{mot}}_{*,*} X_0 & \xrightarrow{(f_0)_{*,*}} & \mathrm{MU}^{\mathrm{mot}}_{*,*} Y_0 \\ \downarrow & & \downarrow \\ \mathrm{MU}^{\mathrm{mot}}_{*,*} I_0 & \xrightarrow{\phi} & \mathrm{MU}^{\mathrm{mot}}_{*,*} J_0. \end{array}$$

Here the existence of the homomorphism ϕ is due to the universal property of injective objects in the category $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$-**Comod**.

We have

$$\mathrm{MU}^{\mathrm{mot}}_{*,*} I_0 = \mathrm{MU}^{\mathrm{mot}}_{*,*}\left(\prod_{k=a}^{b} \Sigma^{k,0} I_{0,k}\right) = \prod_{k=a}^{b} \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{k,0} I_{0,k}),$$
$$\mathrm{MU}^{\mathrm{mot}}_{*,*} J_0 = \mathrm{MU}^{\mathrm{mot}}_{*,*}\left(\prod_{k=a}^{b} \Sigma^{k,0} J_{0,k}\right) = \prod_{k=a}^{b} \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{k,0} J_{0,k}).$$

The Chow–Novikov-degree-k parts of $\mathrm{MU}^{\mathrm{mot}}_{*,*} I_0$ and $\mathrm{MU}^{\mathrm{mot}}_{*,*} J_0$ are given by

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{k,0} I_{0,k}) \quad \text{and} \quad \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{k,0} J_{0,k}).$$

Therefore, the homomorphism ϕ is given by the product of homomorphisms

$$\phi_k \colon \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{k,0} I_{0,k}) \longrightarrow \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{k,0} J_{0,k})$$

for each $k \in [a,b]$.

Since $J_{0,k}$ satisfies the conclusions of Lemma 5.1, we have that

$$\begin{aligned}
[\Sigma^{k,0} I_{0,k}, \Sigma^{k,0} J_{0,k}]_{\widehat{S^{0,0}}/\tau} &\cong [I_{0,k}, J_{0,k}]_{\widehat{S^{0,0}}/\tau} \\
&\cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*} I_{0,k}, \mathrm{MU}^{\mathrm{mot}}_{*,*} J_{0,k}) \\
&\cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{k,0} I_{0,k}), \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{k,0} J_{0,k})),
\end{aligned}$$

where the second isomorphism is given by Lemma 5.3. Therefore, the homomorphism ϕ_k can be realized by a $\widehat{S^{0,0}}/\tau$-linear map

$$g_{0,k} \colon \Sigma^{k,0} I_{0,k} \longrightarrow \Sigma^{k,0} J_{0,k}.$$

Taking the product of $g_{0,k}$ for all $k \in [a,b]$, we define a map $g_0 \colon I_0 \to J_0$. Then g_0 realizes ϕ, and we have the diagram

$$\begin{array}{ccc}
X_0 & \xrightarrow{f_0} & Y_0 \\
\downarrow & & \downarrow \\
I_0 & \xrightarrow{g_0} & J_0.
\end{array}$$

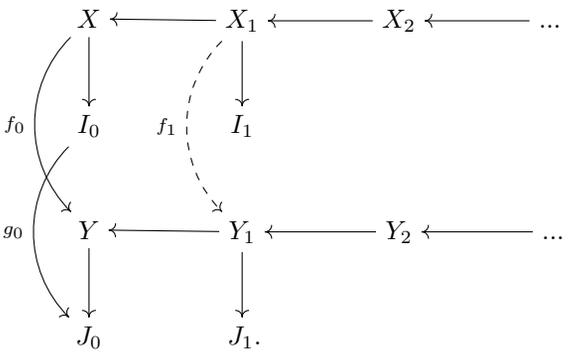
To see that the square commutes up to homotopy, we have

$$\begin{aligned}
[X, J_0]_{\widehat{S^{0,0}}/\tau} &\cong \left[X, \prod_{k=a}^{b} \Sigma^{k,0} J_{0,k}\right]_{\widehat{S^{0,0}}/\tau} \\
&\cong \prod_{k=a}^{b} \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*} X, \mathrm{MU}^{\mathrm{mot}}_{*,*}(\Sigma^{k,0} J_{0,k})) \\
&\cong \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*} X, \mathrm{MU}^{\mathrm{mot}}_{*,*} J_0),
\end{aligned}$$

where the second isomorphism is given by Lemma 5.3.

Therefore, the commutativity of this square follows from the commutativity of the corresponding square in $\mathrm{MU}^{\mathrm{mot}}$-homology.

The commutative diagram in $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{cell}}$ induces a map $f_1\colon X_1 \to Y_1$ between the fibers, so the following diagram commutes, up to homotopy:



Iterating this process produces the desired map of absolute Adams–Novikov towers. □

5.2. The spectral sequence

Every absolute Adams–Novikov tower gives rise to an absolute Adams–Novikov spectral sequence. In the following Theorem 5.6, we identify the E_2-page of the spectral sequence and its abutment. We also show that it does not depend on the absolute Adams–Novikov tower, and converges strongly for objects with bounded Chow–Novikov degree.

THEOREM 5.6. *For $X, Y \in \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{cell}}$, there is an absolute Adams–Novikov spectral sequence*

$$E_2^{s,t,w} \cong \mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y) \Longrightarrow [\Sigma^{t-s,w}X, Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}]_{\widehat{S^{0,0}}/\tau},$$

with differentials

$$d_r\colon E_r^{s,t,w} \longrightarrow E_r^{s+r,t+r-1,w},$$

that does not depend on the absolute Adams–Novikov tower. Here, $Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}$ is the $\mathrm{MU}^{\mathrm{mot}}$-completion of Y. Moreover, this spectral sequence converges strongly and collapses at a finite page.

Proof. The arguments for the existence of the absolute Adams–Novikov spectral sequence and its independence of the absolute Adams–Novikov tower are both standard. Under the hypotheses that both $\mathrm{MU}^{\mathrm{mot}}_{*,*}X$ and $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ are concentrated in bounded Chow–Novikov degrees, the argument for the strongly convergence and collapse at a finite page is similar to the one given in the proof of Theorem 3.2: it follows from degree reasons.

To complete the proof, we only need to identify the abutment.

Let Y/Y_s be the cofiber of the map $Y_s \to Y$ in the absolute Adams–Novikov tower

$$\begin{array}{ccccccc} Y & \longleftarrow & Y_1 & \longleftarrow & Y_2 & \longleftarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ I_0 & & I_1 & & I_2, & & \end{array}$$

and define the limit in $\widehat{S^{0,0}}/\tau$-**Mod**

$$\widehat{Y} = \lim(Y/Y_s).$$

The spectral sequence converges conditionally to

$$[X, \widehat{Y}]_{\widehat{S^{0,0}}/\tau}.$$

See, for example, [6, §5 and §15] for a discussion of convergence issues of the Adams spectral sequence.

To identify it as $[X, Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}]_{\widehat{S^{0,0}}/\tau}$, since X is $\widehat{S^{0,0}}/\tau$-cellular, we only need to show that $\widehat{Y}$ has the same homotopy groups as the $Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}$.

Take $X = \widehat{S^{0,0}}/\tau$. Since $\mathrm{MU}^{\mathrm{mot}}_{*,*}\widehat{S^{0,0}}/\tau = \mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$ is free over itself, we can use the canonical $\mathrm{MU}^{\mathrm{mot}}/\tau$-Adams resolution [62, Definition 2.2.10] for Y in this case. Now we compare the canonical $\mathrm{MU}^{\mathrm{mot}}/\tau$-based Adams–Novikov tower of Y with the absolute Adams–Novikov tower of Y.

As we did in the proof of Proposition 5.5, we have a map of towers from the canonical $\mathrm{MU}^{\mathrm{mot}}/\tau$-based one to the absolute one. The identity map on Y induces a homomorphism from the canonical cobar resolution of $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ to the absolute injective resolution of $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$, so in particular a homomorphism of relative injective resolutions.

This induces a homomorphism from the usual Adams–Novikov spectral sequence to the absolute Adams–Novikov spectral sequence, with an isomorphism on the E_2-page. It is therefore an isomorphism of spectral sequences and we have an isomorphism

$$\pi_{*,*}\widehat{Y} \xrightarrow{\cong} \pi_{*,*}Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}.$$

Since any cellular $\widehat{S^{0,0}}/\tau$-module X can be written in terms of filtered colimits and cofibers of suspensions of $\widehat{S^{0,0}}/\tau$'s, there is an isomorphism

$$[X, \widehat{Y}]_{\widehat{S^{0,0}}/\tau} \xrightarrow{\cong} [X, Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}]_{\widehat{S^{0,0}}/\tau}.$$

Therefore, the absolute Adams–Novikov spectral sequence computes

$$[X, Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}]_{\widehat{S^{0,0}}/\tau}. \qquad \square$$

When Y is harmonic, the isomorphism

$$\pi_{*,*}\widehat{Y} \xrightarrow{\cong} \pi_{*,*}Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}$$

gives the following corollary.

COROLLARY 5.7. *For any* $X, Y \in \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b}_{\mathrm{harm}}$, *there is an absolute Adams–Novikov spectral sequence*

$$E_2^{s,t,w} = \mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y) \Longrightarrow [\Sigma^{t-s,w}X, Y]_{\widehat{S^{0,0}}/\tau},$$

with differentials

$$d_r \colon E_r^{s,t,w} \longrightarrow E_r^{s+r,t+r-1,w},$$

that converges strongly and collapses at a finite page.

Remark 5.8. The above arguments can be applied to more general situations. However, this construction of an absolute Adams–Novikov spectral sequence depends on realizability of categorical injective objects, so the range of situations to which it applies may be rather limited.

In the case of the classical stable homotopy category, for spectra X and Y, there is a conditionally convergent spectral sequence

$$\mathrm{Ext}^{s,t}_{\mathrm{MU}_*\mathrm{MU}}(\mathrm{MU}_*X, \mathrm{MU}_*Y) \Longrightarrow [\Sigma^{t-s}X, Y^{\wedge}_{\mathrm{MU}}],$$

where MU_*X does *not* have to be projective over MU_*. We will discuss this case in a general framework in future work.

5.3. Proofs of Lemma 4.2 and Corollaries 4.7 and 4.8

We give the proofs of Lemma 4.2 and Corollaries 4.7 and 4.8 in this section.

Corollary 4.7 states that, if $X \in \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b,\geqslant 0}_{\mathrm{harm}}$ and $Y \in \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{b,\leqslant 0}_{\mathrm{harm}}$, then the abelian group of homotopy classes of bi-degree $(0,0)$ maps can be computed algebraically by the isomorphism

$$[X, Y]_{\widehat{S^{0,0}}/\tau} \longrightarrow \mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X, \mathrm{MU}^{\mathrm{mot}}_{*,*}Y)$$

that is induced by applying $\mathrm{MU}^{\mathrm{mot}}_{*,*}$.

Proof of Corollary 4.7. The proof is similar to the one of Corollary 3.3.

Consider the E_2-page of the absolute Adams–Novikov spectral sequence. The tri-degrees that converge to the bi-degree $(0,0)$ are of the form $(t,t,0)$, which correspond to $E_2^{s,t,w}=E_2^{t,t,0}$, for $t\geqslant 0$.

By the proof of Theorem 5.6, the t-degrees of all possible non-zero elements in the E_1-page and therefore E_2-page satisfy $t\leqslant d-a+2w=d-a$. Since $\mathrm{MU}^{\mathrm{mot}}_{*,*}X$ and $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ are concentrated in non-negative and non-positive bounded Chow–Novikov degrees, we have $d=a=0$. Therefore, we have $t\leqslant 0$.

Combining both facts, we have established that the only possible non-zero elements in the E_2-page that converge to the bi-degree $(0,0)$ are in

$$E_2^{0,0,0}=\mathrm{Hom}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X,\mathrm{MU}^{\mathrm{mot}}_{*,*}Y).$$

To show that all elements in $E_2^{0,0,0}$ survive in the spectral sequence, note that they are not targets of any non-zero differentials since they are in s-degree zero. Second, all d_r-differentials for $r\geqslant 2$ increase the t-degree. Since the t-degrees of all non-zero elements are non-positive, the elements in $E_2^{0,0,0}$ do not support non-zero differentials. There are no hidden extensions due to degree reasons. This completes the proof. □

Corollary 4.8 states that, given $X,Y\in\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}}$, for any bi-degree (t,w) there is an isomorphism

$$[\Sigma^{t,w}X,Y]_{\widehat{S^{0,0}}/\tau}\cong\mathrm{Ext}^{2w-t,2w,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X,\mathrm{MU}^{\mathrm{mot}}_{*,*}Y).$$

Proof of Corollary 4.8. Consider the E_2-page of the absolute Adams–Novikov spectral sequence. Since both $\mathrm{MU}^{\mathrm{mot}}_{*,*}X$ and $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$ are concentrated in Chow–Novikov degree zero, the E_2-page

$$E_2^{s,t,w}=\mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}\left(\mathrm{MU}^{\mathrm{mot}}_{*,*}X,\mathrm{MU}^{\mathrm{mot}}_{*,*}Y\right)$$

is concentrated in degrees $t=2w$. Since all differentials preserves the motivic weights w, this spectral sequence collapses at the E_2-page. There are no hidden extensions due to degree reasons. Therefore, we have the isomorphism

$$[\Sigma^{t,w}X,Y]_{\widehat{S^{0,0}}/\tau}\cong\mathrm{Ext}^{2w-t,2w,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}X,\mathrm{MU}^{\mathrm{mot}}_{*,*}Y).$$ □

We prove Lemma 4.2, which states that if $\{Y_\alpha\}$ is a filtered system in $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}$ such that each Y_α is harmonic, then the colimit of $\{Y_\alpha\}$ in $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}$ is also harmonic.

Proof of Lemma 4.2. Consider the absolute Adams–Novikov spectral sequence of Theorem 5.6,

$$\begin{aligned}&\mathrm{Ext}^{s,t,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}\widehat{S^{0,0}}/\tau,\mathrm{MU}^{\mathrm{mot}}_{*,*}Y)\\ &\qquad\Longrightarrow[\Sigma^{t-s,w}\widehat{S^{0,0}}/\tau,Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}]_{\widehat{S^{0,0}}/\tau}\cong\pi_{t-s,w}Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}},\end{aligned}$$

in the case that $X=\widehat{S^{0,0}}/\tau$ and $Y=\mathrm{colim}\,Y_\alpha$. As both $\widehat{S^{0,0}}/\tau$ and Y are in $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}^{\heartsuit}$, the E_2-page is concentrated in degrees $t=2w$. Since all differentials preserve the motivic weights w, this spectral sequence collapses at the E_2-page. There are no hidden extensions, due to degree reasons. Therefore, we have the isomorphism

$$\pi_{t,w}Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}\cong[\Sigma^{t,w}\widehat{S^{0,0}}/\tau,Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}]_{\widehat{S^{0,0}}/\tau}\cong\mathrm{Ext}^{2w-t,2w,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}\widehat{S^{0,0}}/\tau,\mathrm{MU}^{\mathrm{mot}}_{*,*}Y).$$

Since $\mathrm{MU}^{\mathrm{mot}}_{*,*}\widehat{S^{0,0}}/\tau\cong\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$ is free over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$, one can use the canonical cobar resolution for $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y$. Since it is functorial and commutes with filtered colimits, the isomorphism

$$\mathrm{colim}\,\mathrm{MU}^{\mathrm{mot}}_{*,*}Y_\alpha\cong\mathrm{MU}^{\mathrm{mot}}_{*,*}(Y)$$

induces an isomorphism

$$\mathrm{colim}\,\mathrm{Ext}^{*,*,*}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}\left(\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau,\mathrm{MU}^{\mathrm{mot}}_{*,*}Y_\alpha\right)\cong\mathrm{Ext}^{*,*,*}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}\left(\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau,\mathrm{MU}^{\mathrm{mot}}_{*,*}Y\right).$$

Therefore, we have the following isomorphisms

$$\begin{aligned}\pi_{t,w}Y&\cong\pi_{t,w}(\mathrm{colim}\,Y_\alpha)\\ &\cong\mathrm{colim}\,\pi_{t,w}Y_\alpha\\ &\cong\mathrm{colim}\,[\Sigma^{t,w}\widehat{S^{0,0}}/\tau,Y_\alpha]_{\widehat{S^{0,0}}/\tau}\\ &\cong\mathrm{colim}\,\mathrm{Ext}^{2w-t,2w,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau,\mathrm{MU}^{\mathrm{mot}}_{*,*}Y_\alpha)\\ &\cong\mathrm{Ext}^{2w-t,2w,w}_{\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau}(\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau,\mathrm{MU}^{\mathrm{mot}}_{*,*}Y)\\ &\cong[\Sigma^{t,w}\widehat{S^{0,0}}/\tau,Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}]_{\widehat{S^{0,0}}/\tau}\\ &\cong\pi_{t,w}Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}},\end{aligned}$$

where the fourth isomorphism is given by Corollary 4.8, since each Y_α is harmonic. The composite is induced by the completion map $Y\to Y^{\wedge}_{\mathrm{MU}^{\mathrm{mot}}}$. This shows that Y is harmonic. □

Remark 5.9. Lemma 4.2 can be generalized to the case when there is a uniform bound on the Chow–Novikov degrees of $\mathrm{MU}^{\mathrm{mot}}_{*,*}Y_\alpha$ for all α. Example 4.1 (3) shows that Lemma 4.2 cannot hold without any bound on Chow–Novikov degrees.

6. Further questions

The category of cellular modules over $\widehat{S^{0,0}}/\tau$ measures the difference between cellular modules over the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-completed motivic sphere spectrum $\widehat{S^{0,0}}$ and cellular modules over the classical p-completed sphere spectrum $\widehat{S^0}$.

Definition 6.1. Let $\widehat{S^{0,0}}$-$\mathbf{Mod}_{\mathrm{fin}}$ be the category of finite cellular modules over $\widehat{S^{0,0}}$, and $\widehat{S^0}$-$\mathbf{Mod}_{\mathrm{fin}}$ be the category of classical finite cellular modules over $\widehat{S^0}$. Let $\widehat{S^{0,0}}$-$\mathbf{Mod}_{\mathrm{fin}}^{\tau\text{-tor}}$ be the full subcategory of $\widehat{S^{0,0}}$-$\mathbf{Mod}_{\mathrm{fin}}$ that is generated by $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}_{\mathrm{fin}}$ under cofibers, i.e. the smallest full subcategory containing objects of $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}_{\mathrm{fin}}$ and closed under taking cofibers.

The following Proposition 6.2 can be proved from Dugger–Isaksen [14, §3.2 and §3.4] and Isaksen [29, Proposition 3.0.2].

Proposition 6.2. *The sequence*

$$\widehat{S^{0,0}}\text{-}\mathbf{Mod}_{\mathrm{fin}}^{\tau\text{-tor}} \longrightarrow \widehat{S^{0,0}}\text{-}\mathbf{Mod}_{\mathrm{fin}} \xrightarrow{\widehat{\mathbf{Re}}} \widehat{S^0}\text{-}\mathbf{Mod}_{\mathrm{fin}}$$

is an exact sequence of stable ∞-categories in the sense of Blumberg–Gepner–Tabuada [5, §5], *where $\widehat{\mathbf{Re}}$ is the p-completed version of the Betti realization functor* ([12, Theorem 1.4]) *explained below.*

Let $\mathbf{Re}$ be the Betti realization functor constructed in Dugger–Isaksen [12, Theorem 1.4]. It is symmetric monoidal and preserves colimits. It was shown by Dugger–Isaksen [14] that $\mathbf{Re}$ sends the motivic Adams tower for $S^{0,0}$ to the classical Adams tower for S^0. Taking the limit, we get a map of E_∞ spectra

$$\mathbf{Re}(\widehat{S^{0,0}}) \longrightarrow \widehat{S^0}.$$

For any $\widehat{S^{0,0}}$-module X, we define the p-completed Betti realization functor to be

$$\widehat{\mathbf{Re}}(X) := \mathbf{Re}(X) \wedge_{\mathbf{Re}(\widehat{S^{0,0}})} \widehat{S^0}.$$

The p-completed Betti realization functor $\widehat{\mathbf{Re}}$ sends $\widehat{S^{0,0}}$ to $\widehat{S^0}$. It is symmetric monoidal, and preserves colimits.

In the sense of Proposition 6.2, our Theorem 1.1 gives a decomposition of the cellular stable motivic category into more classical categories.

In particular, we can apply the non-connective algebraic K-theory functor $\mathbb{K}$ constructed in Blumberg–Gepner–Tabuada [5, §9], and get a cofiber sequence of non-connective algebraic K-theory spectra, since the functor $\mathbb{K}$ sends exact sequences of stable ∞-categories into cofiber sequences:

$$\mathbb{K}(\widehat{S^{0,0}}\text{-}\mathbf{Mod}_{\mathrm{fin}}^{\tau\text{-tor}}) \longrightarrow \mathbb{K}(\widehat{S^{0,0}}\text{-}\mathbf{Mod}_{\mathrm{fin}}) \xrightarrow{\widehat{\mathbf{Re}}} \mathbb{K}(\widehat{S^0}\text{-}\mathbf{Mod}_{\mathrm{fin}}).$$

Since the Betti realization functor admits a section, the above cofiber sequence actually splits:

$$\mathbb{K}(\widehat{S^{0,0}}\text{-}\mathbf{Mod}_{\mathrm{fin}}) \simeq \mathbb{K}(\widehat{S^0}\text{-}\mathbf{Mod}_{\mathrm{fin}}) \vee \mathbb{K}(\widehat{S^{0,0}}\text{-}\mathbf{Mod}_{\mathrm{fin}}^{\tau\text{-tor}}).$$

The spectrum

$$\mathbb{K}(\widehat{S^0}\text{-}\mathbf{Mod}_{\mathrm{fin}})$$

for the p-completed sphere spectrum is described by Bökstedt–Hsiang–Madsen [7].

To understand the spectrum

$$\mathbb{K}(\widehat{S^{0,0}}\text{-}\mathbf{Mod}_{\mathrm{fin}}^{\tau\text{-tor}}),$$

we consider the inclusion functor

$$\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}_{\mathrm{fin}} \longrightarrow \widehat{S^{0,0}}\text{-}\mathbf{Mod}_{\mathrm{fin}}^{\tau\text{-tor}}. \tag{6.1}$$

We propose the following question.

Question 6.3. Does the inclusion functor (6.1) induce an equivalence on non-connective algebraic K-theory spectra?

Our Question 6.3 is an example of the dévissage question for algebraic K-theories. It is known to be false in some situations (see Antieau–Barthel–Gepner [2]).

Let

$$\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}_{\mathrm{fin}}^{\mathrm{ev}}$$

be the subcategory of $\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}}$ on those comodules whose underlying BP_*-module is finitely presented. If the answer to Question 6.3 is yes, then, by the theorem of the heart due to Barwick [4] and Theorem 1.1, we have the following isomorphism for all $i \geqslant 0$ (we require this condition, since Barwick's theorem only applies to the connective K-theories):

$$\mathbb{K}_i(\widehat{S^{0,0}}\text{-}\mathbf{Mod}_{\mathrm{fin}}) \cong \mathbb{K}_i(\widehat{S^0}\text{-}\mathbf{Mod}_{\mathrm{fin}}) \oplus \mathbb{K}_i(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}_{\mathrm{fin}}^{\mathrm{ev}}).$$

If we further regard the category $\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}_{\mathrm{fin}}^{\mathrm{ev}}$ as the category $\mathbf{Coh}(\mathcal{M}_{FG})$ of coherent sheaves over the moduli stack $\mathcal{M}_{FG}$ of formal groups [20], and the answer to Question 6.3 is yes, then for all i there is an isomorphism

$$\mathbb{K}_i(\widehat{S^{0,0}}) \cong \mathbb{K}_i(\widehat{S^0}) \oplus \mathbb{K}_i(\mathbf{Coh}(\mathcal{M}_{FG})).$$

7. $\mathbf{H\mathbb{F}_p^{mot}}$-completion

Let R be an E_∞-algebra in a symmetric monoidal stable ∞-category $\mathcal{C}$ with the unit object S.

Definition 7.1. For any object Z in $\mathcal{C}$, we define its *R-completion*, denoted by $Z^\wedge_R$, as the totalization of the co-simplicial object

$$Z\otimes R^{\otimes\bullet},$$

where the co-face maps are induced by the unit map $S\to R$.

The R-completion $Z^\wedge_R$ in $\mathcal{C}$ has the following properties:

(1) It commutes with finite limits and finite colimits.

(2) It commutes with suspensions and desuspensions.

(3) If Z is an E_∞-algebra in $\mathcal{C}$, then $Z^\wedge_R$ is also an E_∞-algebra in $\mathcal{C}$. Moreover, if Y is an Z-module, then $Y^\wedge_R$ is an $Z^\wedge_R$-module. These are special cases of Corollary 3.2.2.5 in Higher Algebra [41].

We now consider the category $\mathbb{C}$-**mot-Spectra** and the $\mathrm{H\mathbb{F}}_p^{\mathrm{mot}}$-completion of the sphere spectrum $S^{0,0}$:

$$\widehat{S^{0,0}} = S^{0,0\,\wedge}_{\mathrm{H\mathbb{F}}_p^{\mathrm{mot}}}.$$

Both the sphere spectrum $S^{0,0}$ and the Eilenberg–Mac Lane spectrum $\mathrm{H\mathbb{F}}_p^{\mathrm{mot}}$ are E_∞-algebras in $\mathbb{C}$-**mot-Spectra**. Therefore, the $\mathrm{H\mathbb{F}}_p^{\mathrm{mot}}$-completed sphere spectrum $\widehat{S^{0,0}}$ is an E_∞-algebra.

It is a theorem of Hu–Kriz–Ormsby [27], [28] that, over any algebraic closed field of characteristic zero, the $\mathrm{H\mathbb{F}}_p^{\mathrm{mot}}$-completion of the sphere spectrum and the usual p-completion of the motivic sphere spectrum have isomorphic motivic homotopy groups.

For the effect of the $\mathrm{H\mathbb{F}}_p^{\mathrm{mot}}$-completion of the sphere spectrum on homotopy groups, Hu–Kriz–Ormsby [27], [28] pointed out that there is a short exact sequence on homotopy groups of the uncompleted sphere spectrum $S^{0,0}$ and the $\mathrm{H\mathbb{F}}_p^{\mathrm{mot}}$-completed sphere spectrum $\widehat{S^{0,0}}$:

$$0\longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, \pi_{s,w}S^{0,0})\longrightarrow \pi_{s,w}\widehat{S^{0,0}}\longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, \pi_{s-1,w}S^{0,0})\longrightarrow 0.$$

See [27] for a general discussion regarding the effect of homotopy groups with respect to the $\mathrm{H\mathbb{F}}_p^{\mathrm{mot}}$-completion.

Now we consider the cellular motivic spectrum MGL. Recall from [69] that

$$\mathrm{MGL} = \operatorname*{colim}_{n,k\to\infty} \Sigma^{-2n,-n}\mathrm{Thom}(V(k,n)).$$

Here, $V(k,n)$ is the tautological bundle over the Grassmannian $\mathrm{Gr}(k,n)$ for $k \geqslant n$, which is the smooth scheme of complex n-planes in $\mathbb{C}^k$, and $\mathrm{Thom}(V(k,n))$ is its associated motivic Thom spectrum.

Recall that we have defined in §1.1 that

$$\mathrm{MU}^{\mathrm{mot}} = \mathrm{MGL} \wedge_{S^{0,0}} \widehat{S^{0,0}}.$$

By adjunction, for any $\widehat{S^{0,0}}$-module X, its MGL-completion in the category

$$\mathbb{C}\textbf{-mot-Spectra}$$

can be identified as its $\mathrm{MU}^{\mathrm{mot}}$-completion in the category $\widehat{S^{0,0}}$-**Mod**.

The following proposition states that $\mathrm{MU}^{\mathrm{mot}}$ has the same homotopy groups as the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-completion of MGL.

PROPOSITION 7.2. *The natural map*

$$\mathrm{MU}^{\mathrm{mot}} = \mathrm{MGL} \wedge_{S^{0,0}} \widehat{S^{0,0}} \longrightarrow \mathrm{MGL}^{\wedge}_{\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}}$$

induces an isomorphism on $\pi_{*,*}$.

We prove Proposition 7.2 by using [28, Lemma 11]. Recall from [28] that a motivic cellular spectrum X is k-connective, if $\pi_{s,w}X = 0$ for all s and w such that $s - w < k$. A cellular map $f\colon X \to Y$ between cellular motivic spectra is a k-equivalence, if its cofiber is $(k+1)$-connective. Lemma 11 of [28] states that, if X is k-connective, then $X^{\wedge}_{\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}}$ is also k-connective.

For MGL, recall from Schubert calculus (see Griffiths–Harris [22, §1.5] in the classical setting and Wendt [76, Proposition 2.2] for adaption to the motivic setting) that the map

$$\Sigma^{-2n,-n}\mathrm{Thom}(V(k,n)) \longrightarrow \mathrm{MGL}$$

is m-connective, where $m = \min\{n-1, k-n-1\}$.

For any finite cellular motivic spectrum X, we have

$$X \wedge_{S^{0,0}} \widehat{S^{0,0}} \simeq X^{\wedge}_{\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}},$$

since both sides commute with finite colimits.

Because $\mathrm{Thom}(V(k,n))$ is a finite motivic spectrum for all n and k, Proposition 7.2 follows from the following Lemma 7.3.

LEMMA 7.3. *Suppose that X is the colimit of motivic cellular spectra*

$$X_1 \longrightarrow X_2 \longrightarrow \dots .$$

Suppose further that there exists an increasing sequence m_n of natural numbers

$$\lim_{n\to\infty} m_n = \infty$$

such that the map $X_n \to X$ is an m_n-equivalence for all n. Then, the map

$$\operatorname{colim}((X_n)^{\wedge}_{\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}}) \longrightarrow X^{\wedge}_{\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}}$$

induces an isomorphism on $\pi_{*,*}$.

Proof. By assumption, the map $X_n \to X$ is an m_n-equivalence. By [28, Lemma 11] and the above discussion, the map

$$(X_n)^{\wedge}_{\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}} \longrightarrow X^{\wedge}_{\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}}$$

is also an m_n-equivalence.

Taking the colimit, we have that

$$\pi_{*,*} \operatorname{colim}((X_n)^{\wedge}_{\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}}) \cong \operatorname{colim} \pi_{*,*}(X_n)^{\wedge}_{\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}} \cong \pi_{*,*} X^{\wedge}_{\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}}. \qquad \square$$

Part 2. Equivalence of spectral sequences

8. Main theorem of Part 2

The algebraic Novikov spectral sequence is introduced by Novikov [56] and Miller [46]. Ravenel's green book [62] and Andrews–Miller's paper [1] are also good references for this material.

THEOREM 8.1. (Novikov [56], Miller [46]) *There exists a tri-graded spectral sequence with*

$$E_1^{s,i,t} = \operatorname{Ext}^{s,t}_{\mathrm{BP}_*\mathrm{BP}/I}(\mathrm{BP}_*/I, I^i/I^{i+1}),$$
$$d_r \colon E_r^{s,i,t} \longrightarrow E_r^{s+1,i+r,t},$$

converging to

$$\operatorname{Ext}^{s,t}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*).$$

Here, $I = (p, v_1, v_2, \dots)$ is the augmentation ideal of BP_*.

To compare it with the motivic Adams spectral spectral, which is studied by Morel, Dugger–Isaksen and Hu–Kriz–Ormsby [14], [28], [52], we regrade the algebraic Novikov spectral sequence.

Definition 8.2. The *a-filtration* of the algebraic Novikov spectral sequence is

$$a = i + s.$$

The goal of Part 2 of this paper is to prove the following Theorem 8.3.

THEOREM 8.3. *At each prime p, there is an isomorphism of tri-graded spectral sequences between the motivic Adams spectral sequence for $\widehat{S^{0,0}}/\tau$, which converges to the motivic homotopy groups of $\widehat{S^{0,0}}/\tau$, and the regraded algebraic Novikov spectral sequence, which converges to the Adams–Novikov E_2-page for the sphere spectrum.*

The indexes are indicated in the following diagram:

$$\begin{array}{ccc}
\mathrm{Ext}^{s,2w}_{\mathrm{BP}_*\mathrm{BP}/I}(\mathrm{BP}_*/I, I^{a-s}/I^{a-s+1}) & \xrightarrow{\cong} & \mathrm{Ext}^{a,2w-s+a,w}_{A^{\mathrm{mot}}_{*,*}}(\mathbb{F}_p[\tau], \mathbb{F}_p) \\
\Vert & & \Vert \\
\text{Algebraic Novikov SS} & & \text{Motivic Adams SS} \\
\Downarrow & & \Downarrow \\
\mathrm{Ext}^{s,2w}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) & \xrightarrow{\cong} & \pi_{2w-s,w}(\widehat{S^{0,0}}/\tau).
\end{array}$$

Here, $A^{\mathrm{mot}}_{*,*}$ is the motivic mod-p *dual Steenrod algebra.*

9. The equivalence to the motivic Adams spectral sequence

9.1. The algebraic Novikov tower

In this subsection, we write down the algebraic Novikov tower using the newly defined a-filtration explicitly.

Recall from [62, Definition A.1.2.7] that a $\mathrm{BP}_*\mathrm{BP}$-comodule is *relative injective* if it is a direct summand of a $\mathrm{BP}_*\mathrm{BP}$-comodule of the form $\mathrm{BP}_*\mathrm{BP}\otimes_{\mathrm{BP}_*} M$ for some BP_*-module M. Recall from [62, Definition A.1.2.10] that, for the $\mathrm{BP}_*\mathrm{BP}$-comodule BP_*, its *relative injective resolution*

$$\mathrm{BP}_* \longrightarrow C_0^0 \longrightarrow C_0^1 \longrightarrow \dots \tag{9.1}$$

is a long exact sequence in the abelian category of $\mathrm{BP}_*\mathrm{BP}$-comodules, that satisfies the following two conditions:

(1) the long exact sequence (9.1) is split exact as BP_*-modules;
(2) each comodule C_0^s is relative injective.

For now on, we fix such a relative injective resolution C_0^* of BP_* that is concentrated in even internal degrees. Such a relative injective resolution exists (for example the cobar complex).

For $a \geqslant 1$, let C_a^* be the sub cochain complex of C_0^* defined by

$$C_a^s = I^{a-s} C_0^s.$$

Since I is an invariant ideal of BP_*, each C_a^s is a sub $\mathrm{BP}_*\mathrm{BP}$-comodule of C_0^s. It is understood that $I^r = \mathrm{BP}_*$ for $r \leqslant 0$. Therefore, for $s \geqslant a$, we have $C_a^s = C_0^s$.

For $a \geqslant 0$, let Q_a^* be the quotient cochain complex of the inclusion map

$$C_{a+1}^* \xrightarrow{i_a} C_a^*.$$

Therefore, we have a tower of cochain complexes, which induces the following tower in the derived category of $\mathrm{BP}_*\mathrm{BP}$-comodules:

$$\begin{array}{ccccccccc} \mathrm{BP}_* & \xrightarrow{\simeq} & C_0^* & \xleftarrow{i_0} & C_1^* & \xleftarrow{i_1} & C_2^* & \xleftarrow{i_2} & \cdots \\ & & \downarrow{\scriptstyle q_0} & & \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_2} & & \\ & & Q_0^* & & Q_1^* & & Q_2^*. & & \end{array}$$

For $s \geqslant a+1$,

$$Q_a^s = I^{a-s} C_0^s / I^{a-s+1} C_0^s = 0.$$

So, in particular, the cochain complex Q_a^* is bounded. This implies that each cochain complex C_a^* has bounded cohomology. Therefore, although the cochain complexes C_a^* are unbounded, they live in the category $\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}})$.

Applying the functor

$$\mathbf{R}^{*,*} \operatorname{Hom}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, -),$$

where $\mathbf{R}^{*,*} \operatorname{Hom}_{\mathrm{BP}_*\mathrm{BP}}(-,-)$ is the derived homomorphisms in the category

$$\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}}),$$

we get a spectral sequence with the E_1-page

$$\mathbf{R}^{*,*} \operatorname{Hom}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, Q_a^*),$$

converging to

$$\mathbf{R}^{*,*} \operatorname{Hom}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) = \operatorname{Ext}^{*,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*).$$

This is the regraded algebraic Novikov spectral sequence.

9.2. Characterization of Adams towers

Recall that we denote by $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$ the motivic mod-p Eilenberg–Mac Lane spectrum. It is shown by Hu–Kriz–Ormsby [28] and Hoyois [25] that $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$ is cellular. We set

$$\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau := \widehat{S^{0,0}}/\tau \wedge_{\widehat{S^{0,0}}} \mathrm{H}\mathbb{F}_p^{\mathrm{mot}}.$$

Definition 9.1. A tower

$$\begin{array}{ccccccccc} \widehat{S^{0,0}}/\tau & \xrightarrow{\simeq} & X_0 & \xleftarrow{g_0} & X_1 & \xleftarrow{g_1} & X_2 & \xleftarrow{g_2} & \cdots \\ & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\ & & K_0 & & K_1 & & K_2 & & \end{array}$$

in $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}^b_{\mathrm{harm}}$ is a *motivic Adams tower* if the following conditions hold:

(1) each motivic spectrum K_m is a retract of a wedge of suspensions of $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau$;

(2) each map $f_m\colon X_m \to K_m$ induces an epimorphism on the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-cohomology. Or equivalently, each map $g_m\colon X_{m+1} \to X_m$ induces the zero map on the $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$-cohomology.

By the adjunction between modules over $\widehat{S^{0,0}}$ and $\widehat{S^{0,0}}/\tau$, and the fact that $\widehat{S^{0,0}}/\tau$ is Spanier–Whitehead dual to itself up to a bi-degree shift (see [18, Proposition 4.3] for a proof, for example), it is equivalent to check that each map g_m induces the zero map on

$$[-, \mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau]_{\widehat{S^{0,0}}/\tau}$$

in condition (2).

From the general discussions (see [10], [16], [47], [48], for example), all such towers are equivalent to each other in the sense that there exist towers maps that induce canonical isomorphisms on the E_2-pages.

Dugger–Isaksen [14] use the cobar construction to define the motivic Adams spectral sequence for $\widehat{S^{0,0}}/\tau$, which satisfies the two conditions in Definition 9.1. Therefore, the motivic Adams spectral sequence for $\widehat{S^{0,0}}/\tau$ by Dugger–Isaksen [14] is canonically isomorphic to the motivic Adams spectral sequence defined by any motivic Adams tower satisfying the two conditions in Definition 9.1.

Having the regraded algebraic Novikov tower in the category $\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}})$, we use the equivalence of stable ∞-categories in Theorem 4.13 and Proposition 1.10 in Part 1,

$$\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}}) \xrightarrow{\simeq} \mathcal{D}^b(\mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}^{\mathrm{ev}}) \xrightarrow{\simeq} \widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{harm}},$$

to get a tower in the category $\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^b_{\mathrm{harm}}$:

$$\begin{array}{ccccccccc}
\widehat{S^{0,0}}/\tau & \xrightarrow{\simeq} & Y_0 & \longleftarrow & Y_1 & \longleftarrow & Y_2 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & L_0 & & L_1 & & L_2 & &
\end{array}$$

PROPOSITION 9.2. *The above tower is a motivic Adams tower in the sense of Definition 9.1, if the following two conditions are satisfied for the regraded algebraic Novikov tower in the category* $\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}})$:

(1) *each* Q_a^* *is quasi-isomorphic to a retract of a direct sum of shifts of* $\mathrm{BP}_*\mathrm{BP}/I$;

(2) *each map* $q_a\colon C_a^* \to Q_a^*$ *induces an epimorphism on* $\mathbf{R}^{*,*}\operatorname{Hom}_{\mathrm{BP}_*}(-,\mathbb{F}_p)$. *Or, equivalently, each map* $i_a\colon C_{a+1}^* \to C_a^*$ *induces the zero map on* $\mathbf{R}^{*,*}\operatorname{Hom}_{\mathrm{BP}_*}(-,\mathbb{F}_p)$.

Proof. By the following Lemmas 9.3 and 9.4, the two conditions in this proposition correspond to the two conditions in Definition 9.1. □

For the first condition in Proposition 9.2, we identify the $\mathrm{BP}_*\mathrm{BP}$-comodule that corresponds to $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau$ with $\mathrm{BP}_*\mathrm{BP}/I$ under the equivalences of the hearts in Propositions 4.11 and 1.10:

$$\widehat{S^{0,0}}/\tau\text{-}\mathbf{Mod}^{\heartsuit}_{\mathrm{harm}} \xrightarrow[\simeq]{\mathrm{MU}^{\mathrm{mot}}_{*,*}} \mathrm{MU}_*\mathrm{MU}\text{-}\mathbf{Comod}^{\mathrm{ev}} \xrightarrow[\simeq]{\mathrm{BP}_*\otimes_{\mathrm{MU}_*}-} \mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}}.$$

LEMMA 9.3. *Under the equivalences of the hearts,* $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau$ *corresponds to* $\mathrm{BP}_*\mathrm{BP}/I$.

Proof. We have that $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$ is an $\mathrm{MU}^{\mathrm{mot}}$-module, both $\mathrm{MU}^{\mathrm{mot}}$ and $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}$ are cellular, and $\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau$ is free over $\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau$. Then, by Dugger–Isaksen's universal coefficient spectral sequence [13, Proposition 7.7] in the category $\mathrm{MU}^{\mathrm{mot}}/\tau\text{-}\mathbf{Mod}_{\mathrm{cell}}$, we have

$$\mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau \cong \mathrm{MU}^{\mathrm{mot}}_{*,*}\mathrm{MU}^{\mathrm{mot}}/\tau \otimes_{\mathrm{MU}^{\mathrm{mot}}_{*,*}/\tau} \mathbb{F}_p,$$

which is isomorphic to $\mathrm{MU}_*\mathrm{MU}\otimes_{\mathrm{MU}_*}\mathbb{F}_p$ forgetting the motivic weight.

Therefore, under the equivalences in Propositions 4.11 and 1.10, the $\widehat{S^{0,0}}/\tau$-module $\mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau$ corresponds to

$$\begin{aligned}
\mathrm{BP}_*\otimes_{\mathrm{MU}_*}\mathrm{MU}_*\mathrm{MU}\otimes_{\mathrm{MU}_*}\mathbb{F}_p &\cong \mathrm{BP}_*\mathrm{MU}\otimes_{\mathrm{MU}_*}\mathbb{F}_p \\
&\cong \mathrm{BP}_*\mathrm{MU}\otimes_{\mathrm{MU}_*}\mathrm{BP}_*\otimes_{\mathrm{BP}_*}\mathbb{F}_p \\
&\cong \mathrm{BP}_*\mathrm{BP}\otimes_{\mathrm{BP}_*}\mathbb{F}_p \\
&\cong \mathrm{BP}_*\mathrm{BP}/I.
\end{aligned}$$

The first and third isomorphisms follow from the Landweber exactness of BP_*. This completes the proof. □

For the second condition in Proposition 9.2, we have the following lemma.

LEMMA 9.4. *Suppose that X is in the category $\widehat{S^{0,0}}/\tau$-$\mathbf{Mod}^b_{\mathrm{harm}}$ and that $C^*(X)$ is the cochain complex of $\mathrm{BP}_*\mathrm{BP}$-comodules representing the image of X under the equivalence in Theorem 4.13 in Part 1.*

Then, we have

$$[\Sigma^{*,*}X, \mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau]_{\widehat{S^{0,0}}/\tau} \cong \mathbf{R}^{*,*}\,\mathrm{Hom}_{\mathrm{BP}_*}(C^*(X), \mathbb{F}_p),$$

where $\mathbf{R}^{*,*}\,\mathrm{Hom}_{\mathrm{BP}_*}(-,-)$ is the derived homomorphism in the derived category of BP_*-modules.

Proof. We have

$$\begin{aligned}[\Sigma^{*,*}X,\ \mathrm{H}\mathbb{F}_p^{\mathrm{mot}}/\tau]_{\widehat{S^{0,0}}/\tau} &\cong \mathbf{R}^{*,*}\,\mathrm{Hom}_{\mathrm{BP}_*\mathrm{BP}}(C^*(X),\ \mathrm{BP}_*\mathrm{BP}\otimes_{\mathrm{BP}_*}\mathbb{F}_p)\\ &\cong \mathbf{R}^{*,*}\,\mathrm{Hom}_{\mathrm{BP}_*}(C^*(X),\ \mathbb{F}_p).\end{aligned}$$

The first isomorphism follows from Theorem 4.13 and Lemma 9.3, and the second one follows from the adjunction of the derived functor of $\mathrm{BP}_*\mathrm{BP}\otimes_{\mathrm{BP}_*}-$ and the forgetful functor between the derived categories of BP_*-modules and $\mathrm{BP}_*\mathrm{BP}$-comodules. □

In the rest of this section, we check that the two conditions in Proposition 9.2 are satisfied by the regraded algebraic Novikov tower in the category $\mathcal{D}^b(\mathrm{BP}_*\mathrm{BP}\text{-}\mathbf{Comod}^{\mathrm{ev}})$.

9.3. Proof of the first condition

LEMMA 9.5. *Suppose that N is a relative injective $\mathrm{BP}_*\mathrm{BP}$-comodule that is concentrated in even degrees. Then, for any a, $I^aN/I^{a+1}N$ is isomorphic to a retract of a direct sum of shifts of $\mathrm{BP}_*\mathrm{BP}/I$.*

Proof. Without loss of generality, we may assume that N has the form

$$\mathrm{BP}_*\mathrm{BP}\otimes_{\mathrm{BP}_*}M.$$

Because I is an invariant ideal,

$$\frac{I^a\mathrm{BP}_*\mathrm{BP}\otimes_{\mathrm{BP}_*}M}{I^{a+1}\mathrm{BP}_*\mathrm{BP}\otimes_{\mathrm{BP}_*}M} \cong \mathrm{BP}_*\mathrm{BP}\otimes_{\mathrm{BP}_*}I^aM/I^{a+1}M.$$

So, it suffices to show that, for any BP_*/I-module M', $\mathrm{BP}_*\mathrm{BP}\otimes_{\mathrm{BP}_*}M'$ corresponds to a direct sum of shifts of $\mathrm{BP}_*\mathrm{BP}/I$. This is straightforward. □

Now, we prove that the regraded algebraic Novikov tower satisfies the first condition in Proposition 9.2.

PROPOSITION 9.6. (1) *All differentials in the cochain complex Q_a^* are zero, and therefore Q_a^* splits as a direct sum of cochain complexes that are concentrated in one cohomological degree.*

(2) *Each Q_a^s is a retract of a direct sum of shifts of $\mathrm{BP}_*\mathrm{BP}/I$.*

Therefore, the first condition of Proposition 9.2 is satisfied.

Proof. In Q_a^*, all differentials $Q_a^s \longrightarrow Q_a^{s+1}$ have the form

$$I^{a-s}C_0^s/I^{a-s+1}C_0^s \longrightarrow I^{a-s-1}C_0^{s+1}/I^{a-s}C_0^{s+1}.$$

They are all zero, since they are BP_*-linear. The second claim follows from Lemma 9.5 and the definition of Q_a^s. □

9.4. Proof of the second condition

We will use the following Lemma 9.7 in the proof of Proposition 9.8. The proof of Lemma 9.7 is technical. We will postpone it to the last subsection of this section.

LEMMA 9.7. *The homomorphisms*

$$\mathrm{Ext}_{\mathrm{BP}_*}^{*,*}(I^a, \mathbb{F}_p) \longrightarrow \mathrm{Ext}_{\mathrm{BP}_*}^{*,*}(I^{a+1}, \mathbb{F}_p),$$

that are induced by the inclusions $I^{a+1} \to I^a$, are zero for all $a \geqslant 0$.

We now prove that the regrade algebraic Novikov tower satisfies the second condition of Proposition 9.2.

PROPOSITION 9.8. *Each map $i_a \colon C_{a+1}^* \to C_a^*$ induces the zero map on*

$$\mathbf{R}^{*,*}\,\mathrm{Hom}_{\mathrm{BP}_*}(-, \mathbb{F}_p).$$

Proof. Because we are computing derived Hom in the derived category of BP_*-modules, we can first apply a forget functor from $\mathrm{BP}_*\mathrm{BP}$-comodules to BP_*-modules on our complexes.

From the definition of a relative injective resolution, C_0^* splits in the category of BP_*-modules as a direct sum of cochain complexes:

$$C_0^* = \bigoplus_{j \geqslant 0} D_{0,j}^*,$$

where $D^*_{0,0}$ is isomorphic to the cochain complex

$$\mathrm{BP}_* \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots$$

and, for $j \geqslant 1$, $D^*_{0,j}$ is a cochain complex of the form

$$\dots \longrightarrow 0 \longrightarrow N_j \xrightarrow{\mathrm{id}} N_j \longrightarrow 0 \longrightarrow \dots$$

that is concentrated in cohomological degrees j and $j-1$, where N_j is a BP_*-module.

The algebraic Novikov filtration only depends on the underlying BP_*-module structure. So, we have

$$C^*_a = \bigoplus_{j \geqslant 0} D^*_{a,j}$$

where, for $j \geqslant 1$, $D^*_{a,j}$ is the subcomplex

$$\dots \longrightarrow 0 \longrightarrow I^{a-j+1}N_j \longrightarrow I^{a-j}N_j \longrightarrow 0 \longrightarrow \dots .$$

It follows that $D^*_{a,j}$ is quasi-isomorphic to the complex

$$\dots \longrightarrow 0 \longrightarrow I^{a-j}N_j/I^{a-j+1}N_j \longrightarrow 0 \longrightarrow \dots .$$

Now, we consider the maps

$$\mathbf{R}^{*,*}\,\mathrm{Hom}_{\mathrm{BP}_*}(D^*_{a,j}, \mathbb{F}_p) \longrightarrow \mathbf{R}^{*,*}\,\mathrm{Hom}_{\mathrm{BP}_*}(D^*_{a+1,j}, \mathbb{F}_p) \tag{9.2}$$

that are induced by the inclusions

$$D^*_{a+1,j} \longrightarrow D^*_{a,j}.$$

For $j \geqslant 1$, these maps can be identified as (shifts of)

$$\mathbf{R}^{*,*}\,\mathrm{Hom}_{\mathrm{BP}_*}(I^{a-j}N_j/I^{a-j+1}N_j, \mathbb{F}_p) \longrightarrow \mathbf{R}^{*,*}\,\mathrm{Hom}_{\mathrm{BP}_*}(I^{a-j+1}N_j/I^{a-j+2}N_j, \mathbb{F}_p).$$

It is clear that the maps

$$I^{a-j+1}N_j/I^{a-j+2}N_j \longrightarrow I^{a-j}N_j/I^{a-j+1}N_j$$

are all zero. Therefore, the maps in (9.2) are all zero for $j \geqslant 1$.

For $j=0$, we have that $D^*_{a,0}$ is the complex

$$I^a \longrightarrow 0 \longrightarrow \dots,$$

and the corresponding maps in (9.2) can be rewritten as

$$\mathrm{Ext}^{*,*}_{\mathrm{BP}_*}(I^m, \mathbb{F}_p) \longrightarrow \mathrm{Ext}^{*,*}_{\mathrm{BP}_*}(I^{m+1}, \mathbb{F}_p).$$

By Lemma 9.7, they are all zero. Therefore, the maps

$$\mathbf{R}^{*,*}\,\mathrm{Hom}_{\mathrm{BP}_*}(C^*_{m+1}, \mathbb{F}_p) \longrightarrow \mathbf{R}^{*,*}\,\mathrm{Hom}_{\mathrm{BP}_*}(C^*_m, \mathbb{F}_p)$$

are all zero, since they are zero on each direct summand. □

Combining Propositions 9.6 and 9.8, we have shown that the regraded algebraic Novikov tower satisfies the two conditions of Proposition 9.2, and therefore corresponds to a motivic Adams tower for $\widehat{S^{0,0}}/\tau$. This proves that there exists an isomorphism between the regraded algebraic Novikov spectral sequence and the motivic Adams spectral sequence for $\widehat{S^{0,0}}/\tau$.

9.5. Proof of Lemma 9.7

We prove Lemma 9.7 in this subsection.

For any BP_*-module M, we have

$$\mathrm{Ext}^{*,*}_{\mathrm{BP}_*}(M,\mathbb{F}_p)\cong \mathrm{Hom}_{\mathbb{F}_p}(\mathrm{Tor}^{\mathrm{BP}_*}_{*,*}(M,\mathbb{F}_p),\mathbb{F}_p).$$

Lemma 9.7 is implied by its dual statement.

LEMMA 9.9. *The maps*

$$\mathrm{Tor}^{\mathrm{BP}_*}_{*,*}(I^{n+1},\mathbb{F}_p)\longrightarrow \mathrm{Tor}^{\mathrm{BP}_*}_{*,*}(I^{n},\mathbb{F}_p)$$

are zero for $n\geqslant 0$.

Proof. The powers of I filter BP_* as a BP_*-module, and the BP_*-action on the associated graded pieces factors through an $\mathbb{F}_p$-action.

Therefore, we have an associated spectral sequence

$$E_1^{s,t,i}=\mathrm{Tor}^{\mathrm{BP}_*}_{s,t}(I^i/I^{i+1},\mathbb{F}_p)\Longrightarrow \mathrm{Tor}^{\mathrm{BP}_*}_{s,t}(\mathrm{BP}_*,\mathbb{F}_p).$$

The E_1-page can be identified as

$$E(\tau_0,\tau_1,\dots)\otimes\mathbb{F}_p[q_0,q_1,\dots],$$

since

$$\mathrm{Tor}^{\mathrm{BP}_*}_{*,*}(\mathbb{F}_p,\mathbb{F}_p)=E(\tau_0,\tau_1,\dots),$$

where $E(\tau_0,\tau_1,\dots)$ is the exterior algebra over $\mathbb{F}_p$ generated by the τ_j's, with the same internal degrees as the v_j's and homological degree 1, and

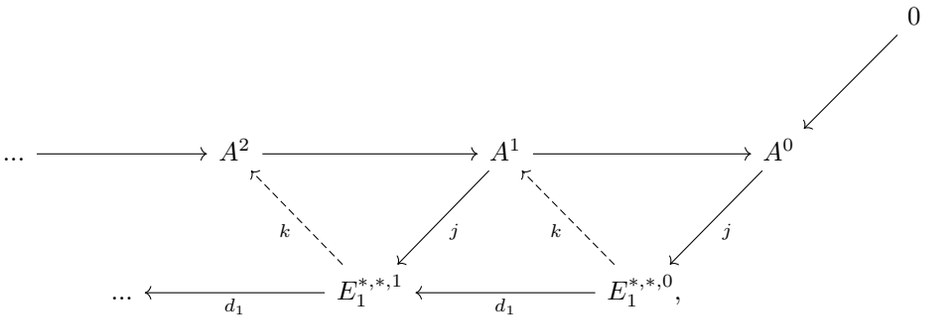
$$\mathrm{gr}^*\mathrm{BP}_*=\mathbb{F}_p[q_0,q_1,\dots],$$

where q_j corresponds to v_j.

Using the Koszul complex, it is straightforward to see that this spectral sequence is multiplicative and that $d_1\tau_n=q_n$. Therefore, its E_2-page is concentrated in degrees $i=0$, and the following sequence is exact

$$0 \longrightarrow \mathrm{Tor}^{\mathrm{BP}_*}_{*,*}(\mathrm{BP}_*, \mathbb{F}_p) \longrightarrow E_1^{*,*,0} \longrightarrow E_1^{*,*,1} \longrightarrow \dots .$$

Now, the exact couple looks like this:



where $A^i=\mathrm{Tor}^{\mathrm{BP}_*}_{*,*}(I^i, \mathbb{F}_p)$, and the bottom line is exact. All we need to show is that all the j maps are injective. This follows from an induction using a diagram chasing argument. □

Remark 9.10. For the polynomial ring with finitely many generators $\mathbb{Z}_p[x_1, x_2, ..., x_t]$, the analogue of Lemmas 9.7 and 9.9 are well known (see [66], for example).

Appendix A. Computation of some classical Adams differentials

In this appendix, we illustrate the power of the isomorphism of spectral sequences in Theorem 8.3, by recomputing certain low filtration and historically more difficult differentials in the range up to the 45-stem at the prime 2. We follow notation in Isaksen's stable stems [29], and Isaksen, the second and third author's more stable stems [32], [33].

When computing non-trivial differentials in the classical Adams spectral sequence, it is usually harder to give proofs for the ones whose sources are in low Adams filtrations. There are at least two reasons for this. Firstly, there are more potential targets that it could hit, so it means more possibilities to check and rule out. Secondly, on the other hand, elements in high Adams filtrations can usually be detected by certain known spectrum in small chromatic height—for instance, elements above the $\frac{1}{3}$-line can be detected by the $K(1)$-local sphere, and many elements around the $\frac{1}{5}$-line can be detected by the spectrum of topological modular forms. This gives ways to compare with Adams spectral sequences of other spectra.

Adams differential	Stem of the source	Filtration of the source
$d_2(h_4)=h_0h_3^2$	15	1
$d_3(h_0h_4)=h_0d_0$	15	2
$d_2(e_0)=h_1^2d_0$	17	4
$d_2(f_0)=h_0^2e_0$	18	4
$d_2(h_5)=h_0h_4^2$	31	1
$d_3(h_0^3h_5)=h_0\Delta h_2^2$	31	4
$d_3(h_2h_5)=h_0p=h_1d_1$	34	2
$d_4(h_3h_5)=h_0x$	38	2
$d_3(e_1)=h_1t$	38	4
$d_2(c_2)=h_0f_1$	41	3

Table 1. Some non-trivial differentials up to the 45-stem.

Up to the 45-stem, we list in Table 1 above ten non-trivial differentials, whose sources are in low Adams filtrations. Five of them are d_2-differentials, four of them are d_3-differentials, and one of them is a d_4-differential.

Historically, the first four of them were proved by May in his thesis, by comparing with Toda's unstable computations. The next two are obtained by the Hopf invariant one problem, and by comparing with the J-spectrum. The elements Δh_2^2 and $h_0\Delta h_2^2$ were historically called r and s, respectively, and there is a non-trivial extension in the May spectral sequence that gives us a relation $s=h_0r$. The last four, except the one on $d_3(e_1)$, were proved by Barratt–Mahowald–Tangora [3] using ad-hoc methods. In fact, the differentials

$$d_3(h_2h_5)=h_0p \quad \text{and} \quad d_2(c_2)=h_0f_1$$

are both closely related to the non-trivial ν-extension from h_4^2 to the element p, and the differential

$$d_4(h_3h_5)=h_0x$$

is closely related to the non-trivial σ-extension from h_4^2 to the element x. For the element e_1, Barratt–Mahowald–Tangora [3] erroneously thought it was a permanent cycle. It was later proved by Bruner [8], using power operations, that it supports a non-trivial differential

$$d_3(e_1)=h_1t.$$

Now, using Theorem 8.3, we compare them with the computations of the motivic Adams spectral sequence of $\widehat{S^{0,0}}/\tau$. All five d_2-differentials are present in the motivic Adams spectral sequence of $\widehat{S^{0,0}}/\tau$. This gives immediate proofs for all of them.

Moreover, the three out of four d_3-differentials except $d_3(h_2h_5)$ are present in the motivic Adams spectral sequence of $\widehat{S^{0,0}}/\tau$. To be careful, one also needs to rule out the possibility of non-zero d_2-differentials in these cases. This can be done by multiplying h_0 to the proposed d_2-differentials and get contradictions.

For the d_3-differential

$$d_3(h_2h_5) = h_1d_1,$$

one can show the following three statements are equivalent, by considering the long exact sequence of motivic homotopy groups associated with the cofiber map of τ.

(1) There is a differential

$$d_3(h_2h_5) = \tau h_1d_1$$

in the motivic Adams spectral sequence of $\widehat{S^{0,0}}$.

(2) In homotopy groups, $\{h_2h_5\}$ maps to $\{h_1d_1\}$ under the quotient map from $\widehat{S^{0,0}}/\tau$ to its top cell $S^{1,-1}$.

(3) There is an η-extension from h_2h_5 to $\overline{h_1^2d_1}$ in $\pi_{*,*}(\widehat{S^{0,0}}/\tau)$, where $\overline{h_1^2d_1}$ is the element in the motivic Adams E_2-page of $\widehat{S^{0,0}}/\tau$ that corresponds to $h_1^2d_1$ in that of the top cell $\widehat{S^{1,-1}}$.

Statement (3) can be checked in the E_∞-page of the motivic Adams spectral sequence for $\widehat{S^{0,0}}/\tau$, which is isomorphic to the classical Adams–Novikov E_2-page. This gives a proof for the d_3-differential in the motivic Adams spectral sequence for $\widehat{S^{0,0}}$, and hence for the classical d_3-differential. Statement (2) is proved by Isaksen in Table 42 of stable stems [29].

At last, the d_4-differential $d_4(h_3h_5)$ is also present in the motivic Adams spectral sequence for $\widehat{S^{0,0}}/\tau$. To pull it back and get the d_4-differential in the motivic sphere, one needs to rule out the possibilities of non-zero d_2's and d_3's. For degree reasons, there are no possible d_2's. To rule out the only d_3 possibility that $d_3(h_3h_5)=x$, since $h_3x=h_0^2g_2$, this would give another d_3-differential by multiplying by h_3:

$$d_3(h_3^2h_5) = h_0^2g_2.$$

However, there is no such d_3 in the motivic Adams spectral sequence for $\widehat{S^{0,0}}/\tau$, which gives a contradiction.

Summarizing, we reprove all ten non-trivial low filtration differentials up to the 45-stem without much effort. In fact, among all non-trivial differentials up to the 45-stem, there is only one that cannot be proved by our motivic $\widehat{S^{0,0}}/\tau$-method:

$$d_3(\Delta h_2^2) = h_1d_0^2.$$

This can be proved by other methods, such as the ad-hoc method by Barratt–Mahowald–Tangora [3], the power operation method by Bruner [9], the method of detection by the spectrum of topological modular forms, and the $\mathbb{R}\mathrm{P}^\infty$-technique in [78].

In the next pages (Figures 1–8), we present Isaksen's charts for the reader's reference of the differentials that are discussed in this appendix.

There are eight charts in total. The first two (Figures 1 and 2) are for the classical Adams spectral sequences. The horizontal degree is $t-s$, i.e., the topological stem, and the vertical degree is s, i.e., the Adams filtration. Each dot is a copy of $\mathbb{F}_2$. Vertical lines and lines of slope 1 and $\frac{1}{3}$ correspond to multiplication by h_0, h_1 and h_2, respectively. Lines of negative slope correspond to differentials. We mark all algebraic generators for completeness.

For the rest of the six charts, we only mark certain low Adams filtration elements and the elements that are relevant to our discussion of differentials in this section.

The second two charts (Figures 3 and 4) are the E_2-pages with d_2-differentials of the motivic Adams spectral sequences for $\widehat{S^{0,0}}/\tau$. Each arrow with slope 1 indicates an infinite h_1-tower, and each arrow with slope -2 is a family of h_1-periodic d_2-differentials.

The third two charts (Figures 5 and 6) are the E_3-pages with d_3- and d_4-differentials of the motivic Adams spectral sequences for $\widehat{S^{0,0}}/\tau$.

The last two charts (Figures 7 and 8) are the E_∞-pages of the motivic Adams spectral sequences for $\widehat{S^{0,0}}/\tau$. The blue lines are non-trivial 2-, η- and ν-extensions.

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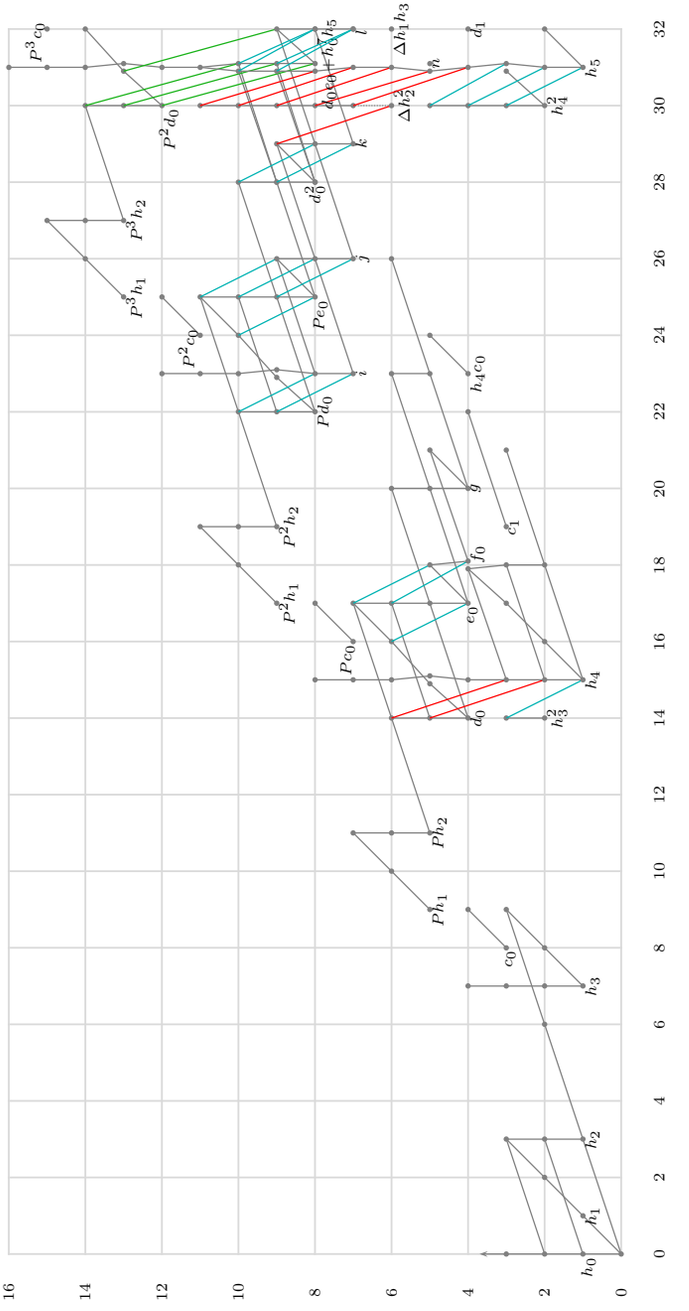

Figure 1. The classical Adams spectral sequence.

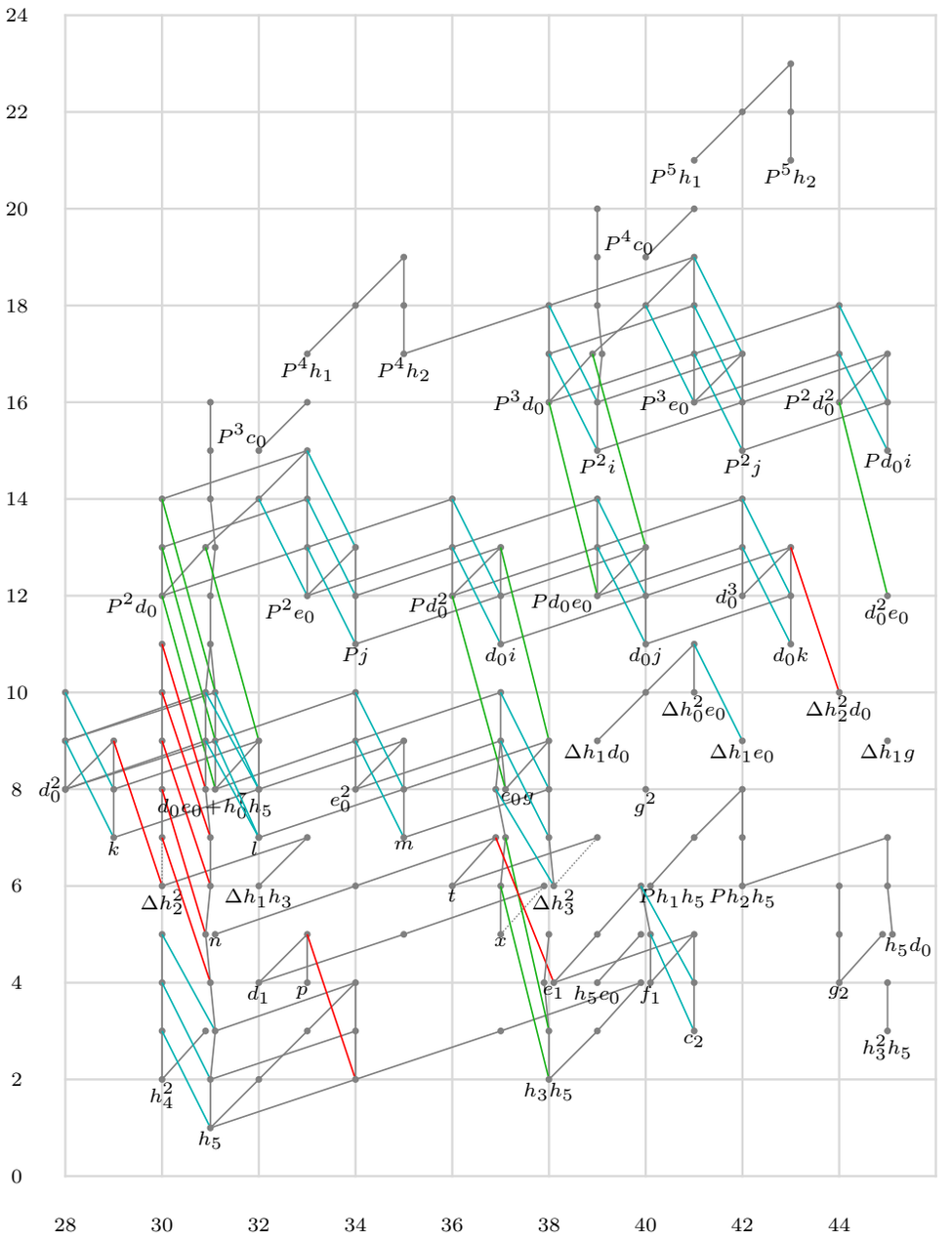

Figure 2. The classical Adams spectral sequence.

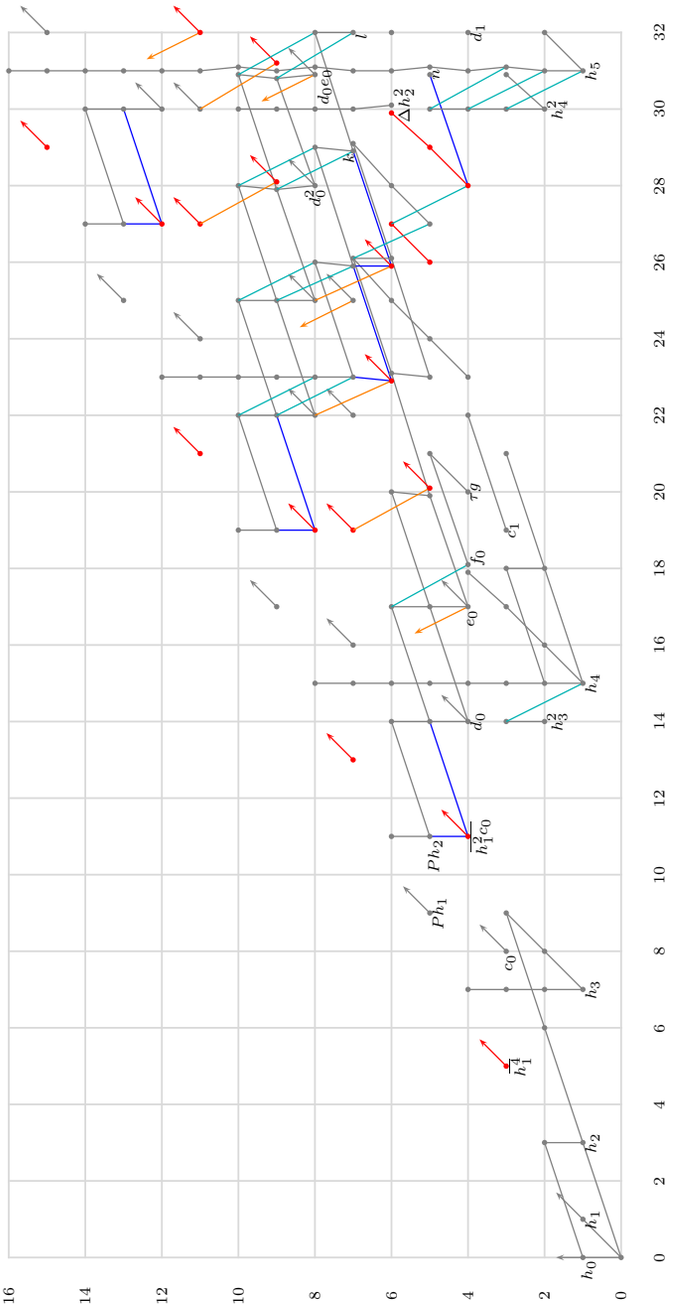

Figure 3. The E_2-page of the motivic Adams spectral sequence for the cofiber of τ.

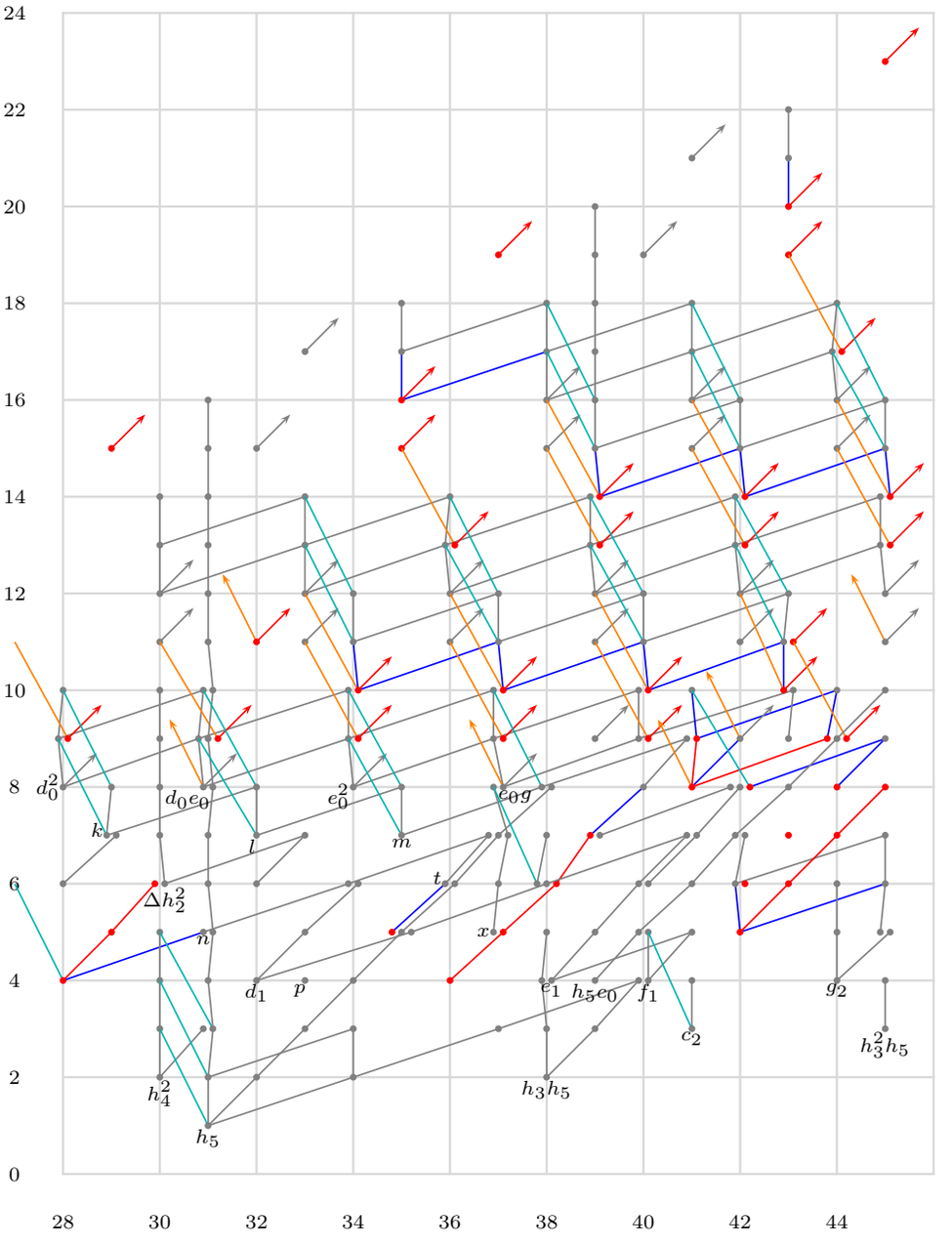

Figure 4. The E_2-page of the motivic Adams spectral sequence for the cofiber of τ.

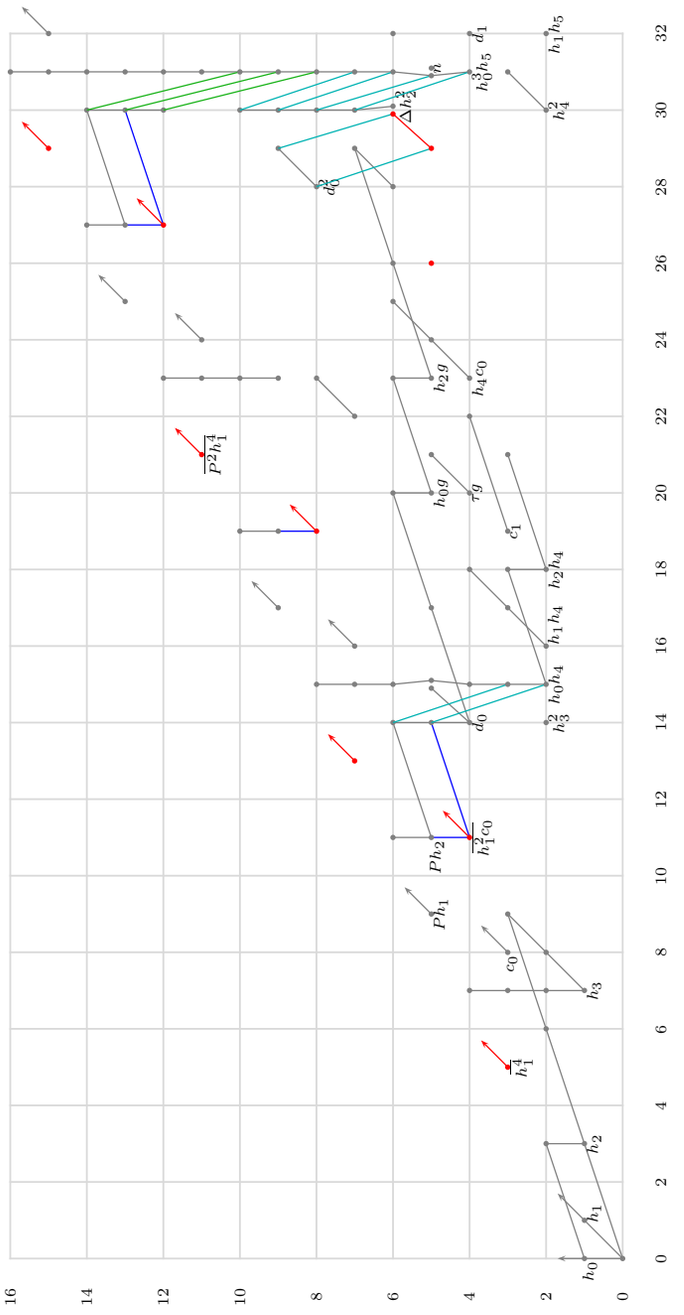

Figure 5. The E_3-page of the motivic Adams spectral sequence for the cofiber of τ.

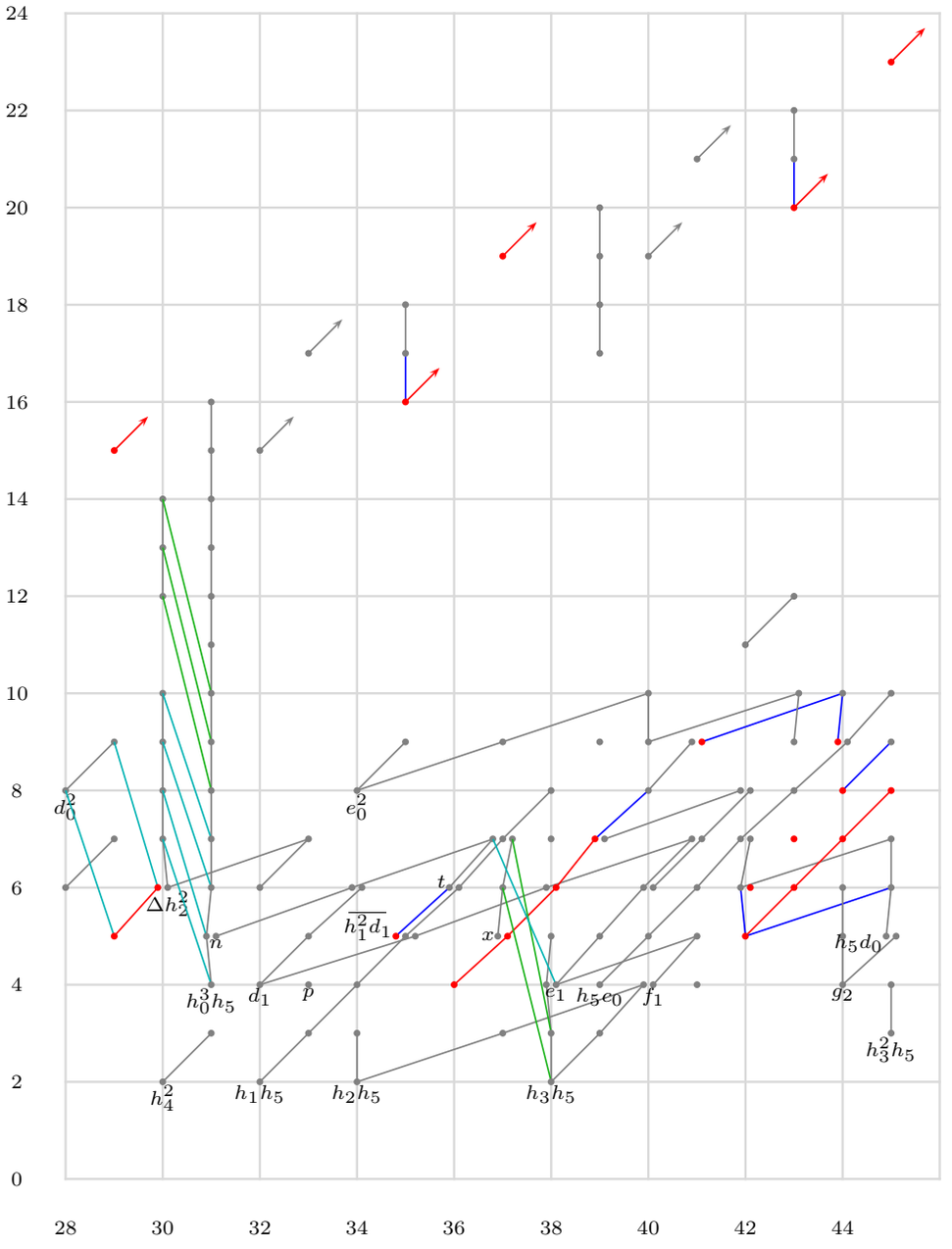

Figure 6. The E_3-page of the motivic Adams spectral sequence for the cofiber of τ.

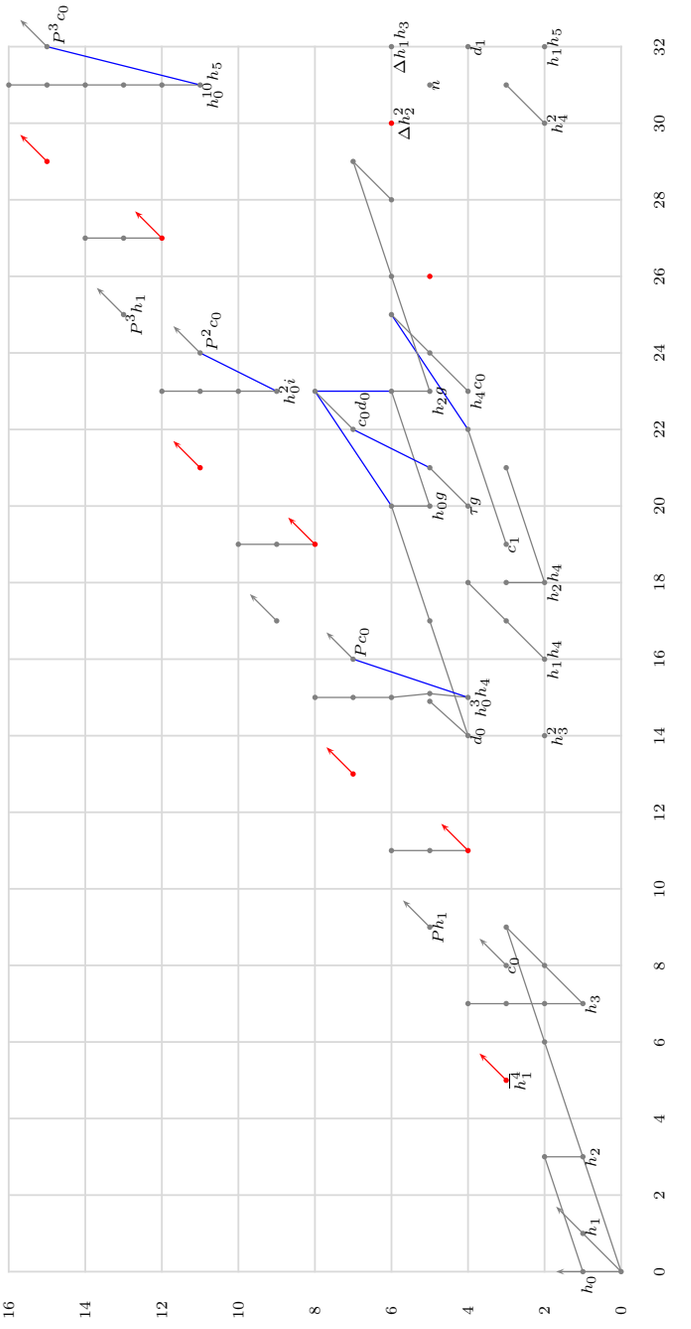

Figure 7. The E_∞-page of the motivic Adams spectral sequence for the cofiber of τ.

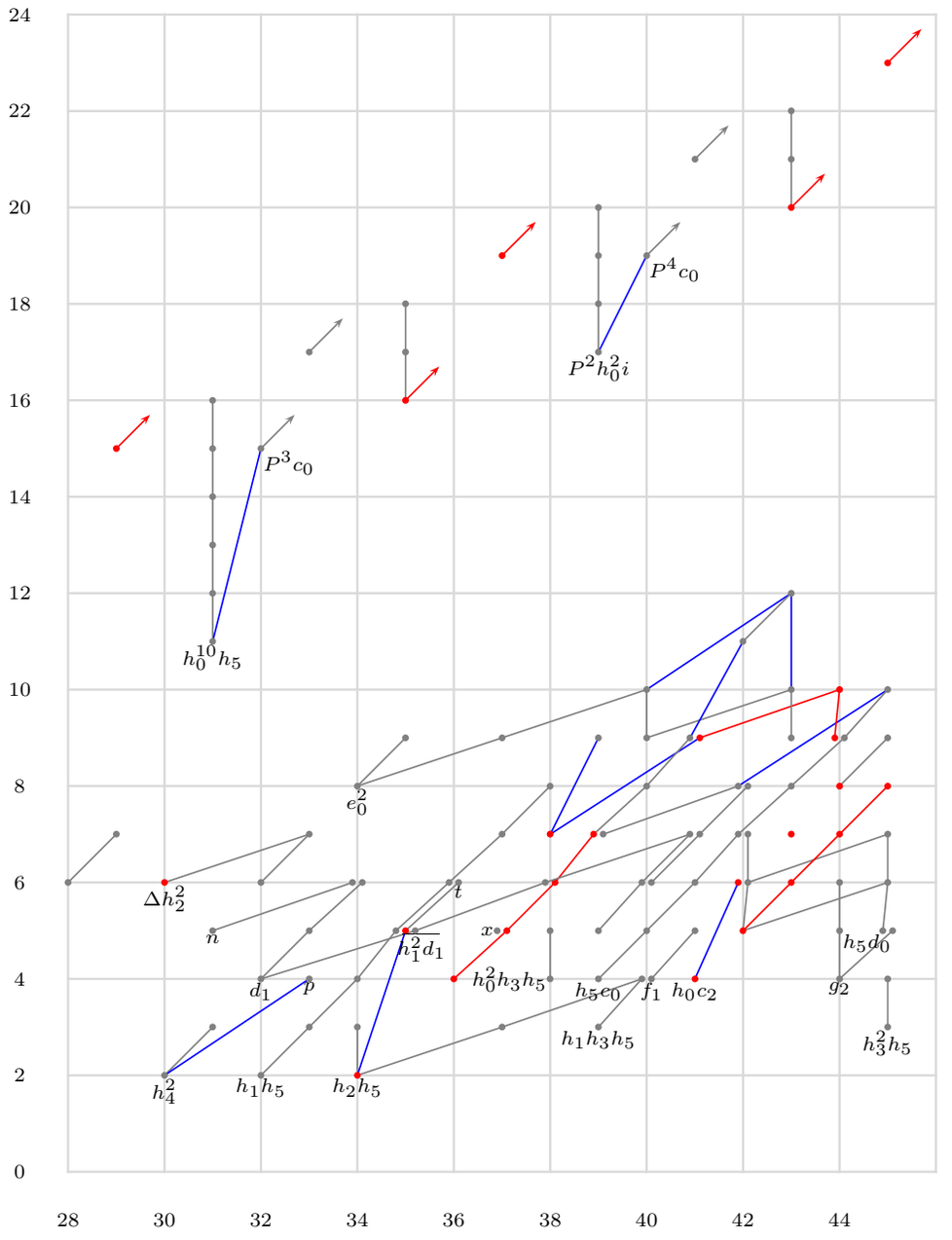

Figure 8. The E_∞-page of the motivic Adams spectral sequence for the cofiber of τ.

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