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Absolute Continuity of the Spectrum of the Periodic Schrödinger Operator in a Cylinder with Robin Boundary Condition

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Abstract. We show that the spectrum of the Schrödinger operator $H = -\Delta + V$ in a smooth cylinder with Robin boundary condition $\partial_\nu u = \sigma u$ is purely absolutely continuous, assuming that the coefficients V and σ are periodic in the axial directions.

Key words: Schrödinger operator in a cylinder, Robin boundary condition, absolutely continuous spectrum, spectral cluster estimates.

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Introduction

Let M be a smooth compact Riemannian manifold with boundary ∂M, and let

$$\Xi = M \times \mathbb{R}^m, \quad k := \dim M, \quad d := \dim \Xi = k + m, \quad m \geqslant 1.$$

We are interested in the spectral type of the Schrödinger operator $H = -\Delta + V$ in the cylinder Ξ. On the boundary $\partial\Xi = \partial M \times \mathbb{R}^m$ we impose the Robin boundary condition

$$\frac{\partial u}{\partial \nu}\bigg|_{\partial\Xi} = \sigma u|_{\partial\Xi}. \tag{0.1}$$

The functions V and σ are assumed to be periodic in the axial directions (see below). Our goal is to show that, under some assuptions on V and σ, the spectrum of the operator H is purely absolutely continuous (see Theorem 1.1 below).

The points of the cylinder Ξ will be denoted by (x, y), where $x \in M$ and $y \in \mathbb{R}^m$. Let Γ be a lattice in $\mathbb{R}^m$:

$$\Gamma = \left\{ l = \sum_{j=1}^m l_j b_j,\ l_j \in \mathbb{Z} \right\}, \tag{0.2}$$

where $\{b_j\}_{j=1}^m$ is some basis in $\mathbb{R}^m$. We will assume the following periodicity conditions to hold:

$$V(x, y + l) = V(x, y), \qquad x \in M,\ y \in \mathbb{R}^m,\ l \in \Gamma, \tag{0.3}$$

$$\sigma(x, y + l) = \sigma(x, y), \qquad x \in \partial M,\ y \in \mathbb{R}^m,\ l \in \Gamma. \tag{0.4}$$

Let

$$\Omega = \left\{ y = \sum_{j=1}^m y_j b_j,\ y_j \in [0, 1) \right\} \tag{0.5}$$

be the elementary cell of Γ. Due to periodicity, the functions V and σ are uniquely determined by their values on $M \times \Omega$ and $\partial M \times \Omega$, respectively.

Let us briefly review some earlier results on the absolute continuity of the spectrum of H. Sufficient conditions for absolute continuity are usually of the form $V \in L_p(M \times \Omega)$ and $\sigma \in L_p(\partial M \times \Omega)$. One can also consider wider classes, such as the Lorentz spaces $L^0_{p,\infty}$, but we will restrict ourselves to the L_p case for simplicity. The case $k = 0$ corresponds to the operator on the whole space (in which case there are no M and σ) and has been extensively studied. In [14] (see also [3]) it was shown that the spectrum of H is absolutely continuous under the "optimal" conditions

$$V \in L_p(\Omega), \qquad p > 1 \text{ for } d = 2, \quad p = d/2 \text{ for } d \geqslant 3.$$

In the case $k = 1$ (where M is a line segment and Ξ is a plane-parallel layer), the absolute continuity of H was established in [12] under the assumptions

$$\begin{aligned} V &\in L_p(M \times \Omega), \qquad p > 1 \text{ for } d = 2, \quad p = 3/2 \text{ for } d = 3, \quad p = d - 2 \text{ for } d \geqslant 4, \\ \sigma &\in L_q(\partial M \times \Omega), \qquad q > 1 \text{ for } d = 2, \quad q = 2 \text{ for } d = 3, \quad q = 2d - 2 \text{ for } d \geqslant 4. \end{aligned}$$

The above assumptions were relaxed in [5] to the "nearly" best possible assumptions

$$V \in L_p(M \times \Omega), \qquad p > 1 \text{ for } d = 2, \quad p = d/2 \text{ for } d \geqslant 3, \tag{0.6}$$

$$\sigma \in L_q(\partial M \times \Omega), \qquad q > d - 1. \tag{0.7}$$

Let us now consider the case $k \geqslant 2$. For the Neumann boundary condition ($\sigma \equiv 0$), absolute continuity was shown in [9] for $V \in L_\infty(M \times \Omega)$. In [8] this condition was relaxed to

$$V \in L_p(M \times \Omega), \qquad p > d/2 \text{ for } d = 2, 3, 4, \quad p > d - 2 \text{ for } d \geqslant 5. \tag{0.8}$$

In [7], assuming that σ does not depend on the axial variables, that is,

$$\sigma(x, y) = \sigma(x),$$

the authors established absolute continuity under the assumptions

$$\sigma \in L_q(\partial M), \qquad q > 1 \text{ for } k = 2, \quad q = k - 1 \text{ for } k \geqslant 3.$$

The case where σ has nontrivial dependence on y remained open. The second author has established absolute continuity in the following special cases:

- $M = [0, a_1] \times \cdots \times [0, a_d]$ is a rectangle, V satisfies (0.6), and σ satisfies (0.7);
- $M = \{x \in \mathbb{R}^k : |x| < R\}$ is a k-dimensional ball, $d \geqslant 3$, V satisfies (0.8), and $\sigma \in L_{4d-8}(\partial M \times \Omega)$;

see [5] and [6], respectively.

In this paper we will establish the absolute continuity of the spectrum of the operator H with coefficient σ of the general form $\sigma = \sigma(x, y)$ in the case where M is an arbitrary compact smooth Riemannian manifold with boundary. Similarly to previous works, our proof will follow the Thomas scheme [13]. In order to establish resolvent estimates for $H(\xi)$ (see Theorem 1.2 below), we will use the spectral cluster estimates for the Laplace operator obtained in [11] and [2]. The idea of using these estimates first appeared in [14]. The boundary estimates from [2] are crucial for considering the Robin boundary condition with nontrivial σ.

Remark 0.1. The case of the Dirichlet boundary condition $\sigma|_{\partial\Xi} = 0$ is easier. In this case, the spectrum of H is absolutely continuous under assumptions (0.8) for a general cylinder (see [8]) and under assumption (0.6) for a rectangular cylinder (see [5]).

1. Statement of the Result

Let M, $\dim M = k$, be a compact smooth Riemannian manifold with boundary, and let $\Xi = M \times \mathbb{R}^m$. Since the case $k = 1$ has been covered by previous works, from now on we will always assume that

$$d = k + m \geqslant 3.$$

Let Γ be a lattice (0.2) in $\mathbb{R}^m$, and let Ω be its elementary cell (0.5). Assume that the real-valued functions $V(x, y)$ and $\sigma(x, y)$ satisfy the periodicity conditions (0.3) and (0.4) and

$$V \in L_{d/2}(M \times \Omega), \qquad \sigma \in L_{d-1}(\partial M \times \Omega). \tag{1.1}$$

Consider the following quadratic form on $L_2(\Xi)$:

$$h[u,u] = \int_{\Xi} (|\nabla u(x,y)|^2 + V(x,y)|u(x,y)|^2)\, dx\, dy + \int_{\partial\Xi} \sigma(x,y)|u(x,y)|^2\, dS(x,y), \qquad \operatorname{Dom} h = H^1(\Xi).$$

Here dS denotes the surface area measure on $\partial\Xi$ and $H^1 \equiv W_2^1$ is the Sobolev space. It is well known that, under (1.1), the quadratic form h is closed and semibounded from below. Therefore, it defines a self-adjoint semibounded operator H on the Hilbert space $L_2(\Xi)$. The operator H will be called the Schrödinger operator in the cylinder Ξ with Robin boundary condition (0.1).

The following theorem is the main result of the paper.

Theorem 1.1. *Let M, Ξ, Γ, Ω, V, σ, and H be defined as above. Assume, in addition, that V satisfies* (0.8) *and σ satisfies*

$$\sigma \in L_q(\partial M \times \Omega), \qquad q > 5/2 \text{ for } d = 3,\ q > 2d - 4 \text{ for } d \geqslant 4. \tag{1.2}$$

Then the spectrum of H is purely absolutely continuous.

It will be convenient to identify Ω with the m-dimensional torus $\mathbb{T}^m = \mathbb{R}^m/\Gamma$. Consider the following quadratic forms that depend on an additional parameter $\xi \in \mathbb{C}\mathbb{C}^m$:

$$h(\xi)[v,v] = \int_{M\times\Omega} (|\nabla_x v|^2 + \langle(\nabla_y + i\xi)v, (\nabla_y + i\bar{\xi})v\rangle + V(x,y)|v|^2)\, dx\, dy, + \int_{\partial M\times\Omega} \sigma(x,y)|v|^2\, dS(x,y), \qquad \operatorname{Dom} h(\xi) = H^1(M \times \mathbb{T}^m).$$

These forms are sectorial in the sense of [4]. Therefore, they define a family of analytic self-adjoint operators $H(\xi)$. The embedding $H^1(M \times \mathbb{T}^m) \subset L_2(M \times \Omega)$ is compact, and, as a consequence, the spectra of $H(\xi)$ are discrete. According to the Thomas criterion (see [13], [10], and [1]), it would suffice to show that the family $H(\xi)$ has no eigenvalues that are constant in ξ. Therefore, Theorem 1.1 is a corollary of the following result.

Theorem 1.2. *Under the assumptions of Theorem* 1.1, *let b_1 be the first basis vector of Γ. For any $\lambda \in \mathbb{C}$ and $\xi \in \mathbb{R}^m$, $\xi \perp b_1$, there exists a $\tau_0 > 0$ such that, for all $\tau > \tau_0$, the operator*

$$H(\tau) := H((\pi + i\tau)b_1 + \xi) - \lambda I$$

is invertible and

$$\|H(\tau)^{-1}\| \leqslant C\tau^{-1}.$$

Rescaling if necessary, we can assume without loss of generality that $|b_1| = 1$.

2. Some Auxiliary Estimates

The following lemma was proved in [5].

Lemma 2.1. *Suppose that $0 < \delta < 1/2$, $b \geqslant 1$, and $|m_\mu| \leqslant b$ for any $\mu \in \mathbb{N}$. Then*

$$\sum_{\mu=1}^{\infty} \frac{\mu^{1-2\delta}}{|(\mu + m_\mu)^2 - \tau^2| + \tau} \leqslant C\tau^{-\delta}$$

for $\tau > 1$.

Let

$$H_0(\tau) = H(\tau)|_{V=0,\,\sigma=0}.$$

In other words, $H_0(\tau)$ is the operator on $L_2(M\times\Omega)$ defined by the quadratic form

$$h_0(\xi)[v,v] = \int_{M\times\Omega} (|\nabla_x v|^2 + \langle(\nabla_y + i((\pi+i\tau)b_1+\xi))v,$$
$$(\nabla_y + i((\pi-i\tau)b_1+\xi))v\rangle)\,dx\,dy, \qquad \operatorname{Dom} h_0(\xi) = H^1(M\times\mathbb{T}^m).$$

Let $\{\lambda_j\}_{j=1}^\infty$ and $\{\varphi_j(x)\}_{j=1}^\infty$ denote the eigenvalues and eigenfunctions of the Laplace operator on M with Neumann boundary condition. One can easily check that the eigenvalues and eigenfunctions of $H_0(\tau)$ are

$$h_{j,n}(\tau) = |n+\pi b_1+\xi|^2 + \lambda_j - \tau^2 + 2i\tau\langle n+\pi b_1, b_1\rangle,$$
$$\varphi_{j,n}(x,y) = \varphi_j(x)e^{i\langle n,y\rangle}, \qquad j\in\mathbb{N},\ n\in\widetilde{\Gamma}, \tag{2.1}$$

where $\widetilde{\Gamma}\subset\mathbb{R}^m$ is the dual lattice:

$$\widetilde{\Gamma} = \Big\{ n = \sum_{j=1}^m n_j\tilde{b}_j,\ n_j\in\mathbb{Z}\Big\}, \qquad \langle b_k,\tilde{b}_j\rangle = 2\pi\delta_{kj}.$$

Since $\langle n, b_1\rangle \in 2\pi\mathbb{Z}$, we have

$$|h_{j,n}(\tau)| \geqslant |\operatorname{Im} h_{j,n}(\tau)| = 2\tau|\langle n,b_1\rangle + \pi| \geqslant 2\pi\tau, \qquad \tau>0. \tag{2.2}$$

We will also need the operator $|H_0(\tau)|^{-1/2}$, which can be defined in the basis (2.1) as the operator of multiplication by $|h_{j,n}(\tau)|^{-1/2}$.

In the following considerations the central object is the spectral projections of the Laplace operator on $M\times\mathbb{T}^m$ with Neumann boundary conditions. Let

$$E_\mu = E_{(-\Delta)}[(\mu-1)^2,\mu^2)$$

denote the spectral projection operator onto the subspace corresponding to the interval $[(\mu-1)^2,\mu^2)$. Note that E_μ and $H_0(\tau)$ commute with each other.

Lemma 2.2. *If $0<\delta<1/2$ and $\tau>1$, then*

$$\sum_{\mu=1}^\infty \mu^{1-2\delta}\|E_\mu|H_0(\tau)|^{-1/2}\|^2 \leqslant C\tau^{-\delta}.$$

Proof. Let $b = |\pi b_1+\xi|$. Due to (2.2), we have

$$\sum_{\mu=1}^{[b]+1} \mu^{1-2\delta}\|E_\mu|H_0(\tau)|^{-1/2}\|^2 \leqslant C\tau^{-1}.$$

Let us estimate the sum over $\mu > [b]+1$. The eigenvalues of the Laplace operator on $M\times\mathbb{T}^m$ are

$$\lambda_j + n^2, \qquad j\in\mathbb{N},\ n\in\widetilde{\Gamma}.$$

The range of E_μ corresponds to the pairs (j,n) satisfying

$$(\mu-1)^2 \leqslant \lambda_j + n^2 < \mu^2.$$

Hence

$$|n+\pi b_1+\xi|^2+\lambda_j \in [(\mu-b-1)^2,(\mu+b)^2),$$

and, for $\mu \geqslant [b]+2$, we have

$$\begin{aligned}\|E_\mu|H_0(\tau)|^{-1/2}\|^2 &= \max_{\lambda_j+n^2\in[(\mu-1)^2,\mu^2)} |h_{j,n}(\tau)|^{-1}\\ &\leqslant \max_{|n+\pi b_1+\xi|^2+\lambda_j\in[(\mu-b-1)^2,(\mu+b)^2)} \frac{\sqrt{2}}{\big|\,|n+\pi b_1+\xi|^2+\lambda_j-\tau^2\big|+\tau}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{\mu=[b]+2}^{\infty} \mu^{1-2\delta}\|E_\mu|H_0(\tau)|^{-1/2}\|^2\\ &\qquad\leqslant \sum_{\mu=[b]+2}^{\infty} \max_{|n+\pi b_1+\xi|^2+\lambda_j\in[(\mu-b-1)^2,(\mu+b)^2)} \frac{\sqrt{2}\,\mu^{1-2\delta}}{\big|\,|n+\pi b_1+\xi|^2+\lambda_j-\tau^2\big|+\tau} \leqslant C\tau^{-\delta},\end{aligned}$$

where we also used Lemma 2.1. □

3. Proof of Theorem 1.2

The key step of the proof is based on estimates of the spectral projections of the Laplace operator.

The following theorem was proved in [11].

Theorem 3.1. *Let N be a compact smooth Riemannian manifold with boundary, and let $d := \dim N \geqslant 3$. Given $\mu \geqslant 1$, let $E_\mu = E_{(-\Delta)}[(\mu-1)^2,\mu^2)$ be the spectral projection of the Neumann Laplacian on N onto the subspace corresponding to the interval $[(\mu-1)^2,\mu^2)$. If*

$$5 \leqslant r \leqslant \infty \quad \text{for } d=3, \qquad 4 \leqslant r \leqslant \infty \quad \text{for } d \geqslant 4,$$

then

$$\|E_\mu f\|_{L_r(N)} \leqslant C\mu^{d/2-d/r-1/2}\|f\|_{L_2(N)} \qquad \text{for all } f \in L_2(N).$$

Moreover, if

$$2 \leqslant r \leqslant 4 \quad \text{for } d \geqslant 4,$$

then

$$\|E_\mu f\|_{L_r(N)} \leqslant C\mu^{d/2-d/r+2/r-1}\|f\|_{L_2(N)} \qquad \text{for all } f \in L_2(N).$$

Remark 3.2. In [11] the estimates of $d \geqslant 4$ were only obtained for $r \geqslant 4$. The estimates in the range $2 \leqslant r \leqslant 4$ can be obtained by interpolating the bound $\|E_\mu f\|_{L_4} \leqslant C\mu^{d/4-1/2}\|f\|_{L_2}$ at $r=4$ with the trivial bound $\|E_\mu f\|_{L_2} \leqslant \|f\|_{L_2}$ at $r=2$.

Remark 3.3. Similar estimates have been obtained in the Dirichlet case.

The following theorem was proved in [2].

Theorem 3.4. *Let N be a compact smooth Riemannian manifold with boundary, and let $\dim N \geqslant 3$. Given $\mu \geqslant 1$, let $E_\mu = E_{(-\Delta)}[(\mu-1)^2,\mu^2)$ be the spectral projection of the Neumann Laplacian on N onto the subspace corresponding to the interval $[(\mu-1)^2,\mu^2)$. If*

$$3 \leqslant s \leqslant \infty, \quad d=3,$$

then

$$\|E_\mu f\|_{L_s(\partial N)} \leqslant C\mu^{1-5/(3s)}\|f\|_{L_2(N)} \qquad \text{for all } f \in L_2(N).$$

If

$$2 \leqslant s \leqslant \frac{2d}{d-1}, \quad d \geqslant 4,$$

then

$$\|E_\mu f\|_{L_s(\partial N)} \leqslant C\mu^{(d-1)/3-(2d-4)/(3s)}\|f\|_{L_2(N)} \qquad \textit{for all } f \in L_2(N).$$

We are now ready to obtain the estimates for $|H_0(\tau)|^{-1/2}$.

Lemma 3.5. *Let*

$$1 \leqslant r < 6 \quad \textit{if } d = 3, \qquad 1 \leqslant r < \frac{2d-4}{d-3} \quad \textit{if } d \geqslant 4.$$

Then there exists a $\delta > 0$ *such that*

$$\||H_0(\tau)|^{-1/2}u\|^2_{L_r(M\times\Omega)} \leqslant C\tau^{-\delta}\|u\|^2_{L_2(M\times\Omega)} \qquad \textit{for all } u \in L_2(M\times\Omega).$$

Proof. Theorem 3.1 with $N = M\times\Omega$ implies that, under the above conditions on r, we have

$$\|E_\mu f\|_{L_r(M\times\Omega)} \leqslant C\mu^{1/2-\delta}\|f\|_{L_2(M\times\Omega)}$$

for some $\delta > 0$. Therefore,

$$\begin{aligned}
\||H_0(\tau)|^{-1/2}u\|_{L_r(M\times\Omega)} &\leqslant \sum_{\mu=1}^{\infty} \|E_\mu|H_0(\tau)|^{-1/2}u\|_{L_r(M\times\Omega)} \\
&\leqslant C\sum_{\mu=1}^{\infty} \mu^{1/2-\delta}\|E_\mu|H_0(\tau)|^{-1/2}u\|_{L_2(M\times\Omega)} \\
&\leqslant C\sum_{\mu=1}^{\infty} \mu^{1/2-\delta}\|E_\mu|H_0(\tau)|^{-1/2}\| \cdot \|E_\mu u\|_{L_2(M\times\Omega)};
\end{aligned}$$

in the last inequality we used the fact that $|H_0(\tau)|^{-1/2}$ commutes with $-\Delta$. Using the Cauchy inequality and Lemma 2.2, we obtain

$$\||H_0(\tau)|^{-1/2}u\|^2_{L_r(M\times\Omega)} \leqslant C\|u\|^2_{L_2(M\times\Omega)}\sum_{\mu=1}^{\infty}\mu^{1-2\delta}\|E_\mu|H_0(\tau)|^{-1/2}\|^2 \leqslant C\tau^{-\delta}\|u\|^2_{L_2(M\times\Omega)}. \qquad \square$$

Lemma 3.6. *Let*

$$1 \leqslant s < \frac{10}{3} \quad \textit{if } d = 3, \qquad 1 \leqslant s < \frac{4d-8}{2d-5} \quad \textit{if } d \geqslant 4.$$

Then there exists a $\delta > 0$ *such that*

$$\||H_0(\tau)|^{-1/2}u\|^2_{L_s(\partial M\times\Omega)} \leqslant C\tau^{-\delta}\|u\|^2_{L_2(M\times\Omega)} \qquad \textit{for all } u \in L_2(M\times\Omega).$$

Proof The argument is similar to that in the proof of Lemma 3.5. Theorem 3.4 with $N = M\times\mathbb{T}^m$ implies that, under the above assumptions on s, we have

$$\|E_\mu f\|_{L_s(\partial M\times\Omega)} \leqslant C\mu^{1/2-\delta}\|f\|_{L_2(M\times\Omega)}$$

for some $\delta > 0$. Therefore, using Lemma 2.2, we obtain

$$\||H_0(\tau)|^{-1/2}u\|^2_{L_s(\partial M\times\Omega)} \leqslant C\bigg(\sum_{\mu=1}^{\infty}\mu^{1/2-\delta}\|E_\mu|H_0(\tau)|^{-1/2}u\|_{L_2(M\times\Omega)}\bigg)^2$$
$$\leqslant C\|u\|^2_{L_2(M\times\Omega)}\sum_{\mu=1}^{\infty}\mu^{1-2\delta}\|E_\mu|H_0(\tau)|^{-1/2}\|^2 \leqslant C\tau^{-\delta}\|u\|^2_{L_2(M\times\Omega)}. \qquad \square$$

Proof of Theorem 1.2. The conditions on the potential V are invariant under adding a constant to V. Therefore, without loss of generality we can assume that $\lambda = 0$. It will be convenient to prove the theorem in the following (equivalent) form: For any $u \in \mathrm{Dom}(H(\tau))$, $\|u\|_{L_2(M\times\Omega)} = 1$, there exists a $v \in \mathrm{Dom}(H(\tau))$, $\|v\|_{L_2(M\times\Omega)} = 1$, such that

$$|(H(\tau)u, v)| \geqslant C\tau, \qquad \tau > \tau_0.$$

Let $H_0(\tau) = \Phi_0(\tau)|H_0(\tau)|$ be the polar decomposition of $H_0(\tau)$. In the basis (2.1), the operator $\Phi_0(\tau)$ is the operator of multiplication by $h_{j,n}(\tau)|h_{j,n}(\tau)|^{-1}$. Let

$$v = \Phi_0(\tau)u.$$

Then

$$(H_0(\tau)u, v) = (|H_0(\tau)|u, u) \geqslant 2\pi\tau,$$

and using (2.2), we obtain

$$(H_0(\tau)u, v) = \||H_0(\tau)|^{1/2}u\|^2_{L_2(M\times\Omega)} = \||H_0(\tau)|^{1/2}v\|^2_{L_2(M\times\Omega)}.$$

We also have

$$|(Vu, v)| \leqslant \|V\|_{L_p(M\times\Omega)}\|u\|_{L_r(M\times\Omega)}\|v\|_{L_r(M\times\Omega)},$$

where $r = \frac{2p}{p-1}$ satisfies the assumptions of Lemma 3.5 by virtue of (0.8). Therefore,

$$|(Vu, v)| \leqslant C\tau^{-\delta}\|V\|_{L_p(M\times\Omega)}\||H_0(\tau)|^{1/2}u\|_{L_2(M\times\Omega)}\||H_0(\tau)|^{1/2}v\|_{L_2(M\times\Omega)}$$
$$= C\tau^{-\delta}\|V\|_{L_p(M\times\Omega)}(H_0(\tau)u, v).$$

Similarly,

$$\left|\int_{\partial M\times\Omega}\sigma u\overline{v}\,dS\right| \leqslant \|\sigma\|_{L_q(\partial M\times\Omega)}\|u\|_{L_s(\partial M\times\Omega)}\|v\|_{L_s(\partial M\times\Omega)},$$

where $s = \frac{2q}{q-1}$ satisfies the assumptions of Lemma 3.6 due to (1.2). Hence

$$\left|\int_{\partial M\times\Omega}\sigma u\overline{v}\,dS\right| \leqslant C\tau^{-\delta}\|\sigma\|_{L_q(\partial M\times\Omega)}(H_0(\tau)u, v).$$

Combining the bounds, we eventually obtain

$$|(H(\tau)u, v)| \geqslant (H_0(\tau)u, v)(1 - C(V, \sigma)\tau^{-\delta}) \geqslant \frac{1}{2}(H_0(\tau)u, v) \geqslant \pi\tau$$

for sufficiently large τ.

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