CONWAY'S WORK ON ITERATION

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ABSTRACT. The paper briefly describes some of John H. Conway's early work related to iteration, in set theory, logic and theory of computation. Topics include his PhD thesis, his book on Regular Algebras and Finite Machines, and his work generalizing the Collatz problem.

In memory of John Horton Conway (1937–2020).

I learned of John H. Conway's work on iteration problems generalizing the Collatz problem in the late 1970's. I first met him while working at the Mathematics Research Center at AT&T Bell Laboratories in the 1980's. John was coming to Bell Labs once a week to work with Neil J. A. Sloane on their book, *Sphere Packings, Lattices and Groups*. At one lunch gathering he explained a group-theoretic restriction on solvability of a particular polyomino tiling problem. This led to joint work ([18]).

At the time I met Conway, he was world-famous. He had constructed new finite simple groups, he had created a theory of (surreal) numbers and games; he had invented new polynomials in knot theory. His personality was outsized, yet he was approachable. I was in awe of him.

John liked to start with simple rules which built up to complicated things. Where did they lead? Sometimes the rules led to complete clear emergent patterns, sometimes to unpredictability and computationally undecidable problems. John liked mathematical objects that looked the same everywhere, having internal structure yet also having large (transitive) symmetry groups. He liked finding new invariants that tell things apart, providing a classification. John liked numbers, and patterns, and experimental computations. He enumerated many knots and links via his theory of tangles, labeling knots with numbers ([9]). With Simon P. Norton, he formulated Monstrous Moonshine, tabulating unexpected patterns of numbers connecting two different fields, the Monster simple group and modular functions, including the j-function ([19]). These patterns were pursued by his student Richard Borcherds, who proved them many years later.

John liked finding a good choice of names, making puns with apt terminology. He replaced FORTRAN with FRACTRAN. To compute (surreal) numbers, play HACKENBUSH; to play the combinatorial game Nim, use *nimbers*. He posed the (as yet unsolved) *thrackle* problem.

This article visits some of John's early work related to iteration. It covers his 1964 PhD thesis on Homogeneous Ordered Sets, his work in the late 1960's on

Regular Algebras, then his 1972 paper motivated by the Collatz problem.

Conway's PhD Thesis

John Conway's thesis supervisor at Cambridge was Harold M. Davenport, a number theorist. Davenport initially suggested that he investigate Waring's problem for fifth powers. Conway avoided working on the topic but eventually delivered a solution to Davenport. According to [47, pp.41-42] Davenport felt it would make a weak thesis. In fact, Davenport had been informed that Jing-Run Chen, a student of Loo-Keng Hua, had just solved this problem, establishing the optimal bound $g(5) = 37$ ([7]).

Conway's response was to write a thesis on a completely different topic, in set theory. It was titled "Homogeneous Ordered Sets", and with it he completed the PhD in 1964.

The thesis extends the work of Cantor and Hausdorff on totally ordered sets (also called linearly ordered sets); Conway terms them "ordered sets". Conway started from several fundamental papers of Hausdorff on ordered sets, written in the period 1904-1908, much of it included in Hausdorff's 1914 monograph [29]; he also cites the 1917 book of Huntington [32]. Hausdorff showed that the set of order types of countable linearly ordered sets have cardinality the power of the continuum; in contrast, the countable well-ordered sets have cardinality $\aleph_1$. Hausdorff showed (assuming the axiom of choice and the generalized continuum hypothesis) the existence of universal linearly ordered sets for each cardinality $\aleph_\alpha$, these being totally ordered sets containing an order-embedded copy of every linearly ordered set of cardinality at most $\aleph_\alpha$ (see [30], [37, Sect. 3]).

Conway's thesis contains a large number of results, requiring extensive terminology; it assumes the axiom of choice. Here we indicate only some of his simplest results.

A totally ordered set H of any cardinality is *homogeneous* if given two elements x, y in H there is an automorphism (i.e. an order-preserving bijection) $\psi : H \to H$ sending $\psi(x) = y$. It is 2-*homogeneous*[1] if given any $x_1 < x_2$ and $y_1 < y_2$ there is such an automorphism $\psi(x_i) = y_i$ for $i = 1, 2$. Totally ordered sets have a complicated structure, depending on the behavior of their "sections" which are partitions $H = \langle L|R \rangle$, where $L < R$, meaning if $\ell \in L, r \in R$, then $\ell < r$, and $L \cup R = H, L \cap R = \emptyset$.) A section may have either no endpoints in H (a "gap") , one endpoint belonging to H, either a left endpoint of R or a right endpoint of L (both are "cuts"), or both a right endpoint L and a left endpoint of R (a "jump").

The Denumerable Homogeneous Sets

In Section 6 of his thesis, Conway classifies countable homogeneous totally ordered sets. The only one that is 2-homogenous is the (order type of the) rationals $(\mathbb{Q}, \leq)$ with its usual order as real numbers (a fact known to Hausdorff). There are uncountably many different order types of countable homogeneous totally ordered

[1] Conway's terminology conflicts with current terminology in model theory, where these concepts would be called 1-o-transitive and 2-o-transitive; the o refers to "order".

sets H, labeled by an arbitrary countable ordinal (hence there are $\aleph_1$ such order types).

The Ruler Sets $\mathbf{R}_\alpha$

In Section 8, Conway defines by transfinite induction a set of "rulers" R_α, for all ordinals α. The "ruler"" construction adds points analogous to Dedekind cuts in the sense that each R_α is a totally ordered set, and at the next stage $R_{\alpha+1}$ it adds all of the sections $< L|R >$ of R_α, while for a limit ordinal β one sets $R_\beta = \bigcup_{\alpha<\beta} R_\alpha$. The thesis has a picture of R_4 (for the finite ordinal $\alpha = 4$) which contains $15 = 2^4 - 1$ elements, labeled 434243414342434, where the numbers indicate the stage at which the new elements are added; one may picture them as marks on a ruler of height $\frac{1}{2^n}$ where n is the label. Conway [8, Sect. 8, Theorem 2] shows that if one stops[2] at ordinals $\gamma = \omega^\beta$ (any ordinal power of ω, the first infinite ordinal) then R_γ is 2-homogeneous. He also shows that for all ordinals α the set R_α is "universal" in the sense of containing embedded copies of all totally ordered order types a of cardinality strictly smaller than that of α. ([8, Sect. 8, Theorem 5]). Furthermore for such a it is *a-homogeneous* in the sense that its order automorphism group acts transitively on all subsets having order type a. (The conclusion is a weaker variant of the "universal" property of Hausdorff, but adding the homogeneity property.)

Intervals in Homogeneous Sets

In Section 10, Conway obtains results on intervals $[x_1, x_2]$ of homogeneous ordered sets. He shows there is an induced total order on order types of intervals in such a set. An order type of one interval I is said to be smaller than that of another interval J if there is an automorphism that moves a copy of I inside a copy of J. A version of the Schröder-Bernstein theorem for ordinals ([8, Sect. 2, Theorems 2 and 3]) implies that if J can also be moved inside a copy of I, then I and J must have the same order type. This ordering (on order types) is a total order because there is always an automorphism to move I so that its left endpoint coincides with the left endpoint of J, and set inclusion then determines which is smaller. He defines an addition operation on the order types of two intervals I and J by moving J by an automorphism to an interval $\sigma(J)$ so that its left endpoint coincides with the right endpoint of I, and defining $I + J$ to be the order type of the interval $I \cup \sigma(J)$; one sees the order type is well-defined. He proves that this addition operation on order-types of intervals is commutative and associative. By lining up copies of an intervals in a row by using automorphisms moving one copy of an interval so its left endpoint coincides with the right endpoint of another copy of the interval, one obtains an order type $I : n$ for the sum of n copies of I. Conway shows using this device one can attach a well-defined ratio of "size" I/J (of order types) to be a nonnegative real number or $+\infty$. Here $I/J = 0$ means I is infinitesimal compared to J, and $I/J = +\infty$ means it is infinitely large, and otherwise the ratio is a well-defined positive real number, obtained by comparing the sizes of $I : m$ and

[2]Theorem 2's hypothesis is: ordinals γ such that $\alpha + \gamma = \gamma$ for all $\alpha < \gamma$.

$J : n$ for various integers $m, n \geq 1$. Conway writes: "The appearance of real numbers 'from nowhere' is rather startling, but it is not as impressive as it looks at first sight, as in most cases the value of I/J is 0 or 1 or ∞; a rather poor selection of real numbers." ([8, p. 62]). For $(\mathbb{Q}, <)$ the ratios are all 1, since by 2-homogeneity all intervals are order-isomorphic.

Conway did not publish his thesis. The work [12] may have been stimulated by it. Work on his thesis helped prepare his mind towards his later discovery of surreal numbers, originally called "numbers" in [13] but renamed "surreal numbers" by Donald Knuth [36]. The class of all surreal numbers $\mathbf{No}$ was constructed by an iterative process proceeding by transfinite induction. The class $\mathbf{No}$ is totally ordered and is homogeneous in a very strong sense: it is (up to isomorphism) the *unique (absolutely) homogeneous universal ordered field* ([22, Theorem 1]). Here *homogeneity* means that every isomorphism between subfields of $\mathbf{No}$ whose universes are sets can be extended to an automorphism of $\mathbf{No}$, and *universality* means every ordered field whose universe is a class in NBG (von Neumann-Bernays-Gödel set theory + AC) can be embedded in $\mathbf{No}$. Other characterizations of $\mathbf{No}$ appear in [20], [21].

Conway's work on Regular Algebras and Finite Machines

Soon after the PhD, in 1966, Conway taught a course on topics in algebra and finite automata. He wrote a book, "Regular Algebra and Finite Machines", published in 1971, growing out of this course. In the book preface he stated that this work stemmed from his interest since 1960 in fundamental work of S. C. Kleene ([35]) on regular languages, and of E. H. Moore ([45]) in recognizing finite automata from their outputs, both of their articles appearing in the 1956 book "Studies in Automata", edited by Claude Shannon and John McCarthy.

In this book Conway reworks earlier results based on his own understanding, making improvements, and introducing his own terminology and notation. Practically every chapter of the book contains new insights, results or problems. Chapters 7 to 11 of the book showcase the work of his first PhD student Donald J. Pilling, presented from Conway's viewpoint; for more on this, see [42]. We focus here on one theme of the book, concerning equational axiom schemes for regular languages, a topic addressed in Chapters 3, 4, 12 and 13. References for formal languages and finite automata include Minsky [44], Eilenberg [23], and Hopcroft and Ullman [31].

Given a finite alphabet $\mathcal{A} = \{a_1., a_2, ..., a_n\}$ let $\mathcal{A}^*$ denote the set of all finite words $\mathbf{w} = a_{i_1}a_{i_2}\cdots a_{i_k}$ with each letter $a_{i_j} \in \mathcal{A}$, including the empty word ε. A *language* L is any subset of $\mathcal{A}^*$, with 0 denoting the empty language and 1 denoting the language containing the empty word, so $1 = \{\epsilon\}$. A *regular language* L is a subset $L \subset \mathcal{A}^*$ accepted by a finite Moore automaton (a kind of non-deterministic finite automaton). (These are often called *rational languages*, see [3], [4]). Let $\mathbf{Reg}(\mathcal{A})$ denote the set of all regular languages in $\mathcal{A}^*$. It is known that $\mathbf{Reg}(\mathcal{A})$ is closed under set union, set intersection, and complement in $\mathcal{A}^*$, so it forms a Boolean algebra of sets. It is closed under the commutative operation, $+$ denoting set union and the noncommutative operation $\cdot$ denoting concatenation of words, with

$$A \cdot B := \{\mathbf{w}_1 \cdot \mathbf{w}_2 : \mathbf{w}_1 \in A \quad \text{and} \quad \mathbf{w}_2 \in B\}.$$

Additionally $\mathbf{Reg}(\mathcal{A})$ is closed under the *Kleene star operation* $*$, a unary operation defined by

$$A^* := 1 + A + A \cdot A + A \cdot A \cdot A + \cdots$$

viewed as an infinite set union of languages. The Kleene $*$-operation encodes the result of an iteration process applied to the words of A, concatenating words of A in any order and in any amount. The $*$-operation can also be characterized as a *least fixed point operator* on languages as a solution to a linear equation: For any E, F, G the set E^*G is the smallest F (in the sense of set inclusion) that satisfies the equation $F = G + E \cdot F$ ([10, p. 27]).

Now consider a universal algebraic structure $\mathbf{X} = \langle 0, 1, +, \cdot, * \rangle$, with two binary operations $+, \cdot$ and a unary operation $*$, with additional constants for each letter in the alphabet $\mathcal{A}$. A *regular expression* E is any well-formed formula in this algebra, for example $E := a(1 + (a^*b)^*)$, with letters $a, b \in \mathcal{A}$. (Such expressions are also termed *rational expressions*.) Each such expression produces a language $L = L(E) \subset \mathcal{A}^*$ obtained by interpreting the operations $+, \cdot, *$ inside the set $\mathcal{A}^*$. *Kleene's theorem* says that the set of all languages in $\mathcal{A}^*$ produced by all regular expressions coincides with the set $\mathbf{Reg}(\mathcal{A})$ of all regular languages.

An important feature of Kleene's theorem is that different regular expressions can yield the same regular language. In Conway's algebraic framework it raises the problem of determining a set of equational axioms on the abstract algebra sufficient to imply exactly the equalities of regular expressions that specify the same regular language; here we call any such equation a *regular identity*. Conway introduced a set of "classical axioms" which are regular identities. The (equational) axioms $(C1) - (C10)$ for the two operations $+$ and $\cdot$ make the structure $\mathbf{X}$ a semiring, with 0 being the additive unit and 1 the multiplicative unit. Under addition, this structure is a commutative monoid (semigroup with identity) that need not be a group.

Further equational axioms address properties of the Kleene $*$-operation with respect to the other two operations, thus specifying properties of iteration. Conway introduced the following three axioms for the $*$-operation:

$$(C11) \qquad (A + B)^* = (A^*B)^*A^* \qquad \text{(sumstar)}$$

$$(C12) \qquad (AB)^* = 1 + A(BA)^*B \qquad \text{(productstar)}$$

$$(C13) \qquad (A^*)^* = A^* \qquad \text{(starstar)}$$

Conway called any structure satisfying the finite set of axioms $(C1) - (C13)$ an $\mathbf{A}$-*algebra*. The idempotency of addition $A + A = A$ is deducible from axioms $(C1) - (C13)$, showing that every $\mathbf{A}$-algebra is a Boolean semiring.

Conway deduced consequences of these axioms. He showed that square matrices of fixed size having entries in an $\mathbf{A}$-algebra, with their usual $+$ and $\cdot$ operations, could be endowed with $*$-operations that made it into an $\mathbf{A}$-algebra.

Redko [46] proved in 1964 that infinitely many equational axioms are needed to generate all regular identities. Therefore the finite set of axioms $(C1) - (C13)$ cannot characterize all regular identities. Conway included a further infinite axiom schema $(C14)$ as part of his “classical axioms", which is:

$$(C14.n) \qquad A^* = A^{n*}A^{<n} \qquad \text{(powerstar)}$$

for $n \geq 2$, with $A^{n*} = (A^n)^*$ and $A^{<n} = 1 + A + A^2 + \cdots + A^{n-1}$, where $A^2 = A \cdot A$, etc. He improved on Redko's result by showing that any complete equational axiom system necessarily must contain infinitely many equational axioms involving two or more variables ([10, p. 118]). It follows that the “classical axioms" $(C1) - (C14)$ cannot be a complete system for regular languages.

In addition, Conway also constructed three further different infinite systems of equational identities satisfied by regular expressions. At the end of Chapter 13 he conjectured that adding each of these systems as axiom systems separately to $(C1) - (C13)$ would result in a complete set of axioms for regular equational identities. One of his first two sets of equational identities quantifies over finite monoids, the second quantifies over all finite simple groups. Conway's third conjecture ([10, p. 118] and [38, p.210]) asserts that $(C1) - (C14)$ together with the infinite family of two-letter identities $(R(n))_{n=2}^{\infty}$ is complete for equations satisfied by regular languages, where

$$(R(n)) \quad (A{+}B)^* = [(A{+}B)(B{+}(AB^*)^{n-2}A)]^* \left(1 + (A+B)\left(\sum_{i=0}^{n-2}(AB^*)^i\right)\right).$$

Conway's book influenced later developments in theoretical computer science. These include:

(1) Conway's axiomatization of a general class of semirings, assuming some subset of axioms $(C1) - (C14)$, proved useful in modeling other kinds of iteration properties in computer science. Bloom and Ésik developed in many papers a theory of *iteration algebras*, presented in book form in [5]. They introduced a notion of *Conway semiring*, which includes as axioms $(C1) - (C10)$ together with certain *Conway identities* for the $*$-operator, including $(C11), (C12)$.
(2) D. Krob [38] proved the first two of Conway's conjectures on completeness of equational axioms for regular languages in 1991. Krob's results were further generalized by Ésik [24], [25]. (Conway's third conjecture seems not to be resolved.)

Conway's work related to the Collatz problem

The Collatz problem concerns the iteration of the *Collatz function*

$$C(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ 3n+1 & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

for positive integers $n \in \mathbb{N}^+$. Collatz conjectured that some iterate $C^{(k)}(n) = 1$, where k depends on n.

Conway reportedly heard of the Collatz problem as an undergraduate at Cambridge University (B.A. 1959). Richard K. Guy told me his son Michael Guy mentioned the Collatz problem to him in the early 1960's when Michael was a student at Cambridge University with Conway, cf. [41], pp. 293-294. For history and work on the Collatz problem, which remains unsolved, see the survey paper [39] and papers in the volume [41], including [6] and [40].

The main result of the 1972 Conway paper ([11]) states:

Theorem. *If $f(n)$ is any computable function, there is a function g such that:*

(1) $\frac{g(n)}{n}$ *is periodic (with rational values)*

(2) $2^{f(n)} = g^{(k)}(2^n)$*, where $k \geq 1$ is the minimal positive value subject to $g^{(k)}(2^n)$ a power of* 2.

The maps $g(n) = a_i n$ are linear maps on each congruence class $i \pmod{m}$ where m is the minimal period given in (1). The iteration $n \mapsto g(n)$ must an integer n to an integer $g(n)$ in order to continue iterating the function g. In consequence the rational number $\frac{g(n)}{n}$ must have its denominator dividing $\gcd(n, m)$, in order for $g(n)$ to be an integer.

The interesting behavior of the iteration appears on restricting the input to positive integers that are powers of primes dividing m, i.e. $n = \prod_{p|m} p^{e_p}$ where each $e_p \geq 0$. Now each exponent e_p serves as an infinite"register", and the computation is really done on the vector of exponents of the set of primes dividing m.

In Conway's theorem, the function $f(\cdot)$ is undefined at values n where there is no $k \geq 1$ with an iterate $g^{(k)}(n)$ that is a power of 2. Here "computable function" means "partial recursive function." In this regard, Conway states the following corollary.

Corollary. *There is no algorithm, which, given a function g with $\frac{g(n)}{n}$ periodic, and given a number n, determines whether or not there is k with $g^{(k)}(n) = 1$. The word "computable" will mean "computable by a Minsky program". This is equivalent to (partial) recursive.*

Conway's paper has no references. It may be helpful to note that a "Minsky program" is one designed for the "counter machines" described in Minsky [44, Section 11.1]. Minsky had previously introduced this machine model in 1961 to show the unsolvability of Post's 'tag' problem ([43]). Since Conway's paper is less than 3 pages, much is left to the reader.

Conway concludes the paper saying: "It is amusing to note that the Theorem contains the Kleene Normal Form Theorem for recursive functions, since the functions $g(n), 2^n$ etc. are obviously primitive recursive." He may have had in mind the one-variable case of the Kleene Normal Form Theorem ([34, Sec. 63, Theorem XIX]). Conway's paper may be summarized thus: Iteration of very simple arithmetic functions can encode universal computation.

Conway returned to this method of computation by iteration in 1987, describing it in [15]. He found a more compact way to describe the computation, describing a subclass of functions $g(n)$ specified by a finite ordered set of positive rational fractions $\{a_i : 1 \le i \le k\}$. For each input integer n one successively multiplies the integer by a_i and accepts the first i such that $a_i n$ is an integer, giving the next value of the iteration. If none of the $a_i n$ are integers, then the computation stops, and the output is undefined. This use of fractions supplied the "program" for computing the function $g(n)$, leading to his name FRACTRAN. He gives a proof that this subclass of functions is sufficient for universal computation.

Richard Guy [26] gave an example of a FRACTRAN program for computing the primes p_k in successive order; given $n = 2^{p_k}$ as input, the first power of 2 it would reach would be $2^{p_{k+1}}$. This program was later titled: "Fourteen fruitful fractions" in their "Book of Numbers" ([17, pp. 147–148]).

The class of maps $g(n)$ does not include the Collatz function, since $x \mapsto 3x + 1$ is an affine map rather than a linear map. The affine map feature has important consequences for the dynamics of iterating the Collatz map. By combining two steps of the Collatz iteration for odd n, the Collatz problem can be rephrased in terms of the $3n + 1$ map,

$$T(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{3n+1}{2} & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

The $3x + 1$ map T extends to a (piecewise affine) map on the 2-adic integers $\mathbb{Z}_2$, and the extended map $T : \mathbb{Z}_2 \to \mathbb{Z}_2$ is a measure-preserving map with respect to the 2-adic measure. This map was shown in [2] to be topologically and metrically conjugate to the one-sided shift map

$$S(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{n-1}{2} & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

which implies that $T(n)$ is ergodic and strongly mixing. The dynamics of $S(\cdot)$ on the nonnegative integers is completely understood; it has $n = 0$ as an absorbing fixed point for all starting values n. The difficulty of the Collatz conjecture is encoded in properties of the 2-adic automorphism doing the conjugation. It maps all non-zero integers to non-integers, conjecturally to rational numbers, cf. [2].

In particular the ergodic property permits "probabilistic" predictions about the behavior of almost all orbits. For the map T it predicts "generic" orbits on $\mathbb{Z}_2$ will have half their iterates odd and half their iterates even, with all finite patterns of even and odd iterates uniformly distributed in the successive iterates. However the positive integers $\mathbb{N}^+$ form a (dense) set of measure zero inside $\mathbb{Z}_2$, and ergodic theory gives no information about its behavior on the iteration on measure zero subsets of $\mathbb{Z}_2$.

In his 2013 paper, "On unsettleable questions" Conway ([16]) expressed the viewpoint that problems like the Collatz problem might be undecidable, and yet not be provably undecidable. He says: "For some of my examples, it might even be that the assertion that they are not provable is not itself provable, and so on."

([16, p. 192]). His examples include the Collatz problem and also a permutation of $\mathbb{N}^+$ which he terms the "*amusical permutation*":

$$h(n) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \,(\mathrm{mod}\, 2) \\ \frac{3n+1}{4} & \text{if } n \equiv 1 \,(\mathrm{mod}\, 4) \\ \frac{3n-1}{4} & \text{if } n \equiv 3 \,(\mathrm{mod}\, 4). \end{cases}$$

The map defines a permutation of $\mathbb{Z}$ which leaves $\mathbb{N}^+$ invariant, whose inverse permutation is

$$g(n) = \begin{cases} \frac{2n}{3} & \text{if } n \equiv 0 \,(\mathrm{mod}\, 3) \\ \frac{4n-1}{3} & \text{if } n \equiv 1 \,(\mathrm{mod}\, 3) \\ \frac{4n+1}{3} & \text{if } n \equiv 2 \,(\mathrm{mod}\, 3). \end{cases}$$

The question is: "*Is the orbit of* $n = 8$ *infinite?*" If it is infinite, then it is two-sided infinite, so the question for $g(\cdot)$ or $h(\cdot)$ is the same. The numerical evidence shows steady growth (on average) of the orbit in both directions on a logarithmic scale, up to 10^{400}, given in Figure 1 of the paper. There are (separate) probabilistic models of the type above for the expected exponential growth rate of an infinite orbit, for the forward orbit (iterating $h(n)$) and the backwards orbit (iterating $g(n)$). The expected growth rate per step of an infinite forward orbit of $h(n)$ is $\sqrt{\frac{9}{8}} \approx 1.06066$, while the expected growth rate per step of an infinite backwards orbit is $\sqrt{\frac{32}{27}} \approx 1.08866$. This difference in exponential growth rates depending on the direction of iteration is the "amusical" property motivating Conway's name (which is also a pun on "amusing"). Experimental data for $n = 8$ agrees with both these predictions.

The "amusical permutation" $h(n)$ has a prehistory. In 1963 Murray Klamkin posed study of the iteration properties of $h(n)$ as an (unsolved) SIAM Review problem ([33]). Klamkin raised the question of whether the orbit of $n = 8$ is infinite, and whether the four known finite cycles on $\mathbb{N}^+$ (with starting points $n = 1, 2, 4, 44$) were the complete list of finite cycles on $\mathbb{N}^+$. Later comments on it were supplied by Daniel Shanks [48] and A. O. L. Atkin [1]. The permutation $g(n)$ was formulated by Collatz in 1932 in his personal notebooks, and study of its iteration was termed the "*original Collatz problem*" in [39, p. 3]).

Concerning the ergodic-theoretic or probabilistic arguments above, which apply to the $3x + 1$ function, Conway [16, p. 194] suggests the new terminology *probvious*, an abbreviation for "probabilistically obvious," for the behavior of such orbits. However he allows the problem to simultaneously be undecidable. So "probvious" does not imply "obvious". We arrive at the dictum of Conway's long-time coauthor Richard K. Guy: "Don't try to solve these problems" ([27]).

In a conversation, Conway raised the question (or expressed the opinion) whether there might exist a piecewise affine map $U(n)$ in the spirit of $T(n)$ which extends to a map $U(\cdot)$ which is provably measure-preserving and ergodic in a similar sense (say on the m-adic integers $\mathbb{Z}_m$ for some integer $m \geq 2$) for which analyzing the long-time features of the iteration restricted to the positive integers $\mathbb{N}^+$ is *provably* undecidable. Candidate questions about long-time features might be: ' Is there *any* periodic orbit?" or "Is there an infinite orbit?"

Concluding Remarks

Conway worked on iteration of many other integer sequences, including his famous "Look and say" sequence ([14]). The simple "look and say" rule, which encodes a pun, seems to have no a priori reason for interesting structure to emerge. Nevertheless Conway showed it leads to a quite complicated but exactly analyzable recursion. In later years he explored many other sequences, often leaving further analysis to others ([28]).

This article reviewed early work of Conway related to iteration and computation. Conway stated key ideas clearly and simply and formulated precise questions and conjectures, leading to further work. He formulated new notation, and was terse, making demands on the reader. He also gave explcit algorithms, computations and examples. Conway's early work contains clues how he came to discover entirely new worlds of mathematics.

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